HOLOMORPHIC SECTIONAL CURVATURE OF FLAG MANIFOLDS VIA KÄHLER REDUCTION

CHING-HSIEN LEE

Abstract. We give an alternative proof that every invariant Kähler metric on a compact generalized flag manifold has positive holomorphic sectional curvature. Invariant Kähler metrics on unitary partial flag manifolds are realized as reductions of Hermitian vector spaces at compact moment-map levels contained in spheres. A curvature identity for linear Kähler quotients gives an explicit positive lower bound. For an inclusion of compact connected Lie groups, we prove that the induced inclusion between adjoint orbits through the same Lie algebra element is holomorphic, isometric, and totally geodesic for the Kähler structures determined by an invariant inner product and its restriction. This reduces the general case to unitary flags. We also obtain a strict estimate for the second fundamental form of the anticanonical projective embedding.

hexagon:2611.00168v1

1. Introduction

A compact generalized flag manifold is a quotient $M = G/H$, where G is a compact connected semisimple Lie group and H is the centralizer of a torus, equipped with a G-invariant complex structure. These manifolds are the Kähler C-spaces of Borel [1]. Ni–Zheng [7, Conjecture 1.8] conjectured that their canonical Kähler–Einstein metrics have positive holomorphic sectional curvature. Li–Petersen [4, Theorem 1.1] prove the stronger statement that every invariant Kähler metric on a compact generalized flag manifold has positive holomorphic sectional curvature. Their proof uses an intrinsic curvature identity. The present paper gives an alternative proof by Kähler reduction and totally geodesic embeddings.

Theorem 1.1. *Every compact generalized flag manifold with an invariant Kähler metric admits a holomorphic isometric totally geodesic embedding into a unitary partial flag manifold with an invariant Kähler metric. Consequently, every invariant Kähler metric on a compact generalized flag manifold has positive holomorphic sectional curvature.*

The proof has two parts. First, an inclusion of compact Lie groups induces a totally geodesic embedding of adjoint orbits when both metrics are defined by the same Lie algebra element and compatible invariant inner products; see Proposition 2.2. Second, the standard Kähler quotient construction realizes every invariant Kähler metric on a unitary partial flag manifold. Its moment-map level lies in a sphere, and Zhang's curvature formula [12, Lemma 3.3] gives a quantitative lower bound. The resulting estimate for arbitrary flag manifolds is stated in (4.2).

The holomorphic isometric embedding of a general flag manifold into a classical flag manifold is established by Loi–Mossa–Zuddas [5, Theorem 1.1]. We prove total geodesy for the particular compatible orbit metrics used here; holomorphic isometry alone would not transfer positivity of holomorphic sectional curvature. The quotient description of unitary flags is standard [3, Section 2.1]; Martin's construction, presented in [2, Section 9.2], identifies such reductions with coadjoint orbits. Curvature computations for linear Kähler quotients appear in [6], and Zhang [12,

Date: November 6, 2026.

2020 *Mathematics Subject Classification.* 53C55, 53C30, 32M10.

Key words and phrases. Generalized flag manifold, holomorphic sectional curvature, Kähler reduction, totally geodesic embedding .

Theorem 3.1] proves positivity for compact moment-map levels. We recall these ingredients to identify the metric exactly and make the lower bound explicit.

Let ∇ denote the Levi–Civita connection and let X_1, X_2, X_3 be smooth real vector fields. Our curvature convention is

$$R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3. \tag{1.1}$$

For a Kähler metric g with complex structure J, we use $\omega(X, Y) = g(JX, Y)$ and define

$$H_g(X) = \frac{g(R(X, JX)JX, X)}{g(X, X)^2} \tag{1.2}$$

for every nonzero real tangent vector X. All tangent vectors in curvature formulas below are real unless explicitly complexified.

2. Invariant metrics and totally geodesic embeddings

2.1. **Kähler structures on adjoint orbits.** Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. For $a \in G$, write Ad_a for the differential of conjugation $b \mapsto aba^{-1}$. Fix an $\mathrm{Ad}(G)$-invariant positive inner product Q on $\mathfrak{g}$ and an element $\xi \in \mathfrak{g}$. Define

$$\begin{aligned} H &= C_G(\xi) = \{a \in G : \mathrm{Ad}_a\,\xi = \xi\}, & \mathfrak{h} &= \mathrm{Lie}(H), \\ L &= \mathrm{ad}\,\xi, & LX &= [\xi, X], \\ \mathfrak{m} &= \mathfrak{h}^{\perp_Q}, & D &= \sqrt{-L^2}\big|_{\mathfrak{m}}. \end{aligned} \tag{2.1}$$

Here $\perp_Q$ denotes orthogonal complement with respect to Q. We have $\mathfrak{h} = \ker L$. Since L is Q-skew-adjoint and has no kernel on $\mathfrak{m}$, the operator D is positive definite. Identify $\mathfrak{m}$ with $T_o(G/H)$, where e is the identity of G and $o = eH$, by sending X to the velocity of $\exp(tX)H$ at $t = 0$. Under the adjoint-orbit identification $aH \mapsto \mathrm{Ad}_a\,\xi$, this velocity becomes $[X, \xi]$.

The following formulas define invariant tensors on G/H:

$$\begin{aligned} \omega_o(X, Y) &= -Q(\xi, [X, Y]) = -Q(LX, Y), \\ J_o &= -LD^{-1}, \\ g_o(X, Y) &= Q(DX, Y). \end{aligned} \tag{2.2}$$

They give a Kähler structure. Indeed, the tensors are $\mathrm{Ad}(H)$-invariant, $J_o^2 = -I$, and $g_o(X, Y) = \omega_o(X, J_oY)$. The Jacobi identity proves that ω is closed. Write $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ for the complexifications. To check integrability, extend L complex-linearly. The $+$i-eigenspace of J_o is the sum of the L-eigenspaces with eigenvalues $-is$, where $s > 0$. Since L is a derivation, brackets add these eigenvalues. This sum is closed under brackets and is stable under $\mathfrak{h}_{\mathbb{C}}$. The invariant almost complex structure is therefore integrable.

For semisimple groups, these formulas recover any prescribed invariant Kähler metric.

Lemma 2.1. *Let $M = G/H$ be a compact generalized flag manifold, and let (J, g, ω) be a G-invariant Kähler structure. For every $\mathrm{Ad}(G)$-invariant positive inner product Q on $\mathfrak{g}$, there is a unique $\xi \in \mathfrak{g}$ such that $H = C_G(\xi)$ and the given tensors satisfy* (2.2).

Proof. Let $p_G : G \to G/H$ be the quotient map and put $\beta = (p_G^*\omega)_e$. The form $p_G^*\omega$ is left-invariant and closed, so β is an alternating Lie algebra two-cocycle. Whitehead's second lemma, $H^2(\mathfrak{g}, \mathbb{R}) = 0$ for trivial real coefficients, gives $\ell \in \mathfrak{g}^*$ such that $\beta(X, Y) = -\ell([X, Y])$; see [11, Corollary 7.8.12]. Write $\ell(X) = Q(\xi, X)$. Then

$$\beta(X, Y) = -Q(\xi, [X, Y]) = -Q([\xi, X], Y). \tag{2.3}$$

Nondegeneracy of ω on G/H gives $\ker\beta = \mathfrak{h}$, so $\mathfrak{h} = \ker\mathrm{ad}\,\xi$. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, the functional ℓ is unique. The $\mathrm{Ad}(H)$-invariance of β consequently implies that H fixes ξ. Thus $H \subset C_G(\xi)$ and

both groups have the same Lie algebra. Both are connected centralizers of tori: for $C_G(\xi)$, use the closure of $\exp(\mathbb{R}\xi)$. Hence $H = C_G(\xi)$. It remains to identify J. Choose a maximal torus contained in H. Invariance makes J preserve each complementary real root plane. On such a plane, choose a Q-orthonormal basis e, f with $Le = bf$ and $Lf = -be$, where $b \neq 0$. Torus invariance and $J^2 = -I$ give $Je = \varepsilon f$ with $\varepsilon \in \{1, -1\}$. Positivity gives $0 < g(e, e) = \omega(e, Je) = -\varepsilon b$. Therefore $J = -LD^{-1}$ on every root plane, and $g(X, Y) = \omega(X, JY) = Q(DX, Y)$. □

2.2. **The inclusion of adjoint orbits.**

Proposition 2.2. *Let $G \subset \widetilde{G}$ be compact connected Lie groups with Lie algebras $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$. Fix an $\mathrm{Ad}(\widetilde{G})$-invariant positive inner product Q on $\widetilde{\mathfrak{g}}$ and $\xi \in \mathfrak{g}$. Equip $G/C_G(\xi)$ and $\widetilde{G}/C_{\widetilde{G}}(\xi)$ with the Kähler structures* (2.2)*, using $Q|_{\mathfrak{g}}$ and Q, respectively. Then*

$$\Phi : G/C_G(\xi) \longrightarrow \widetilde{G}/C_{\widetilde{G}}(\xi) \tag{2.4}$$

given by $\Phi(aC_G(\xi)) = aC_{\widetilde{G}}(\xi)$ is a holomorphic isometric totally geodesic embedding.

Proof. Write $\widetilde{L} = \operatorname{ad}\xi$ on $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}} = \ker \widetilde{L}$, $\widetilde{\mathfrak{m}} = \widetilde{\mathfrak{h}}^{\perp_Q}$, and $\widetilde{D} = \sqrt{-\widetilde{L}^2}|_{\widetilde{\mathfrak{m}}}$. Let P be the Q-orthogonal projection onto $\widetilde{\mathfrak{m}}$. Invariance of Q gives

$$[\mathfrak{g}, \mathfrak{g}^{\perp_Q}] \subset \mathfrak{g}^{\perp_Q}. \tag{2.5}$$

Consequently, both $\mathfrak{g}$ and $\mathfrak{g}^{\perp_Q}$ are preserved by $\widetilde{L}$ and its spectral projections. In particular,

$$\mathfrak{m} = \mathfrak{g} \cap \widetilde{\mathfrak{m}}, \tag{2.6}$$

and P, $\widetilde{D}$, and $\widetilde{D}^{-1}$ preserve the appropriate subspaces of $\mathfrak{g}$. Formula (2.2) now shows that $d\Phi_o$ preserves the complex structure and the metric. Moreover, $G \cap C_{\widetilde{G}}(\xi) = C_G(\xi)$, so Φ is injective; its differential is injective by (2.6). Compactness makes it an embedding. To prove total geodesy, use the reductive decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{m}}$. The Levi–Civita connection is represented in homogeneous frames by the bilinear map $\alpha : \widetilde{\mathfrak{m}} \times \widetilde{\mathfrak{m}} \to \widetilde{\mathfrak{m}}$ satisfying Nomizu's formula [8]:

$$\begin{aligned} 2\widetilde{g}(\alpha(X, Y), V) = {} & \widetilde{g}(P[X, Y], V) + \widetilde{g}(P[V, X], Y) \\ & + \widetilde{g}(X, P[V, Y]). \end{aligned} \tag{2.7}$$

Substituting $\widetilde{g}(X, Y) = Q(\widetilde{D}X, Y)$ and using invariance of Q gives

$$\alpha(X, Y) = \frac{1}{2}P[X, Y] + \frac{1}{2}\widetilde{D}^{-1}P([X, \widetilde{D}Y] + [Y, \widetilde{D}X]). \tag{2.8}$$

If $X, Y \in \mathfrak{m}$, every bracket on the right belongs to $\mathfrak{g}$, and the spectral operators preserve $\mathfrak{g}$. Therefore $\alpha(X, Y) \in \mathfrak{m}$. The normal component of this connection map is the second fundamental form of the subgroup orbit at the base point; it vanishes. Equivariance gives total geodesy everywhere. □

3. Kähler reduction and unitary flags

3.1. **The curvature formula.** Let (V, J_0, g_0, ω_0) be a finite-dimensional flat Hermitian vector space, with $\omega_0(u, v) = g_0(J_0u, v)$. Suppose a compact Lie group K acts linearly and unitarily on V. Write $\mathfrak{k} = \mathrm{Lie}(K)$ and $\eta_V(z) = A_\eta z$ for the fundamental vector field of $\eta \in \mathfrak{k}$. Thus A_η is skew-adjoint and commutes with J_0. Choose an equivariant moment map $\mu : V \to \mathfrak{k}^*$ with

$$\mu^\eta(z) = \mu(z)(\eta) \tag{3.1}$$

and $d\mu^\eta = \iota_{\eta_V}\omega_0$. Let $c \in \mathfrak{k}^*$ be a coadjoint-invariant regular value, assume $Z = \mu^{-1}(c)$ is nonempty, and assume K acts freely on Z. Orthogonal complements in V are taken with respect to g_0. The vertical and horizontal spaces are

$$\begin{aligned} \mathcal{V}_z &= \{A_\eta z : \eta \in \mathfrak{k}\}, \\ \mathcal{H}_z &= (\mathcal{V}_z \oplus J_0\mathcal{V}_z)^\perp, \\ N_z Z &= J_0\mathcal{V}_z. \end{aligned} \tag{3.2}$$

Indeed, $d\mu^\eta_z(u) = g_0(J_0A_\eta z, u)$ identifies the normal space, and coadjoint invariance makes Z invariant under K. In particular, $T_zZ = \mathcal{V}_z \oplus \mathcal{H}_z$, and $\mathcal{H}_z$ is J_0-invariant. The quotient map $\pi : Z \to Z/K$ defines the reduced Kähler structure by

$$\begin{aligned} g_{\mathrm{red}}(d\pi u, d\pi v) &= g_0(u, v), \\ J_{\mathrm{red}} d\pi u &= d\pi(J_0 u) \end{aligned} \tag{3.3}$$

for $u, v \in \mathcal{H}_z$. Denote by II_Z the second fundamental form of the real submanifold $Z \subset V$.

The next identity is Zhang's formula [12, Lemma 3.3 and equation (3.3)], written using II_Z. We recall its derivation to fix the normalization.

Lemma 3.1. *Under the preceding assumptions, every nonzero $u \in \mathcal{H}_z$ satisfies*

$$H_{g_{\mathrm{red}}}(d\pi u) = \frac{4\|\mathrm{II}_Z(u,u)\|^2_{g_0}}{\|u\|^4_{g_0}}. \tag{3.4}$$

Proof. The moment functions are quadratic up to constants, with $\operatorname{Hess}\mu^\eta(u,v) = g_0(J_0A_\eta u, v)$. Differentiating their restrictions to Z gives

$$g_0(\mathrm{II}_Z(u,v), J_0A_\eta z) = -g_0(J_0A_\eta u, v). \tag{3.5}$$

Since the normal vectors $J_0A_\eta z$ span N_zZ, skew-adjointness and complex linearity of A_η imply

$$\begin{aligned} \mathrm{II}_Z(J_0u, J_0u) &= \mathrm{II}_Z(u,u), \\ \mathrm{II}_Z(u, J_0u) &= 0. \end{aligned} \tag{3.6}$$

Let $\mathcal{A}^\pi$ be O'Neill's tensor, defined on horizontal fields by taking the vertical component of their covariant derivative in Z. Parallelness of J_0 and the orthogonal splitting in (3.2) give

$$\mathcal{A}^\pi_u(J_0u) = J_0\mathrm{II}_Z(u,u). \tag{3.7}$$

To see this, take a horizontal extension of u, differentiate its J_0-multiple in V, and project onto $\mathcal{V}_z$. Only the J_0-image of the normal component survives. The flat Gauss equation and (3.6) give the curvature numerator $\|\mathrm{II}_Z(u,u)\|^2_{g_0}$ for the plane spanned by u, J_0u in Z. O'Neill's formula [9] adds $3\|\mathcal{A}^\pi_u(J_0u)\|^2_{g_0}$. Division by $\|u\|^4_{g_0}$ proves (3.4). □

Corollary 3.2. *If $g_0(z,z) = \varrho^2$ on Z for a constant $\varrho > 0$, then $H_{g_{\mathrm{red}}} \geq 4/\varrho^2$.*

Proof. The position vector z is normal to Z, since the squared norm is constant on Z. Differentiating its orthogonality to a tangent extension of u gives

$$g_0(\mathrm{II}_Z(u,u), z) = -\|u\|^2_{g_0}. \tag{3.8}$$

Thus $\|\mathrm{II}_Z(u,u)\|^2_{g_0} \geq \|u\|^4_{g_0}/\varrho^2$ by Cauchy–Schwarz. Apply Lemma 3.1. □

3.2. **The flag quotient.** Fix integers

$$0 = d_0 < d_1 < \cdots < d_r < d_{r+1} = N, \tag{3.9}$$

put $n_j = d_j - d_{j-1}$ for $1 \le j \le r+1$, and choose $c_1, \dots, c_r > 0$. Equip each $\mathbb{C}^d$ with its standard Hermitian inner product. Write A^* for a Hermitian adjoint and $\|A\|_{\mathrm{HS}}^2 = \operatorname{Tr}(A^*A)$ for the squared Hilbert–Schmidt norm. Define

$$V = \bigoplus_{i=1}^{r} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_{i+1}}) \tag{3.10}$$

and $K = \prod_{i=1}^{r} U(d_i)$. For $A = (A_1, \dots, A_r) \in V$ and $X, Y \in T_A V \cong V$, the flat structure is

$$\begin{aligned} J_0 X &= \mathrm{i}X, \\ g_0(X,Y) &= 2\operatorname{Re}\sum_{i=1}^{r} \operatorname{Tr}(X_i^* Y_i), \\ \omega_0(X,Y) &= 2\operatorname{Im}\sum_{i=1}^{r} \operatorname{Tr}(X_i^* Y_i). \end{aligned} \tag{3.11}$$

The unitary action and its infinitesimal action are

$$\begin{aligned} (k \cdot A)_i &= k_{i+1} A_i k_i^{-1}, \\ (\eta_V(A))_i &= \eta_{i+1} A_i - A_i \eta_i, \end{aligned} \tag{3.12}$$

where $k = (k_1, \dots, k_r) \in K$, $\eta = (\eta_1, \dots, \eta_r) \in \mathfrak{k}$, and $k_{r+1} = I_N$, $\eta_{r+1} = 0$. Set $A_0 = 0 : \mathbb{C}^0 \to \mathbb{C}^{d_1}$. An equivariant moment map is given by

$$\mu^\eta(A) = -\mathrm{i}\sum_{i=1}^{r} \operatorname{Tr}\big((A_i^* A_i - A_{i-1} A_{i-1}^*)\eta_i\big) = \tfrac{1}{2}\omega_0(\eta_V(A), A). \tag{3.13}$$

For any linear symplectic action, differentiating the last expression and using infinitesimal invariance of ω_0 gives $d\mu_A^\eta(X) = \omega_0(\eta_V(A), X)$. Equivariance follows from conjugation and invariance of trace. The functional $c(\eta) = -\mathrm{i}\sum_i c_i \operatorname{Tr}(\eta_i)$ is coadjoint-invariant, and its level is

$$Z = \{A \in V : A_i^* A_i - A_{i-1} A_{i-1}^* = c_i I_{d_i} \text{ for } 1 \le i \le r\}. \tag{3.14}$$

The group $U(N)$ acts on V by

$$u \cdot (A_1, \dots, A_r) = (A_1, \dots, A_{r-1}, uA_r). \tag{3.15}$$

The action preserves Z and commutes with K, and therefore acts on Z/K. Write $F(d_1, \dots, d_r; \mathbb{C}^N)$ for the partial flag manifold of chains $F_1 \subset \cdots \subset F_r \subset \mathbb{C}^N$ with $\dim_{\mathbb{C}} F_i = d_i$. Its natural $U(N)$-action sends each F_i to uF_i.

We give the metric-specific details of the standard quotient construction [3, Section 2.1]; compare [2, Section 9.2].

Proposition 3.3. *The level Z is nonempty and compact, c is a regular value, and K acts freely on Z. For $A \in Z$ set $T_i = A_r \cdots A_i : \mathbb{C}^{d_i} \to \mathbb{C}^N$. Then*

$$\begin{aligned} \phi : Z/K &\longrightarrow F(d_1, \dots, d_r; \mathbb{C}^N), \\ [A] &\longmapsto (\operatorname{im} T_1 \subset \cdots \subset \operatorname{im} T_r \subset \mathbb{C}^N) \end{aligned} \tag{3.16}$$

is a $U(N)$-equivariant biholomorphism. Here $[A] = K \cdot A$ and $\operatorname{im} T_i = \{T_i v : v \in \mathbb{C}^{d_i}\}$. Moreover, putting

$$S = \sum_{i=1}^{r} (r - i + 1) c_i d_i, \tag{3.17}$$

one has $g_0(A, A) = 2S$ for every $A \in Z$.

Proof. Equation (3.14) makes $A_i^* A_i$ positive definite, hence every A_i injective. To prove nonemptiness, choose linear isometric inclusions $j_i : \mathbb{C}^{d_i} \to \mathbb{C}^{d_{i+1}}$ and successively define

$$A_i = j_i \circ (A_{i-1} A_{i-1}^* + c_i I_{d_i})^{1/2} \tag{3.18}$$

for $1 \leq i \leq r$, using the positive Hermitian square root. These maps satisfy (3.14). Taking traces there gives

$$\|A_i\|_{\mathrm{HS}}^2 = \sum_{k=1}^{i} c_k d_k \tag{3.19}$$

and summing over i gives $\sum_i \|A_i\|_{\mathrm{HS}}^2 = S$. Thus Z is closed and bounded, hence compact, and $g_0(A, A) = 2S$. If $k \cdot A = A$, the last component gives $A_r k_r^{-1} = A_r$, so $k_r = I_{d_r}$. Descending through the components gives $k_i = I_{d_i}$ for every i. The action is free. The infinitesimal action is therefore injective at every $A \in Z$. Since $d\mu_A^\eta(X) = g_0(J_0 \eta_V(A), X)$, the transpose of $d\mu_A$ is injective, so $d\mu_A$ is surjective. Hence the quotient is a smooth Kähler manifold. To identify its complex structure, let $\Omega \subset V$ be the open set on which all maps A_i are injective, and define $\widetilde{\phi} : \Omega \to F(d_1, \ldots, d_r; \mathbb{C}^N)$ by the same chain of images as in (3.16). The map $\widetilde{\phi}$ is holomorphic, as is the full-rank matrix map to each Grassmannian. Its fibers are the orbits of $K^{\mathbb{C}} = \prod_i GL(d_i, \mathbb{C})$ acting by (3.12): the maps T_i are frames for the subspaces of a flag, and changing these frames gives precisely this action. The group $GL(N, \mathbb{C})$ acts on Ω by left multiplication on A_r. Equivariance of $\widetilde{\phi}$ and transitivity on the flag manifold show that $d\widetilde{\phi}$ is surjective. At $A \in Z$, the tangent space to its fiber is $\mathcal{V}_A \oplus J_0 \mathcal{V}_A$, since $\mathrm{Lie}(K^{\mathbb{C}}) = \mathfrak{k} \oplus \mathrm{i}\mathfrak{k}$. Thus $d\widetilde{\phi}_A$ restricts to a complex-linear isomorphism on $\mathcal{H}_A$. The restriction $\widetilde{\phi}|_Z$ descends to ϕ, which is consequently a local biholomorphism by (3.3). Its image is nonempty and $U(N)$-invariant; transitivity on the flag manifold gives surjectivity. For injectivity, suppose $A, A' \in Z$ define the same flag. Write $A' = h \cdot A$ with $h \in K^{\mathbb{C}}$ and use polar decomposition $h = k \exp(\mathrm{i}\eta)$, where $k \in K$ and $\eta \in \mathfrak{k}$. Both A and $\exp(\mathrm{i}\eta) \cdot A = k^{-1} \cdot A'$ lie in Z. Along $A(t) = \exp(\mathrm{i}t\eta) \cdot A$ the velocity is $A'(t) = J_0 \eta_V(A(t))$, so

$$\frac{d}{dt} \mu^\eta(A(t)) = g_0(J_0 \eta_V(A(t)), J_0 \eta_V(A(t))) \geq 0. \tag{3.20}$$

The endpoint values equal $c(\eta)$, so the derivative vanishes throughout $[0, 1]$. Hence $A(t)$ is constant and $A' = k \cdot A$. Therefore ϕ is a bijective local biholomorphism, as required. □

3.3. **The reduced metric.**

Lemma 3.4. *Under* (3.16)*, the reduced metric is $U(N)$-invariant and satisfies*

$$H_{g_{\mathrm{red}}} \geq \frac{2}{S} > 0. \tag{3.21}$$

Moreover, every $U(N)$-invariant Kähler metric g on $F(d_1, \ldots, d_r; \mathbb{C}^N)$ is realized by positive parameters $c_1, \ldots, c_r$, with $\phi^* g = g_{\mathrm{red}}$ for the corresponding quotient.

Proof. The action (3.15) preserves the flat structure, the level set, and the horizontal spaces; hence g_{red} is invariant. We first identify it exactly. Define

$$a_j = \sum_{k=j}^{r} c_k \tag{3.22}$$

for $1 \leq j \leq r$ and set $a_{r+1} = 0$. Equation (3.14) and equality of the nonzero spectra of $A_i^* A_i$ and $A_i A_i^*$ imply inductively that $A_i A_i^*$ has eigenvalues $\sum_{k=j}^{i} c_k$ of multiplicity n_j for $1 \leq j \leq i$, and zero of multiplicity n_{i+1}. Indeed, the level equation shifts the spectrum of $A_{i-1} A_{i-1}^*$ by c_i, and injectivity contributes exactly $d_{i+1} - d_i$ zero eigenvalues to $A_i A_i^*$. Thus $A_r A_r^*$ has eigenvalues $a_1 > \cdots > a_{r+1}$ with multiplicities $n_1, \ldots, n_{r+1}$. Iterating the level equation also gives

$$A_r A_r^* T_i = T_i (A_{i-1} A_{i-1}^* + a_i I_{d_i}). \tag{3.23}$$

Since T_i is injective, its image is the sum of the eigenspaces for $a_1, \dots, a_i$. Consequently, at the point $[A]$ corresponding to the standard coordinate flag,

$$A_r A_r^* = \operatorname{diag}(a_1 I_{n_1}, \dots, a_{r+1} I_{n_{r+1}}). \tag{3.24}$$

For $X \in \mathfrak{u}(N)$, define $\widehat{X}([A])$ as the velocity of $\exp(tX) \cdot [A]$ at $t = 0$. The moment function for this action is $f_X(A) = \mathrm{i} \operatorname{Tr}(A_r A_r^* X)$, which is K-invariant; use the same notation for its descended function. Differentiating $A_r A_r^*$ along $\widehat{Y}$ gives $[Y, A_r A_r^*]$, and hence

$$\omega_{\mathrm{red}}(\widehat{X}, \widehat{Y}) = df_X(\widehat{Y}) = \mathrm{i} \operatorname{Tr}(A_r A_r^* [X, Y]). \tag{3.25}$$

At the standard flag, the stabilizer is $\prod_j U(n_j)$. Thus each tangent vector $\widehat{X}([A])$ has a unique skew-Hermitian representative X with zero diagonal blocks: the kernel of the differential of the orbit map is exactly the block-diagonal Lie algebra. Let X_{jk} denote its (j, k) block. Local holomorphic coordinates are given by block lower triangular matrices with identity diagonal blocks. The tangent coordinates are therefore X_{jk} for $j > k$, and the representative of $J_{\mathrm{red}} \widehat{X}([A])$ has lower blocks $\mathrm{i} X_{jk}$ and upper blocks $-\mathrm{i} X_{jk}$. Substitution into (3.25) gives

$$g_{\mathrm{red}}(\widehat{X}([A]), \widehat{X}([A])) = 2 \sum_{1 \le j < k \le r+1} (a_j - a_k) \|X_{jk}\|_{\mathrm{HS}}^2. \tag{3.26}$$

For clarity, the (j, j) block of the commutator is $2\mathrm{i} \sum_{k<j} X_{kj}^* X_{kj} - 2\mathrm{i} \sum_{k>j} X_{jk} X_{jk}^*$; these are the blocks that enter the trace. We now realize an arbitrary invariant Kähler metric g. Apply Lemma 2.1 to the transitive $SU(N)$-action with $Q(X, Y) = -\operatorname{Tr}(XY)$. At the standard flag the representing element is $\xi = \mathrm{i} \operatorname{diag}(b_1 I_{n_1}, \dots, b_{r+1} I_{n_{r+1}})$, since it is fixed by the block-diagonal stabilizer. Here $b_j \in \mathbb{R}$ and $\sum_j n_j b_j = 0$. Positivity and the standard complex structure give $b_1 > \cdots > b_{r+1}$, by the same block calculation as in (3.26). Choose $c_j = b_j - b_{j+1}$. The resulting a_j satisfy $a_j = b_j - b_{r+1}$, and a scalar matrix has zero pairing with every commutator. Thus (3.25) agrees with $\phi^* \omega$ at the standard flag and, by invariance, everywhere. The complex structures also agree, so $g_{\mathrm{red}} = \phi^* g$. Finally, Proposition 3.3 gives $g_0(A, A) = 2S$ on Z. Corollary 3.2 proves (3.21) for every choice of positive parameters. □

4. Curvature and projective embeddings

4.1. **The general flag manifold.**

Proof of Theorem 1.1. Write $M = G/H$ and replace G by its effective quotient. Its center is trivial, because the center of a compact connected semisimple group is contained in every maximal torus and hence in H. The adjoint representation on $W = \mathfrak{g}_{\mathbb{C}}$ is therefore faithful. Equipped with the Hermitian extension of the negative Killing form, W is a unitary representation. Regard G as a subgroup of $U(W)$ and choose the positive invariant inner product $Q(X, Y) = -\operatorname{Tr}(XY)$ on $\mathfrak{u}(W)$. Lemma 2.1, applied with $Q|_{\mathfrak{g}}$, supplies $\xi \in \mathfrak{g}$ recovering the given metric and complex structure. Proposition 2.2 provides a holomorphic isometric totally geodesic embedding

$$(M, g) \longrightarrow U(W)/C_{U(W)}(\xi). \tag{4.1}$$

The manifold on the right is a unitary partial flag manifold with the invariant Kähler metric (2.2). Lemma 3.4 gives positive holomorphic sectional curvature on it. The Gauss equation for the totally geodesic embedding identifies its holomorphic sectional curvatures along TM with those of g. This proves the theorem. □

The proof yields a quantitative estimate without a classification by Lie type. In the representation used in (4.1), let $a_1 > \cdots > a_{r+1}$ be the distinct eigenvalues of $-\mathrm{i}\xi$, of multiplicities $n_1, \dots, n_{r+1}$. Put $d_i = \sum_{j=1}^{i} n_j$ and $c_i = a_i - a_{i+1}$. Replacing ξ by $\xi - \mathrm{i} a_{r+1} I_W$ preserves the centralizer, complex structure, and Kähler form of the ambient unitary orbit. This shifted element need not belong

to $\mathfrak{g}$; the shift is used only to identify the ambient metric with the quotient construction. When $r \geq 1$, the estimate is

$$H_g \geq \delta \tag{4.2}$$

where

$$\delta = \frac{2}{\sum_{i=1}^{r}(r-i+1)(a_i - a_{i+1})d_i} > 0. \tag{4.3}$$

This bound depends on the chosen representation and is not asserted to be sharp. For $r = 1$ and $d_1 = 1$, the quotient is projective space with metric $c_1 g_{\mathrm{FS}}$, and the estimate is the exact value $2/c_1$ when $H_{g_{\mathrm{FS}}} = 2$.

Corollary 4.1. *The canonical Kähler–Einstein metric on every Kähler C-space has positive holomorphic sectional curvature.*

Proof. The canonical metric is invariant, so Theorem 1.1 applies. A positive rescaling preserves the sign, since $H_{bg} = b^{-1}H_g$ for $b > 0$. □

4.2. **The projective second fundamental form.** We record the consequence for the projective embedding that induces the canonical metric; compare [10]. Normalize the Fubini–Study form by $\omega_{\mathrm{FS}} = \mathrm{i}\partial\bar{\partial}\log(1+|z|^2)$ with associated real metric g_{FS}. Then $H_{g_{\mathrm{FS}}} = 2$.

Choose a maximal torus of G contained in H and positive roots adapted to the invariant complex structure, so the complementary negative-root spaces represent $T^{1,0}_o M$. Denote the root space for a root α by $\mathfrak{g}_\alpha$. Let Δ^+_M be the complementary positive roots and define

$$\lambda = \sum_{\alpha \in \Delta^+_M} \alpha \tag{4.4}$$

and $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta^+_M} \mathfrak{g}_{-\alpha}$ in $\mathfrak{g}_{\mathbb{C}}$. The dominant integral weight λ vanishes on the Levi simple coroots and is positive on the others. Let V_λ be the irreducible representation of highest weight λ, with a G-invariant Hermitian inner product, and let p be a unit highest-weight vector. Since λ lies in the root lattice, this representation descends to the adjoint group. The stabilizer of $[p]$ is H, so $\iota(aH) = [ap]$ defines a holomorphic embedding in the space $\mathbb{P}(V_\lambda)$ of complex lines. Write $\mathcal{O}(1)$ for its hyperplane bundle and K_M^{-1} for the anticanonical bundle. The isotropy representations on the fibers of $\iota^*\mathcal{O}(1)$ and K_M^{-1} have the same weight $-\lambda$; hence these line bundles agree. Set $\omega = \iota^*\omega_{\mathrm{FS}}$ and $g = \iota^* g_{\mathrm{FS}}$. Then $[\omega] = 2\pi c_1(M)$. Its Ricci form ρ_ω is invariant and has the same class. By the $\partial\bar{\partial}$-lemma, $\rho_\omega - \omega = \mathrm{i}\partial\bar{\partial} f$ for a real function f. Averaging over G makes f constant, without changing its complex Hessian. Therefore $\rho_\omega = \omega$, and the induced metric is exactly the canonical metric with this normalization.

Use the Hermitian norm of V_λ in what follows. Let P_N be the orthogonal projection onto $(\mathbb{C}p \oplus \mathfrak{n}_- p)^\perp$. For $X \in \mathfrak{n}_-$, let u_X be the real tangent velocity at $t = 0$ of the curve whose projective image is $[e^{tX}p]$. Every nonzero X gives $Xp \neq 0$. Use the affine chart $w \mapsto [p + w]$, with $w \in p^\perp$. Weight orthogonality gives $w(t) = e^{tX}p - p \in p^\perp$, with derivatives Xp and X^2p at zero. Let II_ι denote the real second fundamental form of the embedding. The ambient Christoffel symbols vanish at $w = 0$, so normal projection gives

$$\|\mathrm{II}_\iota(u_X, u_X)\|^2_{g_{\mathrm{FS}}} = 2\|P_N(X^2p)\|^2. \tag{4.5}$$

The real Kähler Gauss equation gives

$$H_g(u_X) = 2 - \frac{\|P_N(X^2p)\|^2}{\|Xp\|^4}. \tag{4.6}$$

Indeed, $g(u_X, u_X) = 2\|Xp\|^2$, and the two second-fundamental-form terms in the real Gauss equation have total contribution $4\|P_N(X^2p)\|^2$.

Corollary 4.2. *For the canonical metric normalized by $\rho_\omega = \omega$, let $\delta > 0$ be given by* (4.3)*. Then every $X \in \mathfrak{n}_-$ with $Xp \neq 0$ satisfies*

$$\|P_N(X^2 p)\|^2 \leq (2-\delta)\|Xp\|^4 < 2\|Xp\|^4. \tag{4.7}$$

Proof. Combine (4.2) with (4.6). □

This is a consequence of the reduction and totally geodesic embedding proof, not an independent computation of the Hermitian norm in the highest-weight representation. No estimate for the algebraic projective second fundamental form alone is used.

Disclosure of AI assistance

OpenAI's ChatGPT was used in mathematical exploration, checking computations, and preparing and revising the text.

References

[1] A. Borel, *Kählerian coset spaces of semisimple Lie groups*, Proc. Natl. Acad. Sci. USA **40** (1954), no. 12, 1147–1151. doi:10.1073/pnas.40.12.1147.

[2] D. Burns, V. Guillemin and Z. Wang, *Stability functions*, Geom. Funct. Anal. **19** (2010), 1258–1295. doi:10.1007/s00039-009-0035-1.

[3] R. Donagi and E. Sharpe, *GLSM's for partial flag manifolds*, J. Geom. Phys. **58** (2008), 1662–1692. doi:10.1016/j.geomphys.2008.07.010.

[4] X. Li and P. Petersen, *Holomorphic sectional curvature and the Calabi operator on Kähler C-spaces*, arXiv:2611.07402v1 (2026), posted November 5, 2026.

[5] A. Loi, R. Mossa and F. Zuddas, *Universal embeddings of flag manifolds and rigidity phenomena*, arXiv:2507.23606v1 (2025).

[6] S. Manikandan, *On holomorphic sectional curvature of Kähler quotients*, Arch. Math. **120** (2023), 331–338. doi:10.1007/s00013-022-01825-y.

[7] L. Ni and F. Zheng, *Kähler manifolds and cross quadratic bisectional curvature*, J. Eur. Math. Soc. **27** (2025), 1715–1738. doi:10.4171/JEMS/1405.

[8] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954), no. 1, 33–65. doi:10.2307/2372398.

[9] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), no. 4, 459–469. doi:10.1307/mmj/1028999604.

[10] M. Takeuchi, *Homogeneous Kähler submanifolds in complex projective spaces*, Japan. J. Math. (N.S.) **4** (1978), no. 1, 171–219. doi:10.4099/math1924.4.171.

[11] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[12] S. Zhang, *Positive holomorphic sectional curvature on rational surfaces*, arXiv:2606.23333v1 (2026).