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THE L'VOV–KAPLANSKY CONJECTURE FOR STANDARD POLYNOMIALS

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Abstract. The L'vov–Kaplansky conjecture asks whether the image of a multilinear polynomial on a full matrix algebra is a vector subspace. For the unnormalized standard polynomial s_d over an algebraically closed field K of characteristic zero, the sets of individual matrix values are determined in every degree and order. The image of s_{2r-1} is zero when $n < r$ and all of $\mathrm{M}_n(K)$ when $n \geq r$; the image of s_{2r} is zero when $n \leq r$ and $\mathfrak{sl}_n(K)$ when $n \geq r+1$. The odd-degree proof uses explicit linear slices and Jordan forms. The even-degree argument combines a uniform nilpotent construction and an additive transfer for admissible cores with generic-nullity results of Brassil–Reichstein and Lin–Tang–Zhao, extending a method of Qiyue Tang to obtain actual polynomial values.

1. Introduction and the classification

The L'vov–Kaplansky conjecture concerns the images of multilinear noncommutative polynomials evaluated on full matrix algebras. In the setting of algebraically closed fields of characteristic zero, it asks whether each such image is a vector subspace [7, 6]. The alternating standard polynomials constitute a distinguished family for which one can go further and identify the image explicitly, in all degrees and matrix orders.

For every integer $d \geq 1$ define the *unnormalized standard polynomial*

$$s_d(X_1, \ldots, X_d) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(d)}. \tag{1}$$

The image $\operatorname{Im} s_d(\mathrm{M}_n(K))$ means the set of matrices attained as $s_d(A_1, \ldots, A_d)$ for *one tuple* of inputs; it is not the linear span or Zariski closure of those values. Write $\mathfrak{sl}_n(K) = \{T \in \mathrm{M}_n(K) : \operatorname{tr} T = 0\}$. Matrix units are indexed from zero except when quoting another author's conventions. Throughout the article, K is algebraically closed of characteristic zero unless a statement explicitly declares otherwise.

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Theorem 1.1 (Images of the standard polynomials)**.** *For every algebraically closed field K of characteristic zero and all integers $r, n \geq 1$, the* actual images *are*

$$\begin{aligned} \operatorname{Im} s_{2r-1}(\mathrm{M}_n(K)) &= \begin{cases} \{0\}, & n < r, \\ \mathrm{M}_n(K), & n \geq r, \end{cases} \\ \operatorname{Im} s_{2r}(\mathrm{M}_n(K)) &= \begin{cases} \{0\}, & n \leq r, \\ \mathfrak{sl}_n(K), & n \geq r+1. \end{cases} \end{aligned} \tag{2}$$

The zero branches follow from the Amitsur–Levitzki identity and its standard recursion [2]. In even degree, cyclically rotating a word reverses its permutation sign while preserving its trace, so every value is traceless. The main task is therefore to attain each matrix in the two positive branches of (2).

The lowest-degree cases are well established. The ordinary commutator s_2 attains every traceless matrix [1], and Vitas's cubic image-linearity theorem gives the corresponding s_3 result [15]. The quartic result of Fagundes, de Mello and da Silva dos Santos includes the actual-image assertion for s_4 [4, Theorem 8, Case 1]. Qiyue Tang established the degree-five and degree-six conclusions, including the actual traceless image of s_6 in orders at least four [14, Appendix A, Proposition 11].

Classical antecedents include Kostant's work on standard identities, Swan's graph method and its correction, and results of Rowen and Hutchinson on standard polynomials and matrix identities [8, 12, 13, 10, 5]. Serre studied special matrix values and related norm questions [11]. These works provide context for the exact all-degree image classification undertaken here.

The two parity arguments have different structures. Part I establishes the odd-degree assertion through signed path and cycle evaluations, fixed linear slices and target-dependent Jordan constructions at the minimal size. Part II uses a rank-one nilpotent with surjective full differential to obtain actual values. Its nilpotent and projective-incidence method builds on Tang's Appendix A, while the uniform seed, additive-core transfer and exact degree-eight and degree-ten cases supply further constructions. The general even-degree step uses the generic-nullity theorem of Brassil–Reichstein [3, Theorem 1.2] and Chen Lin, Chenhao Tang and Enhan Zhao [9, Theorem 1.2]. The final appendices give the exact finite checks supporting specified steps.

I. Odd standard polynomials

2. Linear slices and balanced endpoint transfer

Fix $A_1, \ldots, A_{m-1} \in \mathrm{M}_n(K)$ and set

$$L_A(X) = f(A_1, \ldots, A_{m-1}, X). \tag{3}$$

For multilinear f, the map L_A is linear. Thus any linear combination of outputs obtained by varying X *with the same fixed tuple* A is again one value

of f. No additive closure of the unrestricted image is assumed. Simultaneous conjugation of all inputs conjugates the output of s_m.

Lemma 2.1 (Jordan-path criterion)**.** *Suppose a slice image contains all diagonal matrices and the matrix units*

$$E_{i_1 i_2}, E_{i_2 i_3}, \dots, E_{i_{n-1} i_n}$$

for some permutation $(i_1, \dots, i_n)$ *of the indices. Then* $\operatorname{Im} f(\mathrm{M}_n(K)) = \mathrm{M}_n(K)$*, provided* f *is equivariant under simultaneous conjugation.*

Proof. Within this one slice we can prescribe each diagonal entry and the superdiagonal entries along the indicated path independently. A Jordan matrix, including a direct sum of several blocks, has precisely this form after permuting the coordinates, with its link entries chosen to be 1 or 0. Every matrix over K is similar to such a matrix. Conjugating the entire input tuple concludes the proof. □

For later use let

$$V_h = \operatorname{span}\{E_{i,i+h \bmod n} : 0 \le i < n\}. \tag{4}$$

If n is odd, the step-two edges in V_2 contain a Hamiltonian cycle. If n is even, the two parity cycles are linked by E_{03}; an explicit Hamiltonian path is

$$2 \longrightarrow 4 \longrightarrow \cdots \longrightarrow n-2 \longrightarrow 0 \longrightarrow 3 \longrightarrow 5 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow 1. \tag{5}$$

Let f be homogeneous multilinear of degree m and let its final variable be free. In the signed word expansion, denote by $P(A)$ the total fixed-input product coefficient when that variable is at the *last* position; let $Q(A)$ be its counterpart in the *first* position. Both depend polynomially on A.

Theorem 2.2 (Balanced endpoint transfer)**.** *Let* $n = p + q$ *with* $|p - q| \le 1$*. Suppose there are fixed* $(m-1)$*-tuples* $A \in \mathrm{M}_p(K)^{m-1}$ *and* $B \in \mathrm{M}_q(K)^{m-1}$ *such that both slice images contain every diagonal matrix in their respective blocks and* $P(A)$ *and* $Q(A)$ *are invertible. Then* $\operatorname{Im} f(\mathrm{M}_n(K)) = \mathrm{M}_n(K)$*, assuming simultaneous-conjugacy equivariance.*

Proof. Take the block-diagonal fixed tuple

$$C_j(t) = \begin{pmatrix} A_j & 0 \\ 0 & tB_j \end{pmatrix} \qquad (1 \le j < m).$$

Writing the free input in four rectangular blocks yields four independent block-preserving linear maps. On the two diagonal blocks they are L_A and $t^{m-1}L_B$. On a top-right input X, setting $t = 0$ kills terms in which a B factor occurs on the right of X, giving $X \mapsto P(A)X$. On a bottom-left input, the limit is $X \mapsto XQ(A)$. These last maps are invertible for $t = 0$. The corresponding determinants are nonzero polynomials in t. Choose a single $t \in K^\times$ avoiding their finitely many zero sets. This choice is independent of the eventual target matrix.

At this fixed nonzero t, the slice covers all four mixed-block entries and all diagonal entries in either square block. Since $|p - q| \le 1$, the bipartite

graph of entries between the two blocks has a Hamiltonian path alternating between the blocks. Therefore Lemma 2.1 applies. □

For $f = s_{2k+1}$ the two endpoint coefficients are

$$P(A) = Q(A) = s_{2k}(A_1, \dots, A_{2k}). \tag{6}$$

Indeed moving the free letter across the other $2k$ letters changes no sign. The balanced hypothesis is essential: no Hamiltonian path through a complete bipartite graph exists if its two parts differ in size by more than one.

3. Signed trails on a bidirected path

Fix $k \geq 1$, write $r = k + 1$, and consider the ordered list

$$\mathcal{A}_k = (E_{01}, E_{12}, \dots, E_{k-1,k}, E_{k,k-1}, E_{k-1,k-2}, \dots, E_{10}). \tag{7}$$

Set

$$c = (1, \underbrace{2, \dots, 2}_{r-2}, 1)^t,$$
$$H = \operatorname{diag}(c) + cc^t,$$
$$\mathcal{E}_r = \operatorname{diag}(1, -1, 1, \dots, (-1)^{r-1}).$$

For $r = 2$ the middle list is empty. Every nonzero product of matrix units is an oriented Euler trail through their directed edges. This simple observation gives the exact slice coefficients.

Lemma 3.1 (Endpoint and loops)**.** *For the tuple* (7)*,*

$$s_{2k}(\mathcal{A}_k) = \operatorname{diag}((-1)^i c_i)_{0 \leq i < r}, \tag{8}$$

$$s_{2k+1}(\mathcal{A}_k, \operatorname{diag}(x)) = \operatorname{diag}(H\mathcal{E}_r x). \tag{9}$$

Furthermore,

$$\det(H\mathcal{E}_r) = (-1)^{r(r-1)/2} 2^{r-2}(2r-1) \neq 0. \tag{10}$$

Proof. The forward and reverse directed edges form a bidirected path. Every closed Euler ordering is a cyclic rotation of the tour traversing that path forward and returning. Vertex i occurs as a starting vertex c_i times, and the resulting permutations have sign $(-1)^i$. This proves (8).

Insert a diagonal loop at vertex j. For a tour beginning at i, there are $c_j + \delta_{ij}$ loop slots. Incorporating the loop into the fixed ordering gives sign $(-1)^j$ for the added-loop coefficient, so its contribution to output coordinate i is $c_i(c_j + \delta_{ij})(-1)^j$. This is exactly $H\mathcal{E}_r$. Finally the determinant lemma yields

$$\det H = \left(\prod_{i=0}^{r-1} c_i \right) \left(1 + c^t \operatorname{diag}(c)^{-1} c \right) = 2^{r-2}(2r-1).$$

Multiply by $\det \mathcal{E}_r = (-1)^{r(r-1)/2}$. □

Lemma 3.2 (Free chord on a bidirected path)**.** *Embed* (7) *on an interval of indices* $a, a+1, \dots, a+k$*, with* c_j *as above. For* $p \neq q$*, the image of the free matrix unit* E_{pq} *is a multiple of* E_{pq}*. The coefficient, before any external window weight, is*

(11)

position of p, q	*coefficient*
p, q *inside, same parity*	$(-1)^{p-a} c_{p-a} c_{q-a}$
p, q *inside, opposite parity*	0
p *inside,* q *outside*	$(-1)^{p-a} c_{p-a}$
q *inside,* p *outside*	$(-1)^{q-a} c_{q-a}$
p, q *outside*	0.

Proof. The directed degree imbalance of the fixed bidirected path is zero. Adding $p \to q$ forces every connected Euler trail to start at p and end at q, so the output unit is E_{pq}. If the free edge joins a path vertex to an external vertex, it must be first or last; (8) gives the two one-sided coefficients. If both endpoints lie outside, the selected edges are disconnected.

For two inside vertices, assume first $p < q$ and put $d = q - p$. The tails outside $[p, q]$ must be traversed outward and back. Each nonempty tail admits the independent attachment multiplicity c_{p-a} or c_{q-a}. After stripping these tails, an Euler trail is specified by a turning vertex $j \in [p, q]$: go from p to j and back, take the chord $p \to q$, then go from q to j and back. Moving j one edge changes the permutation sign by -1. The initial sign, by inversion counting in the displayed order of (7), is $(-1)^{q-a}$. Summing gives

$$(-1)^{q-a} c_{p-a} c_{q-a} \sum_{h=0}^{d} (-1)^h.$$

The sum vanishes for odd d and equals 1 for even d, in which case p and q have the same parity. For $p > q$, reversing both the directed path and the fixed ordering contributes the same sign $(-1)^k$ twice, giving the identical formula. □

4. Positive path cores and large matrix sizes

We next build one slice with invertible diagonal restriction and invertible even-standard endpoint on every $\mathrm{M}_q(K)$ with $q \geq r$. For $k = r - 1$, define

(12)
$$F_j = \sum_{e=0}^{q-2} (e+1)^{j-1} E_{e,e+1},$$
$$R_j = \sum_{e=0}^{q-2} z_e (e+1)^{j-1} E_{e+1,e}, \qquad z_e = (-1)^{\lfloor e/k \rfloor},$$

for $1 \leq j \leq k$, and order the fixed inputs as $F_1, \dots, F_k, R_k, \dots, R_1$. Put

$$v_k = \prod_{j=1}^{k-1} j!, \qquad \iota_a : K^r \longrightarrow K^q, \quad (x_0, \dots, x_{r-1}) \longmapsto \sum_{i=0}^{r-1} x_i e_{a+i}.$$

Proposition 4.1. *For every $q \geq r$, the diagonal restriction of the final-variable slice associated with* (12) *is*

$$L_{\mathrm{diag}} = G\mathcal{E}_q, \qquad G = v_k^2 \sum_{a=0}^{q-r} \iota_a H \iota_a^t. \tag{13}$$

Its fixed-input even standard polynomial is

$$s_{2k}(F_1, \dots, F_k, R_k, \dots, R_1) = v_k^2 \operatorname{diag}\left((-1)^i \sum_{\substack{0 \leq a \leq q-r \\ a \leq i \leq a+k}} c_{i-a} \right)_{0 \leq i < q}. \tag{14}$$

Both matrices are invertible in characteristic zero.

Proof. Expand the fixed inputs into path units. Alternation among the F_j requires k distinct forward edges, and alternation among the R_j requires k distinct backward edges. A diagonal output has zero net flow through every edge cut, so the forward and reverse edge sets coincide. With only one extra diagonal loop, a disconnected set of edges cannot support one Euler trail. Consequently the k selected edges form a consecutive interval $a, a+1, \dots, a+k-1$.

On a fixed window, the coefficients from the forward and reverse choices are two Vandermonde determinants, both of absolute algebraic value v_k, before reverse signs. Their product is v_k^2. Furthermore

$$\prod_{e=a}^{a+k-1} z_e = (-1)^a :$$

the product is $+1$ at $a = 0$, and shifting the window changes its sign because $z_{a+k}/z_a = -1$. Multiplying the local formula (9) by this sign converts the local alternating factor into the restriction of $\mathcal{E}_q$. Thus the windows sum to (13); (8) similarly yields (14).

There is no assumption that K is ordered. Instead regard these integer matrices first over $\mathbb{R}$. The matrix $H = \operatorname{diag}(c) + cc^t$ is positive definite, as is each of its nonzero embedded quadratic forms. Every index i occurs in at least one interval $[a, a+k]$, so for $x \neq 0$,

$$x^t G x = v_k^2 \sum_{a=0}^{q-r} (\iota_a^t x)^t H (\iota_a^t x) > 0.$$

Therefore $\det G$ is a nonzero integer, and remains nonzero over any field of characteristic zero. All diagonal entries in (14) are nonzero signed positive integers. □

Corollary 4.2. *For $r \geq 2$ and $n \geq 2r$, the image of s_{2r-1} on $\mathrm{M}_n(K)$ is all of $\mathrm{M}_n(K)$.*

Proof. Let $p = \lceil n/2 \rceil$, $q = \lfloor n/2 \rfloor$. Both are at least r; apply Proposition 4.1 to each block and Theorem 2.2, using (6). □

5. Minimal size: the cycle and its maximal-minor bivector

We treat the boundary

$$\operatorname{Im} s_{2n-1}(\mathrm{M}_n(K)) = \mathrm{M}_n(K), \qquad n \geq 3. \tag{15}$$

At this size, a single fixed slice need not serve all targets. The construction instead adjusts the diagonal fixed inputs to each prescribed Jordan matrix.

Fix the n directed cycle edges

$$C_i = E_{i,i+1 \bmod n}, \qquad 0 \leq i < n,$$

followed by $n-2$ diagonal matrices $D_1, \ldots, D_{n-2}$ with diagonal row matrix $D \in \mathrm{M}_{n-2,n}(K)$. Leave the final variable free and denote the resulting slice by L_D. Let

$$A_n = 2\mathbf{1}\mathbf{1}^t - I_n. \tag{16}$$

For a row-deleted coordinate matrix of rank $n-2$, the signed maximal minors of D form a skew matrix $B(D)$, described explicitly in the following lemma. Its scale is fixed by the order of the cycle inputs followed by the diagonal inputs.

Lemma 5.1 (Cycle–bivector formulas)**.** *For $0 \leq i < j < n$, let $D_{\widehat{ij}}$ be the square submatrix obtained by removing columns i, j, and put*

$$\begin{aligned} \gamma_{ij} &= (-1)^{n(n+1)/2+j-i-2}, \\ B_{ji} &= \gamma_{ij} \det D_{\widehat{ij}}, \quad B_{ij} = -B_{ji}, \quad B_{ii} = 0. \end{aligned} \tag{17}$$

Then

$$L_D(\operatorname{diag}(x)) = \operatorname{diag}(A_n B x). \tag{18}$$

If D has rank $n-2$, then B has rank two and is decomposable. Conversely, for independent $v, w \in K^n$, choosing the rows of D to span $\{x : v^t x = w^t x = 0\}$ makes B a nonzero scalar multiple of $vw^t - wv^t$.

For $p \neq q$, let $I(p,q)$ be the inclusive forward cyclic interval $q, q+1, \ldots, p$ and define

$$a_i = \begin{cases} 1, & i = p \text{ or } i = q, \\ 2, & i \in I(p,q) \setminus \{p,q\}, \\ 0, & i \notin I(p,q), \end{cases} \qquad b_i = \mathbf{1}_{i \notin I(p,q)}. \tag{19}$$

The free chord E_{pq} obeys the exact formula

$$L_D(E_{pq}) = -2(a^t B b) E_{pq}. \tag{20}$$

In particular the original forward cycle edges have zero coefficient.

Proof. All coefficients are alternating multilinear expressions in the $n-2$ diagonal rows of D. It is therefore enough to calculate them on coordinate diagonal inputs omitting a pair of vertices $i < j$; expansion in minors then gives the stated general formulas.

For a free loop at i, the cycle and the $n-2$ fixed loops admit closed Euler orderings obtained by rotating a cyclic ordering. The vertex j without a loop occurs once as a starting point and all other vertices twice. Since the total word length is $2n-1$, these rotations have the same sign. Place the diagonal loops in increasing coordinate order after the cycle edges. Inversion counting of the rotation and its loop insertions gives the common factor

$$\gamma_{ij} = (-1)^{n(n+1)/2+j-i-2}.$$

Thus the free loop at i produces $\gamma_{ij}\,\mathrm{diag}(2\mathbf{1} - e_j)$; the loop at j gives the negative corresponding expression with i omitted. The two columns are exactly $A_nB_{\cdot i}$ and $A_nB_{\cdot j}$. This establishes (18) in the coordinate case.

For completeness we record the chord sign calculation in the same minor normalization. An Euler trail containing the free chord $p \to q$ has two possible edge skeletons before the fixed loops are inserted: chord first followed by the forward cycle starting at q, or the cycle from p followed by the chord. A diagonal loop strictly in the inclusive interval $I(p,q)$ has a forced placement, while at either chord endpoint there are two placements. Pairing the skeletons for a pair of omitted loops on the same side of the interval cancels their signs. Moving an omitted loop across an endpoint changes the inversion parity; the resulting complete coefficient table, for the elementary bivector $B_{ij} = -B_{ji}$, is

(21)

positions of omitted i, j $(i < j)$	coefficient of B_{ij}
$i, j \in I$ or $i, j \notin I$	0
i endpoint of I, $j \notin I$	-2
i strictly inside I, $j \notin I$	-4
$i \notin I$, j endpoint of I	2
$i \notin I$, j strictly inside I	4.

For example, take $n = 4$, $p = 2$, $q = 0$ and omit the loop vertices $i = 0, j = 3$, leaving $D_1 = E_{11}$ and $D_2 = E_{22}$. Writing $X = E_{20}$ and keeping the input order $C_0, C_1, C_2, C_3, D_1, D_2, X$, the four admissible Euler words and permutation signs are

$$\begin{array}{ll} C_2C_3C_0D_1C_1D_2X & -1 \\ XC_0D_1C_1D_2C_2C_3 & -1 \\ D_2C_2C_3C_0D_1C_1X & +1 \\ D_2XC_0D_1C_1C_2C_3 & -1. \end{array}$$

Their sum is -2, as in the endpoint/outside row of the table: (17) gives $\gamma_{03} = -1$, hence $B_{03} = 1$. Here is a parameter-uniform inversion check of the table, including cyclic wraps. Write $S = \{0, \dots, n-1\} \setminus \{i, j\}$, and let $\ell = (p-q) \bmod n \in \{1, \dots, n-1\}$. For the first skeleton, with any endpoint loop placed at its earliest opportunity, list the vertices in the order

$$\pi_{p,q} = (p, q, q+1, \dots, p-1, p+1, \dots, q-1),$$

where each interval runs in the forward cyclic direction. Deleting i, j gives the loop ordering $\pi_{p,q}|_S$. Define $\eta(p) = \eta(q) = n$ and $\eta(v) = n - ((v-q) \bmod n)$

otherwise, and write inv for the number of numerical inversions. The sign of this baseline Euler word is

$$\varepsilon = (-1)^{q(n-q)+\mathbf{1}_{p\in S}+\mathrm{inv}(\pi_{p,q}|_S)+\sum_{v\in S}\eta(v)}. \tag{22}$$

The terms count, respectively, the rotation of the labelled cycle, the last-labelled chord crossing the optional early p-loop, inversions among the fixed-loop labels, and the cycle letters following their respective loops. In particular, the endpoint values of η cannot be replaced by its interior formula.

To see that this absolute sign matches the cofactor convention, let z_v be the zero-based position of v in $\pi_{p,q}$. Moving p from position ℓ to the front of the forward rotation starting at q yields

$$\mathrm{inv}(\pi_{p,q}) \equiv q(n-q)+\ell, \qquad \sum_{v=0}^{n-1}\eta(v) = \frac{n(n+1)}{2}+\ell \pmod 2.$$

Deleting v changes inversion parity by $z_v + v$; deleting both $i < j$ requires the additional parity correction $\mathbf{1}_{z_j<z_i}$. Substitution in (22) gives

$$\frac{\varepsilon}{\gamma_{ij}} = (-1)^{1+h(i)+h(j)+\mathbf{1}_{z_j<z_i}}, \quad h(v) = z_v+\eta(v)+\mathbf{1}_{v=p} \pmod 2.$$

The vertex order gives $h(v) \equiv n+1$ for $v \in I(p,q)$ and $h(v) \equiv n$ outside. When i, j lie on opposite sides, $h(i)+h(j) \equiv 1$, and all interval vertices precede the outside vertices in $\pi_{p,q}$. Hence

$$\varepsilon = \gamma_{ij}\,\mathrm{sgn}(a_ib_j - a_jb_i) \quad \text{whenever } a_ib_j - a_jb_i \neq 0. \tag{23}$$

For clarity, the remaining sign summation is finite independently of n. If A is the forward arc from q to p and B its complement, their edge lengths, including forced interior loops, are $\alpha = 2\ell - 1 - u$ and $\beta = 2(n-\ell) - 1 - v$, where u, v count the omitted strict-interior loops in the respective arcs. Let $P, Q \in \{0,1\}$ indicate the presence of endpoint loops at p, q. In the first skeleton the parity changes on moving a present p-loop or q-loop to its late placement are $1+Q+\alpha$ and $\alpha+\beta$; in the second they are $\alpha+\beta+Q$ and $\alpha+1$. The baseline skeletons differ by $\beta(1+Q+\alpha)+Q+\alpha$ inversions. Summing the at most $2^{P+Q+1} \leq 8$ possible words therefore makes both same-side configurations cancel. In an opposite-side configuration the signed sum divided by ε is 2 when the omitted interval vertex is an endpoint, and 4 when it is a strict interior vertex. Together with (23), the complete sum is $2\gamma_{ij}(a_ib_j - a_jb_i)$, namely the table (21) with $B_{ij} = -\gamma_{ij}$. Expansion in the maximal minors proves (20). The degree imbalance of the chord forces output E_{pq}, so no other matrix units occur.

The maximal-minor cofactor construction is the Pluecker bivector of the two-dimensional annihilator of the row space of D. For rank $n-2$, it is nonzero and decomposable, and its image is that annihilator. In particular taking the row space to be $\ker v^t \cap \ker w^t$ gives the claimed multiple of $vw^t - wv^t$. No positivity or nonisotropy assumption on v, w is involved. □

Proposition 5.2 (Every Jordan target at the minimal size)**.** *For each $n \geq 3$ and every lower Jordan matrix $J \in \mathrm{M}_n(K)$, there is a rank-$n-2$ choice of D such that J is contained in the image of the single slice L_D, after a possible simultaneous permutation of coordinates.*

Proof. Write $d \in K^n$ for the diagonal of J and $\tau = \sum_i d_i$. The matrix A_n is invertible: it acts by $2n-1$ on the line $K\mathbf{1}$ and by -1 on its sum-zero complement. If $d \neq 0$, put

$$v = A_n^{-1} d = -d + \frac{2\tau}{2n-1}\mathbf{1}. \tag{24}$$

Select $w \notin Kv$ and take $B = vw^t - wv^t$, up to the nonzero cofactor scalar. The range of $A_n B$ contains d. For a Jordan link E_{pq}, the coefficient in (20) is proportional to

$$(a^t v)(w^t b) - (a^t w)(v^t b). \tag{25}$$

If a, b are independent, this is a nonzero linear functional of w precisely when $(a^t v, b^t v) \neq (0,0)$. The independence holds for the nonforward edges used below: the interval is proper and contains its two endpoints.

Here and below every Jordan block is represented with lower subdiagonal links, and the coordinates may be permuted. For an edge joining equal diagonal entries λ, formula (24) yields

$$\frac{1}{2}a^t v + b^t v = \lambda - \frac{\tau}{2n-1}. \tag{26}$$

Indeed the left side assigns weight one to the two endpoints, and weight one to every other coordinate; the sum is $\sum_i v_i - (v_p + v_q)/2$. Substituting $\sum_i v_i = \tau/(2n-1)$ and the common endpoint entries $v_p = v_q = -\lambda + 2\tau/(2n-1)$ gives the identity.

We consider all possible Jordan diagonals.

(i) $\tau \neq 0$. List the Jordan blocks consecutively and use their ordinary lower links $E_{i+1,i}$. Each relevant inclusive interval has just two vertices. If $a^t v = b^t v = 0$, the two equal endpoint entries of v would sum to zero and so both vanish. The sum of the other entries would vanish too, contradicting $\sum_i v_i = \tau/(2n-1) \neq 0$. For $n \geq 3$ none of these lower links is a forward cycle edge.

(ii) $\tau = 0$ and $d \neq 0$, with no nontrivial zero-eigenvalue Jordan chain. Again use the consecutive Jordan ordering. Every required link has nonzero eigenvalue λ; (26) shows $a^t v$ and $b^t v$ cannot both vanish.

(iii) $\tau = 0$, $d \neq 0$, with at least one nontrivial zero-eigenvalue Jordan chain. Two-color the vertices in every zero-eigenvalue Jordan chain as a path, black and white, so each required zero link joins different colors; assign any zero-eigenvalue singleton to either group. Both colors occur. Select the leading endpoint of one nonzero-eigenvalue Jordan chain as the single vertex in group A, with eigenvalue $\mu \neq 0$, and put the remaining nonzero-eigenvalue vertices in group B, retaining their chain ordering. Since the total trace is zero, the

sum of the eigenvalues in B is $-\mu$, so this group is nonempty. Arrange the cyclic coordinates in the order

$$\text{black zero vertices} \mid A \mid \text{white zero vertices} \mid B. \tag{27}$$

A link with nonzero eigenvalue has nonzero functional by (26); its endpoints are not a forbidden forward cycle edge. A zero Jordan link joins black to white, and the inclusive cyclic interval from its source to its target contains precisely one of A, B among nonzero vertices. Thus $\sum_{i\in I} d_i = \mu$ or $-\mu$. Now $v = -d$ and the zero endpoints contribute nothing, so $a^t v = -2\sum_{i\in I} d_i = \mp 2\mu \neq 0$. This deals simultaneously with several zero blocks and the possible singleton vertices.

(iv) $d = 0$. The target is nilpotent, possibly with several Jordan blocks. Use the consecutive lower-Jordan ordering but choose $v = \mathbf{1}$ instead of (24). The zero diagonal belongs to the range of $A_n B$ automatically. For each required link the inclusive interval has two vertices, so $a^t v = 2$ and $b^t v = n - 2$, and (25) is a nonzero functional of w.

In each case we have finitely many nonzero linear functionals of w whose vanishing must be avoided, together with the proper subspace Kv. Over the infinite field K, their union does not exhaust K^n. Choose w outside it, and select diagonal fixed inputs realizing the decomposable bivector, as in Lemma 5.1. Their coefficients on all required links are nonzero. Solve (18) for an input $\mathrm{diag}(x)$ giving d (with $x = 0$ sufficient in (iv)), then add to *that same free input* the required E_{pq} divided by their coefficients in (20). The result is precisely the permuted Jordan matrix J; the slice introduces no unwanted matrix entries. Finally, conjugation gives the original target. □

Corollary 5.3. *For every $n \geq 3$, (15) holds.*

6. Intermediate matrix sizes: periodic path colors

We now consider $r < n < 2r$. Let $k = r - 1$ and write

$$n = k + t + 1, \qquad 1 \leq t \leq k. \tag{28}$$

First assume $k \geq 3$. On the path with n vertices, give edge $e \to e+1$ the color $e \bmod k$. Put

$$\begin{aligned} F_j &= \sum_{\substack{0\leq e<n-1 \\ e\equiv j \pmod k}} E_{e,e+1}, \\ R_j &= \sum_{\substack{0\leq e<n-1 \\ e\equiv j \pmod k}} (-1)^{\lfloor e/k \rfloor} E_{e+1,e}, \qquad 0 \leq j < k. \end{aligned} \tag{29}$$

Fix the inputs in the order

$$F_0, \ldots, F_{k-1}, R_{k-1}, \ldots, R_0 \tag{30}$$

and denote the final-variable slice by L. Each term in the expansion chooses one forward and one reverse edge of each of the k colors. The colors $0, \ldots, t-1$ occur twice, and the others once.

6.1. **Diagonal restriction and the connected-window rule.** A connected selection with the same forward and reverse edges must be a window of k consecutive edges. Every such window contains each color once. The signs of the permutations of forward and reverse colors cancel; the reversed-edge scalar signs have product $(-1)^a$ when the window starts at a. By Lemma 3.1,

$$L_{\mathrm{diag}} = \left(\sum_{a=0}^{t} \iota_a H \iota_a^t \right) \mathcal{E}_n. \tag{31}$$

The Gram argument in Proposition 4.1 shows this restriction is invertible: H is positive definite over the reals, and every coordinate belongs to one of the $t+1$ windows. Its determinant is a nonzero integer.

We will use ordinary offsets as well as cyclic ones. For $|h| < n$, put

$$\mathcal{U}_h = \mathrm{span}\{E_{i,i+h} : 0 \le i, i+h < n\}.$$

Because (30) contributes k forward and k backward unit steps, inserting $E_{u,u+h}$ gives output in $\mathcal{U}_h$. Thus L preserves *ordinary* offsets, and consequently respects their summands inside a cyclic offset.

A flow comparison across each path cut shows that, if the coefficient from $E_{u,u+h}$ to $E_{i,i+h}$ is nonzero, the multisets of colors on the oriented intervals beginning at u and at i of length $|h|$ must coincide. Removing complete cycles of k colors, a remaining nonzero proper color interval has a unique start residue. Hence

$$h \not\equiv 0 \pmod{k} \quad \Longrightarrow \quad i \equiv u \pmod{k}. \tag{32}$$

This restriction is particularly strong since $n \le 2k+1$: for offsets two and three there are at most two positions in a residue class.

For a coefficient along the diagonal of an ordinary-offset restriction, input and output have the same endpoints, hence forward and reverse selected edge sets agree. Encode the choice of edge from duplicated color $j < t$ by $\epsilon_j = 0$ for its first occurrence and $\epsilon_j = 1$ for its second. The edge indicator along the path is

$$(1-\epsilon_0, \dots, 1-\epsilon_{t-1}, \underbrace{1, \dots, 1}_{k-t}, \epsilon_0, \dots, \epsilon_{t-1}). \tag{33}$$

When $t < k$, this pattern has one plus twice the number of $0 \to 1$ rises of ϵ as its number of connected components; the only permissible connected choices are the ordinary windows. A single free chord cannot make one Euler trail out of three or more components. When $t = k$, there is in addition a two-component selection $\epsilon = 0^a 1^{k-a}$ for each $1 \le a < k$: it takes one edge interval at the far left and another at the far right, separated by a gap of k edges. Such a selection cannot be joined by a chord of length two if $k \ge 3$, nor by a chord of length three if $k \ge 4$. We retain it for the long wrapping offsets below.

6.2. **The nonwrapping offset-two restriction.** Consider $L|_{\mathcal{U}_2}$. By (32), its matrix splits into singleton blocks and 2×2 blocks on the positions $i, i+k$ with $0 \le i \le t-2$. For a diagonal coefficient, the connected-window rule and Lemma 3.2 show that all contributions have sign $(-1)^i$, and their unsigned contributions are positive integers. Some window contains both endpoints of $E_{i,i+2}$. For $k \ge 3$ the multiplicity product in that window is at least two, so

$$|(L|_{\mathcal{U}_2})_{ii}| \ge 2. \tag{34}$$

For a pair $i, i+k$, the off-diagonal coefficient has absolute value one. For example, to carry the free input $E_{i+k,i+k+2}$ into output $E_{i,i+2}$, the cut equations force the unmatched forward edges at $i, i+1$ and reverse edges at $i+k, i+k+1$. Connectivity forces every edge in between. There is then one Euler trail: forward from i to $i+k$, take the free chord, and return backward to $i+2$. Reversing this construction gives the reverse coefficient, again of magnitude one. The two-component exceptional selections cannot intervene because their gap exceeds two.

Thus a paired restriction block has diagonal product of absolute value at least four and off-diagonal product of absolute value one; its determinant is nonzero. Every singleton is nonzero by (34). Hence

$$L(\mathcal{U}_2) = \mathcal{U}_2. \tag{35}$$

6.3. **The wrapping offset-two restriction.** The remaining two units in V_2 have ordinary offset $2-n$. It is convenient to compute the opposite offset $n-2$ and use an anti-involution. Choose a diagonal Z with

$$\frac{Z_{e+1,e+1}}{Z_{ee}} = (-1)^{\lfloor e/k \rfloor} \qquad (0 \le e < n-1), \tag{36}$$

and define $\theta(X) = ZX^tZ^{-1}$. The map θ reverses products and interchanges the forward and reverse colored inputs. Reversing the word of length $2k+1$ contributes $(-1)^k$, and reversing the $2k$ fixed inputs contributes the same sign. Consequently

$$L\theta = \theta L, \tag{37}$$

so opposite ordinary-offset restrictions are conjugate and have the same invertibility status.

There are two positions on $\mathcal{U}_{n-2}$. Its restriction is given by the following complete table (the displayed ordered pairs are diagonal entries):

(38)

case	diagonal entries	off-diagonal entries
$2 \le t < k,\ n$ even	$(4,-4)$	0
$2 \le t < k,\ n$ odd	$(-2,-2)$	0
$t = 1,\ k$ even	$(3,-3)$	each of magnitude ≤ 1
$t = 1,\ k$ odd	$(-2,-2)$	each of magnitude ≤ 1
$t = k$	$(-(2k-1),-(2k-1))$	0.

We give the signed counts because the last row does not follow from the ordinary connected-window rule alone.

For $2 \le t < k$, color conservation prevents mixing between the two positions. Applying (11) to the connected windows gives respectively $1 + 3(-1)^{n-2}$ and $-3 + (-1)^{n-1}$. For $t = 1$, the two window contributions give $\mathbf{1}_{2|k} + 2(-1)^k$ and $-2 - \mathbf{1}_{2|k}$. A possible off-diagonal trail has a turning point at one of k positions and the signs alternate, so its sum is 0 or ± 1. In either case the determinant is nonzero.

If $t = k$, the connected windows contribute -2 to each of the two diagonal coefficients. For the extra selections $\epsilon = 0^a 1^{k-a}$, the left and right bidirected components have lengths a and $k - a$ and the reverse-color sign is $(-1)^{k-a}$. For the long chord $E_{0,2k-1}$, the local starting sign at the left endpoint is positive, and the sign of the right component at its penultimate vertex is $(-1)^{k-a-1}$. Thus every two-end contribution has negative sign after the reverse-color sign is included. Its magnitude is two except for $a = k - 1$, where it is one. For the other long chord $E_{1,2k}$, the left local sign is negative and the right endpoint sign is $(-1)^{k-a}$; again all contributions are negative, with the unique magnitude-one case $a = 1$. More explicitly, for $b = k - a$ the contributions of one selection to the respective diagonal positions are

$$-(2 - \mathbf{1}_{a=k-1}), \qquad -(2 - \mathbf{1}_{a=1}), \quad 1 \le a \le k - 1. \tag{39}$$

The reverse-color sign $(-1)^b$ is included in both expressions. Summing the $k-1$ choices gives $-(2k-3)$ in each position. Including the connected-window contributions -2 gives $-(2k - 1)$; off-diagonal entries vanish by (32), since the two long-chord positions have different residues modulo $k \ge 3$. This proves the last row of (38) uniformly in k, including its signs.

Every restriction in (38) is invertible. By (37), so is the restriction on $\mathcal{U}_{2-n}$. Combining it with (35) yields

$$L(V_2) = V_2. \tag{40}$$

For odd n, (31) and (40) now imply surjectivity by Lemma 2.1.

6.4. **The additional edge for even sizes.** Assume $k \ge 4$. For the ordinary-offset-three restriction, (32) isolates the residue-zero block. Its index set is $\{0\}$ if $t \le 2$, and $\{0, k\}$ if $t \ge 3$. The corresponding matrix is

$$\begin{cases} [-2], & t = 1, \\ [-4], & t = 2, \\ \begin{pmatrix} -5 & \varepsilon \\ \eta & 5(-1)^k \end{pmatrix}, & 3 \le t \le k, \end{cases} \qquad |\varepsilon| = |\eta| = 1. \tag{41}$$

For the row at 0, the windows starting at $1, 2, 3$ contribute $-2, -2, -1$ when present; the window at zero contributes zero, since its endpoints have opposite parity. For the row at k, the first three windows contribute $1, 2, 2$, with common sign $(-1)^k$. The paired off-diagonal coefficients are the unique Euler-trail values ± 1, as for ordinary offset two; the two-end selections have too long a gap. This proves (41). Every block is invertible: in the last case its diagonal product has absolute value 25, while its off-diagonal product has

absolute value 1. Hence the slice contains E_{03}, by choosing the appropriate free input inside the isolated residue-zero block. For even n, the Hamiltonian path (5) completes the argument.

Proposition 6.1. *For $r \geq 5$ and $r < n < 2r$, the image of s_{2r-1} is $\mathrm{M}_n(K)$. The same argument gives this conclusion for $(r, n) = (4, 5)$ and $(4, 7)$.*

Proof. For $k \geq 4$ the diagonal and cyclic offset-two subspaces are in the same fixed slice by (31) and (40); for even n it also contains E_{03} by (41). Apply Lemma 2.1. If $k = 3$, the relevant intermediate odd sizes are $n = 5, 7$; step two is Hamiltonian and the same diagonal and offset-two arguments apply. The even intermediate size $n = 6$ is treated separately below. □

7. Exceptional small matrix orders

The preceding uniform proofs leave s_5 on $\mathrm{M}_4, \mathrm{M}_5$ and s_7 on M_6. We now give concrete slices or a block construction for each. In the two fixed-slice cases, the complete restriction matrices below are exact integer certificates; the recurrences for regenerating every entry are given in Appendix A.

7.1. s_5 **on** M_5**.** Use (29) with $k = t = 2$ and the fixed-input order (30). This case is not covered by the previous proof, which assumed $k \geq 3$. The restrictions to the diagonal and cyclic step-two spaces, in the natural unit bases, are

$$L_0 = \begin{pmatrix} 2 & -2 & 1 & 0 & 0 \\ 2 & -8 & 4 & -1 & 0 \\ 1 & -4 & 10 & -4 & 1 \\ 0 & -1 & 4 & -8 & 2 \\ 0 & 0 & 1 & -2 & 2 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 4 & -1 & 1 & 0 & 0 \\ -1 & -4 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}. \tag{42}$$

Direct exact evaluation gives

$$\det L_0 = 900, \qquad \det L_2 = -594. \tag{43}$$

Both restrictions are invertible. Since step two is Hamiltonian on five vertices, Lemma 2.1 proves

$$\operatorname{Im} s_5(\mathrm{M}_5(K)) = \mathrm{M}_5(K). \tag{44}$$

7.2. s_7 **on** M_6**.** Take the Vandermonde core (12) with $r = 4$ and $q = 6$. Its diagonal restriction has determinant

$$\det L_0 = -31\,911\,936. \tag{45}$$

The cyclic offset-two restriction is

$$L_2 = \begin{pmatrix} -16 & -24 & 24 & -4 & 0 & 0 \\ 24 & -132 & 144 & -24 & 0 & 0 \\ -24 & 144 & 60 & 24 & 0 & 0 \\ 4 & -24 & 24 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & -416 & -36 \\ 0 & 0 & 0 & 0 & -36 & -376 \end{pmatrix}. \tag{46}$$

The exact determinant of this restriction is

$$\det L_2 = 1\,838\,804\,889\,600.$$

For the parity-linking step the cyclic offset-three restriction is

(47)
$$L_3 = \begin{pmatrix} 164 & 44 & 0 & 0 & 0 & 0 \\ 44 & 432 & 100 & 0 & 0 & 0 \\ 0 & 100 & 52 & 0 & 0 & 0 \\ 0 & 0 & 0 & 164 & -44 & 0 \\ 0 & 0 & 0 & -44 & 432 & -100 \\ 0 & 0 & 0 & 0 & -100 & 52 \end{pmatrix}, \quad \det L_3 = 3\,776\,896\,843\,776.$$

All three restrictions belong to the *same* final-variable slice, and all are invertible. Thus it contains every diagonal, all of V_2, and in particular $E_{03} \in V_3$. The even-size Hamiltonian path (5) gives

$$\operatorname{Im} s_7(\mathrm{M}_6(K)) = \mathrm{M}_6(K). \tag{48}$$

7.3. s_5 **on** M_4**: an explicit three-dimensional core.** We use an elementary 3×3 construction. Put

$$A = E_{01}, \quad B = E_{12}, \quad C = E_{20}, \quad D = \operatorname{diag}(-1, 0, 1) \tag{49}$$

inside $\mathrm{M}_3(K)$. Direct expansion of the alternator yields

$$s_4(A, B, C, D) = \operatorname{diag}(-1, 2, -1), \tag{50}$$

and the slice image is the five-dimensional subspace

$$W = \left\{ \begin{pmatrix} \alpha & 0 & z \\ u & \beta & 0 \\ 0 & v & \alpha \end{pmatrix} : \alpha, \beta, u, v, z \in K \right\}. \tag{51}$$

Indeed the specified matrix is obtained from the single free input

$$X = \operatorname{diag}\left(\frac{3\beta - 4\alpha}{5}, \frac{2\beta - \alpha}{5}, 0 \right) + \frac{u}{2} E_{10} + \frac{v}{2} E_{21} - \frac{z}{4} E_{02}. \tag{52}$$

Equations (50) and (52) follow by direct evaluation.

Lemma 7.1. *Every similarity class in* $\mathrm{M}_3(K)$ *meets* W.

Proof. For a cyclic target with characteristic polynomial $x^3 - tx^2 + qx - \delta$, choose $\alpha \in K$ solving

$$2t\alpha - 3\alpha^2 = q, \qquad \beta = t - 2\alpha, \qquad u = v = 1, \qquad z = \delta - \alpha^2 \beta.$$

The matrix in (51) has the required characteristic polynomial. Moreover $\det[e_0, Me_0, M^2 e_0] = 1$, so it is cyclic and the characteristic polynomial identifies its cyclic similarity class. For noncyclic nonscalar classes the elementary representatives $\operatorname{diag}(c, d, c)$ and $cI + E_{10}$, as appropriate to the Jordan partition, lie in W. Scalars plainly do also. □

Embed (49) into the upper-left 3×3 block of $\mathrm{M}_4(K)$, with zero lower-right singleton. Because the even-standard endpoint (50) is invertible, the two off-diagonal rectangular maps are arbitrary, and the upper-left block ranges over W. The lower-right output is always zero. Thus this one slice has image

$$\left\{ \begin{pmatrix} Y & b \\ c & 0 \end{pmatrix} : Y \in W,\ b \in K^{3\times 1},\ c \in K^{1\times 3} \right\}. \tag{53}$$

Let $T \in \mathrm{M}_4(K)$ be nonscalar. There exists v such that v, Tv are independent; otherwise every vector is an eigenvector and T would be scalar. Choose $\phi \in (K^4)^*$ with $\phi(v) = 1$, $\phi(Tv) = 0$. In a basis adapted to $K^4 = \ker \phi \oplus Kv$, the bottom-right coefficient of T is zero. By Lemma 7.1, a similarity change on $\ker \phi$ makes its 3×3 compression a member of W. The other two blocks are unrestricted in (53). Hence every nonscalar T is a single s_5 value, up to conjugation.

To obtain scalars we need a short general observation.

Lemma 7.2 (Traceful multilinear values)**.** *Let f be multilinear on $\mathrm{M}_n(K)$, with K algebraically closed of characteristic zero. If f has an evaluation with nonzero trace, then some evaluation of f is a nonzero scalar matrix. Consequently every scalar matrix is in its actual image.*

Proof. Fix a primitive nth root of unity and the usual clock-and-shift matrices U, V with $UV = \zeta VU$. The n^2 matrices $U^a V^b$, $a, b \bmod n$, form a basis of $\mathrm{M}_n(K)$ and are homogeneous for the $\mathbb{Z}_n^2$ grading. Every product of a fixed tuple of homogeneous basis elements lies in the same graded component. By multilinearity, expanding a traceful evaluation in this basis gives a homogeneous basis tuple on which f has nonzero trace. Every nonzero graded component has trace zero, whereas the degree-zero component is KI. Thus this particular value is a nonzero scalar. Rescaling one input gives any scalar. □

The core (49) is traceful: for example, (51) contains $\mathrm{diag}(1, 0, 1)$, which is a single s_5 value on $\mathrm{M}_3(K)$. Embedding the same evaluation in $\mathrm{M}_4(K)$ preserves its nonzero trace. Lemma 7.2 therefore gives all scalar targets, completing

$$\operatorname{Im} s_5(\mathrm{M}_4(K)) = \mathrm{M}_4(K). \tag{54}$$

8. Proof of the odd-standard assertion

Proof of the odd assertion of Theorem 1.1. If $n < r$, then $2r - 1 \geq 2n + 1$. By Amitsur–Levitzki, s_{2n} is an identity on $\mathrm{M}_n(K)$; the standard recursion expresses every higher alternator through lower ones, proving $s_{2r-1} = 0$.

For $r = 1$, the claim is $s_1(X) = X$. For $r = 2$ and $n = 1$, the alternating cubic vanishes. For $n \geq 2$, the published cubic image-linearity result [15, Theorem 3.5] implies that $\operatorname{Im} s_3$ is a vector subspace. The substitution $s_3(I, A, B) = [A, B]$ gives every traceless matrix by the classical commutator theorem [1], and a traceful 2×2 evaluation, embedded into $\mathrm{M}_n(K)$, gives

a value outside that subspace. For example, $s_3(E_{00}, E_{01}, E_{10}) = \mathrm{diag}(2, 1)$ on $\mathrm{M}_2(K)$. Since the traceless subspace has codimension one, cubic image-linearity now gives the whole $\mathrm{M}_n(K)$.

Suppose $r \geq 3$. The case $n = r$ follows from Corollary 5.3. The range $n \geq 2r$ follows from Corollary 4.2. For $r < n < 2r$, when $r \geq 5$ apply Proposition 6.1; when $r = 4$, it covers $n = 5, 7$ and (48) covers $n = 6$. When $r = 3$, the sizes $n = 4, 5$ are (54) and (44). The cases are partitioned as follows:

Degree parameter	Surjective sizes and argument
$r = 1$	all $n \geq 1$; identity map
$r = 2$	all $n \geq 2$; cubic theorem and direct values
$r = 3$	$n = 3$ cycle, $n = 4, 5$ cases, $n \geq 6$ transfer
$r = 4$	$n = 4$ cycle; $n = 5, 7$ periodic; $n = 6$ case; $n \geq 8$ transfer
$r \geq 5$	$n = r$ cycle; $r < n < 2r$ periodic; $n \geq 2r$ transfer

Together with the identity branch $n < r$, this exhausts every pair of positive integers (r, n). □

Remark 8.1. *The large and periodic ranges use a fixed linear slice, whereas the minimal-size construction chooses a slice for the prescribed Jordan target.*

II. Even standard polynomials

9. Actual images from an attained rank-one nilpotent

The remainder of the article deals with s_{2r}. Throughout this part, its trace-zero property follows directly from cyclic alternation: for any word of length $2r$, moving the first letter to the last preserves the trace but changes the permutation sign by $(-1)^{2r-1} = -1$. Consequently

$$\mathrm{tr}\, s_{2r}(A_1, \ldots, A_{2r}) = 0, \qquad \mathrm{Im}\, s_{2r}(\mathrm{M}_n) \subseteq \mathfrak{sl}_n. \tag{55}$$

The identity $s_{2n} = 0$ on M_n implies the entire zero branch for $n \leq r$. Moreover $s_{2r}(A_1, \ldots, A_{2r-1}, I_n) = 0$. For nonzero images the following local-to-global mechanism is essential. It is the smooth-map formulation of the nilpotent argument in [14, Lemma 2]; the elementary degeneration is included to make the actual-value conclusion explicit.

Lemma 9.1 (Attained-nilpotent actual-image criterion)**.** *Let $F : \mathbb{A}_K^m \to \mathfrak{sl}_n(K)$ be a polynomial map whose image contains zero and is invariant under similarities and multiplication by nonzero scalars. Suppose that for some x the value $R = F(x)$ is nonzero, rank one and square zero, and $dF_x : K^m \to \mathfrak{sl}_n(K)$ is surjective. Then $\mathrm{Im}\, F = \mathfrak{sl}_n(K)$.*

Proof. Both affine spaces are smooth. Surjectivity of the differential makes F smooth at x, and smoothness is open; the restriction to a suitable neighborhood maps onto a Zariski-open subset containing R. In particular, the image contains an open neighborhood U of R in $\mathfrak{sl}_n$.

Let $T \in \mathfrak{sl}_n$ be nonzero. It is nonscalar in characteristic zero. A nonscalar linear transformation has a vector v for which v, Tv are linearly independent: otherwise every line is invariant, forcing the transformation to be scalar. Thus after a similarity $T_{01} \neq 0$. For $t \neq 0$, write

$$D(t) = \operatorname{diag}(t^{-1}, t, 1, \dots, 1), \qquad T(t) = t^2 D(t) T D(t)^{-1}. \tag{56}$$

All entries of $T(t)$ have nonnegative powers of t; its value at $t = 0$ is $T_{01}E_{01}$. Every nonzero rank-one square-zero matrix is similar, up to a nonzero scalar, to R. By similarity and scaling invariance there is therefore an open neighborhood of $T_{01}E_{01}$ contained in the image. Its preimage under the affine-line morphism $t \mapsto T(t)$ contains zero and hence some nonzero t. Thus $T(t)$ is an actual value. Undoing the similarity and scalar t^2 shows that T is an actual value. The zero target is already in the image. □

A standard polynomial is equivariant for simultaneous conjugation; scaling one input scales its output, and setting an input to zero gives zero. Thus every standard evaluation map satisfies the invariance assumptions of Lemma 9.1.

10. A uniform minimal-size seed

The first matrix order at which s_{2r} need not vanish is $n = r + 1$. The following construction treats every such order with one matrix-unit formula, including the full differential required by Lemma 9.1.

Theorem 10.1 (Uniform even-standard seed)**.** *For every $n \geq 2$,*

$$\operatorname{Im} s_{2n-2}(\mathrm{M}_n(K)) = \mathfrak{sl}_n(K). \tag{57}$$

More precisely, the evaluation constructed below has a nonzero rank-one square-zero output and surjective full differential.

Proof. Write $p = n - 2$ and index matrix units by $0, \dots, n-1$. For $n \geq 3$, use the ordered $2p + 2$ inputs

$$\begin{aligned} \mathcal{A} = (&L = E_{00},\ X = E_{01},\\ &S_2 = E_{02}, T_2 = E_{20}, \dots,\\ &S_{n-1} = E_{0,n-1}, T_{n-1} = E_{n-1,0}). \end{aligned} \tag{58}$$

The signed matrix-unit trail calculation gives

$$\begin{aligned} s_{2n-2}(\mathcal{A}) &= (p+1)!E_{01},\\ \partial_{X \leftarrow E_{10}} s_{2n-2}(\mathcal{A}) &= -(p+1)!E_{10},\\ \partial_{X \leftarrow E_{12}} s_{2n-2}(\mathcal{A}) &= -p!E_{12},\\ \partial_{S_2 \leftarrow E_{10}} s_{2n-2}(\mathcal{A}) &= -(p+1)!E_{20},\\ \partial_{S_2 \leftarrow E_{12}} s_{2n-2}(\mathcal{A}) &= p!(E_{11} - E_{22}). \end{aligned} \tag{59}$$

The notation $\partial_{A_j \leftarrow V}$ denotes the directional derivative in the jth argument; by multilinearity it is exactly the value with that argument replaced by V.

Here is a signed-trail derivation valid for all n. A nonzero matrix-unit word is precisely an oriented Euler trail on the labelled directed graph. In

the unmodified tuple, each trail starts at 0, traverses the loop L and the p excursions S_jT_j in any order, and ends with X. The excursions have even length and commute in permutation parity; moving X to the end crosses $2p$ labels. These $(p+1)!$ trails all have positive sign. With $X = E_{10}$ the trail instead begins with X, making its sign negative. With $X = E_{12}$ it starts with X, T_2, traverses the remaining root blocks and ends with S_2; there are $p!$ such trails and the word has odd inversion parity. With $S_2 = E_{10}$ the trail begins with T_2 and traverses the root loop, the new excursion XS_2, and the remaining $p-1$ excursions; the resulting $(p+1)!$ trails are negative.

Finally, for $S_2 = E_{12}$, there is a directed triangle X, S_2, T_2 together with a loop and the other two-edge excursions. Trails starting at 0 or another leaf cancel by exchanging the odd-length loop and triangle. Starting at 1 leaves $p!$ positively signed trails, whereas starting at 2 leaves $p!$ negatively signed trails. This proves the last formula in (59), including its diagonal signs.

Let $\mathcal{D}$ be the image of the *full* differential of s_{2n-2} at $\mathcal{A}$. It lies in $\mathfrak{sl}_n$ by (55). Equivariance supplies every commutator $[Z, E_{01}] \in \mathcal{D}$ (the scalar $(p+1)!$ is nonzero). Consider a change of basis $g = \mathrm{diag}(1, 1, G)$, $G \in \mathrm{GL}_p(K)$, acting on the leaf coordinates. It fixes L and X and sends the p outgoing and p incoming leaf arguments to mutually contragredient linear combinations of their original arguments. The combined transformation on these $2p$ frozen slots has determinant one. The alternating multilinear map, and hence its differential image at the tuple, is therefore invariant under conjugation by g.

The orbits of E_{12} and E_{20} in (59) supply all E_{1j} and E_{i0} for $i, j \geq 2$. The orbit of $E_{11} - E_{22}$ spans

$$\{(\mathrm{tr}\, B)E_{11} - B : B \text{ is supported on the leaf block}\}, \tag{60}$$

because the conjugates of a rank-one diagonal idempotent span its full matrix block: rank-one idempotents include E_{ii} and $E_{ii} + E_{ji}$ for distinct leaf indices. Together with E_{10} and $[Z, E_{01}]$, these directions give every off-diagonal matrix unit and every diagonal difference. Thus $\mathcal{D} = \mathfrak{sl}_n$, as required.

When $n = 2$, take (E_{00}, E_{01}). Its commutator value is E_{01}; replacing E_{01} by E_{10} gives $-E_{10}$ and the conjugation direction gives $E_{00} - E_{11}$. The full differential is $\mathfrak{sl}_2$. In each order the output is a nonzero rank-one square-zero matrix, so Lemma 9.1 proves (57). □

Remark 10.2 (The size-four seed in Tang's construction)**.** *At $n = 4$, the tuple in* (58) *is $(E_{00}, E_{01}, E_{02}, E_{20}, E_{03}, E_{30})$. Qiyue Tang's Appendix A uses $(4E_{11}, E_{12}, E_{13}, E_{14}, E_{31}, E_{41})$ in one-based coordinates* [14, Proposition 11]*. The two coincide after relabelling the indices, interchanging the fourth and fifth slots, and multiplying the first slot by 4; alternating multilinearity accounts for the sign and scale. Thus the size-four instance agrees, up to these changes, with Tang's matrix-unit construction.*

11. Admissible cores and an additive transfer

The seed can be transferred to larger orders by a normal-map criterion that keeps track of the *full differential*, rather than merely of a frozen generic kernel. Fix $r \geq 1$ and write

$$k = 2r - 1. \tag{61}$$

For $B = (B_1, \dots, B_k) \in \mathrm{M}_N(K)^k$ and $u \in K^N$, define

$$\begin{aligned} S &= s_k(B_1, \dots, B_k), \\ L(P) &= s_{k+1}(B_1, \dots, B_k, P), \\ C_i(P) &= s_k(B_1, \dots, B_{i-1}, P, B_{i+1}, \dots, B_k), \\ \Gamma_{B,u}(P) &= \big(L(P), C_1(P)u, \dots, C_k(P)u\big). \end{aligned} \tag{62}$$

Thus $\Gamma_{B,u} : \mathrm{M}_N(K) \to \mathrm{M}_N(K) \oplus (K^N)^k$ is a linear map in P; the frozen data (B, u) vary polynomially. Call the size N *admissible for* r if there exists one such tuple with

$$\det S \neq 0, \qquad \ker \Gamma_{B,u} = \{0\}. \tag{63}$$

Both conditions are polynomial nonvanishing conditions in unrestricted parameters. A witness therefore defines a nonempty Zariski-open subset of the core parameter space.

Lemma 11.1 (Normal-map criterion)**.** *If N is admissible for r, then*

$$\operatorname{Im} s_{2r}(\mathrm{M}_{N+1}(K)) = \mathfrak{sl}_{N+1}(K). \tag{64}$$

Indeed the defining witness yields an attained nonzero rank-one square-zero value at which the full differential is surjective. At the particular border tuple used below, assuming S invertible, this differential condition is equivalent to injectivity of $\Gamma_{B,u}$.

Proof. Embed the fixed matrices and the final input as

$$A_i = \begin{pmatrix} B_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}. \tag{65}$$

Only words ending in Y can survive, because $YA_i = 0$. Since k is odd, the value of the even alternator is

$$s_{k+1}(A_1, \dots, A_k, Y) = \begin{pmatrix} 0 & Su \\ 0 & 0 \end{pmatrix}. \tag{66}$$

If $u = 0$, then $\Gamma = L$ and its kernel contains all B_i (and also the identity), contradicting injectivity. Hence $Su \neq 0$. The value is nonzero, square zero and rank one.

To compute the differential, pair its values with a general matrix using the nondegenerate trace form. Variations of Y in the upper and lower borders give the full upper and lower border directions by left and right multiplication by S and $-S$, respectively. Thus every normal is block diagonal. The differential

values have trace zero, so scalar matrices are normals by (55). Subtracting a scalar we may write a normal uniquely as $\mathrm{diag}(P, 0)$.

Variations of the old $N \times N$ block of Y give $\mathrm{tr}(PL(Q))$ for arbitrary Q. The map L is skew-adjoint for the trace pairing. To see this, collect the cyclic classes in $\mathrm{tr}\, s_{k+2}(B_1, \ldots, B_k, Q, P)$. The length $k+2$ is odd, so a cyclic shift preserves sign; moving Q past the k frozen letters contributes a minus sign. Consequently the trace equals $-(k+2)\,\mathrm{tr}(QL(P))$. Interchanging P, Q in the alternating trace yields

$$\mathrm{tr}(PL(Q)) = -\,\mathrm{tr}(QL(P)). \tag{67}$$

The old-block normal equations are therefore $L(P) = 0$.

Now vary the ith A_i in the lower border by a row $c \in (K^N)^*$. For a nonzero word the unique upper-border factor Y must be immediately followed by this lower-border variation. Remove Y and rotate the remaining k letters cyclically. If a of the original old letters followed the adjacent pair, the removal has sign $(-1)^{a+1}$ and moving the varied letter to its original position has sign $(-1)^a$. The product is -1, and the normal pairing is exactly

$$-c\, C_i(P)u. \tag{68}$$

Because all rows c are available, these normal equations say $C_i(P)u = 0$ for each i. Therefore every normal modulo scalars satisfies $\Gamma_{B,u}(P) = 0$. Conversely, in the pure border tuple (65), these are all the normal equations: other source-block variations yield only upper-border outputs or zero. This also proves the asserted equivalence. If Γ is injective, all normals are scalar, so the full differential has image $\mathfrak{sl}_{N+1}$. Lemma 9.1 now proves (64) as an *attained* image equality. □

A complementary rectangular nonvanishing calculation records the invertibility available for independently varying block-diagonal inputs.

Lemma 11.2 (Mixed-rectangle nonvanishing)**.** *Let $a, b \geq 1$ and $m < 2(a+b)$. For an m-variable standard polynomial, the free rectangular map $\mathrm{M}_{a\times b} \to \mathrm{M}_{a\times b}$ obtained by freezing $m-1$ block-diagonal inputs of sizes a and b is generically invertible as the frozen blocks vary independently.*

Proof. Choose integers $e, f \geq 0$ with $e + f = m - 1$, $e < 2a$ and $f < 2b$; they exist since $m < 2(a+b)$. Specialize the first e frozen labels to the a block and the remaining f to the b block. The only surviving words put the left labels before the free rectangle and the right labels after it, so the map is

$$X \longmapsto (-1)^f s_e(A) X s_f(B), \qquad s_0 = I. \tag{69}$$

Both factors admit invertible evaluations. Indeed, for degree below the Amitsur–Levitzki threshold a standard polynomial is a nonidentity. Expand a nonzero evaluation in the clock-and-shift homogeneous basis of the corresponding matrix algebra; by multilinearity one homogeneous tuple yields a nonzero scalar multiple of an invertible basis element. Thus the rectangular

map is invertible at one specialization, and its determinant is a nonzero polynomial. □

Theorem 11.3 (Additivity of admissible core sizes)**.** *For a fixed $r \geq 1$, if $a, b \geq r$ are admissible core sizes, then $a + b$ is admissible for r.*

Proof. Choose witnesses $(B^{(a)}, u_a)$ and $(B^{(b)}, u_b)$ for the two sizes. For one parameter t set

$$B_i(t) = \operatorname{diag}(B_i^{(a)}, tB_i^{(b)}), \qquad u = \begin{pmatrix} u_a \\ u_b \end{pmatrix}. \tag{70}$$

For every $t \neq 0$, the diagonal endpoint blocks are S_a and $t^k S_b$ and hence are invertible. The old-block L and C_i operators in the second block are respectively scaled by t^k and t^{k-1}, so their individual admissibility kernels remain zero.

On the two off-diagonal rectangular spaces, the map L_t in (62) is polynomial in t. At $t = 0$ the top-right rectangular map is $X \mapsto S_a X$, because only words with the free variable last survive. The bottom-left rectangular map is $X \mapsto -XS_a$, because only words with the free variable first survive and k is odd. Both are invertible. Therefore their determinants are nonzero polynomials in the *same* scalar parameter. As K is infinite, select one common $t \in K^\times$ avoiding their zero sets.

At this fixed t, if $\Gamma_{B(t),u}(P) = 0$, the two mixed L_t equations force the off-diagonal blocks of P to be zero. The remaining $C_i(P)$ are block diagonal, and the normal-map equations reduce to those for the two original cores, with nonzero scalar factors on the second block. Both diagonal blocks of P vanish by the original injectivity hypotheses. Thus $\Gamma_{B(t),u}$ is injective, and $\det s_k(B(t)) \neq 0$. This constructs a single admissible witness of size $a + b$. □

Corollary 11.4 (An infinite congruence family)**.** *For every $r, q \geq 1$,*

$$\operatorname{Im} s_{2r}(\mathrm{M}_{qr+1}(K)) = \mathfrak{sl}_{qr+1}(K). \tag{71}$$

Proof. In Theorem 10.1, set $n = r + 1$, place the exit variable last, and remove its sink coordinate. The old $r \times r$ core has

$$B = (E_{00}, E_{01}, E_{10}, \ldots, E_{0,r-1}, E_{r-1,0}), \qquad u = e_0, \tag{72}$$

with the obvious singleton interpretation when $r = 1$. Counting closed trails through the root loop and the $r - 1$ two-edge excursions gives

$$S = s_{2r-1}(B) = \operatorname{diag}(r!, (r-1)!, \ldots, (r-1)!), \tag{73}$$

which is invertible in characteristic zero. The full differential in Theorem 10.1 is onto $\mathfrak{sl}_{r+1}$; the converse statement in Lemma 11.1 shows that this old core is admissible. For $r = 1$ it follows also directly from $S = 1$ and $C_1(P)u = P$. Add q copies of the admissible size r with Theorem 11.3, then apply Lemma 11.1 in size $qr + 1$. □

12. Exact core certificates in degrees eight and ten

For each r, the uniform construction supplies an admissible core of size r. By additive transfer, to obtain all sizes $N \geq r$, it suffices to establish admissibility for the finite range

$$r, r+1, \ldots, 2r-1. \tag{74}$$

Indeed every $N \geq r$ has the form $N = qr + s$, $q \geq 1$ and $0 \leq s < r$. If $s = 0$, use q copies of r; otherwise use $q-1$ copies of r and one copy of $r+s$. Thus a finite collection of admissible cores extends to every larger core order by Theorem 11.3.

For $r = 4$ and $r = 5$, the seven residual cases have exact integer witnesses. For each case in the table, the corresponding certificate contains an explicit ordered tuple B of integer matrices, an integer column u, $\det S \neq 0$, and the row indices and exact nonzero integer determinant of an $N^2 \times N^2$ submatrix of $\Gamma_{B,u}$. The last two columns give reductions of these full integer minors modulo 101 and 103; their nonzero residues verify nonvanishing in characteristic zero.

r	N	$2r$	$\operatorname{rank}\Gamma$	minor mod101	minor mod103
4	5	8	25	48	57
4	6	8	36	36	25
4	7	8	49	75	58
5	6	10	36	52	34
5	7	10	49	64	45
5	8	10	64	45	60
5	9	10	81	85	7

The certificate names are listed in Appendix B; they encode each pair (r, N) from the table. For example, `certificates/border_r4_N5.json` records $(r, N) = (4, 5)$. The normal map has N^2 columns, so a nonzero N^2-minor proves its injectivity. The invertible endpoint is separately recorded for each case.

Theorem 12.1 (Degrees eight and ten without generic-nullity input)**.** *For every algebraically closed characteristic-zero field K,*

$$\begin{aligned} \operatorname{Im} s_8(\mathrm{M}_n(K)) &= \begin{cases} \{0\}, & n \leq 4, \\ \mathfrak{sl}_n(K), & n \geq 5, \end{cases} \\ \operatorname{Im} s_{10}(\mathrm{M}_n(K)) &= \begin{cases} \{0\}, & n \leq 5, \\ \mathfrak{sl}_n(K), & n \geq 6. \end{cases} \end{aligned} \tag{75}$$

Proof. For $r = 4$, the uniform seed gives admissible size $N = 4$, and the three finite residual sizes in (74) are $N = 5, 6, 7$. For $r = 5$, the uniform seed gives $N = 5$, and the finite residual sizes are $N = 6, 7, 8, 9$. Thus the seven exact certificate witnesses together with Theorem 11.3 make every core size $N \geq r$ admissible, for $r = 4, 5$. Lemma 11.1 yields the actual traceless image for every order $n = N + 1 \geq r + 1$. The other orders vanish by Amitsur–Levitzki. □

Appendix B describes how to check the seven saved certificates. The extension to all matrix orders uses Theorem 11.3.

13. Generic nullity and the all-even classification

We now deduce the remaining even degrees from two precise external generic-kernel theorems. The argument extends the approach to s_6 in Qiyue Tang's Appendix A [14, Proposition 11]; in particular, its use of a rank-one nilpotent, a generic central division algebra and a projective eigenline incidence is directly indebted to that source.

Write

$$\mathcal{L}_{A_1,\dots,A_j}(P) = s_{j+1}(A_1,\dots,A_j,P). \tag{76}$$

The two inputs needed below are as follows. Brassil and Reichstein [3, Theorem 1.2] prove that for j even, $2 \le j \le 2N-2$, and generic j-tuples in M_N, the kernel of (76) has dimension j in characteristic zero. Their theorem is stated more generally over an infinite field with specified characteristic exclusions; only the characteristic-zero specialization is used here. Lin, Chenhao Tang and Zhao [9, Theorem 1.2 and Remark 6.1] prove the complementary odd-frozen statement: for $N \ge 3$, odd j with $3 \le j \le 2N-3$ and any characteristic-zero field, the generic nullity is

$$\dim\ker \mathcal{L}_{A_1,\dots,A_j} = \begin{cases} j+1, & N \text{ even},\\ j+2, & N \text{ odd}. \end{cases} \tag{77}$$

These generic-kernel dimensions provide the input for the following attained-image argument.

Theorem 13.1 (Even-standard classification from generic nullity)**.** *For every algebraically closed field K of characteristic zero and all $r, n \ge 1$,*

$$\operatorname{Im} s_{2r}(\mathrm{M}_n(K)) = \begin{cases} \{0\}, & n \le r,\\ \mathfrak{sl}_n(K), & n \ge r+1. \end{cases} \tag{78}$$

For $r \ge 2$ and $n \ge r+2$, the proof uses the above external Brassil–Reichstein and Lin–Tang–Zhao generic-nullity theorems.

Proof. If $n \le r$, then $2r \ge 2n$, and the standard identity forces the image to be zero. The case $r = 1$ is the classical commutator image theorem [1]. For $r \ge 2$ and $n = r+1$, the positive claim is Theorem 10.1.

Now let $r \ge 2$ and $n \ge r+2$. Set

$$N = n-1, \qquad k = 2r-1; \quad N \ge r+1 \ge 3, \quad 3 \le k \le 2N-3. \tag{79}$$

For each C_i of (62), the number $k-1 = 2r-2$ of frozen inputs is even and satisfies $2 \le k-1 \le 2N-2$. Therefore both external nullity theorems apply in precisely their stated ranges.

Take k independent generic $N \times N$ matrices $B_1,\dots,B_k$ over K, and let D denote the central division algebra of generic matrices generated by them, with center F and degree N. Write E for the field of their independent

matrix-entry indeterminates. The natural generic matrix representation embeds D in $\mathrm{M}_N(E)$ and splits it over E; the center F embeds in E as scalar matrices. The matrices B_i are linearly independent over F, because their vectorized columns are independent already over E (some $k \times k$ minor is a nonzero polynomial, as $k \leq N^2$). Set

$$S = s_k(B_1, \ldots, B_k) \in D. \tag{80}$$

Since $k < 2N$, the standard polynomial is not an identity of M_N; genericity makes $S \neq 0$ and thus $S \in D^\times$. In the generic matrix model, all of the maps L, C_i from (62) are F-linear operators on the N^2-dimensional space D. Their ranks persist under splitting-field extension.

The Brassil–Reichstein theorem gives, simultaneously for every i,

$$\ker C_i = \operatorname{span}_F\{B_j : j \neq i\}. \tag{81}$$

Indeed alternation gives the displayed $(k-1)$-dimensional subspace, and the generic theorem gives exactly this nullity. The generic open conditions for the finitely many C_i intersect in the irreducible generic input space, so one set of generic entries realizes all the ranks at once. The odd-frozen theorem (77) gives

$$W = \ker L / \operatorname{span}_F(B_1, \ldots, B_k), \qquad \dim_F W = \begin{cases} 1, & N \text{ even}, \\ 2, & N \text{ odd}. \end{cases} \tag{82}$$

(The upper bound $\dim W \leq 2$ would suffice below.)

Alternation also gives

$$C_i(B_j) = \delta_{ij} S. \tag{83}$$

The F-linear maps

$$T_i : W \longrightarrow D/F1, \qquad [P] \longmapsto S^{-1} C_i(P) + F1 \tag{84}$$

are consequently well defined. They are *jointly injective*. In fact, if $C_i(P) = b_i S$ for every i, then (83) gives $P - \sum_i b_i B_i \in \bigcap_i \ker C_i = \{0\}$; the last equality follows from (81) and the linear independence of all the B_i. This joint injectivity persists after scalar extension.

Choose an F-linear section $W \to \ker L$ and let $\bar{F}$ be an algebraic closure of F. Fix a splitting identification $D \otimes_F \bar{F} \simeq \mathrm{M}_N(\bar{F})$. For a class $[P] \in W_{\bar{F}}$ the endomorphisms $S^{-1}C_i(P)$ are defined by this section; changing the section adds scalars to them, so their eigenlines are independent of the section. We claim that there is a nonzero column $u \in \bar{F}^N$ which is not a common eigenvector of $S^{-1}C_i(P)$ for *any* nonzero class $[P]$.

If $\dim W = 0$ the claim is immediate. If $\dim W = 1$, joint injectivity means at least one of these matrices is nonscalar; its eigenvectors form a finite union of proper eigenspaces, which does not exhaust $\bar{F}^N$. The substantive case is $\dim W = 2$. In that case form the projective closed incidence

$$\mathcal{I} = \{([P], [u]) \in \mathbb{P}(W_{\bar{F}}) \times \mathbb{P}^{N-1}_{\bar{F}} : u \wedge S^{-1} C_i(P) u = 0 \quad (1 \leq i \leq k)\}. \tag{85}$$

The incidence records *simultaneously* all kernel classes and their common eigenlines.

At the generic point of the projective line $\mathbb{P}(W_{\bar{F}})$, write $P = P_0 + tP_1$ using an F-basis of W. Joint injectivity of (84) shows that some $S^{-1}C_i(P)$ is noncentral in $D \otimes_F F(t)$. This algebra is a division algebra: it embeds in the Laurent-series division ring $D((t))$, with central indeterminate t, and as a finite-dimensional domain over $F(t)$ every nonzero element is invertible. A noncentral element in that division algebra has an irreducible separable minimal polynomial of degree $a \geq 2$ over $F(t)$; its degree divides N. Upon splitting, its reduced characteristic polynomial is the N/a-th power of that minimal polynomial. Its distinct eigenvalues all have eigenspace dimension $N/a \leq \lfloor N/2 \rfloor$. Consequently the generic fiber of (85) has dimension at most $\lfloor N/2 \rfloor - 1$.

At every special geometric point $[P]$, the scalar-extended joint injectivity still ensures that some $S^{-1}C_i(P)$ is nonscalar. Its eigenspaces have dimension at most $N-1$, and hence the corresponding fiber has dimension at most $N-2$. Thus any component of $\mathcal{I}$ dominating the projective line has dimension at most $\lfloor N/2 \rfloor$, and a component over a point has dimension at most $N-2$. Since $N \geq 3$,

$$\dim \mathcal{I} \leq N - 2 < N - 1 = \dim \mathbb{P}^{N-1}. \tag{86}$$

The projection to $\mathbb{P}^{N-1}$ is proper, so its image is a proper closed subset. Choose $[u]$ outside that image. This proves the claim for the *same* u and every nonzero kernel class.

For this column, the normal map $\Gamma_{B,u}$ is injective after the chosen splitting. Indeed $\Gamma(P) = 0$ first implies $P \in \ker L$. If its class in W were nonzero, all $C_i(P)u = 0$ would make $[u]$ a common eigenline, contrary to construction. Therefore $P = \sum_i b_i B_i$; then $C_i(P)u = b_i Su = 0$ for all i forces each $b_i = 0$. We have also retained $\det S \neq 0$.

It remains to obtain a witness over the originally specified field K, rather than merely over an abstract splitting of D. Choose an algebraic closure Ω of the original generic-entry field E containing an F-embedding of $\bar{F}$. The generic matrix representation and the chosen split representation of $D \otimes_F \Omega \simeq \mathrm{M}_N(\Omega)$ differ by an inner automorphism, by Skolem–Noether. Transport the column u under that conjugacy. Equivariance of S, L, C_i shows that $\det S \neq 0$ and $\operatorname{rank} \Gamma = N^2$ hold in the original generic-entry representation over Ω.

Both assertions mean that the product of $\det S$ with a suitable $N^2 \times N^2$ minor of Γ is a *nonzero polynomial* in the generic entries of B and the coordinates of u, with coefficients in K. Because K is infinite, there is one specialization of all these coordinates to K at which this polynomial is nonzero. This gives an admissible core of size $N = n - 1$ over K itself, in the sense of (63). By Lemma 11.1, the full polynomial map s_{2r} attains a nonzero rank-one nilpotent and therefore has actual image $\mathfrak{sl}_n(K)$. This proves (78) in the remaining range. □

Proof of Theorem 1.1. The odd assertion is proved in Section 8. The even assertion follows from Theorem 13.1, including the identity and low-order cases. Both arguments establish equality of sets of individual values: by explicit slices in odd degree and by the attained-nilpotent criterion in even degree. □

Use of AI

OpenAI's GPT-6 Pro assisted substantially in the mathematical investigation, development of proofs, exact computation, source assessment and manuscript preparation. The author is responsible for the mathematical claims, attribution and final presentation.

Appendix A. Exact checks for odd standard polynomials

The programs `code/verify_standard.py` and `code/graded.py` provide finite exact checks of the odd path, cycle, periodic and exceptional-order calculations. The first uses SymPy and evaluates alternating products by a full-matrix subset recurrence and by enumerating nonzero matrix-unit Euler trails. The second computes cyclic-homogeneous slice restrictions by a graded subset recurrence.

For fixed inputs $A_1, \dots, A_m$, let $D(S)$ denote the alternating sum over the variables indexed by $S \subseteq \{1, \dots, m\}$. The full-matrix recurrence is

$$D(\varnothing) = I, \qquad D(S) = \sum_{j \in S} (-1)^{|S| - \mathrm{pos}_S(j)} D(S \setminus \{j\}) A_j, \tag{87}$$

where $\mathrm{pos}_S(j)$ is the position of j in the increasing listing of S. For homogeneous cyclic inputs $A_j = \mathrm{diag}(w_j) P^{g_j}$, the graded calculation tracks the offset of each product and constructs the restriction matrix directly. The two calculations compare every column $L(E_{i,i+h \bmod n})$ for the specified exceptional cases; the generated JSON also includes integer determinants, finite trail counts, periodic pairs and metadata for sample Jordan constructions. The examples cover orders 3 through 8, with direct full-alternator evaluation checked for orders 3 through 5.

With Python 3.10 or later and the packages in `requirements.txt`, run from the manuscript directory:

```
python -m pip install -r requirements.txt
python code/verify_standard.py --output standard_rerun.json
python code/verify_standard.py --extended \
  --output standard_extended_rerun.json
```

Compare the parsed outputs with `certificates/standard_certificates.json` and `certificates/standard_certificates_extended.json`, respectively. The tests use assertions, so `python -O` must not be used. The corresponding finite ranges are described in `REPRODUCIBILITY.md`.

Appendix B. Even-standard integer certificates

The seven even-core certificates, indexed by their pairs (r, N), are the following files:

```
certificates/border_r4_N5.json
certificates/border_r4_N6.json
certificates/border_r4_N7.json
certificates/border_r5_N6.json
certificates/border_r5_N7.json
certificates/border_r5_N8.json
certificates/border_r5_N9.json
```

Each file contains integer matrices $B_1, \dots, B_{2r-1}$, a column u, the nonzero integer determinant of $S = s_{2r-1}(B)$, and a specified $N^2 \times N^2$ minor of the normal map $\Gamma_{B,u}$ with its integer determinant. The nonzero determinant of S and minor of Γ establish admissibility (63) for that core size.

The constructor `code/verify_border_even.py` uses an alternating-product subset recurrence and replaced-input columns, with direct word enumeration for smaller cases. The saved-input checker `code/verify_saved_border.py` uses a different first-letter recurrence to rebuild every normal-map column; it checks $\det S$, the bordered rank-one square-zero value and the recorded minor modulo 101 and 103. A nonzero residue of the integer minor establishes its nonvanishing over characteristic-zero fields. The seed checker `code/verify_minimal_even.py` tests the signed matrix-unit formulas and full differential in specified finite sizes.

For example, run:

```
python code/verify_saved_border.py \
  --input certificates/border_r4_N5.json \
  --output checked_border_r4_N5.json
python code/verify_minimal_even.py \
  --max-n 8 --rank-max-n 6 \
  --output minimal_even_rerun.json
```

The first command applies likewise to the other six certificate files; `REPRODUCIBILITY.md` gives the full list and constructor command. These integer checks establish the listed finite admissible cores. Their extension to infinitely many matrix orders is the symbolic additive transfer of Theorem 11.3, while the general even-degree argument uses the cited generic-nullity theorems.

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