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THE L'VOV–KAPLANSKY CONJECTURE FOR MULTILINEAR POLYNOMIALS OF DEGREES FOUR TO SIX

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Abstract. Let K be an algebraically closed field of characteristic zero. We determine the images of ordinary multilinear polynomials of degrees four, five and six on matrix algebras. For $n \geq 3$, a nonidentity has image $\mathfrak{sl}_n(K)$ when its cyclic class is zero and $M_n(K)$ otherwise. In these degrees the only nonzero identity exception is the standard degree-six line on $M_3(K)$. The proof starts with proper sextics and treats arbitrary sextics by unit restrictions and local trace lifting. An image-preserving product substitution then gives degrees four and five as consequences. We also give cyclic and matrix lifting results applicable in arbitrary degree. The finite calculations are given by explicit matrix-unit identities.

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Introduction

Let K be an algebraically closed field of characteristic zero. For a noncommutative polynomial f, write $\mathrm{Val}_n(f)$ for its set of values on $M_n(K)$. The L'vov–Kaplansky conjecture asks whether this set is a vector space whenever f is multilinear. The question appears under I. V. L'vov's name in [12, Problem 1.98, p. 12]; a survey of matrix polynomial images is [13].

For 2×2 matrices, Kanel–Belov, Malev and Rowen [11, Theorem 2] proved the four possible images $\{0\}$, KI_2, $\mathfrak{sl}_2(K)$ and $M_2(K)$. The degree-three case was proved by Vitas [14, Theorem B]. In degree four, traceless containment for nonzero polynomials on $M_n(K)$, $n \geq 3$, was established by Buzinski and Winstanley [15], with the correction in [16, Theorem 4.1]. Tang's preprint [18, Theorem 1.1] gives a proof of the degree-four conjecture under the same field assumptions. Its quartic image classification agrees with the degree-four part of Theorem A below. The methods share trace-pairing tests, scalar values from clock-and-shift matrices, frozen-input linear maps, and cycle and block constructions with exact minors. Tang's main route uses traceful linear slices of codimension at most two to meet nonscalar conjugacy classes [18, Corollary 2.5], followed by quartic coefficient reductions and transfer from sizes two and three. We first treat proper sextics by studying local fibers and corrected trace terms at rank-one square-zero values. Unit deletions give the ordinary sextic case, from which degrees four and five follow by product substitution. The proof of the proper-sextic theorem is independent of lower-degree image classifications, including Tang's theorem.

Let P_d be the ordinary multilinear component: every word contains each of $x_1, \ldots, x_d$ once, with coefficients in K. Let $\pi : P_d \to C_d$ identify words under cyclic rotation. Its coordinates are the sums of coefficients in the cyclic orbits. Put

$$s_d(x_1, \ldots, x_d) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(d)}.$$

Our main result is the following.

Theorem A. Let $d \in \{4, 5, 6\}$, $f \in P_d$, and $n \geq 3$. Define

$$I_{d,n} = \begin{cases} Ks_6, & (d, n) = (6, 3), \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{Val}_n(f) = \begin{cases} \{0\}, & f \in I_{d,n}, \\ \mathfrak{sl}_n(K), & f \notin I_{d,n} \text{ and } \pi f = 0, \\ M_n(K), & \pi f \neq 0. \end{cases}$$

For $n = 2$, the four image alternatives are determined by actual identity, centrality and trace conditions, as specified in Corollary 8.12.

Here the image is the set of single evaluations, not its linear span or Zariski closure. In particular, dominance alone does not prove the statement. We

construct an open neighborhood of a nonzero rank-one square-zero value and use conjugation and scalar multiplication to recover every nonscalar target. Scalar targets in the traceful case are obtained independently from a homogeneous matrix basis.

The paper is organized as follows. In Section 1, we fix notation and collect results on cyclic words, proper components and local fibers. A polynomial is proper if every single unit deletion annihilates it. The first two upper-triangular symbols divide the proper sextics into a one-slot case, products of three commutators, and a mixed two-slot case. Sections 2–7 treat these cases and prove Theorem 7.16.

In Section 8, we use this theorem and image-preserving Lie substitutions to determine the values of proper polynomials of lower degree. A maximal nonzero unit restriction of a nonproper sextic is proper. Local trace lifting in the original inputs then gives the ordinary sextic theorem. Replacing one input by a product of fresh variables preserves the value set, so the degree-four and degree-five classifications follow in Corollary 8.11. This product substitution is distinct from the Lie substitutions used for proper polynomials.

In Section 9, we prove further lifting results applicable in arbitrary degree. In Section 10, we describe the finite calculations as matrix-unit identities and parameter factorizations. The arguments for arbitrary matrix size are given in the preceding sections.

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1. Preliminaries

Throughout the paper, K is algebraically closed of characteristic zero. All varieties and tangent spaces are over K. Put $\mathcal{H}_n = \{A \in M_n(K) : A_{ii} = 0\}$ and $\mathfrak{sl}_n = \{A \in M_n(K) : \operatorname{tr} A = 0\}$. We use $[A, B] = AB - BA$ and $\operatorname{ad}_A(B) = [A, B]$. The space P_d has basis the $d!$ words $x_{\sigma(1)} \cdots x_{\sigma(d)}$, $\sigma \in S_d$. For $f \in P_d$, set

$$\operatorname{Val}_n(f) = \{f(A_1, \ldots, A_d) : A_i \in M_n(K)\}.$$

The cyclic projection $\pi : P_d \to C_d$ identifies words under cyclic rotation; its coordinates are the sums of the coefficients in the cyclic orbits. A polynomial is *traceful* on M_n if its trace function is not identically zero. A *proper* multilinear polynomial is killed, in the free algebra, when any one input is replaced by 1. We denote the proper subspace of P_d by Γ_d. Properness implies invariance under adding any scalar matrix to an individual input. This definition agrees with the usual multilinear proper component described by products of Lie commutators through the Poincaré–Birkhoff–Witt decomposition [9].

1.1. **Unit deletions and proper components.** For $d \geq 2$, deletion of x_i induces $\partial_i : C_d \to C_{d-1}^{(i)}$, where the superscript means that x_i is omitted. For $f_i = f|_{x_i=1}$ one has $\pi(f_i) = \partial_i \pi(f)$. The cyclic deletion kernel is

$$\mathcal{K}_d = \bigcap_{i=1}^{d} \ker \partial_i \subseteq C_d.$$

It is distinct from the proper word space $\Gamma_d = \bigcap_i \ker(f \mapsto f_i) \subseteq P_d$.

Proposition 1.1. *For a nonempty subset $B \subseteq [d]$, let $\mathcal{L}(B)$ denote the multilinear free Lie component on the variables indexed by B. Let $\Pi_{\geq 2}(d)$ be the set partitions of $[d]$ without singleton blocks. Symmetrized multiplication gives a natural S_d-module isomorphism*

$$\Gamma_d \cong \bigoplus_{\pi \in \Pi_{\geq 2}(d)} \bigotimes_{B \in \pi} \mathcal{L}(B).$$

For $\pi = \{B_1, \ldots, B_r\}$, this map sends a tensor of Lie elements to $\frac{1}{r!} \sum_{\sigma \in S_r} \ell_{\sigma(1)} \cdots \ell_{\sigma(r)}$; it is independent of the ordering of the blocks. Consequently, with $b_0 = 1$,

$$b_d := \dim \Gamma_d = \sum_{\pi \in \Pi_{\geq 2}(d)} \prod_{B \in \pi} (|B| - 1)! = d! \sum_{j=0}^{d} \frac{(-1)^j}{j!}.$$

Proof. The free associative algebra is the enveloping algebra of the free Lie algebra $\mathcal{L}$. In characteristic zero, PBW symmetrization is a natural multigraded vector-space isomorphism $\mathrm{Sym}(\mathcal{L}) \cong U(\mathcal{L})$; see [9, Theorems 0.2 and 0.5; Section 0.4.3]. Its multilinear part is the direct sum over all set partitions, including singleton blocks.

Deleting x_i kills every Lie factor of length at least two containing x_i. On a symmetrized product with singleton factor x_i, deletion removes that factor: its $r+1$ possible positions change the coefficient $1/(r+1)!$ into $1/r!$. Distinct remaining partition components are independent by PBW. Hence the kernel of this deletion is precisely the sum of components without singleton block $\{i\}$. Intersecting over i gives the asserted decomposition. Symmetrization commutes with relabeling, proving equivariance.

For $|B| = r$, the multilinear Lie component has dimension $(r-1)!$. Indeed, fix one variable innermost on the right. The corresponding right-normed brackets span by Jacobi; their associative expansions have distinct words ending in that fixed variable, proving independence. This gives the partition sum. Separating singleton supports in the full multilinear PBW decomposition gives $d! = \sum_{j=0}^{d} \binom{d}{j} b_{d-j}$. Binomial inversion yields the second formula. □

In degrees four and five the dimensions are 9 and 44. In degree six, the block types $(6), (4,2), (3,3), (2,2,2)$ contribute respectively $120, 90, 40, 15$, giving $b_6 = 265$. These are coefficient-space decompositions; they do not permit addition of independently attained polynomial values.

1.2. **Detection by pairwise collisions.** For $d \geq 3$ and $i < j$, let $C_d^{(ij)}$ be the space spanned by cyclic words in which x_i, x_j have been replaced by one letter D. The substitution defines a *collision map*

$$\rho_{ij} : C_d \longrightarrow C_d^{(ij)}.$$

It preserves total degree, although it reduces the number of distinct variables.

Proposition 1.2. *Let $d \geq 3$ and let G be any connected graph on $[d]$. Then*

$$\bigcap_{\{i,j\}\in E(G)} \ker \rho_{ij} = \begin{cases} \{0\}, & d \text{ even}, \\ K\,\pi(s_d), & d \text{ odd}. \end{cases}$$

In particular, this one-dimensional subspace is contained in $\mathcal{K}_d$ whenever d is odd.

Proof. Fix $\tau = (ij)$. A cyclic word with two occurrences of D has exactly two distinct multilinear lifts, interchanged by τ. They are distinct because at least one other letter occurs exactly once: a rotation fixing that letter cannot interchange the two labelled occurrences of D. Consequently, $\ker \rho_{ij} = \{\xi \in C_d : \tau\xi = -\xi\}$. The edge transpositions of a connected graph generate S_d: conjugating an edge transposition along a path produces the transposition between its endpoints. Thus the common kernel is the subspace on which every permutation acts by its sign.

The action of S_d on the cyclic-word basis is transitive. The stabilizer of $\pi(x_1 \cdots x_d)$ is generated by the d-cycle $(1\,2\,\cdots\,d)$. In a sign eigenvector, all coefficients are determined by one coefficient; this is consistent precisely when the sign is trivial on that stabilizer. Since the sign of a d-cycle is $(-1)^{d-1}$, there is no such nonzero vector in even degree, and there is at most one independent such vector in odd degree. For odd d, each cyclic orbit contributes d equal signed terms to $\pi(s_d)$, so this class is nonzero and spans the common kernel.

Finally, for odd d, we perform direct grouping according to the position of x_i to get

$$s_d(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d) = (-1)^{i-1} s_{d-1}(x_1, \ldots, \widehat{x_i}, \ldots, x_d).$$

The cyclic class of the even-degree polynomial on the right is zero: cyclic rotation reverses its alternating sign. Every deletion of $\pi(s_d)$ therefore vanishes. □

Proposition 1.3. *Let $d \geq 3$. Write $\Gamma_d \subseteq P_d$ for the common kernel of all unit deletions. This is distinct from the cyclic deletion kernel $\mathcal{K}_d \subseteq C_d$. For a connected graph G on $[d]$, let S_{ij} be the word substitution $x_i = x_j = D$. Then*

$$\bigcap_{\{i,j\}\in E(G)} \ker S_{ij} = K\,s_d, \qquad \bigcap_{\{i,j\}\in E(G)} \ker(S_{ij}|_{\Gamma_d}) = \begin{cases} K\,s_d, & d \text{ even}, \\ 0, & d \text{ odd}. \end{cases}$$

Thus both this result and Proposition 1.2 require only the $d-1$ edges of a spanning tree.

Proof. A word containing two copies of D has exactly two multilinear lifts, exchanged by (ij). Hence $S_{ij}f = 0$ if and only if $(ij)f = -f$. The edge transpositions generate S_d, and its action on the multilinear word basis is free and transitive. The common kernel is therefore the one-dimensional alternating subspace Ks_d. Grouping the terms of s_d by the position of x_i gives

$$s_d|_{x_i=1} = \begin{cases} 0, & d \text{ even}, \\ (-1)^{i-1} s_{d-1}(x_1, \dots, \widehat{x_i}, \dots, x_d), & d \text{ odd}. \end{cases}$$

The insertion signs alternate, with sum zero in even degree and one in odd degree. The displayed free polynomial in odd degree is nonzero, which proves the restriction to Γ_d. □

These are statements about free words and free cyclic classes. A nonzero cyclic collision implies a traceful matrix specialization only after the matrix identity quotient in Proposition 1.4 has been checked.

1.3. **Trace, identities and hollow matrices.**

Proposition 1.4. *For every $d \geq 1$, rotation to the unique representative beginning with x_d defines a vector-space isomorphism*

$$\Theta_d : C_d \longrightarrow P_{d-1}, \qquad \Theta_d(\pi(x_d w)) = w,$$

where the target uses $x_1, \dots, x_{d-1}$ and $P_0 = K$. Put $h_f = \Theta_d(\pi(f))$. Then

$$\operatorname{tr} f|_{M_n} \equiv 0 \quad \Longleftrightarrow \quad h_f \in \operatorname{Id}(M_n(K)).$$

In particular, if $\mathcal{T}_{d,n} \subseteq C_d$ consists of classes with identically zero matrix trace, then

$$C_d/\mathcal{T}_{d,n} \cong P_{d-1}/\Big(P_{d-1} \cap \operatorname{Id}(M_n(K))\Big).$$

Proof. Every multilinear cyclic word contains x_d exactly once, so rotation gives the stated isomorphism. Cyclicity of trace gives

$$\operatorname{tr} f(A_1, \dots, A_d) = \operatorname{tr}\Big(A_d h_f(A_1, \dots, A_{d-1})\Big).$$

Fix the first $d-1$ inputs and vary A_d. Nondegeneracy of the matrix trace pairing shows that the right side vanishes for every A_d exactly when the displayed value of h_f is zero. This proves the equivalence and identifies the subspace defining the quotient. □

Proposition 1.5. *For $1 \leq d \leq 6$, $f \in P_d$ and $n \geq 3$, the identity $\operatorname{tr} f \equiv 0$ holds on M_n if and only if $\pi f = 0$. There is no nonzero identity in these degrees when $n \geq 4$, nor in degrees at most five when $n = 3$. The ordinary multilinear sextic identity space of M_3 is Ks_6.*

Proof. In Proposition 1.4, the polynomial h_f has degree $d - 1 < 2n$. The minimum-degree part of the Amitsur–Levitzki theorem gives $h_f = 0$ whenever it is an identity [1]. The same bound excludes the asserted polynomial identities. In size three, uniqueness of the multilinear identity in minimum degree six

gives Ks_6; see the recollection preceding Lemma 3.4 in [2]. This characteristic-zero statement is unchanged by extension of the coefficient field. Indeed, the multilinear identity space is the kernel of the finite evaluation matrix on tuples of matrix units, with integer entries. Its rank and the line spanned by the integral coefficient vector of s_6 are the same over every characteristic-zero field. □

Lemma 1.6. *Every traceless matrix over K is similar to a hollow matrix. For a diagonal D with distinct entries and every integer $r \geq 1$, the restriction of ad_D^r to $\mathcal{H}_n$ is invertible.*

Proof. We prove the first assertion by induction on n. A scalar traceless matrix is zero. If T is nonscalar, choose v such that v, Tv are independent and a functional ϕ with $\phi(v) = 1$ and $\phi(Tv) = 0$. In the decomposition $Kv \oplus \ker \phi$, the first diagonal entry is zero and the complementary compression has trace zero. Induction makes its diagonal zero. Finally, $(\mathrm{ad}_D^r Z)_{ij} = (d_i - d_j)^r Z_{ij}$, which proves the second assertion. □

1.4. **Local fibers and polynomial values.** For an ordinary multilinear polynomial f which is not central-valued, put $V_f = \mathfrak{sl}_n$ when $\mathrm{tr}\, f \equiv 0$, and $V_f = M_n$ otherwise. All open subsets below are Zariski open in this target vector space.

Lemma 1.7. *Let $V = M_n$ or $V = \mathfrak{sl}_n$, and let $S \subseteq V$ contain zero and be invariant under simultaneous conjugation and scalar multiplication. If $V = M_n$, assume also that $KI \subseteq S$. If S contains an open neighborhood of a nonzero rank-one square-zero matrix in V, then $S = V$. In particular this applies to the actual value set of an ordinary multilinear polynomial with $V = V_f$: in the traceful case every scalar matrix is an actual value.*

Proof. An ordinary multilinear value set contains zero and is invariant under simultaneous conjugation and scalar multiplication. To verify its scalar condition in the traceful case, choose clock and shift matrices U, V with $UV = \zeta VU$, where ζ is a primitive nth root of unity. The matrices $U^a V^b$ form a homogeneous basis for the $(\mathbb{Z}/n)^2$ grading of M_n; every component has dimension one, and only the degree-zero component KI has nonzero trace. Expand a nonzero-trace evaluation of f multilinearly in this basis. Some basis tuple has nonzero output trace. Every word at this tuple has the same total degree, so the output is a nonzero scalar. Scaling one original input gives every scalar. The same scalar-value argument is recorded in [18, Lemma 2.2].

Let T be nonscalar. It has a similar representative $B = \left(\begin{smallmatrix} u & v \\ w & Z \end{smallmatrix}\right)$ with $v \neq 0$: otherwise every row covector would be an eigenvector for right multiplication by T, forcing T to be scalar. The polynomial curve

$$R(t) = \begin{pmatrix} tu & v \\ t^2 w & tZ \end{pmatrix} = t \begin{pmatrix} t^{-1} & 0 \\ 0 & I \end{pmatrix} B \begin{pmatrix} t & 0 \\ 0 & I \end{pmatrix} \quad (t \neq 0)$$

has a nonzero rank-one square-zero limit. If T is traceless, the whole curve is traceless. All nonzero rank-one square-zero matrices are conjugate, so after a fixed conjugation $R(0)$ lies in the given open neighborhood. Its inverse image

contains a nonzero parameter t. Undoing the scalar and conjugation gives T as one actual evaluation. The assumed scalar condition completes the proof when $V = M_n$. In $\mathfrak{sl}_n$ the only scalar is zero. This also proves the stated ordinary multilinear application. □

Corollary 1.8. *Let $f : (M_n)^d \to V_f$ be ordinary multilinear, and let $f(a) = N$ be nonzero rank-one square-zero. Either of the following suffices for $\mathrm{Val}_n(f) = V_f$:*

(i) *df_a is onto V_f;*
(ii) *$\dim \mathcal{O}_{f^{-1}(N),a} = dn^2 - \dim V_f$.*

One may also use the differential of a simultaneous-conjugation saturation of an actual restricted evaluation family, provided that its image remains in the actual value set and has the required scalar targets.

Proof. A surjective differential between these smooth affine spaces makes f smooth at a, hence open on a neighborhood [4]. In the second case source and target local rings are regular and the source is Cohen–Macaulay. Miracle flatness gives flatness at a [5]. Restricting to the open flat locus [6], openness of flat morphisms locally of finite presentation gives a target neighborhood [7]. A nonempty finite-type fiber over a K-point has a K-point, so this is a neighborhood of actual values. Apply Lemma 1.7. The same smoothness argument applies to a saturated restricted family; the final assertion includes explicitly the invariances and scalar condition used in Lemma 1.7. □

Proposition 1.9. *Let $h = \operatorname{tr} f$ and $g = f - (h/n)I$. Suppose that $f(a) = N$ is nonzero rank-one square-zero and dg_a is onto $\mathfrak{sl}_n$. Then f is flat at a if and only if h is nonzero in the local ring of $g^{-1}(N)$ at a. Either condition implies $\mathrm{Val}_n(f) = M_n$.*

Proof. The local ring $R = \mathcal{O}_{g^{-1}(N),a}$ is a regular local domain of dimension $dn^2 - n^2 + 1$. Since $h(a) = 0$, a nonzero $h|_R$ is a nonunit nonzerodivisor, and $\dim R/(h) = dn^2 - n^2$. If $h|_R = 0$, this dimension instead remains $dn^2 - n^2 + 1$. The fiber of f is cut out in this smooth fiber by $h = 0$. Miracle flatness and the dimension formula for a flat local map give the equivalence. Corollary 1.8 gives actual surjectivity. Completion is faithfully flat, so one may detect nonvanishing by a formal expansion on the smooth fiber. □

In particular, put $K_a = \ker dg_a$ and choose a linear right inverse R_a of dg_a. The formal implicit function theorem gives $\phi(u) = a + u + O(u^2)$ with $g(\phi(u)) = N$ for $u \in K_a$. The relevant expression is $h(\phi(u))$, including these corrections. If $dh_a = 0$, its quadratic and cubic terms are respectively

$$\frac{1}{2}d^2h_a(u,u), \qquad \frac{1}{6}d^3h_a(u,u,u) - \frac{1}{2}d^2h_a\Big(u, R_a d^2 g_a(u,u)\Big).$$

Thus a nonzero uncorrected cubic alone does not establish flatness. All later leading-term arguments either retain this correction or prove that every lower ambient trace derivative vanishes.

1.5. **The first two upper-triangular symbols.** Let

$$W_6 = \operatorname{span}\{[x_i, x_j][x_k, x_l][x_r, x_s] : \{i, j, k, l, r, s\} = \{1, \dots, 6\}\}.$$

The order of the three factors is arbitrary. We use the classical identity theorem for upper-triangular matrices: in characteristic zero, the polynomial identity ideal of UT_r is generated as a T-ideal by a product of r commutators; see [3, Theorem 2.1].

Proposition 1.10. *For $f \in \Gamma_6$, there are scalars $c_s(f)$, $1 \le s \le 6$, such that*

$$f(D, \dots, X_s, \dots, D) = c_s(f)\,\mathrm{ad}_D^5\, X.$$

Write $c(f) = (c_1(f), \dots, c_6(f))$. Then $c(f) = 0$ precisely when f is an identity of UT_2. In this layer, evaluation on UT_3 has the form $u^{\mathsf{T}} A(D) v E_{13}$, where u, v collect the first and second superdiagonal input entries and A has zero diagonal. Furthermore, $A = 0$ precisely when $f \in W_6$.

Proof. Leave slot s free and put all other slots at independent diagonal matrices D_i. The coefficient of an entry X_{rt} is multilinear in the five pairs $(d_i(r), d_i(t))$. Adding a scalar to D_i changes neither the evaluation nor this coefficient, by properness. A linear translation-invariant expression in that pair is a multiple of $d_i(r) - d_i(t)$. Thus the coefficient is $c_s \prod_{i \ne s}(d_i(r) - d_i(t))$. On the diagonal it is zero, since a proper polynomial vanishes on commuting diagonal inputs. Specializing all D_i to D gives the first formula.

In UT_2, a nonzero word contains at most one strictly upper-triangular input factor. Its remaining factors are diagonal. The displayed one-edge coefficients therefore detect the whole value, proving the UT_2 assertion. When all of them vanish, a UT_3 value requires exactly a path $1 \to 2 \to 3$. Multilinearity puts its two edges in distinct original slots and yields $u^{\mathsf{T}} A(D) v E_{13}$, with zero diagonal in A. This symbol vanishes exactly when f is a UT_3 identity. By the stated T-ideal theorem, the multilinear component in the minimum degree six is spanned by the ordered products of three commutators: substitutions of higher degree or multiplication by nonconstant words would increase the degree. This is exactly W_6. □

2. The one-slot case

Theorem 2.1. *Let $f \in \Gamma_6$ and $c(f) \ne 0$. For every $n \ge 3$,*

$$\mathrm{Val}_n(f) = \begin{cases} \mathfrak{sl}_n(K), & \pi f = 0, \\ M_n(K), & \pi f \ne 0. \end{cases}$$

In the traceful case there is a repeated-diagonal, rank-one nilpotent seed a at which dg_a is onto $\mathfrak{sl}_n$ and the corrected trace germ on $g^{-1}(f(a))$ has a nonzero leading term of some order $1 \le r \le 5$. Here $g = f - n^{-1}(\operatorname{tr} f)I$. All coefficients of f are allowed subject only to $c(f) \ne 0$ and the indicated cyclic condition.

Proof. Fix a slot s with $c_s(f) \ne 0$. The repeated-diagonal identity is

$$f(D, \dots, X_s, \dots, D) = c_s(f)\,\mathrm{ad}_D^5\, X.$$

For D with distinct diagonal entries this realizes every hollow matrix. Similarity to a hollow matrix therefore gives $\mathfrak{sl}_n \subseteq \mathrm{Val}_n(f)$. Together with Proposition 1.5, this proves the trace-zero case. Assume from now on that $\pi f \neq 0$.

Step 1: a nonzero cyclic derivative and its first diagonal term. Rotate every word so that x_s comes first, then remove x_s. This produces a multilinear polynomial G in the other five variables with

$$h = \operatorname{tr} f = \operatorname{tr}(x_s G), \qquad G \neq 0.$$

In the free algebra on D and independent variables Y_i $(i \neq s)$, write

$$G((D + tY_i)_{i\neq s}) = \sum_{q=0}^{5} t^q G_q(D; \boldsymbol{Y}).$$

The constant term is zero: G_0 is a scalar multiple of D^5 and $\operatorname{tr}(XG_0) = \operatorname{tr}(c_s \operatorname{ad}_D^5 X) = 0$ for every D, X. Also $G_5 = G(\boldsymbol{Y}) \neq 0$. Let $r \in \{1, \dots, 5\}$ be the smallest index with $G_r \neq 0$.

Step 2: the first term has a detectable off-diagonal value. The free polynomial $[D, G_r]$ is nonzero. Indeed the centralizer of the single free generator D is $K[D]$, whereas G_r has positive total Y-degree. The centralizer assertion follows directly by comparing words: for homogeneous P of positive degree, $DP = PD$ forces every word of P to begin with D; removing that first D and repeating leaves only powers of D.

Moreover $[D, G_r]$ is not an identity of M_n for $n \geq 3$. For $n \geq 4$ its total degree six is less than $2n$. For $n = 3$, use also the uniqueness of the multilinear identity in minimal degree: that degree-six space is Ks_6; see [1] and the recollection before Lemma 3.4 in [2]. If $r \leq 4$, choose a nonzero multihomogeneous component of $[D, G_r]$. It has D-degree $6 - r \geq 2$. Its complete polarization in D is nonzero, is symmetric in the new D variables, and would be a multilinear degree-six identity. It cannot be a nonzero multiple of the alternating s_6. If $r = 5$, $[D, G_5]$ is multilinear in six variables, but all its words have D first or last. Since s_6 has words with D in an interior position, it cannot be a multiple of s_6. This proves the assertion.

The nonvanishing locus of this commutator meets the dense open set where D has distinct eigenvalues. After simultaneous conjugation take D diagonal. Then $G_r(D; \boldsymbol{Y})$ has a nonzero off-diagonal entry. Relabel the basis so that one such entry is $(b, 1)$, $b \neq 1$. Choose $z_2, \dots, z_n$ all nonzero with

$$X = \sum_{j=2}^{n} z_j E_{1j}, \qquad \operatorname{tr}\Big(XG_r(D; \boldsymbol{Y})\Big) \neq 0.$$

This is possible because the displayed trace is a nonzero linear form in the z_j. Let $a_s = X$ and $a_i = D$ for $i \neq s$. Then

$$f(a) = N = c_s \sum_{j=2}^{n} z_j (d_1 - d_j)^5 E_{1j}$$

is nonzero rank-one square-zero, with every indicated row entry nonzero.

Step 3: the leading trace tensor survives restriction to the fiber. Minimality of r gives $d^k h_a = 0$ for $0 \le k < r$. To see this for arbitrary source variations, expand $h = \mathrm{tr}(x_s G)$: a term varying k slots other than s comes from G_k; a term also varying slot s comes from G_{k-1}. Repeated variations of a single slot are zero by multilinearity. The same expansion, with the chosen $\boldsymbol{Y}$, shows that $C = d^r h_a \neq 0$.

Let W be spanned by all slot-s variations and simultaneous conjugation tangents $v_H = ([H, a_i])_i$. The restriction $dg_a|_W$ is onto $\mathfrak{sl}_n$: slot-s variations give all hollow matrices, and the diagonal parts of $[E_{j1}, N]$ give nonzero multiples of $E_{jj} - E_{11}$ for every $j \ge 2$. Consequently $T_a(M_n^6) = \ker dg_a + W$.

The symmetric tensor C annihilates W in any argument. For slot-s directions this follows from $G_{r-1} = 0$. For conjugation directions differentiate $dh_x(v_H(x)) = 0$ a total of $r-1$ times. Its leading term is $C(v_H(a), \cdot, \dots, \cdot)$; all remaining terms involve derivatives of h of order less than r and vanish. This includes $r = 1$ directly. It follows that the restriction of C to $(\ker dg_a)^r$ is still nonzero: otherwise decomposition of each argument modulo W would make C zero everywhere.

The fiber $Y = g^{-1}(N)$ is smooth at a. In formal coordinates on Y,

$$\phi(u) = a + u + O(u^2), \quad u \in \ker dg_a, \qquad h(\phi(u)) = \frac{1}{r!} C(u, \dots, u) + O(u^{r+1}).$$

All lower ambient derivatives vanish, so corrections cannot change this leading term. In characteristic zero a nonzero symmetric tensor has a nonzero diagonal polynomial. Therefore $h|_Y \neq 0$, and Proposition 1.9 gives $\mathrm{Val}_n(f) = M_n$. □

3. The four-parameter collision family

3.1. **Statement and conventions.** Throughout K is algebraically closed of characteristic zero. For $D, E, A, B \in \mathrm{M}_n(K)$ set

$$P = [D, E], \quad Q = [D, A], \quad R = [E, B],$$
$$S = [D, B], \quad T = [E, A], \quad Z = [A, B],$$
$$L = PZ + ZP - QR - RQ + ST + TS = s_4(D, E, A, B),$$
$$C = P^2 Z + ZP^2 - QPR - RPQ + SPT + TPS,$$
$$H_{\alpha,\beta,\gamma}(D, E, A, B) = \alpha PL + \beta LP + \gamma C.$$

Define additionally

$$\mathcal{T} = s_3(P, Q, R) + s_3(P, S, T), \qquad \mathcal{F}_{\alpha,\beta,\gamma,\delta} = H_{\alpha,\beta,\gamma} + \delta \mathcal{T},$$

where s_3 is the standard alternating polynomial in three variables.

Theorem 3.1. *For every $n \ge 3$, the restricted value set of $\mathcal{F}$ on $\det[D, E] \neq 0$ is*

$$\begin{cases} \{0\}, & (\alpha, \beta, \gamma, \delta) = 0, \\ \mathfrak{sl}_n(K), & \delta = 0,\ \alpha + \beta + \gamma = 0,\ (\alpha, \beta, \gamma) \neq 0, \\ \mathrm{M}_n(K), & (\alpha + \beta + \gamma, \delta) \neq (0, 0). \end{cases}$$

We first prove the $\delta = 0$, $\gamma \neq 0$, traceful branch, then treat the remaining coefficient branches.

Matrix units E_{ij} in the mathematical text are one based. Jacobian row and column labels are zero based. All images below consist of single actual evaluations. In particular, addition of unrelated values is never used.

Theorem 3.2. *If $\gamma \neq 0$ and $\alpha + \beta + \gamma \neq 0$, then for every $n \geq 3$,*

$$\{H_{\alpha,\beta,\gamma}(D, E, A, B) : \det[D, E] \neq 0\} = \mathrm{M}_n(K).$$

Normalize $a = \alpha/\gamma$, $b = \beta/\gamma$ and write $F = H_{a,b,1}$. Scaling A removes the overall factor γ. Thus the parameter domain is $a + b + 1 \neq 0$.

3.2. **Smooth points and polynomial values.**

Lemma 3.3. *Suppose that for a fixed size n the restricted map F on $\det P \neq 0$ has a point with value a nonzero rank-one square-zero matrix and differential of rank n^2. Then its image contains every nonscalar matrix.*

Proof. The Jacobian criterion makes the map smooth at the point, and a smooth morphism is open. Consequently its image contains an open neighborhood of the specified nilpotent value. The restricted image is invariant under simultaneous conjugation and under scalar multiplication of A. For every nonscalar target choose a similar representative $W = \left(\begin{smallmatrix} u & v \\ w & X \end{smallmatrix}\right)$ with $v \neq 0$. For $t \neq 0$ define

$$R(t) = t \begin{pmatrix} t^{-1} & 0 \\ 0 & I \end{pmatrix} W \begin{pmatrix} t & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} tu & v \\ t^2 w & tX \end{pmatrix}.$$

Its value at zero is a nonzero rank-one square-zero matrix, hence conjugate to the given value. After this conjugation the inverse image of the open neighborhood is an open subset of $\mathbb{A}^1$ containing zero, so it contains a nonzero t. Scale A by t^{-1} and undo the conjugation. The condition $\det P \neq 0$ is preserved. □

Lemma 3.4. *For every $n \geq 3$ and $a + b + 1 \neq 0$, all scalar matrices are values of F with P invertible.*

Proof. Choose clock and shift matrices U, V with $UV = qVU$, where q is a primitive nth root of unity. Put

$$D = U, \quad E = V, \quad A = U^{-1}, \quad B = U^{-1}V^{-2}.$$

Then $P = (1 - q^{-1})UV$ is invertible. All products in PL, LP, C have total bidegree zero in U, V, so these three matrices are scalar. Cyclicity of trace gives $\operatorname{tr} C = \operatorname{tr} PL = \operatorname{tr} LP$, hence they are equal. Direct multiplication using $UV = qVU$ gives

$$PL = LP = C = -\frac{(q-1)^5(q+1)}{q^2} I_n.$$

The coefficient is nonzero since $q \neq \pm 1$. Thus F is a nonzero scalar at this tuple. Scaling A gives every scalar, including zero. □

3.3. **The three base sizes.** All Jacobians in this section have row-major output order. Their columns are the entries of D, E, A, B, in this order and each in row-major order. Column lists are zero based. All minors use every output row.

3.3.1. *Size three.* Take

$$D = E_{12} + E_{23} + E_{21}, \quad E = E_{21} + 2E_{32}, \quad A = E_{12}, \quad B = E_{31}.$$

Direct multiplication gives

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & -2 \end{pmatrix}, \qquad PL = LP = 0, \qquad C = -6E_{32}.$$

Thus P is invertible and the value is nonzero rank-one square-zero for every a, b. Put $z = a+b+1$, $u = a-2b+1$ and $v = 2a-b-1$. The following five column lists give the indicated exact determinants:

columns	determinant
$(1, 2, 5, 9, 11, 13, 18, 27, 29)$	$-31104\, bz(z+1)$
$(0, 1, 2, 5, 9, 11, 20, 27, 31)$	$-62208\, uz^2v$
$(0, 1, 2, 5, 9, 11, 20, 27, 29)$	$-124416\, bz^2v$
$(1, 2, 5, 9, 11, 13, 18, 27, 31)$	$-15552\, uz(z+1)$
$(1, 2, 5, 9, 11, 13, 20, 27, 29)$	$31104\, bz\, W(a, b)$

where $W(a, b) = 2a^2+ab-5a-b^2-b+4$. Suppose that $z \neq 0$ and the first three determinants vanish. The second and third imply either $v = 0$ or $u = b = 0$; the latter would give $z = 0$. Hence $b = 2a-1$. The first determinant then gives $b = 0$ or $z+1 = 0$, leaving exactly

$$(a, b) = (1/2, 0), \qquad (a, b) = (-1/3, -5/3).$$

At the first point the fourth determinant is -87480; at the second point the fifth is 276480. Thus the Jacobian has rank nine for every allowed parameter. The product rule in Section 10.1 specifies the full matrix. Lemmas 3.3 and 3.4 prove Theorem 3.2 in size three.

3.3.2. *Size four.* Put $J_4 = E_{12} + E_{23} + E_{34} + E_{41}$ and take

$$D = \operatorname{diag}(0, 1, 2, 3), \quad E = J_4, \quad A = E_{11}, \quad B = E_{32}.$$

Then $\det P = 3$, $PL = LP = 0$ and $C = E_{12}$, independently of a, b. The columns

$$(1, 2, 4, 5, 6, 7, 8, 11, 20, 36, 37, 38, 46, 50, 52, 62)$$

give determinant

$$-8(3a+b)(b-1)(a+b-1)(a+b+1)(3a-b-1).$$

Outside its zero locus the assertion follows. The allowed part of that zero locus is contained in four lines. The exact gcds of specified maximal minors on these lines, up to nonzero rational factors, are:

(a,b)	gcd	remaining condition
$(x,1)$	$x+2$	$a+b+1=0$
$(x,1-x)$	1	none
$(x,3x-1)$	x	$a+b+1=0$
$(x,-3x)$	$2x-1$	$a+b+1=0$

Every listed zero is excluded by $a+b+1\neq 0$. Explicit minors and their factorizations are given in Section 10.2. Univariate gcd 1 means that no common zero is possible over any extension of $\mathbb{Q}$; the same observation applies after deleting the trace-zero factor. This verifies every parameter, not merely sampled rational values. The two preceding lemmas prove the size-four assertion.

3.3.3. *Size five.* Put $J_5 = E_{12}+E_{23}+E_{34}+E_{45}+E_{51}$ and take

$$D=\operatorname{diag}(0,1,2,3,4),\quad E=J_5+E_{35},\quad A=E_{11},\quad B=E_{32}.$$

Here $\det P=4$, $PL=LP=0$, and $C=E_{12}$. The columns

$$(1,2,5,6,7,8,9,10,14,15,30,55,56,57,58,\\ 62,66,67,72,77,78,79,80,92,97)$$

give determinant

$$384b(4a+b)^2(b-1)(3b-2)(a+b-1)(a+b+1)\\ \cdot(2a-5b+6)(3a-b-1).$$

The boundary factorizations in Section 10.2 give the following gcds, again up to nonzero rational factors:

(a,b)	gcd	zeros with $a+b+1\neq 0$
$(x,0)$	$(x-2)(x+1)$	$(2,0)$
$(x,-4x)$	$(3x-1)(4x+1)^2$	$(-1/4,1)$
$(x,1)$	$(x+2)(4x+1)^2$	$(-1/4,1)$
$(x,2/3)$	$3x+5$	none
$(x,1-x)$	1	none
$(x,(2x+6)/5)$	$7x+11$	none
$(x,3x-1)$	x	none

For both remaining points replace E by

$$E'=J_5+E_{35}+E_{15}.$$

The equations $\det P=4$, $PL=LP=0$ and $C=E_{12}$ remain true. At $(a,b)=(2,0)$ the columns

$$(1,2,5,6,7,8,9,10,14,15,30,52,55,56,57,\\ 66,67,72,77,78,79,80,83,90,97)$$

give determinant -17043554304. At $(a, b) = (-1/4, 1)$ the columns

$$(2, 5, 6, 7, 8, 9, 10, 14, 15, 22, 30, 52, 55, 56, 57,$$
$$58, 66, 67, 72, 77, 78, 80, 90, 97, 98)$$

give determinant 103950. Hence every allowed parameter is covered. As an additional check, the unmodified size-five seed has determinant 7776000 in the first column list at $(a, b) = (-1, -1)$.

3.4. **The block argument in arbitrary size.**

Lemma 3.5. *Suppose that every matrix in sizes $r_1, \dots, r_k$ is a value of $H_{\alpha,\beta,\gamma}$ with $P = [D, E]$ invertible. If $(\alpha+\gamma, \alpha+\beta, \beta+\gamma) \neq (0, 0, 0)$, the same statement holds in size $r_1 + \cdots + r_k$.*

Proof. Write the target in block form (W_{ij}). For each diagonal block, choose a tuple (D_i, E_i, A_i, B_i) attaining W_{ii} with P_i invertible. Take D, E, B block diagonal. Since H is linear in A, the map on each rectangular entry $A_{ij} = X$ is independent of all other such entries.

Replace D_i by $s_i D_i$ and A_i by $s_i^{-2} A_i$. This keeps W_{ii} unchanged, since the polynomial has degree two in D and degree one in A. Replacing B_i by $B_i + c_i I$ also keeps W_{ii} unchanged. The resulting map on the (i, j) block is

$$M_{ij}(X) + (c_j - c_i) K_{ij}(X),$$

where M_{ij} is independent of all c_i and, with the scaled P_i,

$$K_{ij}(X) = (\alpha + \gamma) P_i^2 X + (\alpha + \beta) P_i X P_j + (\beta + \gamma) X P_j^2.$$

Before scaling, insert $P_i \mapsto s_i P_i$, $P_j \mapsto s_j P_j$. If $\alpha + \gamma \neq 0$, the top s_i-coefficient of $\det K_{ij}$ is nonzero, because $X \mapsto P_i^2 X$ is invertible. If $\beta + \gamma \neq 0$, use the top s_j-coefficient instead. If both coefficients vanish, $\alpha + \beta \neq 0$ and $X \mapsto P_i X P_j$ is invertible. Thus every $\det K_{ij}$ is a nonzero polynomial in the scaling parameters.

Choose the finitely many nonzero s_i simultaneously outside these zero sets. For these fixed choices, each $\det(M_{ij} + (c_j - c_i) K_{ij})$ is a nonzero polynomial in the c_i, with nonzero leading coefficient in $c_j - c_i$. Choose the c_i simultaneously outside their zero sets. Each rectangular equation can now be solved for the prescribed W_{ij}. The block diagonal P remains invertible. □

Proof of Theorem 3.2. Sizes three, four and five were proved above. Every $n \geq 6$ is a sum of threes and fours: $6, 7, 8$ are the initial cases and adding three gives all subsequent sizes. The three coefficients in Lemma 3.5 cannot all vanish unless $\alpha = \beta = \gamma = 0$, in characteristic zero. Apply the lemma. Finally undo the normalization by scaling A. □

3.5. **The remaining traceful two-coefficient pencil.**

Lemma 3.6. *If $\alpha + \beta \neq 0$, then $\alpha PL + \beta LP$ takes every value in $\mathrm{M}_n(K)$ with P invertible, for every $n \geq 3$.*

Proof. Normalize $\alpha = x$, $\beta = 1 - x$. The following seeds give exact parameter coverage in sizes three, four and five. Write $J_n = \sum_{i=1}^{n-1} E_{i,i+1}$ and $E(w) = J_n^2 + \sum_{i=1}^{n-1} w_i E_{i+1,i}$, and take $D = J_n$. The two seeds in each size are as follows:

n	w	A	B
3	$(1, 4)$	E_{13}	$E_{11} + E_{32}$
3	$(5, 4)$	E_{13}	$-E_{12} - E_{21} + E_{33}$
4	$(1, 4, 9)$	E_{14}	$-E_{11} + E_{24} - E_{32} - E_{43}$
4	$(2, 9, 28)$	E_{14}	$-E_{11} + E_{24} - E_{32} - E_{43}$
5	$(1, 4, 9, 16)$	E_{51}	$-E_{11} - E_{41} - E_{42} + E_{52} - E_{55}$
5	$(2, 9, 28, 65)$	E_{51}	$-E_{11} - E_{41} - E_{42} + E_{52} - E_{55}$

Their output values are, in the same order,

$$(10x - 8)E_{13}, \qquad (12 - 27x)E_{13},$$
$$(24x - 30)E_{13} + (63 - 70x)E_{14},$$
$$(357x - 399)E_{13} + (672 - 720x)E_{14},$$
$$(289x - 17)E_{51}, \qquad (4489x - 134)E_{51}.$$

Thus all outputs have rank at most one and square zero. The determinants of P in the three pairs are respectively

$$(-12, 20), \quad (-135, -7448), \quad (-1680, -639730).$$

For each size, Section 10.3 gives two explicit full Jacobian minors, one at each seed, whose gcd is 1. Consequently every parameter has a seed with surjective differential. The proof of Lemma 3.3 applies to a nonzero rank-one output. If a selected output is zero instead, smoothness gives an open neighborhood of zero; every line through zero meets it away from zero, so scalar multiplication gives all matrices directly. Scalar targets are also supplied by the clock construction in Lemma 3.4, whose value for the present pencil is nonzero. Finally Lemma 3.5 and the sums of sizes three and four give all $n \geq 6$. No limiting argument at $\gamma = 0$ is used. □

3.6. **The trace-zero plane.**

Lemma 3.7. *For every $n \geq 3$, the restricted image of*

$$(D, E, A, B) \longmapsto [[D, E], s_4(D, E, A, B)] \qquad \text{on } \det[D, E] \neq 0$$

is $\mathfrak{sl}_n(K)$.

Proof. Put $P = [D, E]$ and $L = s_4(D, E, A, B)$. Choose

$$D = \sum_{i=1}^{n-1} E_{i,i+1}, \qquad E = \sum_{i=1}^{n-1} iE_{i+1,i},$$
$$A = \sum_{i=1}^{n-1} iE_{i,i+1}, \qquad B = \operatorname{diag}(0^2, 1^2, \ldots, (n-1)^2).$$

At this tuple,

$$P = I - nE_{nn}, \qquad L = 4\sum_{i=1}^{n-2} E_{i,i+1} - 2(n-2)E_{n-1,n}.$$

Consequently $\det P = -(n-1) \neq 0$, and

$$[P, L] = -2n(n-2)E_{n-1,n}$$

is nonzero, rank one and square zero. We prove that the differential of $[P, L]$ at this same tuple is onto $\mathfrak{sl}_n$.

We compute the differential in the A directions. Set $w_i = i$ for $1 \leq i < n$, $w_0 = w_n = 0$, $b_i = (i-1)^2$, $t_i = 2i - 1$, and $p_i = w_i - w_{i-1}$. With D, E, B fixed and A replaced by U, write $T(U) = s_4(D, E, U, B)$, and write $\{X, Y\} = XY + YX$. The identity

$$T(U) = \{P, [U, B]\} - \{[D, U], [E, B]\} + \{[D, B], [E, U]\}$$

gives

$$\begin{aligned} T(U)_{ij} = c_{ij}U_{ij} &+ w_j(t_j - t_i)U_{i+1,j+1} \\ &+ w_{i-1}(t_{j-1} - t_{i-1})U_{i-1,j-1}, \end{aligned}$$

where out-of-range entries are zero and

$$c_{ij} = (p_i + p_j)(b_j - b_i) + w_{i-1}t_{i-1} + w_it_i - w_{j-1}t_{j-1} - w_jt_j.$$

In particular this formula gives the displayed L by substituting $U = \sum iE_{i,i+1}$. For $1 \leq i < n$, set $c_i = c_{in}$. Then

$$c_i = -2(n-2) - (n-1-i)\Big((n+2)i + n^2 - n - 4\Big)$$

is a strictly negative integer, hence is nonzero in characteristic zero. Since $c_{ni} = -c_i$ and the upper-left $(n-1)$-block of P is scalar, the same band formula gives exactly

$$d_A[P, L](E_{in}) = nc_iE_{in}, \qquad d_A[P, L](E_{ni}) = nc_iE_{ni}.$$

Thus the differential contains every matrix unit joining the last coordinate to the first $n-1$ coordinates.

Let $Z \in M_n$ annihilate the full differential under the trace pairing. The preceding A directions force both off-diagonal blocks of Z relative to $K^{n-1} \oplus K$ to vanish; equivalently, $[Z, P] = 0$. For all variations of the original inputs, therefore,

$$0 = \operatorname{tr}\Big(Z\, d[P, L]\Big) = \operatorname{tr}\Big([L, Z]dP\Big).$$

The differential $dP = [dD, E] + [D, dE]$ is onto $\mathfrak{sl}_n$. Indeed its trace-pairing annihilator is the common centralizer of D, E. A matrix commuting with the regular nilpotent D is a polynomial in D. If it also commutes with E, it commutes with $P = I - nE_{nn}$; its last column then forces every positive-degree coefficient of that polynomial to vanish. The common centralizer is exactly KI. It follows that $[L, Z]$ is scalar; as a traceless commutator it is zero.

All superdiagonal coefficients of L are nonzero, so L is regular nilpotent. Its centralizer is $K[L]$. A polynomial in L that also commutes with P has all positive-degree coefficients zero, by the same last-column argument. Hence Z is scalar. The annihilator of the full differential is exactly KI, proving that its rank is n^2-1.

Smoothness supplies an open neighborhood of this rank-one square-zero value in $\mathfrak{sl}_n$, still with P invertible. The restricted image is invariant under conjugation and scalar multiplication of A; it contains zero by taking $A=0$. The rank-one degeneration argument of Lemma 3.3, with target $\mathfrak{sl}_n$, now gives every traceless matrix as a single actual value. □

Lemma 3.8. *If $\alpha+\beta+\gamma=0$ and $(\alpha,\beta,\gamma)\neq 0$, then*

$$\{H_{\alpha,\beta,\gamma}(D,E,A,B) : \det P \neq 0\} = \mathfrak{sl}_n(K) \qquad (n\geq 3).$$

Proof. Free cyclic-word expansion gives $\operatorname{tr}(PL)=\operatorname{tr}(LP)=\operatorname{tr} C$, so the image is contained in $\mathfrak{sl}_n$. If $\gamma=0$, the polynomial is a nonzero multiple of $[P,L]$, so Lemma 3.7 gives the assertion in every size. We may therefore suppose $\gamma\neq 0$ and normalize $(\alpha,\beta,\gamma)=(x,-x-1,1)$. The three seeds used in the preceding three-coefficient proof still have outputs $-6E_{32}$, E_{12}, and E_{12}, respectively. Delete the last diagonal output coordinate, identifying $\mathfrak{sl}_n$ with K^{n^2-1}. Section 10.3 gives respectively $2,2,4$ explicit maximal minors whose gcds are all 1. For example, in size three the determinants are

$$-3456(x+1)(3x-1), \qquad -10368x(x+1),$$
$$-10368x(2x+3).$$

Their lack of a common root is immediate. The factorizations and the exclusion of common roots in all three sizes are given in Section 10.3.

Apply the proof of Lemma 3.3 with target $\mathfrak{sl}_n$ and rank n^2-1. The degeneration there remains inside $\mathfrak{sl}_n$, since $\operatorname{tr} R(t) = t\operatorname{tr} W = 0$. Every nonzero traceless matrix is nonscalar in characteristic zero. The value zero is obtained by taking $A=0$, keeping P invertible. This proves the base sizes.

For the block step first make the target hollow by similarity. For completeness, a nonscalar traceless endomorphism has a vector u with u, Tu independent. Choose a complementary hyperplane containing Tu; the first diagonal entry is zero and the complementary compression still has trace zero. Induction makes its diagonal zero; a scalar traceless compression is already zero. Thus every diagonal block of sizes three and four has trace zero. Realize these blocks by the base cases and solve the rectangular equations by Lemma 3.5. Its coefficient condition holds for every nonzero (α,β,γ). This proves the assertion in all larger sizes. □

3.7. **The branch with nonzero fourth coefficient.** Put

$$\mathcal{T} = s_3(P,Q,R)+s_3(P,S,T), \qquad \mathcal{F}_{\alpha,\beta,\gamma,\delta} = H_{\alpha,\beta,\gamma}+\delta\mathcal{T},$$

where $s_3(U,V,W)=UVW+VWU+WUV-UWV-WVU-VUW$. The formula for $\mathcal{T}$ is part of the specification of the family.

Lemma 3.9. *For arbitrary α, β, γ and $\delta \neq 0$, the restricted image of $\mathcal{F}$ is $\mathrm{M}_4(K)$ in size four. For every $n \geq 2$ its restricted image contains all scalar matrices.*

Proof. Set $B = A$ and write

$$\Psi(D, E, A) = s_3([D, E], [D, A], [E, A]).$$

Direct expansion gives $PL = LP = C = 0$ and $\mathcal{T} = 2\Psi$. Thus the restriction is $2\delta\Psi$, independently of the first three coefficients.

For size four use

$$D = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad A = E_{24}.$$

Then $\det P = 3$ and $\Psi = E_{21} - E_{23} + 2E_{24}$. A full 16×16 differential minor of Ψ is -60. Section 10.4 gives the columns and proves their independence by a sequence of homogeneous linear equations. Since the differential of a restriction is a restriction of the full differential, this gives a smooth rank-one square-zero value of $\mathcal{F}$ for every $\delta \neq 0$. The argument of Lemma 3.3 applies to $\mathcal{F}$, which is linear in A and therefore has a cone as its restricted image.

For scalar targets choose $UV = qVU$, $U^n = V^n = I$, and $D = U$, $E = V$, $A = B = U^{-1}V^{-1}$. Exact multiplication yields

$$\Psi(D, E, A) = \frac{3(q-1)^4}{q} I, \qquad \mathcal{F}(D, E, A, A) = \frac{6\delta(q-1)^4}{q} I.$$

This is nonzero for a primitive nth root with $n \geq 2$, and $P = (1 - q^{-1})UV$ is invertible. Scaling B gives every scalar target, including zero, while P stays invertible. □

Lemma 3.10. *Every matrix of size $n \geq 4$ is similar to a block matrix whose first 2×2 diagonal block is scalar.*

Proof. If the minimal polynomial has degree at least four, choose u, Tu, T^2u, T^3u independent. Let $U = \langle u, T^2u \rangle$ and choose a complement containing Tu, T^3u. The compression is zero. If the degree is three, take an invariant cyclic summand of dimension three, a cyclic vector u in it, and an eigenvector w of eigenvalue λ in its invariant complement. The latter exists since $n \geq 4$. On $\langle u, w \rangle$ choose a complement containing $Tu - \lambda u$; the compression is λI_2. If the degree is at most two, an eigenspace has dimension at least two: this is immediate in the semisimple case, while for $(T - \lambda I)^2 = 0$ the kernel has dimension at least $\lceil n/2 \rceil$. Any two-dimensional subspace of that eigenspace works. □

Lemma 3.11. *Suppose the diagonal blocks of a target have been attained by $\mathcal{F}$ with the corresponding P_i invertible. Then the off-diagonal blocks can also be attained, provided $\delta \neq 0$.*

Proof. Use block diagonal D, E, B and solve in the independent blocks of A. If $(\alpha, \beta, \gamma) \neq 0$, the proof of Lemma 3.5 still applies: central shifts of each B_i do not change the $\mathcal{T}$ contribution to any rectangular A_{ij} equation, since $R = [E, B]$ and $S = [D, B]$ remain unchanged.

It remains to treat the pure $\delta\mathcal{T}$ case. On diagonal blocks replace B_i by $B_i - r_i D_i$. This preserves $\mathcal{T}$, since $R_i \mapsto R_i + r_i P_i$, S_i is unchanged, and $s_3(P_i, Q_i, P_i) = 0$. Now shift D_i by $d_i I$. These shifts also preserve all diagonal values. For a rectangular block $X = A_{ij}$ the coefficient of $d_i - d_j$, apart from the nonzero factor δ, is

$$K_{ij}(X) = P_i X R_j - R_i X P_j + [R_i, P_i]X + X[R_j, P_j].$$

After the preceding shear this becomes

$$K^{(0)}_{ij}(X) + (r_j - r_i) P_i X P_j.$$

Since P_i, P_j are invertible, its determinant is a nonzero polynomial in $r_j - r_i$. Choose all r_i simultaneously so that all these operators are invertible. Then choose the d_i outside the finitely many zero sets of the full rectangular determinants, whose top coefficients are nonzero. Solve all equations independently. The commutators P_i have remained invertible throughout. □

3.8. **A flat point in size three.**

Lemma 3.12. *Put*

$$a = (D, E, A) = (E_{13} - E_{21},\ E_{12} + E_{31} - E_{33},\ E_{12}).$$

Then $\det[D, E] = 2$ *and* $\Psi(a) = -E_{32} =: N$. *The morphism* $\Psi : M_3(K)^3 \to M_3(K)$ *is flat at* a. *Consequently its image on* $\det[D, E] \neq 0$ *contains a Zariski-open neighborhood of* N.

Proof. At this tuple we have

$$P = 2E_{11} - E_{13} - E_{22} - E_{33}, \qquad Q = E_{11} - E_{22}, \qquad R = E_{32}.$$

Direct multiplication gives $\det P = 2$ and $\Psi(a) = N$. Let $\pi : M_3 \to K^8$ delete the $(2, 3)$ coordinate and put $\ell(Y) = Y_{21} + Y_{23}$. Then (π, ℓ) is a linear isomorphism. Write $g = \pi\Psi$ and $h = \ell\Psi$.

We first show that $\operatorname{im} d\Psi_a = \ker \ell$. Put $C_0 = E_{12} + E_{32}$, so that $\ell(Y) = \operatorname{tr}(C_0 Y)$. The trace gradients of $\ell(s_3(P, Q, R))$ at this tuple, with respect to P, Q, R, are respectively $0, 0, C_0$. For example, the first is

$$[Q, R]C_0 + C_0[Q, R] + QC_0R - RC_0Q,$$

and the other two are obtained cyclically. Since $[C_0, E] = [C_0, A] = 0$, differentiation of $R = [E, A]$ gives $dh_a = 0$. Thus $\operatorname{im} d\Psi_a \subseteq \ker \ell$. Conjugation equivariance gives

$$\mathcal{O} = [M_3, N] = \langle E_{12}, E_{31}, E_{32}, E_{22} - E_{33} \rangle \subseteq \operatorname{im} d\Psi_a.$$

Four further derivative values are

input variation	$d\Psi_a$ of that variation
$\dot{E} = E_{13}$	$-2E_{22} - E_{33}$
$\dot{E} = E_{23}$	E_{13}
$\dot{A} = E_{11}$	$E_{11} - 3E_{12} - E_{22} + 3E_{31}$
$\dot{A} = E_{23}$	$-E_{13} + E_{21} - E_{23}$.

Their classes form a basis of $\ker \ell/\mathcal{O}$: they supply successively the remaining E_{22}, E_{33} direction, E_{13}, the E_{11} direction, and $E_{21} - E_{23}$. In the first step the two diagonal coefficient vectors have determinant $-3 \neq 0$. It follows that $d\Psi_a$ has rank eight and g is smooth at a.

Consider the two triples

$$u = \left(\frac{3}{2}E_{23}, -E_{11} - \frac{1}{2}E_{13}, -E_{11} + E_{22}\right),$$
$$v = \left(-\frac{15}{8}E_{23}, \frac{13}{8}E_{13} + \frac{1}{2}E_{23}, \frac{5}{4}E_{11}\right).$$

Matrix multiplication gives

$$\Psi(a + tu + t^2 v) = N + t^3 \begin{pmatrix} 5/8 & 0 & 1/8 \\ 3/2 & -15/4 & -1/2 \\ 0 & 0 & -5/2 \end{pmatrix} + O(t^4),$$
$$h(a + tu + t^2 v) = t^3 + O(t^4).$$

Choose eight fixed linear input directions on which dg_a is invertible, and complete them to affine coordinates on M_3^3. The formal implicit function theorem corrects these eight coordinates by $O(t^3)$ to give a formal curve $\widehat{a}(t)$ with $g(\widehat{a}(t)) = g(a)$. Since $dh_a = 0$, this correction changes h only by $O(t^4)$. Thus $h(\widehat{a}(t)) = t^3 + O(t^4)$.

The local ring of $g^{-1}(g(a))$ at a is regular of dimension $27-8 = 19$. The formal curve shows that h is nonzero in its completion, and hence in the local ring itself. Also $h(a) = 0$. This nonzero nonunit in a regular local domain cuts the dimension down to $18 = 27 - 9$. Since (π, ℓ) is a target-coordinate isomorphism, this quotient is the local fiber of Ψ over N. Miracle flatness [5] therefore implies that Ψ is flat at a. Restrict to its open flat locus and to $\det[D, E] \neq 0$. A flat morphism locally of finite presentation is open [7], so its image contains an open neighborhood of N. Every nonempty fiber over a K-point has a K-point, since K is algebraically closed. This is therefore a neighborhood of actual values. All displayed data and calculations are rational, so the argument holds over every algebraically closed field of characteristic zero. □

Proposition 3.13. *For $n = 3$ and every $\delta \neq 0$, independently of α, β, γ,*

$$\{\mathcal{F}_{\alpha,\beta,\gamma,\delta}(D, E, A, B) : \det[D, E] \neq 0\} = M_3(K).$$

Proof. On $B = A$, the first three summands vanish and $\mathcal{F} = 2\delta\Psi$. Lemma 3.12 puts an open neighborhood of $2\delta N$ in the restricted image of $\mathcal{F}$. This image is invariant under simultaneous conjugation and under scalar multiplication, by

scaling the independent input B. The degeneration in the proof of Lemma 3.3 therefore gives every nonscalar target. The clock substitution in Lemma 3.9 gives every scalar target, also with P invertible. □

Proposition 3.14. *If $\delta \neq 0$, then for every $n \geq 3$,*

$$\{\mathcal{F}_{\alpha,\beta,\gamma,\delta}(D, E, A, B) : \det P \neq 0\} = \mathrm{M}_n(K).$$

Proof. The base sizes three and four are Proposition 3.13 and Lemma 3.9. For $n \geq 5$, apply Lemma 3.10. The scalar block of size two is attained by the clock substitution in Lemma 3.9; at $q = -1$ its value is $-96\delta I_2$, and scaling B supplies the prescribed scalar. The complementary block of size $n - 2 \geq 3$ is attained by induction. All input commutators are invertible. Lemma 3.11 supplies the rectangular blocks. Undo the similarity of the target. □

3.9. **Proof of the classification.**

Proof of Theorem 3.1. The zero coefficient vector gives the zero polynomial. When $\delta = 0$ and $\alpha + \beta + \gamma = 0$, Lemma 3.8 proves the stated trace-zero image for every nonzero coefficient vector. When $\delta = 0$ and the sum is nonzero, use Theorem 3.2 if $\gamma \neq 0$ and Lemma 3.6 otherwise. Proposition 3.14 handles $\delta \neq 0$. All statements preserve $\det P \neq 0$. This includes every coefficient vector in size three as well. □

We have $\operatorname{tr} \mathcal{T} = 3 \operatorname{tr}(P[Q, R] + P[S, T])$. The first trace functional $\operatorname{tr}(PL)$ changes sign under $A \leftrightarrow B$, whereas $\operatorname{tr} \mathcal{T}$ is unchanged. Both are nonzero on M_n for $n \geq 3$ by the two clock evaluations. Thus they are independent, and the trace-zero locus is $\delta = 0$, $\alpha + \beta + \gamma = 0$.

The remaining parameter calculations are explicit in Section 10.3; the size-four alternating map is treated in Section 10.4. The commutator branch is covered by the uniform band calculation in Lemma 3.7, and the size-three alternating map by the normal cubic and flatness argument above.

4. Cubic values with one common hollow basis

The following result on a common diagonal will be used in the collision argument.

Theorem 4.1. *Let $n \geq 3$, and write*

$$q(x, y, z) = axyz + bxzy + cyxz + dyzx + ezxy + \ell zyx,$$
$$A = a + d + e, \qquad B = b + c + \ell.$$

Then the set of single evaluations on three traceless matrices is

$$q(\mathfrak{sl}_n^3) = \begin{cases} \{0\}, & q = 0, \\ \mathfrak{sl}_n, & q \neq 0,\ A = B = 0, \\ M_n, & (A, B) \neq (0, 0). \end{cases}$$

Consequently the same classification holds for

$$F_q = q(P, Q, R), \qquad P = [x_1, x_2], \quad Q = [x_3, x_4], \quad R = [x_5, x_6].$$

The same classification also holds for the shared-factor map

$$H_q(D, X, Y, Z) = q([D, X], [D, Y], [D, Z]).$$

For every nonzero q, this map has an actual nonzero rank-one square-zero value with differential onto $\mathfrak{sl}_n$ if $A = B = 0$, and onto M_n otherwise. Thus the common factor imposes no additional image restriction for a cubic, in any size $n \geq 3$.

Proof. Put $D = \operatorname{diag}(d_i)$ and $E = \operatorname{diag}(e_i)$. The multiplier of the first-slot map on E_{ij} is

$$p_{ij} = (a + b)d_j e_j + c d_i e_j + e e_i d_j + (d + \ell) d_i e_i.$$

Here the coefficient e in the third summand is the coefficient of zxy; the subscripts distinguish the diagonal entries of E. If this four-variable polynomial vanishes identically, then $b = -a$, $c = e = 0$, $\ell = -d$, so

$$q(x, y, z) = ax[y, z] + d[y, z]x.$$

Its second-slot multiplier at (D, X, E) is $(ad_i + dd_j)(e_j - e_i)$, which is nonzero unless $a = d = 0$. Thus every nonzero q has an active slot with a nonzero multiplier. Relabel the variables so that this is the first slot, and relabel the coefficients accordingly. This preserves the condition $A = B = 0$.

For $n \geq 3$, the projection from traceless diagonal matrices to any two specified diagonal entries is onto K^2. Therefore each p_{ij}, $i \neq j$, remains a nonzero polynomial after imposing $\sum d_i = \sum e_i = 0$. The product of these polynomials is nonzero on the affine space of two traceless diagonal matrices. Choose D, E there with every $p_{ij} \neq 0$. The first-slot map on hollow matrices is then onto the hollow matrices, so hollow similarity gives $\mathfrak{sl}_n \subseteq q(\mathfrak{sl}_n^3)$. Cyclicity of trace gives

$$\operatorname{tr} q(X, Y, Z) = A \operatorname{tr}(XYZ) + B \operatorname{tr}(XZY).$$

This proves the trace-zero case, including equality rather than just containment. The reverse implication for trace vanishing follows by using the triples (E_{12}, E_{23}, E_{31}) and (E_{12}, E_{31}, E_{23}), all of whose entries are traceless.

Now assume $(A, B) \neq (0, 0)$ and put $S = A + B$. Let $X = \sum_{j=2}^{n} E_{1j}$. At the traceless input (X, D, E) the value

$$N' = q(X, D, E) = \sum_{j=2}^{n} p_{1j} E_{1j}$$

is nonzero rank-one square-zero. The first-slot derivative contains every hollow matrix. Simultaneous conjugation gives all $[H, N']$ in the restricted differential image. Modulo hollow matrices, $H = E_{j1}$ gives $p_{1j}(E_{jj} - E_{11})$. Hence this differential image contains $\mathfrak{sl}_n$. All these are legitimate tangent directions to $\mathfrak{sl}_n^3$.

If $S \neq 0$, additionally choose D, E so that $d_1 e_1 \neq d_2 e_2$. This is a nonempty open condition on the same parameter space (for example use diagonal entries $(1, 0, -1, 0, \ldots)$ for both matrices). Variation of X by $E_{11} - E_{22}$ has output trace $S(d_1 e_1 - d_2 e_2) \neq 0$. If $S = 0$, then $A = -B \neq 0$ and $\operatorname{tr} q = A \operatorname{tr}(X[Y, Z])$. Additionally choose $e_1 \neq e_2$; variation of Y by E_{21} has output trace $A(e_1 - e_2) \neq$

0. In either case the differential of $q : \mathfrak{sl}_n^3 \to M_n$ has rank n^2 at (X, D, E). Smoothness produces an open image neighborhood of N'. The conjugation-and-scaling argument in Lemma 1.7 consequently puts every nonscalar matrix in $q(\mathfrak{sl}_n^3)$.

Scalars can be obtained explicitly. If $S \neq 0$, let T be the cyclic permutation matrix of size n. The three matrices T, T, T^{-2} are traceless for $n \geq 3$, commute, and have product I; their value is SI. If $S = 0$, choose Weyl matrices U, V with $UV = \zeta VU$, where ζ is a primitive nth root of unity. The matrices $U, V, (UV)^{-1}$ are traceless, the three cyclic words have value I, and the three reversed words have value $\zeta^{-1}I$. Thus the value is $A(1 - \zeta^{-1})I \neq 0$. Scaling one input gives every scalar. This proves the restricted cubic classification.

Finally every traceless matrix is a commutator: after making it hollow by similarity, fix a diagonal $H = \operatorname{diag}(h_i)$ with distinct entries and solve $[H, Z] = C$ by $Z_{ij} = C_{ij}/(h_i - h_j)$ for $i \neq j$. The three pairs of variables are disjoint, so the three commutators can be prescribed independently in one six-tuple. Hence $F_q(M_n^6) = q(\mathfrak{sl}_n^3)$, as required for independent pairs. The shared-factor assertion is proved after Lemmas 4.2 and 4.3; it needs compatible common-commutator inputs, which are not furnished by three independent commutator realizations. □

4.1. **Compatible common-commutator lifts.**

Lemma 4.2. *Let $(a, b, c) \neq 0$, $P = [D, E]$, $Q = [D, B]$, and*

$$F = aP^2Q + bPQP + cQP^2, \qquad S = a + b + c.$$

For every $n \geq 3$, F has a rank-one square-zero value with differential onto $\mathfrak{sl}_n$ if $S = 0$, and onto M_n if $S \neq 0$. Consequently its actual image is respectively $\mathfrak{sl}_n$ or M_n.

Proof. First regard P, Q as independent traceless inputs. Choose $P = \operatorname{diag}(p_1, \ldots, p_n)$ generically subject to $\sum p_i = 0$. We require $p_i \neq 0$, all p_i^2 distinct, and

$$k_{ij} = ap_i^2 + bp_ip_j + cp_j^2 \neq 0 \quad (i \neq j).$$

These are nonempty open conditions: any two coordinates on the trace-zero diagonal hyperplane are independent for $n \geq 3$. Put $Q = uv$, where $u = e_1$ and $v = (0, v_2, \ldots, v_n)$ with all $v_j \neq 0$. Then

$$F(P, Q) = N = \sum_{j=2}^{n} k_{1j}v_jE_{1j}.$$

The Q derivative gives every hollow matrix. Conjugation of N gives all traceless diagonal directions modulo the hollow matrices. If $S \neq 0$, a traceless diagonal variation of Q also gives a nonzero trace, since $\operatorname{tr} F = S \operatorname{tr}(P^2Q)$ and the p_i^2 are not all equal. This proves the desired differential assertion for independent P, Q.

We realize this pair as common commutators so that

$$\kappa : (D, E, B) \longmapsto ([D, E], [D, B])$$

is a submersion onto $\mathfrak{sl}_n^2$. Set $V = \ker v$ and choose

$$w_1 = 0, \quad w_2 = \frac{-p_3}{v_2(p_2 - p_3)}, \quad w_3 = \frac{p_2}{v_3(p_2 - p_3)}, \quad w_j = 0 \ (j \geq 4).$$

Thus $vw = 1$ and $vPw = 0$. Relative to $Kw \oplus V$, the first diagonal entry of P is zero and its compression $A = (I - wv)P|_V$ is traceless. Its characteristic polynomial is

$$\det(t - A) = (t - p_2 - p_3) \prod_{j \neq 2,3} (t - p_j).$$

For generic $\boldsymbol{p}$ these roots are distinct. The vectors u and Pw are eigenvectors of A with eigenvalues p_1 and $p_2 + p_3$. Also $\phi = vP|_V \neq 0$ and $\phi(Pw) = -p_2p_3 \neq 0$. Choose an eigenbasis of A and apply the Fourier matrix of order $n - 1$. Its diagonal becomes zero. Both u and Pw now have all coordinates nonzero. Generic rescaling of the eigenvectors makes ϕ nonzero on every new basis vector. Moreover all off-diagonal entries of A are nonzero for generic eigenvalues: they are the nonconstant Fourier coefficients of its generic trace-zero spectrum. Consequently, in this basis, P is hollow with every off-diagonal entry nonzero, and Q has only a first column, all of whose entries below the diagonal are nonzero.

Take D diagonal with any distinct entries in this basis, and solve $[D, E] = P$, $[D, B] = Q$ entrywise. To verify smooth lifting, let (L, M) annihilate $d\kappa$ under the trace pairing. Variation of E, B forces L, M to commute with D, hence they are diagonal. Variation of D gives

$$[E, L] + [B, M] = 0.$$

The first row, using $E_{1j} \neq 0$ and $B_{1j} = 0$ for $j > 1$, forces all diagonal entries of L to agree. The first column, using $B_{j1} \neq 0$, then forces all entries of M to agree. Scalars are zero in the dual of $\mathfrak{sl}_n$, so $d\kappa$ is onto. Thus the preceding differential assertion holds at a realization of the pair by common commutators.

For $S = 0$ the local characterization in $\mathfrak{sl}_n$ gives the whole trace-zero space. For $S \neq 0$ it gives every nonscalar matrix. To include scalars, take $P = \operatorname{diag}(1, \zeta, \ldots, \zeta^{n-1})$ and $Q = P^{-2}$, where ζ is a primitive nth root. Both are traceless for $n \geq 3$ and $F(P, Q) = SI$. The same Fourier basis makes both hollow, so they are common commutators with a regular diagonal D. Scaling B gives all scalars. □

Lemma 4.3. *For every $n \geq 3$ the polynomial*

$$V(D, E, A, B) = s_3([D, E], [D, A], [D, B])$$

has an actual nonzero rank-one square-zero value with differential rank n^2.

Proof. Put $k = n - 1$. In the lower $k \times k$ corner take

$$Y = \sum_{i=1}^{k-1} E_{i,i+1}, \qquad Z = \sum_{i=1}^{k-1} z_i E_{i+1,i}, \qquad H = [Y, Z] = \operatorname{diag}(h_i), \quad h_i = z_i - z_{i-1},$$

where $z_0 = z_k = 0$. Embed them in M_n with first row and column zero, and put $X = \sum_{j=2}^{n} E_{1j}$. Choose the z_i generically so every $h_i \neq 0$. Then

$s_3(X,Y,Z) = XH = N$ is rank-one square-zero with every entry of its first row outside the diagonal nonzero. A diagonal D with distinct entries realizes X, Y, Z independently as $[D, E]$, $[D, A]$ and $[D, B]$.

For a hollow variation U of X the derivative is

$$L(U) = UH + HU + ZUY - YUZ.$$

Its first row and column are arbitrary. On the rth upper diagonal in the lower corner its matrix is tridiagonal, of size $k - r$, with

$$(L_r)_{ii} = h_i + h_{i+r}, \qquad (L_r)_{i,i-1} = z_{i-1}, \qquad (L_r)_{i,i+1} = -z_{i+r}.$$

For $r \geq 2$ this matrix is generically invertible. Indeed set $z_i = t^i$. The diagonal entry has leading degree $i+r$ except at the last position, where it has leading degree $k-1$ and negative leading coefficient. In the tridiagonal determinant every adjacent transposition has strictly smaller degree than the two corresponding diagonal terms. Thus the diagonal product is the unique highest-degree term. For $r = 1$ the row sums vanish, while its upper-left minor of size $k - 2$ has a nonzero leading diagonal product. Hence its rank is $k - 2$. The nonzero upper and lower off-diagonal entries force both endpoints of a left null vector to be nonzero. The lower diagonals have the same ranks: the weighted transpose interchanging Y and Z intertwines L.

It remains to fill one cokernel direction on each first off-diagonal of the lower corner. A variation of Y by the last matrix unit in the first column gives $-e_k x Z$ in that corner, where $x = (1, \ldots, 1)$. It has only a last row, zero diagonal, and nonzero last subdiagonal entry. A variation of Z by the first matrix unit in the first column gives $e_1 x Y$, with only a first row, zero diagonal, and nonzero first superdiagonal entry. These fill the two cokernels. All hollow matrices are therefore in the differential image. Conjugation of N supplies all traceless diagonal directions. Finally, varying Z by the second matrix unit in the first column gives trace $3x_1 = 3 \neq 0$. All these are permitted common-commutator variations or simultaneous conjugation directions in the original inputs, so dV is onto M_n. □

Completion of Theorem 4.1: shared-factor cubics. Write $S = A + B$, with A, B the two cyclic coefficient sums of q. If all three pair identifications in q vanish as polynomials, polarization makes q antisymmetric in every pair of variables. Thus q is a multiple of s_3. For a nonzero multiple, Lemma 4.3 gives the asserted full-rank nilpotent point.

Otherwise relabel the cubic slots so that

$$J(P,Q) = q(P,P,Q) = a'P^2Q + b'PQP + c'QP^2 \neq 0.$$

The sum $a' + b' + c'$ is S. Lemma 4.2 supplies an actual common-commutator point $P = [D, E]$, $Q = [D, B_0]$, with nilpotent value $N = J(P,Q)$, and the required differential if either $S \neq 0$ or $A = B = 0$. Its tangent directions are also tangent directions of H_q on the subspace where the first two commutator inputs agree.

The only remaining case is $S = 0$ and $A = -B \neq 0$. Retain the smooth common-commutator realization from Lemma 4.2. Before its change of basis, P is diagonal with distinct entries and $Q = e_1(0, v_2, \dots, v_n)$, all $v_j \neq 0$. Therefore $[P, Q]$ is nonzero rank-one nilpotent. The common factor D is regular semisimple, so

$$K = [[P, Q], D] \neq 0 :$$

a matrix commuting with D is diagonal in its eigenbasis and cannot be nonzero nilpotent. At the actual shared-factor input $(D, X, Y, Z) = (D, E, E, B_0)$ the value is already N. The restricted map J gives all $\mathfrak{sl}_n$ tangent directions there. Independently vary the first input X by L. Cyclicity gives

$$\operatorname{tr} q([D, L], P, Q) = A \operatorname{tr}([D, L][P, Q]) = A \operatorname{tr}(LK).$$

This is a nonzero linear functional of L. Hence the differential of H_q at this same actual point is onto M_n. No separate realizations of its trace and traceless parts have been combined.

The image of H_q is conjugation invariant and conical, by scaling one of X, Y, Z. The degeneration in Lemma 1.7, applied to the obtained open neighborhood of N, therefore gives every nonscalar target in $\mathfrak{sl}_n$ or M_n, respectively.

For completeness all scalars in the traceful case have a common-factor realization as well. Take a primitive nth root ζ, the cyclic shift T, and a diagonal Λ with $T\Lambda = \zeta\Lambda T$. Set

$$U = T, \qquad V = \Lambda T, \qquad W = (UV)^{-1}.$$

Their cyclic offsets are $1, 1, -2$, all nonzero modulo $n \geq 3$. Thus all three are hollow in this same basis. The cyclic cubic words are I, and the reversed words are $\zeta^{-1}I$, giving

$$q(U, V, W) = (A + \zeta^{-1}B)I.$$

At least one of the two distinct primitive roots ζ, ζ^{-1} makes the coefficient nonzero when $(A, B) \neq (0, 0)$. For any regular diagonal D, solve $[D, X] = U$, $[D, Y] = V$ and $[D, Z] = W$ entrywise. Scaling X gives all scalars in one shared-factor evaluation. Together with the trace-zero identity when $A = B = 0$, this completes the common-D classification in every size $n \geq 3$. □

4.2. **Values on hollow matrices in one basis.**

Corollary 4.4. *Let $n \geq 3$ and $V_n = \{A \in M_n : A_{ii} = 0$ for every $i\}$. For a nonzero multilinear cubic g, the similarity saturation of $g(V_n^3)$ is $\mathfrak{sl}_n$ if g is identically traceless, and M_n otherwise. In particular every traceless target is similar to one actual value on three matrices hollow in the same basis.*

Proof. Apply the shared-factor construction in Theorem 4.1. Its proof gives a smooth nonzero rank-one nilpotent value with the common input D regular semisimple: this is explicit in Lemmas 4.2 and 4.3. Restricting the input space to the open locus where D is regular semisimple preserves smoothness and hence the image neighborhood. This restricted image remains conjugation

invariant and conical, with scaling performed in X, not in D. The local degeneration therefore gives every nonscalar target in the appropriate space. The scalar construction in the proof of Theorem 4.1 also uses regular diagonal D. Diagonalizing D makes each of $[D,X],[D,Y],[D,Z]$ hollow in one common basis. Conversely, any three hollow matrices have these three commutator realizations for any fixed regular diagonal D. This proves precisely the stated saturated image, including the scalar targets in the traceful case. There is no reliance on an ordinary unrestricted cubic image theorem. □

Corollary 4.5. *For any fixed regular diagonal D and $n \geq 3$, the similarity saturation of $g([D,A],[D,B],[D,C])$ has the same full-target classification.*

Proof. For fixed regular diagonal D, the map $\mathrm{ad}_D : V_n \to V_n$ is an isomorphism. Apply the preceding corollary; simultaneous similarity conjugates all inputs, including D. □

5. Double collisions and products of three commutators

5.1. Repeated variables and coefficient detection.

Proposition 5.1. *Let W be an S_m-submodule of the multilinear free associative component or its cyclic-word quotient, and write*

$$W = \bigoplus_{\lambda \vdash m} S^\lambda \otimes M_\lambda.$$

For a partition $\mu \vdash m$ and a set partition $\mathcal{P}$ of type μ, let $T_\mathcal{P}$ identify the variables in each block of $\mathcal{P}$. Then

$$\bigcap_{\mathcal{P} \text{ of type } \mu} \ker T_\mathcal{P} = \bigoplus_{\lambda \not\trianglerighteq \mu} S^\lambda \otimes M_\lambda,$$

where $\lambda \trianglerighteq \mu$ denotes dominance of partitions.

Proof. Let $H_\mathcal{P}$ permute the variables within the blocks of $\mathcal{P}$, and let $e_\mathcal{P} = |H_\mathcal{P}|^{-1} \sum_{h \in H_\mathcal{P}} h$. Complete polarization, separately in the repeated letters, gives

$$T_\mathcal{P}(w) = 0 \quad \Longleftrightarrow \quad e_\mathcal{P} w = 0.$$

Substitution and polarization descend to cyclic words, so this argument applies to both stated spaces. The intersection over all conjugates is an S_m-submodule. On a simple module S^λ it is the whole module precisely when $(S^\lambda)^{H_\mathcal{P}} = 0$, and is zero otherwise. By Young's rule this invariant space has dimension $K_{\lambda\mu}$, which is nonzero precisely when $\lambda \trianglerighteq \mu$; see [8]. The group algebra acts trivially on the multiplicity factor. Equivalently, its two-sided ideal generated by $e_\mathcal{P}$ is the whole matrix algebra on each detected simple factor. The same conclusion therefore holds with arbitrary multiplicities. □

5.2. **Arbitrary mixtures of three pair commutators.** Write W_6 for the span of the ninety ordered products of three disjoint pair commutators. The result of this section applies to one arbitrary linear combination of all ninety products. Define

$$\mathcal{K} = \bigcap_{\sigma} \ker(R_\sigma|_{W_6}), \qquad R_\sigma P = P(D, D, E, E, A, B),$$

where σ runs over the forty-five choices of two disjoint pairs, with the remaining variables relabelled A, B.

Theorem 5.2. *For every $P \in W_6$ and $n \geq 3$, the actual image is*

$$\mathrm{Val}_n(P) = \begin{cases} \{0\}, & P = 0 \text{ or } (n = 3 \text{ and } P \in Ks_6), \\ \mathfrak{sl}_n(K), & \pi P = 0 \text{ and } P \text{ is not an identity of } M_n, \\ M_n(K), & \pi P \neq 0. \end{cases}$$

All coefficients across the fifteen pairings and six factor orders are arbitrary. The double-collision kernel satisfies

$$\dim \mathcal{K} = 11, \quad \pi\mathcal{K} = 0, \qquad \mathcal{K} \simeq 2S^{(2,1,1,1,1)} \oplus S^{(1,1,1,1,1,1)}.$$

Its last summand is Ks_6: it is an identity for $n = 3$, and every nonzero member has image $\mathfrak{sl}_n$ for $n \geq 4$.

The proof uses Lemmas 4.2 and 4.3, together with the collision constructions below. No image assertion is inferred by adding separately obtained polynomial values.

We repeatedly use the following parameter observation. Suppose F_t is polynomial in t and its matrix inputs, $F_0(a) = N$, and $d(F_0)_a$ is onto a target vector space. The map $(t, x) \mapsto (t, F_t(x))$ is smooth at $(0, a)$. Restrict its source to the open set of surjective matrix Jacobians. Its image contains an open neighborhood of $(0, N)$, hence contains (t, N) for some $t \neq 0$. Thus F_t has an actual full-rank point over N. The original map therefore has such a point if, for $t \neq 0$, F_t is obtained from it by invertible input transformations and invertible output transformations fixing N.

Lemma 5.3. *Every double collision of a member of W_6 lies in the span*

$$\mathcal{W} = \langle P^2Z, PZP, ZP^2,\ \mathrm{perm}(P, Q, R),\ \mathrm{perm}(P, S, T)\rangle,$$

with P, Q, R, S, T, Z as in Theorem 3.1. Every traceful member of $\mathcal{W}$ has an open image neighborhood of a rank-one square-zero matrix. Every nonzero trace-zero member has image $\mathfrak{sl}_n$. Both assertions hold for all $n \geq 3$.

Proof. The displayed spanning property follows by listing the three possible pairing types after the collision: two P factors and Z, or one each of P, Q, R, or one each of P, S, T. This is a fifteen-dimensional space. For precision, write its coefficients as a_0, a_1, a_2 and b_τ, c_τ for $\tau = 123, 132, 213, 231, 312, 321$. The four substitutions $A = D, E$ and $B = D, E$ are common-factor cubics from Lemma 4.2. Their joint polynomial kernel is exactly

$$H = \alpha PL + \beta LP + \gamma C + \delta\mathcal{T} + \eta\mathcal{U}, \qquad \mathcal{U} = s_3(P, Q, R) - s_3(P, S, T).$$

This assertion requires only coefficient elimination. The three coefficients of each substituted cubic, with an irrelevant overall sign suppressed in the third row, are

$$\begin{array}{c|ccc} A=D & a_0-c_{132}-c_{312} & a_1-c_{123}-c_{321} & a_2-c_{213}-c_{231} \\ A=E & a_0+b_{123}+b_{213} & a_1+b_{132}+b_{231} & a_2+b_{312}+b_{321} \\ B=D & a_0+b_{132}+b_{312} & a_1+b_{123}+b_{321} & a_2+b_{213}+b_{231} \\ B=E & -a_0+c_{123}+c_{213} & -a_1+c_{132}+c_{231} & -a_2+c_{312}+c_{321} \end{array}$$

Setting these entries to zero gives $\alpha=(a_0+a_1-a_2)/2$, $\beta=(a_1+a_2-a_0)/2$, $\gamma=(a_0+a_2-a_1)/2$, and two free alternating coefficients $\delta+\eta$, $\delta-\eta$. This proves the stated kernel expression directly. It is five-dimensional: the three trace functions of $PL,\mathcal{T},\mathcal{U}$ are independent (as verified by the trace-coordinate matrix below), while the coefficients of $DEDEAB$ and $DEDAEB$ separate $PL-LP$ and $C-PL$ by the matrix $\left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right)$. The same argument also verifies independence of the fifteen spanning products.

If all four polynomial substitutions vanish, use this kernel expression. The first four terms are invariant under $E\mapsto E+tD$, while

$$\mathcal{U}(D,E+tD,A,B)=\mathcal{U}(D,E,A,B)+2tV(D,E,A,B).$$

For $\eta\neq 0$, Lemma 4.3 and the parameter observation applied to $\varepsilon H(D,E+\varepsilon^{-1}D,A,B)$ give a full-rank nilpotent point for H. For nonzero ε the factor ε is absorbed by the invertible substitution $B\mapsto\varepsilon B$. For $\eta=0$, Theorem 3.1 applies. The trace-zero kernel members have $\eta=\delta=0$ and $\alpha+\beta+\gamma=0$, so that theorem also proves their assertion.

Suppose now that one substituted polynomial is nonzero. If H is trace zero, the corresponding common-factor cubic is nonzero and trace zero; Lemma 4.2 supplies every traceless target in the image of H. If H is traceful and one substitution is traceful, the same lemma supplies a full-rank nilpotent point. It remains to consider traceful H for which all four substitutions are trace zero and at least one is a nonzero polynomial. Relabel so that

$$J(D,E,B)=H(D,E,D,B)\neq 0.$$

This is a trace-zero common-factor cubic.

The five independent trace coordinates of $\mathcal{W}$ are

$$\tau_0=\operatorname{tr}P^2Z,\quad \tau_1=\operatorname{tr}PQR,\quad \tau_2=\operatorname{tr}PRQ,\quad \tau_3=\operatorname{tr}PST,\quad \tau_4=\operatorname{tr}PTS.$$

Their independence can be checked by rotating A to the front and reading the coefficients of the remaining words $DDEEB$, $DEDEB$, $DEDBE$, $DEBDE$, $DBDEE$. Their coefficient matrix is

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ -1 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & -1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\qquad \det=1.$$

Write $h = \operatorname{tr} H = \sum k_i \tau_i$. The four trace-zero substitutions give

$$k_0 + k_1 + k_2 = 0, \qquad k_0 - k_3 - k_4 = 0.$$

Let K be the cyclic derivative with respect to A, so $h = \operatorname{tr}(AK)$. Direct cyclic rotation gives

$$K = k_0[B, P^2] + k_1[RP, D] + k_2[PR, D] + k_3[PS, E] + k_4[SP, E].$$

Choose the nilpotent seed for J from Lemma 4.2, retaining its smooth lift, a hollow dense P, and a regular diagonal D in the common basis. Here $S = [D, B]$ is the rank-one matrix denoted Q in that lemma. The eigenvalues of D remain freely selectable.

Put $\theta = k_0 + k_1 - k_3$. Replacing E by $E + \lambda D$ leaves P, S and the map $J(D, E, B)$ unchanged and changes K by

$$\lambda\theta[[S, P], D].$$

If $\theta \neq 0$, this coefficient is nonzero: $[S, P]$ is a nonzero rank-one nilpotent, and cannot commute with the regular semisimple D. Thus one can arrange $K \neq 0$ without changing the nilpotent value of J.

If $\theta = 0$, the trace is a linear combination $h = a \operatorname{tr}(PL) + b \operatorname{tr} \mathcal{T}$, with $(a, b) \neq (0, 0)$. Replacing B by $B + \mu D^2$ still leaves S, hence J, unchanged. The coefficient added to K is

$$\mu\{a([D^2, P^2] + 2[PDP, D]) + 3b[D, [D, P^2]]\}.$$

It is nonzero for a generic choice of the diagonal entries d_i of D. Indeed the (i, j) entry of the first summand, without a, is

$$(d_i - d_j) \sum_{\ell} (d_i + d_j - 2d_\ell) P_{i\ell} P_{\ell j}.$$

For $a \neq 0$, choose distinct i, j, ℓ; dense hollowness of P makes the coefficient of d_ℓ nonzero, while the second summand depends only on d_i, d_j. For $a = 0$, the simple spectrum of P^2 prevents it from commuting with D: otherwise P, a polynomial in P^2, would be both diagonal and hollow. Thus the second summand is nonzero. In either case choose μ so that $K \neq 0$. These diagonal shifts preserve the smooth-lift first-row and first-column conditions of Lemma 4.2.

Return directly to the actual point $(D, E, A, B) = (D, E, D, B)$. Its value is $J(D, E, B) = N$. Varying D, E, B with $A = D$ gives all $\mathfrak{sl}_n$ tangent directions from J. An independent A variation L has output trace $\operatorname{tr}(LK)$, which is nonzero for some L. Consequently dH at this same point is onto M_n. This proves the remaining case. □

Proof of Theorem 5.2. The ninety products form a basis: a word has a unique decomposition into its ordered adjacent pairs, and each product has eight disjoint word terms. As an S_6-module this is the multilinear part of $(\bigwedge^2 V)^{\otimes 3}$. The Pieri rule gives

$$W_6 \simeq S^{(3,3)} \oplus 2S^{(3,2,1)} \oplus 3S^{(2,2,1,1)} \oplus S^{(3,1,1,1)} \oplus S^{(2,2,2)} \oplus 2S^{(2,1,1,1,1)} \oplus S^{(1^6)}.$$

Apply Proposition 5.1 with pattern $(2,2,1,1)$. In the displayed decomposition only $(2,1^4)$ and (1^6) fail to dominate this pattern. Thus the joint kernel consists exactly of these summands. Their dimensions are 5 and 1, so $\dim \mathcal{K} = 2\cdot 5+1 = 11$.

For completeness, its trace vanishing also has a direct cyclic-word explanation. Modulo cyclic permutation of the three factors there are thirty formal products. Each cyclic word of length six appears in exactly two products, obtained by the two possible parities of adjacent pair cuts. Their coefficient matrix is a signed incidence matrix on thirty vertices. The graph is connected: the pair swaps and a one-place cyclic shift generate S_6. Its kernel is therefore at most one-dimensional; the alternating s_6 supplies that kernel. Equivalently, the trace quotient is the cyclic-factor coinvariant space modulo Ks_6. The elementary symmetric/exterior Pieri decompositions

$$\operatorname{Sym}^3(\textstyle\bigwedge^2 V) = S_{(3,3)}V \oplus S_{(2,2,1,1)}V \oplus \bigwedge^6 V, \qquad \bigwedge^3(\bigwedge^2 V) = S_{(3,1,1,1)}V \oplus S_{(2,2,2)}V$$

then identify that quotient as $S^{(3,3)} \oplus S^{(2,2,1,1)} \oplus S^{(3,1,1,1)} \oplus S^{(2,2,2)}$. In particular $\pi\mathcal{K} = 0$ and the cyclic rank is $5+9+10+5 = 29$.

If $\pi P \neq 0$, some double collision H has nonzero cyclic projection: otherwise polarization in the trace quotient and the preceding Young rule would put its cyclic class in a zero joint kernel. Such an H has an open image neighborhood at a rank-one nilpotent by Lemma 5.3. Its image is contained in that of the original multilinear P. The local characterization, including its multilinear scalar lemma, gives $\operatorname{Val}_n(P) = M_n$. If $\pi P = 0$ and $P \notin \mathcal{K}$, some double collision is nonzero and trace zero. Lemma 5.3 gives $\mathfrak{sl}_n$ in the image of P; trace vanishing gives equality. Finally $S^{(1^6)} = Ks_6$ is an identity of M_3 by Amitsur–Levitzki. The following subsections complete $\mathcal{K}$ in size three and in every larger size. □

5.2.1. *Completion of the residual module in size three.* In this subsection only, index the variables by $0,\dots,5$. Let Alt_5 mean the unnormalized signed sum over permutations of $x_1,\dots,x_5$, leaving x_0 fixed, and put

$$\begin{aligned} F_1 &= \operatorname{Alt}_5[x_0,x_1][x_2,x_3][x_4,x_5],\\ F_2 &= \operatorname{Alt}_5[x_2,x_3][x_0,x_1][x_4,x_5],\\ F_3 &= \operatorname{Alt}_5[x_2,x_3][x_4,x_5][x_0,x_1],\\ A &= F_1 - F_3, \qquad B = 2F_2 - F_1 - F_3. \end{aligned}$$

Set $A_0 = A$, $B_0 = B$, and $A_s = -(0\ s)A$, $B_s = -(0\ s)B$ for $s > 0$. These elements have

$$\sum_{s=0}^{5} A_s = \sum_{s=0}^{5} B_s = 0, \qquad \mathcal{K} = \langle A_s, B_s : 0 \le s \le 5\rangle \oplus Ks_6.$$

To prove this basis description, observe that alternation in five variables kills every double collision. The orbit of each alternant is a quotient of the sign-twisted six-point permutation module. The full alternating projections of F_1, F_2, F_3 agree, so A, B have no sign component; the displayed sums vanish. Each

nonzero orbit therefore affords $S^{(2,1^4)}$. The two S_5-alternating vectors A, B are linearly independent: their coefficients at the words 012345 and 120345 are respectively $(4, -4)$ and $(0, 8)$. They therefore generate two different copies of that irreducible module. Together with s_6 their dimension is eleven, equal to $\dim \mathcal{K}$.

Since s_6 vanishes on M_3, it remains to consider

$$P_{a,b} = \sum_{s=0}^{5} a_s A_s + \sum_{s=0}^{5} b_s B_s.$$

Adding a constant to all the a_s, or to all the b_s, does not change this polynomial. Variable permutations act by the same permutation of the coefficient indices and an overall sign.

First fix the five inputs in slots $1, \dots, 5$ to be

$$H,\ K,\ E_{12},\ E_{23},\ E_{31}, \qquad H, K \text{ arbitrary traceless diagonal matrices.} \tag{C}$$

The linear map in slot zero preserves every root space KE_{ij}. Write $m_{ij}(H, K)$ for its multiplier. Expansion of the three commutators, followed by alternation, gives the following exact conditions for a multiplier to vanish identically in H, K:

ij	$m_{ij} \equiv 0$ if and only if
12	$a_0 = a_3$
23	$a_0 = a_4$
31	$a_0 = a_5$
13	$a_1 = a_2,\ b_1 = b_2,\ 2a_0 - a_2 - a_3 - a_4 + a_5 + 3b_3 - 3b_4 = 0$
21	$a_1 = a_2,\ b_1 = b_2,\ 2a_0 - a_2 + a_3 - a_4 - a_5 + 3b_4 - 3b_5 = 0$
32	$a_1 = a_2,\ b_1 = b_2,\ 2a_0 - a_2 - a_3 + a_4 - a_5 - 3b_3 + 3b_5 = 0$

(C0)

These are linear coefficient identities, not checks at a single choice of H, K. For example, expand $H = h_1 \operatorname{diag}(1, -1, 0) + h_2 \operatorname{diag}(0, 1, -1)$ and similarly K; each condition is obtained by setting the four coefficients of $h_i k_j$ to zero.

Suppose the a_s are not all equal. Select an a-value with minimum multiplicity $r \le 3$, put one of its indices in slot zero, and assign the remaining indices to the five fixed slots as follows. If $r = 2$, put the other occurrence and an index with a different a-value in slots $1, 2$. All three remaining a-values differ from a_0, and $a_1 \neq a_2$; thus none of (C0) holds. If $r = 1$ and the five remaining pairs (a_s, b_s) are not all equal, put two different such pairs in slots $1, 2$. Again no multiplier vanishes. If those five pairs are equal, the last three equations in (C0) reduce to $2(a_0 - a_1) \neq 0$. If $r = 3$, the a-values split into two triples. If b is nonconstant on either triple, choose the active index in that triple so that its other two b-values differ, and put these in slots $1, 2$. Otherwise b is constant on each triple; use the other two members of the active triple in slots $1, 2$. The last three equations in (C0) then reduce to the difference of the two a-values, which is nonzero. This covers every nonconstant a, with arbitrary b.

For constant a, remove it using $\sum A_s = 0$ and fix instead

$$H,\ E_{12},\ E_{21},\ E_{23},\ E_{32} \quad \text{in slots } 1, \dots, 5. \tag{P}$$

For pure B the four multipliers on $E_{12}, E_{21}, E_{23}, E_{32}$ vanish identically precisely when, respectively, $b_2 = b_0$, $b_3 = b_0$, $b_4 = b_0$, $b_5 = b_0$. With $t = (b_0 + b_1)/2$, the other two vanish precisely as follows:

$$\begin{array}{c|ll} 13 & b_2 + 3b_3 = 4t, & b_4 + 3b_5 = 4t, \\ 31 & 3b_2 + b_3 = 4t, & 3b_4 + b_5 = 4t. \end{array} \tag{P0}$$

Assume some b-value has multiplicity at most two and place it in slot zero. If its multiplicity is two, put its other occurrence in slot one; then $b_2, \dots, b_5$ all differ from $b_0 = t$. If its multiplicity is one, choose slot one so that the other four values are not all t: if an initial choice fails, interchange that slot with one of the four equal values. In both cases the four remaining values differ from b_0 and are not all t.

Among four numbers not all t, there are two, say u, v, with $u + 3v \neq 4t$ and $3u + v \neq 4t$. Indeed, subtract t. A zero and a nonzero offset give such a pair. If all offsets are nonzero and every pair failed, their ratios would all belong to $\{-3, -1/3\}$, which is impossible already for three offsets in characteristic zero. Put u, v in slots $2, 3$. All six multipliers are now nonzero polynomials by (P0).

In either successful construction, choose diagonal parameters outside the union of the six multiplier zero sets. Varying the single active slot then realizes every hollow 3×3 matrix in one evaluation. Every traceless matrix is similar to a hollow matrix: for a nonscalar matrix choose independent vectors v, Tv to make its first diagonal entry zero, then continue on the remaining trace-zero compression; a scalar trace-zero compression is zero. Conjugation equivariance therefore gives every member of $\mathfrak{sl}_3$ as an actual value of $P_{a,b}$.

The only coefficients left are constant a and two different constant b-values on complementary triples. Modulo constants and scaling, these give ten lines, one for each unordered $3 + 3$ partition. All are variable permutations, up to sign, of

$$G = \frac{1}{12}(B_1 + B_2 + B_3).$$

They are settled by one explicit smooth point. At

$$(x_0, x_1, x_2, x_3, x_4, x_5) = (E_{31}, E_{13}, E_{21}, E_{12}, \operatorname{diag}(0, 1, -1), E_{23}),$$

one has $G = -5E_{23}$. Its differential has the following eight values; the unlisted slots of each tangent vector are zero:

slot	variation	dG
0	E_{31}	$-5E_{23}$
0	E_{32}	$-6E_{13}$
1	$\operatorname{diag}(1, -1, 0)$	$-5E_{21}$
1	E_{32}	$-7E_{31}$
2	E_{31}	$\operatorname{diag}(4, -3, -1)$
2	E_{32}	E_{12}
4	E_{32}	$\operatorname{diag}(-2, 4, -2)$
5	E_{32}	$7E_{32}$

(J)

The six off-diagonal matrix units and two independent traceless diagonals span $\mathfrak{sl}_3$. Thus dG has rank eight over a rank-one square-zero value. Scaling and conjugation normalize that value to E_{12}, and Corollary 1.8 completes these ten lines. If both a and b are constant the polynomial is zero modulo Ks_6. Together with the preceding proof outside $\mathcal{K}$, this proves the size-three classification in Theorem 5.2.

5.2.2. *A single collision in arbitrary size.* Continue the variable numbering $0, \dots, 5$. Let ρ identify slots zero and one, and retain the notation A_s, B_s of Section 5.2.1. Alternation immediately gives

$$\rho P_{a,b} = (a_0 - a_1)\rho A + (b_0 - b_1)\rho B.$$

The terms with distinguished index at least two vanish, and $\rho A_1 = -\rho A_0$, $\rho B_1 = -\rho B_0$. After a variable permutation some such collision is nonzero unless both coefficient vectors are constant. A multiple of s_6 is killed by every collision.

For $U = [D, X]$, $V = [D, Y]$, $W = [D, Z]$, direct alternation gives

$$(aA + bB)(D, D, D^2, X, Y, Z) = -4Q_{a+3b,-a+3b}(D; U, V, W),$$

where

$$Q_{r,s}(D; U, V, W) = \mathrm{Alt}_{U,V,W}\Big(rDUVW + sUDVW - rUVDW - sUVWD\Big). \quad \text{(Q)}$$

One way to verify the identity without a long word expansion is to follow a path through four vertices with diagonal entries d_i, d_j, d_k, d_l. The coefficients of A and B, after removing the three factors $(d_i - d_j)(d_j - d_k)(d_k - d_l)$, are respectively $-4(d_i - d_j - d_k + d_l)$ and $-12(d_i + d_j - d_k - d_l)$. The change of coefficients $(a, b) \mapsto (r, s) = (a + 3b, -a + 3b)$ is invertible.

Lemma 5.4. *For every $(r, s) \neq (0, 0)$ and $n \geq 4$, the actual values of (Q), with D regular diagonal and U, V, W hollow, after simultaneous conjugation, fill $\mathfrak{sl}_n(K)$.*

Proof. Every value is traceless. Indeed

$$Q_{1,0} = \sum_{\mathrm{cyc}} [D, [U, V]]W, \qquad Q_{0,1} = \sum_{\mathrm{cyc}} U[D, [V, W]],$$

and cyclicity of trace reduces both traces to the Jacobi identity.

First suppose $rs \neq 0$. Write $n = 1 + k$, index the lower block by $1, \dots, k$, and take

$$D = \mathrm{diag}(d_0, \dots, d_k), \quad U = \sum_{j=1}^{k} E_{0j}, \quad V = \sum_{i=1}^{k-1} E_{i,i+1}, \quad W = \sum_{i=1}^{k-1} z_i E_{i+1,i}.$$

Use $z_0 = z_k = 0$. The output has only its first row nonzero, with entries

$$N_{0j} = r\{z_j(d_0 - d_{j+1}) - z_{j-1}(d_0 - d_{j-1})\}.$$

An absent boundary term is omitted. All these entries are nonzero for generic parameters, so N is a rank-one square-zero matrix. The U derivative covers the

first row and column: their respective multipliers are the preceding expression and

$$s\{z_i(d_{i+1}-d_0)-z_{i-1}(d_{i-1}-d_0)\}.$$

On the lower off-diagonal band $j-i=h$, the same derivative is tridiagonal. Its diagonal, forward and backward coefficients are

$$\begin{aligned}
C_{ij} &= r\{z_j(d_i-d_{j+1})-z_{j-1}(d_i-d_{j-1})\}+s\{z_i(d_{i+1}-d_j)-z_{i-1}(d_{i-1}-d_j)\},\\
A_{ij} &= -z_j\{r(d_i-d_{j+1})+s(d_{i+1}-d_j)\},\\
B_{ij} &= z_{i-1}\{r(d_i-d_{j-1})+s(d_{i-1}-d_j)\}.
\end{aligned}$$

For $h\geq 2$, specialize $z_i=t^i$. Every diagonal coefficient except the last has unique leading term $rz_{i+h}(d_i-d_{i+h+1})$; the last has leading term $-rz_{k-1}(d_{k-h}-d_{k-1})$. In a tridiagonal determinant every adjacent transposition has strictly smaller t-degree than the corresponding diagonal pair. Thus the leading diagonal product is unique and nonzero. Negative bands follow by transposition and diagonal rescaling of the shifts, interchanging r,s.

On the first upper band the rows sum to zero, and its upper-left $(k-2)$-minor is generically nonzero by the same leading-term argument. Its rank is $k-2$. Both off-diagonals are nonzero, so both endpoints of a left null vector are nonzero. The first lower band has the same properties. Varying V by E_{k0} gives a hollow lower-block output supported in its last row, whose $(k,k-1)$ entry is $-sz_{k-1}(d_0-d_{k-1})\neq 0$. Varying W by E_{10} gives a hollow output in its first row, whose $(1,2)$ entry is $s(d_0-d_2)\neq 0$. These fill the two remaining band cokernels. All hollow output directions are therefore covered. Conjugation of N supplies every traceless diagonal direction modulo the hollow matrices. The saturated derivative is onto $\mathfrak{sl}_n$ over N, and Corollary 1.8 applies.

It remains to treat $rs=0$. Transposition interchanges r,s, and nonzero scaling is harmless, so take $(r,s)=(1,0)$. Here a regular *zero* is available. Set

$$D=\operatorname{diag}(0,1,\dots,k),\quad U=\sum_{i=1}^{k}E_{i0},\quad V=\sum_{i=1}^{k-1}E_{i,i+1},\quad W=V^{\mathsf{T}}. \tag{Z}$$

Since $[V,W]$ is diagonal, (Q) is zero at this point. We show that its derivative in hollow U,V,W alone is onto $\mathfrak{sl}_n$. The U derivative covers the entire first row: its multipliers are -2 for $j<k$ and $k-1$ for $j=k$. Lower-block variations of V,W give only a first column, namely

$$[D,[\dot V,W]+[V,\dot W]]\mathbf{1}\,e_0^{\mathsf{T}}.$$

They cover that column. In fact $\dot W=E_{12}$ gives $2e_1$, $\dot V=E_{k,k-1}$ gives $2e_k$, and $\dot V=E_{i,i+2}$ gives $-e_i+e_{i+1}$ for $1\leq i\leq k-2$. This uses exactly $k\geq 3$.

On each lower-block band of distance $h\geq 2$, the U derivative has determinant $(h-1)^{k-h}$. For the upper band its tridiagonal matrix has diagonal $(-2,\dots,-2,h-1)$, superdiagonal $h+1$ and subdiagonal $-(h-1)$; the determinant formula follows by the two-term tridiagonal recurrence. Reversal gives the lower-band formula. The first upper and lower bands each have rank $k-2$, with cokernel coordinates respectively $(k-1,k)$ and $(2,1)$.

It remains to cover the k lower diagonal entries and these two coordinates; the top diagonal is determined by trace zero. Write $\dot{V} = e_0 v^{\mathsf{T}}$, $\dot{W} = e_0 w^{\mathsf{T}}$. Their lower-block output is

$$[D_-, \mathbf{1}v^{\mathsf{T}}]W - [D_-, \mathbf{1}w^{\mathsf{T}}]V + D_-(W\mathbf{1}v^{\mathsf{T}} - V\mathbf{1}w^{\mathsf{T}}), \quad D_- = \operatorname{diag}(1, \dots, k).$$

Its diagonal entries are

$$t_i = -\mathbf{1}_{i<k} v_{i+1} + i\mathbf{1}_{i\geq 2} v_i - \mathbf{1}_{i\geq 2} w_{i-1} - i\mathbf{1}_{i<k} w_i,$$

and the two cokernel coordinates are $2(v_1 - w_1)$ and $(k-1)(v_k - w_k)$. This map has rank $k+2$. To check it, let a linear relation have diagonal coefficients c_i and two further coefficients. The coefficients of v_1 and w_k first kill the latter two. The coefficients of w_1 and v_2 then give $c_2 = -c_1$ and $c_1 = 2c_2$, hence $c_1 = c_2 = 0$ in characteristic zero. Successive coefficients give all other $c_i = 0$. Together these variations cover $\mathfrak{sl}_n$ at the zero value (Z). Smoothness gives an open image neighborhood of zero. The image is a cone, so its intersection with every target line contains a nonzero point; scaling gives the entire target space. This proves the boundary case as well. □

Applying this lemma to a nonzero single collision proves the assertion for every member of $\mathcal{K} \setminus Ks_6$, including an arbitrary added multiple of s_6, for all $n \geq 4$.

5.2.3. *The standard direction and compatible block construction.* It remains to treat Ks_6. We first compute minors in the base sizes and then prove the result in arbitrary size by a block construction. This argument does not use a general sextic inclusion theorem.

For $q = 3, 4, 5$, use a hub indexed by zero and a lower block of size q. Take

$$Y_0 = \begin{pmatrix} 1 & \mathbf{1}^{\mathsf{T}} \\ 0 & 0 \end{pmatrix}, \qquad Y_j = 0 \oplus B_j \quad (1 \leq j \leq 5),$$

with the five lower matrices listed below. Put $H_q = s_5(B_1, \dots, B_5)$. Multiplication of matrix units gives the exact identities

q	$(B_1, B_2, B_3, B_4, B_5)$	$\mathbf{1}^{\mathsf{T}} H_q$	$\det H_q$	Jacobian minor
3	$(E_{12}, E_{23}, E_{31}, E_{11}, E_{22})$	$(-2, -2, -1)$	-4	1024
4	$(E_{12}, E_{23}, E_{34}, E_{41}, E_{11})$	$(2, 1, 1, 1)$	2	256
5	$(E_{12}, E_{23}, E_{34}, E_{45}, E_{51})$	$(1, 1, 1, 1, 1)$	1	-1

The output is a nonzero rank-one square-zero matrix supported in the hub row. The minors use row-major coordinates with the last diagonal coordinate omitted, which are coordinates on $\mathfrak{sl}_{1+q}$. Section 10.6 gives the column lists and a triangular path proof of their nonzero determinants. Thus local surjectivity, invertibility of H_q, and nonvanishing of a chosen output coordinate are nonempty open conditions on the local star parameters $(B_1, \dots, B_5, r)$.

For arbitrary $n \geq 4$, write $n-1 = q_1 + \cdots + q_t$ with $q_i \in \{3,4,5\}$. On $K \oplus \bigoplus_i K^{q_i}$ use the one star family

$$Y_0 = \begin{pmatrix} 1 & r_1 & \cdots & r_t \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad Y_j = 0 \oplus B_{j,1} \oplus \cdots \oplus B_{j,t} \quad (1 \leq j \leq 5).$$

There is an additional open condition for each directed rectangle between two distinct lower blocks: the Y_0 derivative there must be invertible. Its determinant is a nonzero polynomial. To see this, set all five matrices on the left block to zero and use the displayed base tuple on the right block. The rectangle map becomes $T_{ab}(X) = XH_b$, whose determinant is $(\det H_b)^a \neq 0$. Interchange left and right to verify the reverse rectangle. These specializations establish polynomial nonvanishing; they need not themselves satisfy the local Jacobian conditions.

The full parameter space is affine and irreducible. The finite product of the nonzero local-minor, H_q-determinant, rectangle-determinant and output-coordinate polynomials is nonzero. Hence one actual input satisfies *all* requirements simultaneously. Its output is

$$N = e_0(r_1 H_{q_1}, \ldots, r_t H_{q_t}),$$

again nonzero rank one and square zero. At this same input, a trace-pairing annihilator of the full derivative has its hub rectangles zero: the two Y_0 rectangle maps are $w \mapsto wH_{q_i}$ and $v \mapsto -H_{q_i}v$. The invertible lower rectangle maps then kill all off-diagonal lower blocks of the annihilator. It remains $h \oplus C_1 \oplus \cdots \oplus C_t$. Embed the variations of the local $(1+q_i)$-dimensional Jacobian into the global slots. The only additional output entries lie in rectangles already annihilated; no other lower diagonal block receives a contribution. Local surjectivity onto $\mathfrak{sl}_{1+q_i}$ therefore forces $h \oplus C_i$ to be scalar, so $C_i = hI_{q_i}$ for every i. The entire annihilator is the scalar line. The global derivative is onto $\mathfrak{sl}_n$ over N.

Every s_6 value is traceless, since a cyclic word rotation reverses its sign. Corollary 1.8 now proves $\mathrm{Val}_n(s_6) = \mathfrak{sl}_n$ for every $n \geq 4$. Together with the preceding subsection and the size-three argument, this completes Theorem 5.2 on all of W_6.

6. Traceful proper sextics on three by three matrices

Throughout this section K is algebraically closed of characteristic zero. All polynomials are ordinary noncommutative polynomials with coefficients in K, multilinear in their indicated *original* variables. The image means the set of single values, not its linear span, its additive closure, or its Zariski closure. A polynomial is proper if replacing any original variable by the identity matrix makes it zero. Write $B = UT_3(K)$, $N = E_{13}$, and $\mathfrak{s} = \mathfrak{sl}_3(K)$.

Theorem 6.1. *If $f \in K\langle x_1, \dots, x_6\rangle$ is proper and multilinear and* $\operatorname{tr} f$ *is not identically zero on $M_3(K)^6$, then*

$$f(M_3(K)^6) = M_3(K).$$

The proof separates actual original-slot arguments from full-differential arguments. In particular, the rank of the full differential at a triangular tuple is never used as the rank of an original slot. We first prove the two original-slot results needed below. These results have no degree-six or properness assumption.

6.1. **Division descent and an original slot of rank at least six.**

Lemma 6.2. *Let h be ordinary multilinear, and select an original slot. Its generic full rank and every nonempty Zariski-open condition on its actual backgrounds can be transported to a degree-three, period-three division algebra. After splitting, the transported background is still generic in all of the original matrix entries. If* $\operatorname{tr} h \neq 0$*, the transported slot has nonzero reduced-trace row.*

Proof. Put $F_0 = K((s))((t))$, choose a primitive cube root ζ, and take

$$D_0 = (s,t)_3 = F_0\langle u, v \mid u^3 = s, v^3 = t, vu = \zeta uv\rangle.$$

The nine monomials $u^a v^b$, $0 \le a, b < 3$, have distinct classes of fractional values in the rank-two valuation group modulo $\mathbb{Z}^2$. In a nonzero linear combination its term of least value is unique, and least values add under multiplication. Thus there are no zero divisors. The symbol algebra is central simple of dimension nine, so it is a central division algebra of degree three. Its Brauer period divides three and is not one, hence equals three.

Choose a basis $e_1, \dots, e_9$ of D_0. For every frozen original input use independent variables $z_{i\ell}$ and put

$$F = F_0(z_{i\ell}), \qquad D = D_0 \otimes_{F_0} F, \qquad B_i = \sum_{\ell=1}^{9} z_{i\ell} e_\ell.$$

Pure transcendental extension preserves division: the rational extension embeds in an iterated formal Laurent series division ring over D_0, and finite-dimensionality over its center then makes it division. In particular D still has period three.

Choose a finite splitting field L_0/F_0 before adjoining the $z_{i\ell}$, for example by adjoining cube roots of s, t. Over L_0, the matrices representing e_ℓ are a basis of $M_3(L_0)$. Changing each block $(z_{i1}, \dots, z_{i9})$ to the corresponding nine matrix entries is therefore an invertible linear substitution over L_0, independent of the variables. It preserves whether a polynomial minor is identically zero and takes the generic point to the generic point. It also preserves a nonempty open background condition after base change, including conditions imposed by projective incidence. Finally, the trace row of the selected slot is nonzero whenever $\operatorname{tr} h$ is a nonzero polynomial, by multilinearity in that slot. The same change of coordinates proves the reduced-trace assertion. □

We use one standard Severi–Brauer fact in an explicit form: if a central simple algebra has period three, every divisor on its Severi–Brauer surface has geometric degree divisible by three. This is the Picard–Brauer exact sequence: the obstruction to descending $\mathcal{O}_{\mathbb{P}^2}(1)$ is the Brauer class. The opposite algebra has the same period, so the choice of left or right ideals is immaterial. A source is Colliot-Thélène and Skorobogatov, *The Brauer–Grothendieck group*, Section 6.1, equation (6.1) [10].

Lemma 6.3. *Let D/F be the division algebra just constructed, let E be an algebraic closure of F, and identify D_E with $M_3(E)$. For an F-subspace $T \subset D$ with basis $T_1, \dots, T_k$ consider*

$$M_T(x) = [T_1 x \mid \cdots \mid T_k x], \qquad [x] \in \mathbb{P}^2_E.$$

For $k = 2$ this matrix has generic rank two and only finitely many rank-drop points. For $k = 3$ its generic rank is three. For $k = 4$ its generic rank is three and its rank-drop set is finite. If moreover $1 \in T$ and $k = 4$, its rank is at least two everywhere.

Proof. A nonzero element of D is invertible, so its evaluation column never vanishes. If two independent columns were everywhere dependent, their proportionality coefficient would be a global regular degree-zero function on $\mathbb{P}^2$, hence constant. This contradicts independence. The common divisorial part of their nonzero quadratic minors descends to the Severi–Brauer surface: twisted Galois acts on all columns by one common invertible semilinear left transformation and on x by the corresponding projective transformation. Thus it preserves the rank ideal and its canonical codimension-one divisor. Its degree is at most two, so the period-three fact forces that divisor to be empty. The rank-drop locus is consequently finite.

For three columns choose two first. If the third were generically in their span, its two uniquely determined coefficients would be regular functions of degree zero away from the finite rank-drop locus. Normality of $\mathbb{P}^2$ extends these functions across codimension two. They are constant, again contradicting operator independence.

For four columns, generic rank three follows. If the four cubic maximal minors had a nonconstant greatest common divisor g, its canonical divisor would descend. Its degree must then be three. The signed cofactor vector consequently has the form gc for a constant nonzero $c \in E^4$. The cofactor identity gives $M_T(x)c = 0$ identically, a constant relation among the four operators. This is impossible. There is again no divisorial rank-drop locus.

Finally suppose $1 \in T$ and evaluation had rank one at $[x]$. Then $T_E x = Ex$. The unital F-subalgebra $A \subset D$ generated by T is itself a division algebra: an inverse of a nonzero element is polynomial in that element by its minimal equation. Since D is a finite left A-vector space, $\dim_F A$ divides nine. It is at least four, hence equals nine. Thus $A = D$, whereas T_E would generate a subalgebra preserving Ex. This contradicts $D_E = M_3(E)$. □

Lemma 6.4. *For a four-dimensional F-subspace $U \subset D$, the map*

$$\{R \in M_3(E) : \operatorname{rank} R = 1\} \longrightarrow U_E^*, \qquad R \longmapsto \Big(T \mapsto \operatorname{tr}(TR)\Big)$$

is surjective, including over the zero functional.

Proof. Write $R = xy^t$. Let $\Delta_i(x)$ be the four signed cubic maximal minors of $M_U(x)$. For a prescribed nonzero functional $d = (d_1, \dots, d_4)$, at a rank-three point solvability of $y^t T_i x = d_i$ is equivalent to

$$\sum_i d_i \Delta_i(x) = 0.$$

If this cubic is zero identically, any full-rank point works. Otherwise its projective zero set is a nonempty curve. By Lemma 6.3, this curve cannot be contained in the finite rank-drop locus. At a rank-three point on it the linear system is consistent; its solution has $y \neq 0$ because $d \neq 0$. For $d = 0$, the projective rank-one variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ has dimension four. Four homogeneous linear conditions meet it by the projective dimension theorem. Any representative gives a nonzero rank-one preimage of zero. □

Theorem 6.5. *Let h be ordinary multilinear of any degree on $M_3(K)$, with $\operatorname{tr} h \neq 0$. If one original full slot has generic rank at least six, then $h(M_3(K)^m) = M_3(K)$. In particular one actual background at which that full slot has rank at least six suffices.*

Proof. Use Lemma 6.2 and choose a six-dimensional F-subspace S of the division-slot image on which reduced trace is nonzero. Put $P = S^{\perp}$ for the reduced-trace pairing. Then $\dim P = 3$, $1 \notin P$, and $U = F1 \oplus P$ has dimension four. For any $\lambda, \mu \in E$ prescribe on U_E the functional

$$d(1) = \mu, \qquad d(T) = -\lambda \operatorname{tr}(T) \quad (T \in P_E).$$

Lemma 6.4 supplies a nonzero rank-one R with $\operatorname{tr} R = \mu$ and $\lambda I + R \in S_E$. Nonzero rank-one matrices of a fixed trace are similar: for $\mu \neq 0$ they are similar to $\operatorname{diag}(\mu, 0, 0)$, and for $\mu = 0$ they are rank-one square-zero matrices. Therefore the conjugation saturation of S_E contains every scalar plus a nonzero rank-one matrix, including all translates of the rank-one nilpotent orbit.

It remains to cover cyclic matrices. Set $W = S \cap \ker \operatorname{Trd}$, so $\dim W = 5$, and on W put

$$q(X) = \operatorname{Trd}(X^2), \qquad d(X) = \operatorname{Nrd}(X).$$

The quadratic q is not identically zero: after splitting, the trace form on $\mathfrak{sl}_3$ is nondegenerate of dimension eight, and a totally isotropic subspace has dimension at most four. The cubic d is anisotropic over F because D is division. It is irreducible over F: a reducible homogeneous cubic has a linear factor over F, whose kernel in the five-dimensional W has a nonzero F-point, contradicting anisotropy. In characteristic zero its distinct geometric irreducible factors form one Galois orbit. If the F-defined nonzero quadratic q shared one, it would share all, forcing $d \mid q$. Thus q, d are geometrically coprime. They form a regular sequence on W_E, and their common zero scheme is pure of dimension three.

Since a trace-zero 3×3 matrix has characteristic polynomial $t^3-q(X)t/2-d(X)$, this is exactly the nilpotent section of S_E.

The rank-at-most-one part of this section has dimension at most one. Indeed its nonzero members are xy^t satisfying $y^tTx=0$ for $T\in U_E$. The four-dimensional space U contains 1; its evaluation rank is three outside finitely many $[x]$, and at least two everywhere, by Lemma 6.3. Each exceptional $[x]$ therefore allows at most a one-dimensional vector space of y's. Fixing a representative x gives a single vector line of matrices; there is no additional scaling parameter for x. A finite union of such lines has dimension at most one.

The three-dimensional nilpotent section consequently contains a regular nilpotent N_0. Near it every nilpotent in the section is regular, so the intersection with its orbit has local dimension three. Consider

$$\Phi:\mathrm{GL}_3(E)\times S_E\longrightarrow M_3(E),\qquad (g,s)\longmapsto gsg^{-1}.$$

Its local fiber at (I,N_0) has dimension $3+3=6$: projection to $S_E\cap\mathcal{O}_{N_0}$ has fibers the three-dimensional centralizer of a regular nilpotent. The source and target are smooth of dimensions 15 and 9. Miracle flatness [5] gives flatness at (I,N_0), and openness of a flat morphism of finite presentation [7] gives an actual-image open neighborhood of N_0. Each E-point of this open image has a nonempty finite-type fiber, hence an E-point because E is algebraically closed.

Conjugate this neighborhood to contain $J=E_{21}+E_{32}$. For any companion matrix and $t\neq0$ one has

$$C=\begin{pmatrix}0&0&c_0\\1&0&c_1\\0&1&c_2\end{pmatrix},\qquad t\,\mathrm{diag}(1,t^{-1},t^{-2})C\,\mathrm{diag}(1,t,t^2)=\begin{pmatrix}0&0&t^3c_0\\1&0&t^2c_1\\0&1&tc_2\end{pmatrix}.$$

The displayed curve has value J at zero. Its inverse image of the open neighborhood contains a nonzero parameter. The saturation of S_E is invariant under scalars and conjugation, so undoing these operations puts C itself in it. Every cyclic matrix is similar to such a companion matrix. Every nonscalar noncyclic 3×3 matrix has minimal polynomial of degree two and is scalar plus a nonzero rank-one matrix: the two-root case has multiplicities two and one, and the one-root case is a scalar plus a rank-one square-zero matrix. Those cases were already covered.

Fix now any nonscalar target $T\in M_3(K)$. We have $T=gsg^{-1}$ with $s\in S_E$ and $g\in\mathrm{GL}_3(E)$. Since S lies in one original-slot image, a free-slot input produces s at the same frozen division background. Split and conjugate the *whole original tuple* by g. This is one actual evaluation of h equal to T over E. The affine equations $h(X_1,\dots,X_m)=T$ have coefficients in K and a solution over E, so their ideal is proper. The weak Nullstellensatz gives a solution over K. This descends each complete fixed-target fiber; no specialization of a chosen generic solution is asserted.

Finally traceful ordinary multilinearity supplies all scalars. In the clock–shift basis Z^aX^b, $(a,b)\in(\mathbb{Z}/3\mathbb{Z})^2$, each basis tuple has its value in one homogeneous

line, and only the zero-grade line KI has nonzero trace. Expanding a nonzero-trace tuple multilinearly therefore yields a nonzero scalar value on a basis tuple. Scaling one original input gives every scalar, including zero. This proves the theorem. □

6.2. **A five-dimensional original-slot criterion.**

Theorem 6.6. *Suppose an ordinary multilinear h has one actual original full-slot image $S \subset M_3(K)$ such that*

$$\dim S = 5, \qquad I \notin S, \qquad S \cap \mathfrak{sl}_3(K) \text{ contains no nonzero rank-one matrix.}$$

Then $h(M_3(K)^m) = M_3(K)$.

Proof. The hypotheses force tracefulness. Otherwise $\mathbb{P}(S)$, of dimension four, would meet the projective rank-one nilpotent variety, of dimension three, in $\mathbb{P}(\mathfrak{sl}_3) = \mathbb{P}^7$. This contradicts the stated avoidance. The selected slot has generic rank at least five. If it is at least six, apply Theorem 6.5. Otherwise it is exactly five.

On its rank-five background open set the images form a rank-five subbundle of the trivial matrix bundle. Tracefulness is open. On the traceful open, the trace-zero sections form a rank-four subbundle. Identity exclusion is open, being the nonvanishing of the section of the quotient bundle represented by I. Rank-one avoidance is also open: the closed incidence of the projectivized rank-four bundle with the fixed projective rank-one nilpotent variety has closed image under its proper projection to the background base. The given actual background lies in their common nonempty open set. Hence these *geometric* conditions hold at the generic background and transfer by Lemma 6.2 to a five-dimensional space $S_D \subset D$. This does not assert that an arbitrary split five-space has a division form.

Write S for this division space, $P = S^{\perp}$, $U = F1 \oplus P$, and $W = S \cap \ker \mathrm{Trd}$; their dimensions are $5, 4, 5, 4$, respectively. Over E consider the rank-at-most-one cone C_1 and

$$\rho : C_1 \longrightarrow U_E^*, \qquad R \longmapsto \big(T \mapsto \mathrm{tr}(TR)\big).$$

Its zero fiber is precisely $C_1 \cap W_E = \{0\}$. The five homogeneous linear projection coordinates form a homogeneous system of parameters in the five-dimensional standard graded Segre cone ring: their common zero is its vertex, so their quotient has finite dimension. Graded induction makes that ring finite over the subalgebra generated by the five coordinates. Equality of dimensions then makes these coordinates algebraically independent. Thus ρ is finite and surjective. Every nonzero functional has a nonzero rank-one preimage, but the zero functional has only the zero matrix.

Prescribe $d(1) = \mu$ and $d(T) = -\lambda\,\mathrm{tr}(T)$ for $T \in P_E$. Since $1 \notin S$, the restriction of trace to P is nonzero; hence this functional is zero exactly when $\lambda = \mu = 0$. The preceding projection consequently puts in the saturation of S_E every scalar-plus-rank-one class except the pure rank-one nilpotent class. In

particular, taking $(\lambda, \mu) = (1, -3)$ gives $A = I + R \in W_E$ with $R^2 = -3R$, and therefore

$$\operatorname{tr}(A^2) = 3 - 6 + 9 = 6.$$

Thus $q = \operatorname{Trd}(X^2)|_W$ is nonzero. The norm cubic on W is anisotropic and irreducible over F, and geometrically coprime to q, by the same linear-factor and Galois-orbit arguments as above. The nilpotent section $q = d = 0$ is pure of dimension two. Every nonzero point is regular nilpotent because W_E contains no nonzero rank-one matrix. At any such point the saturation morphism $\mathrm{GL}_3 \times S_E \to M_3$ has local fiber dimension $3 + 2 = 5 = 14 - 9$. Miracle flatness [5] and the companion contraction in the proof of Theorem 6.5 give every cyclic matrix. The degree-two-minimal-polynomial classification gives every other nonscalar matrix except the pure rank-one nilpotent class. Nonzero scalars are also missing from this fixed saturation because $1 \notin S$.

These two missing classes are supplied by the original polynomial at other backgrounds. A noncentral multilinear polynomial attains a nonzero multiple of a matrix unit E_{ij}, $i \neq j$. Indeed conjugate a nonscalar actual value until it has a nonzero off-diagonal entry, and expand all inputs in matrix units. At least one matrix-unit tuple has that off-diagonal output entry nonzero. A nonzero word on such a tuple is a directed path; its total edge imbalance fixes its two endpoints. Thus every nonzero word at that tuple has the same ordered off-diagonal endpoints, and the entire value is a nonzero multiple of E_{ij}. Scaling and conjugating supply the rank-one nilpotent orbit. Noncentrality here follows from the five-dimensional slot. The clock–shift argument supplies all scalars. For every fixed remaining K-target, the extension-field evaluation descends by its whole affine input fiber and the weak Nullstellensatz, exactly as in Theorem 6.5. No addition of values is involved. □

6.3. **The triangular symbol and differential ranks.** Write Γ_6 for the proper multilinear sextic space and W_6 for the span of the ninety ordered products of three disjoint pair commutators. We use the classical upper-triangular identity theorem: $\mathrm{Id}(UT_r)$ is generated as a T-ideal by a product of r pair commutators. In particular UT_3 has no nonzero polynomial identity of degree less than six. One precise reference is Gonzales Gargate and de Mello, *Images of multilinear polynomials on $n \times n$ upper triangular matrices over infinite fields*, Theorem 2.1 [3]. Only this standard PI input is used in the next lemma.

Lemma 6.7. *There are scalars $c_s(f)$ such that for independent diagonal matrices $D_i = \operatorname{diag}(d_{i1}, d_{i2}, d_{i3})$ and $a \neq b$,*

$$f(D_1, \dots, E_{ab}, \dots, D_6) = c_s(f) \prod_{i \neq s} (d_{ia} - d_{ib}) E_{ab}.$$

In particular $f(D, \dots, X_s, \dots, D) = c_s(f) \operatorname{ad}_D^5 X$. Moreover

$$\ker c = \Gamma_6 \cap \mathrm{Id}(UT_2), \qquad \Gamma_6 \cap \mathrm{Id}(UT_3) = W_6.$$

If $c(f) = 0$ and $f \notin W_6$, then on triangular tuples

$$a_i = D_i + u_i E_{12} + v_i E_{23} + w_i E_{13}, \qquad f(a) = u^t A(D) v \, E_{13}, \tag{6.1}$$

where A is a nonzero 6×6 polynomial matrix with $A_{ii} = 0$. Its entries have degree four and do not depend on the diagonal coordinates in either of their two occupied original slots.

Proof. Properness implies invariance under adding a scalar matrix to any input. A word with the sole off-diagonal factor E_{ab} visits only vertices a, b. Its coefficient is multilinear in each pair (d_{ia}, d_{ib}) and invariant under simultaneous translation of that pair. Each pair therefore contributes its difference, leaving a scalar. Diagonal inputs all commute and their value is zero. This proves the first assertions, including the full-slot formula by linearity in X.

For inputs $D_i + z_i E_{12}$ in UT_2, terms with two strictly upper factors vanish. The surviving expression is exactly the sum of the one-exception formulas. It vanishes identically if and only if $c = 0$. The stated T-ideal theorem gives the UT_3 assertion: in multilinear degree six a substituted product of three pair commutators can have only linear substitutions and scalar outside multipliers. Its component is exactly W_6.

If $c = 0$, both adjacent UT_2 corner quotients vanish, so $f(B^6)$ lies in KE_{13}. A single E_{13} factor gives zero by diagonal rigidity. The only possible remaining upper path is $1 \to 2 \to 3$, with an E_{12} in one slot and an E_{23} in another. This gives (6.1), the zero diagonal, and independence from the two occupied diagonal slots. Its matrix is nonzero because $f \notin \mathrm{Id}(UT_3)$. □

Lemma 6.8. *Let all vector spaces in the following assertions be over an infinite characteristic-zero field.*

(1) Suppose a bilinear map $T(u, v)$ takes values in a fixed space $U \subset F^m$ of dimension at most two. Write its coefficients as τ_{ijk}, where k is the output coordinate. If repeated-index coefficients vanish and the nonzero support is cyclic,

$$\tau_{ijk} \neq 0 \iff \tau_{jki} \neq 0 \iff \tau_{kij} \neq 0,$$

then $T = 0$.

(2) Let $B : V \to Z$ and $C : U \to Z$ have the same rank $r \geq 2$. Assume $r \geq 3$, or $r = 2$ and $\operatorname{im} B \neq \operatorname{im} C$. If a bilinear T satisfies $T(u, v) \in \operatorname{span}(Cu, Bv)$ generically, then

$$T(u, v) = \beta(u)Bv + \alpha(v)Cu$$

for fixed linear forms β and α.

Proof. For (1), let $t_{ij} \in U$ be the coefficient vector of $u_i v_j$. If $\tau_{ijk} \neq 0$, cyclic support makes the three coordinate functionals $\epsilon_i, \epsilon_j, \epsilon_k$ nonzero on U. In dimension one, $t_{ij} \neq 0$ cannot be killed by ϵ_i. In dimension two, its annihilation by ϵ_i, ϵ_j makes these two nonzero functionals proportional. The functional ϵ_k, which does not kill t_{ij}, is not proportional to them. But $\tau_{jki} \neq 0$ similarly forces ϵ_j and ϵ_k proportional, a contradiction.

For (2), quotient first by the line FCu for generic u. The induced map B has rank at least two. Two linear maps which are pointwise collinear, one

of rank at least two, are scalar multiples: the first vanishes on the second's kernel by perturbing with two vectors having independent images; the induced endomorphism on the image preserves every line and hence is scalar. Therefore

$$T_u = \beta(u)B + (Cu)\alpha_u.$$

Choose generic u_1, u_2 with Cu_1, Cu_2 independent and $\operatorname{im} B$ not contained in their plane. Such pairs exist under the stated hypotheses. Additivity of T and reduction modulo that plane give additivity of β on generic pairs. Independence of Cu_1, Cu_2 then gives $\alpha_{u_1} = \alpha_{u_2} = \alpha_{u_1+u_2}$. Varying pairs gives a fixed α. Now $T_u - (Cu)\alpha$ is linear in u and is a scalar multiple of the fixed nonzero map B, so the scalar β is linear. The polynomial identities extend from the dense open set. □

Proposition 6.9. *Suppose $f \in \Gamma_6$, $c(f) = 0$, $f \notin W_6$, and $\operatorname{tr} f \neq 0$. On a nonempty open subset of B^6, $f(a) = \lambda E_{13} \neq 0$, the three full differential rows*

$$(df)_{21}, \qquad (df)_{32}, \qquad d(\operatorname{tr} f)$$

are independent, and the full differential has rank seven or eight. Its output 31 *row vanishes identically on B^6.*

Proof. Work over the field $F = K(d_{iq})$ and use the independent u, v, w coordinates. Put $r = \operatorname{rank}_F A \geq 1$ and let I, J be the nonzero row and column supports of A. Path weights show that only input E_{31} directions can produce output 21 or 32. On their amplitude vector z the two rows are

$$B(v) = v^t A_1, \qquad C(u) = (A_2 u)^t,$$

where A_1, A_2 are obtained from A by cyclic permutations of the three diagonal labels. These matrices have rank r and exactly its zero-entry pattern. The rows are generically independent: for $r \geq 2$ the independent parameters u, v can give independent vectors; for $r = 1$, writing $A = \alpha\beta^t$, the zero diagonal forces the nonempty supports I, J to be disjoint, and the two row supports are disjoint. The 31 row is zero: a single negative-root variation could give that weight only through an E_{31} with all other inputs diagonal, and that one-exception coefficient is zero.

At a nonzero value λE_{13}, covariance supplies the four-dimensional orbit tangent

$$\langle E_{13}, E_{12}, E_{23}, E_{11} - E_{33} \rangle \subset \operatorname{im} df.$$

It remains to show that trace is independent of the two negative rows. Its full differential is nonzero generically on B^6: cyclically rotate an original variable x_s to the front in the trace expression and delete it. The resulting cyclic derivative G_s is a nonzero degree-five ordinary polynomial when the cyclic class of f is nonzero. Tracefulness implies that nonzero cyclic class. By the UT_3 identity theorem, G_s cannot vanish on all triangular backgrounds. The nondegenerate trace pairing detects it by a full-matrix variation of x_s.

Suppose nevertheless that the trace row were generically in the span of the $21, 32$ rows. Its derivatives on E_{21}, E_{32} inputs would be zero, since those two

rows vanish there. Its derivatives on upper inputs already vanish. On an E_{31} input, the trace can come either from one background E_{13}, or from one E_{12} and one E_{23}. Vertex permutation identifies the former coefficients with the already vanishing two-vertex trace-cycle coefficients. Thus its remaining row has the form

$$T_k(u,v) = \sum_{i,j} \tau_{ijk}(D) u_i v_j, \qquad T \neq 0.$$

Repeated-index coefficients are zero by ordinary multilinearity. Cyclically permuting the matrix vertices preserves the nonzero support of τ_{ijk} under $(i,j,k) \mapsto (j,k,i)$. The actual coefficients are related by field automorphisms, not asserted equal. Our assumption says $T(u,v) \in \operatorname{span}(B(v), C(u))$.

For $r = 1$, or for $r = 2$ with equal two-dimensional images of the maps B, C, all coefficients of T lie in one fixed space of dimension at most two. Lemma 6.8(1) gives $T = 0$. Otherwise its part (2) gives $T = \beta(u)B(v) + \alpha(v)C(u)$. The coefficient of $u_k v_j$ in output k is $\beta_k (A_1)_{jk}$ because $(A_2)_{kk} = 0$; hence $\beta_k = 0$ for $k \in J$. Similarly $\alpha_k = 0$ for $k \in I$. A nonzero τ_{ijk} must satisfy at least one of

$$\begin{array}{ll} \text{(a)} & i \notin J, \quad j \in I, \quad k \in J, \\ \text{(b)} & i \in J, \quad j \notin I, \quad k \in I. \end{array}$$

These patterns cannot persist through all cyclic orders. If (a) holds for (i,j,k), the next order must satisfy (b), forcing $j \in J$, $k \notin I$, $i \in I$. The third order then satisfies neither pattern. If no cyclic order satisfies (a), all three satisfy (b), forcing an index simultaneously in and outside I. This again gives $T = 0$, a contradiction.

The three rows are therefore independent. They vanish on the four orbit directions just displayed. Thus the full rank is at least seven and at most eight. All nonvanishing and independence conditions hold on one common nonempty open subset of the irreducible triangular parameter space, which has K-points. □

6.4. **Normal Hessians and the fixed lower source space.** The following calculations will be used in both generic-rank branches. They are made over the generic triangular parameter field, and all nonzero conditions can subsequently be realized on a common nonempty open set of K-points. Put

$$W_0 = \{(x_i E_{21} + y_i E_{32})_{i=1}^6\}, \qquad q_{31}(x,y) = \tfrac{1}{2}(d^2 f_a(x+y, x+y))_{31} = y^t A(D^{\mathrm{rev}}) x, \tag{6.2}$$

where D^{rev} reverses vertices $1, 3$. The equality follows from the sole path $3 \to 2 \to 1$. No upper background factor can occur in that quadratic, by root weight. Its bilinear matrix rank is r, and its quadratic rank is $2r$.

There are forced symbol kernels. Conjugating a one-step tuple $D_i + v_i E_{23}$ by $I + tE_{12}$, and then a tuple $D_i + u_i E_{12}$ by $I + tE_{23}$, shows

$$(d_{i1} - d_{i2})_i^t A = 0, \qquad A(d_{i2} - d_{i3})_i = 0.$$

The new E_{13} entries cannot contribute. Consequently the two fixed lower vectors

$$\rho_{21} = ((d_{i1} - d_{i2}) E_{21})_i, \qquad \rho_{32} = ((d_{i2} - d_{i3}) E_{32})_i \tag{6.3}$$

lie in the radical of q_{31} on W_0. They are the lower parts of simultaneous-conjugation tangents and satisfy, modulo KE_{13},

$$df(\rho_{21}) = \lambda E_{23}, \qquad df(\rho_{32}) = -\lambda E_{12}. \tag{6.4}$$

We next check the correction terms in the normal quadratics. Since the 31 row of the full differential vanishes identically on the triangular family, differentiation in any triangular direction gives

$$(d^2 f_a(b,\xi))_{31} = 0 \quad (b \in B^6, \xi \in M_3^6). \tag{6.5}$$

For a moving normal ℓ in a constant-rank triangular family, differentiate $\ell(a)df_a(\xi) = 0$ along $b \in B^6$. The result is

$$\ell(a)d^2 f_a(b,\xi) + (d\ell)_a(b)df_a(\xi) = 0.$$

The second term vanishes when $\xi \in \ker df_a$, and only then is it discarded. Thus

$$\ell(a)d^2 f_a(B^6, \ker df_a) = 0. \tag{6.6}$$

Also $df_a(B^6) = KE_{13}$, because its image is contained in that line and scaling one input differentiates to $f(a) \neq 0$. If a raw lower vector z satisfies $df_a(z) \in KE_{13}$, choose an upper b so $z + b \in \ker df_a$. Equations (6.6) and $f(B^6) \subset KE_{13}$ give

$$\ell d^2 f_a(z,z) = \ell d^2 f_a(z+b, z+b).$$

The same holds for polarizations. Hence lower representatives compute the quadratic terms on the full kernel, without unproved invariance under arbitrary source corrections. Fixing independent output controls and formally eliminating their source coordinates has these same normal quadratic initial forms.

For later use define the two-cycle matrix M by the output 11 coefficient of $u_i x_j$ in the E_{12}, E_{21} corner, with all other inputs diagonal. Set $\mathbf{d}_q = (d_{iq})_{i=1}^6$, the vector of diagonal entries at vertex q. Path expansion gives

$$M(\mathbf{d}_1, \mathbf{d}_2) = A(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_1), \qquad M_{ii} = 0, \qquad M_{22} = \sigma(M)^t, \tag{6.7}$$

where M_{22} denotes the output 22 coefficient matrix, and σ swaps diagonal labels $1, 2$. Properness makes each entry of M a homogeneous degree-four polynomial in the differences $d_{i1} - d_{i2}$. Therefore $\sigma(M) = M$ and $M_{22} = M^t$. Every two-cycle diagonal jet is a vertex permutation of this one.

6.5. **Generic rank seven with vanishing two-cycle jets.**

Proposition 6.10. *Under the hypotheses of Proposition 6.9, suppose the generic triangular differential rank is seven and $M = 0$. Then f is flat at a triangular point over a nonzero multiple of E_{13}.*

Proof. Besides the constant 31 normal, choose a second normal in the form

$$\ell(Y) = a(Y_{11} + Y_{33}) + bY_{22} + cY_{21} + dY_{32}.$$

The orbit directions force this form. Its coefficients can be chosen in $F(u,v)$, independently of w. Indeed upper source directions give only E_{13}; E_{21}, E_{32} directions have no non-upper outputs because their two-cycle diagonal jets vanish; on E_{31} directions the negative rows depend only on u, v. The possible

w-dependent diagonal terms are 13 two-cycle jets, also zero by vertex permutation. Thus the entire non-upper differential matrix is independent of w, and so is its one-dimensional annihilator. The independence of the trace row gives $\delta = a - b \neq 0$: otherwise ℓ would be a linear combination of the three independent rows trace, 21, and 32.

Let $L_w = (df \bmod KE_{13})|_{W_0}$. Since $M = 0$, its image is $\langle E_{12}, E_{23}\rangle$. By (6.4), with $R_0 = \langle \rho_{21}, \rho_{32}\rangle$, one has

$$W_0 = \ker L_w \oplus R_0.$$

Put $q_\ell(z) = \frac{1}{2}\ell d^2 f_a(z,z)$. Its coefficients have w-degree at most one: two negative adjacent-root insertions can contribute to a diagonal output using one E_{13} or two adjacent upper entries, and can contribute to outputs $21, 32$ without any E_{13} factor.

We need the exact projection correction. For $C_T = ([T, a_i])_i$, differentiating covariance gives

$$\ell d^2 f_a(C_T, z) = \ell[T, df_a(z)].$$

The omitted term is in $\operatorname{im} df_a$ and is killed by ℓ. For $T \in \langle E_{21}, E_{32}\rangle$ and a raw $z \in \ker L_w$, the right-hand side vanishes since $df_a(z) \in KE_{13}$ and $[T, E_{13}]$ is strictly upper. Writing $C_T = \rho_T + b_T$ with b_T upper, and correcting z by an upper vector to the full kernel, equation (6.6) shows that the upper part contributes zero as well. Hence R_0 and $\ker L_w$ are orthogonal for q_ℓ. For two conjugation directions, the same covariance identity and the vanishing on two upper directions give

$$[sE_{21} + tE_{32}, [sE_{21} + tE_{32}, E_{13}]] = st\,\mathrm{diag}(1, -2, 1),$$
$$q_\ell(s\rho_{21} + t\rho_{32}) = \lambda\delta st.$$

The upper parts do not change this calculation: their pairing with C_T equals $\ell[T, df(b_T)]$, which is zero since $df(b_T)$ is a multiple of E_{13}.

Write $L_w z = p_w(z)E_{12} + q_w(z)E_{23}$. The projection to its kernel along the *fixed* complement is

$$\Pi_w z = z - \frac{q_w(z)}{\lambda}\rho_{21} + \frac{p_w(z)}{\lambda}\rho_{32}.$$

Orthogonality and the preceding double bracket give the exact formula

$$Q_{\ell,w}(z) := q_\ell(\Pi_w z) = q_\ell(z) + \frac{\delta}{\lambda}p_w(z)q_w(z), \qquad q_{31}(\Pi_w z) = q_{31}(z). \tag{6.8}$$

Both moving-kernel forms have now been pulled back to the same W_0.

The homogeneous w part of p_w is the path $1 \to 3 \to 2$, a bilinear expression in w and y; that of q_w is the path $2 \to 1 \to 3$, bilinear in x and w. Their matrices are vertex-permuted copies of A, so both are nonzero. Choose a generic fixed w_0 for which both are nonzero and put $w = tw_0$. The coefficient of t^2 in (6.8) is

$$\frac{\delta}{\lambda}p^{(1)}_{w_0}(y)q^{(1)}_{w_0}(x), \tag{6.9}$$

a nonzero product of two linear forms. There is no t^2 contribution from q_ℓ.

If $r \geq 2$, the fixed q_{31} has quadratic rank at least four and is irreducible over the algebraic closure of the coefficient field. It cannot be proportional to the product in (6.9). Therefore Q_{ℓ,tw_0} is not a rational-function multiple of q_{31}: comparison of a nonzero coefficient of q_{31} would make that multiplier a polynomial in t of degree at most two, and its leading coefficient would force the prohibited proportionality. The two quadrics are coprime. For $r = 1$, write $A = \alpha\beta^t$ with disjoint supports I, J. The factors of q_{31} involve respectively y_I and x_J, whereas the factors of (6.9) involve y_J and x_I. Neither factor of q_{31} divides that product. Divisibility over the rational function field in t would force divisibility of every polynomial coefficient, so coprimality holds in this case too.

The pullback $W_0 \to \ker L_w$ is surjective, hence the quadrics on $\ker L_w$ are coprime. The same is true on the full differential kernel: a common positive-degree homogeneous factor would restrict either to a common positive-degree factor, or to zero and make both nonzero restricted quadrics zero. Both alternatives are impossible. Choose seven independent output controls. Their fixed-value local fiber is smooth by the Jacobian criterion [4] of dimension $54 - 7 = 47$. Its two remaining equations have these coprime normal quadratic initial forms by Section 6.4. Coprime quadrics form a regular sequence in the associated graded polynomial ring; the regular initial-form criterion gives a regular sequence in the completed local ring. Explicitly, for a germ with nonzero initial form in this regular local ring, the associated graded quotient by the germ is the polynomial ring modulo its initial form. Since the second initial form is a nonzerodivisor in that quotient, its germ is a nonzerodivisor as well; separation and completeness of the maximal-ideal filtration justify passage from successive orders to the germs. Thus the full fiber has dimension $45 = 54-9$. Miracle flatness [5] applies. Clearing denominators and taking the common nonempty open set realizes all calculations at actual K-points. □

6.6. **Generic rank seven with nonzero two-cycle jets.**

Lemma 6.11. *Under the hypotheses of Proposition 6.9, if the generic triangular rank is seven and $M \neq 0$, then $M = M^t$.*

Proof. The rows u^tM, u^tM^t must be generically collinear. Otherwise the 12 corner supplies two independent diagonal directions, and the orbit direction $E_{11} - E_{33}$ supplies a third. Along with the three upper directions and the two independent negative-output rows, they force full rank at least eight. A rank-one $M = ab^t$ cannot satisfy collinearity: $M_{ii} = 0$ makes the nonempty supports of a, b disjoint, so u^tM and u^tM^t lie on two distinct lines. Thus $\operatorname{rank} M \geq 2$. The pointwise-collinearity argument used in Lemma 6.8 gives $M^t = \rho M$ for one scalar in F; transposition gives $\rho^2 = 1$. If $\rho = -1$, the two adjacent corners give the two traceless diagonal lines. Together with the three upper directions they fill a five-dimensional subspace of the kernel of outputs $21, 32$. The image has dimension seven and those two rows are independent, so that kernel is exactly this five-space and trace vanishes on it. Trace would consequently be

a linear combination of the $21, 32$ rows, contrary to Proposition 6.9. Therefore $\rho = 1$. □

Choose original slots $i \neq j$ with $M_{ij} \neq 0$. Then both A_{ij} and A_{ji} are nonzero, since their closed-path specializations are $M_{ij} = M_{ji}$. Keep precisely these two original inputs free, $x_i = X, x_j = Y$, fixing the other four to diagonal matrices. Diagonal parts of either free input give zero: the other input is then the sole off-diagonal exception and $c = 0$. Direct word paths therefore give, for hollow X, Y and distinct r, s, t,

$$F(X, Y)_{rt} = a_{rst} X_{rs} Y_{st} + b_{rst} Y_{rs} X_{st}, \tag{6.10}$$

and for each unordered pair $\{r, s\}$ the diagonal contribution

$$k_{rs}(X_{rs} Y_{sr} + Y_{rs} X_{sr})(E_{rr} + E_{ss}). \tag{6.11}$$

Here the two equal endpoint coefficients use $M = M^t$ and (6.7). The off-diagonal coefficients are vertex-permuted copies of A_{ij}, A_{ji}, and the closed coefficients are copies of M_{ij}. All belong to the *same* polynomial ring in the four frozen diagonal inputs; none depends on a diagonal coordinate in the two occupied slots. Each is nonzero as a polynomial. One actual choice off their finitely many zero loci makes every a_{rst}, b_{rst}, k_{rs} nonzero simultaneously. Equations (6.10)–(6.11) are then an exact restriction of f on full X, Y, not just an equality of triangular symbols. In particular $X = E_{12}, Y = E_{23}$ gives a nonzero rank-one nilpotent actual value.

Proposition 6.12. *Any ordinary traceful multilinear polynomial with the exact actual restriction (6.10)–(6.11), with all displayed weights nonzero, has full image on M_3.*

Proof. Fix hollow Y with all six entries nonzero and write $z_{rs} = X_{rs}/Y_{rs}$. The three pair-diagonal matrices

$$D_1 = E_{11} + E_{22}, \qquad D_2 = E_{22} + E_{33}, \qquad D_3 = E_{33} + E_{11}$$

are independent: their determinant in the diagonal basis is two. Thus zero diagonal output is equivalent to $z_{sr} = -z_{rs}$. With $x = z_{12}, y = z_{23}, z = z_{31}$, the forward off-diagonal zero equations are

$$a_{123} x + b_{123} y = 0, \qquad a_{231} y + b_{231} z = 0, \qquad a_{312} z + b_{312} x = 0.$$

Together with the other three they are six equations whose coefficients are independent of Y, after cancellation of its nonzero entry products. The first two rows are independent, so their common kernel has dimension at most one. Hence the original X-slot has rank five or six. A rank-six instance is covered by Theorem 6.5.

In the remaining case the reduced six-equation matrix has rank two. Its kernel vector (h_{12}, h_{23}, h_{31}) has every coordinate nonzero, since a zero propagates through the three forward equations. Set $h_{sr} = -h_{rs}$. For every Y in the six-edge torus, $F(H \odot Y, Y) = 0$, where $\odot$ is entrywise multiplication. This is

a polynomial identity, hence holds for every hollow Y. The invertible input change gives an alternating bilinear map

$$T(Z,Y) = F(H \odot Z, Y).$$

Its diagonal coefficients are $\gamma_{rs}(Z_{rs}Y_{sr} - Y_{rs}Z_{sr})D_{rs}$, with $\gamma_{rs} = k_{rs}h_{rs} \neq 0$; its off-diagonal entries are

$$T(Z,Y)_{rt} = \eta_{rst}(Z_{rs}Y_{st} - Y_{rs}Z_{st}), \qquad \eta_{rst} = a_{rst}h_{rs} \neq 0.$$

The second equality of coefficients follows from $a_{rst}h_{rs} + b_{rst}h_{st} = 0$.

Orient the cycles as $(12, 23, 31)$ and $(21, 32, 13)$ and write $Z = (x, y), Y = (u, v)$ accordingly. The opposite-cycle output is an invertibly weighted copy of $x \times u$, and the other-cycle output is an invertibly weighted copy of $y \times v$. The diagonal is

$$\sum_{\ell=1}^{3} \gamma_\ell(x_\ell v_\ell - u_\ell y_\ell)D_\ell.$$

At Y_0 with $u = (1,1,1), v = 0$, the image S_{Y_0} is the direct sum of all diagonals and a two-plane in the opposite directed cycle. That plane contains no coordinate axis: its defining linear equation has all coefficients nonzero, as does the plane perpendicular to $(1,1,1)$ before the invertible coordinate weights. It follows that $S_{Y_0} \cap \mathfrak{sl}_3$ contains no nonzero rank-one matrix. In fact two nonzero diagonal entries of a rank-one matrix force both opposite off-diagonal entries nonzero, while this support permits only one. A traceless rank-one member therefore has all diagonal entries zero. A matrix on one directed three-cycle has rank equal to its number of nonzero entries. Rank one would then require a coordinate axis, which the two-plane excludes.

To obtain identity exclusion, take u, v with every coordinate nonzero. Zero off-diagonal output forces $x = \alpha u, y = \beta v$, so the diagonal output is $(\alpha - \beta)\sum \gamma_\ell u_\ell v_\ell D_\ell$. Since $I = (D_1 + D_2 + D_3)/2$, its membership in the image requires

$$\gamma_1 u_1 v_1 = \gamma_2 u_2 v_2 = \gamma_3 u_3 v_3.$$

It fails, for example, at $u = (1,1,1)$, $v = (\gamma_1^{-1}, 2\gamma_2^{-1}, 3\gamma_3^{-1})$. At this point the kernel is precisely KY, and the rank is five.

Because T is alternating, every frozen map has rank at most five (the zero background has rank zero). Thus the rank-five traceful locus Ω is a nonempty open subset of the irreducible hollow Y-space, containing Y_0 and the latter example. Its images form a rank-five bundle and its trace-zero images a rank-four bundle. Identity exclusion is a nonempty open condition. Rank-one avoidance is another nonempty open condition by the proper projective-incidence argument in Theorem 6.6, witnessed by Y_0. They intersect in this same irreducible Ω, and that intersection has an actual K-point. At that *one* background the original X-slot therefore satisfies all three hypotheses of Theorem 6.6. The theorem also handles the possibility that its unrestricted generic rank, away from this diagonal chart, is larger than five, by applying the rank-six theorem. □

6.7. **Generic rank eight and symbol rank at least two.**

Proposition 6.13. *Under the mixed triangular setup, if the full differential has generic rank eight and* $r = \operatorname{rank} A \geq 2$*, the unique corrected* 31 *normal quadratic is nonzero at generic triangular points. Consequently there is a flat point over a nonzero multiple of* E_{13}*.*

Proof. Work first over $F = K(D)$ and then with independent u, v, w. On $W_0 = X \oplus Y$ let $q = q_{31}$ and let T be the diagonal part of the differential. Path weights give

$$T(x, y) = (p_u(x), h_u(x) + k_v(y), s_v(y)). \tag{6.12}$$

Here p_u, h_u depend linearly on u, and k_v, s_v linearly on v, and no w occurs. The two corner maps have the same generic rank $c \in \{0, 1, 2\}$ by vertex permutation. The two vectors (6.3) lie in $\ker T$, are radical for q, and remove the $12, 23$ controls by (6.4). Every vector of $\ker T$ can therefore be moved to $\ker(df \bmod KE_{13})|_{W_0}$ without changing q, and then corrected by an upper vector to the full differential kernel, again without changing q by (6.5). It suffices to find a nonzero restriction of q to $\ker T$, unless an E_{31} direction is explicitly needed below.

Suppose first $r \geq 3$. If $c \leq 1$, then $\operatorname{rank} T \leq 2$. Restricting a quadratic form to codimension at most two lowers its rank by at most four, so $\operatorname{rank}(q|_{\ker T}) \geq 2r - 4 > 0$. If $c = 2$ and $q|_{\ker T} = 0$, restrict to $X_0 \oplus Y_0 = \ker p_u \oplus \ker s_v$. The restrictions h_0, k_0 are both nonzero, and the pure bilinear quadratic q_0 vanishes on $h_0 + k_0 = 0$. Hence $q_0 = (h_0 + k_0)(a(x) + b(y))$. Its pure bidegree $(2, 0)$ part $h_0 a$ and pure bidegree $(0, 2)$ part $k_0 b$ vanish. Therefore $a = b = 0$ and $q_0 = 0$. A bilinear form vanishing on $\ker p_u \times \ker s_v$ has the form

$$q = p_u B + C s_v,$$

so its matrix rank is at most two, a contradiction. This proves the assertion for $r \geq 3$.

Now let $r = 2$ and assume, seeking a contradiction, that the quadratic vanishes on the *entire* full differential kernel generically. In particular it vanishes on $\ker T$. If the corner matrix M is zero, then $T = 0$ and $q \neq 0$, a contradiction. For $M \neq 0$ the endpoint rows p_u, s_v are nonzero. If $c = 2$, the same bidegree argument shows

$$q|_{\ker p_u \times \ker s_v} = 0. \tag{6.13}$$

If $c = 1$, the middle rows are multiples of the endpoint rows and $\ker T = \ker p_u \times \ker s_v$, so the same conclusion holds.

A fixed bilinear form of matrix rank two cannot satisfy (6.13) for independent generic u, v unless the spans of the p_u and the s_v each have dimension one. Indeed, fix generic v. For $y \in \ker s_v$, the functional $q(\,\cdot\,, y)$ belongs to the line Fp_u for every generic u. If those lines span dimension at least two, their intersection is zero, forcing $q(\,\cdot\,, \ker s_v) = 0$ and matrix rank at most one. The symmetric argument applies to s_v. This reasoning can be made over independent transcendental parameter copies, so it is a generic identity over F, not a choice of incompatible specializations. It follows that $\operatorname{rank} M = 1$.

Write $M = ab^t$, with disjoint nonempty supports $I = \operatorname{supp} a$, $J = \operatorname{supp} b$, since $M_{ii} = 0$. The other corner row is obtained from $\sigma(M)^t$, so its fixed support is I while that of the first row is J. Thus both corner jets actually have rank two. We may write

$$p_u = \alpha(u)p, \qquad s_v = \delta(v)s, \qquad q(x,y) = p(x)B(y) + C(x)s(y), \tag{6.14}$$

where p, s, B, C are defined over F, p has support J, s has support I, and α, δ are nonzero. Since q has matrix rank two, B is not proportional to s and C is not proportional to p. The row h_u is independent of p, and k_v of s. Consequently T has rank three; its kernel projects onto all of $\ker p$ in X and all of $\ker s$ in Y.

Introduce the E_{31} source amplitudes z. The two negative rows on them are

$$\beta_v(z) = v^t A(D^{(2,3,1)})z, \qquad \gamma_u(z) = z^t A(D^{(3,1,2)})u.$$

They are independent, since the full image at rank eight is precisely the hyperplane with 31 coordinate zero. Set $Z = \ker \beta_v \cap \ker \gamma_u$, a codimension-two space over $F(u,v)$, independent of w.

The following mixed equivariance identities hold for every full source vector ξ:

$$(d^2 f_a(\rho_{21}, \xi))_{31} = -(df_a \xi)_{32}, \qquad (d^2 f_a(\rho_{32}, \xi))_{31} = (df_a \xi)_{21}. \tag{6.15}$$

They follow by differentiating covariance with E_{21} or E_{32} and taking output 31. The extra differential term has zero 31 coordinate; the upper part of the conjugation tangent contributes nothing by (6.5). Thus both ρ directions are radical for the normal Hessian restricted to $W_0 \oplus Z$. They can remove upper controls without altering any pairing used below. Subsequent upper source corrections also alter no such pairing.

For $z \in Z$, the diagonal part of $df_a(zE_{31})$ is affine-linear in w. Its homogeneous w part occurs only at endpoints $11, 33$ and is the 13 two-cycle jet. Write its vertex-relabeled rank-one matrix as $M^{13} = a^{13}(b^{13})^t$ and let σ_{13} swap labels $1, 3$. These two endpoint terms are

$$\begin{aligned} d_1(w,z) &= (w^t a^{13})((b^{13})^t z), \\ d_3(w,z) &= (w^t \sigma_{13}(b^{13}))(\sigma_{13}(a^{13})^t z). \end{aligned} \tag{6.16}$$

Their two w forms have disjoint supports I, J and are independent. Their corresponding z forms have supports J, I.

Choose a right inverse of the surjective map T over $F(u,v)$, independent of w, and solve

$$T(x(w), y(w)) = -\operatorname{diag}(df_a(zE_{31})).$$

The solutions are affine-linear in w. Their homogeneous parts x_1, y_1 satisfy

$$\alpha p(x_1) = -d_1(w,z), \qquad \delta s(y_1) = -d_3(w,z), \qquad h_u(x_1) + k_v(y_1) = 0.$$

After the control adjustments justified by (6.15), the direction $zE_{31} + x(w) + y(w)$ is in the full kernel. Likewise each $\kappa = (\kappa_x, \kappa_y) \in \ker T$, chosen independently of w, has a full-kernel lift preserving the pairings.

By our contradiction assumption their polarized normal pairing is zero. The raw pairing between zE_{31} and κ contains no w: an E_{31}, E_{21} pair requires an E_{12}

background edge to end at 31, and an E_{31}, E_{32} pair requires an E_{23}. Taking the homogeneous w coefficient in the pairing and using (6.14) therefore gives

$$0 = p(x_1)B(\kappa_y) + C(\kappa_x)s(y_1) = -\frac{d_1(w,z)}{\alpha}B(\kappa_y) - \frac{d_3(w,z)}{\delta}C(\kappa_x). \tag{6.17}$$

Here $p(\kappa_x) = s(\kappa_y) = 0$ explains the absent two polarized terms. The control corrections vanish by (6.15) and (6.5). Independence of the w forms in (6.16), the surjective projections of $\ker T$, and the nonproportionality of B, s and C, p show that both b^{13} and $\sigma_{13}(a^{13})$ annihilate Z. They are independent; hence

$$\operatorname{span}(\beta_v, \gamma_u) = R := \operatorname{span}(b^{13}, \sigma_{13}(a^{13})), \tag{6.18}$$

a fixed plane over F with one nonzero J-supported vector and one nonzero I-supported vector.

Pass to the quotient by this fixed plane. Since β_v and γ_u depend linearly on independent variables, each coefficient vanishes in the quotient. Their coefficient matrices have rank two, so their row spaces equal R. Their path formulas, followed by the inverse diagonal-label field automorphisms, show that both row and column supports of A lie in $I \cup J$, and its row space contains both a nonzero I-supported vector and a nonzero J-supported vector. Only preservation of zero coordinates under field automorphisms is used; no invariance of the plane R is assumed. On the other hand reversing the path in (6.14) yields

$$A = \eta V^t + U\theta^t,$$

where η has precisely support I and θ precisely support J. The zero diagonal forces $V_i = 0$ for $i \in I$ and $U_j = 0$ for $j \in J$. If L is the complement of $I \cup J$, absence of L-columns and L-rows forces $V_\ell = U_\ell = 0$ for $\ell \in L$ (use any nonzero entry of η or θ). Thus A is supported only in $I \times J$. Its row space is wholly J-supported, contrary to (6.18). This contradiction proves the nonvanishing of the full-kernel quadratic when $r = 2$.

Finally choose the eight independent non-31 output controls. Their local fixed-value fiber is smooth by the Jacobian criterion [4] of dimension $54-8 = 46$. The remaining corrected equation has nonzero quadratic initial term, so it is a nonzero nonunit in a regular local domain and cuts dimension by one. The full fiber has the expected dimension 45; miracle flatness applies. All generic conditions hold on a common nonempty open set, providing an actual triangular K-point as claimed. □

6.8. **Original slots for rank-one symbols.**

Proposition 6.14. *If the mixed two-step symbol A has rank one, one actual original full slot has rank at least six. This conclusion is independent of the generic triangular differential rank and of the two-cycle jets.*

Proof. Write $A = \alpha\beta^t$ over F. Its zero diagonal forces disjoint nonempty supports $I = \operatorname{supp}\alpha$ and $J = \operatorname{supp}\beta$. Choose $i \in I, j \in J$. Then the polynomial A_{ij} is nonzero and $A_{ji} = 0$ identically. Keep precisely the original slots

$x_i = X, x_j = Y$ free and fix the other four to diagonal matrices. The polynomial A_{ij} is independent of diagonal coordinates in slots i, j. Thus all six vertex-permuted coefficients

$$c_{rst} = A_{ij}(D^{(r,s,t)}), \qquad r, s, t \text{ distinct},$$

are nonzero in the same polynomial ring of the other four diagonal inputs. Their product is nonzero. Choose one actual background where all six are nonzero simultaneously.

On hollow X, Y, a word path with off-diagonal output (r, t) must use the third vertex s and the two free edges. Its reverse-order coefficient is the corresponding permutation of A_{ji}, which is zero. Hence the exact off-diagonal output is

$$f(\dots, X, \dots, Y, \dots)_{rt} = c_{rst} X_{rs} Y_{st}.$$

Closed two-vertex paths contribute only to diagonal outputs. Freeze this actual Y to have all six hollow entries equal to one and diagonal zero. On the six hollow matrix units of the original X slot, the hollow projection is

$$\begin{aligned} &E_{12} \mapsto c_{123} E_{13}, \quad E_{13} \mapsto c_{132} E_{12}, \quad E_{21} \mapsto c_{213} E_{23}, \\ &E_{23} \mapsto c_{231} E_{21}, \quad E_{31} \mapsto c_{312} E_{32}, \quad E_{32} \mapsto c_{321} E_{31}. \end{aligned}$$

In the order $(12, 13, 21, 23, 31, 32)$ its determinant is

$$-c_{123} c_{132} c_{213} c_{231} c_{312} c_{321} \neq 0.$$

Therefore the *full* original-slot map has rank at least six. The hollow projection only certifies its rank; it is not substituted for the full-slot map in Theorem 6.5. Tracefulness and that theorem give full actual image. Nothing here asserts triangular flatness, which need not hold for rank-one symbols. □

6.9. **Proof of the image theorem in size three.**

Lemma 6.15. *Let $h : M_3(K)^m \to M_3(K)$ be ordinary multilinear and traceful. If it is flat at a point with value $\lambda E_{13} \neq 0$, then its actual image is all of $M_3(K)$.*

Proof. The flat locus is open [6], so restrict to a flat open source neighborhood. A flat morphism of finite presentation is open [7], and its target-open image contains λE_{13}. Every K-point of this open image has a nonempty finite-type fiber, hence a K-point by algebraic closedness. Thus it is an actual-image neighborhood. The image is invariant under scalar multiplication (scale one original input) and conjugation (conjugate all inputs). Conjugating the open set, assume it contains E_{13}.

For a nonscalar target B_0, choose v with $v, B_0 v$ independent; otherwise every vector is an eigenvector and B_0 is scalar. Use $B_0 v$ as first basis vector and v as third. The conjugate C has $C_{13} = 1$. For $t \neq 0$, put

$$\Gamma(t) = t^2 \operatorname{diag}(t^{-1}, 1, t) C \operatorname{diag}(t, 1, t^{-1}).$$

Its entry exponents are $2 + e_i - e_j$ with $(e_1, e_2, e_3) = (-1, 0, 1)$; they are nonnegative and only the 13 exponent is zero. This polynomial curve therefore has $\Gamma(0) = E_{13}$. Its inverse image of the open neighborhood contains some nonzero

t_0. Undoing the scalar multiplication and conjugation gives C and then B_0 as actual values. This treats every nonscalar Jordan type, not just cyclic matrices. The traceful clock–shift argument in the proof of Theorem 6.5 supplies all scalars. □

Proof of Theorem 6.1. If $c(f) \neq 0$, choose an original slot s with $c_s \neq 0$ and an actual regular diagonal D. The slot $c_s \operatorname{ad}_D^5 : M_3 \to M_3$ has exactly the six-dimensional hollow image. Theorem 6.5 applies.

If $c(f) = 0$ and $f \in W_6$, apply the arbitrary-W_6 traceful classification, Theorem 5.2. Its hypothesis of nonzero cyclic class follows from $\operatorname{tr} f \neq 0$. This invokes the theorem for arbitrary simultaneous coefficients across all ninety ordered products, not a sum of independently attained values or a single-pairing special case.

Otherwise Lemma 6.7 supplies a nonzero symbol A. Proposition 6.9 puts the generic full triangular rank R in $\{7, 8\}$. If $r = 1$, use Proposition 6.14, regardless of R. For $R = 7$ and $M = 0$, Proposition 6.10 and Lemma 6.15 apply. For $R = 7$ and $M \neq 0$, Lemma 6.11 and the exact two-free-slot restriction yield Proposition 6.12. Finally, for $R = 8$ and $r \geq 2$, Proposition 6.13 and Lemma 6.15 apply. These alternatives exhaust all possibilities. All conclusions concern actual K-valued original tuples, with every extension-field target descended through its full affine fiber. □

7. Two free original inputs and the mixed proper-sextic layer

Throughout this section K is algebraically closed of characteristic zero, $f \in K\langle x_1, \dots, x_6\rangle$ is ordinary multilinear and proper, and $f \in \operatorname{Id}(UT_2)$ but $f \notin \operatorname{Id}(UT_3)$. Proper means that every original unit substitution is the zero free polynomial. The standard triangular-identity filtration identifies this layer with $c(f) = 0$ and $f \notin W_6$. Write $\mathcal{H}_n$ for the vector space of hollow $n \times n$ matrices. All statements about images concern single actual evaluations over K.

7.1. **Local images and two-input restrictions.**

Lemma 7.1. *Let p be an ordinary multilinear polynomial. If its actual image contains an open neighborhood of a nonzero rank-one square-zero matrix in M_n, and $\operatorname{tr} p$ is not identically zero, then its image is M_n. If p is identically traceless and its image contains such a relative open neighborhood in $\mathfrak{sl}_n$, its image is $\mathfrak{sl}_n$. A smooth evaluation chart, or a flat point of the full evaluation map, over that matrix supplies the required actual open neighborhood.*

Proof. Conjugate and scale the neighborhood to contain E_{12}. For a nonscalar matrix A, choose v with v, Av independent and a basis making the $(1, 2)$ entry of a conjugate B equal to one. The curve

$$t^2 \operatorname{diag}(t^{-1}, t, 1, \dots, 1) B \operatorname{diag}(t, t^{-1}, 1, \dots, 1)$$

extends to $t = 0$ with value E_{12}: all entry exponents are nonnegative and only position $(1, 2)$ has exponent zero. The inverse image of the actual open neighborhood contains some nonzero t. Undoing conjugation and absorbing the

scalar into one original multilinear input gives A. In the traceless case every nonzero target is nonscalar, and zero is obtained by setting one original input to zero.

In the traceful case use the clock–shift homogeneous basis Z^aX^b, $(a,b) \in (\mathbb{Z}/n)^2$. Expand a nonzero-trace evaluation multilinearly in this basis. Some basis tuple has nonzero-trace output. Every word has the same total grade, and only the degree-zero component KI has nonzero trace. Thus p attains a nonzero scalar, and original-slot scaling gives every scalar. Smooth maps and flat locally finitely presented maps are open near the specified point. A nonempty finite-type fiber over a K-point has a K-point, so this openness concerns actual values, not only geometric image closures. □

The triangular restriction has the form

$$f(D_l + u_lE_{12} + v_lE_{23} + w_lE_{13}) = u^{\mathsf{T}}A(D)vE_{13}, \qquad A \neq 0,$$

and $A_{ii} = 0$ by original multilinearity. Choose original slots i, j, leave their inputs X, Y free, and make the other four inputs diagonal.

Lemma 7.2. *The resulting actual map $F_D(X,Y)$ ignores the diagonal parts of X, Y. For distinct matrix vertices its coordinates are*

$$(F_D)_{rt} = \sum_{s\neq r,t} \Big(a_{rst}X_{rs}Y_{st} + b_{rst}Y_{rs}X_{st}\Big), \tag{7.1}$$

$$(F_D)_{rr} = \sum_{s\neq r} q_{rs}\Big(\alpha X_{rs}Y_{sr} + \beta Y_{rs}X_{sr}\Big), \tag{7.2}$$

$$q_{rs} = \prod_{l\notin\{i,j\}} (d_{lr} - d_{ls}) = q_{sr}. \tag{7.3}$$

Here a_{rst}, b_{rst} are vertex relabelings of A_{ij}, A_{ji}. Their closed-path coefficients are respectively $\alpha q_{rs}, \beta q_{rs}$. Each nonzero symbol gives nonzero polynomials for all its distinct-vertex relabelings, and these may be made nonzero simultaneously at one actual diagonal background.

Proof. If one free input is diagonal and the other is a matrix unit, every surviving word visits at most two vertices. After relabeling this is a UT_2 evaluation, hence zero. Linearity proves the first assertion. Two off-diagonal factors can reach (r,t) only along a path $r \to s \to t$, in one of the two orders of the original free slots; closed paths give the diagonal formula. There is no third off-diagonal factor. Properness permits an independent scalar translation in each passive diagonal input. Its closed coefficient is linear in that input's two vertex entries and translation invariant, hence contains their difference. Multiplying the four differences leaves constants α, β. Their number is even, giving $q_{rs} = q_{sr}$. A distinct-vertex relabeling is an injective renaming of independent coordinates. The finite product of the nonzero weights is nonzero in one polynomial ring, so a common actual background exists. □

Lemma 7.3. *An $m \times m$ matrix with zero diagonal and off-diagonal entries $c_{rs}y_{rs}$, where every $c_{rs} \neq 0$ and $m \geq 2$, has nonzero determinant as a polynomial in the independent y_{rs}.*

Proof. A derangement exists. Its determinant term has nonzero coefficient, and different permutations have different directed-edge monomials. Thus that term cannot cancel. □

7.2. **Leading trace tensors and two low-degree detectors.**

Lemma 7.4. *Let $h = \operatorname{tr} f$ and $g = f - hI/n$. Suppose an actual tuple a has $f(a) = N$ nonzero rank-one square-zero. Suppose a source control space W maps onto $\mathfrak{sl}_n$ under dg_a. If, for some $r \geq 1$, all ambient derivatives $d^q h_a$ vanish for $q < r$ and the nonzero tensor $C = d^r h_a$ annihilates W in any argument, then f is flat at a. Consequently its image is M_n.*

Proof. The tangent space is $\ker dg_a + W$, so C remains nonzero on $(\ker dg_a)^r$. The fiber $g^{-1}(N)$ is smooth of dimension $6n^2 - (n^2 - 1) = 5n^2 + 1$. A formal parametrization is $a + u + O(u^2)$ with $u \in \ker dg_a$. Since every lower ambient trace jet vanishes, the leading corrected trace term is exactly $C(u, \dots, u)/r!$, which is nonzero in characteristic zero. Thus the trace germ is a nonzerodivisor in the regular local ring of this fiber. The full f-fiber has local dimension $5n^2$, the expected dimension. Miracle flatness applies to the smooth ambient source and target. Lemma 7.1 finishes the proof. □

Lemma 7.5. *Let $n \geq 4$. Let P be an ordinary multilinear polynomial of degree five, with $k \geq 2$ inputs arbitrary and the other $5 - k$ inputs diagonal. If all values of this restriction are diagonal, the restriction is zero.*

Proof. First let $k < n$. Expand the arbitrary inputs in matrix units. A diagonal word value requires a closed Euler trail on their k labeled directed edges and hence uses at most k vertices. Fix an output vertex r and choose an unused vertex t. Partition the surviving word contributions by their last arbitrary-input edge. If their sum is nonzero, one subtotal is nonzero. Change the terminal vertex of that last edge from r to t. Any surviving path from r to t now uses that edge last, since t has no outgoing arbitrary-input edge. On identifying every passive diagonal entry at t with its entry at r, the resulting off-diagonal coefficient is exactly the chosen subtotal. It is therefore nonzero, a contradiction. This works for every matrix-unit tuple and output coordinate.

Only two situations remain. For $k = 5$, all inputs are arbitrary; diagonal-valuedness and equivariance force scalar values. Thus $[z, P]$ is an identity of degree six on M_n. Since $6 < 2n$, it vanishes freely; the center of a free algebra is K, contradicting positive degree unless $P = 0$. For $n = 4, k = 4$, write the single diagonal input as D. Diagonal-valuedness gives $[D, P(D, Y_1, \dots, Y_4)] = 0$ for diagonal D. Conjugation and density extend this to all D, making it a degree-six identity of M_4 and hence a free identity. The free centralizer of D is $K[D]$, incompatible with positive degree in the Y inputs. □

Lemma 7.6. *Suppose that, for all X, Y and independent diagonal D_l,*

$$\operatorname{tr} f(X, Y; D_1, \dots, D_4) = 0.$$

Expand all four passive inputs as $D_l + tZ_l$, and let H_r be the first nonzero trace coefficient. For $n \geq 4$, after possibly interchanging the two original active slots, H_r is nonzero on the family

$$X = \sum_{t=2}^{n} x_t E_{1t}, \qquad Y = 0 \oplus W, \quad W \in \mathcal{H}_{n-1}.$$

All diagonal backgrounds and perturbations remain variable.

Proof. Choose a nonzero component indexed by r perturbed slots. There are $k = r + 2$ free labeled inputs and $6 - k$ diagonal inputs. Rotate each trace word to start at X. Group coefficients by the cyclic order σ of the free letters and by the allocation of diagonal letters to its k gaps. Diagonal letters within a gap commute; let P_σ be the resulting polynomial in gap-vertex diagonal coordinates. If every such polynomial were zero, every matrix-unit evaluation would be zero: a closed word follows one such cyclic order and identifies some of its gap vertices. Thus some $P_\sigma \neq 0$.

A simple directed k-cycle, with its labeled edges in order σ, has that unique cyclic Euler order and detects P_σ. Whenever $k \leq n$, use it. Of the distinct X, Y edges, at least one choice of row edge has a tail avoided by both endpoints of the other edge; otherwise a cycle of length at least three would have just those two edges. Relabel the row tail as vertex one.

For $n = 4, k = 5$, the single diagonal letter makes P_σ a nonzero linear form in five gap coordinates. Contract one of the three Z-labeled edges to a loop. A directed four-cycle with one loop still has a unique cyclic Euler order. This identifies the two gap coordinates at that edge. A linear form vanishes on this identification precisely when it is a multiple of their difference. The three permitted edges give three distinct difference lines, whose intersection is zero. At least one contraction preserves the coefficient. The X, Y edges remain distinct nonloop cycle edges and can be oriented as required.

For $n = 5, k = 6$, P_σ is a nonzero constant, and contracting any Z edge to a loop suffices. For $n = 4, k = 6$, choose two nonadjacent edges among the four Z edges and contract them. The two loops lie at distinct vertices of a directed four-cycle. Its cyclic Euler order is unique: each loop must be traversed at the visit to its vertex before leaving along the cycle. Starting at a loop vertex and traversing its loop last only rotates the same order. Thus the same constant survives. Again X, Y remain nonloop cycle edges. These cases exhaust $k \leq 6$. They prove polynomial nonvanishing on one of the asserted irreducible row families; no claim that both orientations work is needed. □

Lemma 7.7. *In the situation of Lemma 7.6, let an actual row seed have $H_r \neq 0$. Then every ambient trace derivative below order r vanishes there, and $d^r h$ annihilates both original active-slot tangent spaces and all simultaneous-conjugation tangents.*

Proof. Every passive-slot derivative of smaller order is a coefficient of an identically zero lower H_q. Varying an active slot differentiates such a zero coefficient and uses still fewer passive variations. Repeated variation of a single original slot is zero by multilinearity. The same argument at order r with one active-slot variation proves the asserted annihilation. The nonzero value of H_r makes $d^r h \neq 0$. For the conjugation field $v_C(x) = ([C, x_l])_l$, differentiate the global identity $dh_x(v_C(x)) = 0$ a total of $r-1$ times. Every term except $d^r h(v_C(a), \dots)$ contains a lower derivative of h and vanishes. This also covers $r = 1$ directly. □

7.3. **Directional symbols and zero closed paths.**

Theorem 7.8. *If $A_{ij} \neq 0$ and $A_{ji} = 0$, then every traceful f in the mixed layer has image M_n for every $n \geq 4$.*

Proof. In the exact chart $b = 0$. Choose all $a_{rst} \neq 0$. For a fixed Y, the hollow projection of the original X-slot acts separately in each row r by the $(n-1)$-square matrix

$$(B_r)_{st} = \begin{cases} a_{rst} Y_{st}, & s \neq t, \\ 0, & s = t, \end{cases} \qquad s, t \neq r.$$

Lemma 7.3 and a common nonempty open make all B_r invertible simultaneously. The original full slot then has rank $n^2 - n$, since it kills diagonal inputs. This rank alone is not used as an unproved higher-size surjectivity criterion.

If $\alpha \neq 0$, choose also every $q_{rs} \neq 0$. Let $\sigma(s) = s + 2$ cyclically and set $Y_{s,\sigma(s)} = 1$, all other entries zero. Each source X_{rs}, $s \neq r$, controls just output $(r, \sigma(s))$, with nonzero path coefficient. The only missing position in row r is $(r, \sigma(r))$. No diagonal or first-superdiagonal position of an upper Jordan matrix is missing for $n \geq 3$. Solve those coordinates individually and put all unused source entries zero. This gives every Jordan form exactly, and conjugation gives every target.

Now let $\alpha = 0$. The chart has zero diagonal output. Write $h = \operatorname{tr} f = \operatorname{tr}(XG)$ and expand the other four passive inputs as $D_l + tZ_l$, leaving Y arbitrary. The constant G_0 is zero by the trace pairing, and some $G_r \neq 0$, $1 \leq r \leq 4$, since the top coefficient is the nonzero ordinary quintic cyclic polynomial. By Lemma 7.5, after separating perturbed-slot components, G_r has a nonzero off-diagonal entry. Its nonvanishing and all $\det B_r \neq 0$ hold at a common point of the same irreducible (D, Y, Z) parameter space. Relabel so $(G_r)_{b1} \neq 0$. Choose X on the first hollow row with $\operatorname{tr}(XG_r) \neq 0$ and with every entry of $F_D(X, Y)$ on that row nonzero. Both are complements of finitely many proper linear hyperplanes because B_1 is invertible.

The value N is rank-one square-zero. The original X-slot supplies all hollow directions and the orbit directions $[E_{t1}, N]$ supply the $n-1$ independent traceless diagonal directions. The minimality of r kills every lower ambient trace derivative, including derivatives in Y, and the first tensor kills the X-slot and conjugation controls. The same differentiation argument as Lemma 7.7 applies. Lemma 7.4 proves the assertion. □

Lemma 7.9. *On an index set B of size $m \geq 2$, let all a_{rst}, b_{rst} for distinct indices and all c_{rt} for $r \neq t$ be nonzero scalars. For hollow W, H and vectors x, z set*

$$T_W(H)_{rt} = \sum_{s \neq r,t} (a_{rst} H_{rs} W_{st} + b_{rst} W_{rs} H_{st}), \qquad R_x(z)_{rt} = c_{rt} z_r x_t.$$

On a nonempty open set of (W, x) the map $(H, z) \mapsto T_W(H) + R_x(z)$ is onto $\mathcal{H}_m$.

Proof. For $m = 2$, $T = 0$ and R is onto when both x entries are nonzero. Inductively write $B = B_0 \cup \{v\}$ and specialize $W = W_0 \oplus 0$. Old controls cover old outputs. The new row is independently controlled by H_{vs} through $(a_{vst} W_{st})_{s,t \in B_0}$, and the new column by H_{sv} through $(b_{rsv} W_{rs})_{r,s \in B_0}$. Their determinants are nonzero by Lemma 7.3. These opens meet the old surjectivity open in one irreducible parameter space. First prescribe old outputs, then correct any new-row/column leakage using these cross controls; the latter alter no old output. This exhibits a nonzero maximal minor. Its nonvanishing is open for unrestricted W. □

Theorem 7.10. *Suppose $A_{ij} \neq 0$ and both closed-path coefficients α, β are zero. If f is traceful, its image is M_n for every $n \geq 4$.*

Proof. The directional case is Theorem 7.8. Otherwise choose one diagonal background with all a, b nonzero, and put

$$X = \sum_{t=2}^{n} x_t E_{1t}, \qquad Y = 0 \oplus W, \qquad N_{1t} = \sum_{s \neq 1,t} a_{1st} x_s W_{st}.$$

The value has just its first row. Variations of that row of X give its entire first-row output through a generically invertible weighted hollow matrix. Variations of the first column of X give the entire first-column output through $(b_{rs1} W_{rs})$. Both determinants are nonzero by derangements. Hollow lower-block variations H of X and variations $z_r = \delta Y_{r1}$ give lower hollow output

$$T_W(H)_{rt} + b_{r1t} z_r x_t,$$

which is onto by Lemma 7.9. These variations create no first row or column; possible closed-path diagonal terms are zero. All minors and all $N_{1t} \neq 0$ hold on one nonempty open. Thus the two original slots give the whole hollow target at a rank-one row seed; conjugation gives the remaining traceless diagonals.

The trace is identically zero on this two-input chart. Expand the four passive slots and choose its first nonzero trace coefficient. By Lemma 7.6 it is nonzero on one of these same row families. Swap the active slots if necessary; both directional symbols remain nonzero. Intersect this trace open with all rank and weight opens in (D, W, x, Z). At the resulting actual point, Lemma 7.7 and Lemma 7.4 apply. □

7.4. **Nonzero closed paths and a reduced-target induction.** At a row seed $X = \sum_{t\ge 2} x_t E_{1t}$, $Y = 0 \oplus W$, write $N = \sum_{t\ge 2} p_t E_{1t}$. Variations of the first row and first column of X have only the corresponding first-row/column output, with weighted hollow matrices as above. Let Q_n be the quotient target represented by all diagonal coordinates and the hollow lower $(n-1)$-square block; let Q_n^0 be its trace-zero subspace. Reduced source controls are hollow lower-block variations H of X, $z_r = \delta Y_{r1}$, and orbit directions $o_r = [E_{r1}, N]$. Their diagonal and lower-row contributions are

$$\begin{aligned} z_r:&\quad q_{1r}x_r(\alpha E_{11} + \beta E_{rr}), \qquad (z_r)_{rt} = b_{r1t}x_t \quad (t \ne r,\ t \ge 2),\\ o_r:&\quad p_r(E_{rr} - E_{11}), \qquad (o_r)_{rt} = p_t \quad (t \ne r,\ t \ge 2). \end{aligned}$$

Lemma 7.11. *Suppose both directional symbols are nonzero. If $\beta \ne 0$ and a nonempty open row-seed family has surjective reduced controls onto Q_d for some $d \ge 3$, then such a family exists in every size $n \ge d$. If $\alpha = -\beta \ne 0$, the same assertion holds with Q_n^0. In either case one may impose, at the same actual point, invertibility of both first-row/column slot matrices and $p_t \ne 0$ for every t.*

Proof. Adjoin one lower vertex v and specialize $W = W_0 \oplus 0$, with $x_v \ne 0$. Work only in the reduced target at this specialization. Old controls cover all old reduced entries, possibly with leakage into the new lower row or column. Those are independently controlled by H_{vs}, H_{sv} through the two weighted hollow matrices $(a_{vst}W_{st})$ and $(b_{rsv}W_{rs})$ on the old lower vertex set. Its size is $d-1 \ge 2$, so both determinant polynomials are nonzero. Their opens intersect the old reduced-surjectivity open. These cross controls change neither old reduced entries nor any diagonal entry, because the new row and column of W vanish.

The control z_v has nonzero new diagonal coefficient $\beta q_{1v}x_v$. Its additional 11 entry is absorbed by old controls in the full-target case. In the trace-zero case its diagonal contribution is traceless; after prescribing the new diagonal, the old remaining diagonal target has trace zero and is covered by old controls. Cross controls remove its other contributions. Thus the enlarged reduced target is covered.

This gives a nonzero reduced maximal minor for unrestricted W. This does not imply invertibility of the full first-row/column matrices at $W_0 \oplus 0$. Each of those determinant polynomials is separately nonzero by Lemma 7.3. Their opens, the reduced open, and the open $p_t \ne 0$ intersect in the same irreducible parameter space. Thus the required conditions hold simultaneously in the next size.

For general diagonal-dependent symbols, perform the induction over the rational-function field of all passive diagonal entries. An old nonzero minor remains nonzero after adjoining the new vertex coordinates, and all new relabeled weights are nonzero in that field. The final finite open conditions specialize to a K-point. No possibly exceptional fixed old background is presumed to extend automatically. For a monomial common-D chart with nonzero weights at $D_n = \mathrm{diag}(0, 1, \dots, n-1)$, the same induction may instead keep that D_n fixed. □

Theorem 7.12. *If $\alpha \neq \beta$ for some original pair, every traceful f has image M_n for all $n \geq 4$. If $\alpha = -\beta \neq 0$, the two-input chart has a smooth rank-one value as a map to $\mathfrak{sl}_n$ already for $n \geq 3$.*

Proof. A zero directional symbol is covered by Theorem 7.8. Otherwise all distinct-vertex weights can be chosen nonzero. For the base size three write

$$x_2 = x, \quad x_3 = y, \quad W_{23} = u, \quad W_{32} = v, \quad p = a_{132}yv, \quad q = a_{123}xu.$$

In coordinates $(11, 22, 33, 23, 32)$ the two lower-X directions are nonzero multiples of

$$(0, \alpha, \beta, 0, 0), \qquad (0, \beta, \alpha, 0, 0).$$

The z_2, z_3 and orbit controls are

$$\begin{aligned} z_2 &= (\alpha q_{12}x, \beta q_{12}x, 0, b_{213}y, 0), \\ z_3 &= (\alpha q_{13}y, 0, \beta q_{13}y, 0, b_{312}x), \\ o_2 &= (-p, p, 0, q, 0), \qquad o_3 = (-q, 0, q, 0, p). \end{aligned}$$

If $\alpha^2 \neq \beta^2$, the first two controls span E_{22}, E_{33}. Modulo them, z_2, o_2 span coordinates $(11, 23)$ on a nonempty open: their determinant is

$$\alpha q_{12}a_{123}x^2u + a_{132}b_{213}y^2v \neq 0$$

as a polynomial. Its monomials are distinct, and the second coefficient is nonzero. Then z_3 supplies coordinate 32. Hence Q_3 is covered. After swapping the slots if necessary, $\beta \neq 0$, and Lemma 7.11 gives a smooth full-target rank-one chart in every size. Lemma 7.1 applies.

If $\alpha = -\beta \neq 0$, the lower controls give $K(E_{22} - E_{33})$. Modulo that line in Q_3^0, the same determinant and z_3 give the remaining three directions. The induction gives smooth $\mathfrak{sl}_n$ controls for every $n \geq 3$. When the original polynomial f is traceful, the chart trace still vanishes identically. For $n \geq 4$, Lemma 7.6 detects the first nonzero passive trace jet on this same row family. Intersect its open with all rank conditions and apply Lemmas 7.7 and 7.4. No uncorrected higher derivative is used. □

7.5. **Equal nonzero closed coefficients: rigidity and a common base.**

Theorem 7.13. *If $\alpha = \beta \neq 0$ for an original pair, then the actual image of f is M_n for every $n \geq 4$.*

Proof. Both symbols a, b are nonzero because their closed specializations are nonzero. At a size-three row seed normalize one lower-X control to $(0, 1, 1, 0, 0)$. Put

$$U = \alpha q_{12}x, \quad V = \alpha q_{13}y, \quad C = b_{213}y, \quad E = b_{312}x.$$

The five columns

$$(0, 1, 1, 0, 0), \ (U, U, 0, C, 0), \ (V, 0, V, 0, E), \ (-p, p, 0, q, 0), \ (-q, 0, q, 0, p)$$

have determinant

$$2pq(CE - UV) = 2pqxy(b_{213}b_{312} - \alpha^2 q_{12}q_{13}). \tag{7.4}$$

If its last parenthesis is a nonzero polynomial, intersect its open with the nonzero-weight and row-seed conditions. This gives a smooth M_3 row chart. Lemma 7.11, now with $\beta = \alpha \neq 0$, gives every larger size. Interchanging the two original slots gives the identical test for a.

It remains that both tests vanish identically. For the path $r \to s \to t$ put, for each of the four passive slots,

$$P_l = d_{lr} - d_{ls}, \qquad Q_l = d_{ls} - d_{lt}.$$

Independent scalar translations show $a, b \in K[P_1, Q_1, \dots, P_4, Q_4]$, separately homogeneous of degree one in each pair. These eight gap variables are independent: adjoining the middle entries gives an invertible linear change from the twelve diagonal coordinates. Reversing endpoints sends (P, Q) to $(-Q, -P)$ and hence, at total passive degree four, to (Q, P). Equation (7.4) therefore says

$$a(P,Q)a(Q,P) = \alpha^2 \prod_l P_l Q_l, \qquad b(P,Q)b(Q,P) = \alpha^2 \prod_l P_l Q_l. \tag{7.5}$$

This is an identity in eight independent variables, not on a special one-parameter set.

The polynomial ring is a UFD. Every irreducible factor of a or b must be one of the eight coordinate variables. Separate degree one forces exactly one choice per pair. Closed specialization $Q = -P$ fixes the scalar. Thus, for subsets $S, T \subseteq \{1, 2, 3, 4\}$,

$$a(P,Q) = (-1)^{|S|}\alpha \prod_{l \in S} P_l \prod_{l \notin S} Q_l, \tag{7.6}$$

$$b(P,Q) = (-1)^{|T|}\alpha \prod_{l \in T} P_l \prod_{l \notin T} Q_l. \tag{7.7}$$

Neither subset is empty or full. To prove this, make all four passive diagonals have equal entries at r, s. They commute with $E_{rs} + \varepsilon E_{rr}$, which is semisimple for $\varepsilon \neq 0$. These five commuting semisimple inputs are simultaneously diagonalizable. With the sixth input arbitrary, the UT_2 identity makes the evaluation zero. Polynomial dependence allows $\varepsilon = 0$. Taking the other active edge to be E_{st} proves $a(P, Q) = 0$ when all $P_l = 0$. The analogous argument with the other edge proves vanishing when all $Q_l = 0$. The same holds for b. Consequently $s = |S|, t = |T|$ belong to $\{1, 2, 3\}$.

Now set all four passive diagonal matrices equal to D. The entire chart, including its diagonal entries, becomes exactly

$$F_{s,t}(D, X, Y) = \alpha\Big((-1)^s \operatorname{ad}_D^s(X) \operatorname{ad}_D^{4-s}(Y) + (-1)^t \operatorname{ad}_D^t(Y) \operatorname{ad}_D^{4-t}(X)\Big). \tag{7.8}$$

Indeed all distinct-vertex coefficients agree by (7.6)–(7.7), all closed coefficients agree by specialization, and diagonal source parts vanish. Only two off-diagonal factors are possible in this original chart, so these formulas include every term of f.

Put $\mathcal{A} = \operatorname{ad}_D$ on the hollow matrices, where $D = \operatorname{diag}(0, 1, 2, 3)$. This operator is invertible on that space. Make the invertible changes in the two original

slots

$$U = (-1)^s \mathcal{A}^s X, \qquad V = \mathcal{A}^{4-s} Y, \qquad h = s + t - 4.$$

Their inverses are $X = (-1)^s \mathcal{A}^{-s} U$ and $Y = \mathcal{A}^{s-4} V$. After division by α, equation (7.8) becomes

$$\Phi_h(U, V) = UV + (-1)^h \mathcal{A}^h(V) \mathcal{A}^{-h}(U).$$

We construct the required chart for every integer h. For this calculation number the four vertices $0, 1, 2, 3$, with $d_i = i$. Indices in $\{1, 2, 3\}$ are read cyclically. Take

$$U = E_{01} + E_{02} + E_{03}, \qquad V = E_{12} + E_{23} + E_{31}.$$

Then $\Phi_h(U, V) = U =: N$, a nonzero rank-one square-zero matrix. All hollow U, V variations correspond to variations of the two original slots. Orbit directions arise from simultaneous conjugation of all six original inputs.

The first-row U variations give the whole first row by multiplication by the cyclic permutation matrix V. The first-column U variations give the whole first column by a weighted cyclic permutation whose weights are

$$(-1)^h \left(\frac{d_i - d_{i+1}}{d_{i+1}} \right)^h \neq 0.$$

It remains to control the ten-dimensional quotient consisting of the four diagonal entries and the hollow lower block.

The reverse-edge variations $\dot{U} = E_{i+1,i}$ give $E_{ii} + E_{i+1,i+1}$. Their lower-diagonal coefficient matrix is the unsigned incidence matrix of a triangle, with determinant 2. Hence all three lower diagonal entries can be prescribed independently. Work modulo these directions, and put

$$e_i = d_i - d_{i+1}, \qquad B_i = (-1)^h (e_i / e_{i+1})^h.$$

The forward-edge variations $\dot{U} = \sum_i \xi_i E_{i,i+1}$ have only reverse-edge lower output, with coefficients

$$(L_h \xi)_i = \xi_i + B_i \xi_{i+1}, \qquad \det L_h = 1 + \prod_i B_i = 1 + (-1)^h.$$

Write z_i for the output of $\dot{V} = E_{i0}$ and $o_i = [E_{i0}, N]$. Direct multiplication gives

$$z_i = E_{00} + E_{ii} + \sum_{j \neq i} (d_i/d_j)^h E_{ij}, \qquad o_i = -E_{00} + E_{ii} + \sum_{j \neq i} E_{ij},$$

where both sums range over the lower vertices. The o_i independently prescribe the three forward-edge lower entries. The combinations

$$K_i = z_i - (d_i/d_{i+1})^h o_i$$

leave those entries unchanged. Modulo the lower diagonal entries, K_i has root-diagonal coefficient $R_i = 1 + (d_i/d_{i+1})^h$ and one reverse-edge coefficient

$$C_i = (d_i/d_{i-1})^h - (d_i/d_{i+1})^h.$$

If h is even, L_h is invertible and $R_1 = 1 + 2^{-h} \neq 0$. Thus K_1 prescribes the root diagonal, and L_h removes its reverse-edge output.

If h is odd, L_h has rank two: its first two rows and columns have determinant one, and its left kernel is spanned by $(e_1^{-h}, e_2^{-h}, e_3^{-h})$. After using its image, the remaining coordinates are the root diagonal and the reverse-edge functional $\sum_i e_i^{-h} r_i$. On these coordinates K_1, K_2 have matrix

$$\begin{pmatrix} R_1 & R_2 \\ \ell_1 & \ell_2 \end{pmatrix}, \qquad \begin{aligned} R_1 &= 1 + 2^{-h}, \quad R_2 = 1 + (2/3)^h, \\ \ell_1 &= 2^{-h} - 3^{-h}, \quad \ell_2 = (2/3)^h - 2^h. \end{aligned}$$

Its determinant $R_1\ell_2 - R_2\ell_1$ is nonzero. Indeed R_1, R_2 are positive rational numbers. For $h > 0$ we have $\ell_1 > 0 > \ell_2$, whereas for $h < 0$ the signs reverse. This proves nonvanishing in $\mathbb{Q}$, hence in every characteristic-zero field.

The reduced controls therefore have rank ten for every integer h. Together with the six first-row and first-column controls, they give rank sixteen for the full original evaluation map at the same rank-one square-zero value. These are variations of two original slots together with simultaneous conjugation of the original inputs.

Apply Lemma 7.11 starting at size four. For (7.8) keep $D_n = \mathrm{diag}(0, 1, \dots, n-1)$; all its distinct-vertex and closed weights are nonzero products of powers of distinct differences. This supplies actual smooth rank-one charts for every $n \geq 4$. Lemma 7.1 proves the result. Tracefulness is automatic here from the nonzero symmetric closed pair. □

7.6. **Images in the mixed layer.**

Theorem 7.14. *Let f be ordinary multilinear proper of degree six, an identity of UT_2 but not of UT_3, with nonzero cyclic projection. Then $f(M_n^6) = M_n$ for every $n \geq 4$.*

Proof. The nonzero cyclic derivative has degree five, below $2n$, so f is traceful in each stated size. If every closed coefficient is zero, choose a nonzero A_{ij} and use Theorem 7.10, including its directional subcase. Otherwise choose $M_{ij} \neq 0$. If $M_{ij} \neq M_{ji}$ use Theorem 7.12; if they are equal use Theorem 7.13. The closed coefficients have the common four-difference form, so these are exactly the alternatives $\alpha \neq \beta$ and $\alpha = \beta \neq 0$. This proves the assertion in all cases. □

Theorem 7.15. *Under the same proper and triangular hypotheses, if the cyclic projection is zero, then $f(M_n^6) = \mathfrak{sl}_n$ for every $n \geq 3$.*

Proof. The whole polynomial is trace zero. Equation (7.2) with $X = E_{rs}, Y = E_{sr}$ gives $\beta = -\alpha$ for every pair. If $A_{ij} \neq 0, A_{ji} = 0$, then $\alpha = \beta = 0$. Fix X generically. For each target column the map in the hollow entries of Y has the weighted hollow coefficient matrix $(a_{rst}X_{rs})_{r,s\neq t}$. By derangements all these matrices are invertible simultaneously at one actual background. Thus every hollow target is one chart value, and every traceless target is its conjugate.

If both directions of a selected nonzero pair survive and its closed coefficients vanish, the hollow-control proof of Theorem 7.10 applies already in size three: Lemma 7.9 has base $m = n - 1 = 2$. The two original slots give all hollow directions at a rank-one row seed; conjugation gives all traceless diagonals. No

trace detector is needed because the whole f is traceless. If any closed coefficient is nonzero, the skew branch of Theorem 7.12 gives the same smooth $\mathfrak{sl}_n$ conclusion, with its size-three base and all-size induction. These cases exhaust the nonzero symbol. Lemma 7.1, applied relatively in $\mathfrak{sl}_n$, proves equality. □

7.7. **Completion of the proper classification.**

Theorem 7.16. *Let $f \in \Gamma_6$ and $n \geq 3$. If f is not an identity of M_n, then*

$$\mathrm{Val}_n(f) = \begin{cases} \mathfrak{sl}_n, & \pi f = 0, \\ M_n, & \pi f \neq 0. \end{cases}$$

The identity case is $f = 0$ when $n \geq 4$, and $f \in Ks_6$ when $n = 3$.

Proof. The identity statements and the trace test are Proposition 1.5. If $c(f) \neq 0$, apply Theorem 2.1. If $c(f) = 0$ and $f \in W_6$, apply Theorem 5.2, including its explicit M_3 identity exception. Otherwise Proposition 1.10 puts f in the nonzero mixed layer. In the trace-zero case, Theorem 7.15 applies for every $n \geq 3$. In the traceful case use Theorem 6.1 for $n = 3$ and Theorem 7.14 for $n \geq 4$. These are all cases of the proper filtration. None of these results uses a classification of nonproper sextics or of lower-degree images. □

8. Nonproper sextics: unit restrictions and trace lifting

Throughout this section $n \geq 3$. We use the notation and properness convention of Section 1. We now apply Theorem 7.16, whose proof is independent of this section and of lower-degree image classifications.

We use the standard minimum-degree facts that a nonzero polynomial identity of M_n has degree at least $2n$, and a nonzero polynomial identity of UT_3 has degree at least six. We also use that the centralizer of a free generator D in a free associative algebra is $K[D]$. The geometric steps below use miracle flatness and openness of flat morphisms locally of finite presentation. The only special M_3 image input is the preceding original-slot theorem 6.5.

8.1. **Image-preserving Lie substitutions.**

Lemma 8.1. *Let q be a nonzero proper multilinear polynomial of degree d, where $2 \leq d \leq 5$. There is a proper ordinary multilinear sextic Q such that*

$$\mathrm{Val}_n(Q) = \mathrm{Val}_n(q) \qquad (n \geq 2).$$

Consequently, by Theorem 7.16, for $n \geq 3$ this common image is M_n if q is traceful and is $\mathfrak{sl}_n$ if q has identically zero trace.

Proof. For $r \geq 2$ use the explicit right-normed polynomial

$$L_r(z_1, \ldots, z_r) = [z_1, [z_2, \ldots, [z_{r-1}, z_r] \ldots]].$$

Its image is exactly $\mathfrak{sl}_n$. Indeed, every value is traceless. Conversely, Lemma 1.6 and $L_r(D, \ldots, D, Z) = \mathrm{ad}_D^{r-1} Z$ realize every hollow target and then every traceless target by simultaneous conjugation.

Set $r = 7 - d$ and replace, for example, the first slot of q by L_r on fresh disjoint variables:

$$Q(z_1, \dots, z_r, x_2, \dots, x_d) = q(L_r(z_1, \dots, z_r), x_2, \dots, x_d).$$

It is ordinary multilinear of degree $r+d-1 = 6$ and proper: a unit in a new leaf kills the Lie word, and a unit in an untouched slot kills q. One image inclusion is immediate from substitution. Conversely, for every actual input tuple,

$$q(A_1, A_2, \dots, A_d) = q\left(A_1 - \frac{\operatorname{tr} A_1}{n} I, A_2, \dots, A_d\right).$$

Realize the displayed traceless first input as one L_r-value. This realizes the original q-value as one Q-value, proving equality of actual sets.

For $n \geq 3$, the free nonzero q is an actual nonidentity since $d \leq 5 < 2n$. Image equality makes Q an actual nonidentity too. In particular, when $n = 3$, Q cannot fall into the degree-six identity line. Tracefulness or trace-zero image is preserved. Applying the already proved proper-sextic theorem to this nonidentity Q gives the asserted alternatives. No classification through degree five is being assumed here. □

8.2. **Local trace lifting.**

Lemma 8.2. *Let $f : (M_n)^6 \to M_n$ be ordinary multilinear and traceful, put $h = \operatorname{tr} f$ and $g = f - (h/n)I$. Suppose that at an actual input a the value $N = f(a)$ is nonzero rank-one square-zero. Suppose also that there is a linear control space W in the source tangent space such that $dg_a(W) = \mathfrak{sl}_n$, and an integer $r \geq 1$ satisfying*

$$d^s h_a = 0 \ (s < r), \qquad C = d^r h_a \neq 0, \qquad C(W, -, \dots, -) = 0.$$

Then f is flat at a, and $\operatorname{Val}_n(f) = M_n$.

Proof. The map g is submersive at a and its fiber through a is smooth of local dimension $6n^2-(n^2-1) = 5n^2+1$. Since the source tangent space is $\ker dg_a + W$, the nonzero symmetric tensor C restricts nontrivially to $(\ker dg_a)^r$. In formal local coordinates on the smooth fiber, its parametrization is $a+u+O(u^2)$, with $u \in \ker dg_a$. All lower ambient trace derivatives vanish. Thus the first term of the corrected trace is $C(u, \dots, u)/r!$, which is nonzero in characteristic zero. The trace therefore cuts the regular local fiber by one nonzerodivisor. The local f-fiber has dimension $5n^2$, the expected dimension. Miracle flatness makes f flat at a.

Restrict to a flat source neighborhood. Flat openness supplies a Zariski-open neighborhood of N in the actual image. More precisely, a K-point in this target open has a nonempty fiber of finite type over the algebraically closed field K, hence an actual K-point. Conjugate the neighborhood to contain E_{12}.

If A is nonscalar, it has a conjugate B with $B_{12} = 1$: choose a vector v for which v, Av are independent and take these as the second and first basis vectors. For $e = (-1, 1, 0, \dots, 0)$, the curve

$$t^2 \operatorname{diag}(t^{e_i}) B \operatorname{diag}(t^{-e_i})$$

extends polynomially to $t = 0$ with value E_{12}; every other entry has a strictly positive power of t. Its inverse image of the target open contains some nonzero t. Ordinary multilinearity makes the value set a cone, and simultaneous conjugation preserves it, so undoing these operations gives A itself as an actual value.

For completeness, scalar values are obtained independently. Choose a primitive nth root of unity and the clock–shift homogeneous basis U^aV^b, $(a, b) \in (\mathbb{Z}/n)^2$, of M_n. Each homogeneous component is one-dimensional; only its degree-zero component KI has nonzero trace. Expand a nonzero-trace evaluation of f in this basis. Multilinearity gives a homogeneous-basis tuple whose output has nonzero trace. Its output belongs to the degree-zero component and is a nonzero scalar. Scaling an original slot gives every scalar, including zero. Hence the whole actual image is M_n. □

The tensor hypotheses in Lemma 8.2 will arise as follows. Select one or two original active slots, and keep the other original slots at diagonal or unit base values. Expand *all* passive original slots, including every unit slot, by arbitrary matrix perturbations. If r is the first nonzero trace coefficient on this parameter family, every ambient trace derivative of order below r vanishes at its base tuples. Indeed, a derivative in an active slot differentiates an identically zero lower coefficient, and multilinearity separates the subsets of passive slots. The order-r tensor annihilates every active-slot direction for the same reason. It also annihilates every simultaneous-conjugation tangent: differentiate the identity $dh_x(v_H(x)) = 0$ a total of $r - 1$ times. All terms involving lower trace derivatives vanish, leaving only $C(v_H(a), -, \dots, -) = 0$. These are statements about the full original source, not a source with its unit slots removed.

8.3. **Two elementary trace detectors.**

Lemma 8.3. *Let $n \geq 4$. Restrict an ordinary multilinear polynomial of total degree at most five by leaving at least two inputs arbitrary and making all others diagonal or equal to I. If every restricted value is diagonal, then the restricted polynomial is zero.*

Proof. Delete the fixed unit letters. Let k be the number of arbitrary inputs. First suppose $k < n$. Expand these inputs into labeled matrix units. A nonzero diagonal output requires a closed directed Euler walk with k labeled edges, hence uses at most k vertices. Choose an unused vertex t. For a nonzero output at (s, s), partition its word contributions by the labeled arbitrary edge used last. Some subtotal is nonzero. Redirect the end of that last edge from s to t. In any surviving path from s to t this modified edge must be last, because t has no outgoing arbitrary edge. After setting each fixed diagonal entry at t equal to its entry at s, the resulting off-diagonal coefficient is exactly that nonzero subtotal. This contradicts the assumed diagonal-valuedness.

It remains that $k \geq n$. If all retained inputs are arbitrary, equivariance forces every value to be scalar. The commutator with a fresh letter is then a matrix identity of degree at most six, less than $2n$. It vanishes freely, forcing the original

polynomial into the center K of the free algebra, impossible in positive degree unless it is zero. The only other case is $n = 4$, $k = 4$, with one diagonal letter D. Diagonal output gives $[D, P(D, Y_1, \dots, Y_4)] = 0$ for diagonal D. Equivariance extends this to diagonalizable D, then density to all D. It is a degree-at-most-six identity of M_4, hence a free identity. The free-generator centralizer is $K[D]$, incompatible with positive multilinear degree in the Y_i. Thus this restriction is zero too. □

Lemma 8.4. *Consider a linear combination H of* single traces *of ordinary multilinear words, with k distinct arbitrary letters, including selected X, Y, and j distinct diagonal letters, where $3 \le k \le 6$ and $k + j \le 6$. Fixed unit letters have been deleted. Each retained letter occurs exactly once in every word; products of traces are not included. If H is not identically zero on the indicated arbitrary/diagonal input space, then, after possibly interchanging X, Y, its restriction to*

$$X = \sum_{t \neq 1} x_t E_{1t}, \qquad Y = 0 \oplus W, \qquad W \in \mathcal{H}_{n-1},$$

is nonzero for every $n \ge 3$.

Proof. Rotate each trace word to start with X and group by its cyclic order σ of the k arbitrary letters. Allocating the diagonal letters among the k gap vertices gives a polynomial P_σ, separately linear in the coordinate set of each diagonal letter. Some P_σ is nonzero: otherwise every matrix-unit evaluation, which identifies some gap vertices of one such cyclic walk, would vanish.

Number the free edges in order by $0, \dots, k-1$, with X edge zero and Y edge y. For a partition $a = (a_0, \dots, a_{k-1})$ with $a_i \in \{0, 1, 2\}$, give edge i the matrix unit $E_{a_i, a_{i+1}}$, cyclically. We use partitions for which X, Y are nonloops, one of their tails is avoided by both endpoints of the other, and the only directed labeled Euler circuit beginning at edge zero is $(0, 1, \dots, k-1)$. The tail-avoidance condition gives precisely one of the required row/lower-block orientations. Euler uniqueness isolates P_σ, with its gap coordinates identified according to a.

For $k = 3$, use 012. The following lists suffice for the other values:

k	y	vertex partitions
4	1	0120, 0121, 0122
4	2	0112, 0120, 0121
4	3	0102, 0112, 0122
5	1	01210, 01220, 01221
5	2	01120, 01122, 01210
5	3	01120, 01220, 01221
5	4	01022, 01102, 01122
6	1	012210
6	2	011220
6	3	012210
6	4	011220
6	5	011022

Each graph is a directed triangle with loops, or two directed two-cycles sharing a vertex, with the displayed loops. Its circuit is unique: a loop must be traversed before the last departure from its vertex; at a shared vertex, taking the edge which closes the current outer cycle before its other excursion would leave unused edges unreachable. This also verifies the listed edge order directly. Inspection of the two designated edges gives the required tail-avoidance orientation in each case.

We show that these restrictions detect every coefficient. For $k = 3$, no gap vertices are identified. For $k = 4$, the three single-pair identifications are respectively the edges of the trees

$$\{03, 13, 23\}, \qquad \{12, 03, 13\}, \qquad \{02, 12, 23\}$$

on the four gap vertices. In each diagonal-variable set, the three tree edge differences are linearly independent. The kernel of restriction along an edge e is the ideal generated by that edge difference in each of the j diagonal-variable sets. After an invertible linear change to tree coordinates, the three edge ideals use disjoint sets of variables. Their intersection is their product, which has no nonzero term of total degree below three. But P_σ has total degree $j \leq 2$. It cannot vanish under all three restrictions.

For $k = 5$, the only nonconstant possibility is $P_\sigma = \sum_{i=0}^{4} c_i d_i$. The following five available class sums, for the four successive positions of Y, have determinants $1, 1, -1, -1$ in the coefficients $c_0, \ldots, c_4$:

y	class sums				
1	$c_0 + c_4$	$c_1 + c_3$	c_2	c_1	c_0
2	$c_0 + c_4$	$c_1 + c_2$	c_3	c_0	$c_1 + c_3$
3	$c_0 + c_4$	$c_1 + c_2$	c_3	c_1	c_0
4	$c_0 + c_2$	c_1	$c_3 + c_4$	$c_0 + c_3$	$c_1 + c_2$

Thus simultaneous vanishing forces every c_i to be zero. For $k = 6$, P_σ is constant and any listed graph preserves it. Constants in the smaller cases are preserved as well.

This argument uses at most three vertices. In size $n > 3$, use the same matrix units on three vertices, with zero arbitrary inputs outside that corner. Every word contains arbitrary edges, so other vertices contribute nothing. Thus the very same trace-coefficient detection works in every $n \geq 3$. This is a direct evaluation of the trace expression in size n, not an inference from an M_3 image theorem. □

Restriction matrices. The restrictions in the preceding proof can also be written as matrices. Index the coefficients of P_σ by gap assignments $(t_1, \ldots, t_j) \in \{0, \ldots, k-1\}^j$. For every listed partition a and output monomial $v \in \{0, 1, 2\}^j$, the restriction matrix entry is

$$R_{(a,v),(t_1,\ldots,t_j)} = \prod_{\ell=1}^{j} \mathbf{1}_{a_{t_\ell} = v_\ell}.$$

It has column rank k^j. At maximal j, actual square-minor determinants are $1, 1$ for $(k, j) = (3, 3)$, $1, -1, 1$ for $(4, 2)$, $1, 1, -1, -1$ for $(5, 1)$, and five copies of 1 for $(6, 0)$. The thirty cases, including smaller j, comprise eight, nine, eight and five cases for $k = 3, 4, 5, 6$. These integer minors express the independence of the class sums and the unique labeled Euler orders used in the proof. The passage to arbitrary n is the explicit three-vertex construction in the proof, and the passage from the detector to a full image uses the separate controls and trace lift below.

8.4. **The two-free-slot trace-zero charts.** Let q be a nonzero proper trace-zero unit restriction of f, of degree $d \leq 5$. If all one-edge coefficients of q vanish, then $q \in \mathrm{Id}(\mathrm{UT}_2)$. It is not an identity of UT_3, by the degree bound, so it has a nonzero two-step symbol. Choose two original surviving slots X, Y with directional coefficient $C \neq 0$, and denote the opposite coefficient by E. Freeze all other surviving inputs at independent diagonal matrices and retain the deleted original inputs at I. The resulting actual chart ignores the diagonal parts of X, Y. Indeed a term involving only one arbitrary off-diagonal edge is an upper-triangular two-vertex evaluation and vanishes by the UT_2 identity. For distinct vertices, its exact off-diagonal formula is

$$F(X, Y)_{rt} = \sum_{s \neq r,t} \Big(a_{rst} X_{rs} Y_{st} + b_{rst} Y_{rs} X_{st} \Big). \tag{8.1}$$

Every a_{rst} is a relabeling of the same nonzero polynomial C in the remaining diagonal entries; the b_{rst} are relabelings of E. Because q is trace zero, the closed contribution of $X_{rs} Y_{sr}$ is

$$k_{rs}(E_{rr} - E_{ss}). \tag{8.2}$$

This follows by taking the trace of that single matrix-unit evaluation. It holds in every degree: no even-degree sign convention is imposed. Either all k_{rs} vanish polynomially, or all are nonzero relabelings. A finite product in the common diagonal-background polynomial ring shows that all nonzero path coefficients can be made nonzero simultaneously at one actual background. If $E = 0$, trace zero forces all $k_{rs} = 0$.

Lemma 8.5. *Let B have size $m \geq 2$, with all displayed coefficients nonzero. For hollow W, H and vectors x, z on B, put*

$$T_W(H)_{rt} = \sum_{s \neq r,t} (a_{rst} H_{rs} W_{st} + b_{rst} W_{rs} H_{st}), \qquad R_x(z)_{rt} = c_{rt} z_r x_t \quad (r \neq t).$$

Generically in (W, x), $(H, z) \mapsto T_W(H) + R_x(z)$ is onto $\mathcal{H}_m$.

Proof. For $m = 2$, R_x alone is onto if both entries of x are nonzero. For induction write $B = B_0 \cup \{v\}$ and specialize $W = W_0 \oplus 0$. The old outputs are the smaller map. The new row is controlled by H_{vs} through the weighted hollow matrix with entries $a_{vst} W_{st}$, and the new column by H_{sv} through the matrix with entries $b_{rsv} W_{rs}$. Each determinant is nonzero: a derangement exists on B_0, and different permutations give different monomials in the independent entries of W_0. Their opens meet the old surjectivity open in the same irreducible space.

Prescribe old outputs first using old H, z, then correct the new row and column independently by these cross H entries. Such corrections do not change old outputs. This gives one surjective specialization; a nonzero maximal minor gives the asserted open for unrestricted W. □

Lemma 8.6. *Suppose both C and E are nonzero in (8.1), with the trace-zero closed terms (8.2). There is a nonempty open of actual row seeds*

$$X = \sum_{t\in B} x_t E_{1t}, \qquad Y = 0 \oplus W, \qquad B = \{2, \dots, n\}, \quad W \in \mathcal{H}_{n-1},$$

at which $N = F(X, Y)$ has every first-row off-diagonal entry nonzero, and the two original slot spaces together with simultaneous conjugation map onto $\mathfrak{sl}_n$. The assertion holds in either orientation of the two selected slots.

Proof. The value is a single hollow row:

$$N_{1t} = \sum_{s\in B\setminus\{t\}} a_{1st} x_s W_{st}.$$

The matrix taking x to this row is generically invertible by the derangement argument. The first-row and first-column variations of X likewise give all first-row and first-column outputs on a common nonempty open; the latter matrix has entries $b_{rs1}W_{rs}$.

If every $k_{rs} = 0$, lower-block variations H of X and first-column variations $z_r = \delta Y_{r1}$ give the hollow lower block

$$T_W(H)_{rt} + b_{r1t} z_r x_t.$$

Lemma 8.5 makes this onto. These controls do not create first-row or first-column output. Thus the two slots give all hollow directions. The conjugation directions $[E_{t1}, N]$ have diagonal parts $N_{1t}(E_{tt} - E_{11})$, supplying the missing traceless diagonals.

Now suppose the k_{rs} are nonzero. Work modulo the already controlled first row and column. The target consists of trace-zero diagonals and hollow lower-block entries. The controls are lower hollow H, first-column z, and the conjugation directions E_{r1}. Start with $n = 3$, writing

$$X = xE_{12} + yE_{13}, \qquad W = uE_{23} + vE_{32},$$
$$N = pE_{12} + q_0 E_{13}, \quad p = a_{132}yv, \quad q_0 = a_{123}xu.$$

The lower H controls give the line $K(E_{22} - E_{33})$. Modulo this line, z_2 and the E_{21} conjugation direction have coordinates on $(11, 23)$ respectively

$$(k_{12}x, b_{213}y), \qquad (-p, q_0).$$

Their determinant is

$$k_{12}a_{123}x^2u + a_{132}b_{213}y^2v \neq 0$$

as a polynomial, since the monomials are distinct and both coefficients are nonzero. The z_3 direction then supplies coordinate 32 through $b_{312}x \neq 0$, its other coordinates being removable by the preceding controls. This proves the reduced surjectivity in size three.

For induction add a lower vertex v and specialize $W = W_0 \oplus 0$. The old controls supply the old reduced target. Cross H entries give the new hollow row and column through the two invertible weighted hollow matrices from the preceding induction; they change neither old outputs nor diagonals. The control z_v contributes $k_{1v}x_v(E_{11}-E_{vv})$, nonzero modulo the old diagonal space. Absorb its remaining old trace-zero diagonal part with old controls and its hollow part with cross controls. This adds exactly the required new diagonal direction. The reduced map is therefore onto at one specialization and on a nonempty open for unrestricted W.

Finally intersect this open with the first-row/column determinant opens and with the open where all entries of N in its row are nonzero. All are nonempty opens in the same irreducible parameter space. Since both directional coefficients are nonzero, the argument is unchanged on interchanging the original slots. □

8.5. **The original-input trace lift.**

Proposition 8.7. *Let f be an ordinary multilinear traceful sextic on M_n, $n \geq 3$. Suppose it has a nonzero proper trace-zero unit restriction q of degree $2 \leq d \leq 5$. Then* $\mathrm{Val}_n(f) = M_n$.

Proof. Every unit slot below remains an original input, and all its tangent and higher-jet directions are retained.
A nonzero one-edge coefficient. Properness gives the exact two-vertex formula

$$q(D_\ell + u_\ell E_{12}) = \sum_s c_s u_s \prod_{\ell \neq s} (d_{\ell 1} - d_{\ell 2}) E_{12}.$$

Indeed each passive diagonal variable enters linearly, and its unit substitution kills the coefficient; hence its scalar-coordinate polynomial is divisible by that variable's two-vertex difference. Suppose $c_s \neq 0$. Keep its original slot X free, take every other surviving input equal to a common regular diagonal D, and retain the deleted inputs at I. The exact original X-slot is, up to a fixed harmless sign, $c_s \operatorname{ad}_D^{d-1}$, an isomorphism on $\mathcal{H}_n$. For $n = 3$ this is an original full-slot minor of rank six, so Theorem 6.5 finishes the proof.

For $n \geq 4$ write $h = \operatorname{tr} f = \operatorname{tr}(XG)$, with G multilinear of degree five in the other original inputs. Expand all five of those inputs about their D or I base values as base $+tZ$. The constant coefficient G_0 is zero, since $\operatorname{tr} q = 0$ for every full X. The top coefficient is the original nonzero cyclic gradient, so let $r \geq 1$ be the least nonzero coefficient G_r. It has an off-diagonal value for some regular diagonal D. Otherwise $[D, G_r] = 0$ for diagonal D, hence for every diagonalizable D by covariance and for every D by density. This is a polynomial identity of degree at most six, less than $2n$, and is therefore zero freely. But the centralizer of D is $K[D]$, whereas every component of G_r has positive degree in the independent Z variables. This forces $G_r = 0$, a contradiction. Components with different perturbed-slot subsets can be separated by their Z multidegrees.

Choose one actual regular D and perturbation tuple for this nonvanishing, and relabel so that $(G_r)_{b1} \neq 0$ for some $b \neq 1$. Choose a hollow first-row X

avoiding the trace hyperplane $\operatorname{tr}(XG_r) = 0$ and the finitely many hyperplanes on which one output row entry vanishes. Then $N = f(a)$ is a full nonzero hollow row. The original X-slot gives all hollow controls, and conjugation gives all traceless diagonal controls. The minimal-jet argument preceding Lemma 8.3 supplies exactly the tensor hypotheses of Lemma 8.2, proving full image.

No one-edge coefficient: a one-direction pair. Then $q \in \mathrm{Id}(\mathrm{UT}_2)$ but not $\mathrm{Id}(\mathrm{UT}_3)$, so the two-step chart described above exists. Suppose first that $C \neq 0$ and $E = 0$. All closed terms vanish by trace zero. For fixed hollow Y, the original hollow X-slot acts row by row through matrices

$$B_r = (a_{rst}Y_{st})_{s,t\neq r}, \qquad (B_r)_{ss} = 0.$$

Each determinant is nonzero as a polynomial in Y: derangements exist since $n-1 \geq 2$, and distinct permutations give distinct monomials. The product of all these determinants is nonzero, so one actual Y makes the slot an isomorphism of $\mathcal{H}_n$ onto itself. When $n = 3$ this gives an original rank-six minor, and the rank-six theorem applies.

For $n \geq 4$, again write $h = \operatorname{tr}(XG)$, now retaining Y arbitrary and expanding the other four original slots, including the unit slots, about their independent diagonal or unit backgrounds. Let $r \geq 1$ be the first nonzero coefficient G_r. Each component has at least the two arbitrary letters Y and one perturbed slot, and after deleting fixed unit letters has total degree at most five. Lemma 8.3 implies that G_r has a nonzero off-diagonal value. This nonvanishing open intersects all the determinant opens in the same irreducible (D, Y, Z) space. Choose a row X with $\operatorname{tr}(XG_r) \neq 0$ and every output row entry nonzero. The original slot and orbit again give $\mathfrak{sl}_n$ controls. Minimality kills all lower ambient trace jets, including those varying Y or an original unit input, and makes the first tensor annihilate the control space. Lemma 8.2 applies.

A bidirectional pair. Finally suppose $C, E \neq 0$. Use the row family from Lemma 8.6. On the full two-free-slot chart, $h(X, Y; D, I) = \operatorname{tr} q = 0$ for all full X, Y. Expand *all four* passive original inputs, including the unit inputs. Let $r \geq 1$ be the least nonzero trace coefficient H_r. It exists because the fourth coefficient is the original trace with all passive slots arbitrary. Take a nonzero perturbed-slot component, and delete its unperturbed unit letters. It has $k = r+2 \geq 3$ arbitrary letters, including X, Y, and $j \leq 6-k$ actual diagonal letters. Lemma 8.4 shows that H_r is nonzero on one of the two row orientations. Both orientations have the smooth-control open of Lemma 8.6. On that same irreducible parameter family, intersect the nonzero trace coefficient, the control minor, and the conditions that all output row entries are nonzero. This gives one actual tuple a satisfying every requirement simultaneously.

Let W be the sum of the two active-slot tangent spaces and the orbit space. Its image under dg_a is $\mathfrak{sl}_n$. Minimality of r gives every lower ambient trace derivative zero and a nonzero order-r tensor annihilating both original active slots and every orbit tangent. In particular this is a statement on all $6n^2$ source coordinates, including the inputs whose base value was I. Apply Lemma 8.2. This exhausts the cases. □

8.6. **Completion of the nonproper classification.**

Theorem 8.8. *Let f be a nonproper ordinary multilinear sextic over K. For every $n \geq 3$,*

$$\mathrm{Val}_n(f) = \begin{cases} M_n, & \operatorname{tr} f \not\equiv 0, \\ \mathfrak{sl}_n, & \operatorname{tr} f \equiv 0. \end{cases}$$

Proof. Choose an inclusion-maximal set S of original slots whose unit deletion leaves a nonzero *free* polynomial q. Nonproperness provides a nonempty such set. If all six slots are deleted, the coefficient sum c is nonzero; setting all but one original input to I gives cX, so the image is full. Otherwise maximality makes q proper of positive degree at most five. Degree one is impossible, since a nonzero linear polynomial has nonzero unit deletion. Thus $2 \leq d \leq 5$. The minimum matrix-identity degree separately shows that this free nonzero q is an actual nonidentity of every M_n, $n \geq 3$.

If q is traceful, Lemma 8.1, applied only after the proper sextic theorem, gives $\mathrm{Val}_n(q) = M_n$. Every such value is an actual value of f by the chosen unit substitution. If f is trace zero, then q is trace zero, and the same lemma gives

$$\mathfrak{sl}_n = \mathrm{Val}_n(q) \subseteq \mathrm{Val}_n(f) \subseteq \mathfrak{sl}_n.$$

These are inclusions of actual values, not spans. The only remaining case is traceful f with trace-zero q, exactly Proposition 8.7. □

8.7. **The ordinary sextic theorem and small sizes.**

Theorem 8.9. *The degree-six classification in Theorem A holds for every $n \geq 3$.*

Proof. For proper polynomials this is Theorem 7.16. Every nonproper polynomial is covered by Theorem 8.8. Proposition 1.5 identifies the trace condition with the cyclic projection and specifies the identity exception in each size. □

Lemma 8.10. *Let $1 \leq d \leq e$ and $f \in P_d$. Set*

$$F(x_1, \ldots, x_e) = f(x_1, \ldots, x_{d-1}, x_d x_{d+1} \cdots x_e).$$

Then $F \in P_e$ and $\mathrm{Val}_n(F) = \mathrm{Val}_n(f)$ for every $n \geq 1$. Moreover, $F = 0$ as a free polynomial if and only if $f = 0$.

Proof. Every value of F is a value of f, by multiplying its last $e - d + 1$ inputs. Conversely set $x_{d+1}, \ldots, x_e$ equal to the identity. This gives every value of f as one value of F. Expanding a word of f replaces its unique x_d by the fixed ordered block $x_d \cdots x_e$. Different words give different expanded words, so the coefficient map is injective. This proves the final assertion. □

Corollary 8.11. *Let $d \in \{4, 5\}$ and $f \in P_d$. For every $n \geq 3$,*

$$\mathrm{Val}_n(f) = \begin{cases} \{0\}, & f = 0, \\ \mathfrak{sl}_n(K), & f \neq 0 \text{ and } \pi f = 0, \\ M_n(K), & \pi f \neq 0. \end{cases}$$

Proof. Apply Lemma 8.10 with $e = 6$, after Theorem 8.9. If $f \neq 0$, then f is not an identity on M_n, because $d < 2n$. Equality of actual value sets therefore makes F a nonidentity too; in particular $F \notin Ks_6$ when $n = 3$. Tracefulness is preserved by the same equality. Proposition 1.5 identifies it with the nonvanishing of the cyclic class in each degree. Apply the ordinary sextic classification to F and transfer its value set back to f. □

Corollary 8.12. *Let $d \in \{4, 5, 6\}$ and $f = \sum_{\sigma \in S_d} a_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}$. For $n = 1$ put $c = \sum_{\sigma \in S_d} a_\sigma$, the total word coefficient of f. Its image is $\{0\}$ when $c = 0$ and is K otherwise. For $n = 2$, the image is $\{0\}$ for an identity, KI_2 for a nonidentity central polynomial, $\mathfrak{sl}_2$ for a noncentral trace-zero polynomial, and M_2 for a noncentral traceful polynomial.*

Proof. The size-one evaluation is $cx_1 \cdots x_d$. The size-two assertions are the published ordinary multilinear image theorem of Kanel-Belov, Malev and Rowen [11, Theorem 2], whose field hypotheses are satisfied here. Its alternatives are distinguished by the actual identity, centrality and trace conditions. The free cyclic projection alone cannot replace those conditions on M_2: for example, $s_4(x_1, x_2, x_3, x_4)x_5x_6$ is an identity of M_2 although its cyclic class is nonzero. In degree four the equivalence $\operatorname{tr} f \equiv 0 \iff \pi f = 0$ still holds on M_2: in Proposition 1.4, h_f has degree three, below the identity bound four. This does not decide identity or centrality. In degrees five and six even the trace equivalence fails; x_5s_4 gives the corresponding degree-five example. □

9. Further lifting results

In this section, we give several lifting results applicable in arbitrary degree. We use the notation of Section 1. These results are independent of the proof of Theorem 7.16.

9.1. **Root-space lifting and cyclic compatibility.**

Proposition 9.1. *Let $\Phi(D, A, Z)$ be a word polynomial linear in A. For diagonal D, C, write $\Phi(D, E_{ij}, C) = m(d_i, d_j; c_i, c_j)E_{ij}$, $i \neq j$. If m is a nonzero polynomial, then some nonempty open set of regular diagonal pairs makes this map an automorphism of $\mathcal{H}_n$.*

Proof. The product of the finitely many multipliers and the eigenvalue differences is a nonzero polynomial in the diagonal coordinates. Its nonvanishing gives the assertion. This only controls root-space coordinates; diagonal equations must be solved separately. □

Proposition 9.2. *Let $r \geq 2$, with indices modulo r, and let $(Lz)_i = \alpha_i z_i + \beta_i z_{i-1}$, where all α_i, β_i are nonzero. Put $h = \prod_i (-\beta_i/\alpha_i)$. If $h \neq 1$, then L is invertible. If $h = 1$, then $\operatorname{rank} L = r - 1$, and $Lz = t$ is solvable exactly when*

$$\sum_i \ell_i t_i = 0, \qquad \ell_{i+1} = -\frac{\alpha_i}{\beta_{i+1}} \ell_i,$$

where ℓ is any nonzero solution of this recurrence. In the constant-coefficient case $L = pI + qS$, with $(Sz)_i = z_{i-1}$, its eigenvalues are $p + q\omega$ for $\omega^r = 1$. Thus it is invertible exactly when $p^r \neq (-q)^r$, including cases in which one coefficient is zero.

Proof. The homogeneous equations give $z_i = (-\beta_i/\alpha_i)z_{i-1}$. Going once around the cycle gives $z_0 = hz_0$. If $h \neq 1$, the kernel is zero. If $h = 1$, one nonzero initial coordinate determines a nonzero kernel vector, so the kernel is one-dimensional. The equations for the left kernel are $\ell_i\alpha_i + \ell_{i+1}\beta_{i+1} = 0$. They likewise have a one-dimensional solution space, and orthogonality to this space is necessary and sufficient for membership in $\operatorname{im} L$. Finally, S has the r distinct rth roots of unity as eigenvalues in characteristic zero. Alternatively, the identity

$$(pI + qS)\sum_{k=0}^{r-1} p^{r-1-k}(-q)^k S^k = (p^r - (-q)^r)I$$

gives its inverse whenever the scalar is nonzero. □

Lemma 9.3. *Let $F : M_n(K)^r \to M_n(K)$ be a polynomial map equivariant for simultaneous conjugation, and put $T = F(a)$. With orthogonal complements taken for the trace pairing, one has*

$$[M_n, T] \subseteq \operatorname{im} dF_a, \qquad dF_a \text{ is surjective } \iff Z(T) \cap (\operatorname{im} dF_a)^\perp = 0.$$

If T is regular, meaning its minimal polynomial has degree n, then surjectivity is equivalent to the existence of tangent vectors $v_1, \dots, v_n$ such that

$$\det\Big(\operatorname{tr}(T^{i-1}dF_a(v_j))\Big)_{1\le i,j\le n} \neq 0.$$

No semisimplicity assumption on T is required. Surjectivity implies that the image of F contains a Zariski-open neighborhood of T.

Proof. Differentiating conjugation equivariance gives

$$dF_a([X, a_1], \dots, [X, a_r]) = [X, T].$$

Moreover, $\operatorname{tr}(Y[X, T]) = \operatorname{tr}((TY - YT)X)$, so nondegeneracy of trace gives $[M_n, T]^\perp = Z(T)$. If $W = \operatorname{im} dF_a$, then $W^\perp \subseteq Z(T)$, and the first criterion follows.

For regular T, choose a cyclic vector v. Every commuting matrix is determined by its value on v, which is uniquely $p(T)v$ with $\deg p < n$. Thus $Z(T) = K[T]$. Trace induces a perfect pairing between $M_n/[M_n, T]$ and $Z(T)$. The functionals $[Y] \mapsto \operatorname{tr}(T^{i-1}Y)$ therefore identify that quotient with K^n. Since W already contains $[M_n, T]$, its image in the quotient is full exactly when the displayed determinant is nonzero for some directions. Finally a surjective differential between these affine spaces makes F smooth near a by the Jacobian criterion; openness of smooth morphisms gives the asserted neighborhood [17, Tags 00T6, 00T7 and 056G]. □

For a regular nonsemisimple T the argument uses the quotient pairing, not the generally false direct-sum identity $M_n = [M_n, T] \oplus K[T]$. In particular it applies to a regular nilpotent matrix.

Remark 9.4. The existence of an open subset of values does not prove $F(M_n(K)^r) = M_n(K)$. Scalar matrices and exceptional Jordan classes still require exact constructions. Even containment of all traceless matrices would not settle a traceful image, for which every traceful target must also be attained. The criterion alone therefore does not establish a full-image theorem.

9.2. **Cycle supports in arbitrary degree.** For $q \geq 0$, let

$$\Phi(D, A, Z) = \sum_{a+b+c=q} (r_{abc} D^a A D^b Z D^c + s_{abc} D^a Z D^b A D^c),$$

and put $R(x, y, z) = \sum r_{abc} x^a y^b z^c$ and $S(x, y, z) = \sum s_{abc} x^a y^b z^c$.

Lemma 9.5. *Let $D = \operatorname{diag}(d_1, \ldots, d_n)$, let $\sigma \in S_n$, and choose $a_i \in K^\times$ for $1 \leq i \leq n$. Put $A = \sum_{i=1}^n a_i E_{i,\sigma(i)}$, $y_{ij} = a_i Z_{\sigma(i),j}$, $\tau(i, j) = (\sigma(i), \sigma(j))$. For every i, j, $\Phi(D, A, Z)_{ij} = \alpha_{ij} y_{ij} + \beta_{ij} y_{\tau^{-1}(i,j)}$, where $\alpha_{ij} = R(d_i, d_{\sigma(i)}, d_j)$ and $\beta_{ij} = \frac{a_{\sigma^{-1}(j)}}{a_{\sigma^{-1}(i)}} S(d_i, d_{\sigma^{-1}(j)}, d_j)$. On a τ-orbit $\mathcal{O}$ the determinant is*

$$\Delta_{\mathcal{O}} = \prod_{e \in \mathcal{O}} \alpha_e + (-1)^{|\mathcal{O}|-1} \prod_{e \in \mathcal{O}} \beta_e.$$

Proof. For an A-before-Z word, $(D^a A D^b Z D^c)_{ij} = d_i^a d_{\sigma(i)}^b d_j^c y_{ij}$. For a Z-before-A word, $(D^a Z D^b A D^c)_{ij} = \frac{a_{\sigma^{-1}(j)}}{a_{\sigma^{-1}(i)}} d_i^a d_{\sigma^{-1}(j)}^b d_j^c y_{\tau^{-1}(i,j)}$. Summing gives the entry formula. Ordering the variables cyclically on $\mathcal{O}$ produces a cyclic bidiagonal matrix, whose determinant is the displayed two-product expression. □

If every orbit determinant is nonzero, $Z \mapsto \Phi(D, A, Z)$ is surjective onto $M_n(K)$, since $Z \mapsto (y_{ij})$ is an invertible coordinate change. On an orbit of length at least two with nonzero coefficients, Proposition 9.2 gives the exact compatibility condition when the determinant vanishes. The determinant formula itself also allows zero coefficients and singleton orbits; for a singleton it reads $\alpha + \beta$.

9.3. **Scalar values and smooth unit-deletion lifting.**

Theorem 9.6. *Let f be multilinear and let $\mathcal{I}$ be a nonempty set of r active slots. Assume that f_J is identically traceless on $M_n(K)$ for every nonempty $J \subseteq \mathcal{I}$, where f_J sets exactly those slots equal to the identity. Put $g = f_{\mathcal{I}}$. Suppose an actual tuple Y^0 for the remaining slots satisfies*

$$g(Y^0) = N \neq 0, \quad N^2 = 0, \quad \operatorname{rank} N = 1, \quad \operatorname{im}(dg)_{Y^0} = \mathfrak{sl}_n(K),$$

and the multilinear scalar function $H(X_{\mathcal{I}}) = \operatorname{tr} f(X_{\mathcal{I}}, Y^0)$ is nonzero. Then $f(M_n(K)) = M_n(K)$.

Proof. Scalar targets follow from the scalar-value argument of Lemma 1.7. For a nonscalar target choose a similar representative $T = \left(\begin{smallmatrix} a & b \\ c & D \end{smallmatrix}\right)$ with $b \neq 0$. This is possible because a nonscalar transpose has a non-eigenvector. For $S_t = \operatorname{diag}(t^{-1}, I_{n-1})$ put

$$R(\epsilon) = \epsilon^r S_{\epsilon^r} T S_{\epsilon^r}^{-1} = \begin{pmatrix} \epsilon^r a & b \\ \epsilon^{2r} c & \epsilon^r D \end{pmatrix}.$$

The value at zero is a nonzero rank-one square-zero matrix. Conjugate Y^0 so $g(Y^0) = R(0)$. The complete translated evaluation

$$F(\epsilon, X, Y) = f((I + \epsilon X_i)_{i\in\mathcal{I}}, Y)$$

satisfies $F(0, X, Y) = g(Y)$ and $\operatorname{tr} F(\epsilon, X, Y) = \epsilon^r \operatorname{tr} f(X, Y)$: every other term has a nonempty deletion inside $\mathcal{I}$. No mixed matrix term is discarded.

Fix all but one active input so that H is a nonzero linear functional in that input. Choose X^0 with $H(X^0) = \operatorname{tr} T$ and $d_X H \neq 0$. Writing $\pi_0(Z) = Z - \operatorname{tr}(Z)I_n/n$, consider

$$\pi_0(F(\epsilon, X, Y) - R(\epsilon)) = 0, \qquad \operatorname{tr} f(X, Y) = \operatorname{tr} T.$$

At $(0, X^0, Y^0)$ their Jacobian in X, Y has rank n^2: the first equations have surjective Y-differential dg and zero X-differential, while the last equation has nonzero X-differential. Choose a nonzero n^2-minor of this relative Jacobian. On its nonvanishing locus the solution scheme is standard smooth over the ϵ-line. Its image is open and contains zero, hence contains some $\epsilon \neq 0$ [17, Tags 00T6, 00T7 and 056G]. Algebraic closedness supplies an actual K-point in this nonempty fibre. The trace identity recovers the full equation $F = R(\epsilon)$. Scale one input by ϵ^{-r} and conjugate all inputs to obtain T. □

Corollary 9.7. *Let $g(Y) = f(I, Y)$ be identically traceless. If $g(Y^0)$ is nonzero rank-one square-zero, dg_{Y^0} is onto $\mathfrak{sl}_n$, and $X \mapsto \operatorname{tr} f(X, Y^0)$ is nonzero, then f is surjective on $M_n(K)$. No conditions on the other unit deletions are needed.*

Proof. Apply Theorem 9.6 with the singleton active set. □

9.4. **Kernel sections and compatibility.**

Lemma 9.8. *Let $M(t)$ be a matrix of rational functions regular at $t = 0$, with constant rank r on a Zariski neighborhood of zero. Set $k = \dim\ker M(0)$. If a constant linear map N to K^k restricts to an isomorphism on $\ker M(0)$, then, after shrinking the neighborhood, each $b \in K^k$ determines a unique rational section $x(t)$ satisfying $M(t)x(t) = 0$ and $Nx(t) = b$. It is regular at zero. The same assertion applies to the left kernel by using $M(t)^T$.*

Proof. Select r independent rows of $M(0)$ and append the k rows of N. The resulting square matrix is invertible at zero: its kernel is $\ker M(0) \cap \ker N = 0$. Its inverse is regular wherever its determinant is nonzero. Solve these selected equations with right side $(0, b)$. The selected r rows span the full row space wherever the rank remains r, so the solution satisfies every row equation. This also proves uniqueness. In particular, if $M(-t)$ is obtained by conjugating input and output coordinates by involutions, and the normalization is fixed by the input involution, uniqueness gives the corresponding parity of $x(t)$. □

10. Matrix-unit identities and parameter coverage

We collect the algebraic rules behind the finite-dimensional arguments. Matrix-unit paths determine the coefficient and differential matrices; polynomial common factors describe the exceptional parameters. The extension to

arbitrary matrix size is furnished by the block and band arguments proved in the preceding sections.

10.1. **Word paths and differential coordinates.** For an associative word, matrix multiplication gives

$$E_{i_1 j_1} \cdots E_{i_r j_r} = \left(\prod_{s=1}^{r-1} \delta_{j_s, i_{s+1}} \right) E_{i_1 j_r}. \tag{10.1}$$

Thus an entry of a word evaluation is the sum of the weights of its composable labeled paths. Its trace is obtained by retaining the closed paths. These statements follow by induction from $E_{ij}E_{kl} = \delta_{jk}E_{il}$.

For a polynomial map G in m matrix inputs, order its outputs and each input in row-major order, and order the inputs as $X_0, \dots, X_{m-1}$. Define

$$J_G(X)_{in+j,\, \ell n^2 + an + b} = [t]\, G(X_0, \dots, X_\ell + tE_{a+1,b+1}, \dots, X_{m-1})_{i+1,j+1},$$

where $0 \le i, j, a, b < n$ and $0 \le \ell < m$. Here the matrix-unit subscripts are one based, whereas row and column labels of J_G are zero based. If the output is traceless, we delete row $n^2 - 1$. The remaining coordinates identify $\mathfrak{sl}_n$ with K^{n^2-1}: the omitted diagonal entry is the negative sum of the other diagonal entries.

Every column is determined by the product rules

$$(U\dot{V}) = \dot{U}V + U\dot{V}, \qquad [U,\dot{V}] = [\dot{U}, V] + [U, \dot{V}]. \tag{10.2}$$

For example, for the four-parameter family one first forms

$$\begin{aligned} \dot{P} &= [\dot{D}, E] + [D, \dot{E}], \quad \dot{Q} = [\dot{D}, A] + [D, \dot{A}], \quad \dot{R} = [\dot{E}, B] + [E, \dot{B}], \\ \dot{S} &= [\dot{D}, B] + [D, \dot{B}], \quad \dot{T} = [\dot{E}, A] + [E, \dot{A}], \quad \dot{Z} = [\dot{A}, B] + [A, \dot{B}]. \end{aligned}$$

One then differentiates the displayed expressions for L, C and $\mathcal{T}$ by (10.2). In particular,

$$d(aPL + bLP + C) = a(\dot{P}L + P\dot{L}) + b(\dot{L}P + L\dot{P}) + \dot{C}.$$

Together with (10.1), this specifies every entry of the matrices used below using additions and multiplications in $\mathbb{Q}[x]$.

Lemma 10.1. *Let $J_1(x), \dots, J_s(x)$ be polynomial matrices with r rows. Choose finitely many $r \times r$ minors among these matrices, and let $g(x)$ be their greatest common divisor in $\mathbb{Q}[x]$. At every $\xi \in K$ with $g(\xi) \neq 0$, at least one $J_i(\xi)$ has rank r. In particular, $g = 1$ gives coverage of every parameter.*

Proof. Write the chosen minors as $d_1, \dots, d_t$. The Euclidean algorithm gives polynomials $u_j \in \mathbb{Q}[x]$ with $g = \sum_j u_j d_j$. If every selected minor vanished at ξ, then $g(\xi) = 0$. Conversely, every zero of g is a common zero of all d_j, since g divides each d_j. These identities remain valid in every characteristic-zero field. No restriction to rational parameters is involved. □

10.2. **Boundary parameters for the three-coefficient family.** Let $F = aPL{+}bLP{+}C$. We use the size-four and size-five tuples in Section 3, without the two size-five modifications unless they are explicitly mentioned. Write $J_n(a, b)$ for the full Jacobian of F at the indicated tuple. The following identities give the boundary coverage used there. All column labels are zero based. In each paragraph a common set C_* and the additional sets $U_1, \dots, U_r$ are displayed. Put $C_j = C_* \cup U_j$, in increasing order, and

$$d_j(x) = \det J_n(a(x), b(x))[:, C_j].$$

The notation $p \doteq q$ means $p = cq$ for some $c \in \mathbb{Q}^\times$. Such constants have no effect on a common zero set. Thus the factorizations below retain exactly the information needed for the rank assertion. They follow by substituting the tuples in the product rule above and expanding the specified determinants. Their common factors follow by the Euclidean algorithm in $\mathbb{Q}[x]$.
Size 4: $(a, b) = (x, 1)$.

$$C_* = \{2, 4, 5, 6, 7, 8, 18, 37, 38, 46, 50, 52, 62\}.$$

j	U_j
1	$\{1, 20, 55\}$
2	$\{11, 34, 36\}$

$$d_1 \doteq (x-1)(3x-2)(x+2)^2(3x+1)^2$$
$$d_2 \doteq x(x+2)(3x+2)(x^3-2x^2+x-1)$$

The common factor is $x + 2$, up to a nonzero scalar.
Size 4: $(a, b) = (x, 1 - x)$.

$$C_* = \{1, 2, 5, 6, 7, 8, 37, 38, 46, 50, 52, 55, 62\}.$$

j	U_j
1	$\{4, 18, 20\}$
2	$\{11, 20, 36\}$
3	$\{4, 11, 34\}$

$$d_1 \doteq (x-1)(2x-1)(2x+1)(4x+1)$$
$$d_2 \doteq x(2x-1)(2x+1)$$
$$d_3 \doteq (x-1)x(8x^3-20x^2+9x-3)$$

The common factor is 1, up to a nonzero scalar.
Size 4: $(a, b) = (x, 3x - 1)$.

$$C_* = \{1, 2, 4, 5, 6, 7, 37, 38, 46, 50, 52, 62\}.$$

j	U_j
1	$\{11, 20, 34, 36\}$
2	$\{8, 11, 34, 55\}$
3	$\{8, 18, 34, 36\}$
4	$\{8, 11, 36, 39\}$

$$d_1 \doteq (2x-1)(3x-2)x^2(12x^2+6x-1)$$
$$d_2 \doteq (3x-2)(x-1)^2x^2(12x^2+6x-1)$$
$$d_3 \doteq (x-1)x(2x-1)(12x^2+6x-1)$$
$$d_4 \doteq (x-1)x(2x-1)(3x-2)(6x-1)(84x-25)$$

The common factor is x, up to a nonzero scalar.
Size 4: $(a,b) = (x,-3x)$.

$$C_* = \{4,5,6,7,8,11,37,38,46,50,52,62\}.$$

j	U_j
1	$\{1,20,36,45\}$
2	$\{1,2,34,55\}$
3	$\{2,18,34,36\}$

$$d_1 \doteq (2x-1)(2x+1)(3x+1)(6x-1)(9x+2)$$
$$d_2 \doteq (x-1)(3x+1)(22x-7)x^2(2x-1)^2$$
$$d_3 \doteq (2x-1)(2x+1)(3x+2)(22x-7)x^2$$

The common factor is $\frac{2x-1}{2}$, up to a nonzero scalar.
Size 5: $(a,b) = (x,0)$.

$$C_* = \{1,6,7,8,9,10,14,15,55,56,57,62,\ 63,66,67,72,77,78,80,92,97\}.$$

j	U_j
1	$\{5,30,71,83\}$
2	$\{2,30,79,83\}$
3	$\{2,5,52,79\}$

$$d_1 \doteq (x-2)(x-1)(x+1)(x+3)(x+4)(3x-1)(4x+1)$$
$$d_2 \doteq (x-2)x(x+1)(3x-1)(4x+1)(2x^2+11x+16)$$
$$d_3 \doteq (x-2)(x-1)(x+1)(x+3)x^2(2x^2-3x-1)$$

The common factor is $(x-2)(x+1)$, up to a nonzero scalar.
Size 5: $(a,b) = (x,-4x)$.

$$C_* = \{1,5,6,7,8,9,10,14,15,56,57,62,\ 67,72,77,78,80,83,92,97\}.$$

j	U_j
1	$\{30,55,63,66,71\}$
2	$\{2,52,58,66,79\}$
3	$\{2,52,55,58,79\}$

$$\begin{aligned} d_1 &\doteq (x-2)(x+4)(3x-1)(3x+1)(6x+1)(7x-1)(11x+3) \\ &\quad \cdot (16x+3)(4x+1)^2 \\ d_2 &\doteq (3x-1)(6x+1)(37x-13)(4x+1)^2x^3 \\ &\quad \cdot (48x^3+55x^2-26x-7) \\ d_3 &\doteq (3x-1)(3x+1)(37x-13)(4x+1)^2x^3 \\ &\quad \cdot (2x^2+9x+2) \end{aligned}$$

The common factor is $\frac{(3x-1)(4x+1)^2}{48}$, up to a nonzero scalar.
Size 5: $(a,b)=(x,1)$.

$$C_* = \{1,2,5,6,7,8,9,10,15,27,56,57,\ 58,62,66,67,72,77,78,79,80,92,97\}.$$

j	U_j
1	$\{30,83\}$
2	$\{52,55\}$

$$\begin{aligned} d_1 &\doteq (x-1)(x+2)(3x-2)(4x+1)^3(4x^2+13x-4) \\ d_2 &\doteq x(x+2)(2x+1)(4x+1)^2(8x^3-12x^2+9x-8) \end{aligned}$$

The common factor is $\frac{(x+2)(4x+1)^2}{16}$, up to a nonzero scalar.
Size 5: $(a,b)=(x,\frac{2}{3})$.

$$C_* = \{1,2,5,6,7,8,9,10,14,15,56,57,\ 58,62,67,72,77,78,79,80,84,92,97\}.$$

j	U_j
1	$\{30,83\}$
2	$\{52,55\}$

$$\begin{aligned} d_1 &\doteq (3x+5)(4x+1)(6x+1)(9x-5) \\ &\quad \cdot (702x^3+2115x^2-1449x-1294) \\ d_2 &\doteq (3x-1)(3x+5)(6x^2+3x+2) \\ &\quad \cdot (108x^3-162x^2+33x-47) \end{aligned}$$

The common factor is $\frac{3x+5}{3}$, up to a nonzero scalar.
Size 5: $(a,b)=(x,1-x)$.

$$C_* = \{1,2,6,7,8,9,10,15,56,57,62,67,\ 72,77,78,79,80,92,97\}.$$

j	U_j
1	$\{5,27,30,58,66,83\}$
2	$\{14,30,55,63,66,83\}$
3	$\{5,14,52,58,66,83\}$
4	$\{5,14,30,58,83,84\}$
5	$\{5,14,30,58,66,84\}$

$$\begin{aligned}
d_1 &\doteq (2x-1)(3x-1)(3x+1)(4x+1)(5x+1)(x-1)^2 \\
&\quad \cdot (15x^2+47x-4) \\
d_2 &\doteq (x-2)x(2x-1)(3x-1)(3x+1)(4x+1) \\
&\quad \cdot (15x^2+47x-4) \\
d_3 &\doteq x(3x-1)(4x+1)(x-1)^2(15x^2+47x-4) \\
&\quad \cdot (10x^3-23x^2+13x-4) \\
d_4 &\doteq x(2x-1)(3x+1)(4x+1)(x-1)^2 \\
&\quad \cdot (65x^2+189x+14) \\
d_5 &\doteq x(2x-1)(3x-1)(3x+1)(x-1)^2 \\
&\quad \cdot (60x^3+179x^2+56x+7)
\end{aligned}$$

The common factor is 1, up to a nonzero scalar.
Size 5: $(a,b) = (x, \frac{2x}{5} + \frac{6}{5})$.

$$C_* = \{1, 2, 5, 6, 7, 8, 9, 10, 15, 56, 57, 62,\ 67, 72, 78, 79, 80, 92, 97\}.$$

j	U_j
1	$\{14, 30, 55, 58, 66, 84\}$
2	$\{14, 30, 55, 63, 66, 83\}$
3	$\{27, 30, 55, 58, 66, 83\}$
4	$\{14, 30, 55, 58, 77, 83\}$
5	$\{14, 30, 58, 66, 77, 83\}$
6	$\{14, 52, 55, 58, 66, 83\}$

$$\begin{aligned}
d_1 &\doteq (x+3)(3x+4)(7x+1)(7x+11)(11x+3)(13x-11) \\
&\quad \cdot (2x+1)^3 \\
d_2 &\doteq (2x+1)(3x+4)(4x+1)(7x+1)(7x+11)(11x+3) \\
&\quad \cdot (13x-11)(x-2)^2 \\
d_3 &\doteq (x-2)(x+3)(3x+4)(4x+1)(7x+1)(7x+11)(9x+2) \\
&\quad \cdot (11x+3)(13x-11) \\
d_4 &\doteq (x-2)(x+3)(2x+1)(4x+1)(5x-2)(7x+1)(7x+11) \\
&\quad \cdot (11x+3)(13x-11) \\
d_5 &\doteq (x-2)(x+3)(2x+1)(3x+4)(4x+1)(7x+11) \\
&\quad \cdot (11x+3)(13x-11)(87x^2+82x-337) \\
d_6 &\doteq (x-2)(x+3)(2x+1)(3x+4)(4x+1)(7x+1)(7x+11) \\
&\quad \cdot (592x^3-197x^2+334x-741)
\end{aligned}$$

The common factor is $\frac{7x+11}{7}$, up to a nonzero scalar.

Size 5: $(a,b) = (x, 3x-1)$.

$$C_* = \{1,2,5,6,7,8,9,15,56,57,62,67,\ 72,77,78,79,80,92,97\}.$$

j	U_j
1	$\{14,30,52,55,58,66\}$
2	$\{10,14,52,58,66,83\}$
3	$\{10,27,52,55,58,66\}$
4	$\{10,14,52,55,63,66\}$
5	$\{10,14,52,55,58,83\}$
6	$\{10,14,55,58,59,66\}$

$$\begin{aligned}
d_1 &\doteq (2x-1)(3x-2)(3x-1)(7x-1)(9x-5)(13x-11)\\
&\quad \cdot x^2(22x^2-5x+1)\\
d_2 &\doteq (x-1)x(3x-2)(3x-1)(4x+1)(9x-5)\\
&\quad \cdot (22x^2-5x+1)(53x^3-38x^2-67x+48)\\
d_3 &\doteq (x-1)x(x+3)(2x-1)(3x-1)(7x-1)(9x-5)\\
&\quad \cdot (13x-11)(22x^2-5x+1)\\
d_4 &\doteq (x-2)(x-1)x(2x-1)(3x-2)(7x-1)(9x-5)\\
&\quad \cdot (13x-11)(22x^2-5x+1)\\
d_5 &\doteq (x-1)x(2x-1)(3x-2)(3x-1)(4x+1)\\
&\quad \cdot (15x^2+31x-38)(22x^2-5x+1)\\
d_6 &\doteq (x-1)x(2x-1)(3x-2)(3x-1)(9x-5)(13x-11)\\
&\quad \cdot (152x-45)(7x-1)^2
\end{aligned}$$

The common factor is x, up to a nonzero scalar.

For size four, these common factors vanish only where $a+b+1=0$. For size five, their zeros with $a+b+1 \neq 0$ are exactly $(a,b) = (2,0)$ and $(-1/4,1)$. The replacement $E' = J_5 + E_{35} + E_{15}$ and the two nonzero determinants displayed in Section 3 cover those points. Together with the generic minors there, these identities cover every parameter with $a+b+1 \neq 0$.

10.3. **The traceful and trace-zero pencils.** Let $\Gamma_n(x) = xPL + (1-x)LP$ and $Z_n(x) = xPL - (x+1)LP + C$. The seeds for these pencils are those displayed in Lemmas 3.6 and 3.8, respectively. For Γ_n the seed index $\varepsilon = 0, 1$ selects the first or second displayed seed of size n. For Z_n use the unique seed of that size. For a column list I let Δ_I be the corresponding maximal minor, retaining all output rows for Γ_n and omitting the last diagonal output coordinate for Z_n. The notation $f \doteq g$ means $f = cg$ for some $c \in \mathbb{Q}^\times$. Only these nonzero constant factors are suppressed below. Every determinant is specified by the displayed inputs and columns, so the product rule above determines it without auxiliary data.

The traceful pencil. The following two minors for each size are coprime. Put

$$\begin{aligned} q_2(x) &= 1176x^2 - 1709x + 605, \\ q_4(x) &= 5066718429x^4 - 7630938514x^3 \\ &\qquad + 2738274885x^2 - 52653960x + 340216. \end{aligned}$$

Size 3.

$$\varepsilon = 0, \qquad I_{G31} = \big(0, 3, 4, 5, 6, 15, 18, 19, 24\big),$$
$$\Delta_{I_{G31}} \doteq x(2x+1)(5x-1)(7x-6)(7x-3)(5x-4)^2.$$
$$\varepsilon = 1, \qquad I_{G32} = \big(0, 1, 3, 4, 6, 15, 18, 19, 24\big),$$
$$\Delta_{I_{G32}} \doteq (x-1)(3x-2)(3x+1)(6x-5)(9x-5)(9x-4)^2.$$

Size 4.

$$\varepsilon = 0, \qquad I_{G41} = \big(0, 1, 2, 4, 5, 8, 9, 10, 12, 13, 24, 28, 32, 36, 37, 40\big),$$
$$\begin{aligned} \Delta_{I_{G41}} \doteq{}& (2x+1)(2x+3)(2x+5)(4x-3)(4x-1)(4x+1)(10x-9)(10x-1) \\ &\cdot (14x-9)x^3 q_2(x). \end{aligned}$$
$$\varepsilon = 1, \qquad I_{G42} = \big(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 28, 32, 36, 38, 40, 56\big),$$
$$\begin{aligned} \Delta_{I_{G42}} \doteq{}& (2x+19)(4x+19)(5x-4)(5x-1)(5x+2)(12x+7)(15x-14) \\ &\cdot (15x-1)(17x-19)(17x+2)(47x-28)(x-1)^2. \end{aligned}$$

Size 5.

$$\varepsilon = 0, \qquad I_{G51} = \big(0, 1, 2, 3, 4, 7, 9, 10, 12, 14, 28, 29, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 79\big),$$
$$\begin{aligned} \Delta_{I_{G51}} \doteq{}& (2x-7)(2x-5)(2x-3)(2x+3)(2x+5)(4x-7)(4x-5)(4x+3) \\ &\cdot (6x-7)(15x+1)(17x-16)(17x-9)(17x-1)(19x-16) \\ &\cdot (21x-16)(23x-16)q_4(x). \end{aligned}$$
$$\varepsilon = 1, \qquad I_{G52} = \big(0, 1, 2, 3, 4, 7, 9, 10, 11, 12, 28, 29, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 79\big),$$
$$\begin{aligned} \Delta_{I_{G52}} \doteq{}& (5x-7)(12x-19)(12x+7)(17x-19)(18x-37)(18x+19) \\ &\cdot (30x-37)(30x+7)(35x-37)(35x+2)(63x+2)(67x-65) \\ &\cdot (67x-2)(72x-65)(84x-65)(102x-65)(134x-69) \\ &\cdot (10797x-130)(x-1)^2. \end{aligned}$$

In each size the rational roots of the displayed linear factors in the first minor are disjoint from those in the second. The only nonlinear factors in the first minors are q_2 and q_4, and neither has a rational root. Indeed the discriminant of

q_2 is 74761, strictly between 273^2 and 274^2. Modulo 5, q_4 becomes $-x^4+x^3+1$, whose values at $0,1,2,3,4$ are respectively $1,1,3,2,4$. Its leading coefficient is a unit modulo 5, so a rational root would have denominator prime to 5 and would reduce to a root modulo 5. Thus the two minors have no common zero over $\overline{\mathbb{Q}}$. Bézout's identity over $\mathbb{Q}[x]$ shows that they have no common zero in any characteristic-zero field. This proves the required surjectivity for every parameter, including the zero-output parameters.

The trace-zero pencil. For the same output coordinates, the following minors suffice:

Size 3.

$$I_{Z31} = \big(1,2,5,9,11,13,27,29\big),$$
$$\Delta_{I_{Z31}} \doteq (x+1)(3x-1).$$
$$I_{Z32} = \big(0,1,2,5,9,11,29,31\big),$$
$$\Delta_{I_{Z32}} \doteq x(2x+3).$$

Size 4.

$$I_{Z41} = \big(1,2,4,5,6,7,8,11,34,36,37,38,46,52,62\big),$$
$$\Delta_{I_{Z41}} \doteq (x+2)(2x+1)(4x^2-x+1).$$
$$I_{Z42} = \big(1,2,4,5,6,7,8,18,20,36,37,38,46,52,62\big),$$
$$\Delta_{I_{Z42}} \doteq x(2x-1)(4x+3).$$

Size 5.

$$I_{Z51} = \big(1,2,5,6,7,8,9,10,14,15,30,55,56,57,62,63,66,67,72,78,79,80,92,97\big),$$
$$\Delta_{I_{Z51}} \doteq (x-2)x(3x+5)(x+2)^2(3x-1)^2.$$
$$I_{Z52} = \big(1,2,5,6,7,8,9,10,15,27,30,55,56,57,58,62,66,67,77,78,79,80,92,97\big),$$
$$\Delta_{I_{Z52}} \doteq (x-1)x(x+1)(3x+5)(5x+3)(3x-1)^2.$$
$$I_{Z53} = \big(1,2,5,6,7,8,9,10,14,15,52,55,56,57,58,62,66,67,72,78,80,83,92,97\big),$$
$$\Delta_{I_{Z53}} \doteq (x+1)(x+4)(3x+5)(4x+1)(x+2)^2(10x^3+3x^2-1).$$
$$I_{Z54} = \big(1,2,5,6,7,8,9,10,14,15,30,55,56,57,58,62,67,72,77,78,79,80,92,97\big),$$
$$\Delta_{I_{Z54}} \doteq x(x+1)(x+2)(3x-1)^2(2x^2+3x+4).$$

For size three, the roots of the two displayed minors are disjoint. For size four, the second minor has roots $0, 1/2, -3/4$, and the first is nonzero at each of them. In size five the common roots of the first two minors are exactly $0, 1/3, -5/3$. The third minor is nonzero at 0 and $1/3$, and the fourth is nonzero at $-5/3$. Consequently the displayed minors have no common zero in any characteristic-zero field, again by Bézout's identity over $\mathbb{Q}[x]$. These calculations prove all three trace-zero base sizes. The block arguments in the text, rather than any further finite calculation, supply every larger size.

10.4. **A kernel calculation for the size-four alternating map.** For $\Psi(D, E, A) = s_3(P, Q, R)$, where $P = [D, E]$, $Q = [D, A]$ and $R = [E, A]$, the product rule gives

$$\begin{aligned} d_D\Psi(X) &= s_3([X, E], Q, R) + s_3(P, [X, A], R), \\ d_E\Psi(X) &= s_3([D, X], Q, R) + s_3(P, Q, [X, A]), \\ d_A\Psi(X) &= s_3(P, [D, X], R) + s_3(P, Q, [E, X]). \end{aligned}$$

We prove surjectivity at the size-four tuple displayed in the proof of Lemma 3.9 by solving a homogeneous system. Use the ordered Jacobian columns

$$(0, 1, 5, 8, 9, 13, 29, 32, 33, 34, 35, 37, 40, 41, 42, 43)$$

and let $v_0, \dots, v_{15}$ be their coefficients in a linear relation. Index output rows from zero in row-major order. The displayed product-rule identities and $E_{ab}E_{cd} = \delta_{bc}E_{ad}$ give the following successive consequences of vanishing output:

output row(s)	equation after the preceding relations
$9, 13$	$-v_8 - 4v_{13} = 0, \quad -v_8 + v_{13} = 0$
12	$v_{12} = -v_{14}$
15	$v_9 = 0$
14	$v_7 = 0$
1	$v_{11} = v_{14}$
8	$v_5 = 0$
0	$v_{10} = v_{15}$
5	$v_6 = v_{14}$
10	$3v_{14} = 0$
$2, 11$	$v_1 + 2v_{15} = 0, \quad v_1 + v_{15} = 0$
3	$v_4 = 0$
4	$v_3 = 0$
7	$2v_2 = 0$
6	$v_0 = 0$

The first pair gives $5v_{13} = 0$, hence $v_8 = v_{13} = 0$. The next eight rows give the displayed substitutions and then $v_{14} = 0$. The pair of rows $2, 11$ gives $v_1 = v_{15} = 0$. All sixteen coefficients therefore vanish. Thus the selected columns are independent and $d\Psi$ is onto M_4. This establishes exactly the full-rank fact used in the proof, by integer equations alone.

For completeness the determinant value can be recovered from the same elimination. Order rows as

$$(9, 13, 12, 15, 14, 1, 8, 0, 5, 10, 2, 11, 3, 4, 7, 6)$$

and order the sixteen coefficient positions as

$$(8, 13, 12, 9, 7, 11, 5, 10, 6, 14, 1, 15, 4, 3, 2, 0).$$

The first block has determinant -5. Eliminating it and each subsequent pivot gives scalar pivots

$$(-1, -1, 1, -1, -1, 1, -2, -3, 1, 1, -1, -1, 2, 1).$$

Their product with -5 is -60. Both the row permutation and the column permutation are even, so the original minor is -60.
The normal cubic in size three. At the tuple a and along the curve $a + tu + t^2v$ of Lemma 3.12, the normal functional is $h(Y) = \Psi(Y)_{21} + \Psi(Y)_{23}$. The same multiplication rule gives

$$h(a + tu + t^2v) = t^3 + \frac{31}{16}t^4 - \frac{75}{16}t^5 + \frac{975}{256}t^6.$$

In particular its first nonzero term is cubic. The differential has rank eight, not nine. The eight tangent directions and the formal correction in that lemma show that the cubic remains nonzero on the smooth fiber of the other eight target coordinates. The ensuing local-dimension and flatness argument, rather than a maximal-minor assertion, gives the open neighborhood of actual size-three values.

10.5. **Collision kernels and cyclic coefficients.** The ranks used for W_6 are consequences of the module decomposition and the cyclic-path description, rather than additional size tests. There are fifteen pair partitions and six orders of each triple of commutators. Their ninety products have disjoint supports of eight associative words, so they are linearly independent. By Proposition 5.1, the double-collision kernel is

$$\mathcal{K} \simeq 2S^{(2,1^4)} \oplus S^{(1^6)}, \qquad \dim \mathcal{K} = 2 \cdot 5 + 1 = 11.$$

Consequently the stacked collision map has rank $90 - 11 = 79$.

For the cyclic projection, the thirty products modulo cyclic permutation of their three factors form the vertices of the signed incidence graph described in the proof of Theorem 5.2. Each cyclic word belongs to the two vertices obtained from the two parities of adjacent pair cuts. Connectedness leaves a one-dimensional kernel, generated by s_6. The cyclic rank is therefore $30 - 1 = 29$. Its irreducible constituents have shapes $(3, 3), (2, 2, 1, 1), (3, 1, 1, 1), (2, 2, 2)$, all of which dominate $(2, 2, 1, 1)$. Thus double collisions detect every nonzero cyclic class, and the joint cyclic-collision kernel is exactly the cyclic kernel. In particular $\pi\mathcal{K} = 0$.

The residual alternants A_s, B_s satisfy the two relations $\sum_s A_s = \sum_s B_s = 0$. Each orbit affords $S^{(2,1^4)}$; the coefficients at 012345 and 120345, displayed in Section 5.2.1, distinguish the two copies. Their joint span has dimension ten,

and adjoining s_6 gives dimension eleven. At the exceptional $3+3$ point, the six off-diagonal values in (J) have nonzero coefficients, and its two diagonal values are independent because

$$\det \begin{pmatrix} 4 & -2 \\ -3 & 4 \end{pmatrix} = 10 \neq 0.$$

This proves the asserted rank eight directly.

The same path rule also explains the single-trace detector. A diagonal letter contributes the coordinate of the vertex at which it is inserted. Identifying vertices therefore replaces gap variables by their class sums. For four gaps, the three tree-edge ideals have product of degree at least three, whereas only two diagonal letters are available. For five gaps the five displayed class sums have nonzero determinant, and for six gaps there are no diagonal variables. These are exactly the three cases proved in Lemma 8.4. They concern polynomial identities in all diagonal parameters; the use of a three-vertex corner then works unchanged in every larger matrix algebra.

10.6. **Triangular path matrices for the standard direction.** We give a finite matrix-unit proof of the three base cases used in Section 5.2.3. It involves only triangular matrices and one four-by-four determinant.

For an ordered set $I = \{i_1 < \cdots < i_r\} \subseteq \{0, \dots, 5\}$, write $S_I = s_r(Y_{i_1}, \dots, Y_{i_r})$ and $S_\varnothing = I_n$. The following two identities determine every matrix entry needed below:

$$S_I = \sum_{j=1}^{r} (-1)^{j-1} Y_{i_j} S_{I \setminus \{i_j\}}, \tag{10.3}$$

$$d_k s_6(X) = \sum_{A \sqcup B = \{0,\dots,5\} \setminus \{k\}} \epsilon(A, k, B) S_A X S_B. \tag{10.4}$$

In the second sum both A and B are written increasingly, and $\epsilon(A, k, B)$ is the sign of their concatenation (A, k, B). The first identity groups alternating words by their first letter; the second groups them by the letters before and after the varied letter. Thus they follow directly from the definition of s_6. For matrix units, they can equivalently be read from the path rule

$$E_{a_1 b_1} \cdots E_{a_r b_r} = \left(\prod_{j=1}^{r-1} \delta_{b_j, a_{j+1}} \right) E_{a_1 b_r}.$$

These formulas give both the zero entries and the signs below.

Use precisely the three tuples Y in Section 5.2.3. The closed paths in the five lower matrix units give

$$H_3 = \operatorname{diag}(-2, -2, -1), \qquad H_4 = \operatorname{diag}(2, 1, 1, 1), \qquad H_5 = I_5.$$

For example, in the five-cycle case there is one cyclic traversal starting at each vertex; all five cyclic rotations have positive sign. In the four-cycle case the loop at vertex 1 gives two traversals starting there and one at each other vertex. In the three-cycle case the loops at vertices 1 and 2 give respectively two, two and

one traversals, all of negative sign. Since Y_0 is a hub row, a nonzero six-letter path must start with Y_0. Consequently

$$s_6(Y) = \begin{pmatrix} 0 & \mathbf{1}^{\mathsf{T}} H_q \\ 0 & 0 \end{pmatrix}.$$

Let J_n be the Jacobian at this tuple, with the last diagonal output row deleted. The row and column labels are those of Section 10.1. In this subsection the full matrix coordinates are numbered $0, \dots, n-1$, with the hub at 0. In particular, a column label $kn^2 + an + b$ means the variation $E_{a+1,b+1}$ in slot k. The following lists R_n, C_n are ordered lists, not sets. Applying the path rule to $J_n[R_n, C_n]$ gives upper triangular matrices for $n = 4, 5$:

$$\begin{aligned} R_4 &= (9, 12, 13, 14, 0, 8, 11, 4, 5, 6, 1, 2, 3, 7, 10), \\ C_4 &= (24, 12, 28, 36, 40, 8, 20, 4, 72, 68, 1, 2, 3, 52, 84), \\ \operatorname{diag} J_4[R_4, C_4] &= (2, 1, 1, 2, -1, 2, 1, 2, 2, 2, -2, -2, -1, 2, 2). \end{aligned}$$

$$\begin{aligned} R_5 &= (17, 11, 23, 12, 18, 20, 21, 22, 0, 5, 8, 10, 13, 15, \\ &\qquad 1, 2, 3, 4, 6, 7, 9, 14, 16, 19), \\ C_5 &= (65, 35, 80, 18, 12, 20, 45, 70, 90, 5, 8, 10, 85, 15, \\ &\qquad 1, 2, 3, 4, 130, 55, 105, 30, 16, 60), \\ \operatorname{diag} J_5[R_5, C_5] &= (-1, -1, -2, -1, 1, -1, -1, -1, 1, -2, -2, -1, -1, -1, \\ &\qquad 2, 1, 1, 1, -2, -2, -2, -1, 2, -1). \end{aligned}$$

For $n = 6$, take

$$\begin{aligned} R_6 &= (7, 14, 21, 28, 13, 20, 27, 34, 30, 31, 32, 33, 0, 12, 15, 16, \\ &\qquad 18, 19, 22, 24, 25, 26, 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 17, 23, 29), \\ C_6 &= (21, 28, 14, 7, 48, 90, 132, 174, 30, 66, 102, 138, 168, 12, 120, 156, \\ &\qquad 18, 54, 162, 24, 60, 96, 1, 2, 3, 4, 5, 6, 78, 114, 150, 186, 42, 84, 126). \end{aligned}$$

The matrix is block upper triangular. Its first diagonal block is

$$B = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}, \qquad \det B = -1,$$

and its other diagonal blocks are one by one, in order

$$\begin{aligned} &(-1, -1, -1, -1, -1, -1, -1, -1, 1, -1, -1, -1, -1, -1, -1, -1, \\ &\quad -1, -1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1). \end{aligned}$$

For clarity, the assertion of triangularity means that each derivative in a column of C_n has zero coordinates in all subsequent rows of R_n, apart from the displayed first four-by-four block when $n = 6$. It follows by retaining only composable paths in (10.4); no row operation or division is involved. The four-by-four determinant follows, for instance, by replacing row two by row two plus row one

and row four by row four minus row one, then expanding in the first column. The remaining three-by-three matrix is $\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$, whose determinant is -1.

The products of these diagonal blocks are respectively $1024, -256, 1$. The signs of the row permutations are $+, -, -$, and the signs of the column permutations relative to their increasing orders are all $+$. Hence the minors in increasing row and column order are $1024, 256, -1$. In particular all three differentials have rank $n^2 - 1$. This proves the finite base cases; the compatible hub argument in Section 5.2.3 proves every larger size.

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