hexagon:2611.00185v1

# SINGLETON DOMINANCE REGIONS IN AFFINE CLUSTER ALGEBRAS

QIYUE TANG

Abstract. For an extended exchange matrix of affine type, the dominance region of every point in an extended **g**-vector cone consists of that point alone.

## 1. Introduction

Fomin and Zelevinsky introduced cluster algebras to study canonical bases and total positivity [1]. A seed records a cluster of mutable variables, the frozen variables, and an extended exchange matrix. The columns of this matrix record differences of exponent vectors in the exchange relations. In their work on coefficients, Fomin and Zelevinsky introduced principal coefficients and the grading that defines the **g**-vectors of cluster variables [2, §§3 and 6]. The nonnegative span of the **g**-vectors of the variables in a cluster is its **g**-vector cone.

Qin used the columns of the extended exchange matrix to compare Laurent monomials in a seed [5, Definition 3.1.1]. An exponent is dominated by a second exponent if the first minus the second is a nonnegative integer combination of the columns. A Laurent polynomial pointed at a given exponent has coefficient one there and all its remaining exponents are dominated by it. To compare exponents across seeds, one transports them by piecewise-linear mutation and imposes the comparison in each seed. In the classical setting, Qin's classification of compatibly pointed bases for upper cluster algebras whose extended matrix has full column rank and admits a green-to-red sequence uses this simultaneous order to specify the possible lower terms in a change of basis [6, Theorem 1.2.1]. Such a sequence ends at a seed whose principal coefficient exponent vectors are all nonpositive.

Rupel and Stella turned the simultaneous comparison into a geometric problem by using real nonnegative combinations of the columns [12, §3]. In each seed, translate the column cone by the coordinates of the base point in that seed and pull it back to the initial seed. The intersection of these pullbacks is the dominance region. In coefficient-free rank two, they computed the regions explicitly. In affine rank two, for integer base points, the region is a point on the support of the **g**-vector fan and a line segment at a nonzero base point on the missing ray [12, Theorem 1.2].

Reading and Speyer identified the **g**-vector fan of an acyclic affine seed with the doubled Cambrian fan. Every point outside its support, the union of its cones, lies in the hyperplane annihilating the null root [8, Corollaries 1.3 and 4.9]. Reading and Stella later gave an affine root model for the cluster fan [9] and identified the cluster scattering fan with the mutation fan in affine type [10, Theorem 2.10]. The complement of the support of the **g**-vector fan is the relative interior of the imaginary wall, a closed cone of codimension one.

Reading, Rupel, and Stella computed the dominance regions of points whose mutable coordinates lie in the relative interior of the imaginary wall [7, Theorems 4.1 and 4.56]. This computation also determines coefficients in products of theta functions indexed by points of the wall [11, §4]. For points whose mutable coordinates lie in a **g**-vector cone, Reading, Rupel, and Stella proved the singleton conclusion when the extended exchange matrix has linearly independent columns [7, Theorem 4.26]. Theorem 2.1 proves their Probable Theorem 4.29, which extends the singleton conclusion to arbitrary coefficient extensions.

## 2. Mutation and dominance regions

Let $m$ and $n$ be integers with $m \geq n \geq 2$. An extended exchange matrix is an integer matrix $\widetilde{B} = (b_{ij})$ of size $m \times n$ whose first $n$ rows form a skew-symmetrizable matrix $B$. Thus $DB$ is skew-symmetric for some positive diagonal matrix $D$. The indices $1, \ldots, n$ are mutable and the indices $n+1, \ldots, m$ are frozen. For $s \in \mathbb{R}$, set $[s]_+ = \max(s, 0)$. For a mutable index $k$, mutation

*Date*: November 7, 2026.

2020 *Mathematics Subject Classification.* Primary 13F60. Secondary 05E40.

*Key words and phrases.* Cluster algebras, affine type, dominance regions, **g**-vector fans.

gives $\widetilde{B}' = \mu_k(\widetilde{B})$ with entries $b'_{ij}$, for $1 \le i \le m$ and $1 \le j \le n$, given by

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & i \ne k,\ j \ne k. \end{cases} \tag{1}$$

Write $T$ as a superscript for matrix transpose. Applying (1) to a real column $a \in \mathbb{R}^m$ appended to $\widetilde{B}$ gives the map $\eta_k^{\widetilde{B}^T} : \mathbb{R}^m \to \mathbb{R}^m$ with

$$(\eta_k^{\widetilde{B}^T}(a))_i = \begin{cases} -a_k, & i = k, \\ a_i + [b_{ik}]_+[a_k]_+ - [-b_{ik}]_+[-a_k]_+, & i \ne k. \end{cases} \tag{2}$$

Mutation preserves the skew-symmetrizer of the principal part. Indeed, for $i, j \ne k$ in the principal part, the correction in (1) multiplied by $D_{ii}$ is

$$D_{kk}\big([-b_{ki}]_+[b_{kj}]_+ - [b_{ki}]_+[-b_{kj}]_+\big),$$

by $D_{ii}b_{ik} = -D_{kk}b_{ki}$. This expression is antisymmetric in $i, j$. The entries in row or column $k$ are negated. A second mutation at $k$ restores the original matrix and column. The sign changes reverse, and in the other entries the two correction terms interchange because $b_{ik}$, $b_{kj}$, and $a_k$ have changed sign. For a square exchange matrix, transposition commutes with mutation by interchanging the indices $i$ and $j$ in (1). Negation also commutes with mutation, since it interchanges the two correction terms and negates the entries of row and column $k$.

A word $w = k_1 \cdots k_r$, with $r \ge 0$ and $k_i \in \{1, \dots, n\}$, specifies successive mutations in that order. Write $\widetilde{B}_w = \mu_w(\widetilde{B})$ for the resulting extended matrix, $B_w = \mu_w(B)$ for its principal part, and $\eta_w^{\widetilde{B}^T}$ for the composition of column mutations using the current matrix at each step. The empty word acts as the identity. The inverse column map is the map for the reversed word starting at $\widetilde{B}_w$. If $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection to the mutable coordinates, (2) gives $\pi\eta_k^{\widetilde{B}^T} = \eta_k^{B^T}\pi$, since the first $n$ output coordinates use only the first $n$ rows and $a_k$. Iterating gives $\pi\eta_w^{\widetilde{B}^T} = \eta_w^{B^T}\pi$.

Write $\mathbb{R}^n_{\ge 0}$ for the nonnegative orthant. The dominance region of $\widetilde{\lambda} \in \mathbb{R}^m$ is

$$\mathcal{P}_{\widetilde{\lambda}}^{\widetilde{B}} = \big\{\widetilde{\mu} \in \mathbb{R}^m : \eta_w^{\widetilde{B}^T}(\widetilde{\mu}) - \eta_w^{\widetilde{B}^T}(\widetilde{\lambda}) \in \widetilde{B}_w\mathbb{R}^n_{\ge 0} \text{ for every word } w\big\}.$$

We use the same definition in the square case $m = n$. Projection gives $\pi(\mathcal{P}_{\widetilde{\lambda}}^{\widetilde{B}}) \subseteq \mathcal{P}_{\pi(\widetilde{\lambda})}^{B}$, since it commutes with the column maps and sends $\widetilde{B}_w\alpha$ to $B_w\alpha$ for every $\alpha \in \mathbb{R}^n_{\ge 0}$. Moreover,

$$\eta_w^{\widetilde{B}^T}(\mathcal{P}_{\widetilde{\lambda}}^{\widetilde{B}}) = \mathcal{P}^{\widetilde{B}_w}_{\eta_w^{\widetilde{B}^T}(\widetilde{\lambda})}. \tag{3}$$

Indeed, each test after mutation along $w$ is a test for a word beginning with $w$ in the original region. Applying the same inclusion with the reversed word gives equality.

For an $n \times n$ exchange matrix $A$, index the principal-coefficient cluster pattern by the $n$-regular tree with one edge of each mutable label at each vertex. At the initial vertex $t_0$ the extended matrix is $\binom{A}{I_n}$, where $I_n$ is the $n \times n$ identity matrix. The initial mutable variables $x_1, \dots, x_n$ and frozen variables $y_1, \dots, y_n$ are independent indeterminates over $\mathbb{Q}$. The exchange relations are taken in their rational function field. If the current extended matrix has entries $a_{ij}$ and the current variables are $X_1, \dots, X_{2n}$, with $X_{n+i} = y_i$ for $1 \le i \le n$, mutation at $k$ changes the matrix by (1) and replaces only $X_k$, by $X'_k$ given by

$$X_kX'_k = \prod_{i=1}^{2n} X_i^{[a_{ik}]_+} + \prod_{i=1}^{2n} X_i^{[-a_{ik}]_+}.$$

Give $x_i$ degree $e_i$, the $i$th standard basis vector of $\mathbb{Z}^n$, and give $y_i$ degree minus column $i$ of $A$. The degree of a nonzero quotient of homogeneous polynomials is the numerator degree minus the denominator degree. Equal fractions have equal cross products, so this degree is well-defined. For the current variables write $d_i = \deg X_i$. Initially the degree balance $\sum_{i=1}^{2n} a_{ij}d_i = 0$ holds for each $1 \le j \le n$, since the first $n$ summands give column $j$ of $A$ and the frozen summands give its negative. If it holds at a seed, the two exchange monomials for mutation at $k$ have the same degree

$$h = \sum_{i=1}^{2n} [a_{ik}]_+ d_i = \sum_{i=1}^{2n} [-a_{ik}]_+ d_i.$$

The new variable has degree $h - d_k$. The new degree balance in column $k$ is the negative of the old one, because $a_{kk} = 0$. For $j \ne k$, the correction terms in the rows other than $k$ sum to

$[a_{kj}]_+h - [-a_{kj}]_+h$, since $a_{kk} = 0$. Adding the new term in row $k$ gives

$$\sum_{i\neq k} a_{ij}d_i + [a_{kj}]_+h - [-a_{kj}]_+h - a_{kj}(h - d_k) = \sum_{i=1}^{2n} a_{ij}d_i = 0.$$

This proves homogeneity of every cluster variable. Its degree is its **g**-vector, and $G_t^{A;t_0}$ has these vectors as columns in mutable order.

Let $A_t$ and $C_t^{A;t_0}$ denote the upper and lower $n \times n$ blocks, respectively, of the extended matrix at $t$ in this pattern. The latter is the **c**-matrix. The degree balance in blocks reads

$$G_t^{A;t_0} A_t = A C_t^{A;t_0}. \tag{4}$$

The principal $F$-polynomial of the variable at position $i$ of $t$ is its specialization at $x_1 = \cdots = x_n = 1$ with $y_1, \ldots, y_n$ retained. This specialization is a polynomial by [2, Proposition 3.6]. For the principal part $B$ of $\widetilde{B}$, the extended **g**-vector cone at $t$ is

$$\mathrm{Cone}_t^{\widetilde{B};t_0} = \pi^{-1}(G_t^{B;t_0}\mathbb{R}^n_{\geq 0}).$$

Here $\pi^{-1}$ denotes inverse image under the coordinate projection, and $G_t^{B;t_0}$ comes from the principal-coefficient pattern for $B$. The support of the **g**-vector fan for $A$, written $|\mathcal{G}(A)|$, is the union of $G_t^{A;t_0}\mathbb{R}^n_{\geq 0}$ over all $t$.

The directed graph of $B$ has vertices $1, \ldots, n$ and an arrow $i \to j$ when $b_{ij} > 0$. A source has no incoming arrows, and a sink has no outgoing arrows. Call $B$ acyclic if this graph has no directed cycle. The Cartan companion of $B$ has diagonal entries 2 and off-diagonal entries $-|b_{ij}|$. A connected Cartan companion is affine if a positive diagonal left multiplier makes it symmetric positive semidefinite of corank one. An exchange matrix is of affine type if a mutation word takes it to an acyclic matrix with connected affine Cartan companion.

**Theorem 2.1.** *Suppose that the principal part $B$ of $\widetilde{B}$ is of affine type. If $\widetilde{\lambda} \in \mathrm{Cone}_t^{\widetilde{B};t_0}$ for some seed $t$, then $\mathcal{P}_{\widetilde{\lambda}}^{\widetilde{B}} = \{\widetilde{\lambda}\}$.*

## 3. Separation by the transpose fan

Fix a square exchange matrix $B$. Use the same rooted tree for the principal-coefficient patterns of $B$, $B^T$, and $-B$. For a vertex $t$, let $w$ be the word of edge labels on the path from $t_0$ to $t$ and write $B_t = B_w$. Let $F_{i;t}$, $1 \leq i \leq n$, be the principal $F$-polynomials for $B^T$. Specializing Laurent positivity at $x_1 = \cdots = x_n = 1$ gives nonnegative coefficients [3, Theorem 4.10]. The columns of each **c**-matrix are sign-coherent, meaning that each is entrywise nonnegative or entrywise nonpositive [3, Corollary 5.5]. It follows from [2, Proposition 5.6] that each $F$-polynomial has constant term 1.

For $a \in \mathbb{R}^n$, define the weight of a nonzero polynomial with nonnegative coefficients in $y_1, \ldots, y_n$ to be the minimum of $\sum_{j=1}^n u_j a_j$ over the exponent vectors $u$ with nonzero coefficient. The weight of a quotient of two such polynomials is the difference of their weights. This is well-defined: equal quotients give equal cross products, and the minimum weight of a product is the sum of the minima because the coefficients are nonnegative. The weight of a sum is the minimum of the two weights, first for polynomials and then for quotients by using a common denominator. Write $f_t(a) \in \mathbb{R}^n$ for the vector of weights of $F_{i;t}$, in mutable order. The constant term gives $f_t(a) \leq 0$ componentwise for every $a$. If $a \in \mathbb{R}^n_{\geq 0}$, every monomial has nonnegative weight, so $f_t(a) = 0$.

Define the $Y$-variables for $B^T$ by starting with $Y_j = y_j$ for $1 \leq j \leq n$ and using the rational mutation rule [2, Eq. (2.3)]. At a step where the current principal matrix for the $B$-pattern has entries $b_{ij}$, the $B^T$-pattern has transposed entries, and mutation at $k$ gives

$$Y_k' = Y_k^{-1}, \qquad Y_j' = Y_j Y_k^{[b_{jk}]_+}(1 + Y_k)^{-b_{jk}} \quad (j \neq k).$$

If $a_j$ are their current weights, the new weights are $-a_k$ at $k$ and

$$a_j + [b_{jk}]_+ a_k - b_{jk}\min(0, a_k) = a_j + [b_{jk}]_+[a_k]_+ - [-b_{jk}]_+[-a_k]_+ \qquad (j \neq k).$$

The equality uses $\min(0, a_k) = -[-a_k]_+$ and $s = [s]_+ - [-s]_+$. This is (2), so the weights at the end of $w$ are the entries of $\eta_w^{B^T}(a)$ for initial weights $a$.

The $Y$-separation formula [2, Proposition 3.13 and Remark 3.14] expresses the $j$th $Y$-variable at $t$ as

$$\prod_{i=1}^n y_i^{(C_t^{B^T;t_0})_{ij}} \prod_{i=1}^n F_{i;t}^{(B_t)_{ji}}.$$

The coefficient monomial has exponent vector column $j$ of $C_t^{B^T;t_0}$ [2, Eq. (3.14)]. The exponents of the $F$-polynomials are column $j$ of $B_t^T$. Taking weights gives

$$\eta_w^{B^T}(a) = (C_t^{B^T;t_0})^T a + B_t f_t(a). \tag{5}$$

Tropical duality gives the two identities

$$B_t = (C_t^{B^T;t_0})^T B C_t^{-B;t_0}, \qquad (G_t^{B^T;t_0})^{-1} = (C_t^{-B;t_0})^T. \tag{6}$$

With the sign coherence above, Nakanishi and Zelevinsky [4, Theorem 1.2] give $(G_t^{A;t_0})^{-1} = (C_t^{-A^T;t_0})^T$ for the pattern of any initial exchange matrix $A$. Applied to $B^T$, this is the second identity. Applied to $-B$, it gives $(G_t^{-B;t_0})^{-1} = (C_t^{B^T;t_0})^T$. Equation (4) in the $-B$-pattern gives $G_t^{-B;t_0} B_t = B C_t^{-B;t_0}$ after cancelling the two minus signs. Multiplication on the left by $(C_t^{B^T;t_0})^T$ gives the first identity.

Re-root the square exchange-matrix pattern at $t$, where its matrix is $B_t$, and take principal coefficients there. In that pattern the $F$-term of (5) vanishes on $\mathbb{R}^n_{\geq 0}$. Hence the column map along the reverse path from $t$ to $t_0$ acts there by $(C_{t_0}^{B_t^T;t})^T$. This matrix equals $G_t^{B;t_0}$ by [4, Eq. (1.13)]. Thus the cone at $t$ is the image of the orthant at $t$ under this reverse path map. If the initial seed is moved from $t_0$ across the edge labelled $k$, compose the map from $t$ to $t_0$ with the map along that edge. Adjacent traversals of the same edge cancel by involutivity, so the result is the map along the path from $t$ to the new initial seed. Hence $\eta_k^{B^T}$ carries each cone to the cone of the same seed in the new coordinates. In particular the map from $t_0$ to $t$ takes the cone of $t$ to its initial orthant.

**Lemma 3.1.** *If $\{z \in \mathbb{R}^n : B^T z \in |\mathcal{G}(B^T)|\}$ is dense in $\mathbb{R}^n$, then $\mathcal{P}_\lambda^B = \{\lambda\}$ for every $\lambda \in \mathbb{R}^n_{\geq 0}$.*

*Proof.* Let $\lambda \in \mathbb{R}^n_{\geq 0}$ and $\mu \in \mathcal{P}_\lambda^B$. For the word $w$ from $t_0$ to $t$, membership gives

$$\eta_w^{B^T}(\mu) - \eta_w^{B^T}(\lambda) = B_t \alpha_t \quad \text{for some } \alpha_t \in \mathbb{R}^n_{\geq 0}.$$

Since $f_t(\lambda) = 0$, subtracting the two instances of (5) gives

$$(C_t^{B^T;t_0})^T(\mu - \lambda) = B_t(\alpha_t - f_t(\mu)).$$

Substitute the first identity of (6) on the right and multiply on the left by $((C_t^{B^T;t_0})^T)^{-1}$. This gives

$$\mu - \lambda = B C_t^{-B;t_0}(\alpha_t - f_t(\mu)).$$

If $B^T z$ lies in $G_t^{B^T;t_0}\mathbb{R}^n_{\geq 0}$, the second identity of (6) gives $(C_t^{-B;t_0})^T B^T z \in \mathbb{R}^n_{\geq 0}$. Since $\alpha_t - f_t(\mu) \in \mathbb{R}^n_{\geq 0}$, we obtain

$$z^T(\mu - \lambda) = (\alpha_t - f_t(\mu))^T (C_t^{-B;t_0})^T B^T z \geq 0.$$

Such $z$ form a dense set by hypothesis. Continuity gives the inequality at $z = \lambda - \mu$, so the sum of the squares of the coordinates of $\mu - \lambda$ is at most zero. Thus $\mu = \lambda$. The point $\lambda$ satisfies every dominance test with difference zero. □

**Lemma 3.2.** *For every exchange matrix $B$ of affine type, the set $\{z \in \mathbb{R}^n : B^T z \in |\mathcal{G}(B^T)|\}$ is dense.*

*Proof.* First let $B$ be acyclic with affine Cartan companion $A$. Choose a positive diagonal matrix $D$ such that $DA$ is symmetric positive semidefinite of corank one. Since $D_{ii}|b_{ij}| = D_{jj}|b_{ji}|$ and the nonzero entries $b_{ij}, b_{ji}$ have opposite signs, $DB$ is skew-symmetric. Gaussian elimination over $\mathbb{Q}$ gives a nonzero rational vector $v \in \ker A$ and let $|v|$ denote its coordinatewise absolute value. Since $DA$ is symmetric positive semidefinite with nonpositive off-diagonal entries,

$$0 \leq |v|^T DA|v| \leq v^T DAv = 0.$$

Diagonalization of $DA$ gives $DA|v| = 0$, hence $A|v| = 0$. A zero coordinate in $|v|$ forces its neighbors to have zero coordinates, since the off-diagonal summands in that row of $A|v|$ all have the same weak sign. Connectedness would then give $|v| = 0$. Thus $|v|$ is positive, and clearing denominators and dividing by the greatest common divisor gives the primitive positive integer vector $\delta$ spanning $\ker A$.

Transposition reverses all arrows, so $B^T$ is acyclic. Its Cartan companion is $A^T$, and $D^{-1}A^T = D^{-1}(DA)D^{-1}$ is positive semidefinite of corank one by congruence. The graph is still connected. As $A^T D = DA$, the vector $D\delta$ spans the kernel of $A^T$. The affine Cambrian fan theorem of Reading and Speyer places the complement of $|\mathcal{G}(B^T)|$ in the hyperplane annihilating the null root for $A^T$

[8, Corollaries 1.3 and 4.9]. In simple-root coordinates that null root is proportional to $D\delta$. For $A^T$, let $\alpha_i$ be the simple roots. With the symmetrizer $D^{-1}$, the simple coroots are $D_{ii}\alpha_i$, up to a common scalar [8, §§2.1–2.2]. The vector $\sum_i (D\delta)_i \alpha_i$ therefore has coroot coordinates proportional to $\delta$. The **g**-vector coordinates are in the basis dual to the simple coroots. Thus the hyperplane in these coordinates is $\{a \in \mathbb{R}^n : a^T\delta = 0\}$.

Its inverse image under $z \mapsto B^T z$ is $\{z : z^T B\delta = 0\}$. A source of the connected acyclic graph of $B$ gives a nonzero row with all entries nonnegative. Its product with the strictly positive vector $\delta$ is positive, so $B\delta \neq 0$. For $z$ on this hyperplane and any nonzero $\varepsilon \in \mathbb{R}$, the point $z + \varepsilon B\delta$ lies outside it, because $(B\delta)^T B\delta > 0$. Thus its complement is dense and is contained in the set in the lemma.

To transport density, let $B' = \mu_k(B)$ and define $T_k^B : \mathbb{R}^n \to \mathbb{R}^n$ by leaving $z_j$ fixed for $j \neq k$ and setting

$$(T_k^B(z))_k = -z_k + \min\left(\sum_{i=1}^n [b_{ik}]_+ z_i,\ \sum_{i=1}^n [-b_{ik}]_+ z_i\right).$$

The sums use only coordinates other than $k$ because $b_{kk} = 0$. Mutation negates column $k$, so $T_k^{B'}$ interchanges the sums and applies the same reflection to coordinate $k$. Thus the two continuous maps are inverse.

Write $P$ and $N$ for the two sums, in order, and put $a = B^T z$, so $a_k = P - N$. The $k$th entry of $B'^T T_k^B(z)$ is $\sum_{i\neq k}(-b_{ik})z_i = -a_k$. For $j \neq k$, substitution of (1) gives

$$\begin{aligned}(B'^T T_k^B(z))_j &= \sum_{i\neq k}\big(b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+\big)z_i \\ &\quad - b_{kj}\big(-z_k + \min(P,N)\big) \\ &= a_j + [b_{kj}]_+\big(P - \min(P,N)\big) - [-b_{kj}]_+\big(N - \min(P,N)\big) \\ &= a_j + [b_{kj}]_+[a_k]_+ - [-b_{kj}]_+[-a_k]_+.\end{aligned}$$

The entries agree with (2) for the exchange matrix $B^T$, so $B'^T T_k^B(z) = \eta_k^B(B^T z)$. The map $\eta_k^B$ carries $|\mathcal{G}(B^T)|$ onto $|\mathcal{G}(B'^T)|$ by the change of initial seed proved above. Thus $B^T z$ belongs to the first support exactly when $B'^T T_k^B(z)$ belongs to the second. The homeomorphism $T_k^B$ carries the set in the lemma onto the corresponding set for $B'$, preserving density. Iterating from the acyclic representative proves the claim. □

Lemmas 3.1 and 3.2 prove the square case at points of the initial orthant. If $\lambda$ lies in any **g**-vector cone, use the seed of that cone as the initial seed. The change of coordinates sends $\lambda$ to the initial orthant and carries dominance regions by (3). This proves Theorem 2.1 when $m = n$.

## 4. Frozen rows

Reading, Rupel, and Stella used projection and a choice of representative to pass to arbitrary coefficients in finite type [7, Proof of Theorem 3.1]. The following property of affine mutation classes, also a consequence of [7, Proposition 4.35], gives the corresponding lift here.

**Lemma 4.1.** *Every affine mutation class has a representative $B_0$ with $\ker B_0 \cap \mathbb{R}^n_{\geq 0} = \{0\}$.*

*Proof.* Start with an acyclic representative $B = (b_{ij})$ with affine Cartan companion, and let $\delta$ be its primitive positive integer null vector. Consider the underlying undirected graph with one edge for each nonzero pair of off-diagonal entries. If it contains a simple cycle, the Cartan row equation at each vertex $i$ of the cycle gives

$$2\delta_i = \sum_{j\neq i} |b_{ij}|\delta_j \ \geq\ \delta_{\mathrm{prev}(i)} + \delta_{\mathrm{next}(i)},$$

where $\mathrm{prev}(i)$ and $\mathrm{next}(i)$ are the preceding and following vertices in a cyclic order on the cycle. Summing around the cycle gives equality. Hence equality holds in each row, excluding any other edges incident to cycle vertices. By connectedness the graph is precisely that cycle. If it has no cycle, it is a tree.

At a source $k$, $b_{ik} \leq 0$ and $b_{kj} \geq 0$ for $i, j \neq k$, so both correction terms in (1) vanish. At a sink the inequalities reverse and the same two terms vanish. Mutation at either vertex thus reverses the incident edges and changes no others.

On a tree, any orientation can be reached by such reversals. For a single vertex there is nothing to orient. For a larger tree, delete a leaf and realize the desired orientation of the smaller tree by induction. When a step reverses the former neighbor of the leaf, first reverse the leaf if its edge

prevents that neighbor from being a source or sink. If the smaller tree has just one vertex, no reversals on it are needed. Finally reverse the leaf if its edge still has the wrong direction. In particular we may orient all edges from the vertices at even distance from a fixed vertex to those at odd distance. Let $B_0$ be the resulting matrix. Every row has entries of one weak sign, and connectedness with $n \geq 2$ gives each vertex a neighbor. Thus each equation $(B_0\alpha)_i = 0$ for $\alpha \geq 0$ forces all the coordinates of $\alpha$ at neighbors of $i$ to vanish. This yields $\alpha = 0$.

On a cycle, fix a clockwise order and mark each edge with a plus sign if it is oriented clockwise and a minus sign if it is oriented counterclockwise. Write $p$ and $q$ for the two counts. Acyclicity gives $p, q > 0$. At a vertex where consecutive signs differ, the two edges are both incoming or both outgoing. Reversal exchanges those signs. Adjacent swaps of opposite signs in a linear listing of the edges can sort the signs into a block of plus signs followed by a block of minus signs. Reversing such swaps reaches any prescribed arrangement with the same counts.

For any resulting matrix $B_0$, let $\alpha \in \ker B_0 \cap \mathbb{R}^n_{\geq 0}$. A source or sink forces both its neighbor coordinates to vanish because its two row terms have the same weak sign. In the row at the successor of a vertex with zero coordinate, the term from that vertex vanishes, leaving the term from the vertex two steps ahead, with nonzero coefficient. Thus a zero propagates by steps of two. When the cycle has odd length, stepping by two visits every vertex, so one sign change suffices to force $\alpha = 0$.

For even length, it suffices to have a source or sink in each parity class of vertices, so that zeros propagate through both classes. Number the vertices $0, \ldots, p+q-1$ in clockwise order, with the first sign on the edge joining vertices 0 and 1. If $p, q$ are odd, place $p$ consecutive plus signs and then $q$ consecutive minus signs. Vertices 0 and $p$ are sign changes of opposite parity. If $p, q$ are even, place $p-1$ plus signs, one minus sign, one plus sign, and $q-1$ minus signs, in that order. Vertices 0 and $p-1$ are sign changes of opposite parity. In each orientation the sign changes are sources or sinks, proving the kernel assertion. □

*Proof of Theorem 2.1.* Choose a word $w$ for which $B_w$ satisfies Lemma 4.1. By (3), the dominance region is transported bijectively by $\eta_w^{\widetilde{B}^T}$. The mutable projection of the transported base point is $\eta_w^{B^T}(\pi(\widetilde{\lambda}))$, which belongs to the corresponding $\mathbf{g}$-vector cone in the new coordinates. For any $\widetilde{\mu} \in \mathcal{P}_{\widetilde{\lambda}}^{\widetilde{B}}$, the projection inclusion and the square case in the new seed give

$$\pi\eta_w^{\widetilde{B}^T}(\widetilde{\mu}) = \pi\eta_w^{\widetilde{B}^T}(\widetilde{\lambda}).$$

The test at the empty word in the new seed gives

$$\eta_w^{\widetilde{B}^T}(\widetilde{\mu}) - \eta_w^{\widetilde{B}^T}(\widetilde{\lambda}) = \widetilde{B}_w\alpha \quad \text{for some } \alpha \in \mathbb{R}^n_{\geq 0}.$$

Projecting gives $B_w\alpha = 0$, so $\alpha = 0$ by the choice of $w$. Invert the column map to obtain $\widetilde{\mu} = \widetilde{\lambda}$. The base point belongs to its dominance region by taking difference zero in each test. □

## References


[1] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.
[2] S. Fomin and A. Zelevinsky, *Cluster algebras IV: Coefficients*, Compos. Math. **143** (2007), 112–164.
[3] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), 497–608.
[4] T. Nakanishi and A. Zelevinsky, *On tropical dualities in cluster algebras*, Algebraic Groups and Quantum Groups, Contemp. Math., vol. 565, Amer. Math. Soc., Providence, RI, 2012, pp. 217–226.
[5] F. Qin, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*, Duke Math. J. **166** (2017), 2337–2442.
[6] F. Qin, *Bases for upper cluster algebras and tropical points*, J. Eur. Math. Soc. **26** (2024), 1255–1312.
[7] N. Reading, D. Rupel, and S. Stella, *Dominance regions for affine cluster algebras*, arXiv:2512.02218v2, 2026.
[8] N. Reading and D. E. Speyer, *Cambrian frameworks for cluster algebras of affine type*, Trans. Amer. Math. Soc. **370** (2018), 1429–1468.
[9] N. Reading and S. Stella, *An affine almost positive roots model*, J. Comb. Algebra **4** (2020), 1–59.
[10] N. Reading and S. Stella, *Cluster scattering diagrams of acyclic affine type*, arXiv:2205.05125v5, 2026.
[11] N. Reading and S. Stella, *Theta functions in acyclic affine type*, arXiv:2603.23429v1, 2026.
[12] D. Rupel and S. Stella, *Dominance regions for rank two cluster algebras*, Ann. Comb. **27** (2023), 873–894.



Zhili College, Tsinghua University, Beijing 100084, China
*Email address*: tangqy24@mails.tsinghua.edu.cn