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A LEAN FORMALIZATION OF WALDSCHMIDT'S SUBGROUP THEOREM WITH PERIOD TERM FOR GROUPS WITH ELLIPTIC FACTORS

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Abstract. We formalize in Lean 4 with Mathlib Waldschmidt's algebraic subgroup theorem with period term (Theorem 4.1 of his 1988 Durham paper) for $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$, with E_j elliptic curves with algebraic invariants, under the additional hypothesis $2N \geq d_1 + 2d_0 + \kappa$. The proof uses only Lean's three standard axioms. Along the way we give what we believe are the first formal proofs of Schneider's theorems on elliptic functions and of the Schneider–Lang criterion, and the first Lean library for the Weierstrass σ and ζ functions. Restricted forms of the theorem cover some vector extensions and the Jacobian of $y^5 = x(1-x)$. A companion certificate uses these to prove explicit cases of the four exponentials conjecture: for example, 2^{ϖ} and 3^{ϖ} are not both algebraic, where ϖ is the real period of $y^2 = 4x^3 - 4x$, and likewise $2^{B(1/5,1/5)}$ and $3^{B(1/5,1/5)}$.

1. Introduction

This paper is a sequel to [6], which formalized Roy's strong six exponentials theorem in Lean. That development proved, as its main intermediate result, Waldschmidt's transcendence theorem on linear groups $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1}$ in the form of Roy's Theorem 1 [16], together with Philippon's multiplicity estimate on those groups. Here we leave the linear world and treat groups with elliptic factors.

1.1. **The theorem.** Let G be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ of dimension d, T_G its tangent space at the origin with its $\overline{\mathbb{Q}}$-structure, $\exp_G \colon T_G \to G(\mathbb{C})$ the exponential map, $\Omega(G) = \ker \exp_G$ and $\Lambda(G) = \exp_G^{-1} G(\overline{\mathbb{Q}})$. Write $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2$. Waldschmidt's Theorem 4.1 of [21] is the following.

Theorem 1.1 (Waldschmidt [21, Thm. 4.1, pp. 385–386])**.** *Let $V \subset T_G$ be a subspace of dimension $N < d$, $W \subset V$ a subspace defined over $\overline{\mathbb{Q}}$, and $Y \subset V \cap \Lambda(G)$ a finitely generated subgroup; put $\kappa = \mathrm{rk}_{\mathbb{Z}}(Y \cap \Omega(G))$. Then*

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there is a connected algebraic subgroup $G' \neq G$ defined over $\overline{\mathbb{Q}}$ with $\delta > \tau$ and

$$(\lambda + \delta_1 + 2\delta_2)(d - N) \leq (\delta - \tau)(d_1 + 2d_2 - \kappa), \tag{1.1}$$

where $\delta = \operatorname{codim} G'$, δ_0 and δ_1 are the codimensions of the projections of G' to $\mathbb{G}_{\mathrm{a}}^{d_0}$ and $\mathbb{G}_{\mathrm{m}}^{d_1}$, $\delta_2 = \delta - \delta_0 - \delta_1$, $\lambda = \operatorname{rk}_{\mathbb{Z}} Y/(Y \cap \exp_G^{-1} G')$ and $\tau = \dim W/(W \cap T_{G'})$.

The *period term* is $-\kappa$ on the right of (1.1). For $\kappa = 0$ the theorem is the algebraic subgroup theorem in its usual form, a common source of the six exponentials theorem, Baker's theorem and their elliptic analogues [22, 23]. The refinement by κ matters when the data contain periods that do not span the Lie algebra of an algebraic subgroup, for instance a period of $\mathbb{G}_{\mathrm{m}} \times E$ lying on a line through a logarithm.

1.2. **What is formalized.** In Lean, the main theorem is

```
theorem LSG.W88.theorem41Abelian :  LSG.W88.Theorem41Abelian,
```

where the proposition `Theorem41Abelian` is defined in `LSG/W88/Statement.lean`. It is Theorem 1.1 for

$$G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m, \qquad E_j \colon y^2 = 4x^3 - g_2(\Lambda_j)x - g_3(\Lambda_j),$$

for arbitrary lattices Λ_j with $g_2, g_3 \in \overline{\mathbb{Q}}$, under the extra hypothesis

$$2N \geq d_1 + 2d_0 + \kappa. \tag{1.2}$$

(NR)

Section 2 reproduces the statement. Hypothesis (NR) is the range in which we prove Waldschmidt's Proposition 6.1 (the auxiliary function) with multiplicities along W; it is the condition of Lemma U of [**?**], which supplies the step of [21, Prop. 6.1] that is only indicated in print. (NR) is a genuine restriction (Remark 2.2). Correspondingly, results of this paper must be cited as "W88 Thm. 4.1 under (NR) for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$".

From the main theorem we derive, in Lean, its consequence of [21, §5d] (Theorem 1.1 with the κ-refined hypothesis (5.4)) for $\mathbb{G}_{\mathrm{m}}^{d_1} \times E$ and for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E^r$, and Theorem 1.1 for $\mathbb{G}_{\mathrm{a}}^{k_0} \times \mathbb{G}_{\mathrm{m}}^{k_1} \times B$ with B an arbitrary abelian subvariety of a product of elliptic curves (Section 8).

For groups whose factor G_2 is a push-out of $(E_1 \times \cdots \times E_m)^\natural$ along an algebraic matrix we prove Theorem 1.1 under (NR) and its §5d consequence only in restricted cases, listed in Theorem 9.1 (Section 9). Those results must be cited with their scope, never as "W88 Thm. 4.1 for vector extensions".

For $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times A^r$ with A the Jacobian of $y^5 = x(1-x)$ we prove Theorem 1.1 under (NR) and its §5d consequence for toric-injective hyperplanes and for subgroups Y whose A-components are (rational) periods (Theorem 10.1, Section 10). With the same restrictions we prove the §5d consequence for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times Q$, Q a push-out of powers of the $\mathbb{G}_{\mathrm{a}}$-extension $A_\sigma^\natural$ of A (Theorem 11.1, Section 11).

1.3. **Why it matters here: the applications.** The paper [**?**] proves four exponentials statements at CM periods by applying Theorem 1.1 to explicit groups whose subgroups it lists: for a CM curve with period Ω and $c \in \overline{\mathbb{Q}}^\times$, if $\log\alpha, \log\beta$ are $\mathbb{Q}$-linearly independent then $\alpha^{c\Omega}, \beta^{c\Omega}$ are not both algebraic, and neither are $\alpha^{c\Omega/\pi}, \beta^{c\Omega/\pi}$; for example $2^{\varpi/\pi}$ and $3^{\varpi/\pi}$ are not both algebraic, with $\varpi = \Gamma(1/4)^2/(2\sqrt{2\pi})$. A separate Lean certificate, `four-exponentials-cm-periods` [8], checks these deductions with Waldschmidt's theorem for the relevant groups as explicit hypotheses. The present work discharges those hypotheses whose groups are built from tori, additive groups and powers of one elliptic curve: the certificate's `Thm11Kappa` is our `LSG.W88.thm11Kappa` verbatim, and its `HypStdCMAt` is `LSG.W88.hypStdCMAt`. In the certificate this makes the four exponentials statement at $t = c\varpi/\pi$, for every algebraic $c \neq 0$, and the paper's "Theorem M" for $\Omega = c\varpi$ unconditional (standard axioms only). The statements at $t = c\varpi$ and the two-curve results use groups with universal vector extensions $E^\natural$. Their hypotheses fall within the restricted cases of Section 9, and are discharged there as well; in the certificate these results, too, no longer assume Waldschmidt's theorem (some of them still use other named classical inputs). The statements at the Beta value $B(1/5, 1/5)$ (Theorem A of [**?**] for the Jacobian of $y^5 = x(1-x)$) use the groups of Section 10 and are discharged there. The headline statement at $t = B(1/5, 1/5)$ itself uses the groups of Section 11 and is discharged there, again on toric-injective hyperplanes.

1.4. **First formal proofs.** To reach the main theorem we had to build the classical theory of elliptic functions in Lean beyond what Mathlib v4.35.0 provides (Mathlib has $\wp$, $\wp'$, the invariants g_2, g_3 and the differential equation for a `PeriodPair`). Section 4 describes the library: the Weierstrass σ function as an infinite product, ζ, the quasi-periods η, quasi-periodicity of σ, growth of order two, Legendre's relation, the addition theorem and the dictionary with Mathlib's algebraic group law on Weierstrass curves. On top of it we prove the Schneider–Lang criterion for meromorphic functions of finite order in one variable and Schneider's theorems [18]: if g_2, g_3 are algebraic, then every nonzero period ω and ω/π are transcendental, and $\wp(\alpha)$ is transcendental for algebraic $\alpha \notin \Lambda$.

Prior formalizations. We searched (on 30 October 2026: the web, the Mathlib v4.35.0 sources, and the prior-art collections we keep; the web again, more briefly, on 6 November 2026) for formal proofs of transcendence results about elliptic functions and found none in any proof assistant. The closest work we know: Karatarakis and Wiedijk's Lean proof of the Gelfond–Schneider theorem [5]; the library [14], which proves the Schneider–Lang criterion for $\mathbb{C}^{d_0} \times (\mathbb{C}^\times)^{d_1}$ (exponential functions only), Gelfond–Schneider and the qualitative form of Baker's theorem; an Isabelle/HOL development of elliptic functions and modular forms ($\wp$, the addition theorem, theta functions, the Jacobi triple product, Dedekind's η) [4], whose abstract and keywords mention no transcendence results; and the Lean files of [2], which

prove irrationality-exponent bounds for CM periods with uniformization, theta and CM inputs taken as hypotheses (Diophantine results, not transcendence; no σ, ζ or quasi-periods). With this evidence we believe that the formalization contains the first formal proofs of a transcendence theorem for elliptic functions and of the meromorphic Schneider–Lang criterion, the first Lean development of σ and ζ, and the first formal multiplicity estimate on a non-linear group. The search was not exhaustive, and we would be glad to learn of earlier work.

1.5. **Organization.** Section 2 gives the formal statement and its review. Section 3 describes the architecture. Sections 4–8 describe the three stages and the bridges, with emphasis on the places where the formal proof had to depart from the literature. Section 9 treats the vector-extension cases, Section 10 the Jacobian of $y^5 = x(1-x)$, Section 11 a vector extension of it, Section 12 the explicit consequences, Section 13 the artifact, and Section 14 what has and has not been checked.

2. The formal statement

The statement uses no algebraic-group theory. Everything is expressed on the tangent space $T_G = \mathbb{C}^{d_0} \times \mathbb{C}^{d_1} \times \mathbb{C}^m$, with $\exp_G(a,b,z) = (a, e^b, (\wp_j(z_j), \wp_j'(z_j))_j)$. The elliptic curves are Mathlib's: a `PeriodPair` `L` is a pair of $\mathbb{R}$-independent complex numbers, with lattice `L.lattice`, Weierstrass function `L.weierstrassP` and invariants $\texttt{L.g}_2$, $\texttt{L.g}_3$.

```
abbrev Tan (d0 d1 m : ℕ) := (Fin d0 → ℂ) × (Fin d1 → ℂ) × (Fin m → ℂ)

def Lam (Ls : Fin m → PeriodPair) : Set (Tan d0 d1 m) :=
  {p | AlgVec p.1 ∧ (∀ i, IsAlgebraic ℚ (Complex.exp (p.2.1 i))) ∧
    ∀ j, p.2.2 j ∈ (Ls j).lattice ∨
      IsAlgebraic ℚ ((Ls j).weierstrassP (p.2.2 j))}

-- carrier of the AddSubgroup ‘Per Ls‘:
  {p | p.1 = 0 ∧ (∀ i, ∃ k : ℤ, p.2.1 i = 2 * Real.pi * I * k) ∧
    ∀ j, p.2.2 j ∈ (Ls j).lattice}

def IsAbelianTangent (Ls : Fin m → PeriodPair)
    (t : Submodule ℂ (Fin m → ℂ)) : Prop :=
  (t.restrictScalars ℝ : Submodule ℝ (Fin m → ℂ)) =
    Submodule.span ℝ {z | z ∈ t ∧ ∀ j, z j ∈ (Ls j).lattice}

structure SubgroupData (Ls : Fin m → PeriodPair) where
  h0 : Submodule ℂ (Fin d0 → ℂ)
  U : Submodule ℚ (Fin d1 → ℚ)
  t : Submodule ℂ (Fin m → ℂ)
  h0_rat : RationalOverQbar h0
  t_ab : IsAbelianTangent Ls t
```

Here `Lam` is $\Lambda(G)$ (the lattice disjunct is redundant, because Mathlib's $\wp$ takes the junk value 0 at lattice points, but it is clearer), `Per` is $\Omega(G)$, and a `SubgroupData` describes the connected algebraic subgroup $G' = H_0 \times T_U \times B$ with tangent space $T_{G'} = h_0 \times U^{\perp} \times t$: h_0 is spanned by algebraic vectors, T_U

is the subtorus with Lie algebra $U^{\perp}$ (`torusLie U`), and t is the tangent space of an abelian subvariety $B = t/(t \cap \Lambda)$ of $E_1 \times \cdots \times E_m$. The codimensions are $\delta_0 = d_0 - \dim h_0$, $\delta_1 = \dim_{\mathbb{Q}} U$, $\delta_2 = m - \dim t$, $\delta = \delta_0 + \delta_1 + \delta_2$, and `Proper` means $\delta \neq 0$. Ranks of subgroups are $\operatorname{rk} Y = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} Y$. The theorem is:

```
def Theorem41Abelian : Prop :=
  ∀ (d0 d1 m : ℕ) (Ls : Fin m → PeriodPair),
    (∀ j, IsAlgebraic ℚ (Ls j).g₂ ∧ IsAlgebraic ℚ (Ls j).g₃) →
    ∀ (V W : Submodule ℂ (Tan d0 d1 m)) (Y : AddSubgroup (Tan d0 d1 m)),
      finrank ℂ V < d0 + d1 + m → W ≤ V → RationalOverQbarTan W →
      Y.FG → (∀ y ∈ Y, y ∈ V ∧ y ∈ Lam Ls) →
      -- (NR)
      d1 + 2 * d0 + rk (Y ⊓ Per Ls) ≤ 2 * finrank ℂ V →
      ∃ H : SubgroupData (d0 := d0) (d1 := d1) Ls, H.Proper ∧
        finrank ℂ W - finrank ℂ ↑(W ⊓ H.tan) < H.δ ∧
        (((rk Y - rk (Y ⊓ (H.tan.toAddSubgroup ⊔ Per Ls))
              + H.δ1 + 2 * H.δ2 : ℕ) : ℤ) *
            ((d0 + d1 + m : ℕ) - finrank ℂ V : ℤ) ≤
          ((H.δ : ℤ) - (finrank ℂ W - finrank ℂ ↑(W ⊓ H.tan) : ℕ)) *
            ((d1 + 2 * m : ℕ) - rk (Y ⊓ Per Ls) : ℤ))

theorem LSG.W88.theorem41Abelian : Theorem41Abelian
```

Since G' is connected, $\exp_G^{-1}(G') = T_{G'} + \Omega(G)$, so the first factor is $\lambda + \delta_1 + 2\delta_2$; the second factor of the right side is $\delta - \tau$.

Remark 2.1 (The subgroup family). The theorem quantifies over products $H_0 \times T_U \times B$. For $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times A$ with A an abelian variety, every connected algebraic subgroup is such a product (Goursat, and the vanishing in characteristic 0 of all Hom groups between additive, multiplicative and abelian pieces), and a complex subtorus of A is an abelian subvariety. So the family is exactly Waldschmidt's. Inside the proof, the obstruction subgroup produced by the multiplicity estimate is shown to have this form in Lean (Section 5.6).

Remark 2.2 (Hypothesis (NR)). (NR) is not in [21]. It excludes, for instance, the standard encoding of the Hermite–Lindemann theorem ($G = \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$, $V = W = \mathbb{C}(1, 1)$, $N = 1$, where $1+2 > 2$); adding $\mathbb{G}_{\mathrm{m}}$ factors restores (NR) but makes the conclusion trivially satisfiable by a subtorus. All applications in [**?**] satisfy (NR) with room to spare.

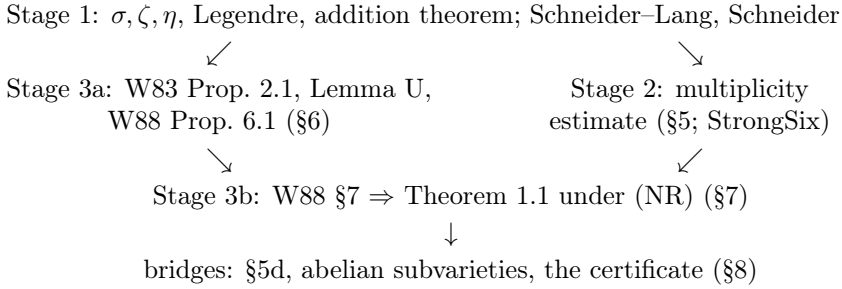
2.1. **Fidelity review.** An AI session that had not written the proof reviewed the statement against the page images of [21, pp. 385–386] and the formulation in [7]. Among other things it checked how Lean elaborates each cast and natural-number subtraction (all $\mathbb{N}$-subtractions are exact, and the three integer subtractions $d - N$, $\delta - \tau$, $d_1 + 2m - \kappa$ are the intended ones) and that the conclusion cannot be met trivially ($G' = G$ is excluded by `Proper`, and $G' = 0$ satisfies (1.1) only when the expected inequality fails). Its verdict was *correct*, with four minor remarks: a docstring reference to a file outside the repository (since fixed), the need to mention (NR) in every

citation, the redundancy of `Proper` given $\tau < \delta$, and the redundancy of the lattice disjunct in `Lam`.

2.2. **Sanity derivations.** As evidence that the statement has teeth, `LSG/W88/Sanity.lean` derives two classical theorems from the *proposition* `Theorem41Abelian` and then instantiates them: `LSG.W88.Sanity.gelfondSchneider` (with $G = \mathbb{G}_{\mathrm{m}}^2$, $V = W = \mathbb{C}(1, b)$, $Y = \mathbb{Z}(x, bx)$, $\kappa = 0$, where (NR) reads $2 \le 2$) and `LSG.W88.Sanity.period_transcendental'`, Schneider's period theorem (with $G = \mathbb{G}_{\mathrm{a}} \times E$, $V = W = \mathbb{C}(1, 1)$, $Y = \mathbb{Z}(\omega, \omega)$). In both, a case analysis shows that every admissible H violates the conclusion. The second derivation reproves a result of Stage 1 by an entirely different route.

3. Architecture

Waldschmidt's proof [21, §§6–7] has two inputs: an auxiliary polynomial P that vanishes to high order along W at many points of $\exp_G(Y)$ with the saving κ (Proposition 6.1, which rests on [20, Prop. 2.1, 2.4]), and Philippon's multiplicity estimate [15], which turns such a P into an obstruction subgroup. Section 7 of [21] combines them. The formalization has three stages and a layer of bridges:



Stage 1 also feeds the multiplicity estimate (the group law of E and its exponential) and Stage 3a (quasi-periodicity of σ). StrongSix [9] is a Lake dependency: Stage 2 reuses its filtered Hilbert functions, Bézout inequality, unmixedness, generic Wüstholz lemma and the classification of algebraic subgroups of linear groups, and the bridges reuse its Theorem 1. Two new block-generic files were added to StrongSix for this purpose; no existing StrongSix file was changed.

Each piece was first stated as a `Prop` interface (`SchneiderLang`, `WeierstrassOrderTwo`, `MultEstAb`, `Theorem41Abelian`) and the deductions between interfaces were proved before their inputs, as in [6]. For example `theorem41Abelian_of : MultEstAb → Theorem41Abelian` was complete before the multiplicity estimate was; the main theorem is then one line, `theorem41Abelian_of LSG.ME.multEstAb`.

4. Stage 1: elliptic functions and Schneider's theorems

4.1. **The Weierstrass library.** The directory `LSG/Weierstrass/` (15 files, about 3,800 lines) defines, for a `PeriodPair`, the series $\zeta(z) = 1/z + \sum_{\omega\neq 0}\big(1/(z-\omega) + 1/\omega + z/\omega^2\big)$ and the product $\sigma(z) = z\prod_{\omega\neq 0}(1 - z/\omega)\exp(z/\omega + z^2/2\omega^2)$. In Lean the factor for $\omega = 0$ evaluates to 1, so the product runs over the whole lattice. The construction uses Mathlib's API for infinite products of holomorphic functions (developed there for the Dedekind η function). The main results are: σ is entire, vanishes exactly on Λ, is odd with $\sigma'(0) = 1$, and $\sigma'/\sigma = \zeta$; $\zeta' = -\wp$; the quasi-periods $\eta(\omega) = \zeta(z+\omega) - \zeta(z)$ are $\mathbb{Z}$-linear on Λ; $\sigma(z+\omega) = \pm e^{\eta(\omega)(z+\omega/2)}\sigma(z)$ (`sigma_add`); σ has order two, and $\wp, \wp'$ are quotients of entire functions of order two by σ^3 (`weierstrassOrderTwo`).

Legendre's relation (`LSG.Weierstrass.legendre`) states $\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1 = \pm 2\pi i$, with the sign of $\mathrm{Im}(\omega_2/\omega_1)$ (Mathlib's `PeriodPair` carries no orientation). Mathlib has Cauchy's theorem for rectangles but no residue theorem for parallelograms. We use only rectangles: for a basis (α, β) with $\tau = \beta/\alpha$ in the upper half-plane, $f(z) = \alpha\zeta(\alpha z) - \alpha\eta(\alpha)z$ is 1-periodic and shifts by a constant under $z \mapsto z + \tau$, and $f = G + 1/z$ with G holomorphic near the rectangle $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{\mathrm{Im}\,\tau}{2}, \frac{\mathrm{Im}\,\tau}{2}]$. The vertical sides cancel, the horizontal sides differ by the constant, $\oint G = 0$ by Cauchy–Goursat, and $\oint dz/z = 2\pi i$ is computed with explicit branches of log (`rectIntegral_inv`). The same trick gives the "sum of residues" fact we need (an elliptic function with at most one simple pole per period is constant) by subtracting $r\zeta(z-a)$ and using Legendre.

The addition theorem (`weierstrassP_add`) and the *degree-two property* ($\wp(z) = \wp(w)$ iff $z \equiv \pm w$, `weierstrassP_eq_iff`) give the *dictionary* (`LSG/Weierstrass/Dictionary.lean`): $\exp_E(z) = (\wp(z), \wp'(z)/2)$ is a homomorphism from $\mathbb{C}$ to the group of points of Mathlib's Weierstrass curve $y^2 = x^3 - \frac{g_2}{4}x - \frac{g_3}{4}$ with its algebraic group law (`expE_add`). Consequences used later: the discriminant is nonzero, and points with coordinates in a field $K \ni g_2, g_3$ form a subgroup.

4.2. **The Schneider–Lang criterion and Schneider's theorems.** The proposition `LSG.SchneiderLang` (`LSG/Interface.lean`) is the one-variable criterion of [11, Ch. III, Thm. 1] for meromorphic functions: if $f_i = g_i/h$ with g_i, h entire of order $\le \rho$, each f_i' is a polynomial over a number field K in the f_j on $\{h \neq 0\}$, and two of the f_i are algebraically independent over K, then only finitely many w with $h(w) \neq 0$ have all $f_i(w) \in K$. The proof `LSG.schneiderLang` (`LSG/SchneiderLang/`, about 1,500 lines) is the classical one: Siegel's lemma over $\mathcal{O}_K$ (Mathlib's `house.exists_ne_zero_int_vec_house_le`), a Schwarz lemma with multiplicities, Cauchy estimates, and Liouville's inequality; it shows that the number of such points is at most $4[K:\mathbb{Q}]\lceil\rho\rceil$.

Schneider's theorems follow by taking $h = \sigma^3$: with $e = \wp(\omega/2)$ for a primitive period ω, the functions $z, \wp, \wp'$ take values in $K = \mathbb{Q}(g_2, g_3, e, \omega)$ at the infinitely many points $(k + \frac{1}{2})\omega$ if ω were algebraic (`LSG.Schneider.period_transcendental`); the functions $\wp, \wp', e^{2\pi i z/\omega}$ do the same if ω/π were algebraic (`period_div_pi_transcendental`); and translates of an algebraic point give `wp_transcendental`.

4.3. **The lemniscatic invariant.** For the certificate we need $g_2(\varpi\mathbb{Z}[i]) = 4$ with $\varpi = \Gamma(1/4)^2/(2\sqrt{2\pi})$ (`LSG.LemniscateIndep.g₂_eq_four_of`, about 600 lines). The proof uses the symmetry $z \mapsto iz$ ($g_3 = 0$), the real branch of $\wp$ on $(0, \varpi/2]$ with the substitution $t = \sqrt{e/\wp}$, and the Beta integral for $2\int_0^1 dt/\sqrt{1-t^4}$. With Schneider's theorem this gives $\varpi/\pi \notin \overline{\mathbb{Q}}$ (`LSG.Bridge.irrational_varpi_div_pi_of`). See Section 14 for the history of this part.

5. Stage 2: the multiplicity estimate with elliptic factors

5.1. **Statement.** `LSG.ME.MultEstAb` (`LSG/ME/MultEstStmt.lean`) is Philippon's multiplicity estimate [15, Thm. 2.1] on $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$ over $\mathbb{C}$, in the shape of StrongSix's `MultEst`. Polynomials live in $\mathbb{C}[X; Y, Z; (U_k, V_k, W_k)_{k\le m}]$, with $E_k \subset \mathbb{P}^2$ in each elliptic block and $Y_j Z_j = 1$ for the torus. Let $n = d_0 + d_1 + m$. If $P \notin I_G$ has block degrees $\le (D_0, D_1)$, is homogeneous of degree D_2 in each elliptic block, and has formal order $> nT$ along W at every point of $\Sigma(n) = \{\sigma_1 + \cdots + \sigma_n : \sigma_i \in S\}$ ($0 \in S$), then there is a proper obstruction G' such that every $F \subset S$ of points pairwise inequivalent modulo G' satisfies

$$T^{\dim W - \dim(W \cap T_{G'})}\, |F|\, D_0^{\dim h_0} D_1^{d_1 - \dim U} D_2^{\dim t} \le C\, D_0^{d_0} D_1^{d_1} D_2^{m}.$$

The obstruction is an `ObstrData`: tangent data (h_0, U, t) with t an abelian tangent space, together with the subgroup $\overline{\exp_G(T_{G'})}^{\mathrm{Zar}}$ and the fact that its Lie algebra is $T_{G'}$. The proof `LSG.ME.multEstAb` produces $C = n^n d_0^{d_0} d_1^{d_1} 4^m 2^n n!\, 4^{d_0} 8^{d_1} 12^m$. $\overline{\mathbb{Q}}$-rationality of h_0 is not part of the output; it is restored in §7.1.

StrongSix follows Roy's affine treatment [23, Ch. 5, 8], which works for linear groups because their invariant derivations and translations preserve degrees. Neither holds with elliptic factors. We kept Roy's architecture (filtered Hilbert functions, Bézout by complete-intersection chains, Wüstholz's lemma by linear algebra, prime ideals instead of components) and changed four things.

5.2. **The model and the addition laws.** Each E_k is the cone over $V^2 W = 4U^3 - g_2 U W^2 - g_3 W^3$ in $\mathbb{C}^3$; the ideal $I_G = (Y_j Z_j - 1, F_k)$ is a prime complete intersection. Translations use a complete system of two addition laws of bidegree $(2, 2)$ on $E \subset \mathbb{P}^2$ (Mathlib's projective addition formula and a Bosma–Lenstra law [1]), proved in Lean to preserve the curve equation, to represent the sum whenever nonzero, and never to vanish simultaneously

(`LSG/Elliptic/AdditionLaws*.lean`). Translation then multiplies elliptic block degrees by 2; this is Philippon's constant c_1.

5.3. **Degree growth of derivations: Philippon's operators.** Roy's main argument builds a chain $L_r = I_G + (\tau_\sigma D_{\mathbf{l}} P)$ from derivation words $D_{\mathbf{l}}$ of length $\leq rT$. On E the invariant derivation, written on the cone in a chart, raises the degree by one, so the generators of L_r have elliptic degree about $D_2 + rT$, and Bézout then bounds everything in degree $D_2 + nT$ instead of D_2. In Waldschmidt's parameters $T = \Delta/\log S$ and $D_2 = \Delta/S^2$, so $T \gg D_2$: the estimate obtained this way is useless. (A first version of the chain was formalized and then discarded for this reason.)

The fix is Philippon's own device [15, §4.1, Déf. 4.1–4.2, Prop. 4.3–4.4]: instead of iterating derivations, differentiate along a curve of fixed degree,

$$\delta_g^\kappa Q = \partial_z^\kappa \, Q\big(\mathrm{law}(X, \mathrm{law}(\mathrm{rep}\, g,\, \exp(zw)))\big)\big|_{z=0}.$$

The point $X + g + \exp(zw)$ is represented by polynomials in X of fixed degree with coefficients that are formal power series in z, so every $\delta_g^\kappa Q$ has elliptic degree $\leq 2D$ for all κ (`phOp_mem_FLE`). Algebraically, $\exp(zw)$ is the formal exponential: $x + zw$ on $\mathbb{G}_{\mathrm{a}}$, $y\exp(zw)$ on $\mathbb{G}_{\mathrm{m}}$, and on E the formal solution $(s(z) : 1 : t(z))$ of the ODE in the V-chart, $s' = -\frac{1}{2} - g_2 st - \frac{3}{2} g_3 t^2$, $t' = -6s^2 + \frac{g_2}{2} t^2$, $s(0) = t(0) = 0$, with coefficients in $\mathbb{Q}[g_2, g_3]$. Proving Philippon's Proposition 4.4 (zeros of $\delta_g^\kappa Q$) needs the formal associativity of the laws along such curves (a uniqueness theorem for formal ODE solutions) and an identification of the formal order with vanishing of derivation words in canonical charts. Derivation words survive only inside Wüstholz's lemma, where they shift the Hilbert function argument by a constant, which does not affect leading terms.

5.4. **Cone vertices: affine slices.** StrongSix works with affine (filtered) Hilbert functions on $K[X, Y, Z]$, where there are no vertices. On the cone model every generator of L_r and of Roy's ideal A is homogeneous of positive degree in each elliptic block. Hence the locus $\{(U_k, V_k, W_k) = 0\}$, of dimension $d_0 + d_1 + 2m - 2$, lies in the affine zero set of every L_r, the pigeonhole on Hilbert dimensions becomes vacuous, a prime of maximal dimension may have no projective zeros at all, and the affine Nullstellensatz fails. In Philippon's multiprojective setting these irrelevant primes have zero Hilbert function; in the affine setting they dominate.

We separate the two uses of ideals. For geometry we pass to *relevant radicals* $I(Z(L))$ (only points with $(U_k, V_k, W_k) \neq 0$ in every block): the pigeonhole, the choice of a relevant prime and Roy's component step are done there (`MainAbChain.lean`). For Hilbert functions we *slice*: choose $\alpha \in \mathbb{C}^m$ and replace an ideal I by $I + (\ell_k - 1)_k$ with $\ell_k = V_k + \alpha_k W_k$. For generic α (avoiding finitely many points, `exists_good_alpha`) the slices of the relevant primes keep their dimension minus m, the sliced I_G is a complete intersection with an explicit standard-monomial bound for its Hilbert function (`SliceIG.lean`), and there is a slice Nullstellensatz $\sqrt{I + (\ell_k - 1)} = I(Z_\alpha(I))$ (`radical_sl`).

Primality of partial slices is proved by homogenizing with powers of ℓ_k, without localization. Bézout and the final count are then carried out for sliced ideals.

5.5. **Projective Wüstholz: a multiplier and glued charts.** Roy's form of Wüstholz's lemma [23, Lemma 8.16, Thm. 8.18] needs derivations D_i and polynomials Q_j in the ideal of the coset union with $D_iQ_j \equiv \delta_{ij}$. On a projective model one only gets $D_iQ_j \equiv \delta_{ij}M$ with M a non-zero-divisor (`wustholzM'`). Moreover a single chart does not work for all cosets: on cosets whose points have a constant 2-torsion coordinate ($V = 0$) the V-chart derivation vanishes. We glue chart-dependent derivations with separating polynomials (`glue`). The sliced version needs a multihomogeneous M (so that `nzd_sl`: M is a non-zero-divisor modulo the sliced ideal), and Euler-corrected derivations $D_i' = D_i - \sum_k (D_i\ell_k)\,\mathrm{Eul}_k$ that preserve the slice (`wustholz_cosets_sl`).

5.6. **The obstruction subgroup.** The stabilizer $H = \{g : g + Z(p) \subseteq Z(p)\}$ of the zero set of a relevant prime is a Zariski-closed subgroup (a Noetherian argument for any ideal, `StabG.lean`). To turn it into data (h_0, U, t) we proved:

- $\exp_G^{-1}(H)$ is a closed subgroup of T_G containing $\Omega(G)$; its real linear part is a complex subspace `lieAlg` (identity theorem), and its image in $\mathbb{C}^m$ meets Λ in a lattice.
- The projection of H to $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1}$ is Zariski closed (projective elimination along $(\mathbb{P}^2)^m$, `isClosed_proj`), hence of the form classified in StrongSix.
- *Zariski tangent space = Lie algebra* (`mem_TZ_iff`). The naive argument ("$I(H)$ is translation invariant, hence D_v-stable") is false as written on the cone, since D_vP is not multihomogeneous (a counterexample: the diagonal of $E \times E$ with $P = U_1W_2 - U_2W_1$); the proof homogenizes D_v with chart forms.
- *Dimension.* The algebraic dimension of $H^0 = \overline{\exp_G(\mathrm{Lie}(H))}$ equals $\dim \mathrm{Lie}(H) + m$ (`hdim_H0`), via a bridge between StrongSix's Hilbert dimension and transcendence degree (`hdim_eq_trdeg`) and the existence of smooth points in characteristic 0 (`exists_smooth_point`): $\leq$ at the identity from the tangent space, $\geq$ at a smooth point.
- *Product decomposition* (`lieAlg_eq_prod`): $\mathrm{Lie}(H) = h_0 \times U^{\perp} \times t$. First, the abelian projection of $\mathrm{Lie}(H)$ is Λ-rational, by a transcendence-degree count through the exponential parametrization. Second, a graph part would give a $\mathbb{C}$-linear form on that rational space with purely imaginary values ($2\pi i\mathbb{Z}$ or 0) on its lattice points, hence zero.

Purely analytic arguments cannot replace the algebraic inputs here: the graph of the Tate uniformization $\mathbb{C}^\times \to E(\mathbb{C})$ is a closed complex Lie subgroup of $\mathbb{G}_{\mathrm{m}} \times E$ that is not algebraic, so no argument using only the closed-subgroup structure of $\exp_G^{-1}(H)$ can yield the product decomposition. The second step above uses that $H^0 \cap (\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1})$ has finitely many components, a consequence of Noetherianity.

The Hilbert function lower bound for H^0 (degree $t_0+t_1+t_2+m$, the extra m from the free cone scalings in blocks not met by t) and the coset count finish the subgroup layer. The assembly (`LSG/ME/MainAb.lean`) follows Roy §8.4.1 with the slicing of §5.4: chain and relevant pigeonhole, relevant prime p, stabilizer and H^0, Roy's ideal A, component step, a good α for all cosets, Wüstholz for the sliced ideals, Bézout for the sliced A, and the final arithmetic.

6. Stage 3a: the auxiliary function with periods

6.1. **W83 Proposition 2.1 in our setting.** The κ-saving comes from [20, Prop. 2.1]: Siegel's lemma is applied with conditions only on a fundamental domain of the periods $\omega_1, \dots, \omega_\kappa$ in a ball of radius r, which saves a factor r^κ in the number of conditions; quasi-periodicity of the auxiliary function then propagates smallness to the whole ball, and an interpolation lemma converts smallness on a point set into smallness on the ball. We formalized this with two simplifications.

(1) In all our applications the periods of G lie over periods of the theta function, so the multiplier $\Psi(z+\omega)/\Psi(z)$ is an explicit exponential given by `sigma_add`. The lower bound for the theta function near the points ([20, Lemma 2.2]) is therefore not needed inside Prop. 2.1.
(2) Waldschmidt's interpolation [20, Lemma 2.3] rests on Masser's interpolation theorem for separated sets in $\mathbb{C}^n$. We avoid it by choosing coordinates of V among the coordinate forms of T_G: these send $\Omega(G)$ into 0, $2\pi i\mathbb{Z}$ or a lattice Λ_j, so the grid can be taken to be a product of one- and two-dimensional progressions, and interpolation is iterated one-variable Lagrange interpolation combined with StrongSix's one-variable Schwarz lemma. For two-dimensional grids the node products are bounded through $|u\alpha + v\beta| \geq c\max(|u|,|v|)$ and $\max(|u|,|v|)^2 \geq \max(|u|,1)\max(|v|,1)$, and the number of period classes in the grid is counted with an adjugate matrix instead of a Smith normal form.

The result `prop21_G` is W83's Prop. 2.1 on a subspace $V \subset T_G$, with one change: the hypothesis $U \geq r^2\log(R/r)$ becomes $c_1 r^2 \log(R/r) \leq U$, because the coordinates $\mathbb{C}^n \cong V$ rescale radii. In the application ($r = \nu^2 S$, $R = \nu^3 S$, $U = \nu^6\Delta$) the constant is absorbed by ν.

6.2. **Lemma U and Proposition 6.1.** Proposition 6.1 of [21] asks for vanishing *along* W at the points of $\exp_G(Y(S))$, $Y(S) = \{\sum h_j y_j : 0 \leq h_j \leq S\}$. Following Lemma U of [**?**], the small values of Q_S on a ball give, through

Cauchy's estimates, small Taylor coefficients along W at each point, and Liouville's inequality makes them vanish. Our formal proof (`lemmaU_core`) differs from that sketch in ways that simplify it.

- *A chart per point.* Instead of a fixed theta coordinate and a decomposition $Y = \mathcal{T} + F$ with translations by F, for every $u \in \mathbb{C}$ some coordinate of $(\sigma^3, \sigma^3\wp, \sigma^3\wp')(u+z)$ is $\geq e^{-c(1+|u|)^2}$ for $|z| \leq \rho_0$ (`exists_chart`, by quasi-periodicity and compactness of a fundamental domain). So the argument runs at every point of $Y(S)$ and the polynomial is Q_S itself.
- *A fixed base point.* The second argument of the addition law is $(\wp, \wp', 1)(z_0 + u)$ with $z_0 = \omega_1/2$, whose Taylor coefficients satisfy the recursion of $\wp'' = 6\wp^2 - g_2/2$ with algebraic initial values; no derivatives are taken at the point $\exp_G(t)$ itself.
- *One variable.* Vanishing along W is reduced to finitely many lines $s \mapsto t + sw$ and recovered by polarization.
- *Sizes* are house and denominator in an abstract number field. Algebraic points $\exp_G(\sum h_j y_j)$ are computed by balanced splitting with whichever addition law is nonzero, which keeps their sizes $\leq e^{cS^2}$.

(NR) enters exactly once, to guarantee $\Delta \geq S^2$, which gives the hypothesis $c_1 r^2 \log(R/r) \leq U$ of Prop. 2.1. The result `LSG.W88.prop61_b` is Prop. 6.1 with $a = 1$ and any integer $b \geq 1$ (Waldschmidt's Remark B: $D_0(\log S)^b = \Delta$, $\Delta^{d-N} = CS^{d_1+2m-\kappa}(\log S)^{bd_0}$), with the auxiliary function nonzero on G and vanishing to order $> \Delta/\log S$ along W on $Y(S)$.

7. Stage 3b: Waldschmidt's §7

`theorem41Abelian_of : MultEstAb → Theorem41Abelian` (`LSG/W88/Sec7*.lean`, about 1,700 lines) follows [21, pp. 396–397].

From analytic to formal vanishing. Prop. 6.1 gives analytic vanishing of $\Psi = \sum p_l M_l \circ \exp_G$ along W; the multiplicity estimate wants formal order of the polynomial P in Philippon's sense. The bridge `ordGt_of_vanishAlong` works only with Taylor coefficients at one point: the Taylor series of the analytic lift of $s \mapsto \exp_G(y + sw)$ in the canonical chart is a formal flow with the right initial value, hence equals the formal exponential by uniqueness of formal ODE solutions (`tps_eq_fflow`); a unit factor relates Ψ and P along the line.

Counting. If $u \in \exp_G^{-1}(G')$ then $Nu \in T_{G'} + \Omega(G)$ for some $N \geq 1$ (`htil_le_span`), using only the fields of `ObstrData`. Hence $(M+1)^\lambda$ points of the grid are pairwise inequivalent modulo G' (`exists_inequiv_grid`).

The inequality. With Prop. 6.1 at $b = 2$ and the multiplicity estimate at $T' = \lfloor \Delta/(2n \log S) \rfloor$, $D_0 = \lfloor \Delta/\log^2 S \rfloor + 1$, $D_1 = \lfloor \Delta/S \rfloor + 1$, $D_2 = D + 1$ (the polynomial is multiplied by $\prod_k W_k$ so that $D_2 \geq 1$), taking logarithms gives $A \log S \leq K + B \log\log S$ with $A = q(\lambda + \delta_1 + 2\delta_2) - e(\delta - \tau)$, $B = 2d_0(\delta - \tau) + q(\tau - 2\delta_0)$, $q = d - N$, $e = d_1 + 2m - \kappa$, and A, B, K bounded independently of S. One large S gives $A \leq 0$, which is (1.1); if $\delta \leq \tau$ one

finds $\delta = \tau = \delta_0 \geq 1$ and $B = -q\delta_0 < 0$, a contradiction. No pigeonhole over S is needed.

7.1. **The $\overline{\mathbb{Q}}$-rationality trick.** The obstruction of `MultEstAb` lives over $\mathbb{C}$, and its additive part h_0 need not be $\overline{\mathbb{Q}}$-rational: a subgroup like $\mathbb{C}\cdot(1,\pi) \subset \mathbb{G}_{\mathrm{a}}^2$ is a perfectly good closed subgroup over $\mathbb{C}$. W88 requires G' defined over $\overline{\mathbb{Q}}$. Rather than descend the stabilizer to $\overline{\mathbb{Q}}$, we replace h_0 by its algebraic part $h_0'' = \mathrm{span}_{\mathbb{C}}\{a \in h_0 : a \text{ algebraic}\}$ (`exists_subgroupData_rat`). This does not change λ, because the $\mathbb{G}_{\mathrm{a}}$-coordinates of points of Y are algebraic, so $y \in T_{G'} + \Omega$ implies $y \in T_{G''} + \Omega$; it leaves δ_1, δ_2 unchanged; and $\delta - \tau$ does not decrease, since $\dim(W \cap T'') \geq \dim(W \cap T) - (\dim h_0 - \dim h_0'')$. As $d_1 + 2m - \kappa > 0$ under (NR), both $\delta > \tau$ and (1.1) transfer.

8. The §5d bridges

Applications often use Waldschmidt's hyperplane consequence [21, §5d]: if V is a hyperplane, Y has rank m and $m > (d_1 + 2d_2 - \kappa)(d - 1 - t)$ ((5.4); $t = \dim W$), then V contains $T_{G'}$ for some connected algebraic subgroup $G' \neq 0$. Waldschmidt's proof chooses a subgroup G' maximizing a ratio $a(G')/e(G')$ (Lemma 5.1), applies Theorem 1.1 to G' with $V' = V \cap T_{G'}$, and concludes with Lemma 5.2. Two difficulties appear in the formalization.

Theorem 1.1 on the maximizer. G' is $\mathbb{G}_{\mathrm{a}}^{k_0} \times \mathbb{G}_{\mathrm{m}}^{k_1} \times B$ with B an abelian subvariety, not a product of curves, and (NR) may fail for G' even if it holds for G. We proved `LSG.CM.thm41Sub`: Theorem 1.1 under (NR) for $\mathbb{G}_{\mathrm{a}}^{k_0} \times \mathbb{G}_{\mathrm{m}}^{k_1} \times B$, B any abelian subvariety of $E_1 \times \cdots \times E_m$. No complex multiplication is needed: for a suitable choice of $\dim B$ coordinates, the coordinate projection $B \to \prod_k E_{\iota(k)}$ is an isogeny, and the theorem transports along its tangent map (logarithms pull back to logarithms, periods have torsion quotient, subgroups push forward with the same δ, τ). When (NR) fails for G' there are two cases. If $B = 0$, G' is linear and StrongSix's Theorem 1 (Waldschmidt's theorem for linear groups, in Roy's form, without (NR)) applies after converting Roy's surjection into a subgroup (`sub41Concl_bot`). If $B \neq 0$, failure of (NR) means $\kappa' \geq d_1' + 2d_2' - 1$, and the periods in V then force $T_B \subset V$, which contradicts the assumption.

Rationality of abelian tangent spaces. The maximizer must be defined over $\overline{\mathbb{Q}}$; for products of copies of one curve (with or without CM) we prove that abelian tangent spaces are $\overline{\mathbb{Q}}$-rational. For different curves this is assumed as a hypothesis `hrat` (`LSG.W88.hypStd_of_rat`); it is true for curves over $\overline{\mathbb{Q}}$, but not proved here.

The resulting endpoints are `LSG.W88.thm11Kappa` ($\mathbb{G}_{\mathrm{m}}^{d_1} \times E$), `LSG.W88.hypStdCM` and `hypStdCMAt` ($\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E^r$). Their statements are byte-for-byte copies of the certificate's hypotheses (`LSG/W88/MStarCopy*.lean`), and we checked that, e.g., `exact LSG.W88.thm11Kappa_of_theorem41 h L d₁` typechecks against the certificate's unmodified definitions.

9. Vector extensions

9.1. **The class and the statements.** The applications MG$^\natural$, MX, MX$'$ and the four exponentials statement at $t = c\varpi$ of [**?**] use groups built from universal vector extensions $E^\natural$ (extensions of E by $\mathbb{G}_{\mathrm{a}}$, with periods $(\lambda, \eta(\lambda))$ for $\lambda \in \Lambda$). We work with the class of groups

$$G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2, \qquad G_2 = \big((E_1 \times \cdots \times E_m)^\natural \times \mathbb{G}_{\mathrm{a}}^r\big)\big/\{(-u, Mu)\},$$

the push-out of $(E_1 \times \cdots \times E_m)^\natural$ along an algebraic matrix $M\colon \mathbb{C}^m \to \mathbb{C}^r$, so $d_2 = \dim G_2 = m + r$. On the tangent space $T_G = \mathbb{C}^{d_0} \times \mathbb{C}^{d_1} \times \mathbb{C}^m \times \mathbb{C}^r$ the exponential of $E^\natural$ is $\exp(z, w) = ((\wp_j(z_j), \wp_j'(z_j)),\ w - \zeta(z))$, with $s = Mw$ on the last factor. Hence $\Lambda(G)$ is described by "$s - M\zeta(z)$ algebraic" (with Lean's ζ, which takes the value $\eta(\lambda)$ at $\lambda \in \Lambda$), and $\Omega(G) = 0 \times 2\pi i\mathbb{Z}^{d_1} \times \{(\lambda, M\eta(\lambda))\}$. The *fibre* is the vector part $0 \times 0 \times 0 \times \mathbb{C}^r$ of G_2. Connected algebraic subgroups are $T_U \times (\tilde{B} + \Gamma)$, with $\tilde{B}$ the image of $B^\natural$ for an abelian subvariety B and Γ a $\overline{\mathbb{Q}}$-rational subspace of $\mathbb{G}_{\mathrm{a}}^{d_0} \times$ fibre; they are not products. The statements are in `LSG/W88/StatementVec.lean`, `V5d2Stmt.lean` and `V5dOnto.lean`; their conclusion is that of `Theorem41Abelian` with $d_2 = m + r$. The general statement `Theorem41Vec` has no further premises.

Two premises concern the curves. `LiftAlg`: if $z \in T_B$ and $\exp(z) \in B(\overline{\mathbb{Q}})$, there is w with $(z, w) \in T(B^\natural)$ and $w - \zeta(z)$ algebraic. `PerTanRat`: $T(B^\natural)$ is $\overline{\mathbb{Q}}$-rational. Both are true for all curves with algebraic invariants ($B^\natural \to B$ is a surjective morphism defined over $\overline{\mathbb{Q}}$), but we have proved them in Lean only for the CM curves used by the applications (`LSG.CM.liftAlg_block`, `perTanRat_block`, and the products $E_i^n \times E_\rho^n$ of the MX bridges).

Theorem 9.1 (Lean; standard axioms)**.** *Let G be in the class above, with $g_2(\Lambda_j), g_3(\Lambda_j)$ and M algebraic.*

(a) *If $r \le 1$, $V \cap$ fibre $= 0$ and the curves satisfy `LiftAlg`, Theorem 1.1 holds under* (NR) *(`LSG.W88.theorem41VecLA`, for every algebraic M; the case of onto M, `theorem41VecLA_onto`, was proved first, by a shorter route).*

(b) *If $d_0 = 0$, $r \le 2$, $\dim(V \cap$ fibre$) \le 1$ and the curves satisfy `LiftAlg` and `PerTanRat`, Theorem 1.1 holds under* (NR) *(`LSG.W88.theorem41Vec2`).*

(c) *The §5d consequence ((5.4) with $d_2 = m + r$, without* (NR)*) holds for $r \le 1$, and for $d_0 = 0$, $r \le 2$, for curves satisfying `LiftAlg` and `PerTanRat` (`LSG.W88.V5d.hyperplaneKappaVec_uncond`, `hyperplaneKappaVec2_uncond`).*

The restrictions on r, d_0 and $V \cap$ fibre are genuine; the general `Theorem41Vec` is not proved. Accordingly these results are to be cited as "W88 Thm. 4.1 under (NR) for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2$, G_2 a push-out of $(\prod E_j)^\natural$ along an algebraic M, in the cases $r \le 1$ with $V \cap$ fibre $= 0$, or $d_0 = 0$, $r \le 2$ and $\dim(V \cap \text{fibre}) \le 1$; for curves satisfying `LiftAlg` (and `PerTanRat`),

true for all curves over $\overline{\mathbb{Q}}$ and Lean-proved for the CM curves used", and never as "W88 Thm. 4.1 for vector extensions".

9.2. **The proof.** The architecture is that of Sections 5–7.

- *A model of* $E^\natural$. The surface $X = V(F,\, WT_3 - VT_1 + 2U^2) \subset \mathbb{P}^4$, with $(U : V : W : T_1 : T_3) = (x : y : 1 : t : yt - 2x^2)$ over affine points, is $E^\natural$ plus a boundary line. Exact linear algebra over $\mathbb{F}_p$ shows that it carries no addition law of bidegree $(2, 2)$ (the missing section is Ut); we found eight laws of bidegree $(3, 3)$ and proved in Lean that they preserve the ideal, represent the sum in all configurations (and analytically, `NLaw_expN_all`), and form a complete system (`NLaw_complete`). The ideal (F, G) is prime (`isPrime_span_FG`).
- *The multiplicity estimate* `LSG.MEV.multEstVec` (`LSG/MEV/`). Every G of the class is the quotient of $\tilde{G}_r = \mathbb{G}_{\mathrm{a}}^{d_0+r} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1^\natural \times \cdots \times E_m^\natural$ by the vector subgroup $K_M = \{(0, -Mv; 0; 0, v)\}$, and polynomials on G are K_M-invariant polynomials on $\tilde{G}_r$. All layers of Section 5 were redone for $E^\natural$ blocks: primality and the complete intersection property of $I_{\tilde{G}}$, the Nullstellensatz despite the boundary plane, translations, the formal exponential and Philippon's operators (degree $\le 3D$), the chain of ideals with its relevant-prime step, the $E^\natural$ curve lemma, the subgroup layer (unconditional classification of Zariski-closed subgroups, $\hdim H^0$, finiteness of cosets, and a Hilbert lower bound matching $\hdim H^0$), the slice layer, and Wüstholz's lemma (homogenized chart derivations, transversals at every coset, coset bounds). The cone over X is singular along $U = V = W = 0$, yet the change of chart for the homogenized derivations is polynomial, with explicit cofactors. When M is onto, the grading of Section 5 (all additive coordinates in one block) gives the estimate. When it is not (for instance $r = 1$, $M = 0$, so that $G_2 = E^m \times \mathbb{G}_{\mathrm{a}}$, a case that W88 §7 needs), the trailing additive coordinates belong to G_2 and must carry the elliptic degree D_2; we use a grading with four blocks (leading and trailing additive coordinates, torus, elliptic). Hilbert *dimensions* do not depend on the grading, so only the degree steps had to be redone (about 2,200 lines).
- *The analytic side*: Prop. 6.1 for the class (`LSG.AuxV.prop61V_b`), using entire coordinates of $\exp_{E^\natural}$ built from σ and the premise $V \cap \text{fibre} = 0$ to keep the grid coordinates discrete (the fibre forms have non-discrete period images when $m \ge 3$). Sizes of fibre values are controlled through the ζ addition formula rather than the $(3, 3)$ laws, whose balanced splitting would give sizes $6^k \approx S^{2.58}$ and break the e^{cS^2} bounds. For case (b), where $V \cap \text{fibre} = \mathbb{C}u$ may be an irrational line, we proved W83's Prop. 2.1 with one polynomial direction (`prop21LatT`): the fibre forms are affine in s, so along u the functions are polynomials of degree $\le D$; the Siegel conditions are imposed at

$D+1$ nodes in that variable and recovered by exact Lagrange interpolation, at the cost of one power of the parameter U.

- *W88 §7 and $\overline{\mathbb{Q}}$-rationality.* The obstruction lives over $\mathbb{C}$, with tangent space $T' = \Gamma_{\text{full}} \oplus U^{\perp} \oplus T(\tilde{B})$. For $r \leq 1$ the fibre part of Γ_{full} is 0 or the whole fibre, hence rational, and replacing Γ_{full} by the span of its algebraic vectors works as in Section 7.1; `LiftAlg` keeps λ unchanged. For $d_0 = 0$, $r \leq 2$, Γ_{full} may be an irrational line in the two-dimensional fibre. Then T' lies between a rational subgroup T_S (its algebraic part) and a rational subgroup T_E (the whole fibre added) with $\dim T_E/T' \leq 1$, and one of the two satisfies the inequality (1.1): if T_E fails, `PerTanRat` puts every algebraic vector of T' into T_S, and T_S works.
- *The §5d engine* follows Waldschmidt's Lemmas 5.1–5.2 as in Section 8, with frames for the class. For $r \leq 1$ the maximizing subgroup may have a non-onto push-out matrix; moving its $\mathbb{G}_{\mathrm{a}}$ fibre into the $\mathbb{G}_{\mathrm{a}}^{d_0}$ factor gives a presentation with $r = 0$, so case (c) for $r \leq 1$ needs only the onto case of (a). For $d_0 = 0$, $r \leq 2$ the engine runs directly on (b).

9.3. **The bridges to the certificate.** From Theorem 9.1 we prove, unconditionally, the certificate's six vector-extension hypotheses (`LSG/W88/V5dOntoBridges.lean`, `V5dSMGn.lean`, `V5dMXPMain.lean`): `Thm11KappaMXG` (Theorem MX), `Thm11KappaMGn` and `Thm11KappaMGnK` for every CM lattice, `Thm11KappaNat` (the four exponentials statement at $t = c\varpi$), `Thm11KappaSMGn` (Theorem S), and `Thm11KappaMXP` (Theorem MX′, which has $r = 2$ and uses case (b)). As in the certificate, the statements for a general CM lattice carry the premise $\kappa_1 = 0$ (`KappaOneZero`), proved there for both lattices used. The CM inputs are proved in Lean: rotation formulas, quasi-periods via Legendre, `LiftAlg` and `PerTanRat` for CM blocks, and the splitting of abelian subvarieties of $E_i^n \times E_\rho^n$ via the irrationality of $\sqrt{3}$. The statements are verbatim copies of the certificate's (`LSG/W88/MStarCopy*.lean`); the certificate, which has this project as a Lake dependency, derives its own unmodified hypotheses from them, so a drift of a copy breaks its build. Two of the CM abelian-variety hypotheses of the certificate are treated in Section 10.

9.4. **Fidelity review.** A second AI session that had not written the proofs reviewed the vector-extension statements against [21, pp. 375, 385–389, 394–395]. It found no fatal or major issue: $\Lambda(G)$, $\Omega(G)$, the $\overline{\mathbb{Q}}$-structure of the push-out and the subgroup family match Waldschmidt's for this class (including completeness of the family), and all casts are safe. As a test of non-vacuity it derived in Lean, from each of (a), (b) and (c), that $\eta(\omega_1) \notin \overline{\mathbb{Q}}\,\omega_1$ for the lemniscatic curve (with $G = E^{\natural}$). Its main remark is the scope of the claim, stated above; the others concern docstrings, the redundancy of `Proper`, and the fact that the lattice case of `LamV` depends on the junk value

of ζ at lattice points (proved in the same file, so a change of definition would break the build).

10. An abelian surface: the Jacobian of $y^5 = x(1-x)$

10.1. **The group and the statements.** Theorem A of [**?**] at the Beta value $B(1/5, 1/5)$ applies Waldschmidt's theorem to $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times A^r$, where A is the Jacobian of the genus-two curve $C\colon y^5 = x(1-x)$, a simple abelian surface with complex multiplication by $\mathbb{Q}(\zeta_5)$. Here A is the Jacobian of C, presented in Lean by its period lattice in the coordinates $\int dX/Y$, $\int X\,dX/Y$ on the model $Y^2 = 4X^5 + 1$ ($X = -y$, $Y = 2x - 1$). Lean computes these periods (Beta values) and proves that they are periods of an analytic exponential map onto Grant's model in $\mathbb{P}^8$ with discrete kernel. The identification of that torus with $\mathrm{Jac}(C)$, and of complex subtori with abelian subvarieties, is classical and not formalized. The final theorems (e.g. `two_three_beta5_uncond` below) are statements about explicit numbers and do not depend on it.

Concretely, $T_G = \mathbb{C}^{d_0} \times \mathbb{C}^{d_1} \times (\mathbb{C}^2)^r$, and $\mathtt{latt} \subset \mathbb{C}^2$ is the lattice spanned by the four periods $\int_{\gamma_e}(dX/Y,\, X\,dX/Y) = (c_0\zeta^e,\, c_1\zeta^{2e})$, $e = 0, \dots, 3$, where $\zeta = e^{2\pi i/5}$, $c_j = \frac{(-1)^j}{5}(1-\zeta^{j+1})B\big(\frac{j+1}{5}, \frac{j+1}{5}\big)$ and γ_e are explicit loops. Lean proves that the four periods are $\mathbb{R}$-independent, that $\mathtt{latt}$ lies in the kernel of the exponential map, and that this kernel lies in $\mathbb{Q}\cdot\mathtt{latt}$; so $\mathtt{latt}$ is a finite-index sublattice of the period lattice, and every statement below depends only on its $\mathbb{Q}$-span. The data are restricted as follows:

$$\Lambda_A = \{(a, b, z) : a \in \overline{\mathbb{Q}}^{d_0},\ e^{b_i} \in \overline{\mathbb{Q}},\ z_k \in \mathtt{latt}\}, \qquad \Omega_A = 0 \times 2\pi i\mathbb{Z}^{d_1} \times \mathtt{latt}^r,$$

and a subspace $V \subset T_G$ is *toric-injective* if every $v \in V \cap \mathbb{Q}\Omega_A$ with zero toric part is 0, i.e. V contains no nonzero $\mathbb{Q}$-period of A^r alone. Connected algebraic subgroups are products $H_0 \times T_U \times B$, with the tangent space t of B characterized by "$t \cap \mathtt{latt}^r$ spans t over $\mathbb{R}$". The statements are in `LSG/W88/StatementAbVar.lean`.

Theorem 10.1 (Lean; standard axioms)**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times A^r$, A as above, so $d_2 = 2r$.*

(a) *Theorem 1.1 holds under* (NR) *for toric-injective V and $Y \subset V \cap \Lambda_A$ (`LSG. W88. theorem41AbVarPer`).*

(b) *The §5d consequence ((5.4) with $d_2 = 2r$, without* (NR)*) holds for toric-injective hyperplanes V and $y_1, \dots, y_m \in V \cap \Lambda_A$ (`LSG. W88. hyperplaneKappaAbVar`); by a common-denominator argument the y_i may be taken in $V \cap \mathbb{Q}\Lambda_A$.*

Scope. Theorem 10.1 is W88 §5d (and Thm. 4.1 under (NR)) for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times A^r$, $A = \mathrm{Jac}(y^5 = x(1-x))$, for toric-injective hyperplanes V and subgroups Y whose A-components are (rational) periods; so $\exp_G(Y) \subset \mathbb{G}_{\mathrm{a}}^{d_0}(\overline{\mathbb{Q}}) \times \mathbb{G}_{\mathrm{m}}^{d_1}(\overline{\mathbb{Q}}) \times A^r_{\mathrm{tors}}$. Both restrictions only weaken Waldschmidt's theorem, and they are all that the certificate's models use. For $G = A^r$ alone

($d_0 = d_1 = 0$) the restricted statement is empty, since toric injectivity forces $V \cap \Lambda_A = 0$; it must not be cited as Waldschmidt's theorem for A^r.

10.2. **The proof.** The architecture is again that of Sections 5–8, but every layer that touched the elliptic curve had to be redone for a surface without a Weierstrass-type uniformization in Mathlib.

- *An explicit model and certificates* (`LSG/Jac/`). We use Grant's model $V(Q) \subset \mathbb{P}^8$ of the Jacobian of a genus-two curve with a rational Weierstrass point (quadrics in Mumford coordinates), with a complete system of addition laws of bidegree $(2, 2)$, the duplication map and the invariant vector fields. The polynomial identities behind these facts (laws preserve the model, flow invariance, the doubling identities, a degree-7 Nullstellensatz showing that duplication does not vanish on $V(Q) \setminus 0$, local complete-intersection systems) are far beyond `ring`. They are produced by Python scripts with exact arithmetic, which are untrusted, and checked in Lean by a normal-form checker modulo the ideal of the model, with Kronecker-packed natural-number accumulators. Each check is a Boolean computation proved by `decide +kernel`, so the kernel evaluates it, and a soundness theorem turns a successful check into the polynomial identity over any commutative ring. There is no `native_decide`. The certificates take about five CPU-hours to check.
- *The analytic exponential* (`LSG/JacAn*/`). We build the genus-two Abel map and a local exponential near a generic pair of points, prove Abel's theorem locally (Euler–Jacobi), extend the exponential to an entire map $\mathbb{C}^2 \to \mathbb{P}^8$ by doubling the germ, prove that it is a homomorphism, that its kernel is discrete and contains `latt` (via path continuation of Abel integrals, including through Weierstrass points), and compute the Beta periods. An entire nowhere-vanishing lift θ, normalized to have an affine factor of automorphy and order two, plays the role of σ; a Liouville argument shows that the addition laws are correct and complete on its image, and a boundary flow-exit argument proves that every nonzero point of the model is $c\,\theta(z)$. Taylor coefficients at algebraic points are algebraic, with explicit size bounds, in three charts that cover the model.
- *A generic multiplicity estimate* (`LSG/MEB/`). The estimate of Section 5 is proved once more for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times B_1 \times \cdots \times B_m$, generic over an interface of *abelian blocks*: an entire lift with discrete period group, an explicit ideal whose zero set is the cone over the image, addition laws correct on the image, chart flows, and Hilbert and local-CI certificates. The interface is semantic: identities that the elliptic proof checked formally modulo the cone become consequences of the laws on the image, which avoids an estimated twenty- to forty-fold increase of the certificate work. The Jacobian is the instance `LSG.JacBlock.aBlock`.

- *The analytic side and §7* (`LSG/AuxFunA/`, `LSG/W88/AbVar*`). For a simple abelian surface, W83's Prop. 2.1 in general needs a several-variable interpolation estimate of Masser type. Toric injectivity avoids it: $\mathbb{Z}$-independent periods in a toric-injective V have $\mathbb{Q}$-independent toric parts, hence are $\mathbb{C}$-independent, and the linear form of Prop. 2.1 used for elliptic curves applies with θ's factor of automorphy as multiplier. Lemma U needs Taylor data only at the origin, because every point of Λ_A has its A-part in `latt`. W88 §7 and the §5d engine are written generically for abelian blocks (`theorem41B_of`); the §5d maximizer must be defined over $\overline{\mathbb{Q}}$, and we prove that abelian tangent spaces of A^r are $\overline{\mathbb{Q}}$-rational using the CM structure (`ratTangentsA`).

10.3. **The bridge to the certificate.** From Theorem 10.1(b) we prove the certificate's hypotheses `AbVar.Thm11KappaA` and `AbVar.Thm11KappaAK` at $y^5 = x(1-x)$ (`LSG.W88.thm11KappaA5`, `thm11KappaAK5`), stated against verbatim copies. The transport rescales the certificate's coordinates, which are dual to dx/y^{4-s}, by the algebraic factors $5(-1)^s$, and checks that the certificate's toric injectivity implies ours and that every nonzero abelian tangent lands in the certificate's subgroup family. In the certificate (which has this project as a Lake dependency) these yield, with no hypothesis and standard axioms only: `two_three_beta5_uncond`, that 2^t and 3^t are not both algebraic for $t = B(1/5,1/5)/\pi$; the four exponentials statements at $cB(1/5,1/5)/\pi$ and $c\pi/B(1/5,1/5)$ for algebraic $c \neq 0$; and the input $(1-\zeta_5)B(1/5,1/5)/\pi \notin \overline{\mathbb{Q}}$ (Wüstholz), here derived from Waldschmidt's theorem.

10.4. **Fidelity review.** A further AI session that had not written the proofs reviewed the statements and the bridge against [21, pp. 375, 385–386, 394–395] and the certificate. It found no fatal or major issue. Its main remark is the scope above (the second restriction, on the A-components, had been omitted from a sentence in [7]); it also verified in Lean that the subgroup condition is not vacuous (a nonzero subspace of $\mathbb{C}^2$ that is not an abelian tangent) and that the restricted statement is empty for A^r alone, and asked for the wording of "A is the Jacobian" used above.

11. A vector extension of the Jacobian

11.1. **The group and the statements.** Theorem MG$^\natural$-A of [**?**] at $B(1/5,1/5)$ gives the headline statement that 2^t and 3^t are not both algebraic for $t = B(1/5,1/5) = \Gamma(1/5)^2/\Gamma(2/5)$ (without the division by π of Section 10). It applies Waldschmidt's theorem to a group with a vector part. Let $A = \mathrm{Jac}(y^5 = x(1-x))$ as in Section 10, and let $A^\natural_\sigma$ be the extension of A by $\mathbb{G}_\mathrm{a}$ whose class is that of the differential of the second kind dx/y; on the model $Y^2 = 4X^5 + 1$ this is the class $[X^3\,dX/Y] = -\frac{1}{5}[dx/y]$, the $\bar\sigma$-eigenline for the action of $\mathbb{Q}(\zeta_5)$. Let Q be the push-out of $(A^\natural_\sigma)^m$

along b^{T}, for b an adapted basis of a number field containing $\mathbb{Q}(\zeta_5)$ (that is, the numbers $\zeta^e b_j$ are $\mathbb{Q}$-linearly independent). Then $T_Q = (\mathbb{C}^2)^m \times \mathbb{C}$, and the periods of Q are the vectors $(\lambda, \sum_j b_j \bar{\eta}(\lambda_j))$, with $\lambda \in \mathtt{latt}^m$ and $\bar{\eta}$ the fibre period.

The statements are in `LSG/W88/AbVarVStmt.lean`. They are generic over a family of *vector blocks* $B_1, \dots, B_m$ (an abelian block of Section 10 together with fibre data) and a matrix $M\colon \mathbb{C}^m \to \mathbb{C}^r$. The data are restricted as in Section 10: $\Lambda_{A^\natural}$ consists of the (a, b, z, s) with $a \in \overline{\mathbb{Q}}^{d_0}$, $e^{b_i} \in \overline{\mathbb{Q}}$, $z_k \in \mathtt{latt}$ and $s - M\bar{\eta}(z) \in \overline{\mathbb{Q}}^r$, so the A^m-component of every point is 0 while its fibre component is an arbitrary algebraic point; and V is *toric-injective* if it contains no nonzero rational period of Q.

Theorem 11.1 (Lean; standard axioms)**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times Q$ with Q as above, so $d_2 = 2m + 1$.*

(a) *The §5d consequence ((5.4), without* (NR)*) holds for toric-injective hyperplanes V and $y_1, \dots, y_\ell \in V \cap \Lambda_{A^\natural}$ (`LSG.W88.hyperplaneKappaAbVarV_aBlockV`).*

(b) *More generally, Theorem 1.1 holds under* (NR) *for $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times (B_1^\natural \times \cdots \times B_m^\natural$ pushed out along M), for every family of vector blocks with analytic and Taylor packages, every onto algebraic M with $r \le 1$, toric-injective V, and Y whose base parts lie in the period lattice (`LSG.W88.theorem41AbVarVPer`).*

(c) *Philippon's multiplicity estimate holds for these groups, generic over vector blocks (`LSG.MEBV.multEstV_holds`).*

Scope. For Theorem MG$^\natural$-A at $B(1/5, 1/5)$ the formal development proves the following special case of W88 §5d: Theorem 11.1(a), that is, W88 §5d for $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times Q$, restricted to

- hyperplanes V that are *toric-injective* (V contains no nonzero rational period of Q);
- subgroups Y whose points have trivial A^m-component (their $\mathbb{G}_{\mathrm{a}}$, $\mathbb{G}_{\mathrm{m}}$ and fibre components are arbitrary algebraic points).

This is implied by W88. The push-out is one-dimensional ($r = 1$) and b is adapted; this is what guarantees that every algebraic subgroup with nonzero abelian part contains the fibre. The generic Props behind Theorem 11.1 are instances of W88 only when M is algebraic and, for (a), when the block family has this fibre property; the proved cases satisfy both. Unlike Section 10, the restricted statement is not empty for $d_0 = d_1 = 0$, since non-torsion points of the fibre are allowed. The Lean proof, which goes through Philippon's multiplicity estimate, is unconditional. The hypothesis of Theorem MG$^\natural$-A is used only at three frame hyperplanes, and each of them is toric-injective. This is proved in Lean from the F-independence of b and the fact that the fibre character $\zeta \mapsto \zeta^4$ is a field embedding.

The group $A_{\bar{\sigma}}^\natural$ is presented in Lean by an explicit projective model over $\mathbb{Q}$ in $\mathbb{P}^{11}$ (Grant's variety, three fibre coordinates T_0, T_4, T_8, Plücker quadrics

and 81 addition laws of bidegree $(3,3)$) and an entire uniformisation $(z,w) \mapsto (\theta_N(z), T(z,w))$ built from theta functions. Its fibre periods are computed in Lean to be $\bar{\eta}([\zeta^e]\gamma) = \frac{3}{5}\zeta^{4e}(1-\zeta^4)B(4/5,4/5)$, i.e. -3 times the periods of $X^3\,dX/Y$. The identification of this model with the extension of $\mathrm{Jac}(C)$ is classical and not formalised. The final four-exponentials statement concerns explicit numbers and does not depend on it.

11.2. **The proof.** The architecture is that of Section 10, with a fibre added to every layer.

- *The analytic group* (`LSG/JacAnV/`). A Liouville argument gives $\partial_i\partial_j \log\theta_{N,0} = -3\wp_{ij} + c_{ij}$ from six small quadric identities. The fibre coordinates are the derivative lifts $T_k(z,w) = \theta_k(z)(w+\ell(z)) - \frac{1}{3}\partial_0\theta_k(z)$. They have the same factor of automorphy as θ_N, with the fibre period $\bar{\eta}$ as translation in w, and their flows are given by rational quadrics. The fibre period formula comes from the path-continuation machinery of Section 10.
- *The model* (`LSG/JacBlockV/`, `LSG/Jac/B6V*`). The fibre addition laws are the polarization of the nine certified laws of A, defined with `pderiv` and proved correct by differentiating them, so no new law certificates are needed. The only new certificates are those of the Hilbert bound: every homogeneous polynomial of degree 41 lies in the ideal of the model plus four linear forms, from eight kernel-checked identities.
- *A generic multiplicity estimate* (`LSG/MEBV/`). The estimate is proved once more, generic over an interface `VBlock` of vector blocks. Its design builds in five corrections found by an exact computer-algebra spike: slicing in the X-variables only (the model is not a local complete intersection on $\{X=0\}$), a Hilbert certificate with linear forms, fibre flows only in their own chart, generators that need not span a saturated ideal, and no requirement on the fibre coordinate of the identity.
- *The analytic side, §7 and §5d* (`LSG/AuxFunAV/`, `LSG/W88/AbVarV*`). Under toric injectivity, W83's Prop. 2.1 again needs only its linear form, and the fibre forms are quasi-periodic with θ's multiplier. The §5d engine needs three properties of the block family, all proved for $A^{\natural}_{\bar{\sigma}}$ from the CM structure: the fibre property above, an algebraic relation between the fibre periods on a rational abelian tangent and on its pivot coordinates, and $\overline{\mathbb{Q}}$-rationality of abelian tangents.

11.3. **The bridge to the certificate.** From Theorem 11.1(a) we prove the certificate's hypotheses `AbVar.Thm11KappaAn` and `AbVar.Thm11KappaAnK` (the latter for $L_{A^{\natural}} = \emptyset$) at $y^5 = x(1-x)$ (`LSG.W88.thm11KappaAn5`, `thm11KappaAnK5`). Like `Thm11KappaA`, these hypotheses are restricted to toric-injective hyperplanes, a restriction the certificate adopted for this purpose (its frame hyperplanes are toric-injective, `perAn_inj`). The

transport rescales by algebraic factors and maps our fibre periods onto the certificate's generators, since $\frac{5}{3}\cdot b_j\frac{3}{5}\zeta^{4e}\bar{P} = b_j\zeta^{(5-1)e}\bar{P}$. In the certificate, `Bridge.two_three_beta5'_uncond` (that 2^t and 3^t are not both algebraic for $t = B(1/5,1/5)$) and `Bridge.wuestholzB5_uncond` ($(1-\zeta^4)B(4/5,4/5)/\pi \notin \overline{\mathbb{Q}}$) follow with standard axioms only.

11.4. **Fidelity review.** A further AI session that had not written the proofs reviewed the statements, the block data and the bridge against [21, pp. 385–386, 394–395] and the certificate. It found no fatal or major issue. Its main remarks are the scope above and that the certificate had yet to adopt the restriction, which it has since done. It proved in Lean, in a scratch file, that the certificate's frame hyperplanes are toric-injective, and it re-ran the three consumers in the certificate under the narrowed hypothesis. It also checked non-vacuity: the fibre period $\bar{\eta}$ is nonzero, the data contain non-torsion fibre points, and the subgroup family of the model is exactly the family of connected algebraic subgroups.

12. Explicit consequences, formally verified

The certificate [8], which has this project as a Lake dependency, derives explicit transcendence statements from the hyperplane theorems of Sections 8, 9, 10 and 11. We list them with their status at its commit `a02679d`. Lean names are in the namespace `CM4E` unless they start with `LSG`; each entry names the sub-namespace once.

Notation. $\mathcal{L} = \{z \in \mathbb{C} : e^z \in \overline{\mathbb{Q}}^\times\}$, $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$, and $\mathcal{L}_E$ is the set of elliptic logarithms of algebraic points of E. The curves are E_i (lattice $\varpi\mathbb{Z}[i]$) and E_ρ (lattice $\Omega_\rho\mathbb{Z}[\rho]$, $\Omega_\rho = \Gamma(1/3)^3/(2^{4/3}\pi)$), and $\eta_\rho = \eta(\Omega_\rho)$. 4E($t$) (`FourExpAt`) says: if $a, b \in \mathcal{L}$ are $\mathbb{Q}$-linearly independent, then e^{ta}, e^{tb} are not both algebraic. A is the Jacobian of $y^5 = x(1-x)$ and $B = B(1/5,1/5) = \Gamma(1/5)^2/\Gamma(2/5)$. Throughout, $c \in \overline{\mathbb{Q}}^\times$.

Derivation. Each statement comes from a group G, a $\overline{\mathbb{Q}}$-rational map $\Psi\colon T_G \to \mathbb{C}^2$ and a line V of transcendental slope. For a number field K of degree n, G'' is the quotient of G^n by the largest connected algebraic subgroup with tangent space in the kernel of $\Psi\circ(z_j)_j \mapsto \sum_j \theta_j z_j$, for a basis θ of $K/\mathbb{Q}$. With $\alpha(H) = 2\dim H - 2d_0(H) - d_1(H)$, $\alpha_{K,\Psi} = \alpha(G'')/n$ and $\kappa = \dim_K(V\cap\Psi(\Omega_K(G)))$, the bound is $\dim_K(V\cap\Psi(\Lambda_K(G))) \le \alpha_{K,\Psi} - \kappa$ for the K-spans Λ_K, Ω_K, and a counterexample would give more K-independent data than that. Table 1 has the numbers for large n, as computed in [**?**, 7]. The Lean proofs do not use this bound: they apply (5.4) of [21, §5d] to a hyperplane of a model of G''.

Status labels. **U**: verified with no hypothesis, that is, a declaration whose only premises belong to the mathematical statement. **U***: every premise of a generic declaration is discharged at this commit, but no single declaration states the composite. **I**: verified with no form of W88 assumed, modulo the named classical inputs listed, which are explicit hypotheses. **W**: verified

Entry	G''	$\alpha_{K,\Psi}$	κ	data	bound	Status
MG	$\mathbb{G}_{\mathrm{m}}^{2n} \times A''$, $A'' = E^n/U$	3	1	3	2	**U**, **I** at Ω_ρ
MG$^\natural$	$\mathbb{G}_{\mathrm{m}}^{2n} \times Q''$, $Q'' = (E^\natural)^n/U$	$3+2/n$	1	3	$2+2/n$	**U**, **I** at Ω_ρ
SMG(a)	$\mathbb{G}_{\mathrm{a}}^2 \times \mathbb{G}_{\mathrm{m}}^{2n} \times A''$	3	1	3	2	$\mathbf{U}^*$
SE(1)	$\mathbb{G}_{\mathrm{a}}^2 \times A_1'' \times A_2'' \times A'''$	3	1	3	2	**I**
MX, deg2(ii)	$(E_i^\natural \times E_\rho^\natural \times \mathbb{G}_{\mathrm{a}}^2)^n/U$	$4+2/n$	2	3	$2+2/n$	**I**
MX′, deg2(i)	ext. of $E_i^n \times E_\rho^n$ by $\mathbb{G}_{\mathrm{a}}^2$	$4+4/n$	2	3	$2+4/n$	**I**
Theorem M	$A'' \times \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{a}}$	2	1	2	1	**U**, **I**
MG-A at B/π	$\mathbb{G}_{\mathrm{m}}^{2n} \times A''$, $A'' \sim A^r$	3	1	3	2	**U**
MG$^\natural$-A at B	$\mathbb{G}_{\mathrm{m}}^{2n} \times Q''$, $Q'' = (A^\natural)^r/U$	$3+2/n$	1	3	$2+2/n$	**U**
X7	$\prod_j A_j'' \times \mathbb{G}_{\mathrm{a}}^2$	4	2	3	2	**W**
X9	$\prod_j A_j'' \times \mathbb{G}_{\mathrm{a}}^2$	4	2	3	2	**N**
X$^\kappa$	$\prod_j A_j'' \times \mathbb{G}_{\mathrm{a}}^2$	2κ	κ	$\kappa+1$	κ	**N**

TABLE 1. The consequences. "Data" counts the period data and the vectors a counterexample would give.

conditionally on a named form of W88 that this project does not prove. **N**: not formalised as stated. Two inputs recur: (g_3) is $g_3(\Omega_\rho\mathbb{Z}[\rho]) \in \overline{\mathbb{Q}}$ (the argument $\mathtt{hg}_3'$; $g_2 = 0$ is proved, and $g_3 = 4$ is classical but not formalised), and (Γ) is $\varpi\Omega_\rho/\pi \notin \overline{\mathbb{Q}}$ ($\mathtt{h}\Gamma$; Wüstholz [24]).

- **MG.** $4\mathrm{E}(c\varpi/\pi)$; for example $2^{\varpi/\pi}, 3^{\varpi/\pi}$ are not both algebraic. For $s = c\varpi/\pi$ and $K \ni i, c$, no $x_1, x_2 \in K\mathcal{L}$ have $sx_j \in K\mathcal{L} + K\mathcal{L}_{E_i}$ with $2\pi i, x_1, x_2$ K-independent. `Unconditional`: `fourExpAt_varpi_div_pi_all`, `two_three`, `mg_K_varpi`, from `LSG.W88.thm11Kappa`, `LSG.W88.hypStdCMAt`: **U**. At Ω_ρ, `fourExpAt_omegaRho_div_pi_all`: **I** $((g_3))$. Arbitrary $K \supseteq \mathbb{Q}(i)$: **N**.
- **MG$^\natural$.** $4\mathrm{E}(c\varpi)$; for example $2^{\varpi}, 3^{\varpi}$ are not both algebraic. `Unconditional`: `fourExpAt_varpi_all`, `two_three_varpi`, the K-form `mgn_K_varpi`, from `LSG.W88.V5d.thm11KappaMGn_uncond`, `thm11KappaNat_uncond`, `thm11KappaMGnK_uncond`: **U**. At Ω_ρ, `fourExpAt_omegaRho_all`, `two_three_omegaRho`: **I** $((g_3))$.
- **SMG(a).** For $s = c\varpi/\pi$ and $y_1, y_2 \in \widetilde{\mathcal{L}}$ with $2\pi i, y_1, y_2$ $\overline{\mathbb{Q}}$-independent, sy_1, sy_2 are not both in $\widetilde{\mathcal{L}}$. `Gen.smg_false` at `Gen.latI`, with `Bridge.thm11KappaSMG_of_std` and `LSG.W88.hypStdCMAt` $(d_0 = 2)$: $\mathbf{U}^*$; the composite occurs inside `Gen.thmS`, hence in `Unconditional.thmS_varpi`. With second row in $\widetilde{\mathcal{L}} + \overline{\mathbb{Q}}\mathcal{L}_{E_i}$: **N**.
- **SE(1).** For $t = c\Omega_\rho/\varpi$ and $u_1, u_2 \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_{E_i}$ with ϖ, u_1, u_2 $\overline{\mathbb{Q}}$-independent, tu_1, tu_2 are not both in $\overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_{E_i} + \overline{\mathbb{Q}}\mathcal{L}_{E_\rho}$. `Unconditional.thmSE_varpi_omegaRho`, from `LSG.W88.hypStd_of_pair`: **I** $((g_3))$. For two non-isogenous CM lattices, `Bridge.thmSE_of_nonIsog` assumes no form of W88 but has the premise `Bridge.CMMulLogE`, proved only for E_i, E_ρ.

- **MX, MX′, deg2.** Let $X \in \Lambda(E_i^\natural)$, $Y \in \Lambda(E_\rho^\natural)$ with base and fibre coordinates (z, w), $a, a' \in \overline{\mathbb{Q}}$, $\beta \in \overline{\mathbb{Q}}^\times$. (MX) $(z(X)+z(Y)+a)\beta\eta_\rho = (w(X)+w(Y)+a')\varpi$ implies $z(X)+z(Y)+a \in \overline{\mathbb{Q}}\varpi + \overline{\mathbb{Q}}\Omega_\rho$, and then $a = 0$, $z(X) \in \mathbb{Q}(i)\varpi$, $z(Y) \in \mathbb{Q}(\rho)\Omega_\rho$. (MX′) $(z(X) + w(Y) + a)\beta\Omega_\rho = (w(X) + z(Y) + a')\varpi$ implies $z(X) + w(Y) + a \in \overline{\mathbb{Q}}\varpi + \overline{\mathbb{Q}}\eta_\rho$ (Proposition D′ of [**?**]; the step to Theorem MX′ of [7] is **N**). (deg2)(i) $\pi, \varpi, \Omega_\rho, \varpi\Omega_\rho$ are $\overline{\mathbb{Q}}$-linearly independent; (ii) so are $1, \varpi, \Omega_\rho, \varpi\Omega_\rho/\pi$. `Unconditional`, all **I**, all with (g_3), (Γ): `mx_K`; `mx` (also `Gen.WuestholzMX`, `Gen.MasserCM` [13]); `mx1`, its corollary on ζ (also `Gen.LaurentZeta` [12]); `mxP_K`; `deg2_i`, `deg2_ii` (also `Gen.WuestholzSix`, which implies (Γ)). From `LSG.W88.V5d.thm11KappaMXG_uncond` ($r = 1$) and `thm11KappaMXP_uncond` ($r = 2$). These statements use hand-written quasi-periods and the coordinates `MX.LogNat`, checked numerically in the certificate's reviews.
- **Theorem M.** If $u \in \overline{\mathbb{Q}}\mathcal{L}_{E_i}$, $\ell \in \widetilde{\mathcal{L}}$ and $i\pi u = c\varpi\ell$, then $u = sc\varpi$, $\ell = si\pi$ with $s \in \overline{\mathbb{Q}}$; hence $\wp(c\varpi/\pi) \notin \overline{\mathbb{Q}}$ (M1), and $\wp(c\pi\varpi) \notin \overline{\mathbb{Q}}$ if $e^{\pi^2} \in \overline{\mathbb{Q}}$ (M3). `Unconditional`: `thmM_varpi`, `m1_varpi`, `m3_varpi`, from `LSG.W88.hypStdCMAt` ($d_0 = 1$): **U**. Also **I**: `thmM_varpi_k` ($s \in \mathbb{Q}(i)$ for $u \in \mathcal{L}_{E_i}$, $c = 1$), `meta_varpi` (`Gen.MasserCM`); `m3'_varpi` (`Gen.HermiteLindemann`); `thmM_omegaRho` ((g_3)). (M2) of [**?**]: **N**.
- **MG-A at B/π.** $4\mathrm{E}(cB/\pi)$ and $4\mathrm{E}(c\pi/B)$; for example $2^t, 3^t$ are not both algebraic at $t = \Gamma(1/5)^2/(\pi\Gamma(2/5))$. Also $P/\pi \notin \overline{\mathbb{Q}}$ for $P = (1-\zeta_5)B$, and the K-form at P with $z_\sigma(\Lambda(A))$ replaced by $\mathbb{Z}P$. `Bridge`: `two_three_beta5_uncond`, `fourExpAt_beta5_uncond`, `fourExpAt_pi_div_beta5_uncond`, `wuestholzA5_uncond`, `mgA5_K_uncond`, from `LSG.W88.thm11KappaA5`, `LSG.W88.thm11KappaAK5`, within the scope of Theorem 10.1: **U**. One of $e^{a\pi^2}, e^{b\pi B}, e^{c\pi^2/B}$ is transcendental ($a, b \in \overline{\mathbb{Q}}^\times$), `Bridge.corA2_beta5_uncond`: **I** (`AbVar.NotLtildeA`, that is, $P \notin \widetilde{\mathcal{L}}$). The K-form with logarithms of A among the data, `AbVar.mgA_K`: **W** (`AbVar.Thm11KappaAK`).
- **MG$^\natural$-A at B.** $4\mathrm{E}(cB)$ and $4\mathrm{E}(c/B)$; for example $2^t, 3^t$ are not both algebraic at $t = B = \Gamma(1/5)^2/\Gamma(2/5)$. Also $\bar{P}/\pi \notin \overline{\mathbb{Q}}$ for $\bar{P} = (1-\zeta_5^4)B(4/5, 4/5)$, and the K-form at $\bar{P}$ with the fibre values replaced by $\mathbb{Z}\bar{P}$. `Bridge`: `two_three_beta5'_uncond`, `fourExpAt_beta5'_uncond`, `fourExpAt_div_beta5_uncond`, `wuestholzB5_uncond`, `mgAn5_K_uncond`, from `LSG.W88.thm11KappaAn5`, `LSG.W88.thm11KappaAnK5`, within the scope of Theorem 11.1: **U**. The K-form with logarithms of $A^\natural$ among the data, `AbVar.mgAn_K`: **W** (`AbVar.Thm11KappaAnK`).
- **X7 and genus three.** On the Jacobian of $y^7 = x(1-x)$, with $P_j = (1-\zeta_7^j)B(j/7, j/7)$ and E with CM by $\mathbb{Q}(\sqrt{-7})$ and period Ω: $(z_1(v_1) + u + q_1)\beta P_3 = (z_3(v_3) + z_2(v_2) + q_2)P_1$ implies $q_1 = q_2 = 0$, $z_j(v_j) \in \overline{\mathbb{Q}}P_j$, $u \in \overline{\mathbb{Q}}\Omega$, and all points are

torsion. Also $1, B(\frac{2}{7},\frac{2}{7}), B(\frac{3}{7},\frac{3}{7}), B(\frac{3}{7},\frac{3}{7})/B(\frac{1}{7},\frac{1}{7})$ are $\overline{\mathbb{Q}}$-linearly independent. `AbVar`: `thmX7`, `thmX7_torsion`, `x7c`, `x7a_AJ`, `x7b_AJ`, with logarithms as a parameter set `LA`: **W**, with hypotheses `Thm11KappaX7`, `ChowlaSelberg7`, `WuestholzBeta7`; except for `x7c` also `WuestholzX7`; for the torsion clause and the Abel–Jacobi forms `x7a_AJ`, `x7b_AJ` also `TorsionX7`. Also **W**: $4\mathrm{E}(cB(\frac{1}{7},\frac{1}{7})/\pi)$ (`fourExpAt_beta7`, hypothesis `Thm11KappaA` at $N = 7$). This project does not cover genus three: it proves no form of these hypotheses, and `dev/M3.md` is a design note with no Lean behind it.

- **X9.** The level-9 analogue of X7, on the CM threefold factor of the Jacobian of $y^9 = x(1-x)$ and E_ρ [7]. **N**: the level-9 data are not in Lean. The generic `AbVar.XSetup.thmXA` (no torsion clause, the period relation as a premise) is **W** (`AbVar.Thm11KappaXA`, `AbVar.WuestholzXA`, `AbVar.WuestholzSplitXA`); this project proves none of these.
- **X$^\kappa$.** The statement with $\kappa = 2^{g-2}$ parallel pairs of slots at level N [7]. **N**, in both projects.

Also with W88 discharged: Theorem S at ϖ (`Unconditional.thmS_varpi`: **I**, `Gen.NotLtilde`), Theorem SE_0 at E_i (`Bridge.thmSE0_varpi`: **U**), and the transcendence of π/ϖ^2 (`Bridge.transcendental_pi_div_varpi_sq`: **U**). Other CM periods and Beta values need their own period data in Lean.

What the labels rest on. For **U** and **I** the group models are internal to the proof: the final statements concern explicit numbers and the kernel checks the deduction, so they do not depend on the reviews of the models. One reads only the definitions in the statements and, for **I**, accepts the inputs. For **W**, the reviews of the certificate and of Section 11 apply. No human expert has reviewed any of this.

Provenance. No new method is involved. Each entry is an instance of [21, Thm. 4.1] or [21, §5d] on a group G'' as above, and the verified entries go through [21, §5d]. For MG, SMG(a), SE(1), MX, MX′, deg2, X7, X9 and X$^\kappa$ the printed weight of Théorème (iii) of [17] already suffices on a product of standard groups (with Wüstholz's theorem for the final statements), and for MG$^\natural$ by an internal check in [**?**]. The proof in [17] reduces to $K = \mathbb{Q}$ and then refers to a refinement of an argument of Roy that is not written out (we could not consult Roy's paper). At B/π and B the printed weight gives no conclusion, and one needs $\alpha_{K,\Psi}$ of [**?**]. Theorem M is classical in substance [**?**]. For the other entries our claim is narrow: we found none of them stated in print, apart from what follows from the conjugation arguments discussed in [**?**], and we believe that the formal proofs here are the first of these statements. The search was not exhaustive, this literature is moving fast and partly AI-generated, and [**?**, **?**] have not been refereed.

13. The artifact

The formalization [10] uses Lean 4 [3] and Mathlib [19], version v4.35.0 (Mathlib commit `56a005855ea17e0b0958f73ec801ae140b0a1679`), with StrongSix [9] as a Lake git dependency pinned to commit `b7f1649`. The default target `LSG` has 737 modules and about 168,500 lines. Table 2 gives the breakdown. The main theorem and the vector-extension results of Section 9 with everything they need take about 78,200 lines, with about 3,450 theorems and lemmas; the abelian chain alone (the main theorem and the abelian §5d bridges) is 133 files and about 35,300 lines. The abelian-surface layer of Section 10 is 323 files and about 67,000 lines, of which about 27,000 lines are generated certificate data. The vector extension of Section 11 adds 110 files and about 23,000 lines.

Verification. The commands are

```
lake exe cache get
lake build
lake env lean Axioms.lean
```

`Axioms.lean` prints `#print axioms` for 63 declarations, among them `LSG.W88.theorem41Abelian`, `LSG.ME.multEstAb`, `LSG.W88.prop61_b`, `LSG.schneiderLang`, the Schneider theorems, `LSG.Weierstrass.legendre`, `LSG.CM.thm41Sub`, `LSG.W88.thm11Kappa`, `LSG.W88.hypStdCM`, the sanity derivations, and the vector-extension endpoints of Section 9 (`LSG.MEV.multEstVec`, the five declarations named in Theorem 9.1 and the six bridges), and those of Section 10 (`LSG.Jac.a1Facts`, `LSG.MEB.multEstB_holds`, the two declarations named in Theorem 10.1 and the two bridges), and those of Section 11 (the three declarations named in Theorem 11.1 and the two bridges); every line reports `[propext, Classical.choice, Quot.sound]`. The source contains no `sorry`, no `axiom` declaration and no `native_decide`. A cold build of the default target, with Mathlib and StrongSix already built and `LEAN_NUM_THREADS=4`, took 3 h 33 min on a Windows machine with 32 logical processors and 31.7 GB of RAM, with no errors. Times per file sum to 14 hours, of which 9.4 are in `LSG/Jac/`: the kernel certificates of Section 10 dominate. Built two at a time they take about five CPU-hours (about 2.7 hours wall-clock), with single files peaking at 4–5 GB of memory; at four at a time we sampled up to 7.6 GB. An earlier attempt with Lake's default concurrency failed with `.olean` read errors, which we suspect but did not show to be memory exhaustion. Outside the certificates, single modules took up to about nine minutes each.

Some vector-extension files contain polynomial identities whose cofactors were produced by Python scripts (`scripts/`); each identity is re-proved in Lean, so the scripts are untrusted inputs. These files are in the default target; the abelian chain imports none of them and uses no generated data.

The certificates of Section 10 are likewise generated by untrusted scripts and rechecked by the kernel.

14. Trust and limitations

What the kernel guarantees. That `LSG.W88.theorem41Abelian` proves the proposition `Theorem41Abelian` from Lean's standard axioms. Whether that proposition is Waldschmidt's theorem is a question about `LSG/W88/Statement.lean`, reproduced in Section 2, and has been checked only by the AI review described there; likewise for the vector-extension statements and the review of Section 9.

Scope. The main theorem has (NR), and its G_2 is a product of elliptic curves (or, by `thm41Sub`, an abelian subvariety of one). The §5d statement for distinct curves assumes `hrat`. For vector extensions only the cases of Theorem 9.1 are proved, and its premises `LiftAlg` and `PerTanRat` are proved in Lean only for the CM curves used; the general `Theorem41Vec` is not proved. For the Jacobian of $y^5 = x(1-x)$ only toric-injective hyperplanes and period A-components are covered, and the identification of the formal torus with the Jacobian is classical, not formalized (Section 10). For its vector extension the same two restrictions apply, with $r = 1$ and an adapted b; the identification of the model in $\mathbb{P}^{11}$ with the extension is classical, not formalized (Section 11). The copies of the certificate's statements are checked by typechecking inside the certificate, which depends on this project; whether its group models describe the intended groups is checked by its own reviews, not here.

Licence history. Early in the project, about 1,500 lines of the ancillary Lean files of [2] (the lemniscatic period identification) were ported for local use. Their licence did not cover those files, so the port was never published; it was deleted and replaced by the independent proof of Section 4, written by an agent that had not seen the ported files. The local history still contains the port, and the repository will be published from a history that excludes it. Two files of `LSG/SchneiderLang/` adapt lemmas from StrongSix (same author, Apache 2.0); no other external Lean code is included.

Reviews. Apart from the Lean kernel, all checking so far was done by AI systems. No human expert has read the statement against [21] or the proofs.

15. AI-assisted formalization

The Lean code, the planning documents and the text of this paper were produced by AI models (Claude, Anthropic) working in Claude Code under the direction of the author, with several agent sessions in parallel coordinated through written plans and interface files. The development git history, which is not published, shows the main theorem and the vector-extension results taking about seventeen and a half hours of wall-clock time (30–31 October 2026). Three practices from [6] were kept: every stage was fixed as a `Prop` interface first; each deduction between interfaces was proved before its inputs; and design problems were recorded in the planning notes when found,

together with the fix. The four problems of Section 5, the irrelevant components in particular, were found by agents whose proofs failed and were then traced to the mathematics, not to Lean. The author is responsible for the intended statements and for the claims made here, and we would be grateful for corrections.

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Directory	Lines	Content
Abelian chain (default target)		
`Weierstrass/`	3,768	σ, ζ, η, order two, Legendre, addition theorem, dictionary
`SchneiderLang/`, `Schneider*.lean`	2,069	Schneider–Lang criterion, Schneider's theorems, interfaces
`LemniscateIndep/`, `Bridge/`	681	$g_2(\varpi\mathbb{Z}[i]) = 4$; statements for the certificate
`Subgroups/`	399	closed subgroups of $\mathbb{R}^n$, subtori
`Elliptic/AdditionLaws*`	965	addition laws on $E \subset \mathbb{P}^2$
`ME/`	16,516	Stage 2: the multiplicity estimate
`AuxFun/` (abelian)	6,470	Stage 3a: W83 Prop. 2.1, Lemma U
`W88/` (abelian)	6,971	statement, Prop. 6.1 (750), §7 (1,664), §5d bridges and copies (4,004), sanity
`CM/Coord`, `CM/Transport`	548	Thm. 1.1 for abelian subvarieties
Vector extensions (Section 9)		
`MEV/`	18,495	multiplicity estimate for the class
`AuxFunV/`, `AuxFun/Prop21LatT*`	5,320	analytic side for $E^\natural$, polynomial fibre direction
`Elliptic/Natural*`	2,887	the model of $E^\natural$ and its addition laws
`W88/` (vector)	12,465	statements, Prop. 6.1, §7, §5d engine, bridges, copies
`CM/` (rest)	1,150	CM toolkit for the bridges
The Jacobian of $y^5 = x(1-x)$ (Section 10)		
`Jac/`	36,518	Grant's model, checkers and soundness (9,623); generated data and certificates (26,895)
`JacAn/`, `JacAn3/`, `JacAn4/`	9,256	Abel map, Beta periods, exponential, θ, laws on the image, surjectivity
`MEB/`	10,579	multiplicity estimate for abelian blocks
`JacBlock/`	3,342	the Jacobian block and its packages
`AuxFunA/`	1,777	Prop. 2.1 under toric injectivity, Lemma U
`W88/AbVar*`	5,376	statements, Prop. 6.1, §7, §5d engine, bridge, copies
The vector extension $A^\natural_{\bar\sigma}$ (Section 11)		
`JacAnV/`	2,244	fibre coordinates, cocycle, fibre laws and periods
`MEBV/`	10,930	multiplicity estimate for vector blocks
`JacBlockV/`	2,208	the block $A^\natural_{\bar\sigma}$, packages, §5d block properties
`AuxFunAV/`	1,976	Prop. 2.1 under toric injectivity, Lemma U
`W88/AbVarV*`, bridge	5,115	statements, Prop. 6.1, §7, §5d engine (4,297); bridge and copies (818)
`Jac/B6V*`	771	Hilbert certificate (degree 41), Wronskian cocycle

TABLE 2. Lines per layer (`wc -l`), counted when each block was completed.