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FOUR EXPONENTIALS AT CM AND BETA PERIODS

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ABSTRACT. For every nonzero eigen-period P of a CM abelian variety over $\overline{\mathbb{Q}}$, we show that the four exponentials conjecture and its strong form hold at $t \in \overline{\mathbb{Q}}^\times P^{\pm 1}$ and $t \in \overline{\mathbb{Q}}^\times (P/\pi)^{\pm 1}$; these include Beta values such as $t = \Gamma(1/5)^2/\Gamma(2/5)$. For an elliptic curve E over $\overline{\mathbb{Q}}$ with complex multiplication, we also show that e^{π^2} and $\wp_E(1/\pi)$ are not both algebraic. Further results concern two CM elliptic curves: for example, $\pi, \varpi, \Omega_\rho, \varpi\Omega_\rho$ are linearly independent over $\overline{\mathbb{Q}}$, where ϖ and Ω_ρ are periods of $y^2 = 4x^3 - 4x$ and $y^2 = 4x^3 - 4$. These are explicit consequences of published theorems of Waldschmidt (1988) and Roy–Waldschmidt (1993), together with classical results of Baker, Schneider, Masser and Wüstholz; the contribution is the statements and the proofs written out at these periods. Cases at specific periods, such as that 2^ϖ and 3^ϖ are not both algebraic, are checked in Lean 4 and reported separately.

1. INTRODUCTION

The *four exponentials conjecture* says: if $\log\alpha, \log\beta$ are linearly independent over $\mathbb{Q}$ and $t \notin \mathbb{Q}$, then $\alpha^t = e^{t\log\alpha}$ and $\beta^t = e^{t\log\beta}$ are not both algebraic. It is known for algebraic t (Gel'fond–Schneider) and open in general [34]. This note records the conjecture, and its strong form, at periods of CM abelian varieties and at these periods divided by π; among them are Beta values. For example:

- 2^ϖ and 3^ϖ are not both algebraic, nor are $2^{\varpi/\pi}$ and $3^{\varpi/\pi}$, where $\varpi = \Gamma(1/4)^2/(2\sqrt{2\pi}) = 2.62205755\ldots$ is the real period of $y^2 = 4x^3 - 4x$ (Corollaries 4.4 and 4.2);
- 2^t and 3^t are not both algebraic at $t = \Gamma(1/5)^2/\Gamma(2/5) = B(\frac{1}{5}, \frac{1}{5})$ and at $t = B(\frac{1}{5}, \frac{1}{5})/\pi$ (Theorem 1.1);
- $(2e)^\varpi$ and 3^ϖ are not both algebraic (Corollary 5.6), and at least one of e^{π^2}, $e^{\pi\varpi}$, $e^{\pi^2/\varpi}$ is transcendental (Corollary 4.7);
- $\wp(\pi/\varpi^2)$ and $\wp(\pi u/\varpi^2)$ are not both algebraic, for $\wp$ the Weierstrass function of $y^2 = 4x^3 - 4x$ and $u \neq 0$ any elliptic logarithm of an algebraic point, e.g. of $(2, \sqrt{24})$ (Example 5.10);
- $\pi, \varpi, \Omega_\rho, \varpi\Omega_\rho$ are linearly independent over $\overline{\mathbb{Q}}$, where $\Omega_\rho = \Gamma(1/3)^3/(2^{4/3}\pi)$ is a period of $y^2 = 4x^3 - 4$ (Corollary 6.7);
- one of e^{π^2}, $\wp(1/\pi)$ is transcendental (Theorem 7.1).

All of this is an explicit application of published general theorems: Waldschmidt's algebraic subgroup theorem with period term [27, Thm. 4.1], its hyperplane case [27, §5d], and the theorem of Roy and Waldschmidt [22, Théorème (iii)], which puts these in a form with number-field

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coefficients (§2). On CM elliptic curves the dimension bounds below are essentially inequality (iii) as printed. On a CM abelian variety of dimension $g \geq 2$ read through one eigen-coordinate they follow from the same count with the weight of the actual quotient group, an inequality inside the proof of [22, p. 364]; we argue from [27, Thm. 4.1] directly on that quotient. We claim no new method. What this note contributes is the explicit statements, the proofs written out at these periods (the group, the map, the line, the period datum, the quotient, and the exclusion of every algebraic subgroup) and a formal verification of many of the statements in Lean 4 (§9; see also [12, §12]). As far as we found, the statements at real t with pairs such as $(2, 3)$ are not in print; §9.1 says what we read and what we could not consult.

Statements. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, $\mathcal{L} = \{z \in \mathbb{C} : e^z \in \overline{\mathbb{Q}}^\times\}$ and $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$. For an elliptic curve $E\colon y^2 = 4x^3 - g_2x - g_3$ with $g_2, g_3 \in \overline{\mathbb{Q}}$ we write $\wp, \zeta$ for its Weierstrass functions, Λ_E for its period lattice, $\eta(\lambda) = \zeta(z+\lambda) - \zeta(z)$ for the quasi-periods, and $\mathcal{L}_E = \{u \in \mathbb{C} : u \in \Lambda_E \text{ or } \wp(u) \in \overline{\mathbb{Q}}\}$. We use two curves throughout:

$$E_i\colon y^2 = 4x^3 - 4x, \qquad \Lambda = \varpi\,\mathbb{Z}[i], \qquad \varpi = \tfrac{\Gamma(1/4)^2}{2\sqrt{2\pi}}, \qquad \eta(\varpi) = \tfrac{\pi}{\varpi};$$
$$E_\rho\colon y^2 = 4x^3 - 4, \qquad \Lambda = \Omega_\rho\mathbb{Z}[\rho], \qquad \Omega_\rho = \tfrac{\Gamma(1/3)^3}{2^{4/3}\pi}, \qquad \eta(\Omega_\rho) = \tfrac{2\pi}{\sqrt{3}\,\Omega_\rho}.$$

Let $A/\overline{\mathbb{Q}}$ be a simple abelian variety with complex multiplication by a CM field F of degree $2g$, with CM type Φ. For an embedding σ of F, an *eigen-differential* is a class $\omega_\sigma \in H^1_{\mathrm{dR}}(A/\overline{\mathbb{Q}})$ with $[x]^*\omega_\sigma = \sigma(x)\omega_\sigma$; it is of the first kind (holomorphic) iff $\sigma \in \Phi$. An *eigen-period* is a nonzero number $P_\sigma = \int_\gamma \omega_\sigma$, $\gamma \in H_1(A, \mathbb{Z})$. By Riemann's bilinear relations $P_\sigma P_{\bar\sigma} \in \pi\overline{\mathbb{Q}}^\times$. For a CM abelian variety that is not simple, every nonzero eigen-period is an algebraic multiple of an eigen-period of a simple factor, of the same kind (§3.1). For $N \geq 3$ and $0 < a, b < N$ with $a + b \neq N$, the Beta value $B(a/N, b/N) = \Gamma(a/N)\Gamma(b/N)/\Gamma((a+b)/N)$ is, up to $\overline{\mathbb{Q}}^\times$, an eigen-period of a CM abelian variety (a factor of the Jacobian of the Fermat curve of degree N), of the first kind iff $a + b < N$ (§3.1).

Theorem 1.1 (Theorem A)**.** *Let P be a nonzero eigen-period of a CM abelian variety over $\overline{\mathbb{Q}}$, let $c \in \overline{\mathbb{Q}}^\times$ and $t \in \{cP,\ c/P,\ cP/\pi,\ c\pi/P\}$. Let $\alpha, \beta \in \overline{\mathbb{Q}}^\times$, fix logarithms $\log\alpha$, $\log\beta$ and put $\alpha^t = e^{t\log\alpha}$, $\beta^t = e^{t\log\beta}$. If $\log\alpha, \log\beta$ are linearly independent over $\mathbb{Q}$, then α^t and β^t are not both algebraic.*

For $g = 1$ the eigen-periods are the periods Ω of a CM curve and their conjugate-eigen quasi-periods, which lie in $\overline{\mathbb{Q}}^\times\pi/\Omega$ (§3.2), so the classes are $\overline{\mathbb{Q}}^\times\Omega^{\pm1}$ and $\overline{\mathbb{Q}}^\times(\Omega/\pi)^{\pm1}$. In particular Theorem 1.1 holds at $t = B(a/N, b/N)^{\pm1}$ and $t = (B(a/N, b/N)/\pi)^{\pm1}$ for $0 < a, b < N$, $a + b \neq N$; there 2^t, 3^t are not both algebraic for any choice of $\log 2$, $\log 3$, since any two such choices are $\mathbb{Q}$-independent.

The *strong four exponentials conjecture* [31, Conj. 1.5] says: if y_1, y_2 are linearly independent over $\overline{\mathbb{Q}}$ and $t \notin \overline{\mathbb{Q}}$, then y_1, y_2, ty_1, ty_2 are not all in $\widetilde{\mathcal{L}}$. Roy's strong six exponentials theorem [20] is the analogous statement with three numbers y_j.

Theorem 1.2 (Theorem S-A)**.** *Let $P \neq 0$ be an eigen-period, of the first or of the second kind, of a CM abelian variety over $\overline{\mathbb{Q}}$, and let $t \in \overline{\mathbb{Q}}^\times P \cup \overline{\mathbb{Q}}^\times P^{-1} \cup \overline{\mathbb{Q}}^\times P/\pi \cup \overline{\mathbb{Q}}^\times \pi/P$. If $y_1, y_2 \in \mathbb{C}$ are linearly independent over $\overline{\mathbb{Q}}$, then y_1, y_2, ty_1, ty_2 are not all in $\widetilde{\mathcal{L}}$. In particular this holds for $t \in \overline{\mathbb{Q}}^\times B(a/N, b/N)^{\pm1}$ and $t \in \overline{\mathbb{Q}}^\times(B(a/N, b/N)/\pi)^{\pm1}$ $(0 < a, b < N,\ a + b \neq N)$.*

The same count, read on a CM curve instead of $\mathbb{G}_{\mathrm{m}}$, gives strong elliptic four exponentials at $t \in \overline{\mathbb{Q}}^\times(\pi/\Omega^2)^{\pm1}$ (Corollary 5.8), and on two non-isogenous CM curves a bilinear statement with no logarithm of $\mathbb{G}_{\mathrm{m}}$ at all (Theorem 6.3).

Part of Theorems 1.1 and 1.2 also follows from the six exponentials theorem, Roy's strong six exponentials theorem and complex conjugation (Diaz [7]; Waldschmidt [33, Cor. 2.7, Thm. 4.1]). Every class $\overline{\mathbb{Q}}^{\times}t$ occurring in them contains a real number (one has $\bar{P} \in \overline{\mathbb{Q}}^{\times}P$), and these arguments together cover every pair at every $t \notin \mathbb{R} \cup i\mathbb{R}$, and, at every t, every pair whose logarithms span a space that is not stable under complex conjugation. They do not apply to real or purely imaginary t — such as $t = \varpi$, ϖ/π, Ω_ρ, or the Beta values themselves — with pairs whose logarithms span a conjugation-stable space, such as $(2, 3)$, which is where our examples lie, nor to Corollary 4.7. Under the period conjecture these relations are not linear among periods, so we do not expect them from the analytic subgroup theorem alone.

Unless stated otherwise, every statement is proved here given the published theorems cited. §8 lists further results without proof; §9 says exactly which statements are checked in Lean and relative to what.

Every result here is a consequence of Waldschmidt's Theorem 4.1 as printed. Every one of them has $W \neq 0$, so each also relies on the one step of its printed proof that [27] indicates by reference to other papers (§2).

2. The published theorems

2.1. **Waldschmidt's subgroup theorem.**

Theorem 2.1 ([27, Thm. 4.1, pp. 385–386])**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2$ be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ of dimension $d = d_0+d_1+d_2 \geq 1$, with projections π_0, π_1 onto the first two factors (d_0, d_1 need not be maximal). Let $V \subset T_G(\mathbb{C})$ be a subspace of dimension $N < d$, $W \subset V$ a subspace defined over $\overline{\mathbb{Q}}$, and $Y \subset V$ a finitely generated subgroup with $\Gamma = \exp_G Y \subset G(\overline{\mathbb{Q}})$; put $\kappa = \mathrm{rk}_{\mathbb{Z}}(Y \cap \ker \exp_G)$. Then there is a connected algebraic subgroup $G' \neq G$ defined over $\overline{\mathbb{Q}}$ such that, with $\delta = \dim G/G'$, $\delta_0 = \dim \mathbb{G}_{\mathrm{a}}^{d_0}/\pi_0(G')$, $\delta_1 = \dim \mathbb{G}_{\mathrm{m}}^{d_1}/\pi_1(G')$, $\delta_2 = \delta - \delta_0 - \delta_1$, $\lambda = \mathrm{rk}_{\mathbb{Z}} \Gamma/(\Gamma \cap G')$ and $\tau = \dim W/(W \cap T_{G'})$, one has $\delta > \tau$ and*

$$(\lambda + \delta_1 + 2\delta_2)(d - N) \leq (\delta - \tau)(d_1 + 2d_2 - \kappa). \tag{1}$$

($G' = 0$ is allowed.) The proof [27, §§6–7] combines an auxiliary function [27, Prop. 6.1] with Philippon's zero estimate [18]. Prop. 6.1 asks for a polynomial that vanishes with multiplicity $\geq T$ along W at the points of $\exp_G(Y(S))$, with the saving κ in the parameters. Waldschmidt says [27, p. 396] that this is proved in [26, Prop. 2.4] when $W = 0$ and the exponent b is 1, and that the estimates for the derivatives are provided by Bertrand's Lemma 7 [25, App. I], and he says to compare with Bertrand [2] and Philippon–Waldschmidt [19]. So for $W \neq 0$ the printed proof only indicates this step. We read [19], which does not state it in this generality (W is a hyperplane there, the group of points is cyclic, and the count has no saving from periods); we could not read [2], and we did not determine whether it covers the step. All our applications have $W \neq 0$, and all but Theorem 7.1 have $\kappa > 0$. We use Theorem 2.1 as printed, and record here how far this one step has been checked independently. Put

$$2N \geq d_1 + 2d_0 + \kappa. \tag{2}$$

This is not a hypothesis of Theorem 2.1; it is the range in which this step has been checked outside print. For $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$, with E_j elliptic curves, Theorem 2.1 under (2) is proved in Lean [11], and so are restricted forms of it for some vector extensions and for the Jacobian of $y^5 = x(1 - x)$ (§9). For other groups, in particular for CM abelian varieties of dimension ≥ 2 in general, we rely on the printed theorem. No human expert has checked this step. Every direct application of Theorem 2.1 below satisfies (2), and each such proof says so. Theorem 6.1 and Proposition 6.6 use Theorem 2.1 through §5d of [27], which applies it also to an auxiliary subgroup; there we rely on §5d as printed (see the proof of Proposition 6.6).

Theorem 2.1 is stated for any d_0 (p. 385); Waldschmidt himself applies its consequence [27, Thm. 1.1] with $d_0 = 2$, $W \neq 0$, $\kappa = 0$ in the proof of [27, Cor. 2.1]. When $d_0 \geq 1$ the printed proof uses an exponent $b > 1$ in the case $\delta = \tau$ (p. 397), which belongs to the same step.

2.2. **The hyperplane case.**

Theorem 2.2 ([27, Thm. 1.1, pp. 375–376, under (5.4) of §5d, p. 394])**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2$ be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ of dimension $d \geq 1$, with d_0, d_1 maximal and $d_2 = d - d_0 - d_1$. Let V be a hyperplane of $T_G(\mathbb{C})$, $W \subset V$ a subspace of dimension $t \geq 0$ defined over $\overline{\mathbb{Q}}$, and $Y = \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_m \subset V$ a subgroup of rank m with $\exp_G Y \subset G(\overline{\mathbb{Q}})$. Put $\kappa = \mathrm{rk}_{\mathbb{Z}}(Y \cap \ker \exp_G)$. If*

$$m > (d_1 + 2d_2 - \kappa)(d - 1 - t), \tag{5.4}$$

then V contains a nonzero algebraic Lie subalgebra of $T_G(\mathbb{C})$ defined over $\overline{\mathbb{Q}}$, that is, $T_{G'} \subset V$ for some connected algebraic subgroup $G' \neq 0$ of G defined over $\overline{\mathbb{Q}}$.

The printed Theorem 1.1 has the hypothesis $m > (d_1 + 2d_2)(d - 1 - t)$, without κ; [27, §5d] proves it under the weaker hypothesis (5.4) (p. 394 prints $\kappa = \ell - m$, a misprint for $m - \ell$). That proof applies Theorem 2.1 to G and then again to an auxiliary subgroup [27, Lemma 5.1]. We use Theorem 2.2 directly only for Proposition 6.6; the Lean proofs of §9 go through it, with the instances they use proved there.

2.3. **The theorem of Roy and Waldschmidt.** Let G be a connected commutative algebraic group over $\overline{\mathbb{Q}}$, T_G its tangent space at the origin with its $\overline{\mathbb{Q}}$-structure, $\Omega(G) = \ker \exp_G$ its periods and $\Lambda(G) = \exp_G^{-1}(G(\overline{\mathbb{Q}}))$ the logarithms of its algebraic points. For a number field K of degree n, $\Omega_K(G)$ and $\Lambda_K(G)$ are the K-spans of $\Omega(G)$ and $\Lambda(G)$ in $T_G(\mathbb{C})$. Fix a basis $\theta = (\theta_1, \ldots, \theta_n)$ of $K/\mathbb{Q}$ and let $p_\theta \colon T_{G^n} = T_G^n \to T_G$, $(z_j) \mapsto \sum_j \theta_j z_j$. Let $d_0(G)$ (resp. $d_1(G)$) be the largest d with a surjection $G \to \mathbb{G}_{\mathrm{a}}^d$ (resp. $G \to \mathbb{G}_{\mathrm{m}}^d$) over $\overline{\mathbb{Q}}$, and $\alpha(G) = 2\dim G - 2d_0(G) - d_1(G)$. Let $\mathcal{E}$ be a finite-dimensional complex vector space with a $\overline{\mathbb{Q}}$-structure, $\Psi \colon T_G(\mathbb{C}) \to \mathcal{E}$ a $\overline{\mathbb{Q}}$-rational linear map, and $\Psi_\theta = \Psi \circ p_\theta$. Let H_0 (resp. H_Ψ) be the largest connected algebraic subgroup of G^n with $p_\theta(T_{H_0}) = 0$ (resp. $T_{H_\Psi} \subset \ker \Psi_\theta$), and put

$$\alpha_K(G) = \alpha(G^n/H_0)/n \quad [22, \text{ p. } 360], \qquad \alpha_{K,\Psi}(G) = \alpha(G^n/H_\Psi)/n;$$

neither depends on θ (a change of basis is an automorphism of G^n), and $\alpha_{K,\Psi}(G) \leq \alpha_K(G)$ [22, p. 364]. Finally $S_K(G)$ is the set of subspaces $p_\theta(T_{G'})$ of T_G, for G' an algebraic subgroup of G^n.

Theorem 2.3 ([22, Théorème, (i)⇒(iii), p. 363])**.** *Let $V \subset \mathcal{E}$ be a complex subspace containing no nonzero $\Psi(S)$, $S \in S_K(G)$, and $W \subset V$ a subspace defined over $\overline{\mathbb{Q}}$. Put $\kappa = \dim_K\big(V \cap \Psi(\Omega_K(G))\big)$. Then*

$$\dim_K\big(V \cap \Psi(\Lambda_K(G))\big) \leq (\alpha_K(G) - \kappa)\dim_{\mathbb{C}}(V/W).$$

By [22, Remarque 1], the condition on V holds whenever V has no nonzero $\overline{\mathbb{Q}}$-rational point (and then $W = 0$); this is the only case we use. The proof has two steps. Step (b) (p. 364) reduces to $K = \mathbb{Q}$, $\Psi = \mathrm{id}$ on G^n/H_Ψ; it is written in full, and the inequality obtained before its last line has $\alpha_{K,\Psi}(G)$ in place of $\alpha_K(G)$. Step (a) (p. 363) is the case $K = \mathbb{Q}$, derived from [27, Thm. 4.1] through Roy's method [21]: Roy shows that the statements of his Proposition 2.1 hold with the period-carrying function a' [21, p. 266], and step (a) outlines how the bound follows from them. For hyperplanes, however, [22, Remarque 2] cites step (a) from [27, Thm. 1.1], i.e. Theorem 2.2; in all our uses V is a line in $\mathbb{C}^2$ and Ψ is onto, so the subspace on G^n/H_Ψ is a hyperplane, and we use nothing from [21]. (Step (b) also applies (a) to $\ker \Psi'$, to get equalities

in its (4); the inequality we use needs only the surjections there, and that instance is in any case Wüstholz's theorem.)

2.4. **The shape of the arguments.** Every result of §§4–6 has the following shape. There are a group G (a product of copies of $\mathbb{G}_\mathrm{a}$, $\mathbb{G}_\mathrm{m}$, CM elliptic curves, CM abelian varieties and their universal vector extensions), a $\overline{\mathbb{Q}}$-rational map $\Psi\colon T_G \to \mathbb{C}^2$ (onto), a line $V = \mathbb{C}(1, s)$ (or $\mathbb{C}(p_1, p_2)$) whose slope is transcendental, and $W = 0$. So the condition of Theorem 2.3 holds by [22, Remarque 1], and $\dim V/W = 1$. A period datum w on V gives $\kappa \geq 1$ (for Propositions 6.4 and 6.6, two Legendre relations give $\kappa \geq 2$), and a counterexample to the statement gives further vectors on V, K-independent of the period data. Their total number is bounded by $\alpha_{K,\Psi}(G) - \kappa$, with $\alpha_{K,\Psi}(G) = \alpha(G'')/n$ for the quotient $G'' = G^n/U$ constructed in each proof ($U = H_\Psi$ there). The table lists the numbers (for $n = [K : \mathbb{Q}]$ large enough; "data" counts the period data together with the vectors a counterexample would give).

result	G''	$\alpha(G'')/n$	κ	data	bound
Thms. 4.1, 4.5 (MG, MG-A)	$\mathbb{G}_\mathrm{m}^{2n} \times A''$	3	1	3	2
Thms. 4.3, 4.6 (MG$^\natural$, MG$^\natural$-A)	$\mathbb{G}_\mathrm{m}^{2n} \times Q''$	$3 + 2/n$	1	3	$2 + 2/n$
Thm. 5.2(a), Thm. 5.5 (SMG, SMG-A)	$\mathbb{G}_\mathrm{a}^2 \times \mathbb{G}_\mathrm{m}^{2n} \times A''$	3	1	3	2
Thm. 5.2(b), SMG$^\natural$-A	$\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}^{2n} \times Q''$	$3 + 2/n$	1	3	$2 + 2/n$
Thm. 5.7 (SE$_0$)	$\mathbb{G}_\mathrm{a} \times A_1 \times A_2 \times Q$	$3 + 2/n$	1	3	$2 + 2/n$
Prop. 6.4 (MX, Cor. 6.7(ii))	$\mathbb{G}_\mathrm{a} \times G_\ell$, $\dim = 2n + 2$	$4 + 2/n$	2	3	$2 + 2/n$
Prop. 6.6 (Cor. 6.7(i))	extension of $E_i^n \times E_\rho^n$ by $\mathbb{G}_\mathrm{a}^2$	$4 + 4/n$	2	3	$2 + 4/n$

In each row the bound is below the number of data for n large enough, which is the contradiction. Here A'' is an abelian variety of dimension $n/2$ *for every* g: reading one eigen-coordinate kills all but $n/2$ dimensions of A^n (Lemma 3.1). That is the point of the abelian case.

For $G = \mathbb{G}_\mathrm{m}^2 \times E$ the printed weight is $\alpha_K(G) = 3$, and Theorem 2.3 gives the bound 2 directly (likewise in the row SMG). In the rows of Propositions 6.4 and 6.6, $\alpha_K(E^\natural) = 2+2/n$ (the largest subgroup of $(E^\natural)^n$ killed by p_θ is a hyperplane of the fibre), so $\alpha_K(E_i^\natural \times E_\rho^\natural \times \mathbb{G}_\mathrm{a}^2) = 4+4/n$, which gives $2 + 4/n < 3$ once $n > 4$. In the other fibre rows the printed weight appears to suffice, using $\Lambda_K(E^\natural) = K\mathcal{L}_E \times K \cdot \mathrm{Fib}$ to read $E^\natural$ through both base and fibre (for Theorem 4.3: $\alpha_K = 4 + 2/n$, $\kappa = 1$, four data including $(\Omega, s\Omega)$). We have only sketched this, and we do not use it. For A simple of dimension $g \geq 2$, however, $\alpha_K(\mathbb{G}_\mathrm{m}^2 \times A) = 2 + g$ and the printed bound gives no contradiction with three data, whereas $\alpha(G'')/n = 3$: there the count needs the inequality inside step (b). In all cases we argue directly: we apply Theorem 2.1 once, to G'', and exclude every connected algebraic subgroup by an explicit case check, with (2) checked at G''. Only Theorem 6.1 (a one-line use of Theorem 2.3) and Proposition 6.6 (Theorem 2.2) are argued otherwise. In these applications V'' is a hyperplane, $N = d - 1$, and $W'' = \ker \Psi''$, so that $\tau = \delta + r' - 2$ where $r' = \dim \Psi''(T_{G'})$; thus only G' with $r' \leq 1$ can satisfy $\delta > \tau$, and then $\delta - \tau = 2 - r'$. Theorem 7.1 is the one case with V'' of codimension 2.

3. CM abelian varieties and universal vector extensions

3.1. **CM abelian varieties.** Let $A/\overline{\mathbb{Q}}$ be a simple abelian variety of dimension g with CM by the CM field F, $[F : \mathbb{Q}] = 2g$, with (primitive) CM type Φ. For $\sigma \in \Phi$ let $z_\sigma\colon \mathrm{Lie}(A) \to \mathbb{C}$ be the $\overline{\mathbb{Q}}$-rational eigen-coordinate dual to a holomorphic eigen-differential ω_σ, so that $z_\sigma([x]v) = \sigma(x) z_\sigma(v)$ for $x \in \mathcal{O}_F$. Since $H_1(A, \mathbb{Q}) = F\gamma_0$ for any $0 \neq \gamma_0 \in \Omega(A)$, we have $z_\sigma(x\gamma_0) = \sigma(x)P_\sigma$ with $P_\sigma = z_\sigma(\gamma_0)$, so $z_\sigma(\Omega(A)) \subset \sigma(F)P_\sigma$.

Reduction to the simple case. A CM abelian variety over $\overline{\mathbb{Q}}$ is isogenous to $\prod_i B_i^{m_i}$ with B_i simple CM abelian varieties of primitive type, with CM fields F_i. Eigen-differentials are taken

with respect to a CM algebra (any choice; the argument does not depend on it), a commutative semisimple $\mathbb{Q}$-subalgebra of $\mathrm{End}^0(A)$ of degree $2\dim A$, maximal commutative, so containing the central idempotents of $\mathrm{End}^0(A)$. An eigen-character is nonzero on exactly one of these, and restricts to an embedding σ_0 of F_i, which acts on $B_i^{m_i}$ diagonally. So an eigen-differential is, on that part, a $\overline{\mathbb{Q}}$-combination of pull-backs $\mathrm{pr}_k^*\omega_{\sigma_0}$ of one σ_0-eigen differential of B_i, of the same kind; every nonzero eigen-period lies in $\overline{\mathbb{Q}}^\times P_{\sigma_0}(B_i)$, and it suffices to prove Theorem 1.1 for simple A of primitive type.

Lemma 3.1 (Lemma K′)**.** *Let $F \subset K$, where K is a number field containing the Galois closure of F, $n = [K : \mathbb{Q}]$, and let $\theta_1, \dots, \theta_n$ be a basis of $K/\mathbb{Q}$. For an embedding $\sigma\colon F \to K$ put $\psi_\sigma(m) = \sum_j \theta_j \sigma(m_j)$ on F^n. Then $(\psi_\sigma)_\sigma\colon F^n \to K^{2g}$ is a $\mathbb{Q}$-linear bijection, each ψ_σ is σ-semilinear and onto K, and for every set Σ of embeddings the common kernel of $(\psi_\sigma)_{\sigma\in\Sigma}$ is an F-subspace of F-dimension $n - |\Sigma|\, n/(2g)$.*

Proof. $F^n \cong F \otimes_{\mathbb{Q}} K$ via $m \mapsto \sum m_j \otimes \theta_j$, and $F \otimes_{\mathbb{Q}} K \cong K^{2g}$ via $x \otimes y \mapsto (\sigma(x)y)_\sigma$, since K contains the Galois closure; the composite is $(\psi_\sigma)_\sigma$. Semilinearity is clear. The common kernel is the preimage of $\{0\}^\Sigma \times K^{2g-|\Sigma|}$, of $\mathbb{Q}$-dimension $(2g - |\Sigma|)n$, i.e. of F-dimension $(2g - |\Sigma|)n/(2g)$. □

For $g = 1$, $F = k$ is an imaginary quadratic field, ψ_{id} and ψ_- are the maps ψ, ϕ below, and Lemma 3.1 becomes the following lemma, which we use throughout for CM elliptic curves.

Lemma 3.2. *Let $k \subset K$ be an imaginary quadratic field inside a number field of degree n, $\theta_1, \dots, \theta_n$ a basis of $K/\mathbb{Q}$, and on k^n put $\psi(m) = \sum \theta_j m_j$, $\phi(m) = \sum \theta_j \bar{m}_j$. Then $(\psi, \phi)\colon k^n \to K \times K$ is a $\mathbb{Q}$-linear bijection;* $\ker\psi$ *and* $\ker\phi$ *are k-subspaces of dimension $n/2$, and* $\ker\psi \cap \ker\phi = 0$.

An *eigen-slot* is a pair (A, σ) with $A/\overline{\mathbb{Q}}$ a simple CM abelian variety and σ an embedding of its CM field; its periods are the eigen-periods P_σ of ω_σ (on $A^\natural$ if $\sigma \notin \Phi$). Two eigen-slots are equal iff the abelian varieties are isogenous and the embeddings correspond under the isogeny. An eigen-slot is of the first kind iff $\sigma \in \Phi$, and of the second kind otherwise. We use the following standard facts.

- (S1) (Standard; see Shimura [24].) $\mathrm{End}^0(A^n) = M_n(F)$, and Hom between non-isogenous simple abelian varieties is 0. Abelian subvarieties of A^n are the B_S, $S \subset F^n$ an F-subspace, with $\dim B_S = g \dim_F S$ and $T(B_S) = \{(\sigma(M)u_\sigma)_{\sigma\in\Phi} : u_\sigma \in \mathbb{C}^{\dim S}\}$ in the eigen-coordinates (M a matrix whose columns span S). On $T(B_S)$ the form $\theta^{\mathsf{T}} z_\sigma$ vanishes iff $\psi_\sigma(S) = 0$, and different $\sigma \in \Phi$ carry independent u_σ.
- (P) Every connected algebraic subgroup of $\mathbb{G}_{\mathrm{a}}^a \times \mathbb{G}_{\mathrm{m}}^b \times B$, B an abelian variety, is a product $(H' \cap \mathbb{G}_{\mathrm{a}}^a) \times (H' \cap \mathbb{G}_{\mathrm{m}}^b) \times B'$; subgroups of $\mathbb{G}_{\mathrm{a}}^a$ are $\overline{\mathbb{Q}}$-subspaces and of $\mathbb{G}_{\mathrm{m}}^b$ rational subtori. (The image of H' in B is an abelian subvariety B', and $H'/(H' \cap M)$, M the linear part, maps to the linear group $M/(H' \cap M)$ trivially; so $H' = (H' \cap M)^0 \times B'$ by dimension, and $H' \cap M$ splits since $\mathrm{Hom}(\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{m}}) = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{a}}) = 0$.)
- (W) (Wüstholz [36].) The numbers 1, π and nonzero periods of pairwise distinct eigen-slots (first or second kind, on one or several simple CM abelian varieties) are linearly independent over $\overline{\mathbb{Q}}$. In particular $P_\sigma \notin \overline{\mathbb{Q}}$ and $P_\sigma/\pi \notin \overline{\mathbb{Q}}$. A nonzero eigen-period of A (resp. a second-kind eigen-period of $A^\natural$) is not in $\overline{\mathbb{Q}}\mathcal{L}$.

All of these follow from Wüstholz's theorem on $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}} \times \prod_i A_i^\natural$ (resp. with $\mathbb{G}_{\mathrm{m}}^r$ added), applied to the point with coordinates 1, $2\pi i$ (resp. the logarithms) and the cycles. For example, $P/\pi \notin \overline{\mathbb{Q}}$: otherwise the point $(\gamma_0, 2\pi i)$ of $\mathrm{Lie}(A \times \mathbb{G}_{\mathrm{m}})$ lies in a $\overline{\mathbb{Q}}$-hyperplane $z_\sigma = a\, x$; the smallest algebraic subgroup whose Lie algebra contains it is $A \times \mathbb{G}_{\mathrm{m}}$ (A is simple and the subgroup is a product by (P)), whose Lie algebra is not in the hyperplane.

Universal vector extension. For $\sigma \notin \Phi$ the eigen-coordinate w_σ lives on $\mathrm{Lie}(A^\natural)$. We use the following facts. The input from the literature is (Mazur–Messing [17]): $\mathrm{Lie}(A^\natural) = H^1_{\mathrm{dR}}(A)^\vee$ functorially, and $A^\natural$ is universal among extensions of A by vector groups (every such extension of B by U is the push-out of $B^\natural$ along a unique linear map $W_B \to U$, where W_B is the fibre of $B^\natural$). The rest is proved by the one-line arguments given.

(F1) $\mathrm{Lie}(A^\natural) = H^1_{\mathrm{dR}}(A)^\vee$ with functorial $\mathcal{O}_F$-action. In the dual eigenbasis (e_τ) the fibre Lie algebra is $\bigoplus_{\tau\notin\Phi} \mathbb{C}e_\tau$, and w_σ ($\sigma \notin \Phi$) is nonzero exactly on e_σ, which lies in the fibre. *Proof:* the Lie algebra sequence of $0 \to V \to A^\natural \to A \to 0$ is the dual of $0 \to H^0(A,\Omega^1) \to H^1_{\mathrm{dR}}(A) \to H^1(A,\mathcal{O}) \to 0$, so the fibre is the annihilator of $H^0(A,\Omega^1) = \bigoplus_{\tau\in\Phi} \overline{\mathbb{Q}}\omega_\tau$.

(F2) For $B \subset A^n$, $B^\natural \to (A^n)^\natural$ is injective on fibres and $\pi^{-1}(B) = B^\natural + V^n$ (V the fibre). *Proof:* on fibres the map is dual to the restriction $H^1(A^n,\mathcal{O}) \to H^1(B,\mathcal{O})$, which is onto because B is a direct factor of A^n up to isogeny (Poincaré reducibility); and $B^\natural$ maps onto B.

(F3) $\mathrm{Ext}(B,\mathbb{G}_{\mathrm{a}}) \cong H^1(B,\mathcal{O})$ and $\mathrm{Hom}(B,\mathbb{G}_{\mathrm{a}}) = 0$: a connected H over B with given intersection W' with the fibre exists iff $W' \supseteq W_B$, and then $H = B^\natural + W'$ is unique. *Proof:* such an H is a splitting over B of $\pi^{-1}(B)/W'$, which by (F2) and universality is the push-out of $B^\natural$ along $W_B \to V^n/W'$; it splits iff this map is 0, i.e. $W_B \subseteq W'$, and the splitting is unique since $\mathrm{Hom}(B, V^n/W') = 0$.

(F4) $T(B^\natural_S) = \bigoplus_\tau \tau(S)\otimes\mathbb{C}$ and $W_{B_S} = \bigoplus_{\tau\notin\Phi} \tau(S)\otimes\mathbb{C}$. *Proof:* under the comparison isomorphism $H^1_{\mathrm{dR}}(B_S)^\vee \cong H_1(B_S,\mathbb{Q})\otimes\mathbb{C}$, with $H_1(B_S,\mathbb{Q}) = S\gamma_0 \subset F^n\gamma_0$, the τ-eigen component is $\tau(S)\otimes\mathbb{C}$; the fibre part is $\tau \notin \Phi$ by (F1).

The periods of $A^\natural$ satisfy $\Omega(A^\natural)\otimes\mathbb{Q} = F\tilde{\gamma}_0$ with $w_\sigma(x\tilde{\gamma}_0) = \sigma(x)P_\sigma$, and the values $w_\sigma(\Lambda(A^\natural))$ ("σ-fibre values") are second-kind abelian integrals $\int \omega_\sigma + \overline{\mathbb{Q}}$.

Beta values (Weil [35]; Rohrlich's appendix to [9]). For $N \geq 3$ and $0 < a, b < N$ with $a + b \not\equiv 0 \bmod N$, $B(a/N, b/N)$ is, up to $\overline{\mathbb{Q}}^\times$, the period of an eigen-differential on the Fermat curve of degree N; the corresponding isotypic factor of the Jacobian is a power of a simple CM abelian variety, and the differential is of the first kind iff $a + b < N$.

3.2. **CM elliptic curves and their universal vector extensions.** For a CM elliptic curve E let $E^\natural$ be its universal vector extension, with base coordinate z and conjugate-eigen fibre coordinate w: $\mathcal{O} = \mathrm{End}(E)$ acts on $T(E^\natural) = H^1_{\mathrm{dR}}(E)^\vee$ by its identity character on z and by the conjugate character on w (the eigenline of H^1_{dR} for $\bar{a}$ is spanned by $x\,dx/y + c\,dx/y$ with $c \in \overline{\mathbb{Q}}$, so these coordinates are rational; this is (F1) for $g = 1$). In these coordinates the periods are $(\lambda, \eta^*(\lambda))$, $\lambda \in \Lambda_E$, where $\eta^*(\lambda) = \eta(\lambda) - \kappa_1\lambda$ satisfies $\eta^*(a\lambda) = \bar{a}\,\eta^*(\lambda)$; the logarithms of points of $E^\natural(\overline{\mathbb{Q}})$ are $(u, \zeta^*(u) + t)$ with $u \in \mathcal{L}_E$, $t \in \overline{\mathbb{Q}}$ and $\zeta^* = \zeta - \kappa_1\,\mathrm{id}$. On E_i and E_ρ, $\kappa_1 = 0$. Since the characters $a \mapsto a, \bar{a}$ of k are K-independent for $K \supseteq k$,

$$\Omega_K(E^\natural) = K\lambda_0 \times K\eta^*(\lambda_0) \qquad (\Lambda_E\otimes\mathbb{Q} = k\lambda_0). \tag{3}$$

We call *fibre values* the w-coordinates of $\Lambda(E^\natural)$, that is, $\zeta^*(u)+\overline{\mathbb{Q}}$ for $u \in \mathcal{L}_E\backslash\Lambda_E$ and $\eta^*(\lambda)+\overline{\mathbb{Q}}$ at lattice points, and write Fib for the set of fibre values. By Legendre and $\eta^*(a\lambda) = \bar{a}\eta^*(\lambda)$, $\Omega\eta^*(\Omega) \in \overline{\mathbb{Q}}^\times\pi$ for every CM period Ω.

4. Four exponentials at CM eigen-periods

This section proves Theorem 1.1: §§4.1–4.2 treat a CM elliptic curve and §4.3 one eigen-coordinate of a CM abelian variety, with the same case checks.

4.1. **Theorem MG.** Throughout this section $E/\overline{\mathbb{Q}}$ has complex multiplication by k, $\Omega \neq 0$ is a period of E, and for a number field K we write $K\mathcal{L}$, $K\mathcal{L}_E$ for the K-spans of $\mathcal{L}$ and $\mathcal{L}_E$. By Baker's theorem $K\mathcal{L} \cong K \otimes_{\mathbb{Q}} \mathcal{L}$, so elements of $\mathcal{L}$ that are $\mathbb{Q}$-linearly independent are K-linearly independent.

Theorem 4.1 (Theorem MG)**.** *Let $c \in \overline{\mathbb{Q}}^\times$ and $s = c\,\Omega/\pi$. For every number field $K \supseteq k$,*

$$\mathcal{S}_K := \{x \in K\mathcal{L} :\ s\,x \in K\mathcal{L} + K\mathcal{L}_E\} \quad \textit{satisfies} \quad \dim_K \mathcal{S}_K \leq 2,$$

and $2\pi i \in \mathcal{S}_K$ when $ic \in K$ (since then $s \cdot 2\pi i = 2ic\,\Omega \in K\mathcal{L}_E$).

Proof. Enlarging K does not change the conclusion (by Baker, K-independence in $K\mathcal{L}$ is preserved under base change), so assume $K \supseteq k(i,c)$; $n = [K:\mathbb{Q}]$ is even (the count below closes for every such n). Let $G = \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \times E$ with coordinates (x,y,z), $\Psi(x,y,z) = (x, y+z)$, $V = \mathbb{C}(1,s)$. Fix a basis $\theta_1, \dots, \theta_n$ of $K/\mathbb{Q}$, put $p_\theta \colon T_{G^n} \to T_G$, $(x_j) \mapsto \sum \theta_j x_j$, and $\Psi_\theta = \Psi \circ p_\theta$.

The group. Connected algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^{2n} \times E^n$ are products $T_L \times B_S$ of a subtorus (given by a subspace $L \subset \mathbb{Q}^{2n}$) and an abelian subvariety (given by a k-subspace $S \subset k^n$). Such a subgroup has tangent space in $\ker \Psi_\theta$ iff $L = 0$ and $\psi(S) = 0$, where $\psi(m) = \sum \theta_j m_j$; by Lemma 3.2, $\ker \psi$ has k-dimension $n/2$. Put $U = B_{\ker \psi}$ and $G'' = G^n/U = \mathbb{G}_{\mathrm{m}}^{2n} \times A''$, $A'' = E^n/U$, of dimension $n/2$. Then $d_0 = 0$, $d_1 = 2n$, $d_2 = n/2$, $d = 5n/2$. The induced map Ψ'' is onto $\mathbb{C}^2$, $V'' = \Psi''^{-1}(V)$ is a hyperplane, and $W'' = \ker \Psi'' \subset V''$ is $\overline{\mathbb{Q}}$-rational of dimension $d-2$. For a connected $G' \subset G''$ put $r' = \dim \Psi''(T_{G'})$; then $\tau = \delta + r' - 2$, so only $r' \leq 1$ can satisfy $\delta > \tau$, and $\delta - \tau = 2 - r'$.

The data. $w = (2\pi i, 2ic\Omega) \in \Psi(\Omega_K(G))$. Suppose $x_1, x_2 \in \mathcal{S}_K$ with $2\pi i, x_1, x_2$ K-independent, and put $v_j = (x_j, sx_j) \in V \cap \Psi(\Lambda_K(G))$. Let Y'' be generated by the images in $T_{G''}$ of elements of $\Lambda(G^n)$ with Ψ_θ-images $D\theta_h w$ (taken in $\Omega(G^n)$, D a common denominator), $\theta_h v_1$, $\theta_h v_2$ ($h = 1, \dots, n$). Then $\operatorname{rk} Y'' = 3n$, $\Gamma = \exp Y'' \subset G''(\overline{\mathbb{Q}})$, and $\kappa = n$: a period in Y'' has first coordinate in $2\pi i K$, so its v-parts vanish. The inequality of Theorem 2.1 reads $\lambda + \delta_1 + 2\delta_2 \leq (2 - r') \cdot 2n$.

No obstruction.

- $G' = 0$: $\lambda = 2n$, so the left side is $2n + 2n + n = 5n > 4n$.
- $r' = 1$ with image $\mathbb{C}(0,1)$ (this covers every G' with an abelian part, since ψ does not vanish on any nonzero abelian subvariety of A''): an element of Y'' lying in $T_{G'} + \Omega(G'')$ has first coordinate in $2\pi i K$, so $\operatorname{rk}(Y'' \cap (T_{G'} + \Omega(G''))) \leq n$ and $\lambda \geq 2n$; the left side is $\geq 3n > 2n$.
- $r' = 1$ with image a line $\mathbb{C}(p,q) \neq \mathbb{C}(0,1)$: then $G' = T_L$ with $\dim L \leq n$. Two K-independent data in $T_{G'} + \Omega(G'')$ would put x_1, x_2 in a K-space of dimension ≤ 2 containing $2\pi i$ (namely $K \cdot 2\pi i + K\pi^2/\Omega$ if $q = 0$, and $K \cdot 2\pi i + K \cdot p\,\Omega/(sp-1)$ otherwise, using $s \notin K$), against the K-independence of $2\pi i, x_1, x_2$. So $\lambda \geq n$, $\delta_1 \geq n$, $\delta_2 = n/2$, and the left side is $\geq 3n > 2n$.
- $r' = 0$ forces $G' = 0$.

So no connected $G' \neq G''$ satisfies the conclusion of Theorem 2.1, a contradiction. Condition (2) reads $5n - 2 \geq 3n$. □

Corollary 4.2 (MG2: four exponentials at $t \in \overline{\mathbb{Q}}^\times \Omega/\pi$)**.** *Let $t \in \overline{\mathbb{Q}}^\times \cdot \Omega/\pi$ and $\alpha, \beta \in \overline{\mathbb{Q}}^\times$ with $\log\alpha, \log\beta$ linearly independent over $\mathbb{Q}$. Then $\alpha^t = e^{t\log\alpha}$ and $\beta^t = e^{t\log\beta}$ are not both algebraic. For example, $2^{\varpi/\pi}$ and $3^{\varpi/\pi}$ are not both algebraic.*

Proof. If $\log\alpha, \log\beta, 2\pi i$ are $\mathbb{Q}$-independent, take $x_1 = \log\alpha$, $x_2 = \log\beta$ in Theorem 4.1: with $2\pi i$ they are three $\mathbb{Q}$-independent logarithms, hence K-independent. Otherwise $m\log\beta = a\log\alpha + 2\pi i\,b$ with integers $m \neq 0$, $b \neq 0$ (after exchanging α, β if necessary). If α^t and β^t are algebraic, so is $e^{2\pi i b t} = (\beta^t)^m(\alpha^t)^{-a}$, i.e. $2\pi i b t = 2ib\,c\,\Omega \in \mathcal{L}$ (with $t = c\Omega/\pi$). Then

$\Omega \in \overline{\mathbb{Q}}\mathcal{L}$, which the analytic subgroup theorem [36] excludes (on $\mathbb{G}_{\mathrm{m}} \times E$, the smallest algebraic subgroup whose tangent space contains (ℓ, Ω) is $\mathbb{G}_{\mathrm{m}} \times E$). □

4.2. **The fibre twin: Theorem MG$^\natural$.** Let $E^\natural$ be read on its conjugate-eigen fibre coordinate w (§3.2), with $\eta^* = \eta^*(\Omega)$; recall that $\Omega\eta^* \in \overline{\mathbb{Q}}^\times \pi$ and that the fibre values are the w-coordinates of $\Lambda(E^\natural)$.

Theorem 4.3 (Theorem MG$^\natural$)**.** *Let $c \in \overline{\mathbb{Q}}^\times$ and $s = c\,\eta^*/\pi \in \overline{\mathbb{Q}}^\times/\Omega$. For every number field $K \supseteq k$, $\dim_K\{x \in K\mathcal{L} :\ s\,x \in K\mathcal{L} + K \cdot (\textit{fibre values})\} \le 2$.*

Proof. As in the proof of Theorem 4.1 we may enlarge K; we take $K \supseteq k(i, c)$ with $n = [K : \mathbb{Q}] \ge 4$ (for $n = 2$ the count below does not close). Let $G = \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \times E^\natural$, $\Psi(x, y, X) = (x,\ y + w(X))$, $V = \mathbb{C}(1, s)$, and θ, p_θ, $\Psi_\theta = \Psi \circ p_\theta$ as before.

The group. There are no nonzero homomorphisms between tori and connected subgroups of $(E^\natural)^n$ in either direction, so connected subgroups of $\mathbb{G}_{\mathrm{m}}^{2n} \times (E^\natural)^n$ are products. A connected $H \subset (E^\natural)^n$ over B_S contains the canonical lift $B_S^\natural$ (since $\mathrm{Hom}(B_S^\natural, \mathbb{G}_{\mathrm{a}}) = 0$), whose tangent space is $\{(\sigma(M)z, \bar\sigma(M)w)\}$; so $\theta^{\mathsf{T}} w$ vanishes on T_H only if $\phi(S) = 0$. Hence the largest connected U with $T_U \subset \ker \Psi_\theta$ is $U = B^\natural_{\ker\phi} + \{\theta^{\mathsf{T}} w = 0\}$, and $G'' = \mathbb{G}_{\mathrm{m}}^{2n} \times Q''$ with $Q'' = (E^\natural)^n/U$, an extension of $A'' = E^n/B_{\ker\phi}$ (dimension $n/2$) by $\mathbb{G}_{\mathrm{a}}$ (coordinate $\varsigma = \theta^{\mathsf{T}} w$). Over an abelian subvariety $B' \ne 0$ of A'', coming from $S' \supsetneq \ker\phi$, we have $\phi(S') \ne 0$, so the image of $B^\natural_{S'}$ contains the fibre. Thus Q'' has no splitting over any $B' \ne 0$, and its connected subgroups are 0, the fibre F, and the $\pi^{-1}(B')$ (dimension $\le n/2 + 1$). Q'' is a quotient of an anti-affine group (no nonconstant regular functions: $\mathrm{Hom}(A^\natural, \mathbb{G}_{\mathrm{a}}) = 0$ by universality, and a character kills the fibre), so $\mathrm{Hom}(Q'', \mathbb{G}_{\mathrm{a}}) = 0$: $d_0 = 0$, $d_1 = 2n$, $G_2 = Q''$, $d_2 = n/2 + 1$. The induced map is $\Psi''(t_1, t_2, q) = (\theta^{\mathsf{T}} t_1, \theta^{\mathsf{T}} t_2 + \varsigma(q))$, V'' is a hyperplane, $W'' = \ker\Psi''$, and $\tau = \delta + r' - 2$ as before.

The data. The period datum is $(2\pi i, 2ic\eta^*) \in \Psi(\Omega_K(G))$ (using $ic \in K$ and that ϕ is onto K). With x_1, x_2 as in Theorem 4.1, $\mathrm{rk}\, Y'' = 3n$ and $\kappa = n$ (a period of G'' has first coordinate in $2\pi i K$). The inequality reads $\lambda + \delta_1 + 2\delta_2 \le (2 - r')(2n + 2)$.

No obstruction. Every G' is $T_L \times H$ with $H \in \{0, F, \pi^{-1}(B')\}$, and $H \ne 0$ forces $\mathbb{C}(0, 1) \subset \Psi''(T_{G'})$.

- $G' = 0$: $\lambda = 2n$, and the left side is $2n + 2n + (n + 2) = 5n + 2 > 4n + 4$.
- Image $\mathbb{C}(0, 1)$ ($H \ne 0$, or L in the second block; $l = \dim L \le n$, $h = \dim H \le n/2 + 1$): elements of Y'' in $T_{G'} + \Omega(G'')$ have first coordinate in $2\pi i K$, so $\lambda \ge 2n$, and the left side is $\ge 2n + (2n - l) + 2(n/2 + 1 - h) \ge 3n > 2n + 2$.
- $H = 0$ and image a line $\mathbb{C}(1, q)$, $q \in K$: a datum X in $T_{G'} + \Omega(G'')$ satisfies $X(s - q) \in K \cdot 2\pi i + K \cdot 2\pi i s$, so $X \in K \cdot 2\pi i + K \cdot 2\pi i/(s - q)$ (here $s \notin K$ because Ω is transcendental). So $\lambda \ge n$, and the left side is $\ge n + n + (n + 2) = 3n + 2 > 2n + 2$.
- $r' = 0$ forces $G' = 0$; $r' = 2$ fails $\delta > \tau$.

The minimum margin is $n - 2$. Condition (2) reads $5n \ge 3n$. □

Corollary 4.4 (MG$^\natural$2: four exponentials at $t \in \overline{\mathbb{Q}}^\times\Omega$)**.** *If $\log\alpha, \log\beta$ are linearly independent over $\mathbb{Q}$ and $c \in \overline{\mathbb{Q}}^\times$, then $\alpha^{c\Omega} = e^{c\Omega\log\alpha}$ and $\beta^{c\Omega} = e^{c\Omega\log\beta}$ are not both algebraic. For example, 2^{ϖ} and 3^{ϖ} (real logarithms) are not both algebraic.*

Proof. Put $x_j = c\Omega\log\alpha_j$ (with $\alpha_1 = \alpha$, $\alpha_2 = \beta$). If $\alpha^{c\Omega}, \beta^{c\Omega}$ are algebraic then $x_j \in \mathcal{L}$, and $sx_j \in \overline{\mathbb{Q}}\log\alpha_j \subset \overline{\mathbb{Q}}\mathcal{L}$. A K-relation among $2\pi i, x_1, x_2$ without the $2\pi i$ term contradicts Baker; one with it gives $0 \ne \ell \in \overline{\mathbb{Q}}\mathcal{L} \cap \overline{\mathbb{Q}}^\times\eta^*$, impossible by the analytic subgroup theorem on $\mathbb{G}_{\mathrm{m}}^r \times E^\natural$ [36]. □

By the symmetry $t \leftrightarrow 1/t$, Corollaries 4.2 and 4.4 give the four exponentials conjecture at every $t \in \overline{\mathbb{Q}}^\times \Omega/\pi \cup \overline{\mathbb{Q}}^\times \pi/\Omega$ and at every $t \in \overline{\mathbb{Q}}^\times \Omega \cup \overline{\mathbb{Q}}^\times/\Omega$, for every CM period Ω.

4.3. **CM abelian varieties: Theorems MG-A and MG$^\natural$-A.** Throughout this subsection A, F, Φ are as in §3.1, F^c is the Galois closure of F, and $K\mathcal{L}$ is the K-span of $\mathcal{L}$. By Baker's theorem $K\mathcal{L} \cong K \otimes_{\mathbb{Q}} \mathcal{L}$, so $\mathbb{Q}$-independent elements of $\mathcal{L}$ are K-independent, and enlarging K does not change the conclusions below.
By the reduction to the simple case in §3.1, we may and do assume A simple of primitive type.

Theorem 4.5 (Theorem MG-A)**.** *Let $\sigma \in \Phi$, $P = P_\sigma \neq 0$, $c \in \overline{\mathbb{Q}}^\times$ and $s = cP/\pi$. For every number field $K \supseteq F^c(i, c)$,*

$$\mathcal{S}_K := \{x \in K\mathcal{L} :\ s\,x \in K\mathcal{L} + K\,z_\sigma(\Lambda(A))\} \quad \textit{satisfies} \quad \dim_K \mathcal{S}_K \leq 2,$$

and $2\pi i \in \mathcal{S}_K$.

Proof. Let $n = [K : \mathbb{Q}]$ (a multiple of $2g$), $G = \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \times A$ with coordinates (x, y, z), $\Psi(x, y, z) = (x,\ y + z_\sigma(z))$, $V = \mathbb{C}(1, s)$, and θ, $p_\theta \colon T_{G^n} \to T_G$, $(x_j) \mapsto \sum \theta_j x_j$, $\Psi_\theta = \Psi \circ p_\theta$ as in the proof of Theorem 4.1 (Φ is the CM type).

The group. By (P) and (S1), connected algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^{2n} \times A^n$ are $T_L \times B_S$, and $T_{T_L \times B_S} \subset \ker \Psi_\theta$ iff $L = 0$ and $\psi_\sigma(S) = 0$. By Lemma 3.1 with $\Sigma = \{\sigma\}$, $\ker \psi_\sigma$ has F-dimension $n - n/(2g)$. Put $U = B_{\ker \psi_\sigma}$, of dimension $gn - n/2$, and $G'' = G^n/U = \mathbb{G}_{\mathrm{m}}^{2n} \times A''$, $A'' = A^n/U$. Then $\dim A'' = n/2$ *for every* g, and $d_0 = 0$, $d_1 = 2n$, $d_2 = n/2$, $d = 5n/2$: the numbers of Theorem 4.1. An abelian subvariety $B \neq 0$ of A'' comes from some $S \supsetneq \ker \psi_\sigma$, so $\psi_\sigma(S) \neq 0$ and $\Psi''(T_B) = \mathbb{C}(0, 1)$; and $\dim B \leq n/2$.

The data. Since ψ_σ is onto K, the θ-span of $z_\sigma(\Omega(A^n))$ is KP, so $w = (2\pi i, 2icP) \in \Psi(\Omega_K(G))$ (using $ic \in K$); in particular $2\pi i \in \mathcal{S}_K$. Suppose $x_1, x_2 \in \mathcal{S}_K$ with $2\pi i, x_1, x_2$ K-independent, and put $v_j = (x_j, sx_j) \in V \cap \Psi(\Lambda_K(G))$. Let Y'' be generated by the images in $T_{G''}$ of lifts of $D\theta_h w$ (in $\Omega(G^n)$, D a common denominator), $\theta_h v_1$, $\theta_h v_2$, $h = 1, \ldots, n$. Then $\operatorname{rk} Y'' = 3n$, $\Gamma = \exp Y'' \subset G''(\overline{\mathbb{Q}})$ and $\kappa = n$: since U is connected, a period of G'' is the image of a period of G^n, so its Ψ''-image has first coordinate in $2\pi i K$, and its v-components vanish. Inequality (1) reads $\lambda + \delta_1 + 2\delta_2 \leq (2 - r') \cdot 2n$.

No obstruction. We need $s \notin K$, i.e. $P/\pi \notin \overline{\mathbb{Q}}$, which is (W). The case check is that of the proof of Theorem 4.1 verbatim, because the only possible dependence on g is through $\dim A''$ and the bound $\dim B \leq \dim A''$, both independent of g: $G' = 0$ gives $5n > 4n$; image $\mathbb{C}(0, 1)$ gives $\lambda \geq 2n$ and left side $\geq 3n > 2n$ (here a subtorus $T_L \subset G'$ lies in the second $\mathbb{G}_{\mathrm{m}}^n$ block, because θ^{T} vanishes on no nonzero rational vector, so $\dim L \leq n$; and $\dim B \leq n/2$); a subtorus with image a line $\mathbb{C}(p, q) \neq \mathbb{C}(0, 1)$ has $\dim L \leq n$, and two K-independent data in $T_{G'} + \Omega(G'')$ would put x_1, x_2 in a 2-dimensional K-space containing $2\pi i$ (using $s \cdot 2\pi i \in KP$ and $s \notin K$), so $\lambda \geq n$ and the left side is $\geq 3n > 2n$; and $r' = 0$ forces $G' = 0$ (maximality of U). The minimum margin is n. Condition (2) reads $5n - 2 \geq 3n$. □

Theorem 4.6 (Theorem MG$^\natural$-A)**.** *Let $\sigma \notin \Phi$, $P = P_\sigma \neq 0$ the corresponding (second-kind) eigen-period, $c \in \overline{\mathbb{Q}}^\times$ and $s = cP/\pi$, so $s \in \overline{\mathbb{Q}}^\times/P_{\bar\sigma}$. For every number field $K \supseteq F^c(i, c)$,*

$$\mathcal{S}_K^\natural := \{x \in K\mathcal{L} :\ s\,x \in K\mathcal{L} + K\,w_\sigma(\Lambda(A^\natural))\} \quad \textit{satisfies} \quad \dim_K \mathcal{S}_K^\natural \leq 2,$$

and $2\pi i \in \mathcal{S}_K^\natural$ (as $s \cdot 2\pi i = 2icP_\sigma \in K\,w_\sigma(\Omega(A^\natural))$).

Proof. Prove it over $K' \supseteq K$ with $[K' : \mathbb{Q}] \geq 4$ and descend by Baker, as in the proof of Theorem 4.3. Let $G = \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \times A^\natural$, $\Psi = (x,\ y + w_\sigma)$, $V = \mathbb{C}(1, s)$. By (F1)–(F4), the largest connected subgroup with tangent space in $\ker \Psi_\theta$ is $U = B^\natural_{\ker \psi_\sigma} + \{\theta^{\mathsf{T}} t_\sigma = 0\}$, where t_σ is the σ-fibre coordinate, of dimension $(gn - n/2) + (gn - 1)$. The quotient $Q'' = (A^\natural)^n/U$ is an extension

of A'' (dimension $n/2$) by $\mathbb{G}_{\mathrm{a}}$; it is a quotient of an anti-affine group, so $\mathrm{Hom}(Q'', \mathbb{G}_{\mathrm{a}}) = 0$, and it does not split over any abelian subvariety $B \neq 0$ of A'' (the image of $B^{\natural}_S$, $S \supsetneq \ker \psi_\sigma$, contains the fibre since $\psi_\sigma(S) \neq 0$). Its connected subgroups are 0, the fibre, and the $\pi^{-1}(B)$, and every nonzero one maps onto $\mathbb{C}(0,1)$. So $G'' = \mathbb{G}_{\mathrm{m}}^{2n} \times Q''$ with $d_0 = 0$, $d_1 = 2n$, $d_2 = n/2+1$, connected subgroups of G'' are $T_L \times H''$, the period datum is $w = (2\pi i, 2icP_\sigma)$, $\kappa = n$, $\mathrm{rk}\, Y'' = 3n$, and the subtorus case needs $s \notin K$, i.e. $P_{\bar\sigma} \notin \overline{\mathbb{Q}}$ (W). This is exactly the situation of Theorem 4.3 (same type of quotient, same dimensions, same subgroup list, same data), with $d_1 + 2d_2 - \kappa = 2n + 2$, so (1) reads $\lambda + \delta_1 + 2\delta_2 \leq (2 - r')(2n + 2)$. Its case table:

G'	r'	left side	right side
0 ($\lambda = 2n$)	0	$5n + 2$	$4n + 4$
$T_L \times H''$ with image $\mathbb{C}(0,1)$ ($\lambda \geq 2n$, $\dim L \leq n$, $\dim H'' \leq n/2 + 1$)	1	$\geq 3n$	$2n + 2$
subtorus with image a line $\neq \mathbb{C}(0,1)$ ($\lambda \geq n$, $\dim L \leq n$, $H'' = 0$)	1	$\geq 3n + 2$	$2n + 2$

and $r' = 0$ forces $G' = 0$. So there is no obstruction iff $n \geq 3$, hence for every admissible n (which is even and ≥ 4: n is a multiple of $2g$, and $[K' : \mathbb{Q}] \geq 4$); the minimum margin is $n - 2$. Condition (2) reads $5n \geq 3n$. □

Proof of Theorem 1.1. By the reduction in §3.1, P is an algebraic multiple of an eigen-period of the same kind of a simple CM abelian variety of primitive type, and the classes in the statement depend only on $\overline{\mathbb{Q}}^\times P$. Write P^+ for a first-kind and P^- for a second-kind eigen-period; $P_\sigma P_{\bar\sigma} \in \pi\overline{\mathbb{Q}}^\times$ gives $\pi/P^+ \in \overline{\mathbb{Q}}^\times P^-$ and $\pi/P^- \in \overline{\mathbb{Q}}^\times P^+$ (for the conjugate slots).

Direct families: $t = cP^+/\pi$ (Theorem 4.5 with $s = t$) and $t = cP^-/\pi$ (Theorem 4.6 with $s = t$). Take $x_1 = \log\alpha$, $x_2 = \log\beta$; if $\alpha^t, \beta^t \in \overline{\mathbb{Q}}$ then $sx_j \in \mathcal{L}$. If $2\pi i, \log\alpha, \log\beta$ are $\mathbb{Q}$-independent they are K-independent (Baker), contradicting the theorem (which also puts $2\pi i$ in the set). Otherwise $m\log\beta = a\log\alpha + 2\pi i b$ with integers $m, b \neq 0$ (after exchanging α, β if necessary), so $e^{2\pi i b t} \in \overline{\mathbb{Q}}$ and $t \cdot 2\pi i \in \overline{\mathbb{Q}}\mathcal{L}$; but $t \cdot 2\pi i = 2ic\,P^+$ (resp. $2ic\,P^-$) is a nonzero algebraic multiple of an eigen-period, which is not in $\overline{\mathbb{Q}}\mathcal{L}$ by (W) (on $A \times \mathbb{G}_{\mathrm{m}}^r$, resp. $A^\natural \times \mathbb{G}_{\mathrm{m}}^r$).

Inverted families: the four exponentials statement at t for all $\mathbb{Q}$-independent pairs is equivalent to the statement at $1/t$ (if $\alpha^t, \beta^t \in \overline{\mathbb{Q}}$, apply the statement at $1/t$ to α^t, β^t, whose logarithms $t\log\alpha, t\log\beta$ are $\mathbb{Q}$-independent). The inverses of the direct classes are $\overline{\mathbb{Q}}^\times \pi/P^+ = \overline{\mathbb{Q}}^\times P^-$ and $\overline{\mathbb{Q}}^\times \pi/P^- = \overline{\mathbb{Q}}^\times P^+$. So all of $t \in \overline{\mathbb{Q}}^\times P^{\pm 1}$, $\overline{\mathbb{Q}}^\times (P/\pi)^{\pm 1}$ are covered, for P of either kind ($1/P^+ \in \overline{\mathbb{Q}}^\times P^-/\pi$, $1/P^- \in \overline{\mathbb{Q}}^\times P^+/\pi$).

Beta values. By §3.1, $B(a/N, b/N)$ is a first-kind eigen-period up to $\overline{\mathbb{Q}}^\times$ if $a + b < N$ and a second-kind one if $a + b > N$. For example, $B(\frac{1}{5}, \frac{1}{5})/\pi$ is covered by Theorem 4.5, and $B(\frac{1}{5}, \frac{1}{5})$ by Theorem 4.6 (inverted). □

4.4. **Three exponentials with π^2.**

Corollary 4.7. *Let $P = P_\sigma$, $\sigma \in \Phi$, be a first-kind eigen-period and $a, b, c \in \overline{\mathbb{Q}}^\times$. At least one of $e^{a\pi^2}$, $e^{b\pi P}$, $e^{c\pi^2/P}$ is transcendental.*

Proof. Suppose all three are algebraic. Take $s = P/\pi$ (Theorem 4.5 with its constant equal to 1), $x_1 = \pi^2$, $x_2 = \pi^2/P$: then $x_1, x_2 \in \overline{\mathbb{Q}}\mathcal{L}$, $sx_1 = \pi P \in \overline{\mathbb{Q}}\mathcal{L}$, $sx_2 = \pi \in \overline{\mathbb{Q}}\mathcal{L}$. It remains to show that $2\pi i, \pi^2, \pi^2/P$ are $\overline{\mathbb{Q}}$-independent. A relation without the π^2/P term contradicts the transcendence of π. Otherwise $\pi/P = a_1 + b_1\pi$ with $a_1, b_1 \in \overline{\mathbb{Q}}$. If $b_1 = 0$ then $P/\pi \in \overline{\mathbb{Q}}$, contradicting (W). If $b_1 \neq 0$ then $P(a' + \pi) = c''\pi$ with $a', c'' \in \overline{\mathbb{Q}}$, so $\pi P = c''\pi - a'P$. Since $\pi P \in \overline{\mathbb{Q}}\mathcal{L}$ and $\pi \in \overline{\mathbb{Q}}\mathcal{L}$, this gives $a'P \in \overline{\mathbb{Q}}\mathcal{L}$, so $a' = 0$ by (W), and then $P = c'' \in \overline{\mathbb{Q}}$, again contradicting (W). □

For $g = 1$ this reads: for every period Ω of a CM curve, one of e^{π^2}, $e^{\pi\Omega}$, $e^{\pi^2/\Omega}$ is transcendental; on $y^2 = 4x^3 - 4x$, one of $e^{\pi^2} \approx 19333.69$, $e^{\pi\varpi} \approx 3779.84$, $e^{\pi^2/\varpi} \approx 43.12$. For $P = B(\frac{1}{5}, \frac{1}{5})$ (up to $\overline{\mathbb{Q}}^\times$): one of e^{π^2}, $e^{\pi\Gamma(1/5)^2/\Gamma(2/5)}$, $e^{\pi^2\Gamma(2/5)/\Gamma(1/5)^2}$. The proof does not use the algebraic independence of π and P, which is not known for abelian periods.

5. Strong four exponentials at CM periods

5.1. **Theorems SMG, SMG$^\natural$ and their abelian forms.** Throughout, $E/\overline{\mathbb{Q}}$ has CM by the imaginary quadratic field k, $\Omega \neq 0$ is a period, and $\eta = \eta^*(\Omega)$ its conjugate-eigen quasi-period (§3.2); Legendre's relation gives $\eta \in \overline{\mathbb{Q}}^\times \pi/\Omega$ (for $y^2 = 4x^3 - 4x$: $\Omega = \varpi$, $\eta = \pi/\varpi$). $\mathcal{L}_E$ and Fib are as in §3.2; for a number field K, $K\mathcal{L}$, $K\mathcal{L}_E$, KFib are K-spans.

Lemma 5.1. *$P_\sigma \notin \widetilde{\mathcal{L}}$ for every eigen-period $P_\sigma \neq 0$.*

Proof. For $\sigma \in \Phi$, a relation $P_\sigma = \beta_0 + \sum \beta_k \log \alpha_k$ puts the vector $(1, \log\alpha, \gamma)$ of $\mathrm{Lie}(\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}^r \times A)$ ($z_\sigma(\gamma) = P_\sigma$, $\gamma \in \Omega(A)$) in a $\overline{\mathbb{Q}}$-hyperplane. By Wüstholz's analytic subgroup theorem [36] the smallest algebraic subgroup through it, which is $H \times T \times A$ since A is simple and $\gamma \neq 0$, has its tangent space in that hyperplane; but z_σ does not vanish on $\mathrm{Lie}(A)$. For $\sigma \notin \Phi$ argue on $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}^r \times A^\natural$: the torus splits off, and the image in $A^\natural$ of the smallest subgroup maps onto A, hence is $A^\natural$ by (F3) (here W_A is the whole fibre); the smallest subgroup is not the graph of a homomorphism $A^\natural \to \mathbb{G}_\mathrm{a}$, since $A^\natural$ is anti-affine and the $\mathbb{G}_\mathrm{a}$-coordinate is 1; so it contains $0 \times A^\natural$, on whose fibre $w_\sigma \neq 0$. □

Theorem 5.2. *Let $c \in \overline{\mathbb{Q}}^\times$, and let $y_1, y_2 \in \widetilde{\mathcal{L}}$ be such that $2\pi i, y_1, y_2$ are linearly independent over $\overline{\mathbb{Q}}$.*

(a) (Theorem SMG) *If $s = c\Omega/\pi$, then sy_1 and sy_2 are not both in $\widetilde{\mathcal{L}} + \overline{\mathbb{Q}}\mathcal{L}_E$.*

(b) (Theorem SMG$^\natural$) *If $s = c\eta/\pi$ (so $s \in \overline{\mathbb{Q}}^\times/\Omega$), then sy_1 and sy_2 are not both in $\widetilde{\mathcal{L}} + \overline{\mathbb{Q}}$Fib.*

Proof. Field. Choose a number field $K \supseteq k(i, c)$ containing all coefficients needed to write $y_j \in K + K\mathcal{L}$ and sy_j in $K + K\mathcal{L} + K\mathcal{L}_E$, resp. $K + K\mathcal{L} + K$Fib, with $[K : \mathbb{Q}] \geq 4$ in (b) (at $n = 2$ the count of (b) fails, with equality, for types Z and F of the table below). The $\overline{\mathbb{Q}}$-independence of $2\pi i, y_1, y_2$ gives their K-independence; no base change is needed (and base change would not obviously preserve K-independence of elements of $\overline{\mathbb{Q}} + K\mathcal{L}$). Let $n = [K : \mathbb{Q}]$ and θ a basis of $K/\mathbb{Q}$.

(a) The group. $G = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m} \times E$ with coordinates (a_1, x, a_2, y, z), $\Psi(a_1, x, a_2, y, z) = (a_1 + x,\, a_2 + y + z)$, $V = \mathbb{C}(1, s)$, $\Psi_\theta = \Psi \circ p_\theta$. By (P) and the structure of E^n, connected subgroups of G^n are $H \times T_L \times B_S$, and $T \subset \ker \Psi_\theta$ iff $H \subset \{\theta \cdot a_1 = \theta \cdot a_2 = 0\}$, $L = 0$, $\psi(S) = 0$ ($\psi(m) = \sum \theta_j m_j$; $\dim_k \ker \psi = n/2$ by Lemma 3.2). So $G'' = \mathbb{G}_\mathrm{a}^2 \times \mathbb{G}_\mathrm{m}^{2n} \times A''$ with $\dim A'' = n/2$: $d_0 = 2$, $d_1 = 2n$, $d_2 = n/2$, $d = 5n/2 + 2$, $d_1 + 2d_2 = 3n$. V'' is a hyperplane and $W'' = \ker \Psi''$ is $\overline{\mathbb{Q}}$-rational of dimension $d - 2$.

Data. $w = (2\pi i, 2ic\Omega) \in \Psi(\Omega_K(G))$, and $v_j = (y_j, sy_j) \in V \cap \Psi(\Lambda_K(G))$, writing $y_j = b_j + x_j$ with $b_j \in \overline{\mathbb{Q}} = \Lambda(\mathbb{G}_\mathrm{a})$. Let Y'' be generated by lifts of $\theta_h w, \theta_h v_1, \theta_h v_2$; Ψ'' is injective on it since $2\pi i, y_1, y_2$ are K-independent, so $m = \mathrm{rk}\, Y'' = 3n$, and $\kappa \geq n$ (in fact $\kappa = n$). With $Y_{G'} = \{y \in Y'' : \exp y \in G'\}$, $\lambda = 3n - \mathrm{rk}\, Y_{G'}$, and an obstruction needs $r' \leq 1$ and

$$(\star) \qquad \lambda + \delta_1 + 2\delta_2 \leq (2 - r')(3n - \kappa) \leq (2 - r') \cdot 2n.$$

Case check. $r' = 0$ forces $G' = 0$: the preimage of G' in G^n is connected (as U is) with tangent space in $\ker \Psi_\theta$, so it lies in U. For $G' = 0$, $(\star)$ reads $6n - \kappa \leq 2(3n - \kappa)$, i.e. $\kappa \leq 0$: excluded, margin $\kappa \geq n$. Let $r' = 1$, $\ell = \Psi''(T_{G'})$, $G' = H \times T_L \times B$ with h, l, b the dimensions. Then $\Psi''(T_H) = T_H$ (so $h \leq 1$, and $h = 1$ forces $\ell = T_H$); $L \neq 0$ forces ℓ to be K-rational and $l \leq n$; $B \neq 0$ forces $\ell = \mathbb{C}(0, 1)$. With $\delta_1 + 2\delta_2 = 3n - l - 2b$, $(\star)$ becomes $\mathrm{rk}\, Y_{G'} + l + 2b \geq 4n$. Since

$y \in Y_{G'}$ implies $\Psi''(y) \in \ell + \Omega_\Psi$, $\Omega_\Psi := \Psi(\Omega_K(G)) = K2\pi i \times (K2\pi i + K\Omega)$ (a K-space), we have $\operatorname{rk} Y_{G'} \le n \dim_K \mathcal{N}$, $\mathcal{N} = \{\beta \in K^3 : \beta_0 w + \beta_1 v_1 + \beta_2 v_2 \in \ell + \Omega_\Psi\}$.

- $\ell = \mathbb{C}(0,1)$: first coordinates give $\beta_1 y_1 + \beta_2 y_2 \in K2\pi i$, so $\dim \mathcal{N} \le 1$ and $\operatorname{rk} + l + 2b \le n + n + n = 3n < 4n$.
- ℓ K-rational, $\ne \mathbb{C}(0,1)$, $L \ne 0$: $\dim \mathcal{N} \le 2$ (Lemma 5.4), so $\operatorname{rk} + l \le 3n < 4n$.
- $L = 0$, $B = 0$, $h = 1$ (ℓ any $\overline{\mathbb{Q}}$-line): $\operatorname{rk} \le 3n$, $l = b = 0$: $3n < 4n$.

No other G' has $r' = 1$. Hence no obstruction, for every $n \ge 2$; the minimum margin is n. Condition (2): $5n + 2 \ge 3n + 4$.

(b) The group. $G = \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_m \times E^\natural$, $\Psi(a, x, y, (z, w)) = (a+x,\ y+w)$ (w the conjugate-eigen fibre coordinate), $V = \mathbb{C}(1, s)$, $s = c\eta/\pi$. As in the proof of Theorem 4.3, $G'' = \mathbb{G}_a \times \mathbb{G}_m^{2n} \times Q''$, where the $\mathbb{G}_a$-part of U is the hyperplane $\{\theta \cdot a = 0\}$ and Q'' is the anti-affine extension of A'' ($\dim n/2$) by its fibre $\mathbb{G}_a$, with no splitting over any abelian subvariety $B \ne 0$: $d_0 = 1$, $d_1 = 2n$, $d_2 = n/2+1$, $d = 5n/2+2$, $d_1 + 2d_2 = 3n+2$. Although $\mathbb{G}_a^n \times (E^\natural)^n$ has non-product subgroups, U is as stated: with $U_0 = \{\theta \cdot a = 0\} \times U_E$, where $U_E = B^\natural_{\ker \phi} + \{\theta^{\mathsf{T}} w = 0\} \subset (E^\natural)^n$ is the group U of the proof of Theorem 4.3, U/U_0 is a connected subgroup of $\mathbb{G}_a \times \mathbb{G}_m^{2n} \times Q''$ with tangent space in $\ker \Psi''$, and by Lemma 5.3 it is 0. The data are $w = (2\pi i, 2ic\eta)$ and $v_j = (y_j, sy_j)$; $m = 3n$, $\kappa \ge n$.

Lemma 5.3. *A connected subgroup of G'' is $T_L \times R$ with $R \subset \mathbb{G}_a \times Q''$ connected and one of: 0; a $\overline{\mathbb{Q}}$-line $\{(\alpha u, \beta u)\}$ in the vector group $\mathbb{G}_a \times \mathbb{G}_a^{\mathrm{fib}}$; $\mathbb{G}_a \times \mathbb{G}_a^{\mathrm{fib}}$; $0 \times Q''_B$; $\mathbb{G}_a \times Q''_B$ ($B \ne 0$, Q''_B the preimage of B).*

Proof. There are no nonzero homomorphisms between tori and $\mathbb{G}_a \times Q''$ or its subgroups, so the torus splits off. A subgroup of Q'' mapping onto $B \ne 0$ contains the fibre (otherwise it would split Q'' over B), so subgroups of Q'' are 0, $\mathbb{G}_a^{\mathrm{fib}}$, Q''_B. If R maps onto $B \ne 0$, its image in Q'' is Q''_B and $R \cap (\mathbb{G}_a \times 0)$ is $\mathbb{G}_a$ or 0; in the latter case R is the graph of a homomorphism $Q''_B \to \mathbb{G}_a$, which is 0 since a nonzero one would split Q''_B over B. If $B = 0$, R is a $\overline{\mathbb{Q}}$-subspace of $\mathbb{G}_a \times \mathbb{G}_a^{\mathrm{fib}}$. □

The diagonal lines $\{(\alpha u, \beta u)\}$ are the new feature compared with (a). With $\delta_0 = 1 - \dim \pi_0(R)$ one has $\delta_1 + 2\delta_2 = 3n + 4 - l - 2\dim R - 2\delta_0$, and for $r' = 1$ an obstruction needs $\operatorname{rk} Y_{G'} + l + 2\dim R + 2\delta_0 \ge 4n + 2$ (using $\kappa \ge n$). The cases:

type	R	image	$\operatorname{rk} Y_{G'}$	l	$\dim R$	δ_0	total
Z	0 ($G' = 0$)	0					$(\star) \iff \kappa \le 2$
F	$0 \times Q''_B$	$\mathbb{C}(0,1)$	$\le n$	$\le n$	$\le n/2+1$	1	$\le 3n+4$
VF	fibre line	$\mathbb{C}(0,1)$	$\le n$	$\le n$	1	1	$\le 2n+4$
T0	0, $L \ne 0$	$\mathbb{C}(0,1)$	$\le n$	$\le n$	0	1	$\le 2n+2$
VK	line, $\alpha \ne 0$, ℓ K-rat.	ℓ	$\le 2n$	$\le n$	1	0	$\le 3n+2$
VQ	line, $\alpha \ne 0$, ℓ not K-rat.	ℓ	$\le 3n$	0	1	0	$\le 3n+2$
TK	0, $L \ne 0$, ℓ K-rat., $\ne \mathbb{C}(0,1)$	ℓ	$\le 2n$	$\le n$	0	1	$\le 3n+2$

Types with $R \supseteq \mathbb{G}_a \times \mathbb{G}_a^{\mathrm{fib}}$ or $R = \mathbb{G}_a \times Q''_B$ have $r' = 2$. For VK and TK, Lemma 5.4 applies with $\Omega_\Psi = K2\pi i \times (K2\pi i + K\eta)$. So there is no obstruction for $n \ge 3$, hence for every even $n \ge 4$ (type Z needs $\kappa \ge 3$, type F needs $n \ge 3$); the minimum margin is $n - 2$, attained by types Z and F; for $n = 2$ both obstruct, with equality, which is why $[K : \mathbb{Q}] \ge 4$. Condition (2): $5n + 2 \ge 3n + 2$. □

Lemma 5.4. *In the proofs above, if ℓ is a K-rational line different from $\mathbb{C}(0,1)$, then $\dim_K \mathcal{N} \le 2$.*

Proof. Otherwise $v_1, v_2 \in \ell + \Omega_\Psi$. If $\ell = \mathbb{C}(1,0)$: $sy_j \in K2\pi i + K\omega_0$ ($\omega_0 = \Omega$, resp. η), so $y_j \in K \cdot 2\pi i + K \cdot \pi^2/\omega_0$ (as $ic \in K$). If $\ell = \mathbb{C}(p,1)$, $p \in K$: $y_j = \lambda_j p + 2\pi i a_j$ and $sy_j = \lambda_j + 2\pi i b_j + \omega_0 c_j$, so $y_j(1-ps) = 2\pi i(a_j - pb_j) - p\omega_0 c_j$; since $\omega_0 = 2\pi i\, s/(2ic) \in K \cdot 2\pi i s$ (the period datum), $y_j \in 2\pi i(K + Ks)/(1-ps)$, where $1 - ps \neq 0$ because $s \notin K$. In both cases y_1, y_2 lie in a 2-dimensional K-space containing $2\pi i$, against the K-independence of $2\pi i, y_1, y_2$. Here $s \notin K$: for (a), $s \in \overline{\mathbb{Q}}$ would give $\Omega \in \overline{\mathbb{Q}} \cdot 2\pi i \subset \widetilde{\mathcal{L}}$, against Lemma 5.1; for (b), $s \in \overline{\mathbb{Q}}^\times/\Omega$ is transcendental (Schneider). □

Now let A be simple with CM by F of primitive type Φ, z_σ ($\sigma \in \Phi$) and w_σ ($\sigma \notin \Phi$) its eigencoordinates on $\mathrm{Lie}(A)$, resp. $\mathrm{Lie}(A^\natural)$, and $P_\sigma \neq 0$ a period of the slot (A, σ). For $g = 1$ the following theorem is Theorem 5.2.

Theorem 5.5 (Theorems SMG-A and SMG$^\natural$-A)**.** *Let $c \in \overline{\mathbb{Q}}^\times$ and $s = cP_\sigma/\pi$. Let $y_1, y_2 \in \widetilde{\mathcal{L}}$ with $2\pi i, y_1, y_2$ linearly independent over $\overline{\mathbb{Q}}$. Then sy_1 and sy_2 are not both in $\widetilde{\mathcal{L}} + \overline{\mathbb{Q}} z_\sigma(\Lambda(A))$ if $\sigma \in \Phi$ (SMG-A), and not both in $\widetilde{\mathcal{L}} + \overline{\mathbb{Q}} w_\sigma(\Lambda(A^\natural))$ if $\sigma \notin \Phi$ (SMG$^\natural$-A).*

Proof of Theorem 5.5. This is the proof of Theorem 5.2(a) (resp. (b)) with E replaced by A read at z_σ (resp. w_σ). Let K be a number field containing the Galois closure of F, i, c and all coefficients, of degree $n \geq 4$. For $\sigma \in \Phi$ take $G = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m} \times A$, $\Psi = (a_1 + x,\ a_2 + y + z_\sigma)$, $V = \mathbb{C}(1, s)$. By Lemma 3.1 the largest connected subgroup of G^n in $\ker \Psi \circ p_\theta$ is $\{\theta \cdot a_1 = \theta \cdot a_2 = 0\} \times 1 \times B_{\ker \psi_\sigma}$, and $G'' = \mathbb{G}_\mathrm{a}^2 \times \mathbb{G}_\mathrm{m}^{2n} \times A''$ with $\dim A'' = n/2$ for every g: these are exactly the numbers $d_0 = 2$, $d_1 = 2n$, $d_2 = n/2$ of the elliptic case. Every abelian subvariety $B \neq 0$ of A'' has $\Psi''(T_B) = \mathbb{C}(0,1)$ and $\dim B \leq n/2$. The period datum is $w = (2\pi i, 2icP_\sigma)$, and $P_\sigma = 2\pi i\, s/(2ic) \in K \cdot 2\pi i\, s$, which is all that the line-subtorus step (Lemma 5.4) uses; $s \notin K$ by Lemma 5.1. The case analysis of Theorem 5.2(a) therefore applies verbatim: no connected subgroup satisfies (1), with margin n. For $\sigma \notin \Phi$ take $G = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \times A^\natural$, $\Psi = (a + x,\ y + w_\sigma)$; by (F1)–(F4) the $A^\natural$-part of G'' is an anti-affine extension Q'' of A'' ($\dim n/2$) by its one-dimensional fibre, with no splitting over any $B \neq 0$, and $\mathrm{Hom}(Q''_B, \mathbb{G}_\mathrm{a}) = 0$, as in the proof of Theorem 4.6. These are the only properties used in Lemma 5.3, the subgroup lemma for $\mathbb{G}_\mathrm{a} \times Q''$ in the proof of Theorem 5.2(b), whose case table then applies verbatim (margin $n - 2$, attained at types Z and F, $n \geq 4$). Condition (2) reads $5n + 2 \geq 3n + 4$, resp. $5n + 2 \geq 3n + 2$. □

Proof of Theorem 1.2. By §3.1 we may assume A simple of primitive type. Strong four exponentials at t is equivalent to the same statement at $1/t$ (replace y_j by ty_j) and is invariant under $t \mapsto \gamma t$, $\gamma \in \overline{\mathbb{Q}}^\times$. By the bilinear relation $P_\sigma P_{\bar\sigma} \in \overline{\mathbb{Q}}^\times \pi$, for a first-kind $Q = P_\sigma$ and $S = P_{\bar\sigma}$ the classes $Q^{\pm 1}, (Q/\pi)^{\pm 1}$ are the classes $S^{\mp 1}, (S/\pi)^{\mp 1}$; so it suffices to treat $t = cP_\sigma/\pi$ for slots of both kinds. If $y_1, y_2 \in \widetilde{\mathcal{L}}$ are independent with $ty_1, ty_2 \in \widetilde{\mathcal{L}}$ and $2\pi i \in \overline{\mathbb{Q}} y_1 + \overline{\mathbb{Q}} y_2$, then $2icP_\sigma = t \cdot 2\pi i \in \widetilde{\mathcal{L}}$, against Lemma 5.1. Otherwise Theorem 5.5 applies. For the Beta values use §3.1. □

Corollary 5.6. (i) *$(2e)^t$ and 3^t are not both algebraic for $t = \varpi$ and for $t = \Gamma(1/5)^2/\Gamma(2/5)$; e^t and 2^t are not both algebraic for $t = \varpi/\pi$, $t = 1/\varpi$, $t = 1/B(\frac{1}{7}, \frac{1}{7})$ and $t = B(\frac{1}{5}, \frac{1}{5})/\pi$; $t \log 2$ and $t \log 3$ are not both in $\widetilde{\mathcal{L}}$ for every $t \in \overline{\mathbb{Q}}^\times \varpi^{\pm 1}$ and $t \in \overline{\mathbb{Q}}^\times B(a/N, b/N)^{\pm 1}$ with $a + b \neq N$.*

(ii) *For every first-kind eigen-period P, at least one of e^{π^2}, $e^{\pi P}$, $e^{P/\pi}$ is transcendental, and at least one of e^{π^2}, $e^{\pi^2/P}$, $e^{1/P}$ is transcendental. More strongly, $\pi^2, \pi P, P/\pi$ are not all in $\widetilde{\mathcal{L}}$, and $\pi^2, \pi^2/P, 1/P$ are not all in $\widetilde{\mathcal{L}}$.*

Proof. (i) Theorem 1.2 with $y = (1 + \log 2, \log 3)$, resp. $y = (1, \log 2)$, resp. $y = (\log 2, \log 3)$ ($\overline{\mathbb{Q}}$-independent by Baker). (ii) Theorem 5.5 with $y = (1, \pi^2)$ ($2\pi i, 1, \pi^2$ are $\overline{\mathbb{Q}}$-independent

since π is transcendental; if $\pi^2 \notin \widetilde{\mathcal{L}}$ there is nothing to prove), at $s = cP/\pi$ ($\sigma \in \Phi$) and at $s = cP_{\bar\sigma}/\pi \in \overline{\mathbb{Q}}^\times/P$ ($\bar\sigma \notin \Phi$). □

5.2. **The elliptic analogue.** In this subsection the strong four exponentials statement is read on a CM curve instead of on $\mathbb{G}_{\mathrm{m}}$. Let E be an elliptic curve over $\overline{\mathbb{Q}}$ with complex multiplication by k, Ω a nonzero period, $\eta = \eta^*(\Omega)$ the conjugate-eigen quasi-period ($\Omega\eta \in \overline{\mathbb{Q}}^\times\pi$). Write $\widetilde{\mathcal{L}}_E = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E$, and Fib_E for the set of fibre values (§3.2; it contains $\overline{\mathbb{Q}}$, the values $\zeta^*(u)$, $u \in \mathcal{L}_E$, and η).

Theorem 5.7 (Theorem SE_0)**.** *Let $c \in \overline{\mathbb{Q}}^\times$, let $y_1, y_2 \in \widetilde{\mathcal{L}}_E$ with Ω, y_1, y_2 linearly independent over $\overline{\mathbb{Q}}$, and $t = c\eta/\Omega \in \overline{\mathbb{Q}}^\times\pi/\Omega^2$. Then ty_1, ty_2 are not both in $\overline{\mathbb{Q}}\mathcal{L}_E + \overline{\mathbb{Q}}\mathrm{Fib}_E$.*

Corollary 5.8 (Strong elliptic four exponentials)**.** *Let $t \in \overline{\mathbb{Q}}^\times(\pi/\Omega^2)^{\pm1}$. If $y_1, y_2 \in \widetilde{\mathcal{L}}_E$ are linearly independent over $\overline{\mathbb{Q}}$, then ty_1 and ty_2 are not both in $\widetilde{\mathcal{L}}_E$. In particular, if $u_1, u_2 \in \mathcal{L}_E$ are linearly independent over k, then $\wp_E(tu_1)$ and $\wp_E(tu_2)$ are not both algebraic.*

Lemma 5.9. *$\eta \notin \widetilde{\mathcal{L}}_E$; in particular $t \notin \overline{\mathbb{Q}}$ in Theorem 5.7.*

Proof. We use Wüstholz's theorem [36]. If $\eta = \beta_0 + \sum \beta_i\ell_i$ with $\ell_i \in \mathcal{L}_E$, the vector $(1, \ell, (\Omega, \eta))$ of $\mathrm{Lie}(\mathbb{G}_{\mathrm{a}} \times E^r \times E^\natural)$ lies in a $\overline{\mathbb{Q}}$-hyperplane. This group is a push-out of $\mathbb{G}_{\mathrm{a}} \times (E^\natural)^{r+1}$: the smallest subgroup through the vector contains the fibre vectors of $B_S^\natural$ for its abelian image B_S, whose last projection is nonzero; and the fibre direction is not in the hyperplane. □

Proof of Corollary 5.8. The statement is invariant under $t \mapsto \gamma t$ ($\gamma \in \overline{\mathbb{Q}}^\times$) and under $t \mapsto 1/t$ (replace y_j by ty_j), and $\overline{\mathbb{Q}}^\times\pi/\Omega^2 = \overline{\mathbb{Q}}^\times\eta/\Omega$. If $\Omega \in \overline{\mathbb{Q}}y_1 + \overline{\mathbb{Q}}y_2$ then $t\Omega \in \widetilde{\mathcal{L}}_E$, i.e. $\eta \in \widetilde{\mathcal{L}}_E$; this is excluded by Lemma 5.9. So Theorem 5.7 applies, since $\widetilde{\mathcal{L}}_E$ lies in its target ($\overline{\mathbb{Q}} \subset \mathrm{Fib}_E$). For the last sentence: $\wp(z) \in \overline{\mathbb{Q}} \cup \{\infty\}$ means $z \in \mathcal{L}_E$, and k-independent elliptic logarithms of a CM curve are $\overline{\mathbb{Q}}$-independent (Masser [16]). □

Proof of Theorem 5.7. Let $K \supseteq k(c)$ contain all coefficients, with $n = [K : \mathbb{Q}] \geq 4$ (enlarge K if necessary, since at $n = 2$ the count fails, with equality, at $G' = 0$ and at image $\mathbb{C}(0, 1)$). Let θ be a $\mathbb{Q}$-basis of K, and $\psi(m) = \sum \theta_j m_j$, $\phi(m) = \sum \theta_j \bar m_j$ on k^n, with kernels of k-dimension $n/2$ (Lemma 3.2). By Lemma 5.9, $t \notin K$.

Let $G = \mathbb{G}_{\mathrm{a}} \times E \times E \times E^\natural$, $\Psi = (a + z_1,\ z_2 + w)$, w the conjugate-eigen fibre coordinate of $E^\natural$ (its base coordinate is not read), $V = \mathbb{C}(1, t)$. All factors are isogenous to powers of E, so G^n is a push-out of $\mathbb{G}_{\mathrm{a}}^n \times (E^\natural)^{3n}$, and its connected subgroups are described by an abelian image B_S ($S \subset k^{3n}$) and a vector part containing the fibre vectors of $B_S^\natural$; since base and fibre parameters of $\mathrm{Lie}(B_S^\natural) = \{(Mu, \bar M u')\}$ are independent, the largest subgroup in $\ker\Psi \circ p_\theta$ is the product $\{\theta \cdot a = 0\} \times B_{\ker\psi} \times B_{\ker\psi} \times (B^\natural_{\ker\phi} + \{\theta^{\mathsf T} w = 0\})$. So $G'' = \mathbb{G}_{\mathrm{a}} \times A_1 \times A_2 \times Q$, $\dim A_i = n/2$, with Q an anti-affine extension of an abelian variety A_3 of dimension $n/2$ by its fibre, with no splitting over any abelian subvariety $\neq 0$ ($d_0 = 1$, $d_1 = 0$, $d_2 = 3n/2 + 1$). The data are $w = (\Omega, c\eta)$ and $v_j = (y_j, ty_j)$: $m = 3n$, $\kappa = n$ (a period in Y'' has first coordinate in $K\Omega$). A connected $G' \subset G''$ is determined by its abelian image and its vector part in $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{a}}^{\mathrm{fib}}$, and contains the fibre of Q as soon as its abelian image projects nontrivially to A_3 (the pull-back of a non-split extension along a surjection stays non-split; this also covers the mixed subgroups between the three E-factors). For $r' = 1$, (1) reads $\mathrm{rk}\, Y_{G'} + 2\dim G' + 2\delta_0 \geq 4n + 2$, and for $G' = 0$ it reads $\kappa \leq 2$. With $r' = 1$ the image of G' is: $\mathbb{C}(0, 1)$ ($\mathrm{rk}\, Y_{G'} \leq n$, $\dim G' \leq n + 1$, $\delta_0 = 1$: total $\leq 3n + 4$); or a K-rational line coming from $A_1 \times A_2$, possibly with a line of $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{a}}^{\mathrm{fib}}$ ($\dim G' \leq n/2 + 1$, $\mathrm{rk}\, Y_{G'} \leq 2n$, because $v_j \in \ell + \Psi(\Omega_K(G))$ for $j = 1, 2$ would give $y_j \in K\Omega + K\Omega^2/\eta$ if $\ell = \mathbb{C}(1, 0)$ and $y_j(1 - pt) \in K\Omega + K\eta = \Omega(K + Kt)$ if $\ell = \mathbb{C}(p, 1)$, by

the period datum $\eta = t\Omega/c$ and $1 - pt \neq 0$ (as $t \notin K$), so that Ω, y_1, y_2 would be K-dependent: total $\leq 3n+2$); or a line of $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{a}}^{\mathrm{fib}}$ with trivial abelian image ($\operatorname{rk} Y_{G'} \leq 3n$, $\dim G' = 1$: total $\leq 3n+2$). So no subgroup satisfies (1) for $n \geq 4$; the tight cases are $G' = 0$ and the subgroups with image $\mathbb{C}(0,1)$ containing A_2 and all of Q, both with margin $n-2$. Condition (2): $3n + 2 \geq 2 + n$. □

Example 5.10. On $E_i\colon y^2 = 4x^3 - 4x$ (period ϖ, $\eta = \pi/\varpi$): for every $u \in \mathcal{L}_{E_i} \setminus \{0\}$, e.g. the logarithm of $(2, \sqrt{24})$, the numbers $\wp_i(\pi/\varpi^2)$ and $\wp_i(\pi u/\varpi^2)$ are not both algebraic, nor are $\wp_i(\varpi^2/\pi)$ and $\wp_i(\varpi^2 u/\pi)$ (Corollary 5.8, i.e. from Theorem 5.7, with $(y_1, y_2) = (1, u)$; $1, u$ are $\overline{\mathbb{Q}}$-independent since $u \neq 0$ is transcendental). Also $\wp_i(\varpi^2 u_1/\pi)$ and $\wp_i(\varpi^2 u_2/\pi)$ are not both algebraic for $\mathbb{Q}(i)$-independent $u_1, u_2 \in \mathcal{L}_{E_i}$.

6. Two CM curves

6.1. **Theorem M.** The single-curve version of the count gives the following theorem, which is classical in substance.

Theorem 6.1 (Theorem M)**.** *Let $E/\overline{\mathbb{Q}}$ have complex multiplication by k, and let $\Omega \neq 0$ be a period. If $u \in \overline{\mathbb{Q}}\mathcal{L}_E$ and $\ell \in \widetilde{\mathcal{L}}$ satisfy $i\pi\, u = \Omega\, \ell$, then $u = s\Omega$ and $\ell = s\, i\pi$ for some $s \in \overline{\mathbb{Q}}$; if $u \in \mathcal{L}_E$ then $s \in k$.*

Corollary 6.2. *On E_i (and analogously on every CM curve): (M1) $\wp(c\varpi/\pi) \notin \overline{\mathbb{Q}}$ for $c \in \overline{\mathbb{Q}}^{\times}$; (M2) $\wp(\varpi \log \alpha/\pi) \notin \overline{\mathbb{Q}}$ for $\alpha \in \overline{\mathbb{Q}}^{\times}$ not a root of unity; (M3) at least one of e^{π^2}, $\wp(\pi\varpi)$ is transcendental; more precisely, if $e^{\pi^2} \in \overline{\mathbb{Q}}$ then $\wp(c\pi\Omega) \notin \overline{\mathbb{Q}}$ for every $c \in \overline{\mathbb{Q}}^{\times}$ and every period Ω of every CM curve; (M3′) for $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and any branch, one of $e^{i\pi \log \alpha}$, $\wp(\varpi \log \alpha)$ is transcendental; (Mη) $\eta(\varpi)\, u \notin \widetilde{\mathcal{L}}$ for $u \in \mathcal{L}_{E_i} \setminus \mathbb{Q}(i)\varpi$.*

Proof of Theorem 6.1. Let $G = E \times \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{a}}$, $\Psi(e, t, a) = (e, t + a)$, $V = \mathbb{C}(\Omega, i\pi)$, $W = 0$, and $K \supseteq k$ containing the algebraic coefficients involved. Then $\alpha_K(G) = 1 + 1 + 0 = 2$; $\Psi(\Omega_K(G)) = K\Omega \times K \cdot 2\pi i$ since $K\Lambda_E = K\Omega$ for CM; so $\kappa = 1$. The condition on V: it has a nonzero rational point iff $\Omega/\pi \in \overline{\mathbb{Q}}$, which is false (Masser [16]; cf. [32, Thm. 26]; also Schneider). Theorem 2.3 gives $\dim_K(V \cap \Psi(\Lambda_K)) \leq 1$, and the period $(\Omega, i\pi)$ lies in it. If $i\pi u = \Omega\ell$ then $(u, \ell) \in V \cap \Psi(\Lambda_K)$, so $(u, \ell) = s(\Omega, i\pi)$ with $s \in K$. If $u \in \mathcal{L}_E$, Masser's theorem ([32, Thm. 27]: elements of $\mathcal{L}_E$ that are k-independent are $\overline{\mathbb{Q}}$-independent) gives $s \in k$. On $G'' = A'' \times \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{a}}$, (2) reads $3n \geq 2n + 2$. □

The corollaries. (M1): $u = c\varpi/\pi$, $\ell = ic$ gives $s = c/\pi \notin \overline{\mathbb{Q}}$. (M2): $\ell = i \log \alpha$ gives $\log \alpha \in \overline{\mathbb{Q}}\pi$, so $\alpha = (-1)^{-is}$ with s algebraic, a root of unity by Gel'fond–Schneider. (M3): if $e^{\pi^2} \in \overline{\mathbb{Q}}$ then $\ell = ic\pi^2 \in \widetilde{\mathcal{L}}$ and $u = c\pi\Omega$ give $s = c\pi \notin \overline{\mathbb{Q}}$. (M3′): $u = \varpi \log \alpha$, $\ell = i\pi \log \alpha$ give $\log \alpha \in \overline{\mathbb{Q}}^{\times}$, contradicting Hermite–Lindemann; (M3) is the case $\alpha = -1$, using $\wp(iz) = -\wp(z)$. (Mη): $\eta(\varpi) = \pi/\varpi$. On a general CM curve $\eta(\Omega) = \kappa_1\Omega + \kappa_2\pi/\Omega$ with $\kappa_2 \neq 0$, and the statement becomes $(\eta - \kappa_1\Omega)u \notin \widetilde{\mathcal{L}}$.

Credit. With $\ell \in \mathcal{L}$, Theorem 6.1 is the CM case, with one row a period, of Waldschmidt's Conjecture 42 [32]. Its case $\ell \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \cdot \pi i$, $u \in \mathcal{L}_E$ follows in one line from Laurent [15, Cor. 3, p. 146], and for $u \in \mathcal{L}_E$ it also follows from Laurent's identity for third-kind periods and the analytic subgroup theorem. For $u \in \mathcal{L}_E$, $\ell \in \mathcal{L}$ the statement $u/\Omega \neq \ell/i\pi$ at non-torsion points is Bertrand's problem on Tate-type parametrisations; according to Waldschmidt [29] it is solved in the CM case in [3, Cor. 3], which we have not read.

6.2. **Theorem MX.** Two non-isogenous CM curves give a statement with no logarithm of $\mathbb{G}_{\mathrm{m}}$ at all. For a CM curve E let $E^{\natural}$ be its universal vector extension and $\Lambda(E^{\natural})$ the logarithms of its algebraic points, written in coordinates (z, w) (base, and conjugate-eigen fibre; §3.2). For E_i,

E_ρ the fibre coordinate can be normalised so that $X = (u, \zeta(u)) \in \Lambda(E^\natural)$ for every $u \in \mathcal{L}_E \setminus \Lambda_E$, and in general $w(X) - \zeta^*(z(X)) \in \overline{\mathbb{Q}}$ for $X \in \Lambda(E^\natural)$ with $z(X) \notin \Lambda_E$, where $\zeta^* = \zeta - \kappa_1 \,\mathrm{id}$ and $\eta^* = \eta - \kappa_1 \,\mathrm{id}$ are the conjugate-eigen versions of ζ and η ($\kappa_1 \in \overline{\mathbb{Q}}$, $\kappa_1 = 0$ for E_i, E_ρ; §3.2).

Theorem 6.3 (Theorem MX)**.** *Let $X \in \Lambda(E_i^\natural)$, $Y \in \Lambda(E_\rho^\natural)$, $a, t \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}^\times$. If*

$$\big(z(X) + z(Y) + a\big) \cdot \beta\, \eta(\Omega_\rho) \;=\; \big(w(X) + w(Y) + t\big) \cdot \varpi,$$

then $a = 0$, $z(X) \in \mathbb{Q}(i)\varpi$ and $z(Y) \in \mathbb{Q}(\rho)\Omega_\rho$; in particular $\exp_{E_i}(z(X))$ and $\exp_{E_\rho}(z(Y))$ are torsion points. In particular, for $u \in \mathcal{L}_{E_i} \setminus \Lambda$ and $u' \in \mathcal{L}_{E_\rho} \setminus \Lambda_\rho$ not both torsion, $(u + u' + a)\,\beta\,\eta(\Omega_\rho) \neq (\zeta_i(u) + \zeta_\rho(u') + t)\,\varpi$.

Proof of Theorem 6.3. The coordinates (z, w) on $E^\natural$, η^*, ζ^* and (3) are as in §3.2. Let $G = E_i^\natural \times E_\rho^\natural \times \mathbb{G}_{\mathrm{a}}^2$, let $K \supseteq \mathbb{Q}(i, \sqrt{-3}, \beta)$ be a number field, $n = [K : \mathbb{Q}]$ (so $4 \mid n$), and let

$$\Psi(x_i, x_\rho, a_1, a_2) = \big(z(x_i) + z(x_\rho) + a_1,\ w(x_i) + w(x_\rho) + a_2\big), \qquad V = \mathbb{C}\, w_1,$$

with $w_1 = (\varpi, \beta\eta_\rho)$, $\eta_i = \eta(\varpi) = \pi/\varpi$, $\eta_\rho = \eta(\Omega_\rho) = 2\pi/(\sqrt{3}\,\Omega_\rho)$, and $w_2 = (\Omega_\rho, \beta\frac{2}{\sqrt{3}}\eta_i)$. By (3), $w_1, w_2 \in \Psi(\Omega_K(G))$ and

$$\Psi(\Omega_K(G)) = \mathcal{P} := (K\varpi + K\Omega_\rho) \times (K\eta_i + K\eta_\rho);$$

$w_1 \parallel w_2$ because $\varpi \cdot \frac{2}{\sqrt{3}}\eta_i = \frac{2\pi}{\sqrt{3}} = \Omega_\rho\eta_\rho$, and w_1, w_2 are K-independent since $\Omega_\rho/\varpi \notin \overline{\mathbb{Q}}$ (periods of non-isogenous curves are $\overline{\mathbb{Q}}$-independent [36]).

Proposition 6.4 (Proposition D)**.** $V \cap \Psi(\Lambda_K(G)) = Kw_1 + Kw_2$.

Granting Proposition 6.4: the point (X, Y, a, t) lies in $\Lambda(G)$ and the hypothesis says that its image lies on V, so $z(X) + z(Y) + a = x\varpi + x'\Omega_\rho$ with $x, x' \in K$. By the linear independence of 1 and elliptic logarithms on non-isogenous curves (Wüstholz [36]), $a = 0$, $z(X) = x\varpi$, $z(Y) = x'\Omega_\rho$, and by Masser's theorem [16] $x \in \mathbb{Q}(i)$, $x' \in \mathbb{Q}(\rho)$. □

Proof of Proposition 6.4. Suppose $v \in V \cap \Psi(\Lambda_K(G))$ with $v \notin Kw_1 + Kw_2$.

Step 1: an elementary lemma. For $L \in \{0,\ \mathbb{C}(1,0),\ \mathbb{C}(0,1)\}$ we have $V \cap (L + \mathcal{P}) = Kw_1 + Kw_2$. Indeed, if $sw_1 \in L + \mathcal{P}$ with $L \neq \mathbb{C}(1,0)$ then $s\varpi \in K\varpi + K\Omega_\rho$, so $s \in K + K\Omega_\rho/\varpi$ and $(\Omega_\rho/\varpi)w_1 = w_2$; if $L = \mathbb{C}(1,0)$ then $s\beta\eta_\rho \in K\eta_i + K\eta_\rho$, so $s \in K + K\eta_i/\eta_\rho$ and $(\eta_i/\eta_\rho)w_1 = (\pi/\eta_\rho, \beta\eta_i) = \frac{\sqrt{3}}{2}w_2$.

Step 2: the group. Fix a basis $\theta_1, \dots, \theta_n$ of $K/\mathbb{Q}$, let $p_\theta \colon T_{G^n} \to T_G$, $(x_j) \mapsto \sum \theta_j x_j$, and $\Psi_\theta = \Psi \circ p_\theta$. Write the coordinates of T_{G^n} as $z_i, w_i, z_\rho, w_\rho, a_1, a_2 \in \mathbb{C}^n$ and let $U \subset G^n$ be the vector subgroup $\{z_i = z_\rho = 0,\ \theta^{\mathsf{T}} a_1 = 0,\ \theta^{\mathsf{T}}(w_i + w_\rho + a_2) = 0\}$, of dimension $4n - 2$, defined over $\overline{\mathbb{Q}}$, with $T_U \subset \ker \Psi_\theta$. Then $G'' := G^n/U = \mathbb{G}_{\mathrm{a}} \times G_\ell$, where G_ℓ is the extension of $A = E_i^n \times E_\rho^n$ by $\mathbb{G}_{\mathrm{a}}$ obtained from the universal vector extension of A by pushing out along $(w_i, w_\rho) \mapsto \theta^{\mathsf{T}} w_i + \theta^{\mathsf{T}} w_\rho$. In the coordinates $s_1 = \theta^{\mathsf{T}} a_1$, z_i, z_ρ, $s_2 = \theta^{\mathsf{T}}(w_i + w_\rho + a_2)$ the map induced by Ψ_θ is

$$\Psi''(s_1, z_i, z_\rho, s_2) = \big(\theta^{\mathsf{T}} z_i + \theta^{\mathsf{T}} z_\rho + s_1,\ s_2\big).$$

We apply [27, Thm. 4.1] (Theorem 2.1) to G'' with $d = 2n + 2$, $d_0 = 1$ ($\pi_0 = s_1$), $d_1 = 0$, $G_2 = G_\ell$, $d_2 = 2n + 1$; $V'' = \Psi''^{-1}(V)$, a hyperplane, so $N = 2n + 1$; $W'' = \ker \Psi''$, rational of dimension $2n$; and Y'' generated by the images of $3n$ elements of $\Lambda(G^n)$ whose Ψ_θ-images are $D\theta_h w_1$, $D\theta_h w_2$ (taken in $\Omega(G^n)$, D a common denominator) and $\theta_h v$, $h = 1, \dots, n$. Since U is a vector group, $\Omega(G'')$ is the image of $\Omega(G^n)$, so $\Psi''(\Omega(G'')) \subset \mathcal{P}$. By Step 1 with $L = 0$, $\kappa = \mathrm{rk}(Y'' \cap \Omega(G'')) = 2n$, and $\Gamma = \exp Y''$ has rank n. The inequality of Theorem 2.1 becomes

$$\lambda + 2\delta_2 \leq (\delta - \tau)(2n + 2), \tag{4}$$

and, with $r = \dim \Psi''(T_{G'})$, one has $\tau = \delta + r - 2$; so an obstruction must have $r \leq 1$.

Step 3: the subgroups of G''. Abelian subvarieties of A are $B = B_i \times B_\rho$ (E_i, E_ρ are non-isogenous), and $B_i \subset E_i^n$ corresponds to a $\mathbb{Q}(i)$-subspace $S_i \subset \mathbb{Q}(i)^n$ with $T_{B_i} = \mathbb{C}\,\sigma(S_i)$, $\dim B_i = \dim S_i$; likewise for E_ρ. With ψ, ϕ as in Lemma 3.2 (for $k = \mathbb{Q}(i)$ and for $k = \mathbb{Q}(\rho)$), $\ker\psi \cap \ker\phi = 0$ and each kernel has k-dimension $n/2$. Because $[a]$ acts on $T(E^\natural)$ by $\mathrm{diag}(a, \bar a)$ and $\mathrm{Hom}(E^\natural, \mathbb{G}_\mathrm{a}) = 0$ (the universal vector extension is anti-affine; [17]), the image in G_ℓ of the universal extension of B is a splitting $s(B) \cong B$ (with tangent space $T_B \times 0$) if $\phi(S_i) = \phi(S_\rho) = 0$, and is $\pi^{-1}(B)$, a non-split extension, otherwise (a splitting would give a nonzero homomorphism from a quotient of $(E_i^\natural)^{b_i} \times (E_\rho^\natural)^{b_\rho}$ to $\mathbb{G}_\mathrm{a}$). A connected subgroup over B either contains the fibre or is a splitting, and a splitting is unique since $\mathrm{Hom}(B, \mathbb{G}_\mathrm{a}) = 0$. It follows that the connected algebraic subgroups of $G'' = \mathbb{G}_\mathrm{a} \times G_\ell$ are exactly

type	G'	condition	$\Psi''(T_{G'})$
A	$\mathbb{G}_\mathrm{a} \times \pi^{-1}(B)$	—	$\mathbb{C}^2$
B	$\mathbb{G}_\mathrm{a} \times s(B)$	$\phi(S) = 0$	$\mathbb{C}(1,0)$
C	$0 \times \pi^{-1}(B)$	—	$\mathbb{C}(0,1)$ if $\psi(S) = 0$, else $\mathbb{C}^2$
D	$0 \times s(B)$	$\phi(S) = 0$	0 if $B = 0$, else $\mathbb{C}(1,0)$
E	$\{(\mu\,\mathrm{pr}_F(g), g) : g \in s(B) \times F\}$, $\mu \neq 0$	$\phi(S) = 0$	$\mathbb{C}(\mu, 1)$ if $B = 0$, else $\mathbb{C}^2$

(with $\phi(S) = 0$ meaning $\phi(S_i) = \phi(S_\rho) = 0$, and similarly for ψ). In type E, $F \cong \mathbb{G}_\mathrm{a}$ is the fibre of $G_\ell \to A$ (coordinate s_2), $\pi^{-1}(B) = s(B) \times F$, and pr_F is the projection to F; so G' is the graph of $\mu\,\mathrm{pr}_F$ into the first factor $\mathbb{G}_\mathrm{a}$ (coordinate s_1), and for $B = 0$ it is $\{s_1 = \mu s_2\}$. In every case allowed by the conditions, $b = \dim B \leq n$.

Step 4: no obstruction. If $\Psi''(T_{G'}) \in \{0, \mathbb{C}(1,0), \mathbb{C}(0,1)\}$ then $\lambda = n$: if $\exp(\sum c_h \tilde y_h) \in G'$ for the lifts $\tilde y_h$ of $\theta_h v$, then $xv \in L + \mathcal{P}$ with $x = \sum c_h\theta_h \in K$, so $x = 0$ by Step 1. With $\delta_0 = 0$ for types A, B, E and $\delta_0 = 1$ for C, D, inequality (4) reads: $5n + 2 \leq 4n + 4$ for $G' = 0$; $5n + 2 - 2b \leq 2n + 2$ for types B and D ($B \neq 0$); $5n - 2b \leq 2n + 2$ for type C with $r = 1$; and $\lambda + 4n + 2 \leq 2n + 2$ for type E with $B = 0$. Since $b \leq n$, none of these holds for $n \geq 3$, and types with $r = 2$ fail $\delta > \tau$. So no connected $G' \neq G''$ satisfies the conclusion of Theorem 2.1, a contradiction; condition (2) reads $4n + 2 \geq 2n + 2$. □

Corollary 6.5. $\Gamma(1/4)^2\,\Gamma(1/3)^3/\pi^{5/2}$ *is transcendental.*

This is immediate from Wüstholz's theorem [36] (a period of E_i and a quasi-period of E_ρ are $\overline{\mathbb{Q}}$-linearly independent), by the explicit formulas for ϖ and Ω_ρ, which give $\varpi\Omega_\rho/\pi \in \overline{\mathbb{Q}}^\times \cdot \Gamma(1/4)^2\Gamma(1/3)^3/\pi^{5/2}$ and $\varpi/\eta_\rho = \frac{\sqrt{3}}{2\pi}\varpi\Omega_\rho$.

6.3. **A twisted variant, and degree two.** Exchanging the roles of base and fibre on E_ρ gives a second statement of the same kind. Let $G = E_i^\natural \times E_\rho^\natural \times \mathbb{G}_\mathrm{a}^2$ and

$$\Psi'(X_i, X_\rho, a_1, a_2) = \big(z(X_i) + w(X_\rho) + a_1,\ w(X_i) + z(X_\rho) + a_2\big), \qquad V' = \mathbb{C}\,w_1', \quad w_1' = (\varpi, \beta\Omega_\rho).$$

With $w_2' = (\eta_\rho, \frac{2\beta}{\sqrt{3}}\eta_i)$ one has $w_2' = (\eta_\rho/\varpi)\,w_1'$ by the two Legendre relations, and for $K \supseteq \mathbb{Q}(i, \sqrt{-3}, \beta)$ both lie in $\Psi'(\Omega_K(G)) = (K\varpi + K\eta_\rho) \times (K\eta_i + K\Omega_\rho)$. The vectors w_1', w_2' are K-independent because $\eta_\rho/\varpi \notin \overline{\mathbb{Q}}$ (Corollary 6.5).

Proposition 6.6 (Proposition D′)**.** *Let $K \supseteq \mathbb{Q}(i, \sqrt{-3}, \beta)$ be a number field of degree $n \geq 8$. Then $V' \cap \Psi'(\Lambda_K(G)) = Kw_1' + Kw_2'$.*

Proof. With θ, p_θ and $\Psi'_\theta = \Psi' \circ p_\theta$ as before, a canonical lift $B^\natural$ of an abelian subvariety $B_{S_i} \times B_{S_\rho}$ has tangent space in $\ker\Psi'_\theta$ only if $\psi(S_\bullet) = \phi(S_\bullet) = 0$, i.e. $B = 0$ (Lemma 3.2). So the largest connected subgroup U' with $T_{U'} \subset \ker\Psi'_\theta$ is the vector subgroup $\{\theta^\mathsf{T}(w_\rho + a_1) = 0,\ \theta^\mathsf{T}(w_i + a_2) = 0\}$, and $G'' = G^n/U'$ is the pushout of $(E_i^n \times E_\rho^n)^\natural$ along $(w_i, w_\rho) \mapsto (\theta^\mathsf{T} w_i, \theta^\mathsf{T} w_\rho)$, an extension of $E_i^n \times E_\rho^n$ by $\mathbb{G}_\mathrm{a}^2$ of dimension $2n + 2$. It is a quotient of an anti-affine group, so

$\mathrm{Hom}(G'',\mathbb{G}_\mathrm{a})=\mathrm{Hom}(G'',\mathbb{G}_\mathrm{m})=0$: $d_0=d_1=0$ (maximal), $d_2=d=2n+2$. Let $\bar\Psi'\colon T_{G''}\to\mathbb{C}^2$ be the map induced by Ψ'_θ, $\bar V'=\bar\Psi'^{-1}(V')$ (a hyperplane) and $W''=\ker\bar\Psi'\subset\bar V'$, $\overline{\mathbb{Q}}$-rational of dimension $t_W=2n$. Suppose $v\in V'\cap\Psi'(\Lambda_K(G))$ with $v\notin Kw'_1+Kw'_2$, and let $Y''\subset\bar V'$ be generated by the images of lifts in $\Lambda(G^n)$ of $D\theta_h w'_1$, $D\theta_h w'_2$ (in $\Omega(G^n)$, D a common denominator) and $\theta_h v$, $h=1,\dots,n$. Since w'_1,w'_2,v are K-independent, their $\bar\Psi'$-images are $\mathbb{Q}$-independent, so $m=\mathrm{rk}\,Y''=3n$; the $2n$ period lifts give $\kappa\ge 2n$, and (2) reads $4n+2\ge 2n$. The proof of Theorem 2.2 also applies Theorem 2.1 to a subgroup $G'\subset G''$ not in $\bar V'$, with $d'_0=d'_1=0$; (2) holds there too. If G' has abelian image of dimension b and vector part of dimension f, then $\kappa'\le 2b$; if $f=0$ the periods in question lie in a proper complex subspace of $T_{G'}$, so $\kappa'\le 2b-2$. The κ-refined Theorem 1.1 of [27, §5d] (Theorem 2.2) applies to $(G'',\bar V',W'',Y'')$: condition (5.4), $m>(d_1+2d_2-\kappa)(d-1-t_W)$, holds since $(4n+4-\kappa)\cdot 1\le 2n+4<3n$ for $n>4$, hence for $n\ge 8$ ($4\mid n$). So $\bar V'$ contains $T_{G'}$ for some connected algebraic subgroup $G'\ne 0$ defined over $\overline{\mathbb{Q}}$. Then $\bar\Psi'(T_{G'})$ is a $\overline{\mathbb{Q}}$-rational subspace of V', which has no nonzero $\overline{\mathbb{Q}}$-rational point ($\Omega_\rho/\varpi\notin\overline{\mathbb{Q}}$); so $T_{G'}\subset\ker\bar\Psi'$, the preimage of G' in G^n is a connected subgroup with tangent space in $\ker\Psi'_\theta$, and $G'=0$ by the maximality of U', a contradiction. □

Propositions 6.4 and 6.6 also apply to data that are periods plus algebraic constants; their single-point forms suffice. This gives statements about products of periods of the two curves.

Corollary 6.7. (i) *The numbers π, ϖ, Ω_ρ, $\varpi\Omega_\rho$ are linearly independent over $\overline{\mathbb{Q}}$; equivalently, so are $\pi,\eta_i,\eta_\rho,\eta_i\eta_\rho$.*

(ii) *The numbers 1, ϖ, Ω_ρ, $\varpi\Omega_\rho/\pi$ are linearly independent over $\overline{\mathbb{Q}}$; equivalently, so are $\pi,\varpi,\eta_\rho,\varpi\eta_\rho$.*

Proof. The equivalences follow by multiplying a relation by $\eta_i\eta_\rho/\pi$, resp. by η_ρ, and using $\varpi\eta_i=\pi$, $\Omega_\rho\eta_\rho=2\pi/\sqrt{3}$. We use that $1,\pi,\varpi,\eta_i,\Omega_\rho,\eta_\rho$ are linearly independent over $\overline{\mathbb{Q}}$ (analytic subgroup theorem [36] on $E_i^\natural\times E_\rho^\natural\times\mathbb{G}_\mathrm{m}\times\mathbb{G}_\mathrm{a}$).

(i) A relation with zero coefficient on $\varpi\Omega_\rho$ is excluded by that independence. A relation $c_1\varpi+c_2\Omega_\rho+c_3\varpi\Omega_\rho=0$ with zero coefficient on π is excluded too: multiplying it by $\pi/(\varpi\Omega_\rho)$ and using $\pi/\varpi=\eta_i$, $\pi/\Omega_\rho=\frac{\sqrt3}{2}\eta_\rho$ gives $c_3\pi+c_2\eta_i+\frac{\sqrt3}{2}c_1\eta_\rho=0$, so $c_1=c_2=c_3=0$. So suppose $\pi=A\varpi\Omega_\rho+B\varpi+C\Omega_\rho$ with $A\ne0$. In the set-up of §6.3 take $\beta=A$, $X=(\varpi,\eta_i)\in\Lambda(E_i^\natural)$, $Y=0$, $a=C/A$, $b=-B$. The point $v=\Psi'(X,Y,a,b)=(\varpi+C/A,\ \eta_i-B)$ of $\Psi'(\Lambda(G))$ lies on $V'=\mathbb{C}(\varpi,A\Omega_\rho)$: $(\varpi+C/A)\,A\Omega_\rho=\pi-B\varpi=(\eta_i-B)\varpi$. By Proposition 6.6, $v=x(\varpi,A\Omega_\rho)+x'(\eta_\rho,\frac{2A}{\sqrt3}\eta_i)$ with $x,x'\in\overline{\mathbb{Q}}$. The first coordinate, $\varpi+C/A=x\varpi+x'\eta_\rho$, and the independence of $1,\varpi,\eta_\rho$ give $C=0$, $x=1$, $x'=0$. Then $\eta_i-B=A\Omega_\rho$, which contradicts the independence of $1,\eta_i,\Omega_\rho$.

(ii) Suppose $\varpi\Omega_\rho/\pi=a+b\varpi+c\Omega_\rho$ (a relation with zero left-hand coefficient is excluded by the independence above). If $c=0$, dividing by ϖ and using $\pi/\varpi=\eta_i$, $\Omega_\rho=2\pi/(\sqrt3\eta_\rho)$ gives $\Omega_\rho=a\eta_i+b\pi$. If $b=0$, multiplying by η_ρ gives $\frac{2}{\sqrt3}\varpi=a\eta_\rho+\frac{2c}{\sqrt3}\pi$. Both contradict the independence above, so $b,c\ne0$. Put $t=-1/c$, $\beta=-\sqrt3\,b/(2c)$, $a_1=a/b$, and take $X=(\varpi,\eta_i+t)\in\Lambda(E_i^\natural)$, $Y=0$. Then $v=(\varpi+a_1,\eta_i+t)\in\Psi(\Lambda(G))$ lies on $V=\mathbb{C}(\varpi,\beta\eta_\rho)$: expanding $(\varpi+a_1)\beta\eta_\rho=(\eta_i+t)\varpi$ with $\varpi\eta_i=\pi$ and $\eta_\rho=2\pi/(\sqrt3\Omega_\rho)$ gives exactly the assumed relation. Proposition 6.4 gives $v=x(\varpi,\beta\eta_\rho)+x'(\Omega_\rho,\frac{2\beta}{\sqrt3}\eta_i)$ with $x,x'\in\overline{\mathbb{Q}}$. The first coordinate and the independence of $1,\varpi,\Omega_\rho$ give $a_1=0$, $x=1$, $x'=0$; then $\eta_i+t=\beta\eta_\rho$, contradicting the independence of $1,\eta_i,\eta_\rho$. □

Both statements are genuinely of degree two: rewriting with the Legendre relations always leaves a product of periods of the two different curves, so under the period conjecture they are not consequences of the analytic subgroup theorem. They are predicted by the period conjecture (by

Chowla–Selberg [9] they concern π, $\Gamma(1/4)$, $\Gamma(1/3)$, whose algebraic independence is open), and they do not follow from known algebraic independence results. As far as we found they are not in print, but they are consequences of [22, Théorème (iii)] as printed (through Propositions 6.4 and 6.6, §2.4) and Wüstholz's theorem. As far as we know they do not follow from known algebraic independence results; they are the claims here that most deserve an expert check. Neither statement contains the constant 1 together with $\varpi\Omega_\rho$, so they do not give $\varpi\Omega_\rho \notin \overline{\mathbb{Q}}$, a case of a conjecture of Waldschmidt [30] that we do not claim.

7. e^{π^2} AND $\wp(1/\pi)$

This is the one application on a subspace of codimension 2. There Theorem 2.1, applied subgroup by subgroup, is stronger than the uniform bound of Theorem 2.3: at $G' = 0$, (1) divides the threshold by $d - N$ [27, (4.2)]. Here $W'' \neq 0$ and G'' has a torus, but $\kappa = 0$ and (2) holds. Since G'' is isogenous to $\mathbb{G}_{\mathrm{a}}^2 \times \mathbb{G}_{\mathrm{m}}^{2n} \times E^{n/2}$, this instance of Theorem 2.1 is covered by the Lean proof of [11]; the identification and the deduction below are not formalised. The proof also uses Schneider's theorem [23] and Chudnovsky's theorem (1976; see [6]) that π and Ω are algebraically independent.

Theorem 7.1 (Theorem P)**.** *Let $E/\overline{\mathbb{Q}}$ have complex multiplication. If $\pi^2 \in \overline{\mathbb{Q}}\mathcal{L}$ (for instance if $e^{a\pi^2}$ is algebraic for some $a \in \overline{\mathbb{Q}}^\times$), then $1/\pi \notin \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E$. In particular, for $a, c \in \overline{\mathbb{Q}}^\times$ and $b \in \overline{\mathbb{Q}}$, $e^{a\pi^2}$ and $\wp_E(c/\pi + b)$ are not both algebraic; e.g. one of e^{π^2}, $\wp_i(1/\pi)$ is transcendental on $y^2 = 4x^3 - 4x$, and one of e^{π^2}, $\wp_\rho(1/\pi)$ on $y^2 = 4x^3 - 4$.*

Proof. Let $K \supseteq k(i, a)$ contain the coefficients expressing $\pi^2 \in \overline{\mathbb{Q}}\mathcal{L}$ and (for a contradiction) $1/\pi \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E$, $n = [K : \mathbb{Q}]$. Let $G = \mathbb{G}_{\mathrm{m}} \times (\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}) \times (\mathbb{G}_{\mathrm{a}} \times E)$ and let Ψ read the slots into $\mathbb{C}^3$: coordinate 1 from $\mathbb{G}_{\mathrm{m}}$, coordinate 2 from $\mathbb{G}_{\mathrm{a}} + \mathbb{G}_{\mathrm{m}}$, coordinate 3 from $\mathbb{G}_{\mathrm{a}} + E$; $V = \mathbb{C}(1, 1/\pi, 1/\pi^2)$, $W = 0$. After the quotient by the largest subgroup in $\ker \Psi_\theta$, $G'' = \mathbb{G}_{\mathrm{m}}^n \times (\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}^n) \times (\mathbb{G}_{\mathrm{a}} \times A'')$ with $\dim A'' = n/2$: $d_0 = 2$, $d_1 = 2n$, $d_2 = n/2$, weight $3n$; V'' has codimension 2 and $W'' = \ker \Psi''$. The period term is $\kappa = 0$: a period vector $(\pi a', \pi b', \Omega c')$ $(a', b', c' \in K$, up to the factor $2i)$ lies on V only if $\pi b' = \pi a'/\pi$, which forces $a' = b' = c' = 0$. The data are $H\colon \lambda = \pi^2 \mapsto (\pi^2, \pi, 1)$ and $X\colon \lambda = \pi \mapsto (\pi, 1, 1/\pi)$, which are K-independent, so $m = 2n$. Modulo periods (multiples of π in coordinates $1, 2$), $H \equiv (\pi^2, 0, 1)$ and $X \equiv (0, 1, 1/\pi)$; since $\Omega \notin \overline{\mathbb{Q}}$ (Schneider) and $\pi\Omega \notin \overline{\mathbb{Q}}$ (Chudnovsky), $1 \notin K\Omega$ and $1/\pi \notin K\Omega$, so the third coordinates are not periods.

Every connected subgroup is (vector part)×(subtorus)×(abelian part), and its image U' is the sum of a $\overline{\mathbb{Q}}$-rational subspace of coordinates $2, 3$, a K-rational subspace of coordinates $1, 2$ and possibly axis 3; with Ψ'' onto $\mathbb{C}^3$ and $W'' = \ker \Psi''$, $\delta - \tau = 3 - \dim U'$, $d - N = 2$, and (1) reads $(\lambda_{U'} + 3n - \operatorname{wt} G') \cdot 2 \leq (3 - \dim U') \cdot 3n$, where $\lambda_{U'} = n \cdot \#\{\text{data not in } U'\text{+periods}\}$ (in this cell the K-rank of the combinations $\alpha H + \beta X$ in U'+periods agrees with the datum-by-datum count). The twelve image types:

G' (image U')	wt	$\lambda_{U'}$	left	right
0	0	$2n$	$10n$	$9n$
torus of coordinate 1 or 2, or the abelian part (an axis)	n	$2n$	$8n$	$6n$
$\mathbb{G}_{\mathrm{a}}$ of coordinate 2 or 3 (an axis)	0	$2n$	$10n$	$6n$
K-line in the $(1,2)$-plane (a subtorus)	n	$2n$	$8n$	$6n$
$\overline{\mathbb{Q}}$-line in the $(2,3)$-plane (a vector subgroup)	0	$2n$	$10n$	$6n$
plane $(1,2)$	$2n$	$2n$	$6n$	$3n$
plane $(1,3)$	$2n$	n (H inside)	$4n$	$3n$
plane $(2,3)$	$2n$	n (X inside)	$4n$	$3n$
K-line in $(1,2)$ + axis 3	$\leq 2n$	$2n$	$\geq 6n$	$3n$
axis 1 + $\overline{\mathbb{Q}}$-line in $(2,3)$; K-line in $(1,2)$ + $\overline{\mathbb{Q}}$-line in $(2,3)$	$\leq n$	$\geq n$	$\geq 6n$	$3n$
$\dim U' = 3$				$\delta \leq \tau$

(On a K-line of the $(1,2)$-plane, H and X modulo periods project to slopes 0 and ∞, with nonzero third coordinates; the axis cases use the two transcendence inputs.) No G' satisfies (1). Condition (2) with $N = d-2$, $d = 5n/2+2$: $5n \geq 2n+4$. So the two data cannot coexist. For the second sentence, if $e^{a\pi^2} \in \overline{\mathbb{Q}}$ and $\wp_E(c/\pi + b) \in \overline{\mathbb{Q}} \cup \{\infty\}$, then $a\pi^2 \in \mathcal{L}$ and $c/\pi + b \in \mathcal{L}_E$, so $1/\pi \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E$. □

Controls and credit. Each datum alone is consistent ($8n \leq 9n$ at $G' = 0$), and a non-CM curve gives $G' = 0$ at equality. Theorem 2.3 on this line allows three data and gives nothing. Replacing E by $\mathbb{G}_{\mathrm{m}}$ gives the classical partner: the strong six exponentials theorem gives that one of e^{π^2}, $e^{1/\pi}$ is transcendental [8, Cor. 1(3)]. As far as we found Theorem 7.1 is not in print. $\wp_E(c/\pi) \notin \overline{\mathbb{Q}}$ alone is not known to us. Numerically, $\wp_i(1/\pi) = 9.889882511\ldots$, $\wp_\rho(1/\pi) = 9.871070987\ldots$.

8. Further results of the same kind

The following are obtained in the same way, by applying Theorem 2.1 to the quotient G'' named in the table, with $W \neq 0$, $\kappa > 0$ and (2). We do not prove them here. The table gives each derivation in one line, as a proof sketch; the proofs, with their case checks, are in [13], except for the row on other pairs of CM curves, which remains a sketch. Nothing above depends on them.

statement	derivation
$\wp(\log 2/\varpi)$ and $\wp(\log 3/\varpi)$ are not both algebraic; likewise at other $t \in \overline{\mathbb{Q}}^\times\{1/\Omega, \Omega^2/\pi, \dots\}$ (Waldschmidt's Conjecture 42 [32, p. 173] at CM ratios)	$\Psi = (x,\, w+z)$ on $\mathbb{G}_{\mathrm{m}} \times E^\natural \times E$; $G'' = \mathbb{G}_{\mathrm{m}}^n \times Q'' \times A''$, $\alpha(G'')/n = 3 + 2/n$, $\kappa = 1$, data 3
$\wp_i(\Omega_\rho/\varpi)$ and $\wp_i(\Omega_\rho u/\varpi)$ are not both algebraic for $u \in \mathcal{L}_{E_i} \setminus \{0\}$ ($t = c\Omega'/\Omega$, $k \neq k'$)	$\Psi = (a_1 + z_1,\, a_2 + z_2 + z')$ on $\mathbb{G}_{\mathrm{a}} \times E \times \mathbb{G}_{\mathrm{a}} \times E \times E'$; $G'' = \mathbb{G}_{\mathrm{a}}^2 \times A_1 \times A_2 \times A'$, $\alpha(G'')/n = 3$, $\kappa = 1$, data 3
If e^{π^2}, $e^{c\pi\Omega} \in \overline{\mathbb{Q}}$ then $\alpha^{\Omega/\pi} \notin \overline{\mathbb{Q}}$ when $2\pi i, \pi^2, \log\alpha$ are $\mathbb{Q}$-independent	Theorem 4.1 with $x_1 = \pi^2$, $x_2 = \log\alpha$ (this row is complete)
$\varpi\Omega_\rho(\zeta_i(u) + t)/(\pi(u + a)) \notin \overline{\mathbb{Q}}$ for $u \in \mathcal{L}_{E_i} \setminus \mathbb{Q}(i)\varpi$, $a, t \in \overline{\mathbb{Q}}$	Theorem 6.3 at $X = (u, \zeta_i(u))$, $Y = 0$; the value 0 is excluded by Laurent [14] (this row is complete)
Theorem 6.3 for other pairs of CM curves; $(\zeta_i(u) + b)/(u + a) \notin \overline{\mathbb{Q}}^\times\Omega_\rho/\varpi$	same proof; Proposition 6.6 with Masser and Wüstholz
1, $B(\frac{2}{7}, \frac{2}{7})$, $B(\frac{3}{7}, \frac{3}{7})$, $B(\frac{3}{7}, \frac{3}{7})/B(\frac{1}{7}, \frac{1}{7})$ are $\overline{\mathbb{Q}}$-linearly independent; analogues at levels 9, 24 and with several pairs	the first-kind relation $B(\frac{1}{7}, \frac{1}{7})B(\frac{2}{7}, \frac{2}{7}) = \frac{\sin(3\pi/7)}{\sin(\pi/7)}B(\frac{3}{7}, \frac{3}{7})B(\frac{1}{7}, \frac{2}{7})$ as a second period on the line (identifying the slots uses Deligne and Koblitz–Ogus); $G'' = \prod_j A_j'' \times \mathbb{G}_{\mathrm{a}}^2$, $\alpha(G'')/n = 4$, $\kappa = 2$, data 3
Strong elliptic four exponentials on eigen-slots, at $t \in \overline{\mathbb{Q}}^\times(P'/P_\sigma)^{\pm 1}$; four exponentials for quasi-logarithms	Theorem 5.7 with A in place of E; fibre coordinates in both targets

9. Formal verification and literature

A Lean 4 certificate [10] (Lean and Mathlib v4.35.0) checks the statements in Table 1, on top of the public project [11], which proves Theorem 2.1 under (2) for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$ and, from it, the instances of Theorem 2.2 used. Every declaration depends only on Lean's standard axioms; there is no `sorry`, no `axiom` declaration and no `native_decide`. Named classical inputs enter as explicit hypotheses, never as axioms. "None" in the last column means that no hypothesis remains, subject to the following caveats.

- The Lean proofs follow the route through Theorem 2.2, not our direct case checks: (5.4) reads $3n > 3n - \kappa$ for Theorem 4.1, $3n > 3n + 2 - \kappa$ for Theorem 4.3, $3n > 4n + 2 - \kappa$ for Proposition 6.4, $3n > 4n + 4 - \kappa$ for Proposition 6.6, with $\kappa = n, n, 2n, 2n$, and the subgroup exclusion then needs only that V has no rational point. So what is checked is the statements, not the text of our proofs.
- The group models in the certificate are hand-written. That a model's logarithms, periods and subgroups are true of the group was checked by reviews, not by Lean. For the statements on explicit numbers with no hypothesis left, this affects only whether the proved hypothesis is an instance of Waldschmidt's theorem, not the statement.
- On $y^2 = 4x^3 - 4$, $g_2 = 0$ is proved, but $g_3(\Omega_\rho\mathbb{Z}[\rho]) \in \overline{\mathbb{Q}}$ is a hypothesis: $g_3 = 4$ is classical and not formalised.
- At $B(\frac{1}{5}, \frac{1}{5})$, Lean presents A by its period lattice on $Y^2 = 4X^5 + 1$ and computes the periods as Beta values; the identification with the Jacobian of $y^5 = x(1 - x)$ is classical and not formalised, and the final statements, about explicit numbers, do not depend on it. The hyperplane statement proved there is restricted to toric-injective hyperplanes (no nonzero period of the abelian part alone in V) and to data whose A-components are periods, which is all those proofs use; likewise for the $\mathbb{G}_{\mathrm{a}}$-extension of A.

- The theorems on $E^\natural$ use hand-coded quasi-periods, checked numerically. Proposition 6.6 goes by a separate route of [11] (vector part of dimension 2).
- All review so far is by AI referees; no human expert has reviewed the definitions or the proofs. The results were found with substantial use of AI tools.

Not formalised: everything not in the table; in particular the $t \leftrightarrow 1/t$ extensions at ϖ, Ω_ρ; Theorems 1.1, 1.2 for other eigen-periods (at $B(\frac{1}{7},\frac{1}{7})/\pi$, Theorem 1.1 is checked conditionally on Theorem 2.2 for the group of genus 3), Theorem 1.2 for $g \geq 2$; the K-forms with logarithms of A, $A^\natural$; the K-span forms of Propositions 6.4, 6.6; the passage to an arbitrary $K \supseteq k$ by Baker's theorem; the mixed second rows of Theorem 5.2; Corollary 5.6; (M2); Corollary 6.5 (a named hypothesis); Lemmas 3.1, 3.2; Theorem 7.1; and the rest of §8 (the level-seven row is checked, `AbVar.x7c`, conditionally on Theorem 2.2 for its groups, Chowla–Selberg at level seven and a Wüstholz input).

9.1. **What we read.** The closest published theorem to our four exponentials statements is Waldschmidt's [31, Thm. 4.2], a mixed statement on $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m} \times E$ without CM whose second row consists of elliptic logarithms; it needs seven columns in general, and five in its period-column case. The nearest to Theorem 6.3 we know is [31, Thm. 4.3], for 2×9 matrices of elliptic logarithms, without quasi-periods. Apart from what follows from [33] as in the introduction, we found no t of the kind of Theorem 1.1 at which the conjecture was stated for all pairs, and no real such t at which it was known for a single pair with conjugation-stable span of logarithms. We looked in Waldschmidt's survey [34] (read in full), in [31, 33] and elsewhere. Roy's Luminy paper [21] (read in full) has only general bounds, not statements at particular periods. We could not consult Bertrand [4], Waldschmidt [28], Baker–Wüstholz [1, §6.2], Bertrand–Laurent [5], Bertrand [3], Reyssat (Bull. SMF 108, 1980; Acta Arith. 41, 1982), Masser (Mém. SMF 2, 1980), Tubbs (J. Number Theory, 1987, 1990), or the MathSciNet citation lists of [22, 27, 15]; any of these might contain some of these statements.

statement	Lean (`CM4E`; `U` = `Unconditional`, `B` = `Bridge`)	hypotheses left
Cor. 4.4 at $\Omega \in \overline{\mathbb{Q}}^\times \varpi$, every c; $2^\varpi, 3^\varpi$	`U.fourExpAt_varpi_all`, `two_three_varpi`	none
Cor. 4.2, same; $2^{\varpi/\pi}, 3^{\varpi/\pi}$	`U.fourExpAt_varpi_div_pi_all`, `two_three`	none
the same at $\Omega \in \overline{\mathbb{Q}}^\times \Omega_\rho$	`U.fourExpAt_omegaRho_all`, `_div_pi_all`, `two_three_omegaRho`	g_3
Thm. 1.1 at $t \in \overline{\mathbb{Q}}^\times B(\frac{1}{5},\frac{1}{5})^{\pm 1}$; $2^t, 3^t$ at $t = \Gamma(\frac{1}{5})^2/\Gamma(\frac{2}{5})$	`B.fourExpAt_beta5'_uncond`, `fourExpAt_div_beta5_uncond`, `two_three_beta5'_uncond`	none
Thm. 1.1 at $t \in \overline{\mathbb{Q}}^\times (B(\frac{1}{5},\frac{1}{5})/\pi)^{\pm 1}$; $2^t, 3^t$ at $t = B(\frac{1}{5},\frac{1}{5})/\pi$	`B.fourExpAt_beta5_uncond`, `fourExpAt_pi_div_beta5_uncond`, `two_three_beta5_uncond`	none
Thm. 4.1 at ϖ, $K \ni i, c$; Thm. 4.3 at ϖ, $K \ni c$, $[K:\mathbb{Q}] \geq 4$	`U.mg_K_varpi`, `mgn_K_varpi`	none
Thms. 4.5, 4.6 at $(1-\zeta_5)B(\frac{1}{5},\frac{1}{5})$, $(1-\zeta_5^4)B(\frac{4}{5},\frac{4}{5})$, with $z_\sigma(\Lambda(A))$, $w_\sigma(\Lambda(A^\natural))$ replaced by $\mathbb{Z}P$	`B.mgA5_K_uncond`, `mgAn5_K_uncond`	none
Thm. 5.2 at ϖ, second rows in $\widetilde{\mathcal{L}}$	`Gen.smg_false`, `Gen.smgn_false` with the bridges	none; at Ω_ρ: g_3
Thm. 1.2 for $g = 1$ at ϖ, Ω_ρ	`U.thmS_varpi`, `thmS_omegaRho`	Wüstholz: $\Omega, \eta \notin \widetilde{\mathcal{L}}$; at Ω_ρ also g_3
Cor. 4.7 on E_i	`U.mg1_varpi_abc`	Wüstholz: $\varpi, \eta(\varpi) \notin \widetilde{\mathcal{L}}$
Cor. 4.7 at $(1-\zeta_5)B(\frac{1}{5},\frac{1}{5})$	`B.corA2_beta5_uncond`	Wüstholz: $P \notin \widetilde{\mathcal{L}}$
Thm. 5.7 on E_i	`B.thmSE0_varpi`, `thmSE0_paper`	none; on E_ρ: g_3
Ex. 5.10: $\wp_i(\pi/\varpi^2)$, $\wp_i(\pi u/\varpi^2)$	`B.thmSE0_varpi_Ltilde`	for $u \notin \overline{\mathbb{Q}}\varpi$: $\varpi, 1, u$ $\overline{\mathbb{Q}}$-independent is the input (Wüstholz); Lean states membership in $\widetilde{\mathcal{L}}_E$
§8: $t = c\Omega'/\Omega$ on (E_i, E_ρ)	`U.thmSE_varpi_omegaRho`	g_3
Thm. 6.1 at $c\varpi$, first conclusion; (M1), (M3) with $\Omega \in \overline{\mathbb{Q}}^\times \varpi$	`U.thmM_varpi`, `m1_varpi`, `m3_varpi`	none; at Ω_ρ (`U.thmM_omegaRho`): g_3
$s \in \mathbb{Q}(i)$ in Thm. 6.1 at ϖ; (Mη); (M3′)	`U.thmM_varpi_k`, `meta_varpi`, `m3'_varpi`	Masser; Masser; Hermite–Lindemann
Thm. 6.3; Prop. 6.4, one v, coefficients in $\overline{\mathbb{Q}}$; §8: the ratio on E_i	`U.mx`, `mx_K`, `mx1`	g_3; Wüstholz ($\varpi\Omega_\rho/\pi \notin \overline{\mathbb{Q}}$; 1 and elliptic logarithms); Masser; for the ratio also Laurent [14]
Prop. 6.6, one v, coefficients in $\overline{\mathbb{Q}}$	`U.mxP_K`	g_3; $\varpi\Omega_\rho/\pi \notin \overline{\mathbb{Q}}$
Cor. 6.7(i), (ii)	`U.deg2_i`, `deg2_ii`	g_3; Wüstholz: $1, \pi, \varpi, \eta_i, \Omega_\rho, \eta_\rho$ $\overline{\mathbb{Q}}$-independent (hΓ, implied by the latter)

TABLE 1. Statements checked in Lean.

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