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HEIGHT-BOUNDED MUTATION TO ACYCLIC QUIVERS ON FOUR VERTICES

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Abstract. We prove Conjecture 5.5 of Armstrong-Williams, Hirst, Jackson and Lee.

1. Introduction

Fomin and Zelevinsky introduced cluster algebras in 2001 as an algebraic setting for total positivity and canonical bases [13]. Their exchange relations are accompanied by mutations of an integer exchange matrix. In the skew-symmetric case, the matrix records a finite quiver without loops or directed two-cycles. Mutation changes its arrows along directed paths of length two.

A quiver is *mutation-acyclic* if a finite sequence of mutations takes it to a quiver without directed cycles. This property connects the combinatorics to hereditary representation theory. Buan, Marsh, Reineke, Reiten and Todorov developed cluster categories of hereditary algebras [14]; Buan, Marsh and Reiten realized matrix mutation through mutation of cluster-tilting objects [15]. Assem, Brüstle and Schiffler described cluster-tilted algebras as relation extensions of tilted algebras [3].

For three vertices, recognition is governed by descent and a cubic invariant. Assem, Blais, Brüstle and Samson gave a combinatorial decision algorithm by mutating opposite a largest multiplicity [16, §2.5]. Beineke, Brüstle and Hille proved that a directed triangle with multiplicities x, y, z is mutation-cyclic exactly when $x, y, z \geq 2$ and $x^2+y^2+z^2-xyz \leq 4$ [17, Theorem 1.2]. Their analysis identifies a fundamental region for the mutation action and relates its integral orbits to the Markov equation. In particular, a mutation-acyclic triangle can be made acyclic without increasing its largest multiplicity.

Every full subquiver of a mutation-acyclic quiver is mutation-acyclic [15, Corollary 5.3]. Felikson and Tumarkin proved that every quiver in the mutation class of an acyclic quiver with at least two arrows between every pair of vertices reaches an acyclic representative by decreasing mutations [18, Corollary 3.5 and Remark 3.6]. Each such mutation lowers an arrow multiplicity and raises none. This gives the initial-height bound in that class, in every rank. Four-vertex quivers with missing edges or unit arrows require other local moves.

Fomin and Neville introduced cyclically ordered quivers and mutation invariants obtained from their unipotent companions [19]. A proper mutation changes both the quiver and its cyclic ordering; a totally proper ordering permits these operations through every mutation sequence. Neville proved that every mutation-acyclic quiver has such an ordering [20, Theorem 1.1]. The associated Markov invariant extends the rank-three cubic. For a fixed vertex set and a fixed value of this invariant there are only finitely many acyclic quivers [20, Corollary 6.7]. These results provide finite sets of possible acyclic endpoints, while a bound on the intermediate quivers is still needed for a finite search.

Armstrong-Williams, Hirst, Jackson and Lee studied four-vertex recognition computationally in their 2024 preprint, revised in 2025 [21]. Their 2026 paper [1] corrects the earlier bounded criterion and proves a general path-height bound. If h is the largest

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initial arrow multiplicity and $m(Q)$ is the Markov invariant in the totally proper class, a mutation-acyclic four-vertex quiver Q reaches an acyclic representative within height $\max\{h, m(Q)\}$. Their resulting decision procedure explores $O(h^{24})$ states. They also show that mutation-acyclicity in ranks at least four has no semialgebraic criterion in the arrow multiplicities.

Conjecture 5.5 of [1] asserts that the term $m(Q)$ can be omitted. The authors checked the conjecture through height three. Thalluri subsequently reported a finite computational verification through height seven, using the authors' decision procedure for mutation-cyclicity certificates [22]. Throughout the paper quivers are ordinary, so their exchange matrices are integral and skew-symmetric. The height $\mathrm{ht}(Q)$ is the largest arrow multiplicity, with height zero for the quiver with no arrows.

Theorem 1.1. *Let Q be a mutation-acyclic quiver on four vertices. There is a finite mutation sequence from Q to an acyclic quiver such that every quiver in the sequence has height at most* $\mathrm{ht}(Q)$.

The theorem gives a decision procedure exploring at most $(2h+1)^6 = O(h^6)$ labelled states. Its proof removes height peaks from a path to an acyclic quiver. Two-vertex mutation orbits provide commuting and pentagon replacements. For the remaining configurations, the relation-extension description produces finitely many possible tilted cuts. Explicit indecomposable modules and path-space calculations exclude those cuts. An endpoint argument also excludes a terminal high segment ending at an acyclic quiver.

2. Mutation and finite-height search

A quiver has a finite labelled vertex set, finitely many arrows, no loops and no oriented cycles of length two. Parallel arrows are allowed. Its exchange matrix $B = (b_{ij})$ is defined by setting b_{ij} equal to the number of arrows $i \to j$ minus the number of arrows $j \to i$. Thus B is skew-symmetric. When the quiver varies, $b_{ij}(Q)$ denotes the signed entry for Q. For four vertices we record it by

$$(b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}).$$

A positive coordinate denotes arrows from the smaller label to the larger label; a negative coordinate denotes arrows in the opposite direction. The height is $\mathrm{ht}(B) = \max_{i<j} |b_{ij}|$. For an integer x, write $[x]_+ = \max(x, 0)$. Mutation at vertex k gives the matrix $B' = \mu_k(B)$ with

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & i, j \neq k. \end{cases}$$

Mutation words are read from left to right: $B123 = \mu_3\mu_2\mu_1(B)$. More generally, for a word $w = k_1 \cdots k_s$, write $\mu_w = \mu_{k_s} \cdots \mu_{k_1}$ and $Qw = \mu_w Q$. The empty word acts as the identity. A permutation π relabels each old vertex i as $\pi(i)$; its action is

$$(B^\pi)_{ij} = b_{\pi^{-1}(i),\pi^{-1}(j)}.$$

Write $\pi(w) = \pi(k_1) \cdots \pi(k_s)$. Substitution in the mutation formula gives

$$(\mu_w Q)^\pi = \mu_{\pi(w)}(Q^\pi).$$

Cycle notation has its usual direction, so (123) sends 1 to 2, 2 to 3, and 3 to 1. A quiver is acyclic if it has no directed cycle. It is mutation-cyclic if it is not mutation-acyclic. Reversing all arrows or permuting vertex labels preserves height and mutation-acyclicity.

Corollary 2.1. *Mutation-acyclicity of four-vertex quivers of height h can be decided by a search through at most $(2h+1)^6$ labelled states.*

Proof. Enumerate the component of Q in the graph whose vertices are skew-symmetric integral matrices of height at most h and whose edges are mutations remaining within that bound. There are at most $(2h+1)^6$ such matrices. If an acyclic matrix is found, the recorded path proves mutation-acyclicity. Conversely, Theorem 1.1 implies that a mutation-acyclic input has an acyclic matrix in this component. Exhausting the finite component therefore decides the question. □

For $h \geq 1$ the number of states is $O(h^6)$. Entries created while testing a mutation have size $O(h^2)$, so each test uses integers with $O(\log(h+1))$ bits. The state bound is polynomial in the numerical height.

3. Cyclic orderings and the Markov invariant

A cyclic ordering $\omega = \langle v_1, v_2, v_3, v_4 \rangle$ is a placement of the vertices on an oriented circle; lists differing by cyclic rotation represent the same ordering. A vertex k is proper if every directed path $i \to k \to j$ meets i, k, j in that clockwise order. A wiggle interchanges adjacent vertices not joined by an arrow. A wiggle class is proper if each vertex is proper in some representative. To mutate properly at k, choose such a representative, mutate at k, and move k clockwise past its incoming neighbours in the mutated quiver without passing its outgoing neighbours. This gives a well-defined wiggle class. A class is totally proper if it remains proper under every finite sequence of these operations.

For $\omega = \langle v_1, v_2, v_3, v_4 \rangle$, set

$$(a, b, c, d, e, f) = (b_{v_1v_2}, b_{v_1v_3}, b_{v_1v_4}, b_{v_2v_3}, b_{v_2v_4}, b_{v_3v_4}).$$

The signed entries use the indicated vertex order. Define

$$M_\omega(Q) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + abd + ace + bcf + def + acdf.$$

The value is independent of cyclic rotation, wiggles and proper mutation. Every mutation-acyclic quiver has a unique totally proper class; for an acyclic quiver it comes from a topological order [1, Theorem 2.2]. We write $m(Q)$ for the value in this class, and abbreviate it to m when the quiver is fixed. The polynomial must be evaluated in that ordering, which need not be the fixed label order used for mutation coordinates. For example, in a topological order all six coordinates are nonnegative, so every term in the displayed formula is nonnegative.

The Pfaffian in the fixed label order is

$$p(Q) = b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23}; \qquad \det B = p(Q)^2.$$

For fixed Q, write $p = p(Q)$. Mutation changes its sign, and relabelling changes it by the sign of the permutation. For connected mutation-acyclic quivers on four vertices, the established inequalities [1, Section 2] are

$$m \geq 3, \qquad m > 2|p(Q)|.$$

We also use the hereditary property that every full subquiver of a mutation-acyclic quiver is mutation-acyclic, the rank-three criterion that a directed triangle is mutation-cyclic precisely when all its multiplicities are at least two and its Markov constant (the sum of their squares minus their product) is at most four, and the four-vertex vortex obstruction [1, Section 2]. A vortex is a quiver with a directed triangle and a fourth vertex joined to all three triangle vertices as a strict source or strict sink.

Lemma 3.1. *Let $Q = (a, b, -c, d, E, f)$ be mutation-acyclic, where $a \geq 0$ and $b, c, f > 0$. Suppose either $d \geq 0$, or $a = 0, d < 0, E \leq 0$. Its totally proper class contains $\langle 1, 2, 3, 4 \rangle$; in particular its Markov value is*

$$m = a^2 + b^2 + c^2 + d^2 + E^2 + f^2 + abd - acE - bcf + dEf - acdf.$$

Proof. The directed triangle 134 restricts a proper class to the three cyclic orders $\langle 1,2,3,4\rangle$, $\langle 1,3,2,4\rangle$, $\langle 1,3,4,2\rangle$. A restriction to a complete triple persists under wiggles, since none of its pairs can be interchanged. If $a=0$, the first and last orders differ by a wiggle at 12. If $d=0$ or $E=0$, the middle order joins them across 23 or 24. Otherwise the complete triple 234 contains the path $2\to 3\to 4$ when $d>0$, or $3\to 4\to 2$ when $d,E<0$; this excludes the middle order. If $a>0$, we have $d\geq 0$. When $d>0$, the complete triple 123 contains $1\to 2\to 3$, leaving only the first order. When $d=0, E\neq 0$, the complete triple 124 contains $4\to 1\to 2$, leaving the first two orders, which are wiggle equivalent at 23. When $d=E=0$, all three are connected by wiggles at $23, 24$. Thus at most one proper class exists. The existence of a totally proper class for a mutation-acyclic quiver identifies it with this class. Substitution in the Markov polynomial gives the formula. □

4. Acyclic endpoints in two-vertex orbits

Lemma 4.1. *Let $B=(b_{ij})$ be the exchange matrix of an acyclic quiver A. Under any single mutation $B'=\mu_k(B)$, one has $|b'_{ij}|\geq |b_{ij}|$ for every pair of vertices.*

Proof. Incident entries change only sign. If an oriented path $i\to k\to j$ exists, an existing arrow between i,j must point from i to j, since the opposite direction would give a directed triangle. The arrows added along the path therefore reinforce the existing arrows. The same argument applies to a path $j\to k\to i$. If neither path exists, the entry is unchanged. □

Lemma 4.2. *Let A be an acyclic quiver and suppose vertices $1,2$ are joined by one arrow. Every quiver obtained from A by mutations only at $1,2$ has height at least $\mathrm{ht}(A)$.*

Proof. Reverse arrows if necessary so that $1\to 2$. The pentagon identity for a unit edge says that the word 12121 interchanges labels $1,2$. Involutivity then reduces every word to one of these five successive states, possibly with the labels interchanged.

For an external vertex v, put $x=|b_{1v}|$ and $z=|b_{2v}|$. Acyclicity allows three signed attachment patterns; zeros are included by $x,z\geq 0$. Successive substitution gives

(b_{1v},b_{2v})	1	12	121	1212	12121
(x,z)	$(-x,z)$	$(-x,-z)$	$(x,-x-z)$	$(-z,x+z)$	(z,x)
$(x,-z)$	$(-x,-z)$	$(-x-z,z)$	$(x+z,-x)$	(z,x)	$(-z,x)$
$(-x,-z)$	$(x,-x-z)$	$(-z,x+z)$	(z,x)	$(z,-x)$	$(-z,-x)$

Every pair contains both initial absolute values or their sum, so its maximum is at least $\max(x,z)$. To check entries between two external vertices, label them $3,4$ in a topological order, so that any arrow between them points $3\to 4$, and write its multiplicity as $f\geq 0$. Set $x=|b_{13}|$, $y=|b_{14}|$, $z=|b_{23}|$, $w=|b_{24}|$. A topological order respecting $1<2$ and $3<4$ has one of the following six forms. Direct substitution in the mutation formula gives the listed 34 entries.

	1	12	121	1212	12121
1234	f	f	f	f	f
1324	f	$f+wz$	$f+wz$	$f+wz$	f
1342	f	f	f	f	f
3124	$f+xy$	$f+wx+wz+xy$	$f+wx+wz+xy$	$f+wz$	f
3142	$f+xy$	$f+xy$	$f+xy$	f	f
3412	f	f	f	f	f

All added terms are nonnegative. Zero arrows are included by setting the corresponding variables to zero. Every initial entry is thus bounded by some entry in each successive state, proving the height assertion. □

5. Tilted cuts and relation potentials

We work over $\mathbb{C}$. For a finite-dimensional complex algebra R and right R-modules M, N, $\mathrm{Hom}_R(M,N)$ is the vector space of R-linear maps and $\mathrm{End}_R(M) = \mathrm{Hom}_R(M,M)$. The group $\mathrm{Ext}^n_R(M,N)$ is computed by applying $\mathrm{Hom}_R(-,N)$ to a projective resolution of M and taking degree-n cohomology. Projective and injective dimension are the shortest lengths of the corresponding resolutions, with value infinity when no finite resolution exists; global dimension is the supremum of the projective dimensions. For a finite-dimensional vector space V, write $V^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$.

A finite-dimensional algebra is hereditary if every module has projective dimension at most one. A basic tilting module over a hereditary algebra H is a direct sum T of pairwise nonisomorphic indecomposable modules with $\mathrm{Ext}^1_H(T,T) = 0$ and with as many summands as H has simple modules. A basic finite-dimensional algebra is *tilted* if it is the endomorphism algebra of such a module. Such an algebra has global dimension at most two and is quasitilted: every indecomposable module has projective dimension at most one or injective dimension at most one [2].

We use right modules; a representation assigns a vector space M_i to vertex i and a linear map $M_i \to M_j$ to each arrow $i \to j$. Path products are read from left to right. We write S_i for the representation with $\mathbb{C}$ at vertex i, zero at every other vertex, and zero arrow maps and $D = \mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$ for vector-space duality.

The path algebra $\mathbb{C}Q$ has the directed paths, including the trivial paths e_i, as a basis; multiplication concatenates composable paths and is zero otherwise. Its arrow ideal $\mathfrak{a}$ is spanned by paths of positive length. A relation is a linear combination of paths with the same start and end, of length at least two. For a relation ideal $I \subseteq \mathfrak{a}^2$, a minimal system of relations means lifts of a basis of $I/(\mathfrak{a}I + I\mathfrak{a})$, chosen separately for each pair of endpoints. This quotient specifies the number of independent minimal relations from each vertex to each other vertex.

For a finite quiver Q, let $\widehat{\mathbb{C}Q}$ be its path algebra completed with respect to the ideal of arrows. A potential is a possibly infinite complex linear combination of directed cycles, considered modulo cyclic rotation. Its cyclic derivative with respect to an arrow a removes each occurrence of a and reads the remaining path beginning just after that occurrence. Thus

$$\partial_a(a_1 \cdots a_s) = \sum_{j : a_j = a} a_{j+1} \cdots a_s a_1 \cdots a_{j-1}.$$

The Jacobian algebra $J(Q,W)$ is the quotient by the closed ideal generated by all cyclic derivatives of the potential W. The order of a nonzero potential is the smallest path length with a nonzero coefficient; the zero potential has order infinity. A potential is rigid if every cycle is cyclically equivalent to an element of that ideal. A right equivalence is a continuous automorphism of the completed path algebra fixing the vertex idempotents; it acts on potentials modulo cyclic rotation.

Proposition 5.1. *If a four-vertex quiver Q is mutation-acyclic, there is an acyclic subquiver Q_0 on the same vertices and a tilted algebra $R = \mathbb{C}Q_0/(\rho_1, \ldots, \rho_t)$ such that Q is obtained by adding an arrow a_j opposite each member ρ_j of a minimal system of relations. The canonical potential*

$$W = \sum_{j=1}^{t} a_j \rho_j$$

is rigid and its Jacobian algebra J is finite-dimensional. Give the arrows of Q_0 degree zero and the added arrows degree one. Then W has degree one, its ordinary path lengths are three or four, and the ideal I_{new} of J generated by the degree-one arrows satisfies $I_{\mathrm{new}}^2 = 0$.

Proof. We use the cluster-category realization theorem [15]: every mutation-acyclic quiver is the ordinary quiver of a cluster-tilted algebra, the endomorphism algebra of a cluster-tilting object in the cluster category of a finite acyclic quiver. Applied to Q, it gives such an algebra A. By [3, Theorems 2.6 and 3.4],

$$A \simeq R \ltimes \operatorname{Ext}^2_R(DR, R)$$

for a tilted algebra R, whose quiver is acyclic and whose global dimension is at most two [12, Corollary 4.2 and Theorem 5.2]; its new arrows reverse a minimal set of relations. Here the trivial extension has underlying vector space $R \oplus \operatorname{Ext}^2_R(DR, R)$ and product $(r, x)(s, y) = (rs, ry + xs)$.

Keller's canonical presentation [4, §6.9 and Theorem 6.12(b)] identifies A with the Jacobian algebra of the displayed potential. Since an old path on four vertices has length at most three, and a minimal relation has length at least two, the claimed path-length bound follows. In this application the Calabi–Yau completion has deformation parameter zero, which has the zero negative-cyclic lift required in [5]. The relations generate an ordinary ideal in the finite-dimensional acyclic path algebra, as required in the corrected construction.

By [6, Lemma 5.2], the relation extension of R is also J/I^2_{new} for this actual ideal of new arrows. Both this quotient and J are isomorphic to A and are finite-dimensional. Equality of their dimensions forces $I^2_{\mathrm{new}} = 0$.

By [7, Corollary 6.8], A also has a rigid potential presentation $J(Q, W')$. Here $[A, A]$ denotes the complex linear span of the commutators $xy - yx$, with $x, y \in A$. The class of W' in $A/[A, A]$ is zero by rigidity. The class of W is zero as well, since its degree-one grading gives, modulo cyclic rotation,

$$W = \sum_{a \text{ new}} a \partial_a W.$$

After a vertex permutation and conjugation by a unit, the algebra isomorphism between the two basic Jacobian presentations fixes their primitive vertex idempotents: complete systems of primitive idempotents lifting the same semisimple quotient are conjugate. Thus it is an isomorphism over the vertex algebra $\prod_{i=1}^4 \mathbb{C} e_i$, where e_i is the trivial path at vertex i. The noncommutative Mather–Yau theorem [8, Theorem 3.5] now applies: both potentials have order at least three, both Jacobian algebras are finite-dimensional over $\mathbb{C}$, and an algebra isomorphism takes their potential classes, both zero, to one another. It follows that W and W' are right-equivalent, so W is rigid. □

The old and new arrow subspaces may initially depend on a choice of arrow bases. A parallel bundle cannot have both an old and a new direction. Indeed, a new arrow $i \to j$ reverses a nonzero old relation from j to i, and any old parallel arrow would close an old directed cycle. Thus every bundle is entirely old or entirely new. The word *cut* below means the set of new bundles. After an arrow change preserving the degree-zero/degree-one grading, both the old algebra and the square-zero new-arrow ideal retain their meanings.

For an old bound quiver algebra R, let $P_i = e_i R$ be the indecomposable projective and $I_i = \operatorname{Hom}_{\mathbb{C}}(Re_i, \mathbb{C})$ the indecomposable injective at vertex i. In the sums below, $\alpha : i \to j$ ranges over arrows and $\rho : i \rightsquigarrow j$ over chosen independent minimal relations with the indicated endpoints. If R has global dimension at most two, its minimal simple resolutions have the forms

$$0 \longrightarrow \bigoplus_{\rho : i \rightsquigarrow j} P_j \longrightarrow \bigoplus_{\alpha : i \to j} P_j \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

$$0 \longrightarrow S_j \longrightarrow I_j \longrightarrow \bigoplus_{\alpha : i \to j} I_i \longrightarrow \bigoplus_{\rho : i \rightsquigarrow j} I_i \longrightarrow 0.$$

The intermediate vertices in these sums are allowed to vary. Write each relation as a sum of its first arrows followed by suffix paths in the projective differential; in the injective differential use prefix paths followed by its last arrows. Applying $\mathrm{Hom}_R(-,M)$ or $\mathrm{Hom}_R(M,-)$ computes $\mathrm{Ext}^2(S_i,M)$ or $\mathrm{Ext}^2(M,S_j)$ as the final cokernel. Indeed $\mathrm{Hom}(P_i,M)=M_i$ and $\mathrm{Hom}(M,I_i)=M_i^*$. Projection onto selected relation coordinates gives a lower bound for this cokernel: surjectivity of the full differential would imply surjectivity of every projected differential. This applies equally to quadratic and cubic relations.

6. The large-edge sink obstruction

We apply Proposition 5.1. An indecomposable representation M with $\mathrm{Ext}^2_R(M,S_j)\neq 0$ and $\mathrm{Ext}^2_R(S_i,M)\neq 0$ has both projective and injective dimension greater than one and therefore excludes a tilted cut.

Lemma 6.1. *Let $T:\mathbb{C}^F\to \mathrm{Mat}_{b\times c}(\mathbb{C})$ be linear, where $b,c\geq 1$ and $F>c$. There exists $v\neq 0$ such that $\operatorname{rank} T(v)<b$.*

Proof. If $b>c$ the assertion is immediate. A nonzero kernel vector also suffices. Otherwise T is injective and $b\leq c$. For $b=1$, injectivity contradicts $F>c$. For $b\geq 2$, the projective variety of matrices of rank at most $b-1$ has codimension $c-b+1$. The projective image of T has dimension $F-1\geq c\geq c-b+1$. The projective dimension theorem forces an intersection with that determinantal variety. □

Lemma 6.2. *The quiver $(a,b,-c,-r,-e,F)$ is mutation-cyclic whenever $a,b,c\geq 1$, $r,e\geq 0$, $r+e>0$ and $F>\max(b,c)$.*

Proof. Denote the bundles by $A:1\to 2$, $B:1\to 3$, $C:4\to 1$, $V:3\to 4$, $R:3\to 2$, $E:4\to 2$, with multiplicities a,b,c,F,r,e. Suppose a tilted presentation exists. No arrow incident with the sink 2 can reverse an old path. By Proposition 5.1, each parallel bundle is entirely old or entirely new. The triangle BVC must lose one bundle; once that bundle is new, neither remaining bundle can be new because its reverse path would use the deleted bundle. There are exactly three cuts, B,C,V.

For cut B, there are b independent relations $\rho_\alpha=\sum_{i,j}t_{\alpha ij}V_iC_j$. Choose a nonzero vector v such that the contracted $b\times c$ matrix $T(v)$ has rank less than b, using Lemma 6.1 and $F>c$. Take a representation with dimension one at every vertex, all $C_j=0$, $V_i=v_i$, one A map equal to one, and one R map equal to one if $r>0$, or one E map equal to one otherwise. Its nonzero-arrow graph is connected, so its endomorphisms are scalars. Resolving the endpoint simples gives

$$\mathrm{Ext}^2_R(M,S_1)=\operatorname{coker}(T(v):\mathbb{C}^c\to\mathbb{C}^b)\neq 0,\qquad \mathrm{Ext}^2_R(S_3,M)=\mathbb{C}^b\neq 0.$$

The second displayed differential vanishes because all C maps vanish.

For cut C, interchange the roles of B,C. Set $B=0$, choose a nonzero V vector whose contracted $c\times b$ relation matrix has rank less than c using $F>b$, and retain the same sink attachments. This again gives a connected thin representation. The two second extension groups are $\mathrm{Ext}^2_R(M,S_4)=\mathbb{C}^c$ and a nonzero cokernel for $\mathrm{Ext}^2_R(S_1,M)$.

For cut V, there are F independent relations along CB. If $F>bc$, the relation space already has insufficient dimension. Otherwise, when $r>0$, set $B=0$ and one map in each of C,A,R equal to one. The nonzero graph is $4\to 1\to 2\leftarrow 3$, so the representation is indecomposable, and

$$\mathrm{Ext}^2_R(S_4,M)=\mathbb{C}^F,\qquad \dim\mathrm{Ext}^2_R(M,S_3)\geq F-b>0.$$

When $r=0$, use $e>0$, set $C=0$, and retain one map in each of B,A,E. The graph is $4\to 2\leftarrow 1\to 3$, and

$$\mathrm{Ext}^2_R(M,S_3)=\mathbb{C}^F,\qquad \dim\mathrm{Ext}^2_R(S_4,M)\geq F-c>0.$$

In these computations the nonzero relation differential has domain of dimension at most b or c, respectively. Every cut therefore contradicts quasitiltedness. □

7. Unit triangles and the small-height base

Lemma 7.1. *A four-vertex quiver containing a directed triangle of unit multiplicities is either a vortex or admits a path of at most two mutations to an acyclic quiver without increasing height.*

Proof. Unless the fourth vertex is a strict source or strict sink on all three triangle vertices, rotate the triangle and, if necessary, reverse all arrows and reflect the triangle to write the quiver as $(1, -1, c, 1, e, -s)$ with $c, e, s \geq 0$. This includes zero attachments: put a zero in the third position when the other two signs agree. Mutation at 3 gives $(0, 1, c - s, -1, e, s)$. For $c \geq s$ this is acyclic with order $1, 3, 2, 4$. Otherwise mutate at 1 to obtain $(0, -1, s - c, -1, e, c)$, acyclic with order $3, 1, 2, 4$. All new magnitudes are bounded by $\max(1, c, e, s)$. □

At height one, a cyclic quiver without a directed triangle must be a chordless unit four-cycle. A single mutation creates a unit triangle without increasing height, so Lemma 7.1 applies. Height zero is immediate. At height two, the classification in [1, Theorem 6.2] gives a height-two path to an acyclic representative for every mutation-acyclic input. This classification is independent of their general height theorem: its positive certificates have length at most three, and its negative certificates use triangle obstructions, vortex obstructions, invariant incompatibility, and the closed mutation orbit of the dreaded torus. Thus the peak argument below may assume initial height at least three and $F \geq 4$.

8. Heavy-edge obstructions

In the following constructions $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and e_1, e_2 are the standard column vectors of $\mathbb{C}^2$; a superscript $*$ denotes the corresponding coordinate functional. A representation is thin on its support if every nonzero vertex space is one-dimensional. If its graph of nonzero arrow maps is connected, its endomorphisms are scalars. The algebra $\mathbb{C}[J]$ is local, with unique maximal ideal generated by J. A finite-dimensional representation with local endomorphism algebra is indecomposable: a nontrivial direct-sum splitting would give a nontrivial idempotent, which a local algebra cannot contain.

Lemma 8.1. *For integers $a \geq 1$, $x \geq 2$, $u \geq 1$, the quiver $(a, x, -1, 0, -u, x)$ is mutation-cyclic.*

Proof. Its triangle has bundles $B : 1 \to 3$, $F : 3 \to 4$, $C : 4 \to 1$, of dimensions $x, x, 1$. The other arrows are $A : 1 \to 2$ and $E : 4 \to 2$. A tilted cut is one of B, C, F. The witnesses in the proof of Lemma 6.2 exclude B using $x > 1$, and exclude F using $u > 0$ and $x > 1$; those parts require only the indicated strict inequalities. For cut C the unique minimal relation has a nonzero coefficient matrix in BF. Choose bases to write it as $\sum_{i=1}^{k} B_i F_i$, where $1 \leq k \leq x$. If $k < x$, take a thin module with $A_1 = E_1 = F_{k+1} = 1$ and all other maps zero. Its nonzero graph is connected and both differentials for the unique relation are zero, so both $\mathrm{Ext}^2(M, S_4)$ and $\mathrm{Ext}^2(S_1, M)$ are nonzero. If $k = x$, take dimensions $(2, 2, 1, 2)$, set $A_1 = E_1 = \mathrm{id}_{\mathbb{C}^2}$, $B_1 = B_2 = e_2^*$, $F_1 = e_1$, $F_2 = -e_1$, and all other maps zero. The relation vanishes. The endomorphism equations give the same matrix $\lambda \,\mathrm{id}_{\mathbb{C}^2} + \mu J$ at vertices $1, 2, 4$ and scalar λ at 3. Thus the module is indecomposable. The two relation cokernels are $(\mathbb{C}^2)^*/\mathbb{C}e_2^*$ and $\mathbb{C}^2/\mathbb{C}e_1$, both nonzero. This excludes the final cut. □

Lemma 8.2. *If $a \geq 2$, $b \geq 1$ and $F > b$, then $(a, b, -1, 0, 1, F)$ is mutation-cyclic.*

Proof. Use bundles $A : 1 \to 2$, $B : 1 \to 3$, $C : 4 \to 1$, $E : 2 \to 4$, $V : 3 \to 4$. Their dimensions are $a, b, 1, 1, F$. The cycles are AEC and BVC. If C is old, an entire bundle

in each of A, E and B, V is new. New A would require $a \geq 2$ independent relations in the one-dimensional path space EC; new V would require $F > b$ relations in CB. Thus the only cut is $E + B$. If C is new, every other bundle is old. For cut $E + B$, basis changes make the quadratic relations $V_i C = 0$ for $1 \leq i \leq b$ and $CA_1 = 0$. The simple at 3 has minimal resolution

$$0 \longrightarrow P_2^b \longrightarrow P_1^b \longrightarrow P_4^F \longrightarrow P_3 \longrightarrow S_3 \longrightarrow 0.$$

The middle map is multiplication by C on the first b components, and its kernel consists of the b copies of the path A_1, since $CA_1 = 0$. No path leaves 2, so that kernel is precisely P_2^b. Applying $\mathrm{Hom}(-, S_2)$ gives $\mathrm{Ext}^3(S_3, S_2) = \mathbb{C}^b$. This contradicts global dimension at most two. For cut C, independent changes of bases put its relation in the form $A_1 E + \sum_{i=1}^{k} B_i V_i$, or omit the first summand, with $0 \leq k \leq b < F$. If both summands occur, use dimensions $(2, 1, 1, 2)$ and maps

$$A_1 = B_1 = e_2^*, \quad A_2 = e_1^*, \quad E = e_1, \quad V_1 = -e_1, \quad V_{k+1} = e_2.$$

Other maps are zero. The relation holds and its endomorphisms are scalars. Its second extension cokernels are $(\mathbb{C}^2)^*/\mathbb{C}e_2^*$ and $\mathbb{C}^2/\mathbb{C}e_1$. If only the BV summand occurs, take the thin module $A_1 = E = V_{k+1} = 1$, with the active B_i, V_i zero. If only $A_1 E$ occurs, so $k = 0$, take $A_2 = B_1 = V_1 = 1$ and $A_1 = E = 0$. These graphs are connected and the relation differentials vanish. The relation cannot have both summands zero. Every case violates quasitiltedness. □

Lemma 8.3. *For $a \geq 2$, $b, c, d, e \geq 1$ and $F > \max\{b + d, c + e, c + d\}$, the quiver $(a, b, -c, d, -e, F)$ is mutation-cyclic.*

Proof. Write the bundles as $A : 1 \to 2$, $B : 1 \to 3$, $C : 4 \to 1$, $D : 2 \to 3$, $E : 4 \to 2$, $V : 3 \to 4$. Their dimensions are a, b, c, d, e, F. Reverse old paths for new A, B, C, D, E, V have respectively types

$$DVC, \quad VC, \quad BV \text{ or } ADV, \quad VE \text{ or } VCA, \quad DV, \quad CB, ED \text{ or } CAD.$$

If V is new every other bundle is old. If V is old and A new, the cut is ABE. If A old, breaking BVC and DVE gives BD, CD, CE; the remaining choice BE leaves $CADV$ old. Thus there are exactly five cuts.

For cut V, take dimensions $(2, 2, 1, 1)$ and $A_1 = \mathrm{id}_{\mathbb{C}^2}$, $A_2 = J$, $C_1 = e_1$, $D_1 = e_2^*$, with every other map zero. All paths CB, ED, CAD vanish, and the endomorphism algebra is $\mathbb{C}[J]$. The F relation coordinates give

$$\dim \mathrm{Ext}^2(M, S_3) \geq F - b - d > 0, \qquad \dim \mathrm{Ext}^2(S_4, M) \geq F - c - e > 0.$$

Indeed each B or D functional is evaluated on a multiple of e_1, and each C or E variation is evaluated by a multiple of e_2^*, so each arrow contributes rank at most one.

For cut CD, the relations are c combinations of BV and d combinations of VE. Take a thin representation with $A_1 = B_1 = 1$, $E = 0$. The BV relations impose at most c linear equations on the vector v of V maps, leaving a space of dimension at least $F - c > e$. By Lemma 6.1, choose nonzero v for which the contracted $d \times e$ matrix $T(v)$ of VE has rank below d. The graph is connected. The two cokernels are $\mathrm{Ext}^2(S_3, M) = \mathbb{C}^d$ and $\mathrm{Ext}^2(M, S_2) = \mathrm{coker}\, T(v)$.

For cut CE, the relations have types BV, ADV (number c) and DV (number e). Set $A_1 = B_1 = 1$, $D = 0$ in a thin module. The first relations leave a subspace of V vectors of dimension at least $F - c > d$. Choose nonzero v such that the contracted $e \times d$ matrix of DV is rank deficient. The module is connected and $\mathrm{Ext}^2(S_2, M)$ is its nonzero cokernel. Projection of $\mathrm{Ext}^2(M, S_4)$ to the e quadratic relation coordinates has zero differential since $D = 0$. The cut BD is dual to this case: pass to the opposite algebra and relabel vertices by $(2, 1, 4, 3)$. The required dimension inequality becomes $F > d + c$, which is among the hypotheses.

For cut ABE, the old chain is $2 \xrightarrow{D} 3 \xrightarrow{V} 4 \xrightarrow{C} 1$. Let $T_E(t)$ and $T_B(t)$ be the $e \times d$ and $b \times c$ matrices obtained by contracting respectively the DV and VC relations with $t \in \mathbb{C}^F$. Impose the e equations from $D_1 = 1$ and choose a nonzero v in their kernel with $\operatorname{rank} T_B(v) < b$; the kernel has dimension at least $F - e > c$. Impose independently the b equations from $C_1 = 1$ and choose nonzero w with $\operatorname{rank} T_E(w) < e$, using $F - b > d$. Since $F \geq 3$, choose $u \notin \operatorname{span}(v, w)$. Define dimensions $(1, 1, 2, 2)$ and

$$D_1 = e_1, \quad C_1 = e_2^*, \quad V_i = \begin{pmatrix} v_i & u_i \\ 0 & w_i \end{pmatrix}.$$

All other maps vanish. The quadratic relations hold by the choices of v, w, and every cubic DVC path is zero. An endomorphism has

$$X_3 = \begin{pmatrix} \lambda_2 & x \\ 0 & \lambda_1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} \lambda_2 & t \\ 0 & \lambda_1 \end{pmatrix}, \quad (\lambda_2 - \lambda_1)u + tw - xv = 0.$$

Thus $\lambda_1 = \lambda_2$, and its endomorphism algebra is $\mathbb{C}$ or the dual numbers according as v, w are independent or dependent. Projecting the VC relation coordinates to evaluation at e_1 gives image $T_B(v)$ and hence $\operatorname{Ext}^2(M, S_1) \neq 0$. Projecting the DV coordinates to their second components gives image $T_E(w)$ and hence $\operatorname{Ext}^2(S_2, M) \neq 0$. Cubic coordinates do not affect either projection. All five cuts are excluded. □

Lemma 8.4. *Let $Q = (a, b, -c, d, E, F)$, with integers $a \geq 2$, $b, c \geq 1$, $d \geq 0$, $0 \leq E < c$, and $a, b, c, d, E < F$. Suppose $ac - E < F$, $b + ad \leq F$, and*

$$m = a^2+b^2+c^2+d^2+E^2+F^2+abd-acE-bcF+dEF-acdF > 2|p|, \quad p = aF-bE-cd.$$

Then Q is mutation-cyclic.

Proof. We first show $F \geq c + d + 2$. If $F \leq c + d + 1$ and $a \geq 3$, the height inequalities give $F \geq 2c + 2$ and $F \geq 3d + 1$, hence $d \geq c + 1$ and $c \geq 2d$, a contradiction. Thus $a = 2$. Define the integer gaps

$$B = b - 1, \quad Y = c - E - 1, \quad Z = F - 2c + E - 1,$$
$$r = d + c - F + 1, \qquad s = F - b - 2d.$$

They are nonnegative: $b \geq 1$, $E < c$, $2c - E < F$, the contrary bound $F \leq c + d + 1$, and $b + 2d \leq F$ give the five inequalities in order. Solving for the original parameters gives

$$d = Y + Z + 1 + r, \quad F = B + 2Y + 2Z + 2r + s + 3,$$
$$c = B + Y + Z + 2r + s + 1, \qquad E = B + Z + 2r + s.$$

Substitution yields

$$\begin{aligned} 2p - m = {} & B^3 + 4B^2Y + 4B^2Z + 5B^2r + 2B^2s + 2B^2 \\ & + 6BY^2 + 11BYZ + 14BYr + 5BYs + 9BY \\ & + 5BZ^2 + 13BZr + 5BZs + 6BZ + 8Br^2 + 6Brs + 9Br + Bs^2 + 4Bs + 2B \\ & + 4Y^3 + 10Y^2Z + 12Y^2r + 4Y^2s + 8Y^2 + 8YZ^2 + 20YZr + 7YZs + 13YZ \\ & + 12Yr^2 + 8Yrs + 18Yr + Ys^2 + 6Ys + 7Y + 2Z^3 + 8Z^2r + 3Z^2s + 4Z^2 \\ & + 10Zr^2 + 7Zrs + 13Zr + Zs^2 + 5Zs + 4Z \\ & + 4r^3 + 4r^2s + 9r^2 + rs^2 + 7rs + 6r + s^2 + 3s + 1. \end{aligned}$$

The right side is positive, contradicting $m > 2|p| \geq 2p$. Also $F > b + d$: use $b + ad \leq F$ if $d > 0$, and $b < F$ otherwise.

Use bundles $A : 1 \to 2$, $B : 1 \to 3$, $C : 4 \to 1$, $D : 2 \to 3$, $E : 2 \to 4$, $V : 3 \to 4$. Reverse old paths and the cycles $AEC, BVC, ADVC$ give exactly the cuts C, AB, AV, EV, BDE. With a zero bundle, delete it and discard a cut whose new arrow has no reverse path.

For cut C, its c relations have types AE, BV, ADV. Let $T(v)$ be the $c \times b$ matrix of the BV part. There is $v \neq 0$ with first column zero and rank below c. To see this, the first-column equations leave dimension at least $F - c$. If $c \geq b$, rank deficiency is automatic. If $c < b$, discard the zero column. The rank-deficient locus in $c \times (b-1)$ matrices has codimension $b - c$, while the projective image dimension, when the map is injective, is at least $F - c - 1 \geq b - c$. For $c = 1$, injectivity is impossible by dimension; a kernel vector suffices. Take dimensions $(2, 2, 1, 1)$ with $A_1 = \mathrm{id}_{\mathbb{C}^2}, A_2 = J, B_1 = e_2^*, D = E = 0, V = v$. The relations hold and the endomorphism algebra is $\mathbb{C}[J]$. The source relation cokernel is that of $T(v)$, so $\mathrm{Ext}^2(S_1, M) \neq 0$. In the opposite cokernel evaluate the c relation functionals at e_1. Only the E variations survive, of rank at most $E < c$, so $\mathrm{Ext}^2(M, S_4) \neq 0$.

For cut EV, choose nonzero $h \in \mathbb{C}^c$ annihilating the E coefficients of the CA relations at A_1. Use dimensions $(2, 2, 1, 1)$ with $A_1 = \mathrm{id}_{\mathbb{C}^2}, A_2 = J, B_1 = e_2^*, D = 0, C_j = h_j e_1$. All relations CA, CB, CAD vanish, and the endomorphism algebra is $\mathbb{C}[J]$. The V relation coordinates give cokernel dimensions at least $F - b - d$ in $\mathrm{Ext}^2(M, S_3)$ and $F - c$ in $\mathrm{Ext}^2(S_4, M)$, both positive.

For cut AB, the relations are b combinations of VC and a combinations of EC, DVC. Write the former as $\rho_\beta = \sum_{i=1}^{F} \sum_{j=1}^{c} t_{\beta ij} V_i C_j$, with $1 \leq \beta \leq b$. Their contraction with $v \in \mathbb{C}^F$ is the $b \times c$ matrix

$$T_B(v)_{\beta j} = \sum_{i=1}^{F} t_{\beta ij} v_i.$$

Choose $\alpha \in (\mathbb{C}^a)^* \setminus \{0\}$ on the a relation coordinates. Since $c > E$, choose $h \in \mathbb{C}^c \setminus \{0\}$ such that contraction with h kills the E coefficients of $\alpha(EC)$. Choose $w \in \mathbb{C}^F \setminus \{0\}$ satisfying the b contracted VC equations and the d equations of $\alpha(DVC)$, using $F > b + d$. Independently choose $v \neq 0$ with $\operatorname{rank} T_B(v) < b$ by Lemma 6.1, since $F > c$. Take $u \in \mathbb{C}^F$ outside their span; $F \geq c + d + 2 \geq 3$. Set dimensions $(1, 1, 2, 2)$, $C_j = h_j e_2^*$, and $V_i = \left(\begin{smallmatrix} v_i & u_i \\ 0 & w_i \end{smallmatrix}\right)$. If $d > 0$ set $D_1 = e_1, E = 0$; if $d = 0, E > 0$ set $E_1 = e_1$. The VC relations evaluate to their contraction at h, w times e_2^* and vanish by the choice of w. Every EC or DVC path is zero, since $C_j e_1 = 0$ and $C_j V_i e_1 = 0$. To check indecomposability when $d > 0$, the D_1 and nonzero C_j maps force

$$X_3 = \begin{pmatrix} \lambda_2 & y \\ 0 & z \end{pmatrix}, \qquad X_4 = \begin{pmatrix} z' & x \\ 0 & \lambda_1 \end{pmatrix}.$$

The equations $X_4 V_i = V_i X_3$, with $v, w \neq 0$, give $z' = \lambda_2$, $z = \lambda_1$, and

$$(\lambda_2 - \lambda_1) u + xw - yv = 0.$$

When $d = 0, E > 0$, the E_1 and C_j maps first give $X_4 = \left(\begin{smallmatrix} \lambda_2 & x \\ 0 & \lambda_1 \end{smallmatrix}\right)$. The lower-left equations for the nonzero vector w force the lower-left entry of X_3 to be zero; its diagonal equations, using $v, w \neq 0$, give λ_2, λ_1. The same displayed equation follows. In both cases $u \notin \mathrm{span}(v, w)$ forces $\lambda_1 = \lambda_2$. The remaining off-diagonal endomorphisms form a zero or one-dimensional square-zero ideal according as v, w are independent or dependent. The endomorphism algebra is therefore local. If $d = E = 0$, the cut has no reverse path for A and is impossible. Projection of $\mathrm{Ext}^2(M, S_1)$ onto first coordinates gives image $T_B(v)$; projection of $\mathrm{Ext}^2(S_2, M)$ by α has zero image by the choices of h, w. Both groups are nonzero.

For cut AV with $d, E > 0$, take a thin module with $C = 0$, $B_1 = D_1 = E_1 = 1$. Its graph is connected. The V relations give $\mathrm{Ext}^2(M, S_3) = \mathbb{C}^F$ and $\dim \mathrm{Ext}^2(S_4, M) \geq F - c > 0$. If $E = 0$, this cut has no reverse path for A. It remains to treat $d = 0, E > 0$. Here $m = a^2 + b^2 + c^2 + E^2 + F^2 - acE - bcF$. We claim $aF - bE > 0$. Otherwise $b < F$ implies

$a < E$, so $E \geq 3$ and $c \geq E+1$. We have

$$a^2 + 2E^2 - acE \leq a^2 + 2E^2 - aE(E+1).$$

The right side is convex in $a \in [2, E]$; its endpoint values are $4-2E < 0$ and $E^2(2-E) < 0$. Hence $a^2 + E^2 - acE < -E^2$. Regard the displayed formula for m as a convex quadratic in b. Our contrary inequality puts $b \in [aF/E, F]$. At these endpoints its values are

$$\begin{aligned} b = aF/E: &\quad c^2 + (a^2 + E^2 - acE)(1 + F^2/E^2) < c^2 - E^2 - F^2 < 0, \\ b = F: &\quad c^2 + (a^2 + E^2 - acE) + (2-c)F^2 < 0. \end{aligned}$$

The second inequality uses $c \geq 4$ and $F > c$. Convexity gives $m < 0$ throughout the interval, contradicting $m > 0$. If $E < a$, take $r = s = 1$, so $as - Er = a - E > 0$ and $Fr - bs = F - b > 0$. Otherwise put $r = 2a^2b, s = 2abE + 1$. The positive integer $aF - bE$ gives

$$as - Er = a > 0, \qquad Fr - bs = b\big(2a(aF - bE) - 1\big) > 0.$$

Moreover $c \geq 3$ and

$$1 < \frac{s}{r} = \frac{E}{a} + \frac{1}{2a^2b} \leq \frac{c-1}{2} + \frac{1}{8} < \frac{c}{2} \leq c - 1.$$

Thus in both cases

$$as - Er > 0, \quad Fr - bs > 0, \qquad r \leq s \leq (c-1)r.$$

There is an explicit indecomposable c-arrow representation $\mathbb{C}^r \to \mathbb{C}^s$: use first and last maps $[\mathrm{id}_{\mathbb{C}^r}; 0]$ and $[J_r; 0]$, where J_r is one nilpotent Jordan block and the zero blocks have $s-r$ rows. Assign the remaining $s-r$ sink basis vectors to distinct columns of the other $c-2$ maps, setting all unused columns to zero; the capacity is $(c-2)r \geq s-r$. The images span $\mathbb{C}^s$. The first map forces the sink endomorphism to restrict to the source endomorphism on the first r coordinates. The last forces the source endomorphism to commute with J_r. The spanning images then determine the sink endomorphism uniquely, so projection to the source component embeds the endomorphism algebra unitally in $\mathbb{C}[J_r]$. Every unital subalgebra of this truncated polynomial algebra is local: a nonzero constant term gives an inverse by a finite geometric series, and a zero constant term gives a nilpotent. Extend this representation by zero at vertices $2, 3$. Every EC, CB relation vanishes and

$$\dim \mathrm{Ext}^2(S_2, M) \geq as - Er > 0, \qquad \dim \mathrm{Ext}^2(M, S_3) \geq Fr - bs > 0.$$

Finally, for cut BDE, its old chain is $3 \xrightarrow{V} 4 \xrightarrow{C} 1 \xrightarrow{A} 2$. Let $T_B(v)$ denote the contraction matrix of this cut's own VC relations, with the same coefficient formula as above. Choose $h \in \mathbb{C}^c \setminus \{0\}$ annihilating the E coefficients of CA at A_1. Impose on $v \in \mathbb{C}^F$ the b contracted VC and d contracted VCA conditions, leaving a space L of dimension at least $F - b - d > 0$. Every $T_B(v)$ kills h. There is a nonzero $v \in L$ with rank below b: if $c \leq b$ this is automatic; otherwise discard the column corresponding to h. If the contraction map has a nonzero kernel, choose v there. Otherwise its projective image has dimension at least $F - b - d - 1 \geq c - b + 1$, exceeding the codimension $c - b$ of the deficient-rank locus in $b \times (c-1)$ matrices. For $b = 1$, injectivity is impossible by dimension. Take $A_1 = \mathrm{id}_{\mathbb{C}^2}, A_2 = J, C_j = h_j e_1, V_i = v_i$ in dimensions $(2, 2, 1, 1)$. All quadratic and cubic relations hold and the endomorphism algebra is $\mathbb{C}[J]$. The VC coordinates give $\mathrm{Ext}^2(M, S_1) = \mathrm{coker}\, T_B(v) \neq 0$. Their second components in $\mathrm{Ext}^2(S_3, M)$ have zero differential, since all C_j land in $\mathbb{C}e_1$. This excludes the final cut. □

9. Tensor and commuting-square obstructions

An arrow space is the complex span of one parallel bundle. We apply Proposition 5.1, and all arrow changes preserve its old/new grading.

The degree-three part of the cyclic Jacobian ideal of a rigid potential is the tangent image of linear changes of arrows: a cubic term in that ideal is a linear combination of an arrow times a derivative of the cubic part. Thus this tangent map is surjective onto the cubic cycle space. The linear arrow-change group consequently has an open orbit containing the cubic part. Every invariant nonempty open rank condition holds there, since the cubic cycle space is irreducible.

We also use path quotients with exactly one arrow from a bundle F omitted by all defining quadratic relations. For fixed endpoints and fixed lengths before and after that arrow, this quotient is the tensor product of the corresponding path quotient before F, the space F, and the path quotient after F. All paths with one F are the direct sum over these length choices. A multiple of a relation lies entirely on one side of F, which proves the factorization.

Lemma 9.1. *For integers $r \geq 2$, $E \geq 1$, and $w \geq 1$, the quiver*

$$(r+1, -1, -rE, r, E, -w)$$

is mutation-cyclic. The conclusion also holds for $r = 1$, $E = 1$, and $w \geq 1$.

Proof. Name its arrow bundles

$$D : 1 \longrightarrow 2, \quad A : 2 \longrightarrow 3, \quad B : 3 \longrightarrow 1,$$
$$E : 2 \longrightarrow 4, \quad C : 4 \longrightarrow 1, \quad F : 4 \longrightarrow 3.$$

Their multiplicities are $r+1, r, 1, E, rE, w$, respectively. Denote their arrow spaces by $D, A, B, \mathcal{E}, C, F$, so that the integer E is the dimension of $\mathcal{E}$. Suppose the quiver is mutation-acyclic and take its canonical tilted potential. The cubic part consists of the tensors for DAB and DEC, and its quartic part consists of $DEFB$. Fix arrow bases D_i $(1 \leq i \leq r+1)$, A_j $(1 \leq j \leq r)$, E_k $(1 \leq k \leq E)$ and C_l $(1 \leq l \leq rE)$, F_ν $(1 \leq \nu \leq w)$, and choose the unique B arrow as a basis. Write the cubic part as

$$W_3 = \sum_{i,j} \alpha_{ij} D_i A_j B + \sum_{i,k,l} \beta_{ikl} D_i E_k C_l.$$

With stars denoting the dual bases, contraction of these arrow tensors defines

$$U : A^* \longrightarrow D, \qquad U(A_j^*) = \sum_i \alpha_{ij} D_i,$$
$$V : C^* \longrightarrow D \otimes \mathcal{E}, \qquad V(C_l^*) = \sum_{i,k} \beta_{ikl} D_i \otimes E_k.$$

The open-rank conditions are

$$\operatorname{rank} U = r, \qquad \operatorname{rank} V = rE, \qquad (\operatorname{im} U \otimes \mathcal{E}) + \operatorname{im} V = D \otimes \mathcal{E}.$$

They are simultaneously nonempty: choose $\operatorname{im} U$ to be a hyperplane in D and U an isomorphism onto it. Choose $\operatorname{im} V$ to contain the complementary E-dimensional tensor space and $(r-1)E$ independent vectors in $\operatorname{im} U \otimes \mathcal{E}$, and choose V an isomorphism onto that image. They therefore hold by rigidity.

Each quartic coefficient in $\operatorname{im} U \otimes \mathcal{E}$ is removed by replacing each A_j by itself plus a suitable linear combination of paths $E_k F_\nu$. Each coefficient in $\operatorname{im} V$ is removed by replacing each C_l by itself plus a suitable linear combination of paths $F_\nu B$. Take preimages homogeneous in the cut grading. The changes have triangular inverses and generate no higher terms, since the quartic part contains neither A nor C. Thus the potential becomes cubic and contains no F. The degree-zero algebra is unchanged by these graded substitutions. Its minimal relation numbers therefore remain equal to the numbers of new arrows. If an F arrow were new, its zero derivative would leave fewer generators than this minimal number. Thus every F arrow is old.

There are precisely four possible cuts:

$$D, \qquad A+E, \qquad B+E, \qquad B+C.$$

If D is new, the two nonzero cubic tensors force A, B, E, C old. If D is old, degree one of the two cubic tensors requires exactly one of A, B and exactly one of E, C to be new. The choice $A+C$ leaves the directed cycle $DEFB$ entirely old and is impossible. These give the four displayed cuts.

Assume first $r \geq 2$. In cut D, use the paths $DEFBD$. The DE quotient has dimension $(r+1)E - rE = E$, and the BD quotient has dimension $(r+1) - r = 1$. Hence this space has dimension $Ew > 0$ and contains two new D arrows. In cut $A+E$, use $EFBDA$. Quotienting BD by the r relations $\partial_A W$ leaves one dimension, and therefore BDA has dimension r. Its only additional relation placement is $B(\partial_B W)$, which already vanishes because the D directions of $\partial_B W$ belong to $\operatorname{im} U$. Thus the space has dimension $Ewr > 0$. In cut $B+E$, the paths EFB have dimension $Ew > 0$ and no relation on either side of F.

In cut $B+C$, use $CDEFB$. The CDE space has initial dimension $rE(r+1)E$. Quotienting by $C(\partial_C W)$ leaves dimension rE^2. The only remaining relation placement is $(\partial_E W)E$, imposing at most E^2 conditions. Indeed its consecutive pairs are CD, DE, so no derivative with other endpoints applies. Its quotient dimension is at least $(r-1)E^2 > 0$. The one-F space therefore has dimension at least $(r-1)E^2 w > 0$. In each case its paths contain two new arrows, contradicting the square-zero ideal.

It remains to handle $r = E = 1$; the following argument works for all $r \geq 1$ when $E = 1$. Write E_1 for the unique arrow in the E bundle. The two rank-r images in D span D and intersect in dimension $r-1$. Homogeneous bases put the cubic potential in the form

$$D_0 A_0 B + \sum_{i=1}^{r-1} D_i(A_i B + E_1 C_i) + D_r E_1 C_r.$$

The cuts remain the four listed above. For cut D, the old algebra has relations $A_0 B$, $A_i B + E_1 C_i$ for $1 \leq i < r$, and $E_1 C_r$. Take the thin module with $B = E_1 = F_1 = 1$ and all A, C and other F zero. Its nonzero support is the connected path $2 \to 4 \to 3 \to 1$, so its endomorphism algebra is $\mathbb{C}$. The simple relation-resolution complexes give a nonzero coordinate $A_0 B$ in $\operatorname{Ext}^2(M, S_1)$, because only the C variations can hit the other coordinates; dually $E_1 C_r$ survives in $\operatorname{Ext}^2(S_2, M)$, since only the A variations hit the other coordinates. Thus this indecomposable has both projective and injective dimension two, impossible for a tilted algebra.

For cut $B+E$, $E_1 F_1 B$ survives. For cut $A+E$, take the quotient setting every arrow to zero except the five arrows E_1, F_1, B, D_r and A_0. The remaining relations are $D_r E_1 = 0$ and $A_0 B = 0$, and $E_1 F_1 B D_r A_0$ survives. For cut $B+C$, take the quotient retaining only C_r, D_0, E_1, F_1, B; the relations are $BD_0 = 0$ and $E_1 C_r = 0$, and $C_r D_0 E_1 F_1 B$ survives. Each path contains two new arrows. This excludes every cut. □

Lemma 9.2. *Let $d \geq 3$, $u \geq 1$, and $v, f \geq 1$ be integers satisfying $v \leq f \leq (d-1)v$. Then*

$$(d, -(d-1), -v, 1, f, u)$$

is mutation-cyclic.

Proof. Use arrow spaces $D : 1 \to 2$, $A : 2 \to 3$, $B : 3 \to 1$, $E : 2 \to 4$, $C : 4 \to 1$, $Z : 3 \to 4$ of dimensions $d, 1, d-1, f, v, u$. Suppose that a canonical tilted potential exists. Its cubic tensors are DAB and DEC, and its quartic tensor is $DAZC$. Fix bases D_i $(1 \leq i \leq d)$, B_j $(1 \leq j \leq d-1)$, E_k $(1 \leq k \leq f)$, C_l $(1 \leq l \leq v)$, Z_ν $(1 \leq \nu \leq u)$, and the unique A arrow. If

$$W_3 = \sum_{i,j} \alpha_{ij} D_i A B_j + \sum_{i,k,l} \beta_{ikl} D_i E_k C_l,$$

the contraction maps, with stars denoting dual bases, are

$$U : B^* \longrightarrow D, \qquad U(B_j^*) = \sum_i \alpha_{ij} D_i,$$

$$V : E^* \longrightarrow D \otimes C, \qquad V(E_k^*) = \sum_{i,l} \beta_{ikl} D_i \otimes C_l,$$

with ranks $d-1, f$, respectively, and

$$(\operatorname{im} U \otimes C) + \operatorname{im} V = D \otimes C.$$

These conditions are nonempty: the projection of V onto the v-dimensional quotient by $\operatorname{im} U \otimes C$ can be surjective since $v \le f < dv$. Replace each B_j by itself plus a linear combination of paths $Z_\nu C_l$, and each E_k by itself plus a linear combination of paths AZ_ν. Preimages under the displayed sum-of-images map choose the coefficients to remove the quartic part. They preserve cut degrees and introduce no higher terms. The graded substitutions preserve the old algebra and its minimal relation numbers. A new Z arrow with zero derivative would lower the number of generating relations below that minimum. Hence all Z arrows are old, and all relations are quadratic and omit Z.

The possible cuts are $D, B+C, A+E, A+C$. If D is new, both cubic tensors force A, B, E, C old. If D is old, exactly one bundle from each of A, B and E, C is new. The choice $B+E$ leaves the cycle $DAZC$ entirely old. Excluding that choice gives the stated list.

In cut D, the one-Z paths $DAZCD$ have dimension $u(dv-f) > 0$: DA has quotient dimension one and CD has quotient dimension $dv-f$. In cut $B+C$, use $BDAZC$. The DA relations leave one D direction, so BDA has dimension $d-1$; the additional placement $(\partial_A W)A$ is already killed by those relations. Hence the quotient has dimension $(d-1)uv > 0$. In cut $A+C$, AZC has dimension $uv > 0$.

For cut $A+E$, use $AZCDE$. In CDE, the relations $(\partial_E W)E$ leave dimension $f(dv-f)$, and $C(\partial_C W)$ impose at most v^2 more conditions. These are all placements, since the consecutive pairs are CD, DE. The quotient dimension is at least

$$f(dv-f) - v^2 > 0.$$

For the strict inequality, the convex quadratic $v^2 + f^2 - dvf$ has value $(2-d)v^2 < 0$ at both endpoints of $[v, (d-1)v]$. Every displayed path space contains two new arrows, contradicting square zero in each cut. □

Lemma 9.3. *Let $2 \le d < e < c < f$ be integers, and put $g = f - c$. If $de - g > \max\{c, g\}$, the quiver $(0, 1, -c, d, -e, f)$ is mutation-cyclic.*

Proof. Write its bundles as $B : 1 \to 3$, $C : 4 \to 1$, $D : 2 \to 3$, $E : 4 \to 2$, $F : 3 \to 4$, of dimensions $1, c, d, e, f$. Suppose it is mutation-acyclic. Its canonical potential is cubic, since every simple directed cycle has length three. Fix the unique B arrow. Up to cyclic rotation its cubic tensor space is

$$F \otimes \big(C \oplus (E \otimes D)\big).$$

A bundle flattening means contraction with the dual of that bundle into the tensor space of the remaining arrows. For C, D, E, F the maps have targets F, $F \otimes E$, $F \otimes D$ and $C \oplus (E \otimes D)$, respectively. Injectivity is a nonempty open rank condition in each case, because

$$c < f, \quad d < ef, \quad e < df, \quad f < c + de.$$

The last inequality follows from $de > c + g = f$. The intersection of these four invariant opens is nonempty, so rigidity gives all four injective flattenings for the actual cubic tensor.

If F is new, the two cubic tensors force B, C, D, E old. If F is old, one of B, C and one of D, E must be new. Thus the cuts are

$$F, \quad BD, \quad BE, \quad CD, \quad CE.$$

The C flattening identifies its image with a c-dimensional subspace $U \subset F$. Choose a basis $u_1, \dots, u_c$ of U, a complementary space G with basis $G_1, \dots, G_g$, and a compatible basis $C_1, \dots, C_c$ such that

$$W = \sum_{i=1}^{c} C_i B u_i + \sum_{\ell=1}^{f} F_\ell q_\ell(E, D), \qquad F_i = u_i, \quad F_{c+a} = G_a.$$

Here $1 \le i \le c$, $1 \le a \le g$, and

$$q_\ell(E, D) = \sum_{j=1}^{e} \sum_{k=1}^{d} \gamma_{\ell jk} E_j D_k, \qquad q_{G_a} = q_{c+a}.$$

The quadratic relations are Bu_i from the C_i derivatives, $\sum_i u_i C_i$ from the B derivative, $C_i B + q_i$ and q_{G_a} from the F derivatives, and the d derivatives of type FE and e derivatives of type DF.

For cut BE, the space BFE has dimension at least $fe - ce - d = eg - d > 0$: its relation multiples are exactly $(\partial_{C_i} W)E$ and $B(\partial_{D_j} W)$. For cut CD, the space DFC has dimension at least $dfc - ec - d = c(df - e) - d > 0$; the placements are $(\partial_{E_k} W)C$ and $D(\partial_B W)$. Both spaces contain two new arrows.

For cut BD, consider length-four paths $BF(CB \text{ or } ED)$. First $Bu_i = 0$ removes the U part, and $C_i B = -q_i$ eliminates the CB endings. The remaining space is $G \otimes E \otimes D$. The remaining relation placements are $BG_a q_{G_b}$ and $B(\partial_{D_j} W)D_k$, imposing at most $g^2 + d^2$ conditions. The placement from $\partial_B W$ is already zero; $\partial_E W$ has incompatible endpoints. Thus the quotient dimension is at least

$$g(de - g) - d^2 \ge de - 1 - d^2 \ge d - 1 > 0.$$

The first inequality uses that $g(de - g)$ is increasing for integers $1 \le g < de/2$. Again every path contains two new arrows. These three cuts contradict square zero.

For the other cuts we use the following hereditary representation terminology. The Auslander–Reiten quiver has indecomposable modules as vertices and irreducible maps as arrows; its translation τ is determined by the almost split sequences $0 \to \tau M \to E \to M \to 0$ for nonprojective M. An indecomposable is preprojective if it has the form $\tau^{-n} P_i$, preinjective if it has the form $\tau^n I_i$, with $n \ge 0$, and regular if it is neither. A module is directing if there is no cycle of nonzero nonisomorphisms between indecomposable modules starting and ending at it. A complete slice is a finite sincere section of an Auslander–Reiten component: it is connected and convex and meets every translation orbit of that component once; sincere means that its summed dimension vector has every coordinate positive. A quasisimple regular module is an indecomposable at the boundary of a regular component, of regular quasi-length one.

Let H be the hereditary path algebra of $1 \to 3 \leftarrow 2$, with d arrows on the second arm. For a representation M, its dimension vector is the column $(\dim M_1, \dim M_2, \dim M_3)^t$. The superscript t denotes transpose. The Euler form $\langle x, y \rangle = x^t E_H y$ satisfies

$$\langle \dim M, \dim N \rangle = \dim \operatorname{Hom}_H(M, N) - \dim \operatorname{Ext}^1_H(M, N).$$

Its Euler matrix, Coxeter matrix, and quadratic form are

$$E_H = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -d \\ 0 & 0 & 1 \end{pmatrix}, \quad \Phi = -E_H^{-1} E_H^t = \begin{pmatrix} 0 & d & -1 \\ d & d^2 - 1 & -d \\ 1 & d & -1 \end{pmatrix}, \quad q_H(x) = x^t E_H x.$$

Put $v = (c, e, de-g)^t$. Then $\Phi v = (g, df-e, f)^t$. The connected mutation-acyclic invariant inequality in the proper cyclic order gives

$$m = 1 + c^2 + d^2 + e^2 + f^2 - f(c + de) > 2(cd - e).$$

Consequently

$$q_H(v) = m - 1 - d^2 > 1,$$

since $cd - e \geq d^2 + d - 1 \geq d^2$. Direct multiplication gives $\Phi^t E_H \Phi = E_H$, hence $q_H(\Phi v) = q_H(v)$. Kac's root theorem [11] excludes an indecomposable H-module of either dimension v or Φv.

We need also exclude decomposable extension modules. We prove $\Phi^k v > 0$ componentwise for every $k \in \mathbb{Z}$. Write $\Phi^k v = (x_k, y_k, z_k)^t$, put $a = d^2 - 1$, and set $\delta_k = dx_k - y_k$, $s_k = x_k + dy_k = z_k + z_{k+1}$. Matrix multiplication gives

$$\delta_k = (-1)^k \delta_0, \qquad z_{k+2} = a z_{k+1} - z_k, \qquad s_{k+2} = a s_{k+1} - s_k.$$

We have $z_0 = de - g > \max(c, g)$ and $z_1 = f = c + g$, whence $1/2 < z_0/z_1 < d$. Therefore

$$\frac{2}{a} < \frac{s_1}{s_0} = \frac{d^2 - z_0/z_1}{1 + z_0/z_1} < a - \frac{2}{a}.$$

For the lower bound use $z_0/z_1 < d$ and

$$\frac{d(d-1)}{d+1} - \frac{2}{d^2 - 1} = \frac{(d-2)(d^2+1)}{(d-1)(d+1)} \geq 0;$$

for the upper bound use $z_0/z_1 > 1/2$ and $(2a+1)/3 \leq a - 2/a$ for $a \geq 3$. It follows that $s_2 > s_0$, $s_{-2} > s_0$, and

$$s_3 - s_1 = \big((a^2 - 2)s_1/s_0 - a\big)s_0 > 0, \quad s_{-3} - s_{-1} = \big((a^2 - 2)(a - s_1/s_0) - a\big)s_0 > 0.$$

Every parity subsequence satisfies the recurrence with coefficient $a^2 - 2 \geq 7$, so the even terms have minimum s_0, and the odd terms on the positive and negative half-lines have minima s_1 and s_{-1}, respectively. Reconstruct the first two coordinates by

$$x_k = \frac{s_k + d(-1)^k \delta_0}{d^2 + 1}, \qquad y_k = \frac{d s_k - (-1)^k \delta_0}{d^2 + 1}.$$

The required base vectors are

$$v_0 = (c, e, z_0)^t, \quad v_1 = (g, df - e, f)^t, \quad v_{-1} = (z_0 - c, dz_0 - e, az_0 - f)^t,$$

all strictly positive, using $z_0 > c > e$ and $z_0 > f/2$. Hence $x_k, y_k > 0$. Also $z_2 > (a-d)f \geq f$ and $z_{-1} > (a-2)z_0 \geq z_0$, and the recurrence proves positivity of every z_k.

For a right H-module M, the one-point extension $H[M]$ is the triangular algebra

$$H[M] = \begin{pmatrix} \mathbb{C} & M \\ 0 & H \end{pmatrix}, \qquad \begin{pmatrix} \lambda & m \\ 0 & h \end{pmatrix} \begin{pmatrix} \lambda' & m' \\ 0 & h' \end{pmatrix} = \begin{pmatrix} \lambda\lambda' & \lambda m' + m h' \\ 0 & hh' \end{pmatrix}.$$

Its added vertex is a source, and its new projective has radical M. The dual construction adds a sink and is called a one-point coextension. A derived hereditary component is called transjective when it contains a shift of a projective module.

We use the one-point-extension criterion of Happel–Slungård [9, Lemma 2]: if $H[M]$ is quasitilted, either M is indecomposable quasisimple regular, or its indecomposable summands are directing and lie on a complete slice. The first case is excluded by $q_H(\dim M) > 1$. In the second, the finite complete slice is transjective [10, Lemma 9 and Corollary 10]. The underlying hereditary graph here is of infinite representation type: its symmetric Cartan matrix has determinant $6 - 2d^2 < 0$. A transjective component in its module category is therefore a preprojective or preinjective component, and all the slice summands belong to that one component. The matrix Φ represents τ on the Grothendieck group of the

bounded derived category. In particular $\Phi \dim P_i = -\dim I_i$. For a preprojective $\tau^{-r}P_i$ and every $n \geq 0$,

$$\Phi^{r+1+n} \dim(\tau^{-r}P_i) = -\dim(\tau^n I_i).$$

These injective translates remain modules because H is connected of infinite type. Dually,

$$\Phi^{-(r+1+n)} \dim(\tau^r I_i) = -\dim(\tau^{-n}P_i).$$

Choose a single exponent beyond the finitely many crossing times. The summed dimension vector is then componentwise nonpositive and nonzero, contradicting positivity of every integral Coxeter iterate proved above. The same argument applies to Φv and, by duality, to H^{op}.

For cut F, the old algebra is the one-point extension of H at source 4. There are c, e arrows from 4 to $1, 2$, and f independent relations among the $c + de$ paths to 3. The extension module is the restriction to H of the radical of the projective at 4. It has c basis paths ending at 1, e ending at 2, and $c + de$ paths ending at 3, modulo the f independent relations. Its dimension vector is therefore $(c, e, c + de - f)^t = v$, which the preceding criterion excludes. For cut CE, the old algebra is a one-point coextension at sink 4 with f arrows from 3 and respectively c, e independent relations BF, DF. Its coextension module is the dual of the restriction of the corresponding left projective, or equivalently is computed from paths to the new sink. At vertex 3 it has dimension f; the c independent BF relations and e independent DF relations leave dimensions $f - c$ and $df - e$ at vertices $1, 2$. Thus its dimension vector is $(f - c, df - e, f)^t = \Phi v$, also excluded. Every cut is impossible. □

10. Height bounds in two-vertex mutation orbits

Lemma 10.1. *Let K be a nonnegative integer. Suppose a connected mutation-acyclic four-vertex quiver has $b_{12} = 0$. Two states of height at most K in its $1, 2$ orbit can be joined within height K.*

Proof. The two mutations commute and are involutions, so their orbit has at most four labeled states. Adjacent low states are joined by one mutation. For opposite low states, choose a high intermediate corner if needed. Each entry incident with 1 or 2 is bounded by K, because one of its low neighbours reverses that entry. Hence only b_{34} can exceed K. Reverse arrows and interchange $3, 4$ to make it positive. Both mutations must lower it, so their attachment paths are $4 \to 1 \to 3$ and $4 \to 2 \to 3$. This normalizes the corner to $(0, b, -c, d, -e, f)$, where $1 \leq b, c, d, e \leq K < f$. Its low neighbours give $|f - bc|, |f - de| \leq K$. The other corner is low unless $bc + de > f + K$. Suppose this failure holds. Lemma 3.1 identifies the transported Markov value, so

$$m = b^2 + c^2 + d^2 + e^2 + f^2 - f(bc + de), \qquad p = be - cd.$$

If $b, c, d, e \geq 2$, interchange the two pairs so $de \leq bc$. Since

$$4(b^2 + c^2) - (bc)^2 \leq 16, \qquad 4(d^2 + e^2) \leq 2(de)^2 \leq 2bcde,$$

the identity

$$\begin{aligned} 4m = {} & 4(b^2 + c^2) - (bc)^2 + 4(d^2 + e^2) - 2bcde \\ & - (2f - bc)(bc + 2de - 2f) \end{aligned}$$

gives $4m \leq 16 - (2de - 1) \leq 9$. The low-neighbour bounds give $bc \leq f + K < 2f$ and $de \geq f - K$. The assumed failure gives $bc + 2de > f + K + de \geq 2f$. Thus both factors are positive integers summing to $2de$, and their product is at least $2de - 1$. This contradicts $m \geq 3$.

The symmetries $(b,c,d,e) \mapsto (c,b,e,d)$ and $(b,c,d,e) \mapsto (d,e,b,c)$ place a unit entry at $b = 1$. Put $u = f - c$, $v = de - f$. Then

$$de = c + u + v, \quad 1 \le c,d,e,u,v \le K, \quad K < c + u, c + v, \quad 2u, 2v < de.$$

If $d = 1$, then $de = e \le K < c + u = f$, contrary to $v > 0$; the same argument excludes $e = 1$. Lemma 3.1 and direct expansion give

$$m = (d - c)^2 + e^2 + 1 - cd(e - 2) - uv, \qquad p = e - cd.$$

If $c = 1$, then $u = v = K$ and $de = 1 + 2K$. Both d, e are odd and at least three; thus $(d^2 - 4)(e^2 - 4) \ge 25$ yields $m \le -K - 1 < 3$. Assume $c \ge 2$. Interchange u, v to have $u \le v$. The identity

$$\begin{aligned}(d - e)^2 + 4 - m = (de - 2v - 1)(u + v - 2) + (u - 1)^2 \\ + 3(u - 1)(v - u) + (v - u)^2\end{aligned}$$

shows $m \le (d - e)^2 + 4$.

If $c < \max(d, e)$, the expression for m and the bounds used in this paragraph are symmetric in d, e, so take $d \ge e$ and $c < d$. The inequalities $K < c + u, c + v$ and $d \le K$ give $u, v \ge d - c + 1$. Hence

$$\begin{aligned}m &= c^2 + d^2 + e^2 + 1 - cde - uv \\ &\le e^2 - 2(d - c) - cd(e - 2) \\ &\le e(4 - e) - 2(d - c) \le 2.\end{aligned}$$

The second inequality uses $cd \ge 2e$ and $e \ge 2$. If $e \le d \le c$, then

$$\begin{aligned}2(cd - e) - (d - e)^2 - 4 &\ge (d - e)^2 + 4e(d - e) + 2(e - 2)(e + 1) \\ &\ge 0,\end{aligned}$$

contradicting $m > 2|p|$. If $d < e = c$, then

$$m = d^2 - (d - 2)e^2 + 1 - uv \le 4 < 2e(d - 1) = 2|p|.$$

For $d \ge 3$ the left side is already negative; for $d = 2$ use $uv \ge 1$. Finally $d < e < c$ gives $g = f - c = u$ and

$$de - g = c + v > K \ge \max(c, g).$$

Lemma 9.3 excludes this last case. Thus the other corner is low. □

Lemma 10.2. *Let K be a nonnegative integer and let Q be a connected mutation-acyclic four-vertex quiver. Two states of height at most K in the orbit of Q generated by mutations at the endpoints of a unit edge can be joined, up to interchanging those endpoints, by at most three mutations within height K.*

Proof. The pentagon relation gives five states modulo interchange of the pair. A shortest route has length at most two except when its intermediate state is high, in which case the other side of the pentagon has length three. It suffices to bound this alternative route. Every entry incident with 1 or 2 is bounded by K, since one of the two low neighbours reverses it. Only b_{34} can be high. Reverse arrows and interchange $3, 4$ to arrange $b_{12} = 1$ and $b_{34} > K$. Both low neighbours must lower that entry, requiring paths $4 \to 1 \to 3$ and $4 \to 2 \to 3$. Thus the high corner is

$$P = (1, b, -c, d, -e, f), \qquad b, c, d, e \ge 1, \quad f > K.$$

Its low neighbours are

$$(-1, -b, c, d, -c - e, f - bc), \qquad (-1, b + d, -c, -d, e, f - de).$$

Hence $b+d, c+e \le K$ and $|f-bc|, |f-de| \le K$. Put $D = bc+de+cd$. The alternative intermediate states are

$$(1, -b, -e, -d, c+e, f-D), \qquad (-1, b, e, -b-d, c, f-D).$$

Only $f-D$ needs attention, and its upper bound is immediate. Suppose $f-D < -K$. Then

$$K+1 \le f \le D-K-1, \qquad D \ge 2K+2,$$

while Lemma 3.1 gives

$$m = 1+(b+d)^2+(c+e)^2-bd-ce+f^2-Df, \qquad p = f+be-cd.$$

First assume $c, d \ge 2$. Write $X = \max(b+d, c+e)$, $Y = \min(b+d, c+e)$. Then $X \ge Y \ge 3$, $bd+ce \ge 4$, and

$$D \ge 2(b+d)+2(c+e)-4 = 2X+2Y-4.$$

Indeed, the difference equals $(b-1)(c-2)+(e-1)(d-2)+(c-1)(d-1)-1 \ge 0$. The interval for f implies

$$f(D-f) \ge (X+1)(D-X-1) \ge (X+1)(X+2Y-5).$$

Therefore

$$m \le 6+4X+Y^2-2XY-2Y-bd-ce \le 2+2Y-Y^2 \le -1.$$

Now let $d = 1$. The cases $b = 1$ and $c = 1$ give $D \le 2K$, so $b, c \ge 2$. Put $S = b+1$ and $U = c+e$, the two attachment sums bounded by K. The convex quadratic $m-2p$ takes its largest endpoint value on $[K+1, D-K-1]$ at $f = K+1$. This endpoint bound decreases as K increases over $\max(S,U) \le K \le (D-2)/2$, since its derivative in K is $2K-D < 0$. Thus take $K = \max(S,U)$.

For $S \ge U$, write $c = B+2$, $e = R+1$, $S = c+e+Z$, where $B, R, Z \ge 0$. Expansion of the resulting bound gives

$$\begin{aligned} m-2p \le\ & -5-11B-6B^2-B^3-10R-9BR-2B^2R-2R^2-BR^2 \\ & -6Z-7BZ-2B^2Z-3RZ-2BRZ-BZ^2 < 0. \end{aligned}$$

For $U \ge S$, its bound is the convex polynomial

$$h(U) = 1+(U-S)^2+(U-S)+c^2+c-Sc(U-1)$$

on $\max(S, c+1) \le U \le c(S-1)-2$. If the left endpoint is S, then $c \le S-1$ gives

$$h(S) \le 1-Sc(S-2) \le -5.$$

For the other possible left endpoint, write $S = B+3$, $c = B+Z+2$, with $B, Z \ge 0$. Then

$$h(c+1) = -5-11B-6B^2-B^3-6Z-8BZ-2B^2Z-Z^2-BZ^2 < 0.$$

At the right endpoint write $S = B+3$, $c = R+2$; expansion gives

$$h(c(S-1)-2) = 1-15B-3B^2-12R-14BR-2B^2R-R^2-BR^2 < 0.$$

The excluded case $B = R = 0$ would make the interval empty; otherwise $B+R \ge 1$. Convexity proves negativity throughout. Each case contradicts $m > 2|p|$. Finally the symmetry $(b, c, d, e) \mapsto (e, d, c, b)$ reduces $c = 1$ to $d = 1$. Hence $f-D \ge -K$, and both alternative intermediates are low. □

11. Removing the first height peak

We use the necessary mutation-acyclicity conditions from Section 3. The local formulas below evaluate m in the totally proper class.

Proof of Theorem 1.1. The small-height cases were treated in Section 7. Assume first that Q is connected and $\mathrm{ht}(Q) \geq 3$. Choose a path from Q to an acyclic quiver minimizing lexicographically (F, N, L), where F is its maximum height, N counts its states of height F, and L is its length. Such a minimum exists because these quantities are nonnegative integers. Suppose $F > \mathrm{ht}(Q)$, put $T = F - 1$, and call states of height at most T low. Let C precede the first high state $P = \mu_i C$. The path has no adjacent identical mutation letters. Moreover, P is not its acyclic endpoint, by Lemma 4.1. Let $j \neq i$ be the next letter.

We will use two elementary estimates. For $1 \leq r, s \leq F - 1$, separate convexity on this rectangle gives

$$r^2 + s^2 - Frs \leq 2 - F.$$

Indeed the maximum is at a corner; the values at $(1,1)$, $(1, F-1)$ and $(F-1, F-1)$ are $2-F$, $2-F$ and $(2-F)(F-1)^2$. If in addition $rs = 2F$, then $r, s \geq 3$. The inequalities $(r-3)(F-1-r) \geq 0$ and its counterpart for s give

$$r^2 + s^2 \leq (F+2)(r+s) - 6F + 6.$$

Since $(r-3)(s-3) \geq 0$, we have $3(r+s) \leq 2F + 9$, and consequently

$$3(r^2 + s^2) \leq 2F^2 - 5F + 36. \tag{1}$$

Every replacement below may identify its endpoint with the original endpoint only after a vertex permutation. Apply that permutation to the entire remaining tail. Heights and the acyclicity of the endpoint are unchanged, so a reduction in the number of high states contradicts the choice of (F, N, L), even if the replacement is longer.

11.1. **A missing edge.** Suppose $b_{ij} = 0$ and relabel the pair as $1, 2$. The low neighbour $\mu_1 P = C$ reverses the entries incident with 1 and leaves every entry incident with 2 unchanged. Thus all these entries are low, and only 34 can have magnitude F. Interchange $3, 4$ to make it positive. Its decrease at 1 forces the path $4 \to 1 \to 3$, giving

$$P = (0, b, -c, d, E, F), \qquad b, c \geq 1, \quad b, c, |d|, |E| \leq T.$$

The sign $d < 0 < E$ is impossible because mutation at 2 would increase the 34 entry. Mutation at 2 leaves the $13, 14$ entries unchanged and reverses the low $23, 24$ entries. Thus a high $\mu_2 P$ must have its 34 entry equal to F or $-F$. The change at that entry is $[-d]_+[E]_+ - [d]_+[-E]_+$. The value $-F$ can occur only when $d > 0, E = -e < 0, de = 2F$. In this rectangle

$$m = (b^2 + c^2 - Fbc) + (d^2 + e^2) - F^2.$$

The preceding estimates give

$$3m \leq 3(2 - F) + (2F^2 - 5F + 36) - 3F^2 = 42 - 8F - F^2 < 0,$$

contradicting $m \geq 3$. Thus

$$P' = (0, b, -c, -d, -E, F), \qquad \mu_1 P' = \mu_2 C \text{ is low.}$$

If mutation at 3 from P stays within height F, the entries $bF - c$ and $E + F[d]_+$ imply $b = 1, d \leq 1$, and $E \leq 0$ when $d = 1$. If $\mu_2 P$ is high, the possibility $d = 1, E < 0$ would instead make its 34 entry $F + E$ low and is excluded. In the remaining cases $\mu_1 P = (0, -1, c, d, E, F - c)$ is acyclic. If $d \leq 0, E \leq 0$, a topological order is $3, 1, 4, 2$. If $d = 0, E > 0$ or $d = 1, E = 0$, a topological order is $2, 3, 1, 4$. These ranges exhaust the possibilities, including zero attachments. The same statement holds for exit 4 by reversing arrows and interchanging $3, 4$. Apply this observation at P'. An exit at a third vertex

gives an acyclic low shortcut $\mu_2 C$; otherwise the next letter is 1, returning low. Neither P nor P' is an endpoint since both contain triangle 134. The word 121 is replaced by 2.

If $\mu_2 P$ is already low, Lemma 10.1 supplies the other low side of the commuting square, with exactly the same endpoint. This removes the high state and completes the missing-edge case.

11.2. **At least two arrows.** Suppose the first pair carries at least two arrows. Relabel the pair as $1, 2$ and reverse all arrows if needed so that $b_{12} = a \geq 2$. Every entry incident with 1 has magnitude at most T, since mutation at 1 reverses it and reaches C. No entry incident with 2 can have magnitude F. Indeed, interchange $3, 4$ to put such an entry at 24. Its cancellation at 1 would force $b_{24} = F$ and $b_{14} < 0$: the change at 24 is $-a[-b_{14}]_+$. Mutation at 2 would then give

$$b_{14}(\mu_2 P) = b_{14}(P) + aF \geq aF - T > F,$$

contrary to the path bound. Thus only 34 can be high. Interchange $3, 4$ so that $b_{34} = F$. For mutation at 1 to lower it, its attachments must be $4 \to 1 \to 3$. We obtain

$$P = (a, b, -c, d, E, F), \quad b, c \geq 1, \quad a, b, c, |d|, |E| \leq T.$$

The low predecessor and the next-state bounds now give

$$|E - ac| \leq T, \quad |F - bc| \leq T, \quad b + ad \leq F \ (d \geq 0).$$

The preliminary sign regions are excluded as follows. If $d < 0 < E$, mutation at 2 increases F. If $d < 0, E \leq 0$ or $d = 0, E < 0$, Lemma 6.2 applies. For $d > 0, E = -e < 0$, the bounds give $F \geq b + 2d > b + d$, $F > 2c + e > c + e$, and $F > \max(2c, 2d) \geq c + d$, so Lemma 8.3 applies. Lemma 8.4 excludes $d \geq 0, 0 \leq E < c$. Thus $d \geq 0, E \geq c$.

The next state is

$$(-a, b + ad, aE - c, -d, -E, F), \qquad b + ad \geq 1, \quad aE - c \geq E \geq 1.$$

Exit 3 creates $(aE - c) + (b + ad)F > F$, while exit 4 creates magnitude $d + EF > F$, except when $d = 0, E = c = 1$. That exception is the heavy-exception obstruction. The next letter must therefore be 1. Thereafter each attachment column to the pair has form $(-X, Y)$ with $1 \leq X \leq Y$. Mutation at 2 followed by interchanging the pair replaces this by $(-Y, aY - X)$, which has the same property. The 34 entry stays F, since the two attachment columns have the same sign pattern. More explicitly, if the columns are $(-X_3, Y_3)$ and $(-X_4, Y_4)$, exit 3 creates $b_{24} = Y_4 + FY_3 > F$, and exit 4 creates $|b_{13}| = X_3 + FX_4 > F$. Each state contains a directed triangle: initially 134, next 124, and subsequently a triangle using the pair and an external vertex. The path can neither return low nor terminate acyclic, a contradiction.

11.3. **A unit edge and the first third vertex.** It remains to consider $|b_{ij}| = 1$. Lemma 10.2 connects two states of height at most K in this orbit, up to interchanging the pair, by at most three mutations within height K. If the high block returns low before a third vertex occurs, apply the lemma with $K = T$ to obtain a low replacement. It cannot terminate at an acyclic state while using only the pair, by Lemma 4.2. Consequently there is a first third letter, normalized to 3 after an alternating prefix in $1, 2$. Reverse all arrows if needed so that $b_{12}(C) = 1$, and write $C = (1, x, c, d, e, f)$.

A prefix of length at least four is shortened by the same two-generator lemma with $K = F$. A prefix 121 is replaced by 21, unless $\mathrm{ht}(C2) > F$. The latter possibility forces

$$C = (1, -b, c, d, -E, F - bc), \quad b > d \geq 1, \ c, E \geq 1,$$

$$bc \leq 2F - 1, \quad bc + dE > 2F, \quad c + E \leq F, \quad c + E = F \text{ or } d(c + E) = 2F.$$

To obtain this normalization, the pentagon identity makes $C21$ a relabeling of $C121$. Every entry of $C2$ except b_{34} is therefore bounded by F. Exchange $3, 4$ so that this unbounded entry is negative. It equals $f - dE$, which forces $d, E > 0$; restoring it at 1

forces $x + d < 0 < c$. Write $x = -b$, so $b > d$. The first high state can then reach F only at 34, giving $f = F - bc$. The three successive states are

$$\begin{aligned} C1 &= (-1, b, -c, d - b, -E, F), \\ C12 &= (1, d, -c - E, b - d, E, F), \\ C121 &= (-1, -d, c + E, b - d, -c, F - d(c + E)). \end{aligned}$$

Their bounds give the displayed alternatives. The directed triangle 123 and the path $1 \to 4 \to 2$ fix the proper class $\langle 1, 4, 2, 3\rangle$. Thus

$$m = 1 + b^2 + c^2 + d^2 + E^2 + cE - bd + F^2 - F(bc + dE), \quad p = F - b(c + E) + cd.$$

If $c + E = F$, then $C121 = (-1, -d, F, b - d, -c, -(d - 1)F)$, so $d \le 2$. Also $b \ge 3$, since $b = 2, d = 1$ would give $bc + dE = F + c < 2F$. For $d = 1$, substituting $E = F - c$ gives

$$m + 2p = F^2 - Fbc - 2F(b - 1) + b^2 - b + c^2 + 2c + 2.$$

The inequality $bc + E > 2F$ implies $bc > F + c$. Using $c^2 - Fc \le -c$ and $b^2 - b \le (F - 2)b$, we obtain

$$m + 2p < c - (F + 2)b + 2F + 2 \le 3F + 1 - (F + 2)b \le -5.$$

For $d = 2$, use $c^2 \le (F - 1)c$ and $b^2 - 2b \le (F - 3)b$ to obtain

$$m + 2p \le -[F(b - 2) - 3]c - (F + 3)b + 2F + 5 \le -2F(b - 2) - 3b + 8 < 0.$$

Both contradict $m > 2|p|$. In the remaining case $c + E < F$ and $d(c + E) = 2F$. The two factors $d, c + E$ are at most $F - 1$, so they are at least three and $F \ge 5$. Moreover

$$m = 1 + d^2 + (c + E)^2 - cE - F^2 + (b - d)(b - Fc).$$

Since $b - d \ge 1$ and $b \le F - 1$,

$$1 + (b - d)b - Fc(b - d) \le 1 + (b - d)(F - 1) - Fc(b - d) \le 0.$$

Applying (1) with its factors equal to $d, c + E$ gives

$$3m \le -F^2 - 5F + 36 - 3cE < 9,$$

contradicting $m \ge 3$. The prefix is therefore 12.

11.4. **The sign partition.** Write $C = (1, x, c, d, e, f)$, with $C1, C12$ high and $\mathrm{ht}(C123) \le F$. The quiver C is not acyclic. The three alternatives below exhaust the signs of x, d. To compute the three successive mutations without changing the coordinates, first write

$$\begin{aligned} C1 = \big(&-1, -x, -c, d - [-x]_+, e - [-c]_+, \\ &f + [-x]_+[c]_+ - [x]_+[-c]_+\big). \end{aligned}$$

The entries of $C12$ are

$$\begin{aligned} b_{12}(C12) &= 1, & b_{13}(C12) &= -x - [-b_{23}(C1)]_+, \\ b_{14}(C12) &= -c - [-b_{24}(C1)]_+, & b_{23}(C12) &= -b_{23}(C1), \\ b_{24}(C12) &= -b_{24}(C1), \end{aligned}$$

$$\begin{aligned} b_{34}(C12) = b_{34}(C1) &+ [-b_{23}(C1)]_+[b_{24}(C1)]_+ \\ &- [b_{23}(C1)]_+[-b_{24}(C1)]_+. \end{aligned}$$

Finally, the three entries not incident with 3 are

$$
\begin{aligned}
b_{12}(C123) &= 1 + [b_{13}(C12)]_+[-b_{23}(C12)]_+ \\
&\quad - [-b_{13}(C12)]_+[b_{23}(C12)]_+, \\
b_{14}(C123) &= b_{14}(C12) + [b_{13}(C12)]_+[b_{34}(C12)]_+ \\
&\quad - [-b_{13}(C12)]_+[-b_{34}(C12)]_+, \\
b_{24}(C123) &= b_{24}(C12) + [b_{23}(C12)]_+[b_{34}(C12)]_+ \\
&\quad - [-b_{23}(C12)]_+[-b_{34}(C12)]_+.
\end{aligned} \tag{2}
$$

The entries incident with 3 are the negatives of their values in $C12$.

First let $x = -b < 0 < d$. If $c > 0$, the first high condition gives $f = F - bc$. For $d < b$, the next state is

$$C12 = (1, d, -c - [-e]_+, b - d, -e, F + (b - d)[e]_+).$$

The second height bound gives $e \leq 0$. For $e \leq 0$ the third 24 entry is $(b - d)F - e$, so $e = 0$ and $b - d = 1$. Its 14 entry is then $dF - c$, whose bound forces $d = 1$. This leaves

$$A_c = (1, -2, c, 1, 0, F - 2c).$$

For $d > b, e \geq 0$, the next state is $C12 = (1, b, -c, b-d, -e, F)$. Its third 14 entry is $bF - c$, forcing $b = 1$, and the sink obstruction applies to the triangle 123, whose unique largest edge is d, and its attached sink 4. For $d > b, e = -E < 0$, put $r = d - b$. Here

$$C12 = (1, b, -c - E, -r, E, F - rE).$$

Since this state is high, either $c + E = F$ or $rE = 2F$. In the latter case $E \leq F - 1$ implies $r \geq 3$, but the third 24 entry is $E - rF < -F$. Hence $c + E = F$. The triangle 123 and path $1 \to 4 \to 2$ fix the proper class $\langle 1, 4, 2, 3\rangle$, in which

$$m = 1 + b^2 + br + r^2 - (b - 2)F^2 - rFE - cE.$$

If $b \geq 3$, use $b + r \leq F - 1$ to get

$$m \leq 1 + (b + r)^2 - F^2 \leq 2 - 2F < 3.$$

If $b = 2$, the exact expression is

$$m = 6 + 2r + r^2 - (r + 1)F - (E - 1)((r + 1)F - E - 1).$$

The final product is nonnegative, and $F \geq r + 3$, so $m \leq 3 - 2r \leq 1$. Hence $b = 1$. Thus $C = (1, -1, c, r + 1, -E, E)$. If $E = 1$, its third state is $(r + 1, -1, -r, r, 1, -w)$ with $w = c - r + 1 \geq 2$, excluded by Lemma 9.1. For $E \geq 2$, the triangle with weights $(r + 1, E, E)$ has Markov constant $(r + 1)^2 - (r - 1)E^2$. If $r = 1$, this is four, contradicting mutation-acyclicity. Thus $r \geq 2$, and the constant must exceed four. Factoring the difference gives

$$(r - 1)(r + 3 - E^2) > 0,$$

so $r \geq E^2 - 2 \geq 2$. Also $d = r + 1 \leq F - 1 = c + E - 1$ gives $c \geq r + 2 - E$. After substituting $F = c + E$, the expression $m - 2(r + 1)c$ is the convex polynomial

$$G(c) = c^2 + ((1 - r)E - 2(r + 1))c + (1 - r)E^2 + r^2 + r + 2.$$

It is negative at both ends of $[r + 2 - E, (r - 1)E]$, since

$$
\begin{aligned}
G(r + 2 - E) &= E^2 - Er^2 - Er - r + 2 \leq -2r^2 - 2r + 4 < 0, \\
G((r - 1)E) &= -(r - 1)E^2 - 2(r^2 - 1)E + r^2 + r + 2 \\
&\leq -3r^2 - 3r + 10 < 0.
\end{aligned}
$$

Here the first bound uses $E^2 \leq r + 2$ and $E \geq 2$, and the second uses $E \geq 2$ and $r \geq 2$. Therefore $c > (r - 1)E$, and $C123 = (r + 1, -1, -rE, r, E, -w)$, $w = c - (r - 1)E \geq 1$, is again excluded. For $d = b$, we have $C12 = (1, b, -c - [-e]_+, 0, -e, F)$. If $e \geq 0$, its third

14 entry $bF - c$ forces $b = 1$, yielding a unit triangle. If $e = -E < 0$, the same entry is $bF - c - E$, forcing $b \le 2$; $b = 1$ is again a unit triangle, while $b = 2$ gives $c + E = F$ and $m = 5 - cE \ge 3$, impossible for $F \ge 4$. Lemma 7.1 applies in the indicated cases.

If instead $c \le 0$, the first state has $b_{34} = f$ and its only possibly high entry is $e + c$. The case $c = 0$ leaves that entry low. Hence $c = -t < 0, e = -v < 0, t + v = F$. Negative f makes C a vortex, so $0 \le f \le T$. When $d < b$, one has $C12 = (1, d, -v, b - d, F, f)$ and the third 24 entry is $F + (b - d)f$, forcing $f = 0$. After reversal, the triangle 123 has unique largest edge b, opposite vertex 2, and the sink 4 is attached to 2 and 1. Lemma 6.2 applies. When $d = b$, the alternative states are

$$C2 = (-1, 0, -t, -b, v, f - bv), \qquad C21 = (1, 0, t, -b, v, f - bv).$$

The lower bound $bv - f \ge 1 - T$ is automatic. If $bv - f \le T$, both displayed states are low, and the pentagon identity gives $C212 = (C12)^{(12)}$ and $C2123 = (C123)^{(12)}$. Thus 2123 removes one high state. If $bv - f \ge F$, the sink obstruction applies at $C2$. When $d > b$, the next state is $(1, b, -v, b - d, F, f - (d - b)F)$; its third 34 magnitude $(d - b)F - f \le F$ and $f \le F - 1$ force $d - b = 1$. The case $f = 0$ is excluded by the sink obstruction. For $f > 0$, put $u = F - f$. If $b = 1$ retain the family $I_{t,u}$ below. For $b \ge 2$,

$$C2 = (-1, 1, -t, -d, v, f - dv), \quad C21 = (1, -1, t, -b, v, -bv - u),$$

$$C123 = (d, -b, -v, 1, f, u).$$

If $f > bv$, then $f \ge bv + 1$ gives $bv + u = bv + F - f \le T$ and

$$1 - v \le f - (b + 1)v \le T.$$

Every entry of the displayed $C2, C21$ is therefore low. The pentagon gives the replacement 2123, ending at $(C123)^{(12)}$, with one fewer high state. If $v > bf$, apply this construction to

$$C' = (1, -b, -u, d, -f, v) = -(C123)^{(13)}.$$

Its original path is the old path reversed and relabeled:

$$C'1 = -(C12)^{(13)}, \quad C'12 = -(C1)^{(13)}, \quad C'123 = -C^{(13)}.$$

Reverse the improved path, reverse all arrows, and relabel its start back to C. This gives the word 1323 from C to $(C123)^{(23)}$, again removing one high state. The tail is transported by (23). If neither inequality holds, $f \le bv$ and $v \le bf$. The same reversal interchanges f, v, so choose the representative with $v \le f \le bv = (d - 1)v$. Lemma 9.2 excludes $C123$.

Second let $x = -b < 0, d = -D \le 0$. Set

$$h = e - [-c]_+, \quad g = f + b[c]_+, \quad z = g + (b + D)[h]_+.$$

The first two states and two entries of the third are

$$C1 = (-1, b, -c, -D - b, h, g),$$

$$C12 = (1, -D, -c - [-h]_+, D + b, -h, z),$$

$$b_{12}(C123) = 1 - D(D + b), \qquad b_{24}(C123) = -h + (D + b)[z]_+.$$

The first high condition is $h = -F$, $g = F$, or $b + D = F$. If $h = -F$, then $c, e < 0$ and $g = z = f$; the third 24 bound $F + (b + D)[f]_+ \le F$ gives $f \le 0$. The order $4, 3, 1, 2$ is topological. If $g = F$, then $c > 0$ and the second bound gives $h \le 0$. The third bound $-h + (b + D)F \le F$ gives $b = 1, D = h = 0$, so $e = 0, c + f = F$ and $3, 1, 2, 4$ is topological. Thus $b + D = F$ remains. Here $D \ge 1$ and $1 - DF \ge -F$ gives $D = 1, b = T$. Finally $-h + F[z]_+ \le F$, with $h \le T$, forces $z \le 1$. The alternative states are

$$C2 = (-1, -T, c + [e]_+, 1, -e, f + [e]_+),$$

$$C21 = (1, T, -c - [e]_+, 1, -e + [c + [e]_+]_+, f + [e]_+ + T[c + [e]_+]_+).$$

If $c > 0$, then $z = f + Tc + F[e]_+ \le 1$ and $f \ge -T$ force $e \le 0$. If $f \ge 0$, the order $3, 1, 4, 2$ is topological. If $f < 0, e < 0$, vertex 2 is a strict sink on the directed triangle 134. Thus $e = 0, f < 0$, and $f + Tc \le 1$ forces $c = 1$, making both states low.

If $c = -t \le 0$ and $f > 0$, the inequality $z = f + F[e - t]_+ \le 1$ gives $f = 1, e \le t$. The third 24 bound is $F + t - e \le F$, so $e = t$. When $t = 0$, the order $3, 1, 2, 4$ is topological; when $t > 0$, vertex 3 is a strict source on the directed triangle 124.

For $f \le 0$, $e \le 0$ is acyclic; take $e > 0$. If $e - t \le 0$, all entries in the two alternative states are bounded differences or $f + e$. If $e - t > 0$, then $z = f + F(e - t) \le 1$ forces $e - t = 1, f = -T$, again making both low. The word 2123 thus improves 123 with permutation (12).

Finally let $x \ge 0$. If $c \ge 0$, every entry of $C1$ in the displayed mutation formula is low. Thus $c = -t < 0$. Put

$$w = f - xt, \qquad z = w + [-d]_+[e - t]_+ - [d]_+[t - e]_+.$$

The first high condition is $e - t = -F$ or $w = -F$. The third-state entries include

$$b_{12} = 1 - (x + [-d]_+)[-d]_+, \quad b_{14} = \min(t, e) - (x + [-d]_+)[-z]_+,$$

$$b_{24} = t - e + [-d]_+[z]_+ - [d]_+[-z]_+.$$

If $e - t = -F$, write $e = -u$, $F = t + u$. For $d = -D < 0$, the third 24 bound is $F + D[w]_+ \le F$, so $w \le 0$. The third 14 bound gives $-(x + D)w \le t$. If $x \ge 1$, then $w > -t$, so $f = xt + w > 0$ and vertex 2 is a strict sink on triangle 134; $x = 0$ gives acyclicity.

For $d \ge 0$, height gives $d \le 1$. If $d = 1$, then $w \ge 0$, and $x \ge 1$ would imply $x(F - w) \le t$ with $F - w \ge xt + 1$, impossible. Thus $x = 0$: $f = 0$ is acyclic and $f > 0$ gives the third detour below. For $d = 0$, $x = 0$ is acyclic. If $x = 1$, then $f \ge 0$, with $f = 0$ acyclic and $f > 0$ giving the second detour. If $x \ge 2$, then $f \ge xt - \lfloor t/x \rfloor > 0$. For $t \ge 2$, use $\lfloor t/x \rfloor \le t/2$ to obtain $f \ge t(x - 1/2) \ge 3t/2 > t$ and $f \ge 2x - 1 > x$. Thus the sink obstruction applies. For $t = 1$, use the sink obstruction when $f > x$, and the equal-sink obstruction when $f = x$.

If $w = -F$ and $e - t > -F$, then $x \ge 1$. When $d = -D < 0$, we have $z = -F + D[e - t]_+$. The second high condition is $x + D = F$ or $z = \pm F$. If $x + D = F$, the third 12 bound gives $D = 1$, and the third 14 bound gives $z \ge -1$. This requires $e - t \ge F - 1$, whereas $e - t \le F - 2$. If $z = -F$, the third 14 entry is at most $T - (x + D)F < -F$. If $z = F$, the third 24 bound gives $(D - 1)F \le e - t \le F - 2$, hence $D = 1$; but $z = F$ would then require $e - t = 2F$.

When $d \ge 0$, $z = -F - d[t - e]_+ \le -F$, so the second bound gives $z = -F$ and $d[t - e]_+ = 0$. The third 14 bound reads $\min(t, e) - xF \ge -F$, forcing $x = 1, e \ge 0$. If $d > 0$, then $e \ge t$ and the third 24 bound becomes $(e - t) + (d - 1)F \le 0$, so $d = 1, e = t$. In that case vertex 3 is a strict sink on triangle 124. The case $d = e = 0$ is acyclic. For $d = 0, e > 0$, use the first detour below.

The three detours replacing 123 are

C	replacement	endpoint permutation
$(1, 1, -t, 0, e, -u)$	3132	(13)
$(1, 1, -t, 0, -u, f)$	3132	(13)
$(1, 0, -t, 1, -u, f)$	3123	(23).

Here $t, u \ge 1$, $t + u = F$, and $1 \le e, f \le T$. In the first case the first two states are $(1, -1, -t, 0, e, u)$ and $(-1, 1, t, -1, e - t, u)$, both low. In the second they are $(1, -1, f - t, 0, -u, -f)$ and $(-1, 1, t - f, -1, -u - \max(t - f, 0), -\min(t, f))$, again low. The last two states in the first case are

$$C313 = (0, -1, F, 1, e - t, -u), \quad C3132 = (0, -1, F, -1, t - e, -u - [t - e]_+).$$

In the second case they are

$$C313 = (0, -1, t-f, 1, -F, \min(t,f)), \quad C3132 = (0, -1, t-f, -1, F, \min(t,f) - F).$$

The pentagon identity gives $C313 = (C13)^{(13)}$. Since $b_{23}(C1) = 0$, the two mutations $2, 3$ commute at $C1$, and direct substitution gives $C3132 = (C123)^{(13)}$ in both cases. The third new state has height F, and the fourth has the height of the old endpoint. The first two new states are low, replacing the two initial high states by one. In the third case the entire sequence is

$$(1, 0, -t, -1, f-u, -f), \quad (-1, 0, t, -1, f-F, -f),$$
$$(1, -1, f-u, 1, F-f, -f), \quad (0, 1, -u, -1, F-f, f),$$

all low. This completes the sign partition.

11.5. **The remaining families.** For $A_c = (1, -2, c, 1, 0, F-2c)$, $1 \le c \le T$, the four high states preceding the forced return are

$$(-1, 2, -c, -1, 0, F), \quad (1, 1, -c, 1, 0, F),$$
$$(1, -1, F-c, -1, F, -F), \quad (-1, 1, c-F, -2, F, -c).$$

They contain directed triangles $134, 134, 234, 123$, respectively, and cannot be endpoints. After 123, exits $2, 4$ create $2F - c, F^2 - 1$; after 1231, they create $2F - c, -1 - F(F - c)$. Thus the path is forced to continue 12313 to a low state. The states along 213 are

$$(-1, -1, c, -1, 0, F-2c), \quad (1, 1, -c, -1, c, F-c), \quad (2, -1, F-2c, 1, c, c-F).$$

They are low because $1 \le c, F - c \le T$ and $|F - 2c| \le T$ already holds at C. The last tuple is the endpoint of 12313 relabeled by (123), so this is a strict improvement.

The second family is

$$I_{t,u} = (1, -1, -t, 2, t-F, F-u), \qquad 1 \le t, u \le T.$$

Its first three states are

$$(-1, 1, t, 1, -F, F-u), \quad (1, 1, t-F, -1, F, -u), \quad (2, -1, t-F, 1, F-u, u).$$

Exactly the first two are high. For $t > u$, the first four states of 21323 are

$$(-1, 1, -t, -2, F-t, 2t-F-u), \quad (1, -1, t, -1, F-t, t-F-u),$$
$$(1, 1, 2t-F-u, 1, -u, F+u-t), \quad (-1, 2, 2t-F-u, -1, u, F-t).$$

For $u > t$, the first four states of 31321 are

$$(-1, 1, -t, -2, F+t-2u, u-F), \quad (1, -1, t, -1, F+t-2u, u-F-t),$$
$$(1, 1, u-F, 1, -u, F+t-u), \quad (-1, 2, u-F, -1, u, F+t-2u).$$

Both fifth states are $(1, -2, F-u, 1, u, t-F) = (I_{t,u}123)^{(123)}$. If $t > u$, then $2 \le F+u-t \le T$ and $2t - F - u = t - (F + u - t)$ is a difference of integers in $[1, T]$. If $u > t$, use $2 \le F + t - u \le T$ and $F + t - 2u = (F + t - u) - u$. Every other entry is $0, \pm 1, \pm 2$ or one of $\pm t, \pm u, \pm(F - t), \pm(F - u)$. Thus both replacement paths are low. For $t = u$ and $F - t \ge 2$, triangle 234 has weights $(2, F - t, F - t)$ and Markov constant four, hence is mutation-cyclic. If $F - t = 1$, the third state is $(2, -1, -1, 1, 1, t)$; interchanging $3, 4$ gives $(2, -1, -1, 1, 1, -t)$, excluded by Lemma 9.1 with $r = E = 1$.

Thus $F \le \mathrm{ht}(Q)$ for connected Q. For disconnected inputs each component has at most three vertices. A cyclic mutation-acyclic triangle with sorted weights $x \le y \le z$ admits a mutation opposite z replacing it by $|xy - z|$ without increasing height: if $x \ge 2$ and $xy \ge 2z$, then $y \le z \le xy/2$. The convex quadratic $x^2 + y^2 + z^2 - xyz$ is at most four at both endpoints: at $z = y$ its value is $x^2 - (x-2)y^2 \le x^2(3 - x) \le 4$, and at $z = xy/2$ it is $4 - (x^2 - 4)(y^2 - 4)/4 \le 4$. This would make the triangle mutation-cyclic. Hence $xy < 2z$, so $|xy - z| < z$. If $x = 1$, then $xy = y \le z$, and again $|xy - z| < z$. Thus the

positive integer sum of multiplicities strictly decreases while the triangle is cyclic. Apply the argument separately to each component.

□

References

[1] K. T. K. Armstrong-Williams, E. Hirst, B. Jackson and K.-H. Lee, *Mutation-acyclicity in rank four is decidable but not semialgebraic*, arXiv:2610.36179v1 (2026).

[2] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **120** (1996), no. 575.

[3] I. Assem, T. Brüstle and R. Schiffler, *Cluster-tilted algebras as trivial extensions*, Bull. London Math. Soc. **40** (2008), 151–162; arXiv:math/0601537.

[4] B. Keller, *Deformed Calabi–Yau completions*, J. Reine Angew. Math. **654** (2011), 125–180; arXiv:0908.3499v5.

[5] B. Keller, *Erratum to "Deformed Calabi–Yau completions"*, arXiv:1809.01126v1 (2018).

[6] I. Assem, J. C. Gatica, R. Schiffler and R. Taillefer, *Hochschild cohomology of relation extension algebras*, arXiv:1507.06142v2.

[7] A. B. Buan, O. Iyama, I. Reiten and D. Smith, *Mutation of cluster-tilting objects and potentials*, Amer. J. Math. **133** (2011), 835–887; arXiv:0804.3813v4.

[8] Z. Hua and G.-S. Zhou, *Noncommutative Mather–Yau theorem and its applications to Calabi–Yau algebras*, arXiv:1803.06128v5.

[9] D. Happel and I. H. Slungård, *On quasitilted algebras which are one-point extensions of hereditary algebras*, Colloq. Math. **81** (1999), 141–152.

[10] I. Assem, T. Brüstle and R. Schiffler, *Cluster-tilted algebras and slices*, J. Algebra **319** (2008), 3464–3479; arXiv:0707.0038.

[11] V. G. Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), 57–92.

[12] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), 399–443.

[13] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529; arXiv:math/0104151.

[14] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), 572–618.

[15] A. B. Buan, R. J. Marsh and I. Reiten, *Cluster mutation via quiver representations*, Comment. Math. Helv. **83** (2008), 143–177; arXiv:math/0412077v2.

[16] I. Assem, M. Blais, T. Brüstle and A. Samson, *Mutation classes of skew-symmetric* 3×3*-matrices*, Comm. Algebra **36** (2008), 1209–1220; arXiv:math/0610627v1.

[17] A. Beineke, T. Brüstle and L. Hille, *Cluster-cyclic quivers with three vertices and the Markov equation*, Algebr. Represent. Theory **14** (2011), 97–112; arXiv:math/0612213v2.

[18] A. Felikson and P. Tumarkin, *Acyclic cluster algebras, reflection groups, and curves on a punctured disc*, Adv. Math. **340** (2018), 855–882; arXiv:1709.10360v3.

[19] S. Fomin and S. Neville, *Cyclically ordered quivers*, Adv. Math. **500** (2026), article 111059; arXiv:2406.03604v3.

[20] S. Neville, *Mutation-acyclic quivers are totally proper*, arXiv:2409.17832v2 (2025).

[21] K. T. K. Armstrong-Williams, E. Hirst, B. Jackson and K.-H. Lee, *Machine learning mutation-acyclicity of quivers*, arXiv:2411.04209v2 (2025).

[22] A. Thalluri, *Conjecture 5.5 of arXiv:2610.36179 checked for rank-4 quivers of height at most 7*, Zenodo (2026), doi:10.5281/zenodo.23872560.

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