Globally periodic rational difference equations of arbitrary order

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November 4, 2026

hexagon:2611.00183v1

Abstract

A rational difference equation of order k with nonnegative coefficients is globally periodic if every solution with positive initial data repeats with a common period. Five such equations have long been known, among them the Lyness equation $x_{n+2} = (1 + x_{n+1})/x_n$, and a problem of Gasull asks whether every globally periodic equation of this kind is obtained from one of the five by rescaling and unfolding. The answer was known to be yes for $k \in \{1, 2, 3, 4, 5, 7, 9, 11\}$. Here I show that it is yes at every order. The proof rests on two ideas. The first is a tropical degeneration: global periodicity forces an associated min-plus recursion to be periodic, and this is refuted, for every choice of coefficients at once, by an unbounded solution of the min-plus recursion linearised along a periodic orbit. The second is the σ-bump, a linearised solution whose existence reduces to a combinatorial condition on residues modulo $2k$; in the hardest case that condition comes down to a lemma on three-term progressions under unit multipliers modulo $2k$. The proof, including its finite computations, has been formalised in the Lean proof assistant.

Disclaimer: This manuscript, and the accompanying Lean formalisation of the proof, have been generated by AI. The validity of the proof has been extensively machine-checked, but the proof is not fully understood by the human author.

1 Introduction

A difference equation is *globally periodic* if all of its orbits are periodic with a common period. The celebrated example is the Lyness equation

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \tag{1}$$

every positive solution of which repeats with period 5 [Lyness, 1942]. Global periodicity is a fragile property, and equations that have it are rare. In a survey of open problems in low-dimensional dynamics, Gasull [2021] records five globally periodic rational difference equations with nonnegative coefficients,

$$x_{n+1} = x_n, \quad x_{n+1} = \frac{1}{x_n}, \quad x_{n+2} = \frac{x_{n+1}}{x_n}, \quad x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \quad x_{n+3} = \frac{1 + x_{n+1} + x_{n+2}}{x_n}, \tag{2}$$

of periods 1, 2, 6, 5 and 8 respectively. The 8-cycle of the last was found by H. Todd [Kocić et al., 1993, Lyness, 1945]. New examples can be made from these in two trivial ways: by *rescaling*, $y_n = \alpha x_n$ with $\alpha > 0$, and by *unfolding*: multiplying every index in an equation of order k by an integer $\ell \geq 1$ gives an equation of order ℓk whose solutions are the interleavings of ℓ solutions

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of the original one, and which is therefore again globally periodic. Gasull's Problem 11 asks whether there are any others. More precisely, consider the equations of order $k \geq 1$,

$$x_{n+k} = \frac{A_0 + A_1 x_n + A_2 x_{n+1} + \cdots + A_k x_{n+k-1}}{B_0 + B_1 x_n + B_2 x_{n+1} + \cdots + B_k x_{n+k-1}}, \tag{3}$$

with $A_i, B_i \geq 0$, $\sum_i A_i > 0$ and $\sum_i B_i > 0$, acting on positive initial data. (Gasull also assumes $A_1^2 + B_1^2 \neq 0$, so that the equation genuinely involves x_n; I do not need this, because global periodicity forces it, as the first step of the proof shows.) Is every globally periodic equation of the form (3) obtained from one of (2) by rescaling and unfolding?

Here I show that the answer is yes at every order.

Theorem 1.1. *Let (3) be globally periodic, that is, suppose there is a $p \geq 1$ such that every positive solution satisfies $x_{n+p} = x_n$. Then, up to multiplying the numerator and denominator by a common positive constant, (3) is one of the following equations, written from the second on with the denominator cleared:*

1. *the shift $x_{n+k} = x_n$;*
2. *$x_{n+k}\, x_n = a$, with $a > 0$;*
3. *$k = 2m$ and $x_{n+2m}\, x_n = b\, x_{n+m}$, with $b > 0$;*
4. *$k = 2m$ and $x_{n+2m}\, x_n = b^2 + b\, x_{n+m}$, with $b > 0$;*
5. *$k = 3m$ and $x_{n+3m}\, x_n = b^2 + b\,(x_{n+m} + x_{n+2m})$, with $b > 0$.*

These are, respectively, the k-fold unfoldings of the first two equations of (2) and the m-fold unfoldings of the last three, after rescaling; so every globally periodic equation of the form (3) is equivalent to one of the five equations of (2).

Conversely, each equation in the list is globally periodic, with periods k, $2k$, $6m$, $5m$ and $8m$ respectively, since it is an unfolding of a rescaling of one of (2); so Theorem 1.1 is a complete classification.

The requirement of a *common* period loses nothing: if every orbit in $(0, \infty)^k$ is periodic, the state map is a continuous bijection, hence a homeomorphism by invariance of domain, and a pointwise periodic homeomorphism of a connected manifold is periodic [Montgomery, 1937]; often it can even be linearised globally [Cima et al., 2012]. (Discontinuous maps can be pointwise periodic without being periodic [Cima et al., 2022].) This remark is not used in the proof.

The question has a long history. Essentially the same question was asked by Graham et al. [1989, Research problem 1.24]; the generalisation that Graham had in mind for his Problem 1343 in *Mathematics Magazine* asks for equations of much the same form [Kocić et al., 1993, Remark 1]. A version of the question is posed by Grove and Ladas [2005, p. 161]; at order two it is Conjecture 6.5.6 of Kulenović and Ladas [2000]. After partial results of Kocić et al. [1993], order two was settled by Barbeau et al. [1995], and Csörnyei and Laczkovich [2001] settled, even for complex coefficients, orders two and three and the reciprocal equations $x_{n+k}x_n = c_0 + \sum_{j=1}^{k-1} c_j x_{n+j}$ in which every c_j is nonzero (the full support). Cima et al. [2004] reduced the problem to a reciprocal family with palindromic coefficients, and showed, using a continuum of 2-periodic points at odd order (Section 6), that the answer is yes for $k \in \{1, 2, 3, 4, 5, 7, 9, 11\}$. The general case has remained open. Different approaches to the global periodicity problem are surveyed by Cima et al. [2016]. Over the complex numbers and with arbitrary coefficients, periodic linear fractional recurrences have been studied from the point of view of birational geometry by Bedford and Kim [2006, 2014] at orders 2 and 3, and Bedford and Kim [2011] show that for every k there are such recurrences of order k and period $4k$; positivity of the coefficients, which is essential here, plays no role there.

The proof has two quite different halves. The first is analytic and short. A surjectivity argument, followed by a Lyapunov function and the derivative at a fixed point, shows that a globally periodic equation is either the shift or of the *reciprocal* form

$$x_{n+k}\,x_n = c_0 + \sum_{j\in S} c_j\,x_{n+j}, \qquad c_0 \geq 0, \quad c_j > 0 \text{ for } j \in S \subseteq \{1,\dots,k-1\}, \tag{4}$$

and linearising at the interior equilibrium shows that the coefficients are palindromic, $c_j = c_{k-j}$. Throughout, $\theta = 1$ if $c_0 > 0$ and $\theta = 0$ if $c_0 = 0$; most of the arguments treat both values at once. Unfolding then lets me assume that the *support* S is primitive, $\gcd(S \cup \{k\}) = 1$. When S has at most two elements, the problem reduces to equations of order at most three, which are settled with the min-plus recursion described below (and, at order three with $c_0 = 0$, with the linearisation), and to a spectral statement about the polynomial $\lambda^k + 1 - \epsilon(\lambda^j + \lambda^{k-j})$, which is settled with Kronecker's theorem on algebraic integers whose conjugates all lie on the unit circle.

The second half, which is most of the paper, deals with supports of three or more elements, and it is combinatorial. The key observation is that global periodicity survives degeneration. If the initial data are taken to be powers δ^{e_i} of a small parameter δ, the exponents of the solution follow a piecewise-linear (min-plus) recursion that depends only on the support and on θ. For the Lyness equation, for example, $x_n \approx \delta^{e_n}$ turns $x_{n+2}x_n = 1 + x_{n+1}$ into $e_{n+2} + e_n = \min(0, e_{n+1})$, and every integer solution of this recursion has period 5, just as the solutions of the equation do. In general a globally periodic equation forces its min-plus recursion to be periodic. Because all the coefficients are positive, there is no cancellation: each term of the solution behaves like a positive multiple of a power of δ, and these leading coefficients follow a recursion of their own, which must also be periodic; this is what settles orders two and three when $c_0 > 0$. The passage to a piecewise-linear recursion is the limiting procedure of Tokihiro et al. [1996], now called ultradiscretisation and familiar from integrable systems [Inoue and Takenawa, 2008, Inoue et al., 2012, Nobe, 2008]; for the full support it was already used by Csörnyei and Laczkovich [2001]. Tropical versions also appear in the theory of cluster algebras, where Fomin and Zelevinsky [2003] use them to prove the periodicity of Zamolodchikov's Y-systems, of which the Lyness equation is the case of type A_2; here the information flows the other way, from the equation to its tropical version.

For larger supports, global periodicity of an equation is refuted, for *every* choice of positive coefficients at once, by exhibiting an unbounded solution of the *tangent recursion*, the linearisation of the min-plus recursion along a periodic solution. The idea is to perturb a periodic orbit of the min-plus recursion slightly. Only the terms that attain the minimum see the perturbation, so it follows a recursion of the same kind, and if the perturbation can grow without bound, the min-plus recursion, and with it the equation, cannot be periodic. For the supports with at least three elements, the periodic orbits used have, except at order 6, a minimum that is always negative, so that the constant term never attains it, and both values of θ are refuted at once.

I construct the unbounded tangent solutions explicitly, case by case (Table 1). Apart from the full support and most supports at odd order, which have tangent solutions of their own, the workhorse is the *σ-bump*: a linearly growing tangent solution supported on two positions in every block of k, whose existence reduces to a purely combinatorial condition on a set of residues modulo $2k$ (Definition 3.11). The hardest case is even order with every element of S odd, where the residues are chosen as a half-arc $\{y : \nu y \bmod 2k \in [0,k)\}$ for a suitable odd multiplier ν, and what is needed is a lemma about three-term arithmetic progressions under unit multipliers modulo $2k$, proved with the geometry of numbers in a plane lattice of determinant $2k$ and Jacobsthal's function.

Several steps finish with a finite computation, for instance at small orders where the general arguments do not yet apply; each is stated precisely in the text and listed in Table 1.

Every finite computation is carried out in exact integer arithmetic, and the whole proof, including these computations, has been formalised and kernel-checked in the Lean proof assistant (Appendix C).

As far as I know, the following ingredients are new: the tangent recursion and the two-scale pinning lemma (Lemma 3.5), which turn an unbounded solution of the linearised min-plus recursion into a proof that the equation is not periodic; the use of such tangent solutions to refute global periodicity for every choice of coefficients at once; the σ-bump and its carriers (the parity, band and half-arc carriers); and the theorem on three-term progressions under unit multipliers (Theorem 9.1). Ultradiscretisation itself is not new, nor is the use of the leading coefficients at orders two and three, which goes back to Csörnyei and Laczkovich [2001]. I am not aware of an earlier use of the derivative of a min-plus map along a periodic orbit to show that the map is not periodic ("linearised" means something different in the ultradiscrete literature, namely straight-line motion on a tropical Jacobian). A by-product is a purely tropical statement: for every palindromic support with at least three elements, and either θ, no power of the min-plus map is the identity (Corollary 10.1).

Section 2 reduces the problem to primitive palindromic supports of the reciprocal form and Section 3 develops the min-plus machinery and the σ-bump; Sections 4–9 treat the cases of Table 1, Section 10 puts them together, and the appendices hold the finite range of Theorem 9.1 and the bounds on Jacobsthal's function, the most mechanical case checks, and a description of the formalisation. A reader who wants only the shape of the argument can read Sections 2 and 3, the statements of the main theorems of the later sections, and Section 10.

2 Reduction to the reciprocal family

Here I show that a globally periodic equation (3) is the shift or of the reciprocal form (8) with palindromic coefficients, and that it is enough to treat supports with $\gcd(S \cup \{k\}) = 1$ (Corollary 2.9).

Most of this is known. The reduction to the reciprocal form and palindromy are the Reduction Theorem (Theorem 2.4) of Cima et al. [2004], and Csörnyei and Laczkovich [2001, Theorem 4] proved $|c_j| = |c_{k-j}|$ for complex coefficients. I follow the route of Cima, Gasull and Mañosas (preimages of a one-parameter family of states, a Lyapunov function, and the derivative at a fixed point), and give self-contained proofs because the linearisation is needed again in Section 4 and palindromy follows from a short period-sum argument. That the linearisation of a periodic recurrence at a fixed point has finite order goes back to Kurshan and Gopinath [1974, Theorem 3.2]; it was used by Kocić et al. [1993], Csörnyei and Laczkovich [2001, Lemma 2] and, with affine normal forms at order two, Rubió-Massegú and Mañosa [2007].

2.1 The state map

Throughout, the coefficients of (3) are nonnegative, with $A_0 + \sum_{i\ge1} A_i > 0$ and $B_0 + \sum_{i\ge1} B_i > 0$. Note that A_{i+1} and B_{i+1} are the coefficients of x_{n+i}, for $0 \le i \le k-1$; in particular A_1 and B_1 multiply the oldest term x_n. Write $\Omega = (0,\infty)^k$, and for $u = (u_1, \dots, u_k) \in \Omega$ put

$$g(u) = \frac{A_0 + \sum_{i=1}^{k} A_i u_i}{B_0 + \sum_{i=1}^{k} B_i u_i}, \qquad F(u) = \big(u_2, \dots, u_k,\, g(u)\big). \tag{5}$$

For $u \in \Omega$ the numerator and denominator of $g(u)$ are both positive, since each contains a positive coefficient multiplied by a positive number. So F maps Ω to itself, every positive initial window $(x_0, \dots, x_{k-1})$ extends to a unique positive solution of (3), and the window $(x_n, \dots, x_{n+k-1})$ of that solution is $F^n(x_0, \dots, x_{k-1})$. The equation is *globally p-periodic* ($p \ge 1$) when every positive solution satisfies $x_{n+p} = x_n$ for all n, and this holds precisely when $F^p = \mathrm{id}$

case	mechanism	results used	finite computations	§
every support	reduction to (8), palindromy, folding	Theorems 2.1, 2.2, 2.3, 2.4, 2.5; Lemma 2.8	none	2
$k = 2$, $S = \{1\}$	leading coefficients	Theorem 4.1, from Theorems 3.2 and 3.3	none	4
$k = 3$, $S = \{1, 2\}$	leading coefficients; for $c_0 = 0$, the linearisation	Theorem 4.2, from Theorems 3.2, 3.3 and (for $c_0 = 0$) 2.4	none	4
two-term supports $\{j, k - j\}$	the linearisation and Kronecker's theorem	Theorem 4.3, from Theorem 2.4, Lemmas 4.4 and 4.5	none	4
full support, $k \geq 4$	a closed-form tangent solution on the carrier $Q = \{1\}$	Theorem 5.1, from Lemmas 5.2 and 5.3 and Theorem 3.6	none	5
odd k, X not a dilated interval	rays along the alternating orbit	Theorem 6.11, from Proposition 6.4, Lemma 6.5, the lemmas of Section 6.3 and Theorem 3.6	none	6.4
odd k, X a dilated interval	a σ-bump	Theorems 6.13 and 3.14, Lemma 6.12	none	6.5
even k, some element of S even	σ-bumps on parity carriers, or on two explicit carriers	Theorem 7.1, from Theorems 7.7 and 7.12 and Corollary 7.15; Theorem 3.14	none	7
even k, all odd, $1 \in S$	σ-bumps on small explicit carriers	Proposition 8.4; Theorem 3.14	none	8
even k, all odd, $1 \notin S$, $\lvert L\rvert = 1$	σ-bumps on half-arc carriers	Theorem 8.8 (all but five supports); Theorem 3.14	$k \leq 22$: five carriers of Table 4	8
even k, all odd, $1 \notin S$, $\lvert L\rvert \geq 2$	σ-bumps on half-arc carriers, with Theorem 9.1	Proposition 8.13, Theorem 8.19, Theorem 9.1; Theorem 3.14	Lemma 8.16; $k \leq 90$ in Theorem 8.19; twelve carriers of Table 4; for Theorem 9.1, the chord table (C1), the chains of Table 5, the certificate (C2)–(C3) at the 127 orders $92 \leq k \leq 2730$ with three or more odd prime factors, and the checks in Lemmas A.3 and A.4	8, 9
$k = 6$, $S = \{1, 3, 5\}$	a tangent certificate	Propositions 3.10 and 3.9; Theorem 3.6	the certificate of Proposition 3.10	3

Table 1: The cases of the proof, the mechanism and results each uses, and the steps that rest on a finite computation ("none" means that the case is proved by hand for all parameters). Every case uses the reduction of the first row. Except in the rows for $\lvert S\rvert \leq 2$, $\lvert S\rvert \geq 3$, and S is not the full support unless stated; θ is arbitrary throughout. At odd order k, X is the set of halved even elements of S (Lemma 6.2); $L = \{s \in S : 2s < k\}$.

on Ω. In particular a globally periodic F maps Ω onto itself. The first step of the reduction, Theorem 2.1, uses only this surjectivity; the second, Theorem 2.2, uses periodicity itself.

2.2 The dichotomy

The idea is to ask for a preimage of a single one-parameter family of states, $(1,\dots,1,t)$ with $t>0$. That alone forces most of the coefficients to vanish.

Theorem 2.1 (dichotomy)**.** *Suppose $F(\Omega)=\Omega$. Then one of the following holds:*

(a) $A_1=0$, $B_0=0$ and $B_i=0$ for all $i\ge 2$, so that $B_1>0$ and (3) reads $B_1x_{n+k}x_n = A_0+A_2x_{n+1}+\cdots+A_kx_{n+k-1}$;

(b) $B_1=0$, $A_0=0$ and $A_i=0$ for all $i\ge 2$, so that $A_1>0$ and (3) reads $x_{n+k} = A_1x_n/\big(B_0+B_2x_{n+1}+\cdots+B_kx_{n+k-1}\big)$.

Proof. Fix $t>0$. A preimage of $(1,\dots,1,t)$ under F must have the form $(u,1,\dots,1)$ with $u>0$ and $g(u,1,\dots,1)=t$. Writing $A'=\sum_{i\ge2}A_i$ and $B'=\sum_{i\ge2}B_i$ (both zero when $k=1$), the hypothesis therefore says that for every $t>0$ there is a $u>0$ with

$$t\,(B_0+B_1u+B')=A_0+A_1u+A'. \tag{6}$$

There are four cases, according to the signs of A_1 and B_1.

If $A_1=B_1=0$, then u drops out of (6), and taking $t=1$ and $t=2$ gives $B_0+B'=A_0+A'=2(B_0+B')$. Hence $B_0+B'=0$, so all the B_i vanish, which is excluded.

If $A_1=0<B_1$, then $tB_1u=A_0+A'-t(B_0+B')$ with $u>0$, so $t(B_0+B')<A_0+A'$ for every $t>0$. Letting $t\to\infty$ gives $B_0+B'\le 0$, that is $B_0=0$ and $B_i=0$ for $i\ge2$: this is case (a).

If $B_1=0<A_1$, then symmetrically $A_1u=t(B_0+B')-A_0-A'$ with $u>0$, so $A_0+A'<t(B_0+B')$ for every $t>0$. Letting $t\to0^+$ gives $A_0+A'\le0$, that is $A_0=0$ and $A_i=0$ for $i\ge2$: this is case (b).

Finally, suppose $A_1>0$ and $B_1>0$, and put $\lambda=A_1/B_1$. At $t=\lambda$ the terms in u cancel from (6), leaving $\lambda(B_0+B')=A_0+A'$. Subtracting this from (6) at $t=\lambda+1$ leaves $(B_0+B')+B_1u=0$ for some $u>0$, which is impossible.

In case (a) the denominator is B_1x_n, so $B_1>0$; in case (b) the numerator is A_1x_n, so $A_1>0$. □

2.3 Branch (b) is the shift

Genuine periodicity, rather than surjectivity, now removes the rest of branch (b). Two simple tools are used. The first is a Lyapunov function: the coordinate sum $\Sigma(u)=\sum_i u_i$ changes under one step by

$$\Sigma(F(u))-\Sigma(u)=g(u)-u_1, \tag{7}$$

so if $g(u)<u_1$ for every $u\in\Omega$ then $\Sigma(F^p(u))<\Sigma(u)$ for every $p\ge1$, and $F^p=\mathrm{id}$ is impossible; the same holds if $g(u)>u_1$ throughout. The second is the derivative at an interior fixed point: if $F^p=\mathrm{id}$ and $F(u^*)=u^*$, the chain rule gives $DF(u^*)^p=I$.

Theorem 2.2 (branch (b) is the shift)**.** *A globally periodic equation of type (b) has $A_1=B_0$ and $B_i=0$ for all $i\ge2$; that is, it is the shift $x_{n+k}=x_n$.*

Proof. Let the equation be globally p-periodic, so that $F^p=\mathrm{id}$ on Ω. In type (b), $g(u)=A_1u_1/\big(B_0+\sum_{i\ge2}B_iu_i\big)$. Write $\beta=\sum_{i\ge2}B_i$.

Suppose first that $\beta=0$. Then $B_0>0$ (the denominator is not zero) and $g(u)=(A_1/B_0)u_1$. If $A_1<B_0$ then $g(u)<u_1$ throughout Ω, and if $A_1>B_0$ then $g(u)>u_1$ throughout; both are excluded by (7). So $A_1=B_0$, which is the conclusion.

Suppose next that $\beta > 0$ and $A_1 \le B_0$. Then for $u \in \Omega$ the denominator exceeds $B_0 \ge A_1$, so $g(u) < u_1$ throughout Ω, which is again excluded.

There remains the case $\beta > 0$, $A_1 > B_0$, which must be shown impossible. Here $x^* = (A_1 - B_0)/\beta > 0$ satisfies $B_0 + \beta x^* = A_1$, so the point $u^* = (x^*, \dots, x^*)$ is fixed by F. The map F is rational, with a denominator that is positive on the open set Ω, so it is smooth there, and $F^p = \mathrm{id}$ gives $DF(u^*)^p = D(F^p)(u^*) = I$ by the chain rule. At u^* the denominator of g equals A_1, so $\partial g/\partial u_1 = 1$ and $\partial g/\partial u_i = -x^* B_i/A_1$ for $i \ge 2$; thus, with $d_i = B_i x^*/A_1 \ge 0$,

$$DF(u^*)\, w = \Big(w_2, \dots, w_k,\ w_1 - \sum_{i \ge 2} d_i w_i \Big).$$

Put $\phi(\lambda) = \lambda^k + \sum_{i \ge 2} d_i \lambda^{i-1}$. Then $\phi(0) = 0$ and $\phi(1) = 1 + \beta x^*/A_1 > 1$, so by the intermediate value theorem there is a $\lambda \in (0,1)$ with $\phi(\lambda) = 1$. The vector $v = (1, \lambda, \dots, \lambda^{k-1})$ then satisfies $DF(u^*)\, v = \lambda v$, since the last coordinate of $DF(u^*)\, v$ is $1 - \sum_{i \ge 2} d_i \lambda^{i-1} = \lambda^k$. Hence $v = DF(u^*)^p v = \lambda^p v$, so $\lambda^p = 1$, which is impossible for $0 < \lambda < 1$. □

2.4 The reciprocal form and its normalisation

Theorem 2.3 (reduction)**.** *Every globally periodic equation of the form (3) is either the shift $x_{n+k} = x_n$ (that is, $A_1 = B_0 > 0$ and every other coefficient is zero), or satisfies $A_1 = B_0 = 0$ and $B_i = 0$ for $i \ge 2$, with $B_1 > 0$. In the second case, dividing through by B_1, its positive solutions are exactly the positive solutions of*

$$x_{n+k}\, x_n = c_0 + \sum_{j \in S} c_j\, x_{n+j}, \qquad c_j > 0 \text{ for } j \in S \subseteq \{1, \dots, k-1\}, \quad c_0 \ge 0, \tag{8}$$

where

$$c_0 = \frac{A_0}{B_1}, \qquad c_j = \frac{A_{j+1}}{B_1} \quad (1 \le j \le k-1), \qquad S = \{\, j \in \{1, \dots, k-1\} : A_{j+1} > 0 \,\}, \tag{9}$$

and $S \neq \emptyset$ or $c_0 > 0$.

Proof. A globally periodic F maps Ω onto itself, so Theorem 2.1 applies. In case (b), Theorem 2.2 gives the shift. In case (a) the equation is $B_1 x_{n+k} x_n = A_0 + \sum_{j=1}^{k-1} A_{j+1} x_{n+j}$, and dividing by $B_1 > 0$ gives (8) with the data (9), after dropping the terms with $A_{j+1} = 0$. The last condition is the hypothesis that the numerator of (3) is not identically zero (recall $A_1 = 0$). □

From now on an equation of the form (8) is described by its *normalised data*: the order k, the support S and the coefficients c_0 and c_j ($j \in S$); it is convenient to put $c_j = 0$ for $j \in \{1, \dots, k-1\} \setminus S$, and $\theta \in \{0, 1\}$ records whether $c_0 > 0$, as in the introduction. The normalisation (9) is a division of numerator and denominator by B_1, not a change of variable, so the normalised equation has the same solutions as the original one, and global periodicity of the one is global periodicity of the other. When $k = 1$ the support is empty and (8) is $x_{n+1} x_n = c_0$.

2.5 The linearisation

The reciprocal family always has an interior equilibrium. Write $s = \sum_{j \in S} c_j$. Since $s + c_0 > 0$, the equation $x^2 = c_0 + sx$ has exactly one positive root,

$$x^* = \tfrac{1}{2}\Big(s + \sqrt{s^2 + 4c_0} \Big), \tag{10}$$

and the constant sequence $x_n = x^*$ is a solution of (8). Put

$$\epsilon_j = \frac{c_j}{x^*} \quad (1 \le j \le k-1), \qquad \text{so that} \qquad \sum_{j=1}^{k-1} \epsilon_j = \frac{s}{x^*} = 1 - \frac{c_0}{x^{*2}} \le 1, \tag{11}$$

with equality exactly when $c_0 = 0$. The *linearisation* of (8) at x^* is the linear recurrence

$$y_{n+k} + y_n = \sum_{j=1}^{k-1} \epsilon_j \, y_{n+j}, \tag{12}$$

whose characteristic polynomial is

$$\chi(\lambda) = \lambda^k + 1 - \sum_{j=1}^{k-1} \epsilon_j \, \lambda^j. \tag{13}$$

Every initial window $(y_0, \dots, y_{k-1})$ extends to a unique solution of (12), and the window advances by the companion matrix Γ,

$$\Gamma(w_1, \dots, w_k) = \Big(w_2, \dots, w_k, \; -w_1 + \sum_{j=1}^{k-1} \epsilon_j w_{j+1} \Big).$$

Theorem 2.4 (linearisation)**.** *Suppose (8) is globally p-periodic. Then:*

(i) $\Gamma^p = I$, so every solution of (12) is p-periodic;

(ii) every complex root λ of χ satisfies $\lambda^p = 1$.

Proof. (i) By Theorem 2.3 the state map of (8) is $F(u) = \big(u_2, \dots, u_k, \, g(u)\big)$ with $g(u) = \big(c_0 + \sum_j c_j u_{j+1}\big)/u_1$, a rational map whose denominator is positive on Ω, and it fixes $u^* = (x^*, \dots, x^*)$. Using $x^{*2} = c_0 + \sum_j c_j x^*$, the partial derivatives of g at u^* are $\partial g/\partial u_1 = -x^{*2}/x^{*2} = -1$ and $\partial g/\partial u_{j+1} = c_j/x^* = \epsilon_j$, so $DF(u^*) = \Gamma$. Since $F^p = \mathrm{id}$ on the open set Ω, the chain rule gives $\Gamma^p = D(F^p)(u^*) = I$.

(ii) If $\chi(\lambda) = 0$, then $v = (1, \lambda, \dots, \lambda^{k-1})$ satisfies $\Gamma v = \lambda v$, since the last coordinate of Γv is $-1 + \sum_j \epsilon_j \lambda^j = \lambda^k$. So $v = \Gamma^p v = \lambda^p v$, and $\lambda^p = 1$. □

2.6 Palindromy

The first structural consequence of Theorem 2.4 is that the coefficients are palindromic. The reason is that a linear map of finite order preserves a positive definite quadratic form, so its inverse is its adjoint. Here the form can be written down explicitly: on p-periodic sequences the *period sum*

$$\langle u, v \rangle = \sum_{n=0}^{p-1} u_n v_n$$

is positive definite, and it is unchanged by shifting both sequences, $\sum_{n<p} u_{n+a} v_{n+a} = \langle u, v \rangle$, since $n \mapsto u_n v_n$ is p-periodic. The recurrence (12) must therefore be the same when it is run backwards, and that is palindromy.

Theorem 2.5 (palindromy)**.** *If (8) is globally periodic, then $c_j = c_{k-j}$ for $1 \le j \le k-1$. In particular the support is palindromic, $S = k - S$; in the notation of (3), $A_i = A_{k+2-i}$ for $2 \le i \le k$.*

Proof. Let the equation be globally p-periodic. By Theorem 2.4(i), every real solution of (12) is p-periodic. For a solution v, put

$$\rho_n = v_n + v_{n+k} - \sum_{j=1}^{k-1} \epsilon_j \, v_{n+k-j}.$$

Let u be another solution, and expand $\sum_{n<p} u_n\rho_n$ into three period sums. Shifting the first by k and the terms of the third by j,

$$\sum_{n<p} u_n v_n = \sum_{n<p} u_{n+k} v_{n+k}, \qquad \sum_{n<p} u_n v_{n+k-j} = \sum_{n<p} u_{n+j} v_{n+k},$$

so that

$$\sum_{n<p} u_n\rho_n = \sum_{n<p} v_{n+k}\Big(u_{n+k} + u_n - \sum_{j=1}^{k-1} \epsilon_j u_{n+j}\Big) = 0,$$

because u solves (12). On the other hand, using (12) for v to replace $v_n + v_{n+k}$, and reindexing the sum $j \mapsto k-j$,

$$\rho_n = \sum_{j=1}^{k-1} \big(\epsilon_j - \epsilon_{k-j}\big)\, v_{n+j}.$$

This is a linear combination of shifts of v, so ρ is itself a solution of (12). Taking $u = \rho$ gives $\sum_{n<p}\rho_n^2 = 0$, so $\rho_n = 0$ for $n < p$, and hence for all n by periodicity.

Now fix $1 \le j \le k-1$ and let v be the solution with initial window $v_j = 1$ and $v_i = 0$ for the other $i < k$. Then $0 = \rho_0 = \epsilon_j - \epsilon_{k-j}$, and multiplying by x^* gives $c_j = c_{k-j}$. The statement about S follows because $S = \{j : c_j > 0\}$, and the statement about the A_i from (9). □

2.7 Known types and primitive supports

The five equations (2) have a simple description in terms of normalised data. Recall that the *ℓ-fold unfolding* of an equation $x_{n+k'} = f(x_n, x_{n+1}, \dots, x_{n+k'-1})$ of order k' is the equation $x_{n+\ell k'} = f(x_n, x_{n+\ell}, \dots, x_{n+(k'-1)\ell})$ of order $\ell k'$; its solutions are exactly the sequences whose ℓ interleaved subsequences $m \mapsto x_{m\ell+r}$ $(0 \le r < \ell)$ each solve the original equation. (Gasull [2021] describes this operation without a name, and uses the word *unfold* for the rescaled family $y_n = \alpha x_n$ instead.)

Definition 2.6. The normalised data (k, S, c) of an equation (8) is of *known type* if one of the following holds:

(K1) $S = \emptyset$;

(K2) $k = 2m$, $S = \{m\}$, and $c_0 = 0$ or $c_0 = c_m^2$;

(K3) $k = 3m$, $S = \{m, 2m\}$, $c_m = c_{2m}$ and $c_0 = c_m^2$.

Lemma 2.7. *If the normalised data of an equation of type (a) is of known type, then the equation is equivalent, under a rescaling $y_n = x_n/\lambda$ $(\lambda > 0)$ and an unfolding, to one of the equations (2). In terms of the coefficients of (3), with m as in Definition 2.6: in case (K1), $A_i = 0$ for all $i \ge 1$; in case (K2), $A_i = 0$ for $i \notin \{0, m+1\}$, and $A_0 = 0$ or $A_0B_1 = A_{m+1}^2$; in case (K3), $A_i = 0$ for $i \notin \{0, m+1, 2m+1\}$, $A_{m+1} = A_{2m+1}$ and $A_0B_1 = A_{m+1}^2$.*

Proof. If x solves (8), then $y_n = x_n/\lambda$ solves $y_{n+k}y_n = c_0/\lambda^2 + \sum_{j\in S}(c_j/\lambda)\, y_{n+j}$. In case (K1), $c_0 > 0$ by Theorem 2.3; with $\lambda = \sqrt{c_0}$ the equation becomes $y_{n+k} = 1/y_n$, the k-fold unfolding of $y_{n+1} = 1/y_n$. In case (K2), $\lambda = c_m$ gives $y_{n+2m}y_n = y_{n+m}$ or $y_{n+2m}y_n = 1 + y_{n+m}$, the m-fold unfoldings of $y_{n+2} = y_{n+1}/y_n$ and of the Lyness equation $y_{n+2} = (1 + y_{n+1})/y_n$. In case (K3), $\lambda = c_m$ gives $y_{n+3m}y_n = 1 + y_{n+m} + y_{n+2m}$, the m-fold unfolding of $y_{n+3} = (1 + y_{n+1} + y_{n+2})/y_n$. The coefficient conditions follow from (9): $c_0 = c_m^2$ reads $A_0/B_1 = A_{m+1}^2/B_1^2$. □

The last step of the reduction removes common factors from the support. If every element of S and k itself are multiples of some $\ell > 1$, then the equation only ever couples terms whose indices agree modulo ℓ, and it splits into ℓ independent copies of an equation of order k/ℓ, as Cima et al. [2004] observe at order four, where they split a solution into its odd and even subsequences. Call S *primitive* (at order k) when $\gcd(S \cup \{k\}) = 1$.

Lemma 2.8 (folding). *Let (8) have normalised data (k, S, c), and let $\ell = \gcd(S \cup \{k\})$ (so $\ell = k$ when $S = \emptyset$). Put $k' = k/\ell$, $S' = \{j/\ell : j \in S\}$, $c'_0 = c_0$ and $c'_{j'} = c_{j'\ell}$ for $j' \in S'$. Then:*

(i) equation (8) is the ℓ-fold unfolding of the equation

$$x_{n+k'}\, x_n = c'_0 + \sum_{j' \in S'} c'_{j'}\, x_{n+j'} \tag{14}$$

of order k', which has the same θ;

(ii) if (8) is globally p-periodic, then so is (14);

(iii) S' is primitive at order k', and if $c_j = c_{k-j}$ for all j, then $c'_{j'} = c'_{k'-j'}$ for all j' (in particular S' is palindromic);

(iv) if the data (k', S', c') is of known type, then so is (k, S, c).

Proof. (i) Every element of S is a multiple of ℓ, and so is k. Writing $n = m\ell + r$ with $0 \le r < \ell$ and $j = j'\ell$, equation (8) at n reads

$$x_{(m+k')\ell+r}\, x_{m\ell+r} = c'_0 + \sum_{j' \in S'} c'_{j'}\, x_{(m+j')\ell+r},$$

which is (14) at m for the subsequence $m \mapsto x_{m\ell+r}$.

(ii) Let z be a positive solution of (14), and put $x_n = z_{\lfloor n/\ell \rfloor}$. Each subsequence $m \mapsto x_{m\ell+r}$ equals z, so by (i) x is a positive solution of (8), and it is therefore p-periodic, hence also $p\ell$-periodic. So $z_{m+p} = x_{m\ell+p\ell} = x_{m\ell} = z_m$ for every m.

(iii) and (iv) are immediate from the definitions: $\gcd(S' \cup \{k'\}) = \gcd(S \cup \{k\})/\ell = 1$, and for (iv) one replaces m by $m\ell$ in Definition 2.6. □

Corollary 2.9. *Suppose that every globally periodic equation (8) whose support S is primitive and whose coefficients satisfy $c_j = c_{k-j}$ for all j has normalised data of known type. Then every globally periodic equation of the form (3) is either the shift $x_{n+k} = x_n$, or is equivalent, under a rescaling and an unfolding, to one of the equations (2), with coefficients as in Lemma 2.7.*

Proof. Let (3) be globally p-periodic. By Theorem 2.3 it is either the shift or has the normalised form (8), which is globally p-periodic. In the second case, Theorem 2.5 gives $c_j = c_{k-j}$, and Lemma 2.8 gives a globally p-periodic folded equation (14) with primitive support and palindromic coefficients. By hypothesis its data is of known type, so by Lemma 2.8(iv) the data of (8) is of known type, and Lemma 2.7 finishes the proof. □

The remaining sections therefore work with a primitive palindromic support S, coefficients $c_j = c_{k-j} > 0$ on S, and either value of θ, and show that global periodicity forces known type.

3 The min-plus map and the σ-bump

Throughout this section the equation is (8), with support $S \subseteq \{1, \dots, k-1\}$ and θ as in the introduction; since the numerator is nonzero, $S \neq \emptyset$ or $\theta = 1$. The section sets up the min-plus recursion and its leading coefficients (Theorems 3.2 and 3.3), the tangent solutions that refute periodicity (Theorem 3.6), and the σ-bump (Theorem 3.14). It ends with a vocabulary for checking the conditions of a σ-bump (Section 3.5) and the mirror lemma (Lemma 3.15), both used in the later sections.

3.1 The min-plus recursion and its periodicity

It is convenient to treat the constant term as a slot. Let

$$S_\theta = S \cup \{0\} \text{ if } \theta = 1, \qquad S_\theta = S \text{ if } \theta = 0,$$

so that $S_\theta \neq \emptyset$. For a sequence $(f_n)_{n\geq 0}$ of real numbers, an index $n \geq 0$ and a slot $j \in S_\theta$, write

$$v_n(f, j) = \begin{cases} 0, & j = 0, \\ f_{n+j}, & j \in S, \end{cases} \qquad m_n(f) = \min_{j \in S_\theta} v_n(f, j).$$

Thus $v_n(f, j)$ is the value read by slot j from the window starting at n, and the constant slot always reads 0. The slots $j \in S$ are called *taps*.

Definition 3.1. An integer sequence $(e_n)_{n\geq 0}$ is a *solution of the min-plus recursion* (for k, S, θ) if

$$e_{n+k} + e_n = m_n(e) = \min\Big(\{0 \text{ if } \theta = 1\} \cup \{e_{n+j} : j \in S\}\Big) \qquad \text{for all } n \geq 0. \tag{15}$$

Every integer window $(e_0, \ldots, e_{k-1})$ extends uniquely to a solution. The *min-plus map* $T : \mathbb{Z}^k \to \mathbb{Z}^k$ is the corresponding window map, $T(e_0, \ldots, e_{k-1}) = (e_1, \ldots, e_{k-1}, m_0(e) - e_0)$. The *attaining sets* of a solution e are

$$A_n = A_n(e) = \{\, j \in S_\theta : \; v_n(e, j) = m_n(e) \,\} \qquad (n \geq 0);$$

ties are allowed, so A_n may have several elements.

Each term of the orbit of (8) started from $x_i = \delta^{e_i}$ behaves like $C_n \delta^{e_n}$ as $\delta \to 0^+$, where the exponents e_n follow the min-plus recursion and the leading coefficients C_n follow a recursion that keeps, in each step of (8), only the terms whose exponent attains the minimum.

Theorem 3.2 (leading coefficients)**.** *Let e be an integer solution of (15). For $\delta > 0$ let $x(\delta) = (x_n(\delta))$ be the solution of (8) with initial window $x_i(\delta) = \delta^{e_i}$ $(0 \leq i < k)$. Define C_n by $C_n = 1$ for $n < k$ and*

$$C_{n+k}\, C_n = c_0\, [0 \in A_n(e)] + \sum_{j \in A_n(e) \cap S} c_j\, C_{n+j} \qquad (n \geq 0), \tag{16}$$

where $[0 \in A_n(e)]$ is 1 if the constant slot attains the minimum and 0 otherwise. Then $C_n > 0$ and

$$\lim_{\delta \to 0^+} x_n(\delta)\, \delta^{-e_n} = C_n \qquad \text{for every } n \geq 0.$$

Proof. First, $x(\delta)$ is a positive solution: the numerator of (8) is a sum of nonnegative terms, at least one of them positive. Likewise $C_n > 0$ by induction, since $A_n(e)$ is nonempty and every term on the right of (16) is a positive coefficient times a positive number.

The limit is proved by induction on n; for $n < k$, $x_n(\delta)\delta^{-e_n} = 1$. Suppose it holds at $n, n+1, \ldots, n+k-1$, and write $M = m_n(e)$. Multiplying (8) at n by δ^{-M} gives

$$x_{n+k}(\delta)\, \delta^{-e_{n+k}} \cdot x_n(\delta)\, \delta^{-e_n} = c_0\, \delta^{-M} + \sum_{j \in S} c_j\, x_{n+j}(\delta)\, \delta^{-e_{n+j}} \cdot \delta^{\,e_{n+j} - M},$$

because $e_{n+k} + e_n = M$. Consider the terms on the right one at a time. For $j \in S$ the exponent $e_{n+j} - M$ is a nonnegative integer, which is 0 exactly when $j \in A_n$; so the term tends to $c_j C_{n+j}$ if $j \in A_n$ and to 0 otherwise. If $\theta = 1$, then $M \leq 0$, and $c_0\delta^{-M}$ tends to c_0 if $M = 0$, that is if $0 \in A_n$, and to 0 otherwise; if $\theta = 0$ the term is absent. The right-hand side therefore tends to the right-hand side of (16), and dividing by $x_n(\delta)\delta^{-e_n} \to C_n > 0$ gives $x_{n+k}(\delta)\delta^{-e_{n+k}} \to C_{n+k}$. □

Theorem 3.3 (periodicity of the degeneration)**.** *Suppose the equation* (8) *is globally p-periodic. Then every integer solution e of the min-plus recursion* (15) *satisfies* $e_{n+p} = e_n$ *for all* $n \geq 0$*; equivalently,* $T^p = \mathrm{id}$ *on* $\mathbb{Z}^k$*. Moreover the leading coefficients of Theorem 3.2 satisfy* $C_{n+p} = C_n$ *for all* $n \geq 0$*.*

Proof. Let $x(\delta)$ and C_n be as in Theorem 3.2. By periodicity $x_{n+p}(\delta) = x_n(\delta)$ for every $\delta > 0$, so

$$\delta^{\,e_{n+p}-e_n} = \frac{x_n(\delta)\,\delta^{-e_n}}{x_{n+p}(\delta)\,\delta^{-e_{n+p}}} \longrightarrow \frac{C_n}{C_{n+p}} \qquad (\delta \to 0^+).$$

The limit is positive and finite, whereas δ^a tends to 0 for an integer $a > 0$ and to infinity for $a < 0$. Hence $e_{n+p} = e_n$, and then the left-hand side is 1, so $C_{n+p} = C_n$. □

The first part of Theorem 3.3 is a condition on (S, θ) alone; the leading coefficients retain a little information about the c_j, which is what is needed at orders 2 and 3. Positivity of the coefficients is essential to both theorems: the terms attaining the minimum cannot cancel. For the full support, Csörnyei and Laczkovich [2001, Lemma 8] pass to the same piecewise-linear recursion (with max in place of min), taking initial data of the form x^{c_i} with exponents c_i that are linearly independent over $\mathbb{Q}$. At orders 2 and 3 the recursion for the leading coefficients is the device of Csörnyei and Laczkovich [2001, Lemma 5], and Theorem 3.2 is its general form.

3.2 Tangent solutions

To use Theorem 3.3 one needs an integer solution of (15) that is not p-periodic. Searching for such orbits directly is awkward, because a non-periodic orbit has to be followed for ever. It is much easier to start from a *periodic* orbit and look at how nearby orbits move. Since T is piecewise linear, nearby orbits are governed by a second recursion of the same kind, read only from the slots that attain the minimum.

Definition 3.4. Let e be an integer solution of (15), with attaining sets $A_n = A_n(e)$. An integer sequence $(d_n)_{n\geq 0}$ is a *tangent solution along* e if

$$d_{n+k} + d_n = \min_{j \in A_n} v_n(d, j) \qquad \text{for all } n \geq 0. \tag{17}$$

The right-hand side of (17) is the one-sided derivative of the minimum: for small $\tau > 0$, $m_n(e+\tau d) = m_n(e) + \tau \min_{j\in A_n} v_n(d, j)$, because slots outside A_n are beaten by a fixed margin. The following lemma, the heart of the method, is the integer version of this observation.

Lemma 3.5 (two-scale pinning)**.** *Let* e *be an integer solution of* (15) *and* d *a tangent solution along* e*. Let* $N \geq 0$*, let* $D \geq 0$ *be an integer with* $|d_n| \leq D$ *for all* $n < N$*, and let* $\kappa > 2D$ *be an integer. Let* g *be the integer solution of* (15) *with initial window* $g_i = \kappa e_i + d_i$ $(0 \leq i < k)$*. Then*

$$g_n = \kappa e_n + d_n \qquad \text{for all } 0 \leq n < N.$$

Proof. By induction on n; the case $n < k$ is the definition of g. Let $k \leq n < N$ and write $n = m + k$, so that the claim holds at $m, m+1, \ldots, m+k-1$, all of which are smaller than N. Then for every slot $j \in S_\theta$,

$$v_m(g, j) = \kappa\, v_m(e, j) + v_m(d, j),$$

and $|v_m(d, j)| \leq D$ (the constant slot reads 0 for every sequence). Put $M = m_m(e)$. For $j \in A_m$ the value is $\kappa M + v_m(d, j) \leq \kappa M + D$. For $j \notin A_m$ the integer $v_m(e, j)$ exceeds M, so $v_m(e, j) \geq M + 1$, and the value is at least $\kappa M + \kappa - D > \kappa M + D$. Hence the minimum over S_θ is attained inside A_m, and

$$m_m(g) = \kappa M + \min_{j\in A_m} v_m(d, j) = \kappa\,(e_{m+k} + e_m) + (d_{m+k} + d_m),$$

by (15) for e and (17) for d. Since $g_{m+k} = m_m(g) - g_m$ and $g_m = \kappa e_m + d_m$, this gives $g_n = \kappa e_n + d_n$. □

Theorem 3.6 (tangent solutions)**.** *Let T be the min-plus map for (S, θ).*

(i) If $T^p = \mathrm{id}$ on $\mathbb{Z}^k$, then every integer tangent solution d along every integer solution e of (15) satisfies $d_{n+p} = d_n$ for all $n \geq 0$; in particular it is bounded.

(ii) Consequently, if some integer solution of (15) carries an unbounded integer tangent solution, then no power of T is the identity on $\mathbb{Z}^k$, and no equation (8) with support S and this θ is globally periodic, for any choice of positive coefficients and any period.

Proof. (i) Fix $n_0 \geq 0$. Take $N = n_0 + p + 1$, an integer $D \geq 0$ bounding $|d_n|$ for $n < N$, and $\kappa = 2D + 1$. By Lemma 3.5 the integer solution g with initial window $\kappa e_i + d_i$ satisfies $g_n = \kappa e_n + d_n$ for $n < N$. As $T^p = \mathrm{id}$, both e and g are p-periodic, so

$$d_{n_0+p} = g_{n_0+p} - \kappa e_{n_0+p} = g_{n_0} - \kappa e_{n_0} = d_{n_0}.$$

A p-periodic sequence is bounded.

(ii) By (i) no power of T is the identity. If an equation with support S and this θ were globally p-periodic, Theorem 3.3 would give $T^p = \mathrm{id}$ on $\mathbb{Z}^k$. □

Definition 3.7. A tangent solution d is a *ray* if there are an integer $q \geq 1$ and a q-periodic integer sequence u, not identically zero, such that $d_{n+q} = d_n + u_n$ for all $n \geq 0$.

A ray grows linearly, since $d_{i+mq} = d_i + m\,u_i$ for all $i, m \geq 0$, and in particular it is unbounded. The certificates below produce rays, as do the σ-bump of Definition 3.11 (with $q = 2k$), the tangent solution for the full support (Section 5, with $q = 8k$) and the tangent solutions at odd order (Remark 6.6).

3.3 Certificates

Theorem 3.6 becomes a finite check when e is periodic and the tangent solution grows linearly. To make the tangent recursion linear one designates, at each step, one attaining slot $w_t \in A_t$ and runs

$$f_{t+k} + f_t = v_t(f, w_t), \tag{18}$$

which is a linear recursion: f is determined by its initial window and the word (w_t), and sums and integer multiples of solutions are solutions. A solution of (18) is a tangent solution provided w_t also minimises $v_t(f, \cdot)$ over A_t at every step. The point of the following definition is that this *margin* condition need only be checked over one period, on two vectors.

Definition 3.8. A *tangent certificate* for (k, S, θ) consists of a period $q \geq 1$, three integer windows $E, V, W \in \mathbb{Z}^k$ with $W \neq 0$, and a word $(w_0, \ldots, w_{q-1})$ of slots in S_θ. Let e be the solution of (15) with initial window E, and let d and u be the solutions of (18) for $0 \leq t < q$ with initial windows V and W. The certificate is *valid* if

1. for every $0 \leq t < q$: $w_t \in A_t(e)$, and $v_t(d, w_t) \leq v_t(d, j)$ and $v_t(u, w_t) \leq v_t(u, j)$ for every $j \in A_t(e)$;
2. the windows return:

$$(e_q, \ldots, e_{q+k-1}) = E, \qquad (d_q, \ldots, d_{q+k-1}) = V + W, \qquad (u_q, \ldots, u_{q+k-1}) = W.$$

Checking validity means running q steps of three explicit integer recursions, which can be done by hand for small q.

Proposition 3.9. *If (k, S, θ) has a valid tangent certificate, then no equation (8) with support S and this θ is globally periodic, for any choice of positive coefficients.*

Proof. Extend the word q-periodically, $w_t = w_{t \bmod q}$, and extend d and u to all $t \geq 0$ by (18). Since e is determined by its window and $(e_q, \ldots, e_{q+k-1}) = E$, e is q-periodic, and so are the attaining sets: $A_{t+q} = A_t$. The shifted sequence $(u_{t+q})_t$ solves (18) (the word is q-periodic) with initial window W, so $u_{t+q} = u_t$ for all t. Likewise $(d_{t+q})_t$ and $(d_t + u_t)_t$ both solve (18) with initial window $V + W$, so $d_{t+q} = d_t + u_t$. By induction,

$$d_{t+mq} = d_t + m\, u_t \qquad (t, m \geq 0).$$

Now let $n = t + mq$ with $0 \leq t < q$. Then $A_n = A_t$, $w_n = w_t$, and for every slot j, $v_n(d, j) = v_t(d, j) + m\, v_t(u, j)$ (both sides are 0 for the constant slot). Multiplying the margin for u at time t by $m \geq 0$ and adding the margin for d gives $v_n(d, w_n) \leq v_n(d, j)$ for every $j \in A_n$. Together with $w_n \in A_n$ and (18) this says that d satisfies (17): it is a tangent solution along e.

Finally, $W \neq 0$, so $u_i \neq 0$ for some $i < k$, and $d_{i+mq} = d_i + m\, u_i$ is unbounded in m. Theorem 3.6 completes the proof. □

In this paper certificates are needed only once, for the support $\{1, 3, 5\}$ at order 6, which carries no σ-bump (see Section 8). A single certificate serves both values of θ.

Proposition 3.10. *Let $k = 6$ and $S = \{1, 3, 5\}$. For both $\theta = 0$ and $\theta = 1$, a valid tangent certificate is given by $q = 15$,*

$$E = (-1, 0, -1, 0, 0, -1), \qquad V = (-1, -1, 0, 0, 0, 1), \qquad W = (0, -1, -1, 0, -1, 1),$$

and

$$(w_t)_{t=0}^{14} = 5, 1, 5, 1, 3, 5, 1, 5, 5, 3, 5, 1, 5, 1, 3.$$

Consequently no equation (8) of order 6 with support $\{1, 3, 5\}$ is globally periodic, for either θ and any positive coefficients.

The orbit e is periodic with period 5, repeating $-1, 0, -1, 0, 0$, so it returns to E after 15 steps.

Proof. The consequence follows from Proposition 3.9, so only validity needs checking. The run is listed in Table 2. Each row t gives the first entries e_t, d_t, u_t of the three windows at time t, the minimum $m_t(e)$ over the taps, the taps $j \in S$ attaining it and the designated slot w_t. For $\theta = 0$ one checks, row by row, the recursions for e, d and u and the margins of Definition 3.8 over the listed taps, and that the windows at time 15 are E, $V + W = (-1, -2, -1, 0, -1, 2)$ and W.

Now let $\theta = 1$. Every minimum $m_t(e)$ in the table is at most 0, so adding the constant slot, which reads 0, does not change the orbit e, and it changes the attaining set only at $t = 3, 8, 13$, where $m_t(e) = 0$ and slot 0 joins the attaining set. At these three times the designated tap reads $(d_{t+w_t}, u_{t+w_t}) = (0, -1)$, $(-2, -1)$ and $(-3, -2)$ respectively, all at most the value 0 read by the constant slot, so the margins still hold. The rest of the check is unchanged. □

3.4 Carriers and the σ-bump

Certificates have to be found one support at a time. For most supports a single explicit construction does the job instead, and it reduces the dynamics to a combinatorial condition on residues: the periodic orbit takes only the values -1 and 0, arranged anti-periodically with period $2k$, and the unbounded tangent solution is supported on two residues modulo k.

Definition 3.11. Let $k \geq 2$ and $S \subseteq \{1, \ldots, k-1\}$.

t	e_t	d_t	u_t	m_t	attaining taps	w_t
0	-1	-1	0	-1	$\{5\}$	5
1	0	-1	-1	-1	$\{1\}$	1
2	-1	0	-1	-1	$\{3,5\}$	5
3	0	0	0	0	$\{1,3,5\}^*$	1
4	0	0	-1	-1	$\{1,3\}$	3
5	-1	1	1	-1	$\{5\}$	5
6	0	2	1	-1	$\{1\}$	1
7	-1	1	0	-1	$\{3,5\}$	5
8	0	1	1	0	$\{1,3,5\}^*$	5
9	0	0	-1	-1	$\{1,3\}$	3
10	-1	1	1	-1	$\{5\}$	5
11	0	0	0	-1	$\{1\}$	1
12	-1	-1	-1	-1	$\{3,5\}$	5
13	0	-2	-1	0	$\{1,3,5\}^*$	1
14	0	-3	-2	-1	$\{1,3\}$	3

Table 2: The certificate of Proposition 3.10 for $k = 6$, $S = \{1,3,5\}$ ($q = 15$). Times $15,\dots,20$ carry the windows $e = (-1,0,-1,0,0,-1)$, $d = (-1,-2,-1,0,-1,2)$ and $u = (0,-1,-1,0,-1,1)$. At the starred rows $m_t = 0$, and when $\theta = 1$ the constant slot also attains the minimum.

1. A set $P \subseteq \mathbb{Z}/2k$ is *anti-periodic* if $n+k \in P \iff n \notin P$ for every n. It is determined by $Q = P \cap \{0,\dots,k-1\}$: for $0 \le i < k$, $i \in P$ iff $i \in Q$, and $k+i \in P$ iff $i \notin Q$. For an integer $n \ge 0$, "$n \in P$" means that the residue of n modulo $2k$ lies in P. An anti-periodic set P, or the set Q that codes it, is also called a *carrier*.

2. The *orbit* of P is the sequence $E_n = -1$ if $n \in P$ and $E_n = 0$ otherwise.

3. Let $\sigma \in S$. The pair (σ, P) is a *σ-bump for S* if P is anti-periodic and

   (B1) $P \cap (\{0\} \cup S) = \{\sigma\}$;

   (B2) for every $0 < i < k$ there are $j, j' \in S$ with $i+j \in P$ and $i+j' \notin P$ (membership of P refers to residues modulo $2k$), such that neither $i+j$ nor $i+j'$ is congruent to 0 or to σ modulo k.

   When such a pair exists, S is said to *carry a σ-bump*. The two taps j and j' in (B2) are necessarily distinct, since one of the positions $i+j$, $i+j'$ lies in P and the other does not; this is used, at $i = 1$, in the proof of Lemma 3.13.

4. The *bump* with parameter σ is the integer sequence defined, for $n = \ell k + i$ with $\ell \ge 0$ and $0 \le i < k$, by

$$d_{\ell k} = \begin{cases} -\ell, & \ell \text{ even}, \\ \ell+1, & \ell \text{ odd}, \end{cases} \qquad d_{\ell k+\sigma} = \begin{cases} 2, & \ell \text{ even}, \\ -2, & \ell \text{ odd}, \end{cases} \tag{19}$$

   and $d_{\ell k+i} = 0$ for $i \neq 0, \sigma$.

The orbit of a carrier is a solution of the min-plus recursion as soon as every window of taps meets the carrier, and then the constant slot plays no role.

Lemma 3.12. *Let $P \subseteq \mathbb{Z}/2k$ be anti-periodic, and suppose that for every $n \ge 0$ some $j \in S$ has $n+j \in P$. Then the orbit E of P is a solution of the min-plus recursion (15) for both values of θ, with $m_n(E) = -1$ and*

$$A_n(E) = \{\, j \in S : \ n+j \in P \,\} \qquad \text{for every } n \ge 0.$$

In particular the constant slot never attains the minimum, and the tangent recursion (17) along E is the same for $\theta = 0$ and $\theta = 1$.

Proof. Since E takes only the values -1 and 0, and some tap reads -1, it follows that $m_n(E) = -1$ for every n, whether or not the constant slot (which reads 0) is present, and that the attaining sets are as stated. By anti-periodicity exactly one of n and $n+k$ lies in P, so $E_{n+k} + E_n = -1 = m_n(E)$. □

Conditions (B1) and (B2) are constraints on the k bits of Q, and checking them involves no dynamics at all. Their meaning is as follows. (B1) says that at the start of each block of length k the orbit E attains its minimum at the single tap σ (in even blocks) or at every tap except σ (in odd blocks). (B2) says that inside a block, whichever parity of block one is in, some attaining tap reads a position where the bump vanishes.

Lemma 3.13. *Let (σ, P) be a σ-bump for S. Then for every $n \geq 0$ some $j \in S$ has $n + j \in P$, so the orbit E of P is a solution of the min-plus recursion as in Lemma 3.12. The bump d of (19) is a tangent solution along E, for both values of θ, and $d_{2mk} = -2m$ for every $m \geq 0$.*

Proof. First, every window meets P. Write $n = \ell k + i$ with $0 \leq i < k$. Shifting by an even multiple of k preserves membership of P, and shifting by an odd multiple reverses it. If $i = 0$ and ℓ is even, $n + \sigma \in P$ by (B1). If $i = 0$ and ℓ is odd, take $j \in S \setminus \{\sigma\}$, which exists because the two taps provided by (B2) at $i = 1$ are distinct; by (B1) $j \notin P$, so $n + j \in P$. If $0 < i < k$, (B2) gives j with $i + j \in P$ and j' with $i + j' \notin P$, so $n + j \in P$ when ℓ is even and $n + j' \in P$ when ℓ is odd.

Next, the left-hand side of (17). For $n = \ell k + i$ a direct check from (19) gives

$$d_{n+k} + d_n = \begin{cases} 2, & i = 0 \text{ and } \ell \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Indeed, at $i = 0$ the sum is $d_{(\ell+1)k} + d_{\ell k} = (\ell + 2) - \ell = 2$ for ℓ even and $-(\ell + 1) + (\ell + 1) = 0$ for ℓ odd; at $i = \sigma$ it is $-2 + 2 = 0$ or $2 - 2 = 0$; and elsewhere both terms vanish.

Then, the negative entries of d lie off P. They are $d_{\ell k}$ with $\ell \geq 2$ even, at positions $\equiv 0 \pmod{2k}$, and $d_{\ell k+\sigma}$ with ℓ odd, at positions $\equiv k + \sigma \pmod{2k}$. By (B1), $0 \notin P$ and $\sigma \in P$, so $k + \sigma \notin P$ by anti-periodicity. Hence

$$d_n \geq 0 \qquad \text{whenever } n \in P. \tag{20}$$

Now compute the right-hand side of (17), using the attaining sets $A_n = \{j \in S : n + j \in P\}$ of Lemma 3.12; the constant slot never attains, so the computation is the same for both θ. Let $n = \ell k + i$.

- $i = 0$, ℓ even. Then $n + j \in P$ iff $j \in P$, so $A_n = \{\sigma\}$ by (B1), and the minimum is $d_{\ell k+\sigma} = 2$.
- $i = 0$, ℓ odd. Then $n + j \in P$ iff $j \notin P$, so $A_n = S \setminus \{\sigma\}$ by (B1). This set is nonempty, and every read is $d_{\ell k+j}$ with $j \neq 0, \sigma$, which is 0.
- $0 < i < k$. Every $j \in A_n$ reads $d_{n+j} \geq 0$ by (20). On the other hand (B2) gives a tap $j \in A_n$ with $i + j \not\equiv 0, \sigma \pmod{k}$ (the tap j of (B2) if ℓ is even, the tap j' if ℓ is odd), and for it $d_{n+j} = 0$ because $n + j \equiv i + j \pmod{k}$. So the minimum is 0.

In every case the right-hand side equals the left-hand side. Finally $d_{2mk} = -2m$ by (19). □

Theorem 3.14. *If $S \subseteq \{1, \ldots, k-1\}$ carries a σ-bump for some $\sigma \in S$, then no equation (8) with support S is globally periodic, for either value of θ and for any choice of positive coefficients.*

Proof. By Lemma 3.13 the orbit E of P is an integer solution of (15), and the bump d is an integer tangent solution along it; both statements hold for $\theta = 0$ and $\theta = 1$. Since $d_{2mk} = -2m$, d is unbounded, and Theorem 3.6 applies. □

The full support never carries a σ-bump; it is treated in Section 5.

3.5 Reading a carrier

Conditions (B1) and (B2) are checked many times in Sections 7 and 8, so I fix a vocabulary for them. Let $S \subseteq \{1, \dots, k-1\}$, and fix $\sigma \in S$ and an anti-periodic set P, coded by $Q = P \cap \{0, \dots, k-1\}$, so that for $0 < y < k$

$$y \in P \iff y \in Q, \qquad y + k \in P \iff y \notin Q. \tag{21}$$

Since (B1) forces $0 \notin Q$, I shall describe carriers by Q and speak of the pair (σ, Q), for which (B1) reads $Q \cap S = \{\sigma\}$. An *index* is an integer i with $0 < i < k$, and, as before, a tap is an element $j \in S$. At the index i the tap j *reads the position* $p = i + j$, an integer with $0 < p < 2k$, whose *residue* is $y = p \bmod k$. The tap is *deleted* at i if $y = 0$ or $y = \sigma$, and otherwise it *survives*; a surviving tap *reads* 1 if $p \in P$ and *reads* 0 if not. So (B2) says that at every index some surviving tap reads 1 and some surviving tap reads 0, and when this holds at a particular index i I shall say that the *clause at* i holds. The tap, and the position it reads, are *unwrapped* if $p < k$, so that $y = p$, and *wrapped* if $p > k$, so that $y = p - k$; the tap with $p = k$ is deleted. By (21), a surviving tap reads

$$\mathbf{1}[\, y \in Q \,] \text{ if it is unwrapped}, \qquad \mathbf{1}[\, y \notin Q \,] \text{ if it is wrapped}. \tag{22}$$

A position p, or a residue y with $0 < y < k$, is a *pin* if its residue lies in S, and *free* if its residue lies outside $S \cup \{0\}$. A pin read by a surviving tap has its residue in $S \setminus \{\sigma\}$, outside Q, and (22) gives

$$\text{an unwrapped pin reads 0}, \qquad \text{a wrapped pin reads 1}. \tag{23}$$

The reads of pins are therefore fixed by S and σ alone, and choosing a carrier amounts to choosing which free residues to put in Q.

As an example, take $k = 6$, $S = \{1, 2, 4, 5\}$, $\sigma = 1$ and $Q = \{1, 3\}$, so that $P = \{1, 3, 6, 8, 10, 11\}$ modulo 12. Then $Q \cap (\{0\} \cup S) = \{1\}$, which is (B1). The only free residue is 3, and the carrier puts it in Q. The following table names, at each index, one surviving tap reading each value, with the position it reads; a star marks a wrapped tap.

index i	1	2	3	4	5
tap reading 1	$2 \to 3$	$1 \to 3$	$5 \to 8^*$	$4 \to 8^*$	$5 \to 10^*$
tap reading 0	$1 \to 2$	$2 \to 4$	$1 \to 4$	$1 \to 5$	$4 \to 9^*$

The unwrapped reads of 1 are of the free residue $3 \in Q$, and the wrapped ones are pins; the reads of 0 are unwrapped pins, except at $i = 5$, where the wrapped tap 4 has residue $3 \in Q$. So (B2) holds, and $(1, Q)$ is a σ-bump; Figure 1 shows the orbit and the bump in space and time.

Consequently, by Theorem 3.14, no equation $x_{n+6}x_n = c_0 + c_1x_{n+1} + c_2x_{n+2} + c_4x_{n+4} + c_5x_{n+5}$ with positive c_1, c_2, c_4, c_5 is globally periodic.

3.6 The mirror

Let $S \subseteq \{1, \dots, k-1\}$ be palindromic. If Q satisfies (B1) for σ, its reflection $Q' = k - Q$ satisfies (B1) for $k - \sigma$, and the reflection turns a read of 1 at the index $k - i$ for $(k - \sigma, Q')$ into a read of 0 at the index i for (σ, Q). This halves the work: to check (B2) for (σ, Q) it is enough to find a surviving tap reading 1 at every index, both for (σ, Q) and for $(k - \sigma, Q')$.

Lemma 3.15 (the mirror)**.** *Let S be palindromic, $\sigma \in S$, and let $Q, Q' \subseteq \{1, \dots, k-1\}$ satisfy $y \in Q' \iff k - y \in Q$ for $0 < y < k$. If at the index $k - i$ the tap j' survives for $(k - \sigma, Q')$ and reads v, then at the index i the tap $k - j'$ survives for (σ, Q) and reads $1 - v$.*

$i=$	0	1	2	3	4	5
$\ell=0$		2				
$\ell=1$	2	−2				
$\ell=2$	−2	2				
$\ell=3$	4	−2				
$\ell=4$	−4	2				
$\ell=5$	6	−2				

Figure 1: The σ-bump of the example at $k = 6$, with $S = \{1, 2, 4, 5\}$, $\sigma = 1$ and $Q = \{1, 3\}$. Row ℓ holds the positions $n = 6\ell+i$, $0 \le i < 6$, so time runs left to right and then down. Shaded cells are the positions in the anti-periodic set P, where the orbit E equals -1; elsewhere $E = 0$. Each row is the complement of the one above, so the pattern repeats every two rows, that is, with period $2k = 12$. The numbers are the nonzero values of the bump d of (19); blank cells have $d_n = 0$. In column 1 the bump alternates between 2 and -2, while in column 0 it grows without bound, $d_{12m} = -2m$. Every negative entry lies in an unshaded cell, as (20) requires.

Proof. Let P and P' be the anti-periodic sets of Q and Q'. The tap $k-j'$ lies in S by palindromy. At i it reads the position $p = i+k-j'$, and at $k-i$ the tap j' reads the position $p' = k-i+j'$, so $p+p' = 2k$. Hence $p \equiv -p' \pmod{k}$, and since $\sigma \equiv -(k-\sigma)$, the tap $k-j'$ is deleted at i exactly when j' is deleted at $k-i$. For the reads, let $r = p' \bmod k$, which is nonzero as j' survives. If $p' = r$, then $p = (k-r)+k$, so by (21) $p \in P \iff k-r \notin Q \iff r \notin Q' \iff p' \notin P'$. If $p' = r+k$, then $p = k-r$ and $p \in P \iff k-r \in Q \iff r \in Q' \iff p' \notin P'$. Either way the two reads are complementary. □

4 Supports with at most two elements

This section deals with the supports with at most two elements. By Theorem 2.5 a globally periodic equation of the form (8) has a palindromic support, and by Lemma 2.8 it suffices to treat primitive supports. A palindromic support with at most two elements is $\emptyset$, $\{k/2\}$ or $\{j, k-j\}$ with $1 \le j < k/2$, since $k/2$ is the only index paired with itself. Primitivity then leaves three cases:

- $S = \emptyset$, which is primitive only at $k = 1$. There the reciprocal form is $x_{n+1}x_n = c_0$ with $c_0 > 0$, which is of known type, so there is nothing to prove.
- $S = \{k/2\}$, which is primitive only at $k = 2$: the support $S = \{1\}$ of order two (Theorem 4.1).
- $S = \{j, k-j\}$ with $\gcd(j, k) = 1$. The case $k = 3j$ forces $j = 1$, and is the support $\{1, 2\}$ of order three (Theorem 4.2); every other case is settled by Theorem 4.3.

At orders two and three with $c_0 > 0$ neither the linearisation nor the min-plus recursion can decide the question: at order two the characteristic polynomial $\lambda^2 - \epsilon_1\lambda + 1$ has roots of unity whenever $\epsilon_1 = 2\cos(2\pi q/p)$ for integers q and $p \ge 1$, and with $\theta = 1$ the min-plus recursion is itself periodic, with period 5 at order two and 8 at order three [Csörnyei and Laczkovich, 2001, Theorem 9]. What decides it is the second half of Theorem 3.3: once round a cycle of the min-plus recursion the leading coefficients are multiplied by a fixed factor, which must be 1.

4.1 Order two

At $k = 2$ with $S = \{1\}$ the equation (8) reads

$$x_{n+2}x_n = c_0 + c_1 x_{n+1}, \qquad c_1 > 0, \quad c_0 \geq 0. \tag{24}$$

The substitution $x_n = c_1 y_n$ turns it into

$$y_{n+2}y_n = \alpha + y_{n+1}, \qquad \alpha = \frac{c_0}{c_1^2} \geq 0, \tag{25}$$

and it preserves positivity and periodicity of solutions. For $\alpha = 1$ this is the Lyness equation [Lyness, 1942], of period 5; for $\alpha = 0$ it has period 6. The following theorem is known: Kocić et al. [1993] showed that the periods 5 and 6 occur only for $\alpha = 1$ and $\alpha = 0$, and the theorem itself was proved by Barbeau et al. [1995] on the whole real plane, by Csörnyei and Laczkovich [2001] with complex coefficients, and by Cima et al. [2004, Theorem 3.1]. I sketch the proof, which is essentially that of Csörnyei and Laczkovich [2001, Lemma 5] and of a remark of Barbeau et al. [1995, Section 3], because it is the model for order three, where the details are given. The formalisation (Appendix C) contains a complete proof along the lines of the sketch.

Theorem 4.1 (order two)**.** *If* (24) *is globally periodic, then* $c_0 = 0$ *or* $c_0 = c_1^2$.

Sketch of proof. Let $c_0 > 0$, so that $\alpha > 0$ and $\theta = 1$. The min-plus recursion $e_{n+2} + e_n = \min(0, e_{n+1})$ started from $(e_0, e_1) = (1, 0)$ runs through the 5-cycle $1, 0, -1, -1, 0$. Following the leading coefficients (16) of Theorem 3.2 once round this cycle, with $c_1 = 1$ and constant term α, multiplies C_{5s+1} by α: the first two steps give $C_{5s+4} = 1/C_{5s+1}$, and at $n = 5s + 4$ the constant slot alone attains the minimum, so $C_{5s+6} = \alpha/C_{5s+4} = \alpha\, C_{5s+1}$. Hence $C_{5s+1} = \alpha^s$. By Theorem 3.3 the period p is a multiple of 5, and $C_{p+1} = C_1$ gives $\alpha^{p/5} = 1$, that is $\alpha = 1$. □

4.2 Order three

At $k = 3$ with $S = \{1, 2\}$, Theorem 2.5 gives $c_1 = c_2$ for a globally periodic equation, so the equation is

$$x_{n+3}x_n = c_0 + c_1(x_{n+1} + x_{n+2}), \qquad c_1 > 0, \quad c_0 \geq 0. \tag{26}$$

The substitution $x_n = c_1 y_n$ turns it into

$$y_{n+3}y_n = \alpha + y_{n+1} + y_{n+2}, \qquad \alpha = \frac{c_0}{c_1^2} \geq 0. \tag{27}$$

For $\alpha = 1$ this is the period-8 equation of (2). The following theorem is also known. With complex coefficients it follows from Theorems 1 and 4 of Csörnyei and Laczkovich [2001], and with nonnegative coefficients it is the case $m = 1$ of Cima et al. [2004, Theorem 4.1]; Kocić et al. [1993] had shown that the period 8 occurs exactly when $c_0 = c_1^2$. For $c_0 > 0$ the proof below is that of Csörnyei and Laczkovich [2001, Lemma 5(ii)], written in the language of Theorem 3.2.

Theorem 4.2 (order three)**.** *If* $x_{n+3}x_n = c_0 + c_1 x_{n+1} + c_2 x_{n+2}$, *with* $c_1, c_2 > 0$ *and* $c_0 \geq 0$, *is globally periodic, then* $c_1 = c_2$ *and* $c_0 = c_1^2$. *In particular* $c_0 > 0$.

Proof. Suppose the equation is globally p-periodic with $p \geq 1$. By Theorem 2.5, $c_1 = c_2$, so it has the form (26). The two values of θ are handled separately.

First suppose $c_0 > 0$, so that (27) is globally p-periodic with $\alpha > 0$. The argument follows the sketch for order two. The min-plus recursion is $e_{n+3} + e_n = \min(0, e_{n+1}, e_{n+2})$, and the solution with initial window $(e_0, e_1, e_2) = (1, 0, -1)$ is 8-periodic, with

n mod 8	0	1	2	3	4	5	6	7
e_n	1	0	-1	-2	-2	-1	0	1
A_n	$\{2\}$	$\{2\}$	$\{1,2\}$	$\{1\}$	$\{1\}$	$\{0,1\}$	$\{0\}$	$\{0,2\}$

.

The period is exactly 8, so Theorem 3.3 gives $p = 8r$ with $r \geq 1$. If $C_{8s} = \xi$, $C_{8s+1} = \eta$ and $C_{8s+2} = \zeta$, the eight steps $n = 8s, \dots, 8s+7$ of (16) give in turn

$$C_{8s+3} = \frac{\zeta}{\xi}, \quad C_{8s+4} = \frac{\zeta}{\xi\eta}, \quad C_{8s+5} = \frac{\eta+1}{\xi\eta}, \quad C_{8s+6} = \frac{1}{\eta}, \quad C_{8s+7} = \frac{\eta+1}{\zeta},$$
$$C_{8s+8} = \frac{\xi(\alpha\eta+1)}{\eta+1}, \quad C_{8s+9} = \alpha\eta, \quad C_{8s+10} = \alpha\zeta.$$

For example, at $n = 8s+2$ both taps attain, so $C_{8s+5}C_{8s+2} = C_{8s+3} + C_{8s+4}$. Since $C_1 = 1$, induction gives $C_{8s+1} = \alpha^s$, and Theorem 3.3 gives $\alpha^r = C_{p+1} = C_1 = 1$. Hence $\alpha = 1$, that is $c_0 = c_1^2$.

Now suppose $c_0 = 0$, so the equation is $x_{n+3}x_n = c_1(x_{n+1} + x_{n+2})$. Its positive equilibrium is $x^* = 2c_1$, so $\epsilon_1 = \epsilon_2 = \frac{1}{2}$ in (11), and the characteristic polynomial of Theorem 2.4 is

$$\chi(\lambda) = \lambda^3 - \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\lambda + 1 = (\lambda+1)\big(\lambda^2 - \tfrac{3}{2}\lambda + 1\big).$$

By Theorem 2.4 every root of χ is a p-th root of unity. Let λ be a root of the quadratic factor. Its discriminant is negative, so its two roots are λ and $\bar{\lambda}$, and $\lambda\bar{\lambda} = 1$; hence $\lambda + \lambda^{-1} = \frac{3}{2}$. But λ is a root of unity, hence an algebraic integer, and so is λ^{-1}; so $\frac{3}{2}$ would be a rational algebraic integer, that is, an integer. This contradiction shows that $c_0 = 0$ is impossible. □

4.3 Two-term supports

The remaining small supports are the two-term supports $\{j, k-j\}$ at every order.

Theorem 4.3 (two-term supports)**.** *Let $1 \leq j$, $2j < k$, $\gcd(j,k) = 1$ and $k \neq 3j$, and let $S = \{j, k-j\}$. Then no equation (8) with support S is globally periodic, for either value of θ.*

Note that $\gcd(S \cup \{k\}) = \gcd(j,k)$, so the hypothesis $\gcd(j,k) = 1$ is exactly primitivity of S; and for primitive S the condition $k \neq 3j$ only excludes $(k,j) = (3,1)$, the order-three support of Theorem 4.2.

Throughout this subsection, for $t \in \mathbb{C}$ I write

$$\chi_t(\lambda) = \lambda^k + 1 - t\,(\lambda^j + \lambda^{k-j}), \tag{28}$$

which is the characteristic polynomial of Theorem 2.4 for the support $\{j, k-j\}$ with $\epsilon_j = \epsilon_{k-j} = t$. Let $\eta_1, \dots, \eta_k$ be its roots, listed with multiplicity, and let $s_r = \sum_{i=1}^{k} \eta_i^r$ be their power sums.

The first step is to compute a few power sums. Because χ_t has only four terms, Newton's identities collapse to a short recursion.

Lemma 4.4 (power sums)**.** *Let $1 \leq j$ and $2j < k$. Then*

1. *for $1 \leq r < k-j$, $s_r = j\,t^{r/j}$ if $j \mid r$, and $s_r = 0$ otherwise;*
2. *$s_{k-j} = (k-j)\,t + t\,s_{k-2j}$;*
3. *$s_k = t\,(s_{k-j} + s_j) - k$.*

Proof. Write $\chi_t(\lambda) = \lambda^k + a_1\lambda^{k-1} + \cdots + a_k$. Since $1 \leq j < k-j < k$, the only nonzero coefficients are $a_j = a_{k-j} = -t$ and $a_k = 1$. Newton's identities $s_r + a_1 s_{r-1} + \cdots + a_{r-1}s_1 + r\,a_r = 0$, for $1 \leq r \leq k$, therefore read

$$s_r = t\,s_{r-j}\,[j < r] + t\,s_{r-(k-j)}\,[k-j < r] + r\,t\,[r = j] + r\,t\,[r = k-j] - k\,[r = k],$$

where $[\,\cdot\,]$ is 1 if the condition holds and 0 otherwise. For $r < k-j$ only the first and third terms can be present, so $s_r = t\,s_{r-j}$ for $j < r < k-j$, $s_j = jt$ and $s_r = 0$ for $r < j$; part 1 follows by induction on r. Parts 2 and 3 are the cases $r = k-j$ and $r = k$, using $k - (k-j) = j$. □

The power sums replace a delicate analysis of where the roots of χ_t leave the unit circle: if every $|\eta_i| = 1$ then $|s_r| \le k$ for every r, and for a suitable r the power sum s_r is an explicit function of t that grows too fast.

Lemma 4.5 (interval bound)**.** *Let $1 \le j$, $2j < k$, $\gcd(j,k) = 1$ and $k \ne 3j$, and let $t \in \mathbb{C}$. If every root of χ_t has modulus 1, then t is real, $|t| \le 2$ and $t \le \sqrt{2}$.*

Proof. First, t is real. The polynomial χ_t is self-reciprocal, $\lambda^k \chi_t(1/\lambda) = \chi_t(\lambda)$, and $\chi_t(0) = 1$, so

$$\chi_t(\lambda) = \lambda^k \chi_t(1/\lambda) = \prod_i (1 - \eta_i \lambda) = \prod_i (-\eta_i) \prod_i \big(\lambda - \eta_i^{-1}\big) = \prod_i \big(\lambda - \eta_i^{-1}\big),$$

since $\prod_i (-\eta_i) = \chi_t(0) = 1$. So the inverses η_i^{-1} are the roots of χ_t again, with multiplicity. As $|\eta_i| = 1$, $\eta_i^{-1} = \bar{\eta}_i$, and therefore $s_j = \sum_i \eta_i^{-j} = \sum_i \bar{\eta}_i^{j} = \bar{s}_j$. By Lemma 4.4, $s_j = jt$, so $t = \bar{t}$ is real.

Second, $|s_r| \le \sum_i |\eta_i|^r = k$ for every r. Suppose $j \ge 2$. Then $0 < k - 2j < k - j$, and $j \nmid k - 2j$, since otherwise $j \mid k$ and $j = \gcd(j,k) = 1$. By Lemma 4.4, $s_{k-2j} = 0$, $s_{k-j} = (k-j)t$, $s_j = jt$ and

$$s_k = t\big((k-j)t + jt\big) - k = k\,(t^2 - 1).$$

So $k|t^2 - 1| \le k$, whence $t^2 \le 2$, and $|t| \le \sqrt{2}$.

Suppose instead $j = 1$. Then $k > 2$ and $k \ne 3$, so $k \ge 4$, and Lemma 4.4 gives $s_{k-2} = t^{k-2}$ and $s_{k-1} = (k-1)t + t\,s_{k-2} = (k-1)t + t^{k-1}$. From the first, $|t|^{k-2} \le k \le 2^{k-2}$ (as $k \ge 4$), so $|t| \le 2$. From the second, if $t > 1$ then $s_{k-1} > (k-1) + 1 = k$, which is impossible; so $t \le 1 \le \sqrt{2}$. □

The last ingredient is Kronecker's theorem [Kronecker, 1857]: if L is a number field and $\zeta \in L$ is a nonzero algebraic integer with $|\varphi(\zeta)| \le 1$ for every field embedding $\varphi : L \to \mathbb{C}$, then ζ is a root of unity; for a modern account see Bombieri and Gubler [2006, Theorem 1.5.9].

Proof of Theorem 4.3. Suppose that an equation (8) with support $S = \{j, k-j\}$ is globally p-periodic. By Theorem 2.5, $c_j = c_{k-j}$, so $\epsilon_j = \epsilon_{k-j} =: \epsilon$, and (11) gives $2\epsilon = \epsilon_j + \epsilon_{k-j} \le 1$. Thus

$$0 < \epsilon \le \tfrac{1}{2},$$

and by Theorem 2.4 every root η_i of χ_ϵ is a p-th root of unity. The idea of the proof is that $\epsilon = \zeta + \zeta^{-1}$ for a root of unity ζ. The bound $0 < \epsilon \le \frac{1}{2}$ forces ζ to have order at least 9, and then some conjugate of ϵ is larger than $\sqrt{2}$, which Lemma 4.5 forbids.

Step 1: ϵ is $\zeta + \zeta^{-1}$ for a root of unity ζ. The coefficient of λ^{k-j} in $\chi_\epsilon = \prod_i (\lambda - \eta_i)$ is $-\epsilon$, and it is also $(-1)^j$ times the j-th elementary symmetric function of $\eta_1, \dots, \eta_k$. Roots of unity are algebraic integers, so ϵ is an algebraic integer. Let ζ be a root of $z^2 - \epsilon z + 1$; then ζ is an algebraic integer (a root of a monic polynomial whose coefficients are algebraic integers), $\zeta \ne 0$, and $\epsilon = \zeta + \zeta^{-1}$. Let $L = \mathbb{Q}(\zeta, \eta_1, \dots, \eta_k)$, a number field containing ϵ. Let $\varphi : L \to \mathbb{C}$ be an embedding. Applying φ to the coefficients of $\chi_\epsilon = \prod_i (\lambda - \eta_i)$ gives $\chi_{\varphi(\epsilon)} = \prod_i (\lambda - \varphi(\eta_i))$, and each $\varphi(\eta_i)$ is again a p-th root of unity, so of modulus 1. By Lemma 4.5, $r = \varphi(\epsilon)$ is real with $|r| \le 2$. Now $\varphi(\zeta)$ is a root of $z^2 - rz + 1$, a real quadratic with discriminant $r^2 - 4 \le 0$; its two roots are therefore complex conjugates (or equal to ± 1) with product 1, so $|\varphi(\zeta)| = 1$. By Kronecker's theorem, ζ is a root of unity.

Step 2: the order of ζ. Let n_0 be the order of ζ. Since $|\zeta| = 1$, $\zeta^{-1} = \bar{\zeta}$, so $\epsilon = 2\cos(2\pi b/n_0)$ for some integer b coprime to n_0, and I may take $0 \le b \le n_0/2$. Since cos is decreasing on $[0, \pi]$, with $\cos(5\pi/12) = (\sqrt{6} - \sqrt{2})/4 > \frac{1}{4}$ and $\cos(\pi/2) = 0$, the condition $0 < \epsilon \le \frac{1}{2}$ forces

$$\frac{5}{24} < \frac{b}{n_0} < \frac{1}{4},$$

that is, $5n_0 < 24b < 6n_0$. If $n_0 \le 8$ then $24b < 48$, so $b = 1$ (as $b = 0$ is excluded), and $5n_0 < 24 < 6n_0$ has no integer solution. Hence $n_0 \ge 9$.

Step 3: a large conjugate. The cyclotomic polynomial Φ_{n_0} is irreducible over $\mathbb{Q}$, so it is the minimal polynomial of ζ, and $e^{2\pi i/n_0}$ is another of its roots. There is therefore a field embedding of $\mathbb{Q}(\zeta)$ into $\mathbb{C}$ sending ζ to $e^{2\pi i/n_0}$, and since $\zeta \in L$ it extends to an embedding $\varphi : L \to \mathbb{C}$. Then

$$\varphi(\epsilon) = \varphi(\zeta) + \varphi(\zeta)^{-1} = e^{2\pi i/n_0} + e^{-2\pi i/n_0} = 2\cos(2\pi/n_0),$$

and, as in Step 1, Lemma 4.5 gives $2\cos(2\pi/n_0) \le \sqrt{2}$. But $n_0 \ge 9$ gives $0 < 2\pi/n_0 < \pi/4$, so $2\cos(2\pi/n_0) > 2\cos(\pi/4) = \sqrt{2}$. This contradiction proves the theorem. □

5 The full support

The full support $S = \{1, \dots, k-1\}$ never carries a σ-bump. Indeed, (B1) forces $Q = \{\sigma\}$, and then at $i = k-1$ every position $i + j$ with $j \in S$ lies in $\{k, \dots, 2k-2\}$, where the only position that can lie outside P is $k + \sigma$, which (B2) excludes. So the full support has to be treated on its own. It is also the classical case: the full-support recursion, in which every term of the window appears in the numerator, was settled, even for complex coefficients, by Csörnyei and Laczkovich [2001]. They also passed to its piecewise-linear shadow and showed, by exhibiting a non-periodic real solution at every order other than these, that it is periodic only at order 2 and, with $c_0 > 0$, at order 3 (their Theorems 9–11). The periods of the shadow at order 4 with $c_0 > 0$ are determined by Linero Bas and Nieves Roldán [2021], and, with $c_0 > 0$, every solution with monotone initial data has period $3k - 1$ [Golomb and Callan, 1992]. The carrier $Q = \{1\}$ satisfies (B1) with $\sigma = 1$ but fails (B2), so the bump is not a tangent solution along it; but every window of the full support meets this carrier, so Lemma 3.12 applies, and here I give a different unbounded tangent solution along its orbit, written down in closed form and checked by hand.

Theorem 5.1. *Let $k \ge 4$. No equation (8) with full support $S = \{1, \dots, k-1\}$ is globally periodic, for either value of θ and for any choice of positive coefficients.*

The bound $k \ge 4$ is sharp: at $k = 2$ and $k = 3$ the full supports $\{1\}$ and $\{1, 2\}$ carry the period-6, Lyness and period-8 equations of (2).

5.1 The carrier

Throughout this section $k \ge 4$ and $S = \{1, \dots, k-1\}$. Let $P \subseteq \mathbb{Z}/2k$ be the anti-periodic set coded by $Q = \{1\}$ (Definition 3.11), that is

$$P = \{1,\ k\} \cup \{k+2, k+3, \dots, 2k-1\},$$

and let E be its orbit. Write every $n \ge 0$ as $n = \ell k + i$ with $\ell \ge 0$ and $0 \le i < k$, and call ℓ the *block* and i the *column* of n.

Lemma 5.2. *The orbit E is a solution of the min-plus recursion for both values of θ, with $m_n(E) = -1$, and its attaining sets are*

$$A_{\ell k+i}(E) = \begin{cases} \{1\}, & \ell \text{ even},\ i = 0, \\ \{k-i\} \cup \{k+2-i, \dots, k-1\}, & \ell \text{ even},\ i \ge 1, \\ \{2, \dots, k-1\}, & \ell \text{ odd},\ i = 0, \\ \{1, \dots, k-1-i\} \cup \big(\{k+1-i\} \cap S\big), & \ell \text{ odd},\ i \ge 1. \end{cases}$$

r	a_r $(i=0)$	b_r $(i=1)$	c_r $(2\le i\le k-2)$	z_r $(i=k-1)$	a_r+a_{r+1}	b_r+b_{r+1}	c_r+c_{r+1}	z_r+z_{r+1}
0	-1	0	$-m$	0	0	1	1	1
1	1	1	$m+1$	1	1	1	0	0
2	0	0	$-m-1$	-1	0	0	0	0
3	0	0	$m+1$	1	1	1	1	1
4	1	1	$-m$	0	1	0	0	0
5	0	-1	m	0	0	0	0	1
6	0	1	$-m$	1	1	1	1	1
7	1	0	$m+1$	0	0	0	0	0

Table 3: The ray of Lemma 5.3. Left: the value of d_n at $n=8km+rk+i$, by column class. Right: the left-hand side $d_{n+k}+d_n$ of the tangent recursion, by column class; for $r=7$ the next block is block 0 with m replaced by $m+1$.

Proof. Let $1\le j\le k-1$. If ℓ is even, $n+j\equiv i+j \pmod{2k}$, with $1\le i+j\le 2k-2$. When $i+j<k$ this lies in P only if $i+j=1$, that is $i=0$ and $j=1$; when $i+j\ge k$ it lies in P exactly when $i+j=k$ or $i+j\ge k+2$. If ℓ is odd, $n+j\equiv k+i+j \pmod{2k}$. When $i+j<k$ this lies in P exactly when $i+j\ne 1$, which for $j\ge 1$ means $i\ge 1$ or $j\ge 2$; when $i+j\ge k$ the residue is $i+j-k\in\{0,\dots,k-2\}$, which lies in P only if $i+j=k+1$. This gives the four sets $\{j\in S : n+j\in P\}$ listed, and each of them is nonempty (it contains 1, $k-i$, 2, and 1 or 2 respectively). So Lemma 3.12 applies. □

In words, a tap j at $n=\ell k+i$ reads column $i+j$ of block ℓ if $i+j<k$, and column $i+j-k$ of block $\ell+1$ otherwise. So in an even block the attaining taps read column 1 of the same block (if $i=0$), or column 0 and columns $2,\dots,i-1$ of the next block (if $i\ge 1$). In an odd block they read columns $i+1,\dots,k-1$ of the same block (columns $2,\dots,k-1$ if $i=0$), and also column 1 of the next block when $i\ge 2$. For example, at $k=5$ the columns read by the attaining taps are as follows, a prime marking a column of the next block:

column i of n	0	1	2	3	4
ℓ even	1	$0'$	$0'$	$0',2'$	$0',2',3'$
ℓ odd	$2,3,4$	$2,3,4$	$3,4,1'$	$4,1'$	$1'$

Here the middle columns are 2 and 3.

5.2 The ray

Write the block as $\ell=8m+r$ with $m\ge 0$ and $0\le r<8$, so that $n=8km+rk+i$. Define d_n by Table 3: its value depends on r, on m, and on the column only through the four classes $i=0$, $i=1$, $2\le i\le k-2$ (the *middle* columns) and $i=k-1$, which are distinct and nonempty because $k\ge 4$. Write a_r, b_r, c_r and z_r for the entries of row r in these four classes. Only the middle entries depend on m: they are $f-m$ in even blocks and $f+m$ in odd blocks, for a constant f given by the row. Block 8, which is block 0 of the next group, has $a_8=a_0$, $b_8=b_0$, $z_8=z_0$ and $c_8=-(m+1)$. So d is a ray (Definition 3.7) with $q=8k$: passing from m to $m+1$ changes only the middle entries, by -1 in even blocks and by $+1$ in odd ones.

Lemma 5.3. *The sequence d of Table 3 is a tangent solution along E, for both values of θ, and $d_{8km+2k+2}=-1-m$ for every $m\ge 0$.*

Proof. Let $n=\ell k+i$ with $\ell=8m+r$. The left-hand side $d_{n+k}+d_n$ is the sum of the entries of rows r and $r+1$ in the class of column i. In the middle class the terms in m cancel, and the four sums are the right-hand half of Table 3.

Now the right-hand side, the minimum of d_{n+j} over $j\in A_n(E)$. Note first that in every odd row $c_r\ge a_r$ and $c_r\ge z_r$, since $m\ge 0$. By the description after Lemma 5.2, the middle columns

are read only from odd blocks, and always alongside a column whose entry is no larger: in an even block ℓ at $i \geq 3$ they are read from block $\ell+1$ together with its column 0, and in an odd block at $i \leq k-3$ they are read together with column $k-1$ of the same block. So the middle reads never lower the minimum and can be ignored, and the right-hand side is:

- for r even: b_r if $i = 0$, and a_{r+1} if $i \geq 1$;
- for r odd: z_r if $i \in \{0, 1\}$, $\min(z_r, b_{r+1})$ if $2 \leq i \leq k-2$, and b_{r+1} if $i = k-1$.

The tangent recursion (17) therefore reduces to the identities

$$\begin{aligned} &r \text{ even:} \quad a_r + a_{r+1} = b_r, \quad b_r + b_{r+1} = c_r + c_{r+1} = z_r + z_{r+1} = a_{r+1}; \\ &r \text{ odd:} \quad a_r + a_{r+1} = b_r + b_{r+1} = z_r, \quad c_r + c_{r+1} = \min(z_r, b_{r+1}), \quad z_r + z_{r+1} = b_{r+1}. \end{aligned}$$

These are read off Table 3. For the even rows $r = 0, 2, 4, 6$ the values $(b_r; a_{r+1})$ are $(0;1)$, $(0;0)$, $(1;0)$ and $(1;1)$, and for the odd rows $r = 1, 3, 5, 7$ the values $(z_r; \min(z_r, b_{r+1}); b_{r+1})$ are $(1;0;0)$, $(1;1;1)$, $(0;0;1)$ and $(0;0;0)$; each matches the corresponding sums. By Lemma 5.2 the attaining sets, and hence this computation, are the same for both θ.

Finally, column 2 is a middle column because $k \geq 4$, so $d_{8km+2k+2} = c_2 = -1 - m$. □

Proof of Theorem 5.1. By Lemma 5.2 the orbit E is an integer solution of the min-plus recursion, and by Lemma 5.3 it carries an unbounded integer tangent solution, for both values of θ. Theorem 3.6 applies. □

6 Odd orders

At odd order the palindromic supports have a rigid shape, and this section uses it to dispose of every one of them except the full support.

Theorem 6.1 (odd orders)**.** *Let k be odd, and let $S \subseteq \{1, \ldots, k-1\}$ be primitive and palindromic with $|S| \geq 3$ and $S \neq \{1, \ldots, k-1\}$. Then no equation (8) with support S is globally periodic, whatever its positive coefficients and for either value of θ.*

Write $k = 2h+1$. Each pair $\{j, k-j\}$ contains exactly one even number, so a palindromic support is determined by its even elements; halving them gives a set $X \subseteq \{1, \ldots, h\}$ with

$$S = \{\, 2x,\ k - 2x \ :\ x \in X \,\}. \tag{29}$$

Two quite different mechanisms are needed, according to whether or not X is a *dilated interval*, that is $X = \mu \cdot \{1, \ldots, M\}$ for some integers $\mu, M \geq 1$. If X is not a dilated interval (Section 6.4), an unbounded tangent solution along a fixed periodic orbit, which alternates between low values at odd positions and high values at even ones, can be written down in closed form; whether it exists turns on rigidity statements about translates and reflections of X modulo k, which fail only for dilated intervals. If X is a dilated interval (Section 6.5), the support carries a σ-bump. The first mechanism cannot simply be replaced by the second. At $k = 7$ there are exactly two primitive palindromic supports with a non-dilated X, $\{1, 2, 5, 6\}$ and $\{1, 3, 4, 6\}$, and neither carries a σ-bump, as one can check by running through the four choices of σ and the 2^7 anti-periodic sets; this explains the choice of method, but it is not used in the proof and is not formalised. The tangent solutions along the alternating orbit give a short uniform argument instead, and the σ-bump is used only where rigidity fails, for the dilated intervals.

The alternating orbit has a precedent in the work of Cima et al. [2004]. At odd order a palindromic equation has a one-parameter family of 2-periodic solutions, equal to t at even and $(st + c_0)/(t - s)$ at odd positions, where $t > s = \sum_{j \in S,\, j \text{ even}} c_j$; along it the eigenvalues of the derivative of the second iterate must be constant roots of unity [Cima et al., 2004,

Proposition 2.6], which gives polynomial conditions on the coefficients that they solve at $k = 5, 7, 9, 11$. Taking $t - s$ a fixed multiple of δ^3 and letting $\delta \to 0^+$ gives the 2-periodic min-plus solution equal to 0 at even and -3 at odd positions, and the alternating orbit below changes it at two positions in each period $2k$. The mechanism here is different, however: the tangent recursion along the alternating orbit depends only on S, so it disposes of every choice of coefficients at once.

Throughout, $k = 2h+1$ is odd, and residues modulo k are identified with their representatives in $\{0, \dots, k-1\}$; for a residue n and a set X of residues, $n + X$ and $n - X$ denote the translate and the reflection of X taken modulo k. The set X always lies in the lower half $\{1, \dots, h\}$, and two elementary facts about such sets are used repeatedly: for $x, y \in X$,

$$x + y \not\equiv 0 \pmod{k}, \qquad \text{and} \qquad -x \equiv k - x \in \{h+1, \dots, 2h\} \text{ lies outside } X \cup \{0\}, \tag{30}$$

because $2 \le x + y \le 2h < k$. I write $\mathbf{1}[\,\cdot\,]$ for the indicator of a statement, equal to 1 if it is true and 0 otherwise, and $\mathbf{1}_U(y) = \mathbf{1}[\,y \in U\,]$ for the indicator function of a set U.

Lemma 6.2 (coordinates)**.** *Let $S \subseteq \{1, \dots, k-1\}$ be palindromic, and put $X = \{s/2 : \ s \in S \text{ even}\}$. Then $X \subseteq \{1, \dots, h\}$, (29) holds, and $|S| = 2|X|$. In particular $|S| \ge 3$ forces $|X| \ge 2$, and then S contains at least two odd elements.*

Proof. By palindromy S is the union of pairs $\{j, k-j\}$, each consisting of one even element $2x \le 2h$ and one odd element $k - 2x$, and these $2|X|$ numbers are distinct. □

6.1 The alternating orbit and the reduced recursion

The periodic orbit used below alternates, low on odd positions and high on even ones, except for a single raised value at one odd position of each period $2k$, compensated a half-period later. For $n \ge 0$ put

$$E_n = \begin{cases} -2 & n \text{ odd},\ n \equiv 1 \pmod{2k}, \\ -3 & n \text{ odd},\ n \not\equiv 1 \pmod{2k}, \\ -1 & n \text{ even},\ n \equiv k+1 \pmod{2k}, \\ \ \ 0 & n \text{ even},\ n \not\equiv k+1 \pmod{2k}. \end{cases} \tag{31}$$

I call E the *alternating orbit.*

Lemma 6.3 (the alternating orbit)**.** *Let S be palindromic with at least two odd elements. Then E is a $2k$-periodic integer solution of the min-plus recursion with support S, for either value of θ: $E_{n+k} + E_n = -3$ for every n, the minimum over the slots at time n is -3, and it is attained exactly at the taps*

$$A_n = \{\, j \in S \ : \ n + j \text{ odd},\ n + j \not\equiv 1 \pmod{2k} \,\}; \tag{32}$$

the constant slot, when $\theta = 1$, never attains.

Proof. As k is odd, n and $n + k$ have opposite parities, and $n \equiv 1$ exactly when $n + k \equiv k + 1 \pmod{2k}$; so the two exceptional values -2 and -1 are always paired, and $E_{n+k} + E_n = -3$ in every case. Every value of E is at least -3, with equality exactly at the odd positions other than those $\equiv 1 \pmod{2k}$, and the constant slot is worth $0 > -3$. It remains to see that A_n is nonempty. If n is even, A_n consists of the odd taps j with $n + j \not\equiv 1$; there are at least two odd taps and at most one of them has $n + j \equiv 1 \pmod{2k}$. If n is odd, the same argument applies to the even taps, of which there are as many as odd ones. □

The tangent recursion along E is

$$d_{n+k} + d_n = \min_{j \in A_n} d_{n+j} \qquad (n \ge 0), \tag{33}$$

and every tap in A_n reads d at an odd position. So the values of d at odd positions should determine everything, and they do. Write $v_t = d_{2t+1}$, and use the coordinates (29). At the odd time $n = 2t+1$ the attaining taps are the even ones, $j = 2x$, which read $d_{2(t+x)+1} = v_{t+x}$ and are absent when $2(t+x)+1 \equiv 1 \pmod{2k}$, that is when $t+x \equiv 0 \pmod{k}$. At the even time $n = 2t+1+k$ they are the odd ones, $j = k-2x$, which read v_{t+k-x} and are absent when $t-x \equiv 0 \pmod{k}$. This motivates the two *read sets*, of the time $2t+1$ and of the time $2t+1+k$ half a period later,

$$R_+(t) = \{\, t+x \;:\; x \in X,\ t+x \not\equiv 0 \,\}, \qquad R_-(t) = \{\, t+k-x \;:\; x \in X,\ t-x \not\equiv 0 \,\}, \tag{34}$$

with congruences modulo k; these are sets of positive integers. Modulo k, $R_+(t)$ lies in the translate $t+X$ and $R_-(t)$ in the reflection $t-X$, which explains the signs. Each is nonempty when $|X| \geq 2$, since at most one $x \in X$ can satisfy $t \pm x \equiv 0$. Note that no read ever lands on a multiple of k: the tap that would have done so is exactly the one that is absent. The *reduced recursion* is

$$v_{t+k} - v_t = \min_{y \in R_-(t)} v_y - \min_{y \in R_+(t)} v_y \qquad (t \geq 0). \tag{35}$$

Proposition 6.4 (reduction to the odd positions)**.** *Let S be palindromic with $|S| \geq 3$, with X as in Lemma 6.2. If the reduced recursion (35) has an unbounded integer solution $(v_t)_{t \geq 0}$, then no equation (8) with support S is globally periodic, for any positive coefficients and either θ.*

Proof. Define d_n for $n \geq 0$ by $d_{2t+1} = v_t$ at odd positions, and at even positions by

$$d_n = \min_{j \in A_n} d_{n+j} - d_{n+k} \qquad (n \text{ even});$$

the right-hand side involves only odd positions ($n+j$ for odd j, and $n+k$), so this is a definition. Then (33) holds at every even n by construction. At an odd time $n = 2t+1$ the position $n+k$ is even, so by the definition and the description of the attaining taps above,

$$d_{n+k} = \min_{j \in A_{n+k}} d_{n+k+j} - d_{n+2k} = \min_{y \in R_-(t)} v_y - v_{t+k},$$

and hence $d_{n+k} + d_n = \min_{R_-(t)} v - v_{t+k} + v_t = \min_{R_+(t)} v = \min_{j \in A_n} d_{n+j}$ by (35). So d is an integer solution of the tangent recursion along the alternating orbit E of Lemma 6.3, which applies because $|X| \geq 2$; and d is unbounded because v is. The conclusion is Theorem 3.6. □

6.2 Rays

The unbounded solutions of the reduced recursion used below are *rays of the reduced recursion*: they satisfy $v_{t+k} = v_t + \Delta(t \bmod k)$ for a fixed function Δ on $\mathbb{Z}/k$, not identically zero. Only residues modulo k matter for the read sets (34): the residues of $R_+(t)$ and $R_-(t)$ are

$$\bar{R}_+(r) = (r+X) \setminus \{0\}, \qquad \bar{R}_-(r) = (r-X) \setminus \{0\}, \qquad r = t \bmod k, \tag{36}$$

the punctured translate and the punctured reflection of X. For a set $U \subseteq \mathbb{Z}/k$, say that r is *reflection-blocked* if $\bar{R}_-(r) \subseteq U$, and *translate-blocked* if $\bar{R}_+(r) \subseteq U$.

Lemma 6.5 (indicator rays)**.** *Let $|X| \geq 2$ and $U \subseteq \mathbb{Z}/k \setminus \{0\}$, and define $\Delta : \mathbb{Z}/k \to \mathbb{Z}$ by*

$$\Delta(r) = \mathbf{1}[\, r \text{ reflection-blocked} \,] - \mathbf{1}[\, r \text{ translate-blocked} \,]. \tag{37}$$

Suppose that 0 is translate-blocked and not reflection-blocked, that no $r \neq 0$ is translate-blocked, and that every reflection-blocked r lies in U. Suppose also the vanishing condition*: every blocked read set, that is $\bar{R}_-(r)$ for r reflection-blocked and $\bar{R}_+(r)$ for r translate-blocked, contains a residue at which Δ vanishes. Then*

$$v_t = \mathbf{1}_U(t \bmod k) - \mathbf{1}[\, t \equiv 0 \,] + \lfloor t/k \rfloor\, \Delta(t \bmod k) \qquad (t \geq 0)$$

solves the reduced recursion (35), and $v_{mk} = -1-m$, so v is unbounded.

Proof. The hypotheses say that $\Delta(0) = -1$, that $\Delta \geq 0$ elsewhere, and that $\Delta = 0$ outside $U \cup \{0\}$. Clearly $v_{t+k} - v_t = \Delta(t \bmod k)$, and $v_{mk} = -1 - m$ because $0 \notin U$. Now look at the value of v at a read y, whose residue $\rho = y \bmod k$ is never 0, by (34). If $\rho \notin U$ then $\Delta(\rho) = 0$ and $v_y = 0$. If $\rho \in U$ then $v_y = 1 + \lfloor y/k \rfloor \Delta(\rho) \geq 1$, with equality when $\Delta(\rho) = 0$. Hence the minimum of v over a read set is 0 if some read lies outside U, and is exactly 1 if all reads lie in U, that is, if the read set is blocked; in that case the vanishing condition supplies a read with $\Delta(\rho) = 0$. So the right-hand side of (35) is $\mathbf{1}[\,r \text{ reflection-blocked}\,] - \mathbf{1}[\,r \text{ translate-blocked}\,]$ with $r = t \bmod k$, which is $\Delta(r)$. □

Remark 6.6 (the tangent solution is a ray). This remark is not used in the proof of Theorem 6.1. For a solution v as in Lemma 6.5, the tangent solution d built from v in the proof of Proposition 6.4 is a ray in the sense of Definition 3.7, with $q = 2k$. At odd positions, $d_{2t+1+2k} = v_{t+k} = v_t + \Delta(t \bmod k)$. At an even position n, write $n + k = 2\tau + 1$; there d_n is the minimum of v over a read set, less $d_{n+k} = v_\tau$. By the proof of Lemma 6.5, that minimum is 0 or 1 and depends only on the residues read, which do not change when n increases by $2k$; so $d_{n+2k} = d_n - \Delta(\tau \bmod k)$. Thus $d_{n+2k} = d_n + u_n$ with u $2k$-periodic, and u is not identically zero since $\Delta(0) = -1$.

The whole difficulty is to choose U so that the blocking hypotheses hold, and for that one needs to know when a punctured translate or reflection of X can lie inside X or a slight enlargement of it.

6.3 Rigidity of translates and reflections

Say that X is *symmetric about* $j \in \mathbb{Z}/k$ if $j - X \subseteq X$ (equivalently $j - X = X$), and that $W = X \cup \{0\}$ is *symmetric about* z if $z \in X$ and $z - x \in X$ for every $x \in X \setminus \{z\}$ (equivalently $z - W = W$). The first three statements below conclude that X is a dilated interval.

Lemma 6.7 (translate rigidity)**.** *Let $\emptyset \neq X \subseteq \{1, \ldots, h\}$ and $\zeta \not\equiv 0 \pmod{k}$. If $\zeta + X \subseteq X \cup \{0\}$ modulo k, then X is a dilated interval.*

Proof. Take $0 < \zeta < k$. If $\zeta \leq h$, then for $x_{\max} = \max X$ the integer $x_{\max} + \zeta$ lies in $(x_{\max}, 2h]$, so modulo k it is neither 0 nor an element of X; hence $\zeta > h$. Put $\mu = k - \zeta \in \{1, \ldots, h\}$, so that translation by ζ is $x \mapsto x - \mu$ modulo k. For $x \in X$ with $x < \mu$, the residue $x - \mu + k$ lies in $\{h + 1, \ldots, 2h\}$ and is neither 0 nor in X; so every $x \in X$ has $x \geq \mu$ and $x - \mu \in X \cup \{0\}$. By induction every element of X is a multiple of μ, and stepping down from $\max X = M\mu$ by μ stays inside X until it reaches 0; so $X = \mu \cdot \{1, \ldots, M\}$. □

Lemma 6.8 (translates of a symmetric set)**.** *Let $\emptyset \neq X \subseteq \{1, \ldots, h\}$ be symmetric about j, and let $\zeta \not\equiv 0$. If $\zeta + X \subseteq X \cup \{0, j\}$, then X is a dilated interval.*

Proof. The translate can leave X only at 0 and at j. It cannot leave at both: if $x_1 + \zeta \equiv 0$ and $x_2 + \zeta \equiv j$ with $x_1, x_2 \in X$, then $y = j - x_2 \in X$ and $y \equiv \zeta \equiv -x_1$, against (30). If it never leaves at j, Lemma 6.7 applies directly. If it leaves only at j, reflect: for $y \in X$ put $x = j - y \in X$. Either $x + \zeta \in X$, and then $y - \zeta = j - (x + \zeta) \in X$; or $x + \zeta \equiv j$, and then $y - \zeta \equiv 0$. So $-\zeta + X \subseteq X \cup \{0\}$, and Lemma 6.7 applies to $-\zeta$. □

Lemma 6.9 (uniqueness of the centre of W)**.** *Let $\emptyset \neq X \subseteq \{1, \ldots, h\}$. If $W = X \cup \{0\}$ is symmetric about two distinct residues z and z', then X is a dilated interval.*

Proof. Put $\zeta = z' - z \not\equiv 0$; I check $\zeta + X \subseteq X \cup \{0\}$ and apply Lemma 6.7. For $x = z$, $x + \zeta = z' \in X$. For $x \neq z$, $y = z - x$ lies in X, and $x + \zeta = z' - y$; this is 0 if $y = z'$, and lies in X otherwise, by the symmetry of W about z'. □

Finally, reflection blocking against X itself is always a symmetry.

Lemma 6.10 (reflection blocking)**.** *If $(r - X) \setminus \{0\} \subseteq X$, then X is symmetric about r or W is symmetric about r.*

Proof. If $r \notin X$ then no $r - x$ vanishes, and $r - X \subseteq X$. If $r \in X$ then $r - x \not\equiv 0$ for every $x \in X \setminus \{r\}$, so $r - x \in X$ for all such x, which is the symmetry of W about r. □

6.4 The non-dilated sets

Theorem 6.11 (non-dilated sets)**.** *Let S be palindromic with $|S| \geq 3$, and suppose that the set X of Lemma 6.2 is not a dilated interval. Then no equation (8) with support S is globally periodic, for any positive coefficients and either θ.*

Proof. By Proposition 6.4 and Lemma 6.5 it suffices to exhibit U satisfying the hypotheses of Lemma 6.5. Here $|X| \geq 2$ by Lemma 6.2. If X is symmetric about some j, then $j \neq 0$, since $-X$ is disjoint from X by (30), and $j \notin X$, since otherwise $0 = j - j \in X$; take $U = X \cup \{j\}$. Otherwise take $U = X$. In both cases 0 is translate-blocked, since $\bar{R}_+(0) = X$, and no $r \neq 0$ is translate-blocked: blocking would mean $r + X \subseteq U \cup \{0\}$, and Lemma 6.7 or Lemma 6.8 would make X a dilated interval. It remains to find the reflection-blocked residues and to check the vanishing condition.

$U = X \cup \{j\}$. Reflection blocking happens exactly at $r = j$, which lies in U. Indeed $\bar{R}_-(j) = j - X = X \subseteq U$. At $r = 0$, $\bar{R}_-(0) = -X$ has at least two elements, none in X, so they cannot all equal j. At $r \notin \{0, j\}$, put $\zeta = r - j \not\equiv 0$; for $y \in X$, the residue $x = j - y$ lies in X and $y + \zeta = r - x$, so reflection blocking at r would give $\zeta + X \subseteq X \cup \{0, j\}$, and Lemma 6.8 would make X a dilated interval. So $\Delta = \mathbf{1}_{\{j\}} - \mathbf{1}_{\{0\}}$, and the two blocked read sets are $\bar{R}_-(j) = X$ and $\bar{R}_+(0) = X$, on which Δ vanishes.

$U = X$. The residue 0 is not reflection-blocked, since $-X$ is disjoint from X. If r is reflection-blocked, Lemma 6.10 makes X or W symmetric about r; the first is excluded, so W is symmetric about r, which puts r in $X = U$. By Lemma 6.9 there is at most one such r. So Δ vanishes on all of X except at most one point, and $\bar{R}_+(0) = X$, having at least two elements, contains a zero of Δ. If z is reflection-blocked, then for $x \in X$ with $x \neq z$ the read $z - x \in \bar{R}_-(z)$ is neither 0 nor z, so $\Delta(z - x) = 0$; such an x exists as $|X| \geq 2$, so $\bar{R}_-(z)$ also contains a zero of Δ. This is the vanishing condition. □

6.5 Dilated intervals

When $X = \mu \cdot \{1, \ldots, M\}$ the support is

$$S = \{\, 2\mu i,\ k - 2\mu i \ :\ 1 \leq i \leq M \,\}, \qquad 2\mu M \leq 2h < k, \tag{38}$$

and the rigidity lemmas of Section 6.3 no longer rule out blocking. These supports carry a σ-bump with σ even. Since k is odd, every residue modulo $2k$ is uniquely of the form $2u$ or $2u + k$ with $u \in \mathbb{Z}/k$, so an anti-periodic set is the same thing as a function on $\mathbb{Z}/k$, and (B1) and (B2) become conditions on translates and reflections of X.

For $s \in \mathbb{Z}/k$, the translate $s + X$ and the reflection $s - X$ are called the two *runs* at s.

Lemma 6.12 (carriers on $\mathbb{Z}/k$)**.** *Let S be palindromic, with X as in Lemma 6.2, and let $x_0 \in X$ and $\sigma = 2x_0$. For $\gamma : \mathbb{Z}/k \to \{0, 1\}$ let P_γ be the set of residues modulo $2k$ of the form $2u$ with $\gamma(u) = 1$ or $2u + k$ with $\gamma(u) = 0$. Then P_γ is anti-periodic, every anti-periodic set is P_γ for exactly one γ, and (σ, P_γ) is a σ-bump for S if and only if*

(F1) $\gamma(0) = 0$, $\gamma(x_0) = 1$, $\gamma(x) = 0$ *for* $x \in X \setminus \{x_0\}$, *and* $\gamma(-x) = 1$ *for* $x \in X$;

(F2) for every $s \in \mathbb{Z}/k \setminus \{0\}$, the values $\gamma(u)$ for u in the run $s + X$ and $1 - \gamma(u)$ for u in the run $s - X$, with $u \notin \{0, x_0\}$, include both 0 and 1.

Proof. As k is odd, $u \mapsto 2u$ is a bijection from $\mathbb{Z}/k$ onto the even residues modulo $2k$, and $u \mapsto 2u+k$ onto the odd ones. Write $g(n) = \mathbf{1}[\,n \in P\,]$ for a set P of residues modulo $2k$. For $P = P_\gamma$, $g(2u) = \gamma(u)$ and $g(2u+k) = 1-\gamma(u)$, so P_γ is anti-periodic. Conversely, if P is anti-periodic, then with $\gamma(u) = g(2u)$ one gets $g(2u+k) = 1-\gamma(u)$, so $P = P_\gamma$, and γ is determined by P.

(B1). The position $0 = 2\cdot 0$ lies in P_γ iff $\gamma(0) = 1$, the even tap $2x$ lies in P_γ iff $\gamma(x) = 1$, and the odd tap $k-2x = 2(-x)+k$ lies in P_γ iff $\gamma(-x) = 0$. So $P_\gamma \cap (\{0\} \cup S) = \{\sigma\}$ says exactly (F1).

(B2). Let $0 < i < k$, and let $s \in \mathbb{Z}/k$ be the residue with $2s \equiv i \pmod k$; then $s \neq 0$, and $i \mapsto s$ is a bijection from $\{1,\dots,k-1\}$ onto $\mathbb{Z}/k \setminus \{0\}$. By the Chinese remainder theorem $i \equiv 2s \pmod{2k}$ if i is even and $i \equiv 2s+k$ if i is odd. So for $x \in X$ the tap $2x$ reads the position $i+2x \equiv 2(s+x)$ or $2(s+x)+k$, and the tap $k-2x$ reads $i+k-2x \equiv 2(s-x)+k$ or $2(s-x)$. When i is even the reads are therefore $\gamma(s+x)$ and $1-\gamma(s-x)$, and when i is odd they are the complements of these. The tap $2x$ is deleted when $i+2x \equiv 0$ or $i+2x \equiv \sigma = 2x_0 \pmod k$, that is when $s+x \equiv 0$ or x_0, since 2 is a unit modulo k; likewise the tap $k-2x$ is deleted when $s-x \equiv 0$ or x_0. Complementing every read does not change whether both values occur, so (B2) at i is (F2) at s. □

A unit modulo k carries the conditions (F1) and (F2) from one set to another. Indeed, let μ be a unit modulo k, let $X' = \mu X$ and $x_0' = \mu x_0$, and put $\gamma'(u) = \gamma(\mu^{-1}u)$. The map $u \mapsto \mu^{-1}u$ fixes 0, carries x_0' to x_0 and X' to X, and carries the runs $s \pm X'$ to the runs $\mu^{-1}s \pm X$, with $\mu^{-1}s \neq 0$ when $s \neq 0$. So if γ satisfies (F1) and (F2) for (X, x_0), then γ' satisfies them for (X', x_0'). This reduces the dilated intervals to $X = \{1,\dots,M\}$ and $x_0 = 1$, where the following explicit function works.

Theorem 6.13 (the interval carrier)**.** *Let $k = 2h+1$ be odd and $M \geq 2$ with $2M+3 \leq k$, and represent the residues modulo k by $u \in [-h, h]$. Define $\gamma : \mathbb{Z}/k \to \{0,1\}$ by*

$$\gamma(0) = 0, \quad \gamma(1) = 1, \quad \gamma(u) = 0 \ (2 \leq u \leq M), \quad \gamma(-u) = 1 \ (1 \leq u \leq M),$$

$$\gamma(u) = \mathbf{1}[\,u \equiv M+1 \pmod 2\,], \qquad \gamma(-u) = 1-\gamma(u) \qquad (M < u \leq h).$$

Then γ satisfies (F1) and (F2) for $X = \{1,\dots,M\}$ and $x_0 = 1$.

Proof. Here $h \geq M+1 \geq 3$, so the five sets $\{0\}$, $[1,M]$, $[-M,-1]$, $I_+ = [M+1,h]$ and $I_- = [-h,-M-1]$ partition $[-h,h]$, and γ is well defined. Condition (F1) holds by construction. Call the residues 0 and 1 *excluded*; they are the residues removed in (F2). The function γ has three properties.

(i) $\gamma(-u) = 1-\gamma(u)$ for $2 \leq u \leq h$: for $u \leq M$ both sides are 1, and for $u > M$ this is the definition.

(ii) γ changes value between M and $M+1$, between $-M-1$ and $-M$, and between h and $h+1 \equiv -h$. Indeed $\gamma(M) = 0$ and $\gamma(M+1) = 1$, $\gamma(-M) = 1$ and $\gamma(-M-1) = 0$ by (i), and $\gamma(-h) = 1-\gamma(h)$ by (i).

(iii) γ takes both values on any two consecutive residues of I_+, since exactly one of them is $\equiv M+1 \pmod 2$, and likewise of I_-, by (i).

Also $\gamma(-1) = 1 \neq 0 = \gamma(2)$.

Going round $\mathbb{Z}/k$, the residues fall into the consecutive arcs $[2,M]$, I_+, I_-, $[-M,-1]$, followed by the excluded pair $\{0,1\}$ and then $[2,M]$ again. By (ii) and the last remark, γ changes value between the last residue of each of the four arcs and the first residue of the next (across the excluded pair, between -1 and 2). So if γ is constant on the non-excluded elements of a run R of M consecutive residues, then R meets only one of the four arcs, since an arc of fewer

than k residues that meets two of them contains a boundary pair. It cannot lie in $I_{\pm}$, by (iii), as $M \geq 2$; so $R \subseteq [-M, 1]$ or $R \subseteq [0, M]$.

Suppose now that (F2) fails at $s \neq 0$. Then γ is constant on the non-excluded elements of both runs. The run $s + \{1, \ldots, M\}$ lies in $[-M, 1]$ or in $[0, M]$, which forces $s \in [-M-1, 1-M]$ or $s \in \{-1, 0\}$; the run $s - \{1, \ldots, M\}$ likewise forces $s \in \{0, 1, 2\}$ or $s \in \{M, M+1\}$. All these residues lie in $[-M-1, M+1]$, whose $2M+3 \leq k$ elements are distinct modulo k. The first list lies in $[-M-1, 0]$ and the second in $\{0\} \cup [1, M+1]$, so the only residue in both is 0. This contradicts $s \neq 0$. □

6.6 Proof of Theorem 6.1

Proof of Theorem 6.1. Let $k = 2h+1$ and S be as in the theorem, and let X be as in Lemma 6.2, so that $|X| \geq 2$. If X is not a dilated interval, Theorem 6.11 applies. Otherwise $X = \mu \cdot \{1, \ldots, M\}$, and S is (38), with $M = |X| \geq 2$ and $2\mu M \leq 2h$. Every element of S is $2\mu i$ or $k - 2\mu i$, so $\gcd(\mu, k)$ divides every element of $S \cup \{k\}$, and primitivity gives $\gcd(\mu, k) = 1$. If $\mu = 1$ and $M = h$, then S is the full support, which is excluded. If $\mu = 1$ and $M \leq h-1$, then $2M + 3 \leq k$. If $\mu \geq 2$, then $4M \leq 2\mu M < k$, and $2M + 3 \leq 4M < k$ since $M \geq 2$. In either of the last two cases Theorem 6.13 gives a function satisfying (F1) and (F2) for $\{1, \ldots, M\}$ and 1; multiplying its argument by μ^{-1} modulo k, as above, gives one for X and $x_0 = \mu$, and Lemma 6.12 turns it into a 2μ-bump for S. Theorem 3.14 now shows that no equation with support S is globally periodic, for any positive coefficients and either θ. □

7 Even orders: supports with an even element

Throughout this section the order k is even. With Theorem 3.14, the following theorem disposes of every support at even order, other than the full support, that is not all-odd. Reads are described in the vocabulary of Section 3.5, in particular (22) and (23), and Lemma 3.15 turns reads of 1 into reads of 0.

Theorem 7.1. *Let k be even and let $S \subseteq \{1, \ldots, k-1\}$ be primitive and palindromic, not the full support, and containing an even element. Then S carries a σ-bump (σ, P).*

The natural carriers at even order are the *parity carriers*: σ together with every free residue that has the parity of σ. Because k is even, the reflection $y \mapsto k - y$ preserves parity, and the argument splits along the two parity classes. Under (B1) the reads of the pins are fixed by (23), and a parity carrier puts every free position of one class into Q, so an unwrapped tap that lands on a free position of that class reads 1. A clause can therefore fail, for σ or (by a reflection) for $k - \sigma$, only at an index i where every unwrapped tap of the right class lands back in S, so that S is almost closed under adding i. If $a = \min S$ is even, a failure of the odd class forces a gap in the even part of S, and taps reading into that gap make the even class work for every even σ (Theorem 7.7). If a is odd and both classes fail for every σ, the failures force a rigid progression structure (Theorem 7.12), which carries one of two explicit carriers at $\sigma = a$ (Lemmas 7.13 and 7.14). On a first reading one can take Lemmas 7.9, 7.10 and 7.11 on trust.

7.1 Parity carriers

Write S_0 and S_1 for the even and odd elements of S, and refer to the residue of an integer modulo 2 as its *class*. Since k is even, $s \equiv k - s \pmod 2$ for every integer s, so S_0 and S_1 are each palindromic, and a nonempty class S_c has largest element $k - \min S_c$. If S is primitive then $S_1 \neq \emptyset$, since otherwise 2 would divide every element of $S \cup \{k\}$.

Definition 7.2. For $\sigma \in S$, the *parity carrier* of σ is

$$Q_\sigma = \{\sigma\} \cup \{\, y \in \{1, \ldots, k-1\} : \; y \notin S, \; y \equiv \sigma \pmod 2 \,\}.$$

An index i is *silent* for Q_σ if no surviving tap reads 1 at i. For an index i and a class c, write $C_c(i)$ for the statement

$$j \in S, \quad i+j<k, \quad i+j \equiv c \pmod 2 \quad \Longrightarrow \quad i+j \in S. \tag{39}$$

Lemma 7.3 (reads of a parity carrier)**.** *Let k be even, S palindromic, $\sigma \in S$, and let c be the class of σ.*

(a) $Q_\sigma \cap S = \{\sigma\}$, so (B1) holds.

(b) $y \in Q_{k-\sigma} \iff k-y \in Q_\sigma$ for $0<y<k$.

(c) A surviving tap whose position has residue y reads 1 if and only if either it is unwrapped, y is free and $y \equiv c$; or it is wrapped and $y \in S$ or $y \not\equiv c$.

(d) If (σ, Q_σ) is not a σ-bump, then for $\sigma' = \sigma$ or $\sigma' = k-\sigma$ (so $\sigma' \in S$ and $\sigma' \equiv \sigma$) some index is silent for $Q_{\sigma'}$.

(e) If i is silent for Q_σ, then $C_c(i)$ holds, and every wrapped tap at i reads a position with residue in Q_σ.

Proof. (a) is the definition. (b) holds because $y = k-\sigma \iff k-y = \sigma$, because y is free exactly when $k-y$ is (palindromy), and because $y \equiv k-y$ and $k-\sigma \equiv \sigma$. (c) is (22), using that a surviving tap has $y \neq \sigma$.

(d) By (a), the pair fails (B2) at some index i. If no surviving tap reads 1 at i, take $\sigma' = \sigma$. Otherwise no surviving tap reads 0 at i, and then by (b) and Lemma 3.15 no surviving tap reads 1 at the index $k-i$ for $Q_{k-\sigma}$; take $\sigma' = k-\sigma$. In both cases $\sigma' \equiv \sigma$ since k is even.

(e) Let $j \in S$ with $i+j<k$ and $i+j \equiv c$. If $i+j = \sigma$ then $i+j \in S$. Otherwise the tap j survives, and since it does not read 1, (c) says that $i+j$ is not free. For a wrapped tap whose position has residue y: if $y = \sigma$ then $y \in Q_\sigma$; otherwise the tap survives, and since it does not read 1, y is free and $y \equiv c$. □

For the rest of the section, S is palindromic at even k with an element of each class, $a = \min S$, and b is the least element of the class not containing a. Then $\max S = k-a$, the class of b has largest element $k-b$, and $b \le k-b$. Since also $a<b$,

$$2a<k, \qquad a+b<k. \tag{40}$$

The next three lemmas say where a silent index can be. Near the start, the taps $k-a$ and $k-\min S_c$ read unwrapped positions above the largest element of their class, which are free.

Lemma 7.4 (the top taps)**.** *Let $\sigma \in S$, of class c. An index i is not silent for Q_σ if either*

(i) $i<a$ and $i \equiv a+c \pmod 2$; or

(ii) i is even and $i < \min S_c$.

Proof. (i) The tap $k-a$ reads the position $y = i+k-a$, with $k-a<y<k$. So it is unwrapped, y is free because it exceeds $\max S$, and $y \equiv i+a \equiv c$. Being free, $y \neq \sigma$, so the tap survives, and it reads 1 by Lemma 7.3(c). (ii) Put $m = \min S_c$. The tap $k-m$ reads the position $y = i+k-m$, with $k-m<y<k$, and $y \equiv m \equiv c$ since i and k are even. As y is of class c and exceeds $\max S_c = k-m$, it is free, and the tap reads 1 as before. □

Beyond a the wrapped taps take over. Every element of Q_σ has the class of σ, and the wrapped taps $k-a$ and $k-b$ then confine a silent index to one parity and to the interval $(a,b]$.

Lemma 7.5 (the parity dichotomy)**.** *Let $\sigma \in S$, of class c, and let $i>a$ be silent for Q_σ. Then $i \equiv a+c \pmod 2$ and $i \le b$.*

Proof. All elements of Q_σ have class c. The tap $k-a$ wraps at i (as $i > a$), and its position has residue $i-a$, so by Lemma 7.3(e) $i - a \equiv c$, that is $i \equiv a + c$. If $i > b$, the tap $k-b$ also wraps, and in the same way $i \equiv b + c$; but $b \not\equiv a$. □

Finally, a containment $C_c(i)$ at an odd index empties an interval of the other class.

Lemma 7.6 (the gap)**.** *Let i be odd with $C_c(i)$, and suppose $S_c \neq \emptyset$. Then S has no element of class $1-c$ in the open interval $(i,\ i + \min S_c)$.*

Proof. Let $s \in S$ have class $1-c$ and $s > i$. Then $k - s \in S$ has class $1-c$, $i + (k-s) < k$ and $i + (k-s) \equiv c$ since i is odd. By $C_c(i)$, $i + k - s \in S_c$, so $i + k - s \leq \max S_c = k - \min S_c$, that is $s \geq i + \min S_c$. □

7.2 Even minimum

Theorem 7.7 (parity carriers at even minimum)**.** *Let k be even and S primitive and palindromic with $a = \min S$ even. If (σ_1, Q_{σ_1}) is not a σ-bump for some odd $\sigma_1 \in S$, then (σ_0, Q_{σ_0}) is a σ-bump for every even $\sigma_0 \in S$. In particular S carries a σ-bump on a parity carrier.*

Proof. Since S is primitive, $S_1 \neq \emptyset$, so $b = \min S_1$ exists, and $a < b$. By Lemma 7.3(d) some index i_1 is silent for $Q_{\sigma'}$, with $\sigma' \in S$ odd. Lemma 7.4 with $c = 1$ excludes $i_1 < a$ odd (by (i), as $a+1$ is odd), and $i_1 \leq a$ even (by (ii), as $\min S_1 = b > a$). So $i_1 > a$, and Lemma 7.5 gives i_1 odd and $i_1 \leq b$. By Lemma 7.3(e) $C_1(i_1)$ holds, and Lemma 7.6 gives

$$S_0 \cap (i_1,\ i_1 + b) = \emptyset. \tag{41}$$

Now fix an even $\sigma \in S$. I claim that every index i has a surviving tap reading 1 for Q_σ. There are four cases.

$i < a$, i even. Lemma 7.4(i) applies, as $i \equiv a$.

$i < a$, i odd. The tap b reads the unwrapped position $i + b < a + b < k$. The position $i + b$ is even and $i_1 \leq b < i + b < a + b < i_1 + b$, so by (41) it is not in S. It is therefore free, of class 0, and not σ, and the tap reads 1.

$i = a$. Suppose a is silent. Then $C_0(a)$ holds by Lemma 7.3(e). Let s be the largest element of $Z = \{y \in S_0 :\ y + i_1 < k\}$, which contains a because $a + i_1 \leq a + b < k$. By $C_1(i_1)$, $s + i_1 \in S$, and it is odd, so $s + i_1 \leq \max S_1 = k - b$. Then $a + s \leq k - b - (i_1 - a) < k$, so $C_0(a)$ puts $a + s \in S_0$; and $a + s + i_1 \leq a + k - b < k$. So $a + s \in Z$, against the maximality of s.

$i > a$. Suppose i is silent. By Lemma 7.5 i is even and $i \leq b$, so $i < b$ since b is odd; and $C_0(i)$ holds. Let x be the largest element of S_0 with $x \leq i_1$ (such elements exist, as $a < i_1$). Then $x + i < i_1 + b \leq 2b \leq k$, so $C_0(i)$ gives $x + i \in S_0$. By the maximality of x, $x + i > i_1$, so $x + i$ lies in the interval (41), a contradiction.

This proves the claim. The same claim for the even element $k - \sigma$, with Lemma 7.3(b) and Lemma 3.15, gives a surviving tap reading 0 at every index for Q_σ. With Lemma 7.3(a), (σ, Q_σ) is a σ-bump.

For the last sentence: either (σ, Q_σ) is a σ-bump for some odd σ, or it fails at $\sigma_1 = b$ and then (a, Q_a) is one. □

7.3 Odd minimum: where the failures sit

From now on $a = \min S$ is odd and $S_0 \neq \emptyset$, so that $b = \min S_0$ in the notation above. Then $a < b \leq k - b$, $\max S_1 = k - a$ and $\max S_0 = k - b$. Note that $a + b < k$, and that an odd number at most b is less than b.

Lemma 7.8 (localisation)**.** *(a) If $\sigma \in S$ is even and i is silent for Q_σ, then $C_0(i)$ holds and either $i = a$, or i is odd with $a < i < b$.*

(b) If $\sigma \in S$ is odd and i is silent for Q_σ, then $C_1(i)$ holds and either i is odd with $i \le a$, or i is even with $a < i \le b$.

Proof. The containments are Lemma 7.3(e). (a) An even $i < a$ is excluded by Lemma 7.4(ii), as $i < a < b = \min S_0$; an odd $i < a$ by Lemma 7.4(i), as $i \equiv a$. If $i > a$, then Lemma 7.5 makes i odd with $i \le b$, hence $i < b$. (b) An even $i < a$ is excluded by Lemma 7.4(i), as $i \equiv a+1$; and $i = a$ is odd. If $i > a$, then Lemma 7.5 makes i even with $i \le b$. □

So a silent index of the even class sits at the end $i = a$ or in the odd part of the inner range (a, b), and a silent index of the odd class at an odd end $i \le a$ or in the even part of the inner range $(a, b]$. When both classes fail, three of the four pairings are impossible, and the fourth is rigid.

Lemma 7.9 (no inner failure of the even class)**.** *Let i_0 be odd with $a < i_0 < b$ and $C_0(i_0)$, and let i_1 satisfy $C_1(i_1)$, with either i_1 odd and $i_1 \le a$, or i_1 even and $a < i_1 \le b$. This is impossible.*

Proof. By Lemma 7.6 with $c = 0$, S has no odd element in $(i_0,\ i_0 + b)$.

If i_1 is odd and $i_1 \le a$, then $b + i_1 < 2b \le k$ is odd, so $C_1(i_1)$ puts it in S; but $i_0 < b < b + i_1 < b + i_0$.

If i_1 is even, let x be the largest odd element of S with $x \le i_0$ (such elements exist, as $a < i_0$). Then $x + i_1 < 2b \le k$ is odd, so $x + i_1 \in S$ by $C_1(i_1)$, and $x + i_1 > i_0$ by the maximality of x. It cannot lie in the gap, so $x + i_1 \ge i_0 + b$, which with $x \le i_0$ forces $i_1 = b$. But then $a + b \in S$ in the same way, and $i_0 < a + b < i_0 + b$. □

The pairing of the two end failures is excluded by primitivity, and this is where the full support appears as the one obstruction. This proof and the next use an elementary observation. Say that a finite set W with largest element w is *closed under the step* $r > 0$ if $y + r \in W$ for every $y \in W$ with $y \ne w$. Then, for every $y \in W$, the sequence $y, y + r, y + 2r, \ldots$ stays in W until it reaches w, which it must do because W is bounded by w; so $r \mid w - y$, and $y, y + r, \ldots, w$ all lie in W.

Lemma 7.10 (two end failures)**.** *Let S be primitive, and suppose $C_0(a)$ and $C_1(i_1)$ hold with i_1 odd and $i_1 \le a$. Then S is the full support.*

Proof. Put $r = a + i_1$. Composing the two containments makes each class closed under a step, and the two steps turn out to be the same.

The odd class has step r. Let $y \in S_1$ with $y < k - a$. Then $a + y < k$ is even, so $a + y \in S_0$ by $C_0(a)$, whence $a + y \le k - b$. Then $a + y + i_1 < k$ (as $i_1 < b$) is odd, so $y + r \in S_1$ by $C_1(i_1)$. So S_1 is closed under the step r, with top $k - a$; every $y \in S_1$ has $r \mid k - a - y$, and in particular $r \mid k - 2a$.

$b = r$. The position $k - b + i_1 < k$ is odd, so it lies in S_1 by $C_1(i_1)$, and $k - b + i_1 \le k - a$. If the inequality were strict, $C_0(a)$ would put the even position $a + i_1 + k - b < k$ in S_0, above $\max S_0 = k - b$. So $k - b + i_1 = k - a$, that is $b = r$.

The even class has step b. Let $y \in S_0$ with $y < k - b$. Then $y + i_1 < k$ is odd, so $y + i_1 \in S_1$ by $C_1(i_1)$; and $y + i_1 < k - b + i_1 = k - a$, so $a + y + i_1 < k$ is even and lies in S_0 by $C_0(a)$. That is, $y + b \in S_0$. So S_0 is closed under the step b, with top $k - b$, and every $y \in S_0$ has $b \mid k - b - y$; at $y = b$ this gives $b \mid k$.

Collapse. Now $b \mid k$ and $b = r \mid k - 2a$, so $b \mid 2a$; since $b > a$, $b = 2a$, and $a \mid k$. Every odd $s \in S$ has $2a \mid k - a - s$ and every even $s \in S$ has $2a \mid k - 2a - s$, so a divides every element of $S \cup \{k\}$, and primitivity gives $a = 1$. Then $r = 2$, and the chain from $a = 1$ puts every odd number in $[1, k - 1]$ in S; and for even $y \in (0, k)$, $y - 1 \in S$ is odd with $1 + (y - 1) < k$, so $y \in S$ by $C_0(1)$. So S is full. □

Lemma 7.11 (end against inner range)**.** *Suppose $C_0(a)$ and $C_1(i_1)$ hold with i_1 even and $a < i_1 \le b$. Then $i_1 = b$, $b \mid k - 2a$, and $S_1 = \{a,\ a + b,\ a + 2b,\ \ldots,\ k - a\}$.*

Proof. By Lemma 7.6 with $i = a$ and $c = 0$, S has no odd element in $(a, a+b)$. The position $a + i_1 \le a + b < k$ is odd, so it lies in S by $C_1(i_1)$, hence $a + i_1 \ge a + b$ and $i_1 = b$. Now let $y \in S_1$ with $y < k - a$. Then $k - y \in S_1$ exceeds a, so $k - y \ge a + b$, that is $y + b \le k - a$; and $y + b$ is odd, so it lies in S_1 by $C_1(b)$. So S_1 is closed under the step b, with top $k - a$: every element of S_1 is $\equiv k - a \pmod b$, and the whole progression from a to $k - a$ lies in S_1. Hence $b \mid k - 2a$ and S_1 is that progression. □

Theorem 7.12 (the both-failing structure)**.** *Let k be even and S primitive and palindromic, not the full support, with $a = \min S$ odd and $S_0 \neq \emptyset$; put $b = \min S_0$. Suppose that (σ, Q_σ) is not a σ-bump for any $\sigma \in S$. Then*

$$k = 2a + Mb \quad \text{for an integer } M \ge 1, \qquad a < b < 2a,$$

$$S_1 = \{a + mb : \ 0 \le m \le M\}, \qquad 2a + mb \in S \ \text{ for } 0 \le m < M.$$

Proof. By Lemma 7.3(d) applied at $\sigma = b$, some index i_0 is silent for Q_{σ_0} with σ_0 even, and at $\sigma = a$ some index i_1 is silent for Q_{σ_1} with σ_1 odd. Lemma 7.8 locates both, and Lemma 7.9 rules out $i_0 \neq a$. So $C_0(a)$ holds.

Next, let $\sigma \in S$ be odd and let i be any index silent for Q_σ. By Lemma 7.8(b) and Lemma 7.10 (with $C_0(a)$ and since S is not full), i is even with $a < i \le b$, and then Lemma 7.11 gives

$$i = b, \qquad b \mid k - 2a, \qquad S_1 = \{a, a + b, \dots, k - a\}. \tag{42}$$

Applying this to i_1 gives the description of S_1, with $M = (k - 2a)/b$. Here $M \ge 1$ because $2a < k$ by (40). For $0 \le m < M$, the odd element $a + mb$ has $a + (a + mb) \le 2a + (M - 1)b < k$, so $C_0(a)$ puts $2a + mb$ in S. At $m = 0$, $2a \in S_0$, so $b \le 2a$.

It remains to exclude $b = 2a$; suppose it holds. If $M = 1$ then $k = 4a$, $S_1 \subseteq \{a, 3a\}$ and $S_0 \subseteq [b, k - b] = \{2a\}$, so a divides every element of $S \cup \{k\}$. Primitivity gives $a = 1$, $k = 4$, and $S \supseteq \{a, b, a + b\} = \{1, 2, 3\}$ is full, against the hypothesis. If $M \ge 2$, then $3a = a + b \in S_1$, and by Lemma 7.3(d) some index i_3 is silent for Q_{σ_3} with $\sigma_3 \in \{3a, k - 3a\}$. By (42) $i_3 = b = 2a$. But there the tap $k - a$ is wrapped and reads the pin $k + a$, of residue a, and $a \neq \sigma_3$ (as $k - 3a = a$ would mean $k = 4a$, that is $M = 1$). So the tap survives and reads 1 by (23), and i_3 is not silent. □

7.4 Carriers for the both-failing supports

The structure of Theorem 7.12 leaves the even elements of S strictly inside $(b, k - b)$ largely undetermined. Both carriers below are built to be insensitive to them.

Lemma 7.13 (the case $M = 1$)**.** *Let a be odd and b even with $a < b < 2a$, put $k = 2a + b$, and let $S \subseteq \{1, \dots, k - 1\}$ be palindromic with $a, b \in S$, with every odd element in $\{a, a + b\}$ and every even element in $[b, 2a]$. Then $\sigma = a$ with*

$$Q = \{a, \ 3a, \ b - a\} \cup \{\, y \in \{1, \dots, k - 1\} : \ y \text{ even}, \ y \notin S \,\}$$

is a σ-bump.

Proof. By palindromy $a + b = k - a$ and $2a = k - b$ lie in S. The positions $3a < k$ and $b - a > 0$ are odd and differ from a and $a + b$ (since $b \neq 2a$), so $Q \cap S = \{a\}$, which is (B1). An odd position lies in Q exactly when it is one of a, $3a$, $b - a$; an even position lies in Q exactly when it is free, and in particular every even position in $(0, b) \cup (2a, k)$ lies in Q.

For (B2) it suffices, at each index i, to name two surviving taps among a, b, $2a$, $a + b$ whose reads differ. The indices fall into twelve ranges, six of even and six of odd indices, cut out by the numbers a, b, $2a$, $a + b$, $2b$ and $3a$, and in each range the same two taps serve. The table of these taps, and the check that they survive and read different values, are in Section B.1. □

For $M \geq 2$ the carrier is the *band carrier*: a together with every free position strictly between a and $k-a$.

Lemma 7.14 (the case $M \geq 2$)**.** *Let S have the structure of Theorem 7.12 with $M \geq 2$, and let*

$$\mathcal{B} = \{\, y \notin S : \; a < y < k-a \,\}.$$

Then $\sigma = a$ with $Q = \{a\} \cup \mathcal{B}$ is a σ-bump.

The idea is that at an index $i > a$ the tap $k-a$ reads the position $i-a$, wrapped, and so reads 1 unless $i-a$ lies in $\mathcal{B}$; when it does, the gaps of S, which are shorter than a, supply an element j of S just below i, and the tap $k-j$ reads a position below $\mathcal{B}$. The mirror then gives the reads of 0.

Proof. (N1) For $1 \leq m \leq M$, $mb = k-(2a+(M-m)b) \in S$, by palindromy and the last clause of the structure.

(N2) An odd $y \in [a, k-a]$ with $y-a$ not a multiple of b is free, since $S_1 = \{a+mb\}$.

(N3) If $a < x \leq k-a$, some $j \in S$ has $x \leq j < x+a$.

For (N3), I first show that every $s \in S$ with $s < k-a = a+Mb$ has an element of S in $(s, s+a]$. Write $s = qb+r$ with $0 \leq r < b$. If $r < a$, then $q \leq M$ (as $qb \leq s < a+Mb < (M+1)b$), and $a+qb \in S$ lies in $(s, s+a]$. If $r \geq a$, then $qb+a \leq s < a+Mb$ gives $q < M$, and $(q+1)b \in S$ by (N1); it lies in $(s, s+a]$ because $(q+1)b - s = b-r \leq b-a < a$. Now, given x, let s be the largest element of S below x (there is one, a). Then $s < k-a$, so some $j \in S$ lies in $(s, s+a]$, and $j \geq x$ by the maximality of s, while $j \leq s+a < x+a$.

The set $\mathcal{B}$ consists of free positions, so $Q \cap S = \{a\}$, which is (B1). By palindromy $\mathcal{B}$ is symmetric under $y \mapsto k-y$, so $Q' = \{k-a\} \cup \mathcal{B}$ is the reflection of Q.

Reads of 1. Let $z \in \{a, k-a\}$ and $Q^z = \{z\} \cup \mathcal{B}$; I show that every index i has a surviving tap reading 1 for (z, Q^z). As $z \in S$, $z \notin \mathcal{B}$, and a tap whose position lies in $\mathcal{B}$ survives.

- If $i < a$ and i is even, the tap a reads $i+a$, which is odd, lies in $(a, 2a) \subset (a, k-a)$, and has $(i+a)-a = i \in (0, b)$. By (N2) it lies in $\mathcal{B}$, and the tap, being unwrapped, reads 1.
- If $i < a$ and i is odd, the tap b reads $i+b$, which is odd, lies in $(b, a+b) \subset (a, k-a)$, and has $(i+b)-a \in (b-a, b) \subset (0, b)$. So it lies in $\mathcal{B}$ and reads 1.
- If $i = a$, the tap $2a \in S$ reads $3a$, which is odd, has $3a-a = 2a \in (b, 2b)$, and lies in $(a, k-a)$, since $2a < 2b \leq Mb$. So it lies in $\mathcal{B}$ and reads 1.
- If $i > a$, the tap $k-a$ is wrapped, with residue $y = i-a \in (0, k-a)$. If $y \notin \mathcal{B}$ and $y \neq z$, it survives and reads 1.
- If $i > a$ and $i-a = z$, then $z \neq k-a$, so $z = a$ and $i = 2a$. The tap $k-b$ is wrapped (as $2a > b$), with residue $2a-b \in (0, a)$, which is neither in $\mathcal{B}$ nor z; so it survives and reads 1.
- If $i > a$ and $i-a \in \mathcal{B}$, then $a < i-a < k-a$, and (N3) with $x = i-a$ gives $j \in S$ with $i-a \leq j < i$. The tap $k-j$ is wrapped, with residue $y = i-j \in (0, a]$, which is not in $\mathcal{B}$. If $y = z$, then $z \leq a < k-a$ forces $z = a$, so $j = i-a \in \mathcal{B}$, which is impossible as $j \in S$. So $y \neq z$, and the tap survives and reads 1.

With $z = a$ this gives a read of 1 at every index for (a, Q). With $z = k-a$ at the index $k-i$, Lemma 3.15 gives a surviving read of 0 at i for (a, Q). So (B2) holds. □

Corollary 7.15 (the both-failing supports)**.** *Under the hypotheses of Theorem 7.12, S carries a σ-bump at $\sigma = a$.*

Proof. If $M \geq 2$ this is Lemma 7.14. If $M = 1$, then $k = 2a+b$, the odd elements of S are a and $a+b$, and every even element s has $b \leq s$ and $b \leq k-s$, so $s \in [b, 2a]$. This is Lemma 7.13. □

7.5 Proof of Theorem 7.1

Proof of Theorem 7.1. Let $a = \min S$. If a is even, Theorem 7.7 gives a σ-bump on a parity carrier. If a is odd, either (σ, Q_σ) is a σ-bump for some $\sigma \in S$, or S satisfies the hypotheses of Theorem 7.12, since it is primitive, palindromic, not full and has an even element; and then Corollary 7.15 gives a σ-bump at $\sigma = a$. □

By Theorem 3.14, no equation (8) whose support satisfies the hypotheses of Theorem 7.1 is globally periodic, for either value of θ.

8 Even orders: all-odd supports

This section treats the supports at even order all of whose elements are odd, the hardest supports for the σ-bump, because the clauses of Definition 3.11 then split by parity (Section 8.1).

Theorem 8.1 (all-odd supports)**.** *Let k be even, and let $S \subseteq \{1, \dots, k-1\}$ be primitive and palindromic, with every element odd and $|S| \geq 3$. Suppose that $(k, S) \neq (6, \{1, 3, 5\})$. Then S carries a σ-bump.*

The support $\{1, 3, 5\}$ at $k = 6$ carries no σ-bump at all: condition (B1) leaves only $\sigma \in \{1, 3, 5\}$ and the two free positions 2 and 4 of Q, and in each of the twelve cases (B2) fails. It is treated separately in Proposition 3.10.

Supports containing 1 are handled in Section 8.2 by small explicit carriers. For the others, with $a = \min S \geq 3$, the carrier is a *half-arc*: a free position y is put in Q when the phase νy modulo $2k$ (shifted by $t \in \{0, k\}$ if y is odd) lies in the lower half $[0, k)$, for a suitable odd multiplier ν. At the index i the reads of free positions are then governed by the translate $\nu i + \nu S$ of the phase set, and the clause at i can fail only if that translate lies in one half of the circle, apart from pins and deleted taps. At odd indices there are no pins, and the phases of S, being symmetric under $z \mapsto k - z$, are spread too evenly to fit into one half once ν separates two elements of S below $k/2$ (Section 8.3). At even indices, for a unit ν, failures for both choices of t would make νS contain a whole coset of a subgroup of $\mathbb{Z}/2k$, which is impossible (Section 8.5); the one gap is the index $2a$, closed by a three-point condition (T) on ν, which a lemma on three-term progressions, Theorem 9.1, supplies (Section 8.6). The supports $\{a, k/2, k-a\}$, with a single element below $k/2$, get an explicit ν (Section 8.4), and seventeen supports at orders $k \leq 42$ escape these arguments and are given a carrier in a table. On a first reading one can skip the case-by-case proofs and the finite checks.

8.1 The parity split

Throughout this section k is even, $S \subseteq \{1, \dots, k-1\}$ is palindromic, and reads are described in the vocabulary of Section 3.5, in particular (22) and (23).

Suppose now that every element of S is odd. Then at an odd index every tap reads an even position, and at an even index an odd one. Two consequences are used repeatedly. First, the clauses at odd indices involve only the even elements of Q, and those at even indices only the odd ones. Second, at an odd index no position is a pin, and since σ is odd the only deleted tap is $k - i$ (when it lies in S), which reads the position k. At an even index no position is $\equiv 0 \pmod{k}$, and the deleted taps are those reading σ or $\sigma + k$.

8.2 Supports containing 1

When $1 \in S$ the taps 1 and $k - 1$ read the two neighbours of every position, and very small carriers suffice.

Lemma 8.2 (a three-point carrier)**.** *Let k be even and S palindromic and all-odd, with $1 \in S$, and let $b \in S$ with $5 \le b \le k/2$. Then $\sigma = b$ and $Q = \{b-1, b, b+1\}$ form a σ-bump for S.*

Proof. (B1) holds because $b \pm 1$ are even and $0 \notin Q$. Note that $k \ge 2b \ge 10$, and that $1, b, k-b, k-1 \in S$; only these four taps are used.

Even indices. Here every position is odd, and the only odd element of Q is $b = \sigma$. So a surviving unwrapped tap reads 0 and a surviving wrapped tap reads 1. For a 1: the tap $k-1$ reads $i+k-1$, which is wrapped and is deleted only if $i+k-1 = k+b$, that is $i = b+1$; at $i = b+1$ the tap $k-b$ reads $k+1$, wrapped and not deleted. For a 0: the tap 1 reads $i+1 < k$, deleted only if $i+1 = b$; at $i = b-1$ the tap b reads $2b-1 < k$, not deleted.

Odd indices. Here every position is even and the only deleted position is k. For a 1: at $i \ge 3$ the tap $k-1$ reads the wrapped position $i+k-1$, of residue $i-1$, which reads 1 unless $i-1 = b \pm 1$. The remaining cases are $i = 1$, where the tap b reads $b+1 \in Q$; $i = b$, where the tap 1 reads $b+1$; and $i = b+2$, where the tap $k-b$ reads $k+2$, of residue $2 \notin Q$ (as $b-1 \ge 4$). For a 0: unless $i = k-1$ the tap 1 reads the unwrapped position $i+1$, which reads 0 unless $i+1 = b \pm 1$. The remaining cases are $i = k-1$ and $i = b$, where the taps b and $k-1$ respectively read $k+b-1$, of residue $b-1 \in Q$; and $i = b-2$, where the tap b reads $2b-2 < k$, not in Q because $b \ge 5$. □

Lemma 8.3 (a carrier when $3 \in S$)**.** *Let $k \ge 8$ be even and S palindromic and all-odd, with $1, 3 \in S$. Then $\sigma = 3$ and $Q = \{2, 3, 6\}$ form a σ-bump for S.*

Proof. Since $k \ge 8$, $Q \subseteq \{1, \dots, k-1\}$, and (B1) holds because 2 and 6 are even. Only the taps 1, 3, $k-3$ and $k-1$ are used.

Even indices. As in Lemma 8.2, every position is odd and the only odd element of Q is σ, so a surviving tap reads 0 if unwrapped and 1 if wrapped. For a 1: the tap $k-1$ reads the wrapped position $i+k-1$, of residue $i-1$, which is deleted only at $i = 4$; there the tap $k-3$ reads $k+1$, wrapped and not deleted. For a 0: the tap 1 reads $i+1 < k$, deleted only at $i = 2$; there the tap 3 reads $5 < k$, not deleted.

Odd indices. Every position is even, the only deleted position is k, and the even elements of Q are 2 and 6. So a surviving tap reads 1 exactly when it is unwrapped with residue in $\{2, 6\}$, or wrapped with residue outside $\{2, 6\}$. For a 1: at $i = 1$ the tap 1 reads 2; at $i = 3$ the tap 3 reads 6; at $i = 7$ the tap $k-3$ reads $k+4$, of residue 4; and at every other odd index, which has $i \ge 5$ and $i \ne 7$, the tap $k-1$ reads $k+i-1$, of residue $i-1 \notin \{0, 2, 6\}$. For a 0: at $i = 1$ the tap 3 reads 4; at $i = 5$ and at $i = k-1$ the taps $k-3$ and 3 respectively read $k+2$, of residue 2; and at every other odd index, which has $i \le k-3$ and $i \ne 1, 5$, the tap 1 reads $i+1 < k$, of residue $i+1 \notin \{0, 2, 6\}$. □

Proposition 8.4 (all-odd supports containing 1)**.** *Let k be even and $S \subseteq \{1, \dots, k-1\}$ palindromic and all-odd, with $1 \in S$, $|S| \ge 3$ and $(k, S) \ne (6, \{1, 3, 5\})$. Then S carries a σ-bump.*

Proof. If $3 \in S$, then $k \ge 8$: for $k \le 6$ the odd elements of $\{1, \dots, k-1\}$ are among $1, 3, 5$, so $|S| \ge 3$ would force the excluded support $(6, \{1, 3, 5\})$. So Lemma 8.3 applies. If $3 \notin S$, then $k-3 \notin S$ either, by palindromy, and S has an element other than 1 and $k-1$; that element or its mirror is an odd $b \in S$ with $5 \le b \le k/2$, and Lemma 8.2 applies. □

8.3 Half-arc carriers and the odd indices

From here on $1 \notin S$. Write $a = \min S$, so that $a \ge 3$ and $\sigma = a$ throughout, and write $L = \{s \in S : 2s < k\}$ for the lower half of S. Since S is palindromic, $S = L \cup (k - L)$, together with $k/2$ when $k/2 \in S$; and $a \in L$ unless $S = \{k/2\}$.

For integers ν, y and z write

$$H(z) = \mathbf{1}[\, z \bmod 2k < k \,], \qquad H_\nu(y) = H(\nu y), \tag{43}$$

where $z \bmod 2k$ denotes the representative in $\{0, \dots, 2k-1\}$. So $H_\nu(y)$ records in which half of $\mathbb{Z}/2k$ the *phase* νy lies. Two elementary facts about the phases are used throughout. If ν is odd then $\nu k \equiv k \pmod{2k}$, so for every integer z and y

$$H(z+k) = 1 - H(z), \qquad H_\nu(y+k) = 1 - H_\nu(y). \tag{44}$$

If also y is odd, then νy is odd, so $\nu y \not\equiv 0, k \pmod{2k}$, and

$$H_\nu(-y) = 1 - H_\nu(y), \qquad H_\nu(k-y) = H_\nu(y), \tag{45}$$

the second because $\nu(k-y) \equiv k - \nu y$, and $z \mapsto k - z$ maps each of the open halves $U = \{1, \dots, k-1\}$ and $V = \{k+1, \dots, 2k-1\}$ of $\mathbb{Z}/2k$ to itself.

Definition 8.5 (half-arc carrier)**.** Let ν be odd and $t \in \{0, k\}$, and for an integer y put $t_y = 0$ if y is even and $t_y = t$ if y is odd. The *half-arc carrier* $Q(\nu; t)$ of S consists of a together with every $y \in \{1, \dots, k-1\} \setminus S$ with $H(\nu y + t_y) = 1$.

So the even free positions follow the half-arc of ν, and the odd ones the same half-arc shifted by t. By construction $Q \cap (\{0\} \cup S) = \{a\}$, so (B1) holds at $\sigma = a$. The point of the construction is that a tap reading a free position reads the half-arc pattern, wrapped or not.

Lemma 8.6 (free reads)**.** *Let $Q = Q(\nu; t)$ with ν odd, and let $0 < p < 2k$ be a free position. Then $p \in P$ if and only if $H(\nu p + t_p) = 1$.*

Proof. If $p < k$ this is the definition, since $p \neq a$. If $p > k$, put $p' = p - k$, which has the parity of p and lies outside $S \cup \{0\}$. Then $p \in P$ iff $p' \notin Q$ iff $H(\nu p' + t_p) = 0$ iff $H(\nu p' + t_p + k) = 1$, by (44), and $\nu p' + k \equiv \nu p \pmod{2k}$. □

The clauses at odd indices read only even positions, and all of these are free except the deleted position k. They are settled by the following lemma, whose idea is that the phases of S are spread round the circle $\mathbb{Z}/2k$ too evenly to fit into one half, even after one of them is removed.

Lemma 8.7 (odd indices)**.** *Let S be all-odd, let ν be odd and $t \in \{0, k\}$, and suppose that $H_\nu(u) \neq H_\nu(v)$ for some $u, v \in L$. Then, for $\sigma = a$ and $Q = Q(\nu; t)$, the clause at every odd index holds.*

Proof. Let i be odd. As noted in Section 8.1, the only deleted tap is $k - i$ (if it lies in S), and every other tap j reads a free even position $p = i + j$, whose read is $H(\nu p) = H(\nu i + \nu j)$ by Lemma 8.6. So the clause fails exactly when the phases $\nu j \bmod 2k$ of the surviving taps all lie in $\{-\nu i, \dots, -\nu i + k - 1\}$ or all in $\{-\nu i + k, \dots, -\nu i + 2k - 1\}$.

First I locate the phases. They are odd, so each lies in U or in V, and by (45) the tap $k - j$ has phase $k - \nu j$, which lies in the same half as νj. Since $H_\nu(u) \neq H_\nu(v)$ both halves occur. Let x_- be the least phase in U, attained at the tap u_0, and y_- the least in V, attained at v_0. The taps $k - u_0$ and $k - v_0$ have the phases $k - x_- \in U$ and $3k - y_- \in V$, so $2x_- \leq k$ and $2y_- \leq 3k$. Going round the circle, each of the four phases $x_- \leq k - x_- < y_- \leq 3k - y_-$ is followed by the next at distances $k - 2x_-$, $x_- + y_- - k$, $3k - 2y_-$ and $x_- + y_- - k$ respectively. Each distance is at most k, and equals k only if $x_- = k/2$ and $y_- = 3k/2$; so at most $k - 1$ consecutive residues avoid all four phases, and $k - 1$ only in that case. If m consecutive residues contain all four phases, the other $2k - m$ residues are consecutive and avoid them, so $2k - m \leq k - 1$. Hence:

(a) no k consecutive residues contain all four phases;

(b) if $k + 1$ consecutive residues contain them, then $x_- = k/2$ and $y_- = 3k/2$, and then every phase is $k/2$ or $3k/2$ (a phase $z \in U$ has $z \geq x_-$ and $k - z \geq x_-$, and similarly in V).

If $k-i \notin S$, nothing is deleted and the clause holds by (a). Suppose instead that $s_0 = k-i \in S$. Then $\nu i \equiv k - \nu s_0$, and by (44) the tap $j \neq s_0$ reads $1 - H(\nu j - \nu s_0)$. If these reads were all 0, every phase would lie in the k consecutive residues $\{\nu s_0, \dots, \nu s_0 + k - 1\}$, which (a) excludes. If they were all 1, every phase other than νs_0 would lie in $\{\nu s_0 + k, \dots, \nu s_0 + 2k - 1\}$, and with νs_0 itself every phase would lie in the $k+1$ consecutive residues $\{\nu s_0 + k, \dots, \nu s_0 + 2k\}$; so (b) gives every phase in $\{k/2, 3k/2\}$, and then the tap $j \neq s_0$ reads 0 when $\nu j \equiv \nu s_0$ and 1 otherwise. The taps u and $k-u$ are distinct (as $2u < k$) and share one phase; v and $k-v$ share the other. At most one of the four taps is s_0, so the pair with the phase of s_0 contains a surviving tap reading 0, and the other pair a surviving tap reading 1. This contradicts the assumption that the reads were all 1. □

8.4 Supports with one element below $k/2$

If $1 \notin S$ and $L = \{a\}$, then $|S| \geq 3$ forces $S = \{a, k/2, k-a\}$, with $k/2$ odd. Here Lemma 8.7 does not apply, since L has a single element, and a specific multiplier is used instead.

Theorem 8.8 (three-element supports)**.** *Let $k = 2h$ with h odd, and $S = \{a, h, k-a\}$ with $3 \leq a < h$ odd and $\gcd(a, h) = 1$. Let $\beta \in \{2, 4\}$ with $\beta \equiv a + h \pmod 4$, suppose that $3\beta < h$ (that is, $h \geq 7$ if $\beta = 2$ and $h \geq 13$ if $\beta = 4$), and let $\nu \equiv a^{-1}(3h + \beta) \pmod{2k}$. Then $\sigma = a$ and $Q = Q(\nu; 0)$ form a σ-bump for S.*

Proof. All residues in this proof are modulo $2k = 4h$. Since a is odd and prime to h, it is a unit, so ν is well defined and $\nu a \equiv 3h + \beta$. As $a^2 \equiv 1 \pmod 4$ and $3h + \beta \equiv 3h + a + h \equiv a \pmod 4$, one has $\nu \equiv a(3h + \beta) \equiv 1 \pmod 4$; so ν is odd and $\nu h \equiv h$. The phases of the three taps are therefore

$$\nu a \equiv 3h + \beta, \qquad \nu h \equiv h, \qquad \nu(k-a) \equiv k - \nu a \equiv 3h - \beta.$$

(B1) holds by construction. By Lemma 8.6, a tap reading a free position p reads $H(\nu p) = H(\nu i + \nu j)$, where j is the tap.

Deletions and pins. The deleted positions are k, a and $k + a$; the pins are h and $k - a$, which read 0, and $k + h$ and $2k - a$, which read 1 (the pins a and $k + a$ are deleted). Solving $i + j = p$ for $j \in S$, p in this list and $0 < i < k$ shows that a tap is deleted or reads a pin only at the seven indices

$$a, \quad h, \quad k-a, \quad h-a, \quad k-2a, \quad h+a, \quad 2a.$$

Generic indices. At any other index all three taps read free positions, with phases $\nu i + \{3h + \beta,\ h,\ 3h - \beta\}$. Going round the circle, the three points $h < 3h - \beta < 3h + \beta$ cut it into arcs of lengths $2h - \beta$, 2β and $2h - \beta$, all less than $2h$, so no half $\{c, \dots, c + 2h - 1\}$ contains all three phases, and the reads are not all equal.

The special indices. At each of the seven special indices one names a tap reading 0 and a tap reading 1, using the rule that the phase of a position is the sum of the phases of the index and the tap, and that the position reads 1 exactly when its phase lies in $\{0, \dots, 2h - 1\}$. The seven cases, and the check that every tap named survives, are in Section B.2. □

8.5 The even indices with a unit multiplier

Now let ν be a unit modulo $2k$. Multiplication by ν is then a bijection of $\mathbb{Z}/2k$, and it is convenient to work entirely with phases. Put

$$\Phi = \{\nu s \bmod 2k : s \in S\}, \qquad \alpha = \nu a \bmod 2k.$$

The elements of Φ are odd, and $\Phi = k - \Phi$ by (45). Moreover H_ν is non-constant on S exactly when Φ meets both U and V.

Let $t \in \{0, k\}$ and $Q = Q(\nu; t)$. Let $0 < i < k$ be even, and put $\phi = \nu i \bmod 2k$. Then ϕ is even, and $\phi \notin \{0, k\}$ because ν is a unit and $i \not\equiv 0, k \pmod{2k}$; the same holds for $\phi + k$. The tap j reads the odd position $p = i + j$, of phase $\nu p \equiv \phi + \nu j$. Since $p > j \geq a$, the tap is deleted exactly when $p = a + k$, that is, when the phase of p is $\alpha + k$. Because $p \mapsto \nu p$ is a bijection, p is an unwrapped pin iff $\nu p \in \Phi$, and a wrapped pin iff $\nu p \in \Phi + k$. So a surviving tap whose position has phase z reads

$$0 \text{ if } z \in \Phi; \qquad 1 \text{ if } z \in \Phi + k; \qquad \text{otherwise } H(z + t) = \begin{cases} \mathbf{1}[z \in U], & t = 0, \\ \mathbf{1}[z \in V], & t = k, \end{cases} \tag{46}$$

by Lemma 8.6; here z is odd, so it lies in U or in V. The phases read at i form the *read set* $\phi + \Phi$, less the point $\alpha + k$ when that point is the phase of a deleted tap. For a residue τ consider the two conditions

$$C_U(\tau): \quad (\tau + \Phi) \cap U \subseteq \Phi, \qquad C_V(\tau): \quad (\tau + \Phi) \cap V \subseteq \Phi.$$

Lemma 8.9 (failing read sets)**.** *With the notation above, suppose that the clause at the even index i fails.*

(i) If every surviving tap reads 1, then $C_U(\phi + k)$ holds when $t = 0$, and $C_V(\phi + k)$ when $t = k$.

(ii) If every surviving tap reads 0, then $C_U(\phi)$ holds when $t = 0$, and $C_V(\phi)$ when $t = k$, unless some tap is deleted at i and $\alpha + k$ lies in U (for $t = 0$), respectively in V (for $t = k$).

(iii) If some tap is deleted at i and $i \neq 2a$, then some surviving tap reads 1.

Proof. (i) Take $t = 0$. Let $x \in \Phi$ with $z = \phi + x \in V$. If z is the deleted point then $z = \alpha + k \in \Phi + k$. Otherwise z reads 1; by (46) it is not in Φ, and since $z \notin U$ it must lie in $\Phi + k$. So $(\phi + \Phi) \cap V \subseteq \Phi + k$, and adding k gives $C_U(\phi + k)$. The case $t = k$ is the same with U and V exchanged.

(ii) Take $t = 0$. Let $x \in \Phi$ with $z = \phi + x \in U$, and suppose z is not the deleted point; this is automatic unless a tap is deleted and $\alpha + k \in U$. Then z reads 0, so $z \notin \Phi + k$, and as $z \in U$ (46) forces $z \in \Phi$. The case $t = k$ is the same.

(iii) The deleted tap is $j_0 = a + k - i$. The tap $k - a \in S$ reads $i + k - a = k + (k - j_0)$, a wrapped pin, since $k - j_0 \in S$. It is deleted only if $k - j_0 = a$, that is $i = 2a$. So for $i \neq 2a$ it reads 1. □

Two failures of opposite type, one in U and one in V, are incompatible. The reason is that together they force Φ to contain a whole coset of a subgroup, which a set of the form νS cannot do.

Lemma 8.10 (no coset)**.** *Let ν be a unit modulo $2k$ and $S \subseteq \{1, \ldots, k - 1\}$. Then $\Phi = \nu S$ contains no coset $c + g\mathbb{Z}/2k\mathbb{Z}$ with $g \mid 2k$ and $0 < g < 2k$.*

Proof. Multiplication by ν^{-1} is an automorphism of $\mathbb{Z}/2k$, so it maps such a coset onto a coset $s_0 + g\mathbb{Z}/2k\mathbb{Z}$, which would then lie in S (read modulo $2k$). Since $g \mid 2k$ and $g < 2k$, $g \leq k$. The integers $s_0, s_0 + g, s_0 + 2g, \ldots$ start below k and increase in steps of at most k, so one of them lies in $\{k, \ldots, 2k - 1\}$; its residue is in the coset but not in S. □

Lemma 8.11 (two failures force a coset)**.** *Let k be even, and let Φ be a set of odd residues modulo $2k$ with $\Phi = k - \Phi$ that meets both U and V. Let ϕ, ϕ' be even residues not in $\{0, k\}$, and suppose that $C_U(\phi)$ and $C_V(\phi')$ hold. Then Φ contains a coset $c + g\mathbb{Z}/2k\mathbb{Z}$ with $g \mid 2k$ and $0 < g < k$.*

Proof. Reductions. $C_U(\phi)$ implies $C_U(-\phi)$: if $x \in \Phi$ and $z = x - \phi \in U$, then $k - x \in \Phi$ and $\phi + (k - x) = k - z \in U$, so $k - z \in \Phi$ and $z \in \Phi$. Likewise $C_V(\phi')$ implies $C_V(-\phi')$, since $z \mapsto k - z$ also maps V to itself. So I may take $0 < \phi, \phi' < k$. The set $\Phi + k$ again consists of odd residues, satisfies $\Phi + k = k - (\Phi + k)$ and meets U and V, and $C_U(\tau)$ for Φ is $C_V(\tau)$ for $\Phi + k$ and conversely. So passing from Φ to $\Phi + k$ swaps C_U and C_V: the hypotheses $C_U(\phi)$ and $C_V(\phi')$ for Φ become $C_U(\phi')$ and $C_V(\phi)$ for $\Phi + k$, which are the hypotheses of the lemma for $\Phi + k$ with ϕ and ϕ' interchanged. A coset in $\Phi + k$ gives one in Φ, by adding k. So if $\phi > \phi'$ I may replace Φ by $\Phi + k$, and I may therefore also take $\phi \le \phi'$.

Translation into $(0, k)$. Put $Y_+ = \Phi \cap U$ and $Y_- = \{z : z + k \in \Phi \cap V\}$, two nonempty sets of odd integers in $(0, k)$. By $\Phi = k - \Phi$ both are closed under $z \mapsto k - z$ (for Y_-: $k - (m + k) \equiv (k - m) + k$). The hypotheses give four rules:

(G1) if $p \in Y_+$ and $p + \phi < k$, then $p + \phi \in Y_+$ (by $C_U(\phi)$);

(G2) if $m \in Y_-$ and $m < \phi$, then $\phi - m \in Y_+$ (the point $2k - m \equiv k - (m + k)$ of Φ moves to $\phi - m \in U$);

(G3) if $m \in Y_-$ and $m + \phi' < k$, then $m + \phi' \in Y_-$ (by $C_V(\phi')$);

(G4) if $p \in Y_+$ and $p < \phi'$, then $\phi' - p \in Y_-$ (the point $k - p$ of Φ moves to $k + \phi' - p \in V$).

Call a set Π of integers in $(0, k)$ *n-periodic* if $p \in \Pi$, $0 < z < k$ and $z \equiv p \pmod n$ imply $z \in \Pi$. The proof uses one mechanism twice.

The engine. Let Π be n-periodic, $n > 0$, and suppose that for every $p \in \Pi$ there is $p' \in \Pi$ with $p' \equiv p + f \pmod n$. Then Π is $\gcd(n, f)$-periodic. Indeed, write $g = \gcd(n, f)$ and choose $y \ge 0$ with $yf \equiv g \pmod n$. If $p \in \Pi$, $0 < z < k$ and $z \equiv p \pmod g$, then $z \equiv p + jg \pmod n$ for some $j \ge 0$; applying the hypothesis jy times gives $p'' \in \Pi$ with $p'' \equiv p + jyf \equiv z \pmod n$, and so $z \in \Pi$.

Step 1: Y_+ is γ-periodic, where $\gamma = \gcd(\phi, \phi')$. By (G1) and the symmetry of Y_+ ($p - \phi = k - ((k - p) + \phi)$), Y_+ is closed under $\pm\phi$ inside $(0, k)$, so it is ϕ-periodic. Let $p \in Y_+$ and $p_0 = p \bmod \phi$; it is odd, so $0 < p_0 < \phi$ and $p_0 \in Y_+$. Let $r \ge 0$ be largest with $p_0 + r\phi < \phi'$. By (G1), $p_0 + r\phi \in Y_+$, and by (G4), $m = \phi' - p_0 - r\phi \in Y_-$. Now $0 < m \le \phi$, and m is odd, so $m < \phi$, and (G2) gives $\phi - m \in Y_+$. Since $(\phi - m) + \phi' = p_0 + (r + 1)\phi \equiv p \pmod \phi$, every $p \in Y_+$ has some $p' \in Y_+$ with $p' + \phi' \equiv p \pmod \phi$. Applying this to $k - p$ and reflecting, every $p \in Y_+$ has $p'' = k - p' \in Y_+$ with $p'' \equiv p + \phi' \pmod \phi$. The engine with $n = \phi$, $f = \phi'$ proves Step 1.

Step 2: if $0 < z < k$ is odd and $z + p \equiv 0 \pmod \gamma$ for some $p \in Y_+$, then $z \in Y_-$. Let $z_0 = z \bmod \phi'$, which is odd, so $0 < z_0 < \phi'$. Then $\phi' - z_0 \equiv -z \equiv p \pmod \gamma$, so $\phi' - z_0 \in Y_+$ by Step 1, and (G4) gives $z_0 \in Y_-$. Applying (G3) repeatedly, $z = z_0 + q\phi' \in Y_-$.

Step 3: Y_+ is g-periodic, where $g = \gcd(\gamma, 2k)$. Let $p \in Y_+$, and let $z = (k + p) \bmod \gamma$; it is odd, so $0 < z < \gamma \le \phi$. Since $(k - z) + p \equiv 0 \pmod \gamma$, Step 2 gives $k - z \in Y_-$, hence $z \in Y_-$, and (G2) gives $\phi - z \in Y_+$, hence $k - \phi + z \in Y_+$. As $\gamma \mid \phi$, $k - \phi + z \equiv k + z \equiv p + 2k \pmod \gamma$. The engine with $n = \gamma$, $f = 2k$ proves Step 3.

Step 4: the coset. The number g is even, divides $2k$, and $0 < g \le \phi < k$. Fix $c \in Y_+$ and let w be any residue $\equiv c \pmod g$, represented in $[0, 2k)$; it is odd. If $w < k$ then $w \in Y_+ \subseteq \Phi$ by Step 3. If $w > k$, put $z = w - k$, so that $-z \equiv k - c \pmod g$. Let $z' = (-z) \bmod \gamma$, which is odd with $0 < z' < \gamma < k$. Since $z' \equiv k - c \pmod g$ and $k - c \in Y_+$, Step 3 gives $z' \in Y_+$, and $z + z' \equiv 0 \pmod \gamma$, so $z \in Y_-$ by Step 2; that is, $w \in \Phi$. So $c + g\mathbb{Z}/2k\mathbb{Z} \subseteq \Phi$. □

The coset lemma disposes of every pair of failures except one: the all-0 failure of a read set that has lost its deleted point. By Lemma 8.9(iii) this can only happen at the index $2a$. The following condition on ν, which asks for an element of L and a free position read at the index $2a$, both in the half opposite to a, prevents it there.

Definition 8.12 (the three-point condition)**.** Let $1 \notin S$ and $a = \min S$. A unit ν modulo $2k$ *satisfies the three-point condition* (T) if

(Ta) some $x \in L \setminus \{a\}$ has $H_\nu(x) \neq H_\nu(a)$, and

(Tb) some $j \in S \setminus \{k-a\}$ with $(2a+j) \bmod k \notin S$ has $H_\nu(2a+j) \neq H_\nu(a)$.

Proposition 8.13 (the half-arc carrier under (T))**.** *Let S be all-odd with $1 \notin S$, and let ν be a unit satisfying (T). Then for $t = 0$ or for $t = k$, $\sigma = a$ and $Q(\nu; t)$ form a σ-bump for S.*

Proof. (B1) holds by construction, and the clauses at odd indices hold for both values of t by Lemma 8.7, applied with $u = x$ from (Ta) and $v = a$. It remains to show that for one of the two values of t the clause holds at every even index.

By (Ta), $2a < 2x < k$, so $2a$ is an index, and Φ meets U and V. Suppose that both choices of t fail, at the even indices i_0 (for $t = 0$) and i_1 (for $t = k$).

I claim first that the failure at i_0 gives $C_U(\tau)$ for some even $\tau \notin \{0, k\}$. By Lemma 8.9(i) and (ii) this holds with $\tau \in \{\nu i_0, \nu i_0 + k\}$, unless every surviving tap at i_0 reads 0, some tap is deleted there, and $\alpha + k \in U$. In that case $i_0 = 2a$ by Lemma 8.9(iii). Moreover $\alpha \in V$, that is $H_\nu(a) = 0$. Take j from (Tb). The tap j is not deleted at $2a$, since $2a + j \neq a + k$, and it reads a free position. By Lemma 8.6 its read is $H(\nu(2a+j)) = H_\nu(2a+j) = 1$, a contradiction. In the same way the failure at i_1 gives $C_V(\tau')$ for some even $\tau' \notin \{0, k\}$. Here the exceptional case has $\alpha + k \in V$, so $H_\nu(a) = 1$, and the tap j reads $H(\nu(2a+j)+k) = 1 - H_\nu(2a+j) = 1$.

Lemma 8.11 now puts a coset $c + g\mathbb{Z}/2k\mathbb{Z}$, $0 < g < k$, inside Φ, and Lemma 8.10 says that there is none. □

8.6 The three-point condition

It remains to find a unit satisfying (T). For $k > 90$ there are two routes. Either some $x \in L \setminus \{a\}$ gives (T) through a unit that witnesses the progression $2a - x$, a, x, provided $x < 2a$ or $x - 2a \notin S$ (Lemma 8.15(i)); or L is the chain $a, 3a, \ldots, (2m+1)a$, and Theorem 9.1, applied to the rescaled triple $(k; 1, 2m+3)$, supplies the unit. Triples with $\gcd(a, x, k) > 1$ may have no witness, but these are rescaled copies of the 59 failures at orders up to 90 listed in Lemma 8.16 (see Lemma 8.17), and Lemma 8.18 shows that they cannot block both routes, except for $a = 15$ at $k = 120$, which is settled by hand. The orders $k \leq 90$ are a finite check.

Definition 8.14 (witnesses and admissible triples)**.** Let k be even and let a, x be integers. A unit ν modulo $2k$ *witnesses* the triple $(k; a, x)$ if

$$H_\nu(2a - x) = H_\nu(a) \neq H_\nu(x), \tag{47}$$

and $(k; a, x)$ *has a witness* if some unit does. The triple is *admissible* if $a < x$ are odd positive integers with $2x < k$, $x \neq 3a$ and $\gcd(a, x, k) = 1$. (The triple with $x = 3a$ never has a witness, since then $2a - x = -a$ and $H_\nu(-a) = 1 - H_\nu(a)$ for every odd ν, by (45).)

Note that $2a - x$, a and x form a three-term progression, so a witness separates the progression after its middle term but not before it. The main input of this section is Theorem 9.1, proved in Section 9: *every admissible triple $(k; a, x)$ with $k > 90$ has a witness.*

Lemma 8.15 (the progression route)**.** *Let S be primitive and all-odd with $1 \notin S$.*

(i) Let $x \in L \setminus \{a\}$, and suppose that $x < 2a$ or $x - 2a \notin S$. Then every unit ν witnessing $(k; a, x)$ satisfies (T).

(ii) Suppose that $|L| \geq 2$ and that every $x \in L \setminus \{a\}$ has $x > 2a$ and $x - 2a \in S$. Then $\gcd(a, k) = 1$, and there is $m \geq 1$ with $2(2m+1)a < k$, $(2i+1)a \in S$ for $0 \leq i \leq m$, and $(2m+3)a \notin S$.

Proof. (i) (Ta) holds with this x, since $H_\nu(x) \neq H_\nu(a)$. For (Tb) take $j = k-x \in S$, which differs from $k-a$. If $x < 2a$, then $(2a+j) \bmod k = 2a-x \in (0,a)$, which is not in S by the minimality of a. If $x > 2a$ (equality is excluded by parity), then $2a+j = k-(x-2a) < k$, which lies in S only if $x-2a$ does. Finally, by (44), $H_\nu(2a+j) = H_\nu(2a-x+k) = 1-H_\nu(2a-x) = 1-H_\nu(a)$.

(ii) Each $x \in L \setminus \{a\}$ has $x-2a \in L$, and descending from x by steps of $2a$ stays in L until it reaches a. So every element of L is $\equiv a \pmod{2a}$, in particular divisible by a. Let $f = \gcd(a,k)$, which is odd. It divides every element of L, every $k-x$ with $x \in L$, and $k/2$ (as f is odd and divides k). So f divides $\gcd(S \cup \{k\}) = 1$. Let $(2m+1)a$ be the largest element of L; $m \geq 1$ since $|L| \geq 2$, and the descent shows $(2i+1)a \in S$ for $0 \leq i \leq m$. Suppose $y = (2m+3)a \in S$. Then $2y < k$ contradicts the maximality, $2y = k$ gives $a \mid k$, and $2y > k$ puts $k-y$ in L, so $a \mid k-y$ and again $a \mid k$. But $a \geq 3$ and $\gcd(a,k) = 1$. □

Up to order 90 some admissible triples have no witness, and these failures propagate upwards by scaling.

Lemma 8.16 (small orders)**.** *Let $(n;a,x)$ be admissible with $n \leq 90$. Then $(n;a,x)$ has a witness unless it is one of the* 59 *triples*

n	(a,x)
12	(1,5) (3,5)
16	(1,5) (1,7) (3,7)
20	(1,7) (3,5) (7,9)
22	(1,5) (1,9) (5,7)
24	(1,5) (1,11) (3,7) (5,7) (5,9) (7,9)
28	(1,7) (3,13) (7,9)
30	(1,11) (1,13) (7,13)
36	(1,7) (1,15) (1,17) (5,11) (5,13) (7,17) (11,15)
40	(1,5) (1,15) (3,15) (5,17) (7,19) (9,13) (15,19)
42	(1,5) (1,9) (1,17) (3,19) (5,7) (11,15) (11,19) (13,15)
48	(3,11) (15,23)
60	(1,23) (11,13) (13,27) (19,21)
70	(1,17) (3,25) (9,13) (13,15)
90	(1,33) (17,21) (19,41) (23,39)

Denote this list by $\mathcal{F}$.

Proof. This is a finite check: for every admissible triple with $n \leq 90$ not in $\mathcal{F}$, a search over the units ν modulo $2n$ finds a witness. □

Lemma 8.17 (lifting)**.** *If $(n;a,x)$ has a witness, then so does $(gn;ga,gx)$ for every $g \geq 1$. Consequently, let $k > 90$ and let $(k;a,x)$ satisfy every condition of admissibility except perhaps $\gcd(a,x,k) = 1$. If $(k;a,x)$ has no witness, then, with $g = \gcd(a,x,k)$, the triple $(k/g;a/g,x/g)$ lies in $\mathcal{F}$.*

Proof. Let ν be a unit modulo $2n$ witnessing $(n;a,x)$. Reduction $\mathbb{Z}/2gn \to \mathbb{Z}/2n$ maps units onto units, so there is a unit ν' modulo $2gn$ with $\nu' \equiv \nu \pmod{2n}$. For every integer y, $\nu'\,gy \bmod 2gn = g\,(\nu y \bmod 2n)$, which is less than gn exactly when $\nu y \bmod 2n < n$. So $H_{\nu'}(gy)$ at order gn equals $H_\nu(y)$ at order n, and ν' witnesses $(gn;ga,gx)$.

For the second part, g divides the odd number a, so it is odd; hence k/g is even, and $(k/g;a/g,x/g)$ is admissible. It has no witness, by the first part. If $k/g > 90$ this contradicts Theorem 9.1, so $k/g \leq 90$, and Lemma 8.16 puts the triple in $\mathcal{F}$. □

Above order 90 the lifted failures can still occur, but only in a way that the three-point condition can absorb.

Lemma 8.18 (failures above order 90)**.** *Let $k > 90$ be even, and let S be primitive and all-odd with $1 \notin S$ and $(k, a) \neq (120, 15)$. Suppose that every $x \in L \setminus \{a\}$ for which $(k; a, x)$ has a witness satisfies $x > 2a$ and $x - 2a \in S$; that is, route (i) of Lemma 8.15 is available for no x. Then $(k; a, x)$ has a witness for every $x \in L \setminus \{a, 3a\}$.*

Proof. Suppose that $(k; a, x_0)$ has no witness for some $x_0 \in L \setminus \{a, 3a\}$; I derive a contradiction. Let $\mathcal{M}$ be the set of orders n such that $(n; a', x') \in \mathcal{F}$ for some a', x' with $n \mid k$ and $a = (k/n)a'$, and let $\ell = \operatorname{lcm} \mathcal{M}$. For $y \in L \setminus \{a, 3a\}$ write $g_y = \gcd(a, y, k)$. If $(k; a, y)$ has no witness, then Lemma 8.17 gives $(k/g_y; a/g_y, y/g_y) \in \mathcal{F}$, so $k/g_y \in \mathcal{M}$. In particular $\mathcal{M}$ is nonempty (take $y = x_0$). Every element of $\mathcal{M}$ divides k, so $\ell \mid k$; write $k = \ell f$. For every $n \in \mathcal{M}$, $n \mid \ell$, so f divides k/n; hence $f \mid g_y$ whenever $(k; a, y)$ has no witness. In particular $f \mid g_{x_0} \mid a$, so f is odd.

I claim that f divides every $y \in L$, by induction on y. For $y = a$ this was just shown. For $y \neq a$ there are three cases. If $(k; a, y)$ has a witness, then $y - 2a \in L$ by hypothesis, $f \mid y - 2a$ by induction, and so $f \mid y$. If it has none and $y = 3a$, then $f \mid a \mid y$. If it has none and $y \neq 3a$, then $f \mid g_y \mid y$. Since f is odd and divides k, it also divides $k/2$ and every $k - y$; so f divides $\gcd(S \cup \{k\}) = 1$. Hence $\ell = k$.

It remains to see that $\ell = k$ is impossible. Each $n \in \mathcal{M}$ comes from a failure $(n; a', x') \in \mathcal{F}$ with $a'/n = a/k$, so all the orders in $\mathcal{M}$ carry failures with one and the same ratio a'/n. Inspecting $\mathcal{F}$, a ratio occurs at two different orders only three times: $3/12 = 7/28$, $3/24 = 5/40$ and $1/16 = 3/48$. So either $\mathcal{M}$ is a single order, and $\ell \leq 90$, or ℓ divides $\operatorname{lcm}(12, 28) = 84$, $\operatorname{lcm}(24, 40) = 120$ or $\operatorname{lcm}(16, 48) = 48$. As $\ell = k > 90$, this leaves $k = 120$ with $a/k = 1/8$, that is, $a = 15$, which is excluded. □

Theorem 8.19 (the three-point condition)**.** *Let S be primitive and all-odd, with $1 \notin S$ and $|L| \geq 2$, and not one of the supports listed in Table 4. Then some unit ν modulo $2k$ satisfies (T).*

Proof. Orders $k > 90$. If some $x \in L \setminus \{a\}$ has a witness for $(k; a, x)$ and satisfies $x < 2a$ or $x - 2a \notin S$, Lemma 8.15(i) gives (T). Otherwise, if $(k, a) \neq (120, 15)$, Lemma 8.18 applies, and every $x \in L \setminus \{a, 3a\}$ has a witness, so it satisfies $x > 2a$ and $x - 2a \in S$; so does $x = 3a$, since $a \in S$. So Lemma 8.15(ii) applies. It gives $\gcd(a, k) = 1$ and $m \geq 1$ with $(2i + 1)a \in S$ for $i \leq m$, $2(2m + 1)a < k$ and $(2m + 3)a \notin S$. As $a \geq 3$, $2(2m + 3) < 6(2m + 1) \leq 2(2m + 1)a < k$, so the triple $(k; 1, 2m + 3)$ is admissible, and by Theorem 9.1 some unit ν_1 witnesses it: $H_{\nu_1}(-(2m+1)) = H_{\nu_1}(1) \neq H_{\nu_1}(2m+3)$. By (45), $H_{\nu_1}(2m+1) = 1 - H_{\nu_1}(-(2m+1)) \neq H_{\nu_1}(1)$. Since a is odd and prime to k, it has an inverse $\bar{a}$ modulo $2k$. Put $\nu = \nu_1 \bar{a}$, a unit, so that $\nu\, ya \equiv \nu_1 y \pmod{2k}$ and $H_\nu(ya) = H_{\nu_1}(y)$ for every integer y. Let $j = (2m + 1)a$, which lies in $L \setminus \{a\}$. Then (Ta) holds with $x = j$, since $H_\nu(j) = H_{\nu_1}(2m + 1) \neq H_{\nu_1}(1) = H_\nu(a)$. For (Tb), $j \neq k - a$ and $2a + j = (2m + 3)a \leq 2(2m + 1)a < k$, so $(2a + j) \bmod k = (2m + 3)a \notin S$, and $H_\nu(2a + j) = H_{\nu_1}(2m + 3) \neq H_{\nu_1}(1) = H_\nu(a)$.

The order $k = 120$ with $a = 15$. Suppose again that route (i) of Lemma 8.15 is available for no x, so that every $x \in L \setminus \{a\}$ with a witness has $x > 30$ and $x - 30 \in S$. Let $x \in L \setminus \{15, 45\}$ have no witness. By Lemma 8.17, $(120/g; 15/g, x/g) \in \mathcal{F}$ with $g = \gcd(15, x, 120)$, and $g \neq 1$ since the orders in $\mathcal{F}$ are at most 90. The only entries of $\mathcal{F}$ of the forms $(40; 5, \cdot)$, $(24; 3, \cdot)$ and $(8; 1, \cdot)$ are $(40; 5, 17)$ and $(24; 3, 7)$, so $x \in \{35, 51\}$. Every element of L therefore lies in $\{15, 35, 45, 51\}$, by induction: an $x \in L$ with a witness has $x - 30 \in L$, and $x - 30 \in \{15, 35, 45, 51\}$ with $x < 60$ forces $x = 45$. If $35 \notin L$, then 3 divides every element of L, hence every element of S (as $k/2 = 60$ is even, $S = L \cup (k - L)$), contradicting primitivity; so $35 \in L$, and in the same way, with 5 in place of 3, $51 \in L$. Now the unit $\nu = 11$ satisfies (T), with residues modulo 240. First $H_{11}(15) = 0$, as $11 \cdot 15 = 165$. (Ta) holds with $x = 51$, since $11 \cdot 51 \equiv 81$. For (Tb), if $45 \notin S$ take $j = 15$, so that $2a + j = 45 \notin S$ and $11 \cdot 45 \equiv 15$; if $45 \in S$ take $j = k - 35 = 85$, so that $2a + j = 115 \notin S$ (as $5 < a$) and $11 \cdot 115 \equiv 65$.

Orders $k \leq 90$. These are a finite check; since $1 \notin S$ and $|L| \geq 2$, $k \geq 12$. For each such k it has three steps.

k	S	ν	t	k	S	ν	t
10	$\{3,5,7\}$	3	0	22	$\{9,11,13\}$	7	0
14	$\{5,7,9\}$	5	k	24	$\{5,9,15,19\}$	3	k
16	$\{3,7,9,13\}$	3	0	30	$\{7,13,17,23\}$	3	0
18	$\{7,9,11\}$	5	k	30	$\{7,13,15,17,23\}$	3	0
22	$\{3,9,13,19\}$	11	k	36	$\{5,13,23,31\}$	3	k
22	$\{3,9,11,13,19\}$	3	0	40	$\{3,9,31,37\}$	5	k
22	$\{5,7,15,17\}$	11	0	40	$\{7,19,21,33\}$	3	0
22	$\{5,11,17\}$	5	k	40	$\{9,13,27,31\}$	11	0
				42	$\{5,15,27,37\}$	3	k

Table 4: The supports of Lemma 8.20, with the parameters (ν, t) of a half-arc carrier $Q(\nu;t)$ that, with $\sigma = \min S$, forms a σ-bump. The multiplier ν is an odd residue modulo $2k$, not necessarily a unit. Five have three elements, of the form $\{a, k/2, k-a\}$ with $3\beta \geq k/2$ in the notation of Theorem 8.8, and the other twelve have $|L| \geq 2$.

1. Let F_k be the set of the pairs $(a, 3a)$ with $6a < k$, which have no witness by (45), together with the pairs (ga', gx') with g odd and $(k/g; a', x') \in \mathcal{F}$; for instance $F_{44} = \{(1,3),(3,9),(5,15),(7,21)\}$. For every odd pair $a < x < k/2$ not in F_k, a witness of $(k;a,x)$ is found by a search over the units ν modulo $2k$.

2. By Lemma 8.15(i), S then satisfies (T) unless every $x \in L \setminus \{a\}$ has $(a,x) \in F_k$ or $x - 2a \in L$. The lower halves L with this property are enumerated, for each odd $a \geq 3$ (as $1 \notin S$) with $2a < k$, starting from $L = \{a\}$ and adding the candidates x in increasing order. Each is completed to a support with and without $k/2$ (when $k/2$ is odd), and the supports that are not primitive or have $|L| < 2$ are discarded. This leaves 411 supports.[1]

3. For each of these 411 supports except the twelve with $|L| \geq 2$ in Table 4, a unit ν satisfying (T) is recorded.

□

8.7 Proof of Theorem 8.1

The seventeen supports that escape the arguments above are listed in Table 4, each with a half-arc carrier. The multiplier ν is a residue modulo $2k$, and it need not be a unit. Indeed, for the supports $\{3,9,13,19\}$ and $\{5,7,15,17\}$ at $k = 22$ and $\{7,13,17,23\}$ at $k = 30$ there is no half-arc carrier with a unit multiplier: for every unit ν modulo $2k$ and both $t \in \{0,k\}$, the pair $(\min S, Q(\nu;t))$ fails to be a σ-bump. Several entries of the table use non-units; for these three supports, with $\nu = 11$, 11 and 3, no unit would do.

Lemma 8.20 (seventeen small supports)**.** *For each of the seventeen supports S of Table 4, with the listed ν and t, $\sigma = \min S$ and $Q(\nu;t)$ form a σ-bump.*

Proof. This is a finite check of (B1) and (B2) for seventeen explicit pairs (σ, Q). □

Proof of Theorem 8.1. If $1 \in S$, Proposition 8.4 applies. Assume $1 \notin S$, and let $a = \min S \geq 3$; then $a \in L$, since otherwise $S = \{k/2\}$. If S is listed in Table 4, Lemma 8.20 applies. Otherwise there are two cases.

$|L| \geq 2$. Theorem 8.19 gives a unit ν satisfying (T), and Proposition 8.13 gives the σ-bump $(a, Q(\nu;t))$ with $t \in \{0,k\}$.

[1] Counts such as this one, and the numbers of triples and orders quoted in Section 9 and Appendix A, describe the size of the finite checks. They are not themselves formalised; what is formalised is that every case met by each check passes it (Appendix C).

$|L| = 1$. Then $L = \{a\}$, and since $|S| \geq 3$, $S = \{a, k/2, k-a\}$ with $h = k/2$ odd. Any common divisor of a and h divides a, h, $k-a$ and k, so $\gcd(a,h) = 1$ by primitivity, and $3 \leq a < h$. Let β be as in Theorem 8.8. If $3\beta \geq h$, then $h \leq 11$, and (h,a) is one of $(5,3)$, $(7,5)$, $(9,7)$, $(11,5)$ and $(11,9)$, all with $\beta = 4$ (for $\beta = 2$ the only candidate order, $h = 5$, has no suitable a); so S would be listed in Table 4. Hence $3\beta < h$, and Theorem 8.8 gives the σ-bump. □

9 Three-term progressions under unit multipliers

The all-odd case of Section 8 needs one purely arithmetic fact, about how multiplication by a unit moves a short arithmetic progression around the cycle $\mathbb{Z}/2k$. Cut $\mathbb{Z}/2k$ into the two halves $\{0,\dots,k-1\}$ and $\{k,\dots,2k-1\}$. Given odd $a < x$ with $x < k/2$, so that both lie in the first quarter $\{0,\dots,k/2-1\}$ of the cycle, is there a unit ν that sends a and $2a-x$ to the same half and x to the other? Note that $2a-x$, a and x form a three-term progression with difference $e = x-a$, so the question asks for a unit that separates the progression after its middle term but not before it. Here I show that, when $\gcd(a,x,k) = 1$, the answer is yes at every even order $k > 90$, with one obvious exception.

The proof is by the geometry of numbers. The pairs $(\nu a, \nu e)$, reduced modulo $2k$, are the points of a planar lattice of determinant $2k$, and a lattice point with ν a unit in one of four open triangles yields a witness. The triangles cover a quarter of the square $[0,2k)^2$, which contains $2k$ lattice points, so one expects about $k/2$ lattice points in them; the difficulty is to make this rigorous for every lattice, including very skewed ones whose points lie on a few widely spaced lines, and to find a unit point. Along a lattice line the multipliers form an arithmetic progression, so Jacobsthal's function $J(n)$ bounds how far one must go along a line to reach a unit. A short vector of the lattice then settles the question: either it is long enough that a whole band of lines crosses the incircle of a triangle, or it is one of finitely many short directions, for which chords inside the triangles are tabulated once and for all. Above order 2730, elementary bounds on $J(n)$ and this chord table finish the proof (Section 9.6). At orders $92 \leq k \leq 2730$ with at most two odd prime factors a direct argument applies (Section 9.7), and at the other orders up to 2730 a finite certificate does (Appendix A). On a first reading one can skip the proofs of the lemmas and Appendix A.

9.1 Statement

The functions H and H_ν are those of (43), and witnesses and admissible triples are as in Definition 8.14. For an integer y write $\langle y \rangle \in \{0,\dots,2k-1\}$ for its residue modulo $2k$, so that $H(y) = 1$ exactly when $\langle y \rangle < k$; recall also that $H(y+k) = 1 - H(y)$, by (44).

Theorem 9.1 (three-term progressions)**.** *Every admissible triple $(k; a, x)$ with $k > 90$ has a witness.*

Both restrictions are genuine. If $x = 3a$ then $2a - x = -a$, and since νa is odd it is neither 0 nor k modulo $2k$, so $H_\nu(-a) = 1 - H_\nu(a)$ for every ν. The bound on k cannot simply be dropped either: the triple $(90; 1, 33)$ satisfies every other condition, but it has no witness, as a direct check of the 48 units modulo 180 shows.

It is convenient to work with the pair (a,e), $e = x-a$, and to allow any integer pair. Since $2a - x = a - e$ and $x = a + e$, a unit ν witnesses $(k; A, A+E)$ exactly when

$$H\big(\nu(A-E)\big) = H(\nu A) \neq H\big(\nu(A+E)\big), \tag{48}$$

and I shall say that a pair (A,E) of integers *has a witness* if some unit ν satisfies (48). Since H depends only on residues modulo $2k$, this depends only on A and E modulo $2k$. Theorem 9.1 states that $(a, x-a)$ has a witness.

Throughout the rest of the section $(k; a, x)$ is admissible with $k > 90$, unless stated otherwise, and $e = x - a$. Then

$$a \text{ odd}, \quad e \text{ even}, \quad 0 < e, \quad a + e < k/2, \quad e \neq 2a, \quad \gcd(a, e, 2k) = 1. \tag{49}$$

The last condition holds because a prime dividing a, e and $2k$ is odd (as a is odd), so it divides k, a and $x = a + e$.

9.2 The target region

The condition (48) only involves the residues $\langle \nu A \rangle$ and $\langle \nu E \rangle$, so it describes a region of the square $[0, 2k)^2$. It is enough to use four open triangles inside that region. Let

$$\Delta_1 = \{(X, Y) \in \mathbb{R}^2 : \ k/2 < X < k, \ |Y - k/2| < X - k/2\}, \qquad \Delta_2 = \Delta_1 + (k, 0),$$
$$\Delta_3 = \{(X, Y) \in \mathbb{R}^2 : \ 0 < X < k/2, \ |Y - 3k/2| < k/2 - X\}, \qquad \Delta_4 = \Delta_3 + (k, 0),$$

and $R = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \subset (0, 2k)^2$. The triangle Δ_1 has vertices $(k/2, k/2)$, $(k, 0)$ and (k, k); it is right-angled and isosceles, with legs $k/\sqrt{2}$ and hypotenuse k, so its inradius is

$$\rho = \tfrac{1}{2}\big(k/\sqrt{2} + k/\sqrt{2} - k\big) = \tfrac{1}{2}(\sqrt{2} - 1)\, k, \tag{50}$$

and its incentre is $O = (k - \rho, k/2)$. The other three triangles are congruent to it. Figure 2 shows the four triangles inside the square, together with the lattice of the next subsection.

Lemma 9.2 (the target region)**.** *If* $\gcd(\nu, 2k) = 1$ *and* $(\langle \nu A \rangle, \langle \nu E \rangle) \in R$*, then* ν *satisfies* (48)*, so* (A, E) *has a witness.*

Proof. Write $X = \langle \nu A \rangle$ and $Y = \langle \nu E \rangle$, so that $H(\nu A) = H(X)$ and $H(\nu(A \pm E)) = H(X \pm Y)$. If $(X, Y) \in \Delta_1$ then $k - X < Y < X$, so $0 < X - Y < 2X - k < k$ and $k < X + Y < 2X < 2k$; hence $H(X - Y) = 1 = H(X)$ and $H(X + Y) = 0$. If $(X, Y) \in \Delta_3$ then $k + X < Y < 2k - X$, so $X - Y$ lies in $(2X - 2k, -k)$, which is $(2X, k)$ modulo $2k$, and $X + Y$ lies in $(k + 2X, 2k)$; hence again $H(X - Y) = 1 = H(X)$ and $H(X + Y) = 0$. Finally, replacing X by $X + k$ adds k to each of X and $X \pm Y$, which reverses all three values of H and so preserves the pattern $H(X - Y) = H(X) \neq H(X + Y)$. This covers Δ_2 and Δ_4. □

9.3 The lattice

The pairs $(\nu a, \nu e)$ modulo $2k$ are the points of a lattice. It is simplest to define it by a single congruence:

$$\Lambda = \{w = (w_1, w_2) \in \mathbb{Z}^2 : \ 2k \mid w_1 e - w_2 a\}.$$

Since $\gcd(a, e, 2k) = 1$, fix integers ξ_1, ξ_2, ξ_3 with $\xi_1 a + \xi_2 e + 2k\xi_3 = 1$, and define the *label* of $w \in \mathbb{Z}^2$ as $\lambda(w) = \xi_1 w_1 + \xi_2 w_2$. For $w, w' \in \mathbb{R}^2$ write $w \times w' = w_1 w_2' - w_2 w_1'$ and $w \cdot w'$ for the dot product.

Lemma 9.3 (labels)**.** *(a)* Λ *is a subgroup of* $\mathbb{Z}^2$ *containing* $2k\mathbb{Z}^2$ *and* $(\nu a, \nu e)$ *for every* $\nu \in \mathbb{Z}$*, and every* $w \in \Lambda$ *satisfies* $w \equiv \lambda(w)\,(a, e) \pmod{2k}$*. In particular* $\Lambda = \mathbb{Z}(a, e) + 2k\mathbb{Z}^2$*, and* λ *is additive.*

(b) If $w, w' \in \Lambda$ *then* $2k \mid w \times w'$*.*

(c) If $w \in \Lambda$*,* $\gcd(\lambda(w), 2k) = 1$ *and* $(\langle w_1 \rangle, \langle w_2 \rangle) \in R$*, then* (a, e) *has a witness.*

Proof. (a) The first claims are clear. If $w_1 e - w_2 a = 2kc$ then

$$w_1 - \lambda(w)a = w_1(\xi_2 e + 2k\xi_3) - \xi_2 a w_2 = \xi_2 \cdot 2kc + 2k\xi_3 w_1,$$

and similarly $w_2 - \lambda(w)e = -\xi_1 \cdot 2kc + 2k\xi_3 w_2$. (b) By (a), $w \times w' \equiv \lambda(w)\lambda(w')\,(a, e) \times (a, e) = 0$ modulo $2k$. (c) By (a), $\langle \lambda(w)a \rangle = \langle w_1 \rangle$ and $\langle \lambda(w)e \rangle = \langle w_2 \rangle$, so Lemma 9.2 applies with $\nu = \lambda(w)$. □

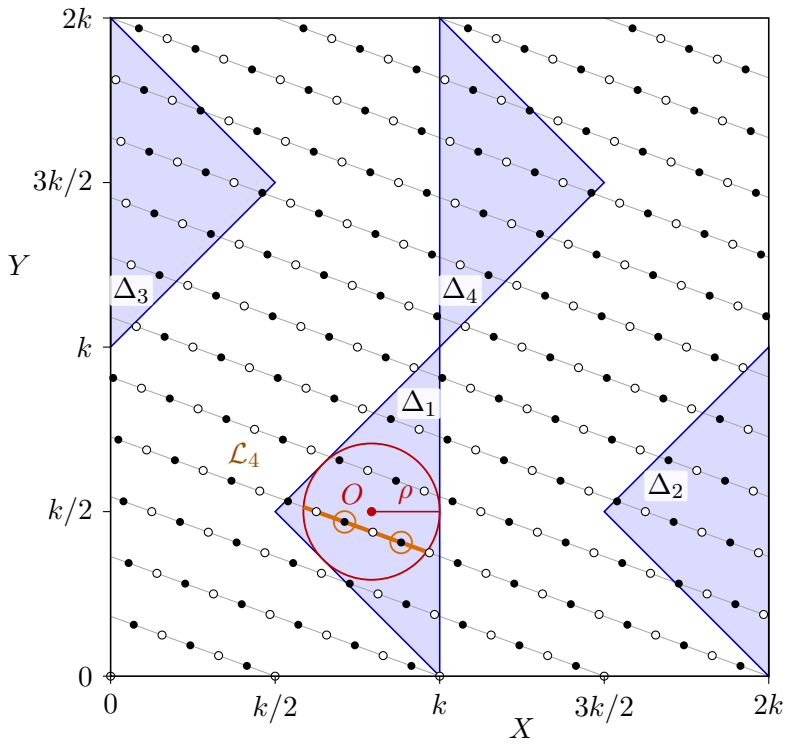


Figure 2: The geometry of Lemma 9.2 and Proposition 9.8, drawn to scale for $k = 128$ and $(a, e) = (9, 20)$. The square is $[0, 2k]^2$ and the shaded triangles are $\Delta_1, \dots, \Delta_4$. The dots are the $2k$ points $(\langle \nu a \rangle, \langle \nu e \rangle)$ of Λ in the square, filled for the unit points (ν odd) and open for the others. A shortest vector is $v = (11, -4)$, of length $l = \sqrt{137}$, and $w = (a, e)$ has $v \times w = 2k$; the grey lines are the lines $\mathcal{L}_0, \dots, \mathcal{L}_{14}$, a distance $2k/l \approx 21.9$ apart. The circle is the incircle of Δ_1, with centre O and radius ρ. The line $\mathcal{L}_4$ passes within $0.06\,k$ of O, and its chord across the disc (thick) contains the points with $\nu = 208, 67, 182, 41$; the two unit points among them (circled) show that $(9, 20)$ has a witness. Here $g = 1$ and $n = 2$, and both hypotheses of Proposition 9.8 hold.

Call $w \in \Lambda$ a *unit point* if $\gcd(\lambda(w), 2k) = 1$. By Lemma 9.3(c), the task is to find a unit point of Λ in $R + 2k\mathbb{Z}^2$. The next lemma collects what a shortest vector of Λ provides; part (b) is Hermite's bound in dimension two.

Lemma 9.4 (a shortest vector)**.** *Let v be a nonzero vector of Λ of least Euclidean length $l = |v|$, let $g = \gcd(v_1, v_2)$ and $u = (p, q) = v/g$. Then:*

(a) there is $w \in \Lambda$ with $v \times w = 2k$;

(b) $3l^4 \le 16k^2$;

(c) $g \mid 2k$, gq is even, and $mu \notin \Lambda$ for $0 < m < g$;

(d) $g \mid \lambda(v)$, and every common divisor of $\lambda(v)$ and $2k$ divides g;

(e) $\gcd(\lambda(w), g) = 1$ for every w as in (a).

Proof. (a) Write $v_1 e - v_2 a = 2k\varphi$ and $c = \gcd(v_1, v_2, \varphi)$. Then $v/c \in \Lambda$ is nonzero, so $c = 1$ by minimality. Take integers μ_1, μ_2, μ_3 with $\mu_1 v_1 + \mu_2 v_2 + \mu_3 \varphi = 1$ and put $w = (-2k\mu_2 + \mu_3 a,\ 2k\mu_1 + \mu_3 e)$. Then $w \in \Lambda$ (it is $\mu_3(a, e)$ plus a vector of $2k\mathbb{Z}^2$), and $v \times w = 2k(\mu_1 v_1 + \mu_2 v_2) + \mu_3(v_1 e - v_2 a) = 2k$.

(b) Replacing w by $w + tv$ with $t \in \mathbb{Z}$ does not change $v \times w$, so I may assume $|v \cdot w| \le l^2/2$. Since $w \ne 0$, $|w| \ge l$. Lagrange's identity $(v \times w)^2 + (v \cdot w)^2 = |v|^2|w|^2$ gives $4k^2 \ge l^4 - l^4/4$.

(c) Let $g' = \gcd(2k, g) = \zeta_1 2k + \zeta_2 g$. Then $g'u = \zeta_1(2ku) + \zeta_2 v \in \Lambda$ is nonzero, so $g'|u| \ge l = g|u|$, and $g' = g$; thus $g \mid 2k$. If $0 < m < g$ then mu is nonzero and shorter than v, so $mu \notin \Lambda$. Finally $v_2 a \equiv v_1 e \pmod{2k}$ with e and $2k$ even, so $v_2 a$ is even, and since a is odd, $v_2 = gq$ is even.

(d) $\lambda(v) = g(\xi_1 p + \xi_2 q)$. If c divides $\lambda(v)$ and $2k$, then by Lemma 9.3(a) it divides v_1 and v_2, hence g.

(e) For every $z \in \mathbb{R}^2$, $(v \times w)\, z = (z \times w)\, v + (v \times z)\, w$ (both sides are linear in z and agree at $z = v$ and $z = w$). Take $z = (a, e) \in \Lambda$: by Lemma 9.3(b) both $z \times w$ and $v \times z$ are multiples of $2k = v \times w$, so $(a, e) = c_1 v + c_2 w$ with $c_1, c_2 \in \mathbb{Z}$. Applying the additive map λ, $1 - 2k\xi_3 = \xi_1 a + \xi_2 e = c_1\lambda(v) + c_2\lambda(w)$. Since g divides $2k$ and $\lambda(v)$, it is prime to $\lambda(w)$. □

Replacing (v, w) by $(-v, -w)$ changes none of these properties, so I may and do assume that u is *normalised*: $p > 0$, or $p = 0 < q$. Eight short directions never occur.

Lemma 9.5 (eight excluded directions)**.** *With $v = gu$ as in Lemma 9.4 and $u = (p, q)$ normalised, $(g; p, q)$ is none of*

$$(1;1,0),\ (1;1,2),\ (2;0,1),\ (2;1,0),\ (2;1,1),\ (2;1,-1),\ (2;1,-2),\ (4;0,1).$$

Proof. Each case says that $2k$ divides $g(pe - qa)$, and each contradicts (49). For $(1;1,0)$: $2k \mid e$, but $0 < e < k$. For $(1;1,2)$: $2k \mid e - 2a$ with $|e - 2a| < k$, so $e = 2a$. For $(2;0,1)$: $k \mid a$, but $0 < a < k$. For $(2;1,0)$: $k \mid e$. For $(2;1,\pm1)$: $k \mid e \mp a$, which is odd while k is even. For $(2;1,-2)$: $k \mid e + 2a$, but $0 < e + 2a = x + a < k$. For $(4;0,1)$: $k/2 \mid a$, but $0 < a < k/2$. □

9.4 Lattice lines and Jacobsthal's function

For $n \ge 1$, *Jacobsthal's function* $J(n)$ is the least m such that every m consecutive integers include one prime to n [Jacobsthal, 1960a,b,c]. It depends only on the set of primes dividing n, and $J(n') \le J(n)$ whenever every prime factor of n' divides n. Deep asymptotic bounds are known, such as $J(n) \ll \omega(n)^2 \log^2 \omega(n)$, with $\omega(n)$ the number of distinct prime factors of n [Iwaniec, 1978], but the elementary bounds proved below suffice here.

Fix $v = gu$ and w as in Lemma 9.4, and write $\lambda_v = \lambda(v)$ and $\lambda_w = \lambda(w)$. For $j \in \mathbb{Z}$ let

$$\mathcal{L}_j = \{jw + tv : \ t \in \mathbb{R}\}$$

(see Figure 2). Since $v \times (jw + tv) = 2kj$, the line $\mathcal{L}_j$ lies at signed distance $2kj/l$ from $\mathcal{L}_0$; consecutive lines are $2k/l$ apart. The points $jw + iv$ with $i \in \mathbb{Z}$ lie in Λ, a distance l apart, and their labels $j\lambda_w + i\lambda_v$ form an arithmetic progression.

Lemma 9.6 (unit points on a line)**.** *Let $n \geq 1$ divide $2k$, with $\gcd(n, g) = 1$ and every prime factor of $2k$ dividing n or g (for instance, n the product of the primes of $2k$ that do not divide g). If $\gcd(j, g) = 1$, then among any $J(n)$ consecutive points $jw + iv$ of $\mathcal{L}_j$ there is a unit point.*

Proof. A common divisor of λ_v and n divides $2k$, so it divides g by Lemma 9.4(d), and it is 1 since $\gcd(n, g) = 1$. Choose ι with $\iota\lambda_v \equiv 1 \pmod{n}$. Then $j\lambda_w + i\lambda_v \equiv \lambda_v(i + \iota j\lambda_w) \pmod{n}$, and as i runs through $J(n)$ consecutive integers so does $i + \iota j\lambda_w$; so for one of them $j\lambda_w + i\lambda_v$ is prime to n. A prime ℓ dividing g divides λ_v (Lemma 9.4(d)) but neither j nor λ_w (Lemma 9.4(e)), so it does not divide $j\lambda_w + i\lambda_v$. Every prime of $2k$ divides n or g, so this label is prime to $2k$. □

Lemma 9.7 (chords)**.** *Let n be as in Lemma 9.6, $\gcd(j, g) = 1$, Δ one of $\Delta_1, \dots, \Delta_4$ and $z \in \mathbb{Z}^2$. If $t_0 < t_1$, $t_1 - t_0 > J(n)$ and $jw + tv \in \Delta + 2kz$ for all $t \in (t_0, t_1)$, then (a, e) has a witness.*

Proof. The integers $i_0, \dots, i_0 + J(n) - 1$ with $i_0 = \lfloor t_0 \rfloor + 1$ lie in $(t_0, t_0 + J(n)] \subset (t_0, t_1)$. By Lemma 9.6 one of the points $jw + iv$ with such i is a unit point. It lies in $\Delta + 2kz$, and since $\Delta \subset (0, 2k)^2$ its residues modulo $2k$ lie in $\Delta \subset R$. Now apply Lemma 9.3(c). □

9.5 Two ways to find a chord

The first way uses a band of lines through the incircle of Δ_1, and works when the shortest vector is long.

Proposition 9.8 (the band)**.** *With $v = gu$ as in Lemma 9.4 and n as in Lemma 9.6, suppose that*

$$J(g) < 0.102\, l \qquad \text{and} \qquad J(n)\, l < 0.3604\, k.$$

Then (a, e) has a witness.

Proof. Let D be the open disc of radius ρ about the incentre O of Δ_1; it lies in Δ_1. Put $\tau = (v \times O)/2k$, so that O is at signed distance $2k\tau/l$ from $\mathcal{L}_0$. The $J(g)$ consecutive integers $\lfloor \tau - J(g)/2 \rfloor + 1, \dots, \lfloor \tau - J(g)/2 \rfloor + J(g)$ include one, j, prime to g, and every one of them satisfies $|j - \tau| \leq J(g)/2$. So the distance from O to $\mathcal{L}_j$ is at most $(2k/l)\, J(g)/2 = J(g)k/l < 0.102\, k$. The line $\mathcal{L}_j$ therefore meets D in an open segment of length greater than $2\sqrt{\rho^2 - (0.102\, k)^2}$. By (50), $\rho^2 = \frac{1}{4}(3 - 2\sqrt{2})k^2 > 0.04289\, k^2$, and $(0.102\, k)^2 = 0.010404\, k^2$, so the segment is longer than $2\sqrt{0.03248}\, k > 0.3604\, k > J(n)\, l$. In terms of the parameter t of $\mathcal{L}_j$ (in which the lattice points sit at the integers) the segment has length greater than $J(n)$, and it lies in Δ_1. Lemma 9.7 completes the proof. □

The second way handles the short directions. Its key feature is that the geometry does not depend on k, a or e once everything is scaled by $2k$. Scale the triangles: let $\widehat{\Delta}_i = \Delta_i/2k$, so that

$$\widehat{\Delta}_1 = \{(X, Y) : \ \tfrac{1}{4} < X < \tfrac{1}{2},\ |Y - \tfrac{1}{4}| < X - \tfrac{1}{4}\}, \qquad \widehat{\Delta}_2 = \widehat{\Delta}_1 + (\tfrac{1}{2}, 0),$$
$$\widehat{\Delta}_3 = \{(X, Y) : \ 0 < X < \tfrac{1}{4},\ |Y - \tfrac{3}{4}| < \tfrac{1}{4} - X\}, \qquad \widehat{\Delta}_4 = \widehat{\Delta}_3 + (\tfrac{1}{2}, 0).$$

For a primitive $u = (p, q)$, an integer $g \geq 1$ and $j \in \mathbb{Z}$, consider the line

$$\widehat{\mathcal{L}}_j(g, u) = \Big\{ \frac{j}{g|u|^2}\,(-q, p) + s\,u : \ s \in \mathbb{R} \Big\}.$$

If $v = gu$ and w are as in Lemma 9.4, then $(-q, p) \cdot w = u \times w = 2k/g$, and decomposing w along u and $(-q, p)$ gives

$$\frac{jw + tv}{2k} = \frac{j}{g|u|^2}\,(-q, p) + \Big(s_* + \frac{g}{2k}\,t\Big)u, \qquad s_* = \frac{j\,(u \cdot w)}{2k|u|^2}. \tag{51}$$

So $\mathcal{L}_j/2k = \widehat{\mathcal{L}}_j(g, u)$, and a change of s by ψ is a change of t by $2k\psi/g$.

Definition 9.9. A *chord witness* for $(g; p, q)$ consists of an integer j with $\gcd(j, g) = 1$, an index $i \in \{1, 2, 3, 4\}$, a vector $z \in \mathbb{Z}^2$ and rationals $s_0 < s_1$ such that the point of $\widehat{\mathcal{L}}_j(g, u)$ with parameter s lies in $\widehat{\Delta}_i + z$ for every $s \in (s_0, s_1)$. Its *length* is $\psi = s_1 - s_0$.

A chord witness is checked by evaluating finitely many signs. Each $\widehat{\Delta}_i + z$ is the intersection of three open half-planes $\{f > 0\}$ with f affine. If, for each of the three functionals, $f \geq 0$ at the two points with parameters s_0 and s_1 and $f > 0$ at one of them, then $f > 0$ on the open segment between them, because along the segment f is a combination of its two end values with positive weights. These are nine rational sign conditions.

Proposition 9.10 (a tabulated chord)**.** *With $v = gu$ as in Lemma 9.4 and n as in Lemma 9.6, suppose $(g; p, q)$ has a chord witness of length ψ with $2k\psi > J(n)\,g$. Then (a, e) has a witness.*

Proof. By (51), $jw + tv \in \Delta_i + 2kz$ for all t in an open interval of length $2k\psi/g > J(n)$. Lemma 9.7 applies. □

When neither proposition applies directly, the lattice can be pinned down completely. Since $gu \in \Lambda$, the pair (a, e) is, up to a unit, one of at most g^2 explicit pairs.

Lemma 9.11 (lattice classes)**.** *Let $v = gu$ be as in Lemma 9.4 and $n_0 = 2k/g$. There are a unit η modulo $2k$ and integers $0 \leq s, t < g$ (unrelated to the line parameters above) such that, with $(A, E) = (p + n_0 s,\ q + n_0 t)$,*

$$\eta\,(a, e) \equiv (A, E) \pmod{2k}.$$

Moreover $\gcd(A, E, 2k) = 1$, and $2k \nmid m(pE - qA)$ for $0 < m < g$. In particular, if (A, E) has a witness then so does (a, e).

Proof. By Lemma 9.4(d), $\lambda_v = \lambda(v) = g\lambda^*$, and $\gcd(\lambda^*, n_0) = 1$: if c divides both then gc divides $2k$ and λ_v, hence g, so $c = 1$. By Lemma 9.3(a), $g(p, q) = v \equiv g\lambda^*(a, e) \pmod{2k}$, so $\lambda^*(a, e) \equiv (p, q) \pmod{n_0}$. Every unit modulo n_0 lifts to a unit modulo $2k$: take $\eta = \lambda^* + n_0 r$, with r the product of the primes of $2k$ that divide neither n_0 nor λ^*. Indeed, let ℓ be a prime dividing $2k$. If $\ell \mid n_0$, then $\eta \equiv \lambda^* \not\equiv 0 \pmod{\ell}$, as $\gcd(\lambda^*, n_0) = 1$. If $\ell \nmid n_0$ and $\ell \mid \lambda^*$, then $\ell \nmid r$, and $\eta \equiv n_0 r \not\equiv 0$. If ℓ divides neither, then $\ell \mid r$, and $\eta \equiv \lambda^* \not\equiv 0$. Then $\eta a = p + n_0 K_1$ and $\eta e = q + n_0 K_2$ for some integers K_1, K_2, and with s, t the residues of K_1, K_2 modulo g one gets $\eta(a, e) \equiv (A, E) \pmod{n_0 g}$, as claimed.

If c divides A, E and $2k$, it divides ηa and ηe, hence $\eta(\xi_1 a + \xi_2 e + 2k\xi_3) = \eta$, and $c = 1$. If $2k \mid m(pE - qA)$, then $2k \mid m\eta(pe - qa)$, so $2k \mid m(pe - qa)$, that is $mu \in \Lambda$; this is impossible for $0 < m < g$ by Lemma 9.4(c). For the last statement, if a unit ν satisfies (48) for (A, E), then $\nu\eta$ is a unit, and it satisfies (48) for (a, e) because $\nu\eta(a, e) \equiv \nu(A, E) \pmod{2k}$. □

The two conditions on (A, E) in Lemma 9.11 are used in reverse: a class (s, t) for which $\gcd(A, E, 2k) > 1$, or for which $2k \mid m(pE - qA)$ for some $0 < m < g$, cannot arise from an admissible (a, e) with shortest vector gu, and needs no witness.

9.6 The chord table and orders above 2730

The two propositions of the last subsection settle every large order. The band (Proposition 9.8) needs a long shortest vector, and the short ones are covered once and for all by a table of chords.

Let $\mathcal{V}$ be the set of triples $(g;p,q)$ with $g \geq 1$, (p,q) primitive and normalised, gq even, $g^2(p^2+q^2) \leq 6400$, and $(g;p,q)$ not one of the eight triples of Lemma 9.5. By Lemma 9.4(c) and Lemma 9.5, a normalised shortest vector $v = gu$ of Λ with $l \leq 80$ gives a triple $(g;p,q) \in \mathcal{V}$ with $g \mid 2k$. The following finite computation is the first part of the certificate for Theorem 9.1; the other two parts, (C2) and (C3), are in Appendix A.

(C1) *The chord table.* For each of the 5012 triples of $\mathcal{V}$ a chord witness is given and its nine sign conditions are verified (after clearing denominators). That the table lists exactly the elements of $\mathcal{V}$ is checked by enumerating all $g \leq 80$ and all (p,q) with $0 \leq p \leq 80/g$ and $|q| \leq 80/g$. Only lower bounds on chord lengths are used, so the witnesses need not be the longest chords. Every entry has

$$g \leq 339.8\,\psi, \tag{52}$$

where ψ is the length of its chord witness; the extreme entry is $(76;1,0)$, with $\psi = 17/76$ and $g/\psi = 339.76\ldots$.

Say that an even order k satisfies the *tail condition* with bound m if

$$17.8\,m^2 \leq k \qquad \text{and} \qquad 169.9\,m < k, \tag{53}$$

that is, $178\,m^2 \leq 10k$ and $1699\,m < 10k$ in integers. The first inequality is what the band (Proposition 9.8) needs, $J(n)\,l < 0.3604\,k$, after Hermite's bound $l^2 \leq 4k/\sqrt{3}$; the second is what a tabulated chord (Proposition 9.10) needs, $2k\psi > J(n)\,g$, after (52). The bounds on Jacobsthal's function used below are proved in Section A.2 (Corollary A.5).

Proposition 9.12. *Let k be even and satisfy the tail condition (53) with bound $m = J(2k)$. Then Theorem 9.1 holds at the order k. The tail condition with $m = J(2k)$ holds for every even $k > 2730$.*

Proof. For the last statement, write $J = J(2k)$. By Corollary A.5(a), $17.8\,J^2 \leq k$. If $J \geq 10$ this gives $169.9\,J < 178\,J \leq 17.8\,J^2 \leq k$, and if $J \leq 9$ then $169.9\,J < 1530 < k$.

Now let $v = gu$ be a normalised shortest vector of Λ, and n the product of the primes of $2k$ not dividing g, so that n satisfies the conditions of Lemma 9.6 and $J(n) \leq J$. By hypothesis, $17.8\,J^2 \leq k$ and $k > 169.9\,J$.

Short vectors, $l^2 \leq 6400$. Then $(g;p,q) \in \mathcal{V}$ by Lemma 9.4(c) and Lemma 9.5, and (52) gives $2k\psi \geq kg/169.9 > Jg \geq J(n)g$, so Proposition 9.10 applies.

Long vectors, $l > 80$. By Corollary A.5(b), either $J(g) \leq 8 < 0.102 \cdot 80 < 0.102\,l$, or $J(g) < 0.102\,g \leq 0.102\,l$. By Lemma 9.4(b), $l^2 \leq 4k/\sqrt{3}$, so

$$J(n)^2 l^2 \leq \frac{k}{17.8} \cdot \frac{4k}{\sqrt{3}} < 0.12975\,k^2,$$

and $J(n)\,l < 0.36021\,k$. So Proposition 9.8 applies. □

9.7 Orders with at most two odd primes

When k has at most two odd prime factors, Theorem 9.1 has a direct proof, with no lattice. It covers 1193 of the 1320 orders $92 \leq k \leq 2730$, so that the finite certificate is needed only at the 127 orders with three or more odd prime factors. The idea is that one term of the progression is then a unit; normalising it to 1 leaves a one-parameter problem, which explicitly located multipliers solve. Throughout, if c is a unit and ν satisfies (48) for the pair (cA, cE), then νc satisfies it for (A,E); so (A,E) has a witness as soon as (cA,cE) does.

case	chain $m_1^{i_1}\, m_2^{i_2} \cdots$	K
$3, 5 \nmid k$	$9^0\, 5^0\, 3^0$	92
$3 \mid k$; $5, 7, 11, 13, 19 \nmid k$	$11^0\, 7^0\, 5^0\, 19^3\, 13^2\, 7^1$	96
$5 \mid k$; $3, 7, 11 \nmid k$	$9^0\, 7^0\, 27^3\, 11^1\, 3^0$	100
$15 \mid k$	$11^0\, 7^0\, 19^2\, 11^1\, 49^8\, 43^7\, 31^5\, 19^3\, 13^2\, 7^1$	120
$21 \mid k$	$11^0\, 19^1\, 17^1\, 5^0\, 19^3\, 23^4\, 17^3\, 11^2$	126
$33 \mid k$	$13^0\, 25^1\, 7^0\, 5^0\, 13^2\, 7^1$	132
$39 \mid k$	$11^0\, 19^1\, 17^1\, 5^0\, 19^3\, 23^4\, 17^3\, 11^2$	156
$57 \mid k$	$11^0\, 7^0\, 5^0\, 29^5\, 13^2\, 7^1$	114
$35 \mid k$	$9^0\, 17^1\, 23^2\, 13^1\, 11^1\, 3^0$	140
$55 \mid k$	$9^0\, 7^0\, 19^2\, 17^2\, 3^0$	110

Table 5: The chains used in the proof of Proposition 9.14, for even $k \geq 92$ with at most two odd prime factors; m^i stands for the pair (m, i). In each case K is the least such order, so $k \geq K$, and every multiplier is a unit.

Lemma 9.13 (two normal forms)**.** *Let k be even.*

(a) If m is a unit with $0 < m < k$ and $k + m < \langle mD \rangle < 2k - m$, then m satisfies (48) for the pair $(1, D)$.

(b) If ν is a unit with $0 < \nu < k$, $\langle \nu A \rangle = r < k$ and $\langle 2r - \nu \rangle \geq k$, then ν satisfies (48) for the pair $(A, A - 1)$.

Proof. (a) Write $Y = \langle mD \rangle$. Then $H(m) = 1$. Next, $m(1 + D) \equiv m + Y$ with $k + 2m < m + Y < 2k$, so $H(m(1 + D)) = 0$; and $m(1 - D) \equiv m - Y + 2k$ with $2m < m - Y + 2k < k$, so $H(m(1-D)) = 1$. (b) Here $H(\nu) = 1 = H(\nu A)$, and $\nu(2A-1) \equiv 2r-\nu$, so $H(\nu(2A-1)) = 0$. □

The multipliers for a unit middle term come in *chains*. A chain is a list $(m_1, i_1), \ldots, (m_N, i_N)$ of odd multipliers m_j and integers $i_j \geq 0$, with a threshold K, such that

$$\frac{2i_1 + 1}{m_1} + \frac{1}{K} \leq \frac{1}{8}, \qquad \frac{2i_{j+1} + 1}{m_{j+1}} + \frac{1}{K} < \frac{2i_j + 2}{m_j} - \frac{1}{K}, \qquad \frac{2i_N + 2}{m_N} - \frac{1}{K} > \frac{1}{2}, \tag{54}$$

the middle condition for $j = 1, \ldots, N - 1$. The pair (m_j, i_j) serves when D/k lies roughly in the interval $\big((2i_j + 1)/m_j,\ (2i_j + 2)/m_j\big)$, where $m_j D$ falls in the window $\big((2i_j + 1)k,\ (2i_j + 2)k\big)$; the three conditions say that, with a margin $1/K$, these intervals overlap and together cover $[1/8, 1/2]$. Table 5 lists ten chains, one for each pattern of small primes dividing k; their conditions (54) are checked in exact rational arithmetic.

Proposition 9.14 (two odd primes)**.** *Let $k \geq 92$ be even, with at most two odd prime factors. Then every admissible triple $(k; a, x)$ has a witness.*

Proof. The units modulo $2k$ are the odd integers prime to the odd primes of k, of which there are at most two. So if three odd integers are such that no prime factor of k divides two of them, one of them is a unit. Write $e = x - a$, so that (49) holds. An odd prime of k dividing two of $a - e$, a, $a + e$ divides a and e, which (49) excludes, so one of these three is a unit.

The case where a is a unit. Let $\bar{a}$ be its inverse modulo $2k$ and $D = \langle \bar{a}e \rangle$, which is even. It suffices to find a witness for $(1, D)$. Since $aD \equiv e$, the values $D = 0, 2, k - 2, k, k + 2, 2k - 2$ would give $e \equiv 0, 2a, k - 2a, k, k + 2a, -2a \pmod{2k}$, which (49) excludes (as $0 < e$, $e \neq 2a$ and $e + 2a < k$). By Lemma 9.13(a) it suffices to find a unit m with $0 < m < k$ and

$$(2i + 1)k + m < mD < (2i + 2)k - m \qquad \text{for some integer } i. \tag{55}$$

If $k < D < 2k$, then $m = 1$ works, with $i = 0$. For odd m, if m satisfies (55) for $k - D$ with some i, then it satisfies it for D with i replaced by $(m - 3)/2 - i$, since $mD = mk - m(k - D)$. So it may be assumed that $4 \leq D \leq k/2$.

If $8D \le k$, every m in the interval $I = \big(k/(D-1),\, 2k/(D+1)\big)$ has $m(D-1) > k$ and $m(D+1) < 2k$, which is (55) with $i = 0$, and $m < 2k/5$. The length of I is $k(D-3)/(D^2-1)$, which exceeds 6: for $D \le 11$ because $k \ge 92$, and for $D \ge 12$ because $k \ge 8D$ and $8D(D-3) - 6(D^2-1) = 2D(D-12) + 6 > 0$. So I contains three consecutive odd integers. Their pairwise differences are 2 and 4, so no odd prime divides two of them, and one of them is a unit.

If $8D > k$, take the chain of Table 5 for the case of k; the cases are exhaustive, and in each the multipliers are units, because k has at most two odd primes. The first multiplier satisfies the left inequality of (55), as $m_1 D > m_1 k/8 \ge (2i_1+1)k + m_1 k/K \ge (2i_1+1)k + m_1$, since $k \ge K$. If m_j satisfies the left inequality but not the right one, then $D/k \ge (2i_j+2)/m_j - 1/k$, and (54) gives the left inequality for m_{j+1}. The right inequality cannot fail for m_N, as $D/k \le 1/2$. So some m_j satisfies (55).

The case where $a - e$ is a unit. Let $y = 2a - x = a - e$, $\bar{y}$ its inverse and $A = \langle \bar{y}a \rangle$. Then $\bar{y}e \equiv A - 1$, so it suffices to find a witness for $(A, A-1)$. It may be assumed that a is not a unit. Let $f = \gcd(A, 2k) = \gcd(a, k)$. It is odd and $1 < f \le a < k/2$, so k/f is even and at least 4. Write $A = fA_0$ and $M = 2k/f$, so that A_0 is prime to M, and let $\nu_0 \in (0, M)$ be its inverse modulo M. Every $\nu \equiv c\nu_0 \pmod{M}$ has $\nu A \equiv fc \pmod{2k}$. The *lifts* $\nu \equiv \nu_0 \pmod{M}$ are odd and prime to M, and by Lemma 9.13(b) with $r = f$, every lift that is a unit and lies in $(2f, k)$ is a witness, since $\langle 2f - \nu \rangle = 2k + 2f - \nu \ge k$. A prime of k dividing a lift does not divide M, so it divides f.

If $f \ge 15$, then $2f + 3M \le k$, so the interval $(2f, k)$ contains three lifts ν, $\nu + M$ and $\nu + 2M$; their pairwise differences divide $2M$, to which ν is prime, so one of them is a unit. If $5 \le f \le 13$, then f is a power of a single prime ℓ, and $2f + 2M \le k$ (using $k \ge 92$ when $f \in \{5, 7\}$), so $(2f, k)$ contains two lifts ν and $\nu + M$; a lift that is not a unit is divisible by ℓ, and ℓ cannot divide both, as ν is prime to M. If $f = 3$ and $9 \mid k$, then $3 \mid M$, so every lift is prime to 3 and a unit, and $2f + M \le k$ gives one in $(6, k)$.

There remains $f = 3$ with $3 \nmid M$, so that $k = 3q$ with q even, $3 \nmid q$ and $q \ge 32$, and $M = 2q$. A number prime to M is then a unit when it is prime to 3, and ν_0 is odd. If $6 < \nu_0 < q$, the lifts ν_0 and $\nu_0 + M$ both lie in $(6, k)$, and one is prime to 3. If $\nu_0 > q$ and $3 \nmid \nu_0$, take $\nu = \nu_0$. If $\nu_0 > q$ and $3 \mid \nu_0$, take $c = q - 3$ and ν the residue of $c\nu_0$ modulo M, so that $\nu A \equiv 3c = k - 9$. Here ν is prime to M, since a common prime factor of $q - 3$ and q divides 3. As ν_0 is odd, $q\nu_0 \equiv q \pmod{M}$, so $\nu \equiv q - 3\nu_0$, and as $q < \nu_0 < 2q$, ν is $5q - 3\nu_0$ or $7q - 3\nu_0$, which is prime to 3. Then $\langle 2(k-9) - \nu \rangle = 2k - 18 - \nu \ge k$, since $\nu < M \le k - 18$. Finally, if $\nu_0 < 6$, the lift $\nu_0 + M$ lies in $(6, k)$, and is a witness unless it is divisible by 3. In that case $\nu_0 \in \{1, 5\}$ and $q \equiv \nu_0 \pmod{3}$. Let $s = 5$ if $5 \nmid k$; if $5 \mid k$, then $5 \mid M$, so $\nu_0 = 1$, and let $s = 11$, which does not divide k, as 3 and 5 do. Take $c = q - s$ and $\nu = q - s\nu_0 \equiv c\nu_0 \pmod{M}$, so that $\nu A \equiv 3c = k - 3s$. A common prime factor of ν and q divides $s\nu_0$, and none does; $\nu \equiv \nu_0(1 - s) \not\equiv 0 \pmod{3}$; and $2k - 6s - \nu \ge k$, since $2q \ge 64 \ge s(6 - \nu_0)$. In each case Lemma 9.13(b) gives the witness.

The case where $a + e$ is a unit. Suppose finally that neither a nor $a - e$ is a unit, so that $y' = a + e = x$ is. With $\bar{y}'$ its inverse and $A' = \langle \bar{y}'a \rangle$, $\bar{y}'e \equiv 1 - A'$, so it suffices to find a witness for $(A', 1 - A')$, whose progression is $2A' - 1$, A', 1. Let $A = A' + k$. For a unit ν, which is odd, $H(\nu A) = 1 - H(\nu A')$ and $\nu(2A - 1) \equiv \nu(2A' - 1)$, so ν satisfies (48) for $(A, A-1)$ exactly when it does for $(A', 1 - A')$. Since A' is odd and k even, $\gcd(A, 2k) = \gcd(A', 2k) = \gcd(a, k)$, and the previous case, applied to A, gives the witness. □

Proof of Theorem 9.1. If k has at most two odd prime factors, Proposition 9.14 applies. Otherwise, if $k > 2730$, Proposition 9.12 applies, and if $k \le 2730$, k is one of the 127 orders of the finite certificate of Proposition A.1 in Appendix A. □

For $92 \le k \le 2730$ the theorem can also be confirmed directly, independently of the certificate: a direct search over the units ν modulo $2k$ finds a witness for each of the 100 435 222 admissible triples $(k; a, x)$ in this range.

10 Proof of the main theorem

By Corollary 2.9 it is enough to show the following: if an equation (8) is globally periodic, its support S is primitive and its coefficients satisfy $c_j = c_{k-j}$, then its normalised data is of known type (Definition 2.6). So let such an equation be given. Since $c_j = c_{k-j}$, its support is palindromic.

$|S| \le 2$. By the case analysis at the start of Section 4, either $S = \emptyset$, which is case (K1); or $k = 2$ and $S = \{1\}$, where Theorem 4.1 gives case (K2); or $k = 3$ and $S = \{1, 2\}$, where Theorem 4.2 gives case (K3); or Theorem 4.3 shows that the equation is not globally periodic, a contradiction.

$|S| \ge 3$. I show that no such equation is globally periodic, which leaves only the three cases above. If S is the full support $\{1, \dots, k-1\}$, then $k \ge 4$ and Theorem 5.1 applies. So suppose S is not the full support. If k is odd, Theorem 6.1 applies. If $k = 6$ and $S = \{1, 3, 5\}$, Proposition 3.10 applies. Otherwise k is even and $(k, S) \ne (6, \{1, 3, 5\})$, and S carries a σ-bump: by Theorem 7.1 if S has an even element, and by Theorem 8.1 if every element of S is odd. In either case Theorem 3.14 shows that no equation with support S is globally periodic.

So every globally periodic equation (8) with primitive support and palindromic coefficients has normalised data of known type. By Corollary 2.9, a globally periodic equation (3) is therefore the shift or has the coefficients listed in Lemma 2.7, and these are exactly the equations listed in Theorem 1.1. This completes the proof of Theorem 1.1. □

The argument for supports with three or more elements never uses the coefficients, and it therefore also proves a statement about the min-plus recursion alone, extending the result of Csörnyei and Laczkovich [2001] for the full support (Section 5).

Corollary 10.1 (the min-plus map is not periodic)**.** *Let $S \subseteq \{1, \dots, k-1\}$ be palindromic with $|S| \ge 3$, and let $\theta \in \{0, 1\}$. Then no power of the min-plus map T of Definition 3.1 is the identity: for every $p \ge 1$ there is an integer window $w \in \mathbb{Z}^k$ with $T^p(w) \ne w$.*

Proof. Suppose first that S is primitive. The proof of Theorem 1.1 above provides an integer solution e of the min-plus recursion (15) for (k, S, θ) that carries an unbounded integer tangent solution d: for the full support by Lemma 5.2 and Lemma 5.3; at odd order by the construction in the proof of Proposition 6.4, with the solutions of the reduced recursion given in the proof of Theorem 6.11, or by a σ-bump; for $\{1, 3, 5\}$ at order 6 by the construction in the proof of Proposition 3.9; and otherwise by a σ-bump and Lemma 3.13. None of these constructions involves the coefficients c_j, so Theorem 3.6(ii) applies.

If S is not primitive, let $\ell = \gcd(S \cup \{k\})$, and let S' and k' be as in Lemma 2.8; then S' is primitive and palindromic, with $|S'| = |S| \ge 3$. As in the proof of Lemma 2.8(ii), if z is an integer solution of the min-plus recursion for (k', S', θ), then $x_n = z_{\lfloor n/\ell \rfloor}$ is one for (k, S, θ). So if $T^p = \mathrm{id}$, every such x is p-periodic, and $z_{m+p} = x_{m\ell+p\ell} = x_{m\ell} = z_m$; that is, the min-plus map for (k', S', θ) has p-th power the identity, against the primitive case. □

Since $\mathbb{Z}^k \subset \mathbb{R}^k$, the min-plus map is not periodic on $\mathbb{R}^k$ either. Note that the corollary asserts only that T is not periodic. Unlike the theorems of Csörnyei and Laczkovich for the full support, it does not exhibit a single non-periodic orbit, since the window w depends on p. The hypothesis $|S| \ge 3$ cannot simply be dropped: with $\theta = 1$ the min-plus maps for $S = \{1\}$ at order 2 and $S = \{1, 2\}$ at order 3 are periodic, with periods 5 and 8 (Section 4).

The corollary leaves the two-term supports $\{j, k-j\}$ open, since the proof of Theorem 1.1 settles them with the linearisation. For $k = 4$ and $S = \{1, 3\}$ the tangent certificates (Definition 3.8) with $E = (-1, -1, -1, -1)$ and

$$\theta = 1: \quad q = 11,\ V = (-1, -1, 0, -1),\ W = (0, 0, 1, 1),\ (w_t) = 1, 1, 1, 3, 0, 1, 3, 3, 3, 3, 1,$$
$$\theta = 0: \quad q = 66,\ V = (-1, -1, 0, 0),\ W = (4, 0, 0, 0),\ (w_t) = 1^6 3^5 1^6 3^5 1^6 3^6 1^5 3^6 1^5 3^6 1^5 3^5,$$

(1^6 stands for six consecutive entries 1) are valid along the 11-periodic solution with window E, so, by the argument above, no power of the min-plus map for $(4, \{1, 3\}, \theta)$ is the identity, for either θ. Whether this holds for every two-term support other than $\{m, 2m\}$ at order $3m$ with $\theta = 1$, where the map is periodic by unfolding the order-3 case, and which non-palindromic supports have a periodic min-plus map, I leave open.

Acknowledgements

This proof was AI-generated with fairly limited input from the human author. The models used were Claude Opus 5.5 and Fable 5.1.

A The finite range of the progression theorem

This appendix completes the proof of Theorem 9.1. It gives the certificate for the 127 orders $92 \le k \le 2730$ with at least three odd prime factors, the only ones not covered by Proposition 9.14 or Proposition 9.12, and proves the two bounds on Jacobsthal's function used in Section 9.6 for the orders above 2730. The notation is that of Section 9.

A.1 The orders up to 2730 with three or more odd primes

By Lemma 9.4(b), (c) and Lemma 9.5, the normalised shortest vector of Λ at an order $k \le 2730$ gives a triple in $\mathcal{V}$ with $g \mid 2k$ and $3l^4 \le 16k^2$: indeed $l^4 \le 16 \cdot 2730^2/3$ gives $l^2 \le 6304 < 6400$, so $l \le 80$. Besides the chord table (C1) of Section 9.6, the finite certificate has two more parts.

(C2) *Jacobsthal values.* A list of 224 pairs (n, m) with $J(n) \le m$. Each is checked over one period: for every residue y modulo n some $0 \le i < m$ has $\gcd(y + i, n) = 1$.

(C3) *The per-order certificate.* For each of the 127 even k with $92 \le k \le 2730$ that have at least three odd prime factors, either (C2) contains a pair (n, m) with n the product of the primes of $2k$ such that k satisfies the tail condition (53) with bound m (the *tail alternative*), or, for each $(g; p, q) \in \mathcal{V}$ with $g \mid 2k$ and $3\big(g^2(p^2 + q^2)\big)^2 \le 16k^2$, one of the following is verified:

- (i) a pair (n, m) from (C2) with $n \mid 2k$, $\gcd(n, g) = 1$, every prime of $2k$ dividing ng, and $2k\psi > mg$, where ψ is the length of the chord witness in (C1); or
- (ii) for each of the g^2 pairs $0 \le s, t < g$, with $(A, E) = (p + \frac{2k}{g}s,\ q + \frac{2k}{g}t)$: a unit $\nu < 100$ satisfying (48), or $\gcd(A, E, 2k) > 1$, or an m with $0 < m < g$ and $2k \mid m(pE - qA)$.

The tail alternative of (C3) holds at 65 of these 127 orders, and alternative (ii) is needed only at 18 orders, all at most 924.

Proposition A.1. *Theorem 9.1 holds at the* 127 *even orders* $92 \le k \le 2730$ *with at least three odd prime factors.*

Proof. If the tail alternative of (C3) holds, then $J(2k) = J(n) \le m$ and k satisfies the tail condition with bound m, hence with bound $J(2k)$, and Proposition 9.12 applies. Otherwise, let $v = gu$ be a normalised shortest vector of Λ. As noted above, $(g; p, q) \in \mathcal{V}$, $g \mid 2k$ and $3l^4 \le 16k^2$, so (C3) applies to it. In case (i), $J(n) \le m$ and $2k\psi > J(n)g$, and Proposition 9.10 shows that (a, e) has a witness; the conditions on n are those of Lemma 9.6. In case (ii), let (s, t) be the class given by Lemma 9.11. Its two exclusions are ruled out by that lemma, so the certificate gives a unit ν showing that (A, E) has a witness, and then so does (a, e), by Lemma 9.11. □

A.2 Elementary bounds on Jacobsthal's function

This subsection proves Corollary A.5, the two inequalities for Jacobsthal's function used in Proposition 9.12, by a union bound and a progression trick.

Lemma A.2. *(a) Suppose every m consecutive integers include at least c that are prime to N, and $\mathcal{T}$ is a finite set of primes with $\sum_{p\in\mathcal{T}}\lceil m/p\rceil < c$. Then $J(n)\le m$ for every $n\ge 1$ whose prime factors divide N or lie in $\mathcal{T}$.*

(b) If $\gcd(f,N)=1$ then $J(fN)\le f\,J(N)$.

Proof. (a) Among m consecutive integers at most $\lceil m/p\rceil$ are multiples of p. So fewer than c of them are divisible by a prime of $\mathcal{T}$, and one of the c integers prime to N escapes; it is prime to n.

(b) Given a run $y, y+1, \dots, y+fJ(N)-1$, choose $0\le r<f$ with $y+r\equiv 1 \pmod f$, and consider $y_i = y+r+fi$ for $0\le i<J(N)$, all in the run. Each y_i is prime to f. With f' an inverse of f modulo N, $f'y_i \equiv f'(y+r)+i \pmod N$ runs through $J(N)$ consecutive integers, so some y_i is prime to N, hence to fN. □

Let $\varpi_1<\varpi_2<\cdots<\varpi_8$ be the primes $17,19,23,29,31,37,41,43$ and $\Pi_r=\varpi_1\cdots\varpi_r$ (with $\Pi_0=1$), and recall that $\omega(n)$ is the number of distinct prime factors of n.

Lemma A.3 (at most eight prime factors)**.** *Let $n\ge 1$ with $\omega(n)\le 8$. Let B be the product of the primes $p\le 13$ dividing n, and r the number of primes $p>13$ dividing n. Then $B\Pi_r\le n$, and there is an m with $J(n)\le m$ such that*

(i) if B is even, $178\,m^2 \le 10\max\bigl(B\Pi_r,\ B(\lfloor 2730/B\rfloor+1)\bigr)$;

(ii) $m\le 8$, or $1000\,m < 102\,B\Pi_r$.

The maximum in (i) is the better of two lower bounds for an order k to which the lemma will be applied: $k\ge B\Pi_r$, and, as k is a multiple of B above 2730, $k\ge B(\lfloor 2730/B\rfloor+1)$. So (i) gives the first inequality of the tail condition, and (ii) gives $J(g)\le 8$ or $J(g)<0.102\,g$, as needed for the band.

Proof. The i-th smallest prime above 13 is ϖ_i, so the primes $p_1<\cdots<p_r$ of n above 13 satisfy $p_i\ge\varpi_i$. Hence $n\ge B\,p_1\cdots p_r\ge B\Pi_r$, and $\sum_i\lceil m/p_i\rceil\le\sum_i\lceil m/\varpi_i\rceil$ for every m.

The rest is a finite table. For each of the 64 sets $\mathcal{P}$ of primes ≤ 13, with product B, and each r with $0\le r\le 8-|\mathcal{P}|$, there is an entry; this gives 384 pairs (B,r). For each the table supplies integers m and c such that

(P1) every m consecutive integers include at least c that are prime to B;

(P2) $\sum_{i=1}^{r}\lceil m/\varpi_i\rceil < c$;

(P3) (i) and (ii) hold for this m.

Condition (P1) is checked by sliding a window of length m over one period $\{0,\dots,B-1\}$ of the starting point; (P2) and (P3) are arithmetic. The largest m needed is 84. Given the table, (P1), (P2) and the inequality above give exactly the situation of Lemma A.2(a), with $N=B$ and $\mathcal{T}$ the primes of n above 13, so $J(n)\le m$. □

Lemma A.4 (at least nine prime factors)**.** *Let n have $\omega=\omega(n)\ge 9$ distinct prime factors. Let B be the product of the primes $p\le\omega$ dividing n, and R the product of the remaining r primes of n. Then $BR\le n$, $J(n)\le B(r+1)$, and*

$$178\,B(r+1)^2\le 10R, \qquad 1000(r+1)<102R.$$

Proof. Every prime of R exceeds $\omega \geq r$, so $r+1$ consecutive integers include at most one multiple of each, and Lemma A.2(a) (with $N=1$, $c=m=r+1$) gives $J(R) \leq r+1$. Since $\gcd(B,R)=1$, Lemma A.2(b) gives $J(BR) \leq B(r+1)$, and $J(n)=J(BR)$.

For the inequalities, the r primes of R are distinct integers above ω, so $R \geq (\omega+1)(\omega+2)\cdots(\omega+r)$. The other $\omega - r$ primes are at most ω, so B divides the primorial $\omega\# = \prod_{p\leq\omega} p$ and $r \geq \omega - \pi(\omega)$. If $\omega \leq 63$, it suffices to check that $178\,(\omega\#)(r+1)^2 \leq 10\prod_{i=1}^{r}(\omega+i)$ and $1000(r+1) < 102\prod_{i=1}^{r}(\omega+i)$ for every $9 \leq \omega \leq 63$ and $\omega - \pi(\omega) \leq r \leq \omega$, a finite computation. If $\omega \geq 64$, use the classical bound $\omega\# \leq 4^{\omega}$ and the count $\pi(\omega) \leq \omega/2+1$ (the odd primes up to ω are distinct odd numbers in $[3,\omega]$). The bound on $\omega\#$ holds for $\omega \leq 2$ by inspection and follows by induction: for even $\omega > 2$, $\omega\# = (\omega-1)\#$; for $\omega = 2m+1$, the primes in $(m+1, 2m+1]$ all divide $\binom{2m+1}{m}$, which is at most 4^m because it and $\binom{2m+1}{m+1}$ are equal and sum to at most 2^{2m+1}, so $(2m+1)\# \leq (m+1)\#\,4^m \leq 4^{m+1}4^m$. Then $r \geq \omega/2 - 1 \geq 31$, $\omega \leq 2r+2$, $\prod_{i=1}^{r}(\omega+i) \geq 64^r$ and $B \leq 4^{\omega} \leq 16\cdot 16^r$, so $178B(r+1)^2 \leq 16^r \cdot 2848(r+1)^2 \leq 16^r \cdot 10 \cdot 4^r = 10\cdot 64^r$, using $2848(r+1)^2 \leq 10\cdot 4^r$ for $r \geq 8$ (true at $r=8$, and the right side grows faster). The second inequality is then immediate. □

Corollary A.5. *(a) Every even $k > 2730$ satisfies $178\,J(2k)^2 \leq 10k$, that is, $17.8\,J(2k)^2 \leq k$.*

(b) Every $g \geq 1$ satisfies $J(g) \leq 8$ or $1000\,J(g) < 102\,g$.

Proof. (a) Note $J(2k) = J(k)$. If $\omega(k) \leq 8$, apply Lemma A.3 to $n=k$. Then B is even, and $k \geq B\Pi_r$; also k is a multiple of B exceeding 2730, so $k \geq B(\lfloor 2730/B \rfloor + 1)$. So $178J(k)^2 \leq 178m^2 \leq 10k$. If $\omega(k) \geq 9$, Lemma A.4 gives $178J(k)^2 \leq 178B^2(r+1)^2 \leq 10BR \leq 10k$. (b) If $\omega(g) \leq 8$, Lemma A.3(ii) gives $J(g) \leq m \leq 8$ or $1000J(g) \leq 1000m < 102B\Pi_r \leq 102g$. If $\omega(g) \geq 9$, Lemma A.4 gives $1000J(g) \leq 1000B(r+1) < 102BR \leq 102g$. □

B Detailed case checks

This appendix contains two elementary case checks that complete proofs in the main text.

B.1 The case $M = 1$ of the both-failing supports

This completes the proof of Lemma 7.13, in the notation of that proof and with the description of Q given at its start.

At each index i the table names two taps among a, b, $2a$, $a+b$ whose reads differ, both surviving. It lists them in twelve numbered rows, with the residue of the position each reads (not the position itself, as in the example of Section 3.5), a star marking a wrapped tap; reads follow from (22) and (23).

	index i	tap reading 1	tap reading 0
	i even		
(1)	$i < b$	$2a \to i+2a \in (2a,k)$	$a \to i+a$, odd
(2)	$i = b$	$b \to 2b \in (2a,k)$	$a \to a+b \in S$
(3)	$b < i < a+b$, $i \neq 2a$	$a+b \to^* i-a$, odd	$a \to i+a$, odd
(4)	$i = 2a$	$a \to 3a$	$2a \to^* 2a-b \in (0,b)$
(5)	$i > a+b$, $i \neq 2b$	$a \to^* i-a-b \in (0,a)$, odd	$b \to^* i-2a \in (0,b)$
(6)	$i = 2b$	$a+b \to^* 2b-a$, odd	$b \to^* 2b-2a \in (0,b)$
	i odd		
(7)	$i < a$	$a+b \to i+a+b \in (2a,k)$	$2a \to i+2a \in (2a,3a)$, odd
(8)	$i = a$	$2a \to 3a$	$a \to 2a \in S$
(9)	$a < i < a+b$	$a \to i+a \in (2a,k)$	$a+b \to^* i-a \in (0,b)$
(10)	$i = a+b$	$a+b \to^* b \in S$	$b \to^* b-a$
(11)	$i > a+b$, $i \neq 3a$	$b \to^* i-2a \in (b-a,b)$, odd	$a \to^* i-a-b \in (0,a)$
(12)	$i = 3a$	$a+b \to^* 2a \in S$	$a \to^* 2a-b \in (0,b)$

Positions not described as odd or as elements of S are even, and every even position named lies in $(0,b)\cup(2a,k)$, so it is free and lies in Q. An element of S other than a reads 0 unwrapped and 1 wrapped. For the odd positions: among the reads of 1, the unwrapped odd position is $3a\in Q$ and the wrapped ones $(i-a,\ i-a-b,\ 2b-a,\ i-2a)$ are not in $\{a,3a,b-a\}$; among the reads of 0, the unwrapped odd positions $(i+a,\ i+2a)$ are not in $\{a,3a,b-a\}$ and the wrapped one is $b-a\in Q$.

These memberships are checked with $a<b<2a$ and $k=2a+b$. In row (1) $i+a\in(a,a+b)$ is not $3a$ as $i<b<2a$, and exceeds $b-a$. In row (3) $i-a\in(b-a,b)$, so it is not $b-a$ or $3a$, and it is not a as $i\neq 2a$; and $i+a\in(a+b,k)$ is not $3a$ as $i\neq 2a$. In row (5) $i-a-b\in(0,a)$ is not $b-a$ because $i\neq 2b$. In row (6), $2b-a$ is not a, $3a$ or $b-a$ since $b\neq a,2a,0$. In row (7), $i+2a\in(2a,3a)$. In row (11), $i-2a\in(b-a,b)$ is not a because $i\neq 3a$, and is less than $3a$. Since $2a<a+b<3a<k$ and $a+b<2b<k$, the cases are exhaustive, and every tap marked with a star does wrap while the others do not. Finally, no residue named is 0 or $\sigma=a$, so every tap named survives.

This proves (B2), and with it Lemma 7.13.

B.2 The special indices of the three-element supports

This completes the proof of Theorem 8.8, in its notation (residues modulo $4h$; the phases, pins and the rule for reads are as stated in that proof).

In each case below, every position named that is not called a pin is free, and no tap used is deleted; this is checked after the list. The inequality $3\beta<h$, a hypothesis of the theorem, puts every phase below strictly inside one half.

- $i=a$, phase $3h+\beta$. The tap a reads $2a$, of phase $6h+2\beta\equiv 2h+2\beta$, a 0; the tap h reads $a+h$, of phase $4h+\beta\equiv\beta$, a 1.
- $i=h$, phase h. The tap a reads $a+h$, a 1; the tap $k-a$ reads $k+h-a$, of phase $4h-\beta$, a 0.
- $i=k-a$, phase $3h-\beta$. The tap h reads $k+h-a$, a 0; the tap $k-a$ reads $2k-2a$, of phase $6h-2\beta\equiv 2h-2\beta$, a 1.
- $i=h-a$, phase $\nu h-\nu a\equiv 2h-\beta$. The tap a reads the pin h, a 0; the tap $k-a$ reads $3h-2a$, of phase $5h-2\beta\equiv h-2\beta$, a 1.
- $i=k-2a$, phase $2h-2(3h+\beta)\equiv -2\beta$. The tap a reads the pin $k-a$, a 0; the tap h reads $3h-2a$, of phase $h-2\beta$, a 1.
- $i=h+a$, phase β. The tap $k-a$ reads the wrapped pin $k+h$, a 1; the tap a reads $h+2a$, of phase $3h+2\beta$, a 0.
- $i=2a$, phase $2h+2\beta$. The tap a reads $3a$, of phase $5h+3\beta\equiv h+3\beta$, a 1; the tap h reads $h+2a$, of phase $3h+2\beta$, a 0.

The positions $2a$, $a+h$, $k+h-a$ and $2k-2a$ are even and $\not\equiv 0 \pmod k$, hence free. The positions $3h-2a$, $h+2a$ and $3a$ are odd and lie in $(0,2k)$; reduced modulo k, equality with a, h or $k-a$ would force one of $h=a$, $h=3a$, $h=2a$ or $3a=2h$, and these are excluded by $a<h$, $\gcd(a,h)=1$ (with $a\geq 3$) and the oddness of a and h. None of these seven positions, nor the pins h, $k-a$, $k+h$ and $2k-a$, is k, a or $k+a$, by the same comparisons, so none of the taps used is deleted.

So the clause holds at each of the seven special indices, which completes the proof of Theorem 8.8.

C The formalisation

The proof of Theorem 1.1 has been formalised in the Lean 4 proof assistant, using the Mathlib library. The formal statement is, in the notation of (3),

> *if every positive solution of* (3) *satisfies* $x_{n+p} = x_n$ *for some fixed* $p \geq 1$*, then* (3) *is one of the equations listed in Theorem 1.1*,

with each of the five cases written out as explicit conditions on the coefficients A_i, B_i, exactly as in the theorem. The formal proof has no hypotheses beyond those of the theorem, and it depends on no axioms other than the three standard ones of Lean's logic (propositional extensionality, the axiom of choice and the soundness of quotients). The converse, that each of the equations listed in Theorem 1.1 is globally periodic, is formalised as well, so that the formal statement is an equivalence: (3) is globally periodic if and only if it is one of the listed equations. The main theorem and the equivalence are, respectively,

`PeriodicDifference.Problem11.problem11_complete` and
`PeriodicDifference.Problem11.problem11_iff`.

The tropical Corollary 10.1 is formalised too, as

`PeriodicDifference.Trop.trop_nonperiodic`,

which I state for the window map of Definition 3.1, with the constant slot 0 added to S when $\theta = 1$. The formalisation and the data for its finite computations are at `https://github.com/johnrudge/periodic_difference`.

The formalisation follows the structure of this paper result by result, including the period-sum proof of Theorem 2.5. In a few places the formal proof takes a different, usually more elementary, route:

- *Linearisation* (Theorem 2.2 and Theorem 2.4). Instead of the chain rule for F^p, the formal proof uses an exact identity $F(u^* + \epsilon w) = u^* + \epsilon\, G_\epsilon(w)$, where G_ϵ is an explicit map that depends continuously on ϵ and $G_0 = DF(u^*)$ is linear. Global periodicity gives $G_\epsilon^p = \mathrm{id}$ for $\epsilon > 0$, and letting $\epsilon \to 0$ gives $G_0^p = \mathrm{id}$.

- *Order three with* $c_0 = 0$ (Theorem 4.2). Instead of the algebraic-integer step, a root λ of $\lambda^2 - \frac{3}{2}\lambda + 1$ satisfies $\lambda^m = \alpha_m \lambda + \beta_m$ with rational α_m, β_m and $2^{m-1}\alpha_m$ an odd integer for $m \geq 1$; so $\alpha_m \neq 0$ and, as λ is not real, $\lambda^m \neq 1$.

- *Small orders in Theorem 4.3.* Step 2 of the proof (that $n_0 \geq 9$) is replaced by the same kind of expansion: with $\zeta^2 - \epsilon\zeta + 1 = 0$, $\zeta^r = P_r(\epsilon)\zeta + Q_r(\epsilon)$ for polynomials P_r, Q_r, and $\zeta^r = 1$ would force $P_r(\epsilon) = 0$, which is checked to be impossible for $1 \leq r \leq 8$ and $0 < \epsilon \leq \frac{1}{2}$.

- *Odd indices* (Lemma 8.7). Instead of the argument with the gaps between phases, the formal proof is local: if $H_\nu(u) \neq H_\nu(v)$, then at every odd index the four taps u, $k-u$, v, $k-v$, less the deleted one, already read both values, which is a case analysis on the residues νi, νu and νv modulo $2k$.

- *The band* (Proposition 9.8). To stay in rational arithmetic, the incircle of Δ_1 is replaced by a slightly smaller disc inside Δ_1, of radius $0.207\,k$ about a rational point close to the incentre. With the same band half-width $0.102\,k$ its chords are longer than $0.36024\,k$ rather than $0.3604\,k$, and the bound $J(n)\,l < 0.36021\,k$ obtained in the proof of Proposition 9.12 still suffices.

Every finite computation in the paper is carried out in exact integer arithmetic inside Lean's kernel by the `decide` tactic, so no step of the proof relies on compiled code or on an external program. In some the kernel finds the required objects itself by search (or none are needed); in the others they were produced outside Lean and are imported as data, which the kernel only checks:

found by the kernel	imported as data and checked
units witnessing the triples of Lemma 8.16 outside $\mathcal{F}$, and in step 1 of the proof of Theorem 8.19 outside F_k (search over the units modulo $2n$ or $2k$)	the seventeen carriers of Table 4
the enumeration of lower halves in step 2 of that proof	the certificate of Proposition 3.10
the ratios of $\mathcal{F}$ in Lemma 8.18, and the entries of $\mathcal{F}$ used at $k = 120$ in Theorem 8.19	the list $\mathcal{F}$ and the sets F_k
the orders of Proposition 9.14 (a divisor test), and the per-order check (C3) for Theorem 9.1 at the others, which looks up the pairs of (C2) and finds the units $\nu < 100$ of alternative (ii) by search	the units satisfying (T) in step 3 of the proof of Theorem 8.19
the check $9 \le \omega \le 63$ in Lemma A.4	the chord table (C1), the Jacobsthal pairs (C2), and the table of Lemma A.3

The case tables of Appendix B and the chains of Table 5 hold for all values of the parameters, and they are formalised as case analyses.

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