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Anomalous regularization of passive scalars up to the Obukhov–Corrsin exponent

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Abstract

We consider the advection–diffusion equation $\partial_t\theta + \mathbf{b}\cdot\nabla\theta - \kappa\Delta\theta = 0$ on the two-dimensional torus $\mathbb{T}^2$, where $\mathbf{b}$ is a divergence-free velocity field constructed as in [AV25], Hölder continuous with an exponent $\alpha < 1/3$ and built from shear flows alternating in direction at infinitely many scales. We prove anomalous regularization: for every s below the Obukhov–Corrsin exponent $\frac{1}{2}(1-\alpha)$, with $\mathbf{b}$ depending on s, the solutions are bounded in $C^{0,s}(\mathbb{T}^2)$ at positive times by a constant times the $L^2(\mathbb{T}^2)$ norm of the initial data, uniformly in the diffusivity κ in a union of bands accumulating at 0. As a consequence, we obtain anomalous dissipation at a uniform rate: solutions dissipate a fixed fraction of their variance in unit time, so that the variance decays exponentially, with constants independent of κ, of the initial datum and of the initial time. The proof combines iterated quantitative homogenization with large-scale regularity: at each scale the solution is compared with a solution of the homogenized equation of that scale. Since the derivatives consumed by each comparison are regenerated at every scale, the Hölder norm is carried from the coarsest scale to the finest by induction. Each of our main results, together with its proof, is fully formalized in the Lean 4 proof assistant.

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1. Introduction

We study the advection–diffusion equation

$$\begin{cases} \partial_t \theta + \mathbf{b} \cdot \nabla \theta - \kappa \Delta \theta = 0 & \text{in } (t_{\mathsf{in}}, \infty) \times \mathbb{T}^2 , \\ \theta(t_{\mathsf{in}}, \cdot) = \theta_{\mathsf{in}} & \text{on } \mathbb{T}^2 \end{cases} \tag{1.1}$$

on the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, for a divergence-free velocity field $\mathbf{b}$, a diffusivity $\kappa > 0$, an initial time $t_{\mathsf{in}} \in \mathbb{R}$ and an initial datum $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$. The solution θ is a passive scalar, carried by $\mathbf{b}$ and diffused with the molecular diffusivity κ. Since $\nabla \cdot \mathbf{b} = 0$, the transport term conserves the $L^2(\mathbb{T}^2)$ norm, and diffusion alone decreases it: for every $t > t_{\mathsf{in}}$,

$$\|\theta(t,\cdot)\|^2_{L^2(\mathbb{T}^2)} + 2\kappa \|\nabla \theta\|^2_{L^2((t_{\mathsf{in}},t)\times\mathbb{T}^2)} = \|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)} .$$

We are interested in small κ and in velocity fields which, like the velocity of a turbulent fluid, are continuous but rough: $\mathbf{b}$ is Hölder continuous with an exponent $\alpha < 1$. The question is how regular the solutions are, uniformly in κ.

For the velocity of a turbulent fluid, whose increments over a distance scale like the power $1/3$ of the distance in Kolmogorov's theory [Kol41], Obukhov [Obu49] and Corrsin [Cor51] predicted by a scaling argument that the increments of the scalar scale like the same power $1/3$, over a range of distances that depends on κ. The same argument predicts the exponent $\frac{1}{2}(1-\alpha)$ when the increments of the velocity scale like the power α [AV25, Section 1.3]. We call $\frac{1}{2}(1-\alpha)$ the *Obukhov–Corrsin exponent* of a velocity field which is Hölder continuous with exponent α.

In [AV25] we constructed, for every $\alpha < 1/3$, a multiscale "fractal" velocity field $\mathbf{b}$ which is continuous in time with values in $C^{0,\alpha}(\mathbb{T}^2)$. For this field, the solutions θ^κ of (1.1) with diffusivity κ, initial time $t_{\mathsf{in}} = 0$ and mean-zero initial data $\theta_{\mathsf{in}} \in H^1(\mathbb{T}^2)$ exhibit anomalous dissipation,

$$\inf_{\kappa \in \mathcal{K}_{\mathrm{AV}}} \kappa \|\nabla \theta^\kappa\|^2_{L^2((0,1)\times\mathbb{T}^2)} \geq c \, \|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)} . \tag{1.2}$$

Here $\mathcal{K}_{\mathrm{AV}}$ is a countable union of bands accumulating at 0, the permissible diffusivities of [AV25, (3.43)], which is the set $\mathcal{K}$ of (2.18) for the value of q fixed in [AV25], and the constant $c > 0$ depends on the ratio $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}/\|\nabla\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ [AV25, Theorem 1.1 and the discussion after it]. In this paper we prove that, for every exponent s below $\frac{1}{2}(1-\alpha)$ and a velocity field $\mathbf{b}$ of the same construction depending on s, the solutions are uniformly Hölder continuous with exponent s at positive times, for every datum in $L^2(\mathbb{T}^2)$, uniformly in κ in the corresponding union of bands. Consequently, for this velocity field, an estimate of the form (1.2) holds for every mean-zero datum in $L^2(\mathbb{T}^2)$ and from every initial time, with a constant c independent of the datum (Theorem B below).

1.1 Main results

Construction of the velocity field. The velocity field of [AV25] is built level by level. Level $m \geq 1$ has a wavelength ε_m, and the wavelengths decrease supergeometrically: $\varepsilon_{m-1}/\varepsilon_m \geq \Lambda$ for a large integer Λ, and ε_m is of the order of ε_{m-1}^q for $m \geq 2$, for a parameter $q > 1$. Level m adds to the velocity field $\mathbf{b}_{m-1}$ of the coarser levels short pulses of a shear flow of wavelength ε_m and speed of order ε_m^α, alternating between the two coordinate directions and written in the Lagrangian coordinates of $\mathbf{b}_{m-1}$. A fixed cutoff function ζ of time switches the pulses on and off. The limit $\mathbf{b}$ of the velocity fields $\mathbf{b}_m$ is periodic in space and time, divergence free, and Hölder continuous with exponent α in space and time. The velocity field of this paper is that of [AV25] with one change: the

parameter q, which [AV25] fixes in terms of α, is here a free parameter, taken close to 1 depending on α and on the exponent s of the main result (Section 2).

With $\beta := 1+\alpha$ and $\gamma := (q-1)\beta/(q+1)$, the diffusivities that we allow are the *permissible diffusivities* of [AV25, (3.43)] for the present q: the union $\mathcal{K}$ of the bands

$$\mathcal{K}_M = \left[\tfrac{1}{2}\varepsilon_M^{\beta-\gamma},\, 2\varepsilon_M^{\beta-\gamma}\right], \qquad M \geq 1,$$

defined in (2.18), which are disjoint and accumulate at 0. The exponent γ corrects β in the exponent of the effective diffusivities, described in the strategy below; since $q(\beta-\gamma) = \beta+\gamma$, a diffusivity in $\mathcal{K}_M$, $M \geq 2$, is of the order of $\varepsilon_{M-1}^{\beta+\gamma}$.

Interior regularity. Our main result is Theorem A, stated precisely in Section 3. Let $\alpha \in (0, 1/3)$ and $s \in (0, \frac{1}{2}(1-\alpha))$, let $q > 1$, close to 1 depending on α and s, and the cutoff function ζ be as in Section 2, and let Λ be an integer at least a threshold $\Lambda_0(\alpha, s, q, \zeta) \geq 128$. Then the velocity field $\mathbf{b}$ of Section 2, which depends on s through q, has the following property: for every $0 < t_0 < T$ there exists $C(\alpha, s, q, \zeta, \Lambda, t_0, T) < \infty$ such that, for every $t_{\mathsf{in}} \in \mathbb{R}$, $\kappa \in \mathcal{K}$ and $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$, every weak solution of (1.1) on $(t_{\mathsf{in}}, t_{\mathsf{in}}+T) \times \mathbb{T}^2$ satisfies

$$\|\theta\|_{L^\infty([t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T]; C^{0,s}(\mathbb{T}^2))} \leq C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)},$$

where $C^{0,s}(\mathbb{T}^2)$ is the space of Hölder continuous functions with exponent s, normed by (3.1). Theorem A is the anomalous regularization of the title: the solution operator maps $L^2(\mathbb{T}^2)$ into $C^{0,s}(\mathbb{T}^2)$ at every positive time, with a bound that does not degenerate as $\kappa \to 0$ in $\mathcal{K}$. The exponent s may be taken as close as desired to the Obukhov–Corrsin exponent, but as s increases to $\frac{1}{2}(1-\alpha)$, the bounds (2.1) force q down to 1.

Uniform anomalous dissipation. Theorem B of Section 3 follows from Theorem A alone. In the setting of Theorem A, there exists $c(\alpha, s, q, \zeta, \Lambda) > 0$ such that, for every $t_{\mathsf{in}} \in \mathbb{R}$, $\kappa \in \mathcal{K}$ and $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$, with $(\theta_{\mathsf{in}})_{\mathbb{T}^2}$ the mean of θ_{in}, every weak solution of (1.1) satisfies

$$\kappa\|\nabla\theta\|^2_{L^2((t_{\mathsf{in}}, t_{\mathsf{in}}+1)\times\mathbb{T}^2)} \geq c\left\|\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\right\|^2_{L^2(\mathbb{T}^2)},$$

and, for every $t \geq t_{\mathsf{in}}$,

$$\left\|\theta(t,\cdot) - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\right\|_{L^2(\mathbb{T}^2)} \leq 2\exp(-c(t-t_{\mathsf{in}}))\left\|\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\right\|_{L^2(\mathbb{T}^2)}.$$

Over every unit of time, every solution thus dissipates a fixed fraction of its variance. The proof is short: by Theorem A, the solution operator over a unit of time maps $L^2(\mathbb{T}^2)$ boundedly into $C^{0,s}(\mathbb{T}^2)$, and a positive operator with this property which preserves constants and integrals contracts the functions of mean zero, with a fixed contraction constant (Lemma 9.4).

Local regularity. Theorem C, stated and proved in Section 9, localizes Theorem A. Consider a solution of the advection–diffusion equation with $\kappa \in \mathcal{K}$, in the sense of distributions and with finite energy, in a space-time tube that follows the trajectories of the velocity field $\mathbf{b}_m$ of the first m levels over the time $8h_{m+1}$ (with h_{m+1} the renewal time of level $m+1$, Subsection 2.2), with no lateral data prescribed. Theorem C bounds its norm in $C^{0,s}$ on the trajectories of a smaller concentric disc, from a waiting time on, by a constant times its root mean square over the tube, and the constant depends neither on κ nor on the final time of the tube. The proof compares the solution with the solution on the whole torus from a cut-off datum, to which Theorem A applies.

Vanishing diffusivity. Corollary D, also in Section 9, passes to the limit $\kappa \to 0$. Let κ tend to 0 along a sequence $\mathcal{K}'$ in $\mathcal{K}$, and let θ^κ be the solutions, from a common initial time t_{in}, with data $\theta^\kappa_{\mathsf{in}}$ bounded in $L^2(\mathbb{T}^2)$. A subsequence converges locally uniformly on $(t_{\mathsf{in}}, \infty) \times \mathbb{T}^2$ to a continuous solution θ_* of the transport equation $\partial_t \theta_* + \nabla \cdot (\mathbf{b}\theta_*) = 0$, in the sense of distributions. The limit takes the weak limit of the data along the subsequence as its initial value, weakly in $L^2(\mathbb{T}^2)$, satisfies the bound of Theorem A with $\sup_{\kappa \in \mathcal{K}'} \|\theta^\kappa_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ in place of $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$, and is Hölder continuous in time with exponent $s/(1+s)$ on every set $[t_{\mathsf{in}} + t_0, t_{\mathsf{in}} + T] \times \mathbb{T}^2$. Corollary D(b) is the local form, in the tubes of Theorem C. The conclusion is subsequential: for the velocity field of [AV25] with $\alpha \geq 1/67$, vanishing diffusivity does not select a unique limit [AV25, Proposition 5.5]; see also [CCS23, HT25].

1.2 Prior results on anomalous dissipation and the Obukhov–Corrsin exponent

If $\mathbf{b}$ is Lipschitz continuous in space, the transport equation $\partial_t \theta + \mathbf{b} \cdot \nabla \theta = 0$ is well posed, its solutions conserve the $L^2(\mathbb{T}^2)$ norm, and, by the energy identity, the dissipation $\kappa \|\nabla \theta^\kappa\|^2_{L^2((t_{\mathsf{in}}, t_{\mathsf{in}}+1) \times \mathbb{T}^2)}$ tends to zero with κ [AV25, Section 1.1]. *Anomalous dissipation*, the failure of this limit for rougher velocity fields, was first discussed by Obukhov [Obu49]; it is a basic assumption of the phenomenological theories of scalar turbulence, and it is supported by strong experimental and numerical evidence [SS00, War00, DSY05]. Rigorous examples include the white-in-time stochastic model of [Kra68, BGK98], for which a quantitative proof of anomalous dissipation is given in [Row24], and deterministic constructions. Those of [DEIJ22, CCS23, EL24] are built on alternating shear flows or on mixing, up to a singular time at which all the dissipation takes place [CCS23, p. 4], [EL24, Section 1.2.2], [HCR25a, Remark 2.1]; those of [HCR25a, HCR25b] iterate a construction of this type and dissipate at countably many times. Related constructions exist for forced fluid equations [BDL23, BCC$^+$24, Che24, HPZZ24]. Autonomous examples were obtained in [JS26a, JS26b] (see also [BBDLM26]); the second of these uses the relation between spontaneous stochasticity and scalar dissipation established in [DE17]. The velocity field of [AV25] has infinitely many active scales at almost every time, and the proof of [AV25, Theorem 1.1] homogenizes these scales one at a time, from the scale at which molecular diffusion acts up to the unit scale, where the effective diffusivity is of order one.

The Obukhov–Corrsin exponent is a threshold for anomalous dissipation. As observed in [AV25, Section 1.3], the commutator estimate of [CET94] shows that, for a velocity field which is continuous in time with values in $C^{0,\alpha}(\mathbb{T}^2)$, solutions which are bounded uniformly in κ, continuously in time, in a Hölder space of exponent larger than $\frac{1}{2}(1-\alpha)$ dissipate no variance in the limit $\kappa \to 0$; see [DEIJ22, Theorem 4] for the scalar statement. Below the exponent, Colombo, Crippa and Sorella [CCS23] obtained anomalous dissipation together with uniform Hölder bounds in the full range of exponents, for a particular datum and in L^2 in time [AV25, Sections 1.1 and 1.3]. A construction with anomalous dissipation for every nonzero smooth mean-zero datum and sharp time-integrated regularity bounds was subsequently given in [EL24, Theorem 2]. In contrast, the estimate of Theorem A is in L^∞ in time at positive times and holds for every datum in $L^2(\mathbb{T}^2)$, uniformly in $\kappa \in \mathcal{K}$.

Regularization from rough data has also been proved for other velocity fields. In [HCR25b, Theorem 1.5], the deterministic construction mentioned above gives a positive Sobolev gain from data in $L^2(\mathbb{T}^2)$, on the time interval $[0,1]$ from the initial time 0, measured in L^2 in time and uniformly in the diffusivity. For rough white-in-time Kraichnan velocities, anomalous Sobolev regularization was proved in [GGM26], and expected Sobolev bounds at each positive time for L^2 data were obtained in [DGP25, Theorem 1.3]. On the torus, uniform-in-diffusivity estimates for the expected regularity yield bounds on the stationary scalar spectrum in [Row25]. The Sobolev threshold in these white-in-time models is $1-\alpha$, rather than the exponent $\frac{1}{2}(1-\alpha)$ for time-regular velocities [Row25,

Remark 1.4]. For the elliptic, time-independent problem in self-similar random incompressible flows, in a perturbative regime, iterated homogenization gives Hölder estimates uniform in the molecular diffusivity, with a random constant determined by a minimal scale [ABRK26, Theorem C]. For regularization in nonlinear models, see [CZ16, Dri22].

Burczak, Székelyhidi and Wu [BSW24] combined the iterated homogenization of [AV25] with the convex integration methods used in the proof of Onsager's conjecture [Ise18, BDLSV19], and obtained anomalous dissipation for every nonzero mean-zero datum in $H^1(\mathbb{T}^3)$ by Hölder continuous weak solutions of the unforced incompressible Euler equations on $\mathbb{T}^3$, with any exponent below $1/3$ [BSW24, Theorem 1.1]. In [BSW26] they proved anomalous dissipation for mean-zero data with more than $\frac{1}{2}(1+\alpha)$ derivatives in $L^2(\mathbb{T}^2)$, a class larger than $H^1(\mathbb{T}^2)$, together with uniform Hölder bounds, along a sequence of diffusivities tending to 0. Their velocity fields are divergence free on $\mathbb{T}^2$, with space-time Hölder exponent $\alpha < 1/3$, and built, as they explain in [BSW26, Section 1.3], essentially as in [AV25, BSW24]. Given $s < \frac{1}{2}(1-\alpha)$, their Hölder bound reads, in our notation, for mean-zero data $\theta_{\mathsf{in}} \in H^{1+s'}(\mathbb{T}^2)$ with $s' > s$,

$$\|\theta\|_{L^\infty([0,1];C^{0,s}(\mathbb{T}^2))} \leq C\|\theta_{\mathsf{in}}\|_{H^{1+s'}(\mathbb{T}^2)} ,$$

with C independent of the datum and of the diffusivity along the sequence [BSW26, Theorem 1.1].

In [BSW26, Section 1.2.1], the authors write that they "do not follow the 'large scale regularity' approach suggested" in [AV25]. Of a "naive approach to regularity", they write that its second difficulty, an order-one background transport against a vanishing diffusivity at the larger scales, "seems to be a fundamental feature of our setup, prohibiting the use of 'large-scale regularity' techniques" [BSW26, Section 1.3]. Their proof transfers regularity from coarser homogenized equations to finer ones, the mechanism described in [AV26, Section 4.1], by high-order corrector expansions and estimates for systems of forced equations [BSW26, Theorems 6.10 and 7.2], started separately for the Littlewood–Paley pieces of the datum [BSW26, Section 8.3].

For smooth data, their bound gives, for a velocity field depending on the exponent, the uniform Hölder regularity below the Obukhov–Corrsin exponent sought in [AV25, Section 1.3]. The interior regularization announced at the end of [AV25, Section 1.3] is a bound of a different kind: a bound at positive times of the Hölder norm of the solution by the norm of the datum in $L^2(\mathbb{T}^2)$, for every datum in $L^2(\mathbb{T}^2)$, with a constant that tends to infinity as the waiting time tends to 0. The bound of [BSW26] propagates regularity: it holds up to the initial time, and its right side requires a datum which is already Hölder continuous.

Theorem A proves this interior regularization for every $s < \frac{1}{2}(1-\alpha)$, with the velocity field constructed here depending on s. Its proof carries out the large-scale regularity program announced in [AV25, Section 1.3], by the induction described in [AV26, Section 4.1]. The comparison of two levels, in $L^2(\mathbb{T}^2)$ at a high power of ε_{m-1}, is read in $C^{0,s}(\mathbb{T}^2)$ through representations, which also measure derivatives at each wavelength, and the derivatives that the correctors consume are regenerated at every level (Proposition 5.3); both are explained in the strategy below.

The constants in the lower bounds of [AV25, Theorem 1.1], [BSW24, Theorem 1.1] and [BSW26, Theorem 1.1] depend on the datum through a ratio of its L^2 norm to a Sobolev seminorm of positive order, while the constant of Theorem B is independent of the datum, of the initial time and of $\kappa \in \mathcal{K}$. Independence from the datum was announced in [AV25, Section 1.3] and [AV26, Section 4.1] as a consequence of uniform regularity. Among deterministic velocity fields, dissipation uniform over the initial datum was obtained by a different construction in [HCR25a, Corollary 1.3], which gives total dissipation in the limit $\kappa \to 0$ for every mean-zero datum in $L^2(\mathbb{T}^2)$; for white-in-time velocity fields, see [Row24, Row25]. A related construction gives accelerating exponential decay for its time-periodic extension [HCR25b, Corollary 1.2]. These estimates are stated from the initial time 0, for velocity

fields constructed on a unit time interval, whose dissipation takes place at singular times [HCR25b, Section 1.1]. Theorem B holds from every initial time $t_{\mathsf{in}} \in \mathbb{R}$, for a velocity field which is periodic and Hölder continuous in space and time, with infinitely many active scales at almost every time.

1.3 Strategy of the proof

The proof of Theorem A rests on nine propositions and one lemma, all stated in Part I; their logical dependence is shown in Figure 3.1. We next outline the main ideas of the proof, and postpone the technical estimates to Part II.

Reduction to finitely many scales. Let $\kappa \in \mathcal{K}_M$. The levels of $\mathbf{b}$ much finer than level M do not matter: their stream functions are much smaller than κ, a perturbation of the stream function of size $\kappa\varpi$ moves the solution by at most $\varpi\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ in $L^2(\mathbb{T}^2)$, and a Hölder estimate for small perturbations of the heat equation turns this into a bound in $C^{0,s}(\mathbb{T}^2)$. It therefore suffices to prove the bound of Theorem A for the velocity field $\mathbf{b}_{M+n}$, for a fixed number n of extra levels and for M large; the finitely many remaining bands follow from the same Hölder estimate directly (Section 4).

Effective diffusivities and three kinds of level. For $\kappa \in \mathcal{K}_M$, the *effective diffusivities* $\kappa_M \coloneqq \kappa, \kappa_{M-1}, \dots, \kappa_0$ of [AV25] are defined by a recursion: κ_{m-1} is the diffusivity that the solution feels once the pulses of level m, at diffusivity κ_m, have been averaged out (Subsection 2.5). By Taylor dispersion [Tay53] (see also [FP94, MK99]), the pulses of level m, whose stream function has amplitude ε_m^β, add about $\frac{9}{80}\varepsilon_m^{2\beta}/\kappa_m$ to the diffusivity (Proposition 2.1(a)). When the enhancement dominates, $\kappa_{m-1}\kappa_m \asymp \varepsilon_m^{2\beta}$, and the scaling $\kappa_m \asymp \varepsilon_m^{\beta+\gamma}$ is consistent with this exactly when $q(\beta-\gamma) = \beta+\gamma$, which defines γ. For $\kappa \in \mathcal{K}_M$ this scaling holds at every level $1 \le m < M$, independently of κ (Proposition 2.1(b)). The restriction to the bands comes from an instability of the recursion, which, according to [AV25, Section 3.3], does not appear to be an artifact of the proof. The Péclet number $\varepsilon_m^\beta/\kappa_m$ then sorts the levels finer than a fixed top level m_0 into three kinds, crossed as described below:

levels	Péclet number $\varepsilon_m^\beta/\kappa_m$	how they are crossed
$m_0 < m < M$	$\asymp \varepsilon_m^{-\gamma} \gg 1$	homogenization cycles; $\kappa_{m-1} \gg \kappa_m$
$m = M$	$\asymp \varepsilon_M^{\gamma} \ll 1$	the terminal cycle; $\kappa \le \kappa_{M-1} \le C\kappa$
$M < m \le M+n$	$\varepsilon_m^\beta/\kappa \le C\varepsilon_{m-1}^{(q+2)\gamma} \ll 1$	weak passages, at the diffusivity κ

From L^2 to L^∞, scale by scale. The proof of anomalous dissipation in [AV25] compares the solutions of the equations of the successive levels, with velocity field $\mathbf{b}_m$ and diffusivity κ_m. For $1 \le m < M$ and a datum of unit norm, the energy identity at the diffusivity κ_m shows that the gradient of the solution of level m is at most of order $\kappa_m^{-1/2} \asymp \varepsilon_m^{-(\beta+\gamma)/2}$ in $L^2((t_{\mathsf{in}},\infty)\times\mathbb{T}^2)$ (Proposition 2.1(b)). If this bound held in L^∞ at every level $1 \le m < M$, the solution would be Hölder continuous with the exponent

$$1 - \frac{\beta+\gamma}{2} = \frac{1-q\alpha}{1+q} \nearrow \frac{1-\alpha}{2} \qquad \text{as } q \downarrow 1\,.$$

This heuristic (Subsection 5.1) is where the Obukhov–Corrsin exponent comes from, and why q is taken close to 1. The proof turns the L^2 information carried by the effective diffusivities into L^∞ information, one scale at a time. This is the mechanism of large-scale regularity in homogenization: solutions are Lipschitz on scales larger than the microscopic scale, uniformly in the scale separation,

for periodic [AL87] and random [AS16, AKM19, GNO20] coefficients. In the quantitative argument the solution is compared on each scale, as in the classical proof of Schauder estimates, with a solution of the homogenized equation, whose regularity improves the *excess*, the normalized L^2 distance of the solution to the affine functions, on the next smaller scale. For a Hölder estimate with exponent $s < 1$, as here, the L^2 oscillation of the solution about its mean, divided by the scale to the power s, plays this role. Here this idea must be repeated across the infinitely many scales of the problem, as proposed in [AV26, Section 4.1] (compare [ABRK26], where on each scale the solution is compared with a harmonic function): the homogenized equation of level m is the equation of level $m-1$, which is again fractal and must be homogenized in its turn. From one level to the next, separation of scales gains $(\varepsilon_m/\varepsilon_{m-1})^{1-s}$ and the enhancement costs $(\kappa_{m-1}/\kappa_m)^{1/2}$, the amplitude of the correctors of level m relative to ε_m; their product is at most $C\varepsilon_{m-1}^{\mu}$ for $2 \le m \le M$, with the exponent $\mu = (q-1)(1-\frac{\beta+\gamma}{2}-s)$ of (2.3) and (5.1), which is positive exactly when $s < 1-\frac{\beta+\gamma}{2}$.

The main difficulty. At every level, the velocity field $\mathbf{b}_{m-1}$ of the coarser levels is of order one, while the diffusivity is small at the fine levels: $\kappa_m \asymp \varepsilon_m^{\beta+\gamma}$ for $1 \le m < M$. The construction removes this transport from the analysis of a level. The pulses of level m are written in the Lagrangian coordinates of $\mathbf{b}_{m-1}$, renewed on blocks of time of length h_m, the *renewal time* of level m, which is short compared with the time scale $\varepsilon_{m-1}^{2-\beta}$ of $\mathbf{b}_{m-1}$. Within a renewal block the coordinates are close to rigid, and the coarse flow carries the fine shear along instead of sweeping across it (Section 2). In these coordinates the coarse transport disappears, and the Laplacian becomes a divergence-form operator whose coefficient, the *conductivity*, is within $C\varepsilon_{m-1}^{2\delta}$ of the identity, where $\delta > 0$ is the small exponent of (2.3). The correctors and the whole analysis of level m are written in these coordinates. The other difficulty is that the correctors consume derivatives, while the datum is only in $L^2(\mathbb{T}^2)$ and the induction cannot lose derivatives at each of infinitely many levels: the derivatives are regenerated at every level, within one period of its pulses.

Two dual cascades. Fix M and $\kappa \in \mathcal{K}_M$. By the definition of the effective diffusivities, homogenizing the pulses of level $m \le M$ turns the operator $\partial_t + \mathbf{b}_m \cdot \nabla - \kappa_m \Delta$ of level m into the operator of level $m-1$; for $m > M$, the operator of level m has κ in place of κ_m. The proof is an induction on the levels made of two dual cascades (Subsection 5.2), both run by Proposition 5.2 between the levels $M-1$ and m_0. The *homogenization cascade* goes from fine to coarse, *up the levels* and forward in time: the *entry* (Proposition 5.1) crosses the levels $M+n, \dots, M$, and then one *cycle* at a time (Proposition 5.3), each in a short interval of time, crosses the levels $M-1, \dots, m_0+1$, up to a fixed top level m_0, independent of M and κ, at which the solution is smooth at a fixed scale. Each cycle compares the solution of a level with one of the homogenized equation, the equation of the next coarser level; this cascade carries the *marked amplitude*, a weighted L^2 norm taken in L^∞ in time (Subsection 5.2). The *regularity cascade* goes from coarse to fine, back down the levels at one fixed time: as in large-scale regularity, it transfers a Hölder bound from the homogenized equation to the heterogeneous one, one finer level at a time. Each cascade needs the other: the first produces the homogenized solutions with which the second compares, and the second turns the L^2 control of the first into a Hölder bound.

Families of coupled equations. The comparison of a level with the next coarser one leaves error terms that are not small and that carry derivatives of the coarse solution. They become new unknowns, each solving the equation of its level with a source made of smoothed derivatives of earlier unknowns, so that the object passed from one level to the next is a coupled system of equations with the operator of its level, a *family* (Subsection 5.4). The marked amplitude of a solution of a

family, which weights the L^2 norms of its *components*, the unknowns of the system, is multiplied per cycle by at most one plus a constant times a fixed positive power of ε_{m-1}; since $\varepsilon_m \le \Lambda^{-m}$, these factors have a bounded product, uniformly in the number of levels.

The entry. At the levels finer than M, the enhancement of a level is small compared with κ. A *weak passage* (Subsection 15.2) therefore keeps the diffusivity κ, carries the enhancement as a source of the coarser family (Subsection 5.4), and regains derivatives on the quiet margins, the pulse-free intervals of length τ'_m at both ends of each block of time of length h_m on which the pulses of level m are renewed (Section 2). Over a margin, molecular diffusion reaches the wavelength ε_{m-1}. Level M also has a small Péclet number, but its margins are too short for this, $\kappa\tau'_M \ll \varepsilon^2_{M-1}$, so it is crossed by a *terminal cycle* (Subsection 15.3), which passes to κ_{M-1} and produces the first family of the homogenization cascade (Proposition 5.2), the *seed* (Subsection 5.4). The weak passages and the terminal cycle form the entry (Section 15).

One homogenization cycle. A cycle has two parts. The first, the *preparation* (Section 12), regenerates derivatives: within one period $4h_m$ of the pulses, the solution of level m becomes, up to a small error, smooth at the *effective diffusion length* $\rho(m) = (\kappa_{m-1}h_m)^{1/2}$, which lies between the two wavelengths, $\varepsilon_m \ll \rho(m) \ll \varepsilon_{m-1}$ (Proposition 2.1(c) and Lemma 7.1(c)). The proof uses the Bloch transform, the Fourier transform adapted to periodic coefficients. For the periodic elliptic theory of corrector expansions and Bloch approximation, see [BLP78, CV97, COV02]. In the Lagrangian coordinates of a renewal block, with the conductivity frozen at a point, the operator of level m is periodic with period ε_m, and Bloch waves split its evolution into evolutions on the period cell, one for each frequency ξ. The principal branch of these evolutions, the one that continues the constants, has, to leading order, the symbol $-\xi \cdot \mathbf{J}^{\kappa_m}_m(t)\xi$, where the *averaged flux* $\mathbf{J}^{\kappa_m}_m(t)$ is the diffusivity matrix produced by the pulses through Taylor dispersion, enhanced along the direction of the active pulses (2.16). Since every renewal block contains pulses of both directions, the real part of this symbol, integrated over a block, is at most a negative multiple of $\kappa_{m-1}h_m|\xi|^2$, in every direction and uniformly in the level: the principal branch is damped like a heat flow at the effective diffusion length. At the small frequencies, the part of each of these evolutions off the branch decays at the rate κ_m/ε^2_m of the cell problem, and at the remaining frequencies the whole evolution does.

The second part, the *comparison* (Section 13), builds from the solution of the next family, the *successor* (Subsection 5.4), a *lift* to level m, in three steps (Subsection 5.5). It adds each new component of the successor to its parent component (Subsection 5.4), with a small weight. It undoes the oscillation in time of the averaged flux, which alternates between the two directions of the pulses and has the average $\kappa_{m-1}I$ over a period: it applies the truncated exponential of the second-order operator $\Gamma_m(t) : \nabla^2$, where the *covariance* $\Gamma_m(t)$ is the accumulated difference between $\mathbf{J}^{\kappa_m}_m$ and $\kappa_{m-1}I$. Since $|\Gamma_m(t)| \le C\varepsilon^\delta_{m-1}\rho(m)^2$, each power of the covariance gains ε^δ_{m-1} on functions that are smooth at the scale $\rho(m)$, so the exponential can be truncated. Finally, it adds the correctors of the pulses, which oscillate at the wavelength ε_m. The errors that are not small become the sources of the next family; the rest is a residual of the size of a fixed power of ε_{m-1}, which the energy estimate for families converts into a bound on the difference between the solution and the lift. A cycle that starts at a time t_{c} is prepared by a time t_{pr} within $4h_m$ of t_{c}, and the comparison holds from t_{pr} on (Proposition 5.3).

Representations and the Hölder norm. The regularity cascade carries the Hölder norm by *representations*: decompositions of a solution of a family into a smooth base and oscillating profiles, each measured at its wavelength (Subsection 5.6); the size of a representation plays the role of the

excess. In terms of representations, a cycle is a *passage* (Proposition 5.4): from a representation of the solution of level $m-1$ at the wavelength ε_{m-1}, it builds one of the solution of level m at the wavelength ε_m, at the same time. The old profiles are copied and never differentiated again; the new ones are the parts of the lift and the error of the comparison. A passage multiplies the size of a representation by at most one plus a constant times a fixed positive power of ε_{m-1}, and adds at most a constant times ε_{m-1} times the marked amplitude of the solution; the balance ε_{m-1}^{μ} above pays for the correctors. Running the two cascades bounds the size of the representation at the finest level, and with it the $C^{0,s}(\mathbb{T}^2)$ norm, by the L^2 norm of the datum, uniformly in the number of levels (Proposition 5.2). At the finest level, the lifts through the entry recover θ from the solution of the seed (Section 15).

Energy estimates and paired smoothing. The families are coupled systems of parabolic equations, and their energy does not grow when the coupling is small (Proposition 6.2). The sources that a homogenization cycle creates are smoothed by products of averages over two thin ellipses, whose long axes come from the two orientations of the pulses; such pairs smooth in every direction, uniformly in their eccentricity (Proposition 6.1). For the sources of the seed the two ellipses are round discs.

1.4 Lean formalization

A formalization of this paper in the Lean 4 proof assistant is available at

`https://github.com/scottnarmstrong/AVAnomalousRegularization`.

It contains Theorems A, B and C, Corollary D and every statement of Part I; the propositions and lemmas of Part II follow from its formal statements, directly or by elementary steps, or are proved, where they are used, inside its formal proofs. A correspondence table, published with the repository, maps every numbered statement and display of the paper to the formal statements that prove it. Some results are stated here in a weaker form than their formal versions: for example, where the paper bounds a norm in L^∞ in time, the formal statement gives the bound at every time. Theorems A, B and C and Corollary D are also stated in a self-contained form, whose definitions use only Mathlib and the statement definitions of the formalization of [AV25], with formal proofs connecting these statements to the development. The formal proofs follow the arguments given here; in some places the paper combines formally proved statements by elementary steps, such as an induction or a sequence of inequalities, which the formal proof combines differently. The formalization uses Lean 4 and Mathlib at version v4.35.0-rc2 and builds on the formalization of [AV25], available at

`https://github.com/scottnarmstrong/AVAnomalousDiffusion`.

It contains no unproved placeholders and no custom axioms: the main theorems depend only on the three standard axioms of Lean 4 (`propext`, `Classical.choice` and `Quot.sound`).

1.5 Use of artificial intelligence

The main ideas of this work were developed by the authors several years ago; the program was announced in [AV25, Section 1.3] and described in [AV26, Section 4.1]. Starting from a detailed outline of the proof written by the authors, a very long and unpolished manuscript (612 pages of slop), in which every detail of the proof was filled in, was first produced by the authors together with AI agents (Claude Opus 5.5 and GPT-6.1 Sol). This manuscript was then formalized in Lean

by AI agents directed by the authors: Claude Opus 5.5 orchestrated the development throughout, and the Lean code was written by Claude Opus 5.5 and Claude Sonnet 5.5, with parts written by GPT-6.1 Sol and GPT-6 Luna. Only then was the present paper written. Its first draft was written by Claude Opus 5.5 following the formalization, and Claude Sonnet 5.5 and GPT-6-Astra checked its statements against the formal ones and reviewed it. The authors checked this draft and rewrote parts of it themselves; the rest was rewritten by Claude Opus 5.5 agents under the direction of the authors, over several rounds of revision and audit.

1.6 Organization of the paper

Part I (Sections 2–6) is a detailed outline of the proof, which can be read by itself. Section 2 constructs the velocity field and states the properties of its effective diffusivities, Section 3 states Theorems A and B, and Section 4 reduces Theorem A to a velocity field with finitely many scales. Section 5 explains the two dual cascades, defines families, lifts and representations, and states the entry, the induction on the levels, one cycle and one passage. Section 6 states the two tools. Part II (Sections 7–15) states the notions used in Part I in full and proves the statements of Part I that are not proved there. Its sections treat the properties of the construction and of the effective diffusivities, relying on [AV25], and the flows and renewal charts (Section 7); the two tools (Section 8); the reduction and the main results, including the statements of Theorem C and Corollary D (Section 9); the cell problems of the pulses, which give the correctors and the averaged flux (Section 10); admissible families (Section 11); the two parts of one cycle, the preparation and the comparison (Sections 12 and 13); the passage and the induction on the levels (Section 14); and the entry (Section 15).

Part I. A map of the proof

2. Construction of the velocity field

In this section we fix the parameters, define the scales, the velocity field $\mathbf{b}$ and the effective diffusivities, and state their properties. The scales, shear streams, cutoff function and recursion are those of [AV25], except for the parameter q, which [AV25, (2.2)] fixes as $q = \frac{1}{2}\big(1 + \frac{2-\beta}{2(\beta-1)}\big)$. Equation and statement numbers of [AV25] refer to its arXiv version, arXiv:2305.05048v3.

2.1 Parameters

Throughout we fix

- $\alpha \in (0, 1/3)$, the Hölder exponent of the velocity field $\mathbf{b}$; we also set $\beta := 1 + \alpha$;
- $s \in (0, \frac{1}{2}(1-\alpha))$, the Hölder exponent of the solutions in Theorem A;
- q satisfying
$$1 < q < \min\Big\{\frac{17}{16}, \frac{1-\alpha}{2\alpha}, \frac{1-s}{\alpha+s}\Big\}, \tag{2.1}$$
the supergeometric scale separation parameter; such a q exists because $\alpha < 1/3$ and $s < \frac{1}{2}(1-\alpha)$, and the last two bounds make the exponents δ and μ below positive;
- an integer $\Lambda \geq 128$, the minimal scale separation;

- an even cutoff function $\zeta \in C_c^\infty((-\frac{2}{3},\frac{2}{3}))$ with

$$0 \le \zeta \le 1\,, \qquad \sum_{k\in\mathbb{Z}} \zeta(\cdot - k) = 1\,, \qquad \int_{\mathbb{R}} \zeta(t)^2\,\mathrm{d}t = \frac{9}{10}\,, \tag{2.2}$$

  which switches the pulses on and off; the last condition fixes the constant $\frac{9}{80}$ of Proposition 2.1, and such a ζ exists (Subsection 7.1).

Together with these we fix the exponents

$$\begin{aligned} \gamma &\coloneqq \frac{(q-1)\beta}{q+1}\,, \qquad \delta \coloneqq \frac{1}{4}(q-1)\Big(1 - \frac{(2q+1)\beta}{2q+2}\Big)\,, \\ \mu &\coloneqq (q-1)(1-s) - q\gamma\,, \qquad \eta \coloneqq \min\Big\{\frac{\delta}{8}, \frac{\mu}{4}\Big\}\,. \end{aligned} \tag{2.3}$$

The exponent γ solves $q(\beta-\gamma) = \beta+\gamma$ and gives the scaling of the effective diffusivities (Proposition 2.1(b)), and the small exponent $\delta > 0$ separates the time scales of Subsection 2.2. The exponents $\mu > 0$ and $\eta > 0$ are used from Section 5 on, μ as the exponent of the cost (5.2) of one level.

We write $x \asymp y$ if $C^{-1}y \le x \le Cy$ for a constant C depending only on α, q, ζ, and, for $m \ge 2$, we write $x \ll y$, or $y \gg x$, if $x \le C\varepsilon_{m-1}^c y$ for constants $c > 0$ and C depending only on α, q, ζ. Throughout, $|\cdot|$ is the absolute value of a number, the Euclidean norm of a vector, and the operator norm, for the Euclidean norms, of a matrix, or of a multilinear map regarded as a linear map on arrays. We denote by $(g)_E \coloneqq \fint_E g$ the average of a function g over a set E of finite positive measure. On $\mathbb{R}^2$ and $\mathbb{C}^2$ we also write $|\cdot|_\infty$ for the largest modulus of a coordinate and, for a 2×2 matrix A, $|A|_\infty$ for its operator norm for $|\cdot|_\infty$, the largest sum of the moduli of the entries of a row. For a map or a vector field X, ∇X is its Jacobian matrix, $(\nabla X)_{ii'} = \partial_{i'} X_i$.

2.2 Scales

For $m \ge 1$, level m has the wavelength ε_m, the renewal time h_m, the margin time τ'_m and the pulse time τ_m, defined by $\varepsilon_0 \coloneqq 1$ and [AV25, (2.8), (2.12)–(2.14)]

$$\begin{aligned} \varepsilon_m &\coloneqq \big\lceil \Lambda^{q^m/(q-1)} \big\rceil^{-1}, \qquad h_m \coloneqq \frac{2^{-25}}{\lceil \varepsilon_{m-1}^{-(2-\beta+2\delta)} \rceil}\,, \\ \tau'_m &\coloneqq \frac{h_m}{4\lceil \varepsilon_{m-1}^{-\delta}\rceil + 1}\,, \qquad \tau_m \coloneqq \frac{h_m}{(4\lceil \varepsilon_{m-1}^{-\delta}\rceil + 1)^2}\,. \end{aligned} \tag{2.4}$$

Then, for every $m \ge 2$, $\varepsilon_m \asymp \varepsilon_{m-1}^q$, and, for every $m \ge 1$, $\varepsilon_{m-1}/\varepsilon_m \ge \Lambda$, $4\lceil \varepsilon_{m-1}^{-\delta}\rceil + 1 \asymp \varepsilon_{m-1}^{-\delta}$ and

$$h_m \asymp \varepsilon_{m-1}^{2-\beta+2\delta}\,, \qquad \tau'_m \asymp \varepsilon_{m-1}^{2-\beta+3\delta}\,, \qquad \tau_m \asymp \varepsilon_{m-1}^{2-\beta+4\delta}\,, \tag{2.5}$$

so that $\tau_m \ll \tau'_m \ll h_m$ for $m \ge 2$; the ratios τ'_m/τ_m and h_m/τ'_m both equal $4\lceil \varepsilon_{m-1}^{-\delta}\rceil + 1$.

2.3 Pulses and the slow cutoff

The slots of level m are indexed by $k \in \mathbb{Z}$, and slot k carries the shear stream [AV25, (2.19)–(2.20)]

$$\psi_{m,k}(y) \coloneqq \begin{cases} \varepsilon_m^\beta \sin(2\pi y_1/\varepsilon_m) & \text{if } k \in 4\mathbb{Z}+1, \\ \varepsilon_m^\beta \sin(2\pi y_2/\varepsilon_m) & \text{if } k \in 4\mathbb{Z}+3, \\ 0 & \text{if } k \in 2\mathbb{Z}, \end{cases} \tag{2.6}$$

so even slots are empty and the odd ones, the *pulses*, alternate between the two directions. Slot k is active for a time of about τ_m around $k\tau_m$, through the fast cutoff

$$\zeta_{m,k}(t) \coloneqq \zeta\Big(\frac{t - k\tau_m}{\tau_m}\Big). \tag{2.7}$$

The renewal blocks are the intervals $[(l - \frac{1}{2})h_m, (l + \frac{1}{2})h_m]$, $l \in \mathbb{Z}$; slot k lies in the block of

$$l_{m,k} \coloneqq \Big\lfloor \frac{k\tau_m + \frac{1}{2}(h_m - \tau_m)}{h_m} \Big\rfloor . \tag{2.8}$$

This is the label for which [AV25, (2.18)] holds: the slot $[(k - \frac{1}{2})\tau_m, (k + \frac{1}{2})\tau_m]$ lies in the block of $l_{m,k}$ (Lemma 7.1(d)). The slow cutoff of the block of l is

$$\widehat{\zeta}_{m,l}(t) \coloneqq \mathsf{H}\Big(\frac{t - (l - \frac{1}{2})h_m}{\tau'_m} - 1\Big)\, \mathsf{H}\Big(\frac{(l + \frac{1}{2})h_m - t}{\tau'_m} - 1\Big), \tag{2.9}$$

where H is the smooth step with $\mathsf{H} = 0$ on $(-\infty, 0]$, $\mathsf{H} = 1$ on $[1, \infty)$ and

$$\mathsf{H}(r) \coloneqq \frac{\exp(-1/r)}{\exp(-1/r) + \exp(-1/(1-r))} \qquad \text{for } 0 < r < 1\,.$$

The slow cutoff vanishes within τ'_m of the ends of the block and equals one at distance at least $2\tau'_m$ from them. Each block therefore has a pulse-free *quiet margin* of length τ'_m at both ends, and the envelope of slot k is

$$\widetilde{\zeta}_{m,k}(t) \coloneqq \widehat{\zeta}_{m,l_{m,k}}(t)\, \zeta_{m,k}(t)\,. \tag{2.10}$$

Our slow cutoff (2.9) is one explicit choice among those of [AV25, (2.29), (2.31)], since $\widehat{\zeta}_{m,l} = \widehat{\zeta}_{m,0}(\cdot - lh_m)$, it has the support and the plateau stated above, and its derivatives satisfy the bounds of Lemma 7.1(d). Figure 2.1 draws one block.

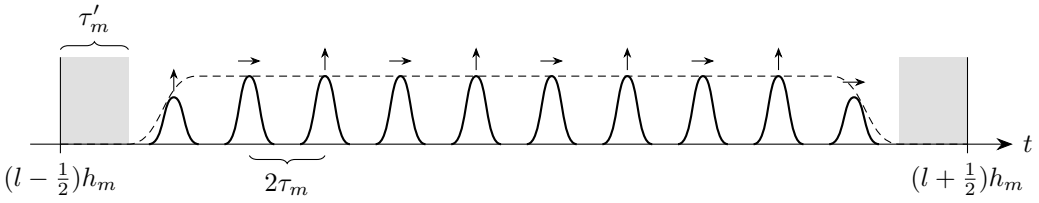


Figure 2.1: One renewal block of level m, in time (not to scale: a block holds $(4\lceil \varepsilon_{m-1}^{-\delta} \rceil + 1)^2$ slots). Shaded: the quiet margins, of length τ'_m. Dashed: the slow cutoff $\widehat{\zeta}_{m,l}$ of (2.9). Solid: the envelopes $\widetilde{\zeta}_{m,k}$ of (2.10) of the pulses k, which are $2\tau_m$ apart; the even slots are empty. The arrow above a pulse is its direction of flow $e_{(k)}$ (Subsection 2.5): vertical for $k \in 4\mathbb{Z} + 1$, horizontal for $k \in 4\mathbb{Z} + 3$.

2.4 The velocity field

Set $\phi_0 \coloneqq 0$ and $\nabla^\perp \coloneqq (-\partial_2, \partial_1)$. For $m \geq 1$, let $X_{m-1}(t, x, t')$ denote the position at time t of the trajectory of $\mathbf{b}_{m-1} = \nabla^\perp \phi_{m-1}$ through x at time t', that is, the solution of

$$\begin{cases} \partial_t X_{m-1}(t, x, t') = \mathbf{b}_{m-1}\big(t, X_{m-1}(t, x, t')\big) & \text{in } \mathbb{R}\,, \\ X_{m-1}(t', x, t') = x\,, \end{cases} \tag{2.11}$$

and define [AV25, (2.36)]

$$\phi_m(t,x) \coloneqq \phi_{m-1}(t,x) + \sum_{k\in\mathbb{Z}} \widetilde{\zeta}_{m,k}(t)\,\psi_{m,k}\big(X_{m-1}(l_{m,k}h_m,x,t)\big), \qquad \mathbf{b}_m \coloneqq \nabla^\perp\phi_m\,. \tag{2.12}$$

The level-m pulses are thus written in the Lagrangian coordinates of the coarser flow, renewed at the centre lh_m of each block: within a block the variable is the position at time lh_m of the trajectory of $\mathbf{b}_{m-1}$, so the coarse flow carries the fine shear along instead of sweeping across it. The coarse velocity field satisfies $\|\nabla\mathbf{b}_{m-1}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le C\varepsilon_{m-1}^{\beta-2}$ with an absolute constant C, and the renewal time is short on the time scale $\varepsilon_{m-1}^{2-\beta}$:

$$\varepsilon_{m-1}^{\beta-2}\,h_m \le 2^{-25}\varepsilon_{m-1}^{2\delta}\,. \tag{2.13}$$

Within a block, the gradient of the flow of $\mathbf{b}_{m-1}$ differs from the identity by at most $C\varepsilon_{m-1}^{2\delta}$, with an absolute constant C, so the flow barely deforms the shear [AV25, Corollary 2.4]. The coordinates change only at the block boundaries, where the slow cutoff has switched the pulses off.

The velocity fields $\mathbf{b}_m$ are smooth, $\mathbb{Z}^2$-periodic in space and 1-periodic in time. The stream functions ϕ_m converge to a stream function ϕ with $\nabla^\perp\phi = \mathbf{b}$, where $\mathbf{b}$ is the pointwise limit of the $\mathbf{b}_m$, and there exists $C(\alpha,q,\zeta,\Lambda) < \infty$ such that, for every $m \ge 0$,

$$\|\phi-\phi_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le C\varepsilon_{m+1}^{\beta}\,, \qquad \|\mathbf{b}-\mathbf{b}_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le C\varepsilon_{m+1}^{\alpha}\,. \tag{2.14}$$

The velocity field $\mathbf{b}$ is bounded, $\mathbb{Z}^2$-periodic in space, 1-periodic in time, divergence free, and Hölder continuous with exponent α jointly in time and space: there exists $C(\alpha,q,\zeta,\Lambda) < \infty$ such that, for every $t,t' \in \mathbb{R}$ and $x,y \in \mathbb{R}^2$,

$$|\mathbf{b}(t,x)-\mathbf{b}(t',y)| \le C\big(|t-t'|+|x-y|\big)^\alpha\,. \tag{2.15}$$

This is the regularity stated in [AV25, (1.3)] for the choice of q in [AV25]. This $\mathbf{b}$ is the velocity field of Theorem A; it depends on $\alpha, q, \Lambda, \zeta$.

2.5 Effective diffusivities

For $\nu > 0$, the *averaged flux of the level-m pulses* at diffusivity ν is the matrix

$$\mathbf{J}_m^\nu(t) \coloneqq \nu I + 2\pi^2\varepsilon_m^{2\beta-2} \sum_{k\in 2\mathbb{Z}+1} \widetilde{\zeta}_{m,k}(t)\Big(\int_{-\infty}^t \exp\Big(-\frac{4\pi^2\nu(t-t')}{\varepsilon_m^2}\Big)\widetilde{\zeta}_{m,k}(t')\,\mathrm{d}t'\Big)\,e_{(k)}\otimes e_{(k)}\,, \tag{2.16}$$

where $e_{(k)} \coloneqq e_2$ for $k \in 4\mathbb{Z}+1$ and $e_{(k)} \coloneqq e_1$ for $k \in 4\mathbb{Z}+3$. This is the spatial average of the flux of the correctors of the level-m pulses [AV25, (3.15)], written explicitly as in [AV25, (3.17)–(3.18)]: the shear dispersion of the pulses. Pulse k enhances diffusion in the direction $e_{(k)}$ of its flow, and the time integral averages its envelope over the past at the rate $4\pi^2\nu/\varepsilon_m^2$ at which the diffusivity ν damps the modes $\sin(2\pi y_i/\varepsilon_m)$. For a pulse held on forever, the time integral equals $\varepsilon_m^2/(4\pi^2\nu)$ and the enhancement is $\varepsilon_m^{2\beta}/(2\nu)$, Taylor's dispersion formula for the steady shear $\psi_{m,k}$. The matrix $\mathbf{J}_m^\nu(t)$ is diagonal, satisfies

$$\nu I \le \mathbf{J}_m^\nu(t) \le \Big(\nu + \frac{\varepsilon_m^{2\beta}}{2\nu}\Big)I\,,$$

and equals νI when no pulse is active. Since the number h_m/τ_m of slots in a block is odd, a shift of time by $2h_m$ maps slot k to slot $k + 2h_m/\tau_m \equiv k+2 \pmod 4$ and exchanges the two diagonal

entries; a shift by $4h_m$ leaves the matrix unchanged. Its average over every interval of length $4h_m$ is the same scalar matrix [AV25, (3.16)]: for every $t \in \mathbb{R}$,

$$\fint_t^{t+4h_m} \mathbf{J}_m^\nu(t')\,\mathrm{d}t' = \overline{K}_m(\nu)\, I\,, \qquad \overline{K}_m(\nu) := \frac{1}{2}\fint_0^{4h_m} \operatorname{tr} \mathbf{J}_m^\nu(t')\,\mathrm{d}t'\,. \tag{2.17}$$

The *permissible diffusivities* are the union $\mathcal{K} := \bigcup_{M\ge 1} \mathcal{K}_M$ of the bands [AV25, (3.43)]

$$\mathcal{K}_M := \left[\tfrac{1}{2}\varepsilon_M^{\beta-\gamma},\, 2\varepsilon_M^{\beta-\gamma}\right], \tag{2.18}$$

which are disjoint and accumulate at 0. For $\kappa \in \mathcal{K}_M$, we call M the *terminal level*, and the *effective diffusivities* $\kappa_M, \kappa_{M-1}, \dots, \kappa_0$ are defined by the recursion [AV25, (3.42)]

$$\kappa_M := \kappa\,, \qquad \kappa_{m-1} := \overline{K}_m(\kappa_m)\,, \quad m = M, M-1, \dots, 1\,. \tag{2.19}$$

Thus κ_{m-1} is the diffusivity the solution feels once the pulses of level m, at diffusivity κ_m, have been averaged out.

Proposition 2.1 (Effective diffusivities and scales)**.** *There exist $\Lambda_0(\alpha, q, \zeta) \ge 128$ and $C(\alpha, q, \zeta) < \infty$ such that, for every $\Lambda \ge \Lambda_0$, $M \ge 1$, $\kappa \in \mathcal{K}_M$ and $1 \le m \le M$, the following hold.*

(a) *At each level the recursion adds about $\frac{9}{80}\varepsilon_m^{2\beta}/\kappa_m$:*

$$\left|\kappa_{m-1} - \kappa_m - \frac{9}{80}\,\frac{\varepsilon_m^{2\beta}}{\kappa_m}\right| \le C\,\frac{\varepsilon_m^{2\beta}}{\kappa_m}\Big(\frac{\varepsilon_m^2}{\kappa_m \tau_m} + \varepsilon_{m-1}^{\delta}\Big).$$

(b) *If $m < M$, then $\kappa_m \asymp \varepsilon_m^{\beta+\gamma}$.*

(c) *If $m \ge 2$, then*

$$\kappa_m \le \kappa_{m-1}\,, \qquad \frac{\varepsilon_m^2}{\kappa_m \tau_m} \le C\varepsilon_{m-1}^{4\delta}\,, \qquad (\kappa_{m-1} h_m)^{1/2} \asymp \varepsilon_{m-1}^{1+\gamma/2+\delta}\,, \qquad \Big(\frac{\kappa_{m-1}}{\kappa_m}\Big)^{1/2} \le C\varepsilon_{m-1}^{-q\gamma}\,.$$

If moreover $m < M$, then $(\kappa_{m-1}/\kappa_m)^{1/2} \asymp \varepsilon_{m-1}^{-q\gamma}$, and, if instead $m = M$, then $\kappa_{m-1} \le C\kappa_m$.

Part (a) is [AV25, Lemma 3.3], and part (b) is the bound (3.45) of [AV25, Lemma 3.4] for $1 \le m < M$, both for the choice of q in [AV25]. *Idea of the proof.* (a) averages (2.16) over a period: a quarter of the slots enhances each direction, each by about $\int_{\mathbb{R}} \zeta(t)^2\,\mathrm{d}t = \frac{9}{10}$ times Taylor's enhancement $\varepsilon_m^{2\beta}/(2\kappa_m)$, which gives $\frac{9}{80}$. Part (b) follows by induction on m, going up the levels from $\kappa_M \in \mathcal{K}_M$. Part (c) follows from (b), (2.18), (2.5), the inequality $\overline{K}_m(\nu) \ge \nu$ of Lemma 7.10(iii) and Lemma 7.1(a), (c).

Let $M \ge 2$ and $\kappa \in \mathcal{K}_M$. Since $q(\beta-\gamma) = \beta+\gamma$, we have $\kappa \asymp \varepsilon_{M-1}^{\beta+\gamma}$, while $\kappa_m \asymp \varepsilon_m^{\beta+\gamma}$ for $1 \le m < M$ by (b), independently of κ. By (c), the enhancement is strong below the terminal level, $\kappa_{m-1} \gg \kappa_m$ for $2 \le m < M$, and bounded at level M, $\kappa_M \le \kappa_{M-1} \le C\kappa_M$.

The scales of one level. Let $2 \le m < M$. On the time scale $\varepsilon_{m-1}^{2-\beta}$ of the coarse velocity field $\mathbf{b}_{m-1}$ (before (2.13)), the cell diffusion time ε_m^2/κ_m of level m, the pulse time, the margin time and the renewal time are $\varepsilon_{m-1}^{8\delta}$, $\varepsilon_{m-1}^{4\delta}$, $\varepsilon_{m-1}^{3\delta}$ and $\varepsilon_{m-1}^{2\delta}$ times this time scale:

$$\frac{\varepsilon_m^2}{\kappa_m} \asymp \varepsilon_{m-1}^{2-\beta+8\delta}\,, \qquad \tau_m \asymp \varepsilon_{m-1}^{2-\beta+4\delta}\,, \qquad \tau_m' \asymp \varepsilon_{m-1}^{2-\beta+3\delta}\,, \qquad h_m \asymp \varepsilon_{m-1}^{2-\beta+2\delta}\,,$$

by (2.5), Proposition 2.1(b) and the identity $q(2-\beta-\gamma) = 2-\beta+8\delta$ of Lemma 7.1(c). The effective diffusion length over one renewal time, $\rho(m) := (\kappa_{m-1} h_m)^{1/2}$, lies between the two wavelengths:

$$\varepsilon_m \asymp \varepsilon_{m-1}^{q\gamma+3\delta}\rho(m)\,, \qquad \rho(m) \asymp \varepsilon_{m-1}^{1+\gamma/2+\delta}\,, \qquad \text{so} \qquad \varepsilon_m \ll \rho(m) \ll \varepsilon_{m-1}\,,$$

by Proposition 2.1(c) and the identity $q = 1+4\delta+(q+\frac{1}{2})\gamma$, which is Lemma 7.1(c) with $2-\beta = 1-\alpha$.

Why the time scales have these sizes. Three requirements fix them. The renewal time h_m is short on the time scale $\varepsilon_{m-1}^{2-\beta}$ of $\mathbf{b}_{m-1}$, by (2.13), so that the Lagrangian coordinates stay close to rigid. The pulse time τ_m is long compared with the cell diffusion time ε_m^2/κ_m of level m, by Proposition 2.1(c), so that each pulse homogenizes. The margins of length τ_m', where the coordinates are renewed, are long compared with a pulse and short compared with a block.

In the notation of [AV25], the renewal time h_m is τ_m'', the number $\varepsilon_m^{\beta-2}$ is a_m, the label $l_{m,k}$ is l_k, and the envelope $\widetilde{\zeta}_{m,k}$ is the product $\widehat{\zeta}_{m,l_k}\zeta_{m,k}$ of the two cutoffs. The argument $X_{m-1}(lh_m,x,t)$ of the shear is written $X_{m-1}^{-1}(t,x,lh_m)$ there, and the matrix $\mathbf{J}_m^\nu$ and the number $\overline{K}_m(\nu)$ are written $\mathbf{J}_m^\kappa$ and $\overline{\mathbf{K}}_m^\kappa$, with $\kappa=\nu$. The effective diffusivities are the renormalized diffusivities of [AV25], and the exponent $\beta-\gamma$ of the bands is written $\frac{2\beta}{q+1}$ there. Section 7 proves the properties stated in this section: Lemma 7.1 the relations between the scales, (2.5), (2.13) and the identities between the exponents; Proposition 7.3 the bound on $\nabla\mathbf{b}_{m-1}$, (2.14) and (2.15); Lemma 7.7(a) the bound on the gradient of the flow within a block, for $m\geq 2$; Lemma 7.10 the properties of $\mathbf{J}_m^\nu$; and Proposition 2.1 is proved in Subsection 7.5.

3. Main results

For a $\mathbb{Z}^2$-periodic function g we write

$$\|g\|_{C^{0,s}(\mathbb{T}^2)} := \sup_{\mathbb{T}^2}|g| + [g]_{C^{0,s}(\mathbb{T}^2)}\,, \qquad [g]_{C^{0,s}(\mathbb{T}^2)} := \sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|_{\mathbb{T}^2}^s}\,, \tag{3.1}$$

for the Hölder norm and seminorm, where $|x-y|_{\mathbb{T}^2}$ is the distance on the torus. A weak solution of the initial-value problem below is a function $\theta\in L^\infty((t_{\mathsf{in}},t_{\mathsf{in}}+T);L^2(\mathbb{T}^2))$ with $\nabla\theta\in L^2((t_{\mathsf{in}},t_{\mathsf{in}}+T)\times\mathbb{T}^2)$, weakly continuous from $[t_{\mathsf{in}},t_{\mathsf{in}}+T]$ to $L^2(\mathbb{T}^2)$, which solves the equation in the sense of distributions and takes the initial value. A weak solution on (t_{in},∞) is a function that, for every $T>0$, is a weak solution on $(t_{\mathsf{in}},t_{\mathsf{in}}+T)$.

Theorem A (Interior regularity)**.** *Let $\alpha\in(0,1/3)$, $s\in(0,\frac12(1-\alpha))$, q satisfy (2.1), and let ζ be an even function in $C_c^\infty((-\frac23,\frac23))$ satisfying (2.2). There exists $\Lambda_0(\alpha,s,q,\zeta)\geq 128$ such that, for every integer $\Lambda\geq\Lambda_0$, the velocity field $\mathbf{b}$ of Section 2 has the following property. For every $0<t_0<T$ there exists $C(\alpha,s,q,\zeta,\Lambda,t_0,T)<\infty$ such that, for every $t_{\mathsf{in}}\in\mathbb{R}$, $\kappa\in\mathcal{K}$ and $\theta_{\mathsf{in}}\in L^2(\mathbb{T}^2)$, every weak solution θ of the initial-value problem*

$$\begin{cases}\partial_t\theta+\mathbf{b}\cdot\nabla\theta-\kappa\Delta\theta=0 & \text{in } (t_{\mathsf{in}},t_{\mathsf{in}}+T)\times\mathbb{T}^2\,,\\ \theta(t_{\mathsf{in}},\cdot)=\theta_{\mathsf{in}} & \text{on } \mathbb{T}^2\end{cases} \tag{3.2}$$

satisfies

$$\|\theta\|_{L^\infty([t_{\mathsf{in}}+t_0,t_{\mathsf{in}}+T];C^{0,s}(\mathbb{T}^2))}\leq C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,. \tag{3.3}$$

The constant C is independent of κ, and $\frac12(1-\alpha)$ is the Obukhov–Corrsin exponent of a $C^{0,\alpha}$ velocity field [Obu49, Cor51]; Subsection 5.1 explains where it enters the proof.

Theorem A also implies that the variance is dissipated at a rate independent of $\kappa\in\mathcal{K}$.

Theorem B (Uniform anomalous dissipation)**.** *Let α, s, q, ζ and Λ_0 be as in Theorem A, and let $\Lambda\geq\Lambda_0$ be an integer. For the velocity field $\mathbf{b}$ of Section 2, there exists $c(\alpha,s,q,\zeta,\Lambda)>0$ such that, for every $t_{\mathsf{in}}\in\mathbb{R}$, $\kappa\in\mathcal{K}$ and $\theta_{\mathsf{in}}\in L^2(\mathbb{T}^2)$, every weak solution θ of the initial-value problem*

$$\begin{cases}\partial_t\theta+\mathbf{b}\cdot\nabla\theta-\kappa\Delta\theta=0 & \text{in } (t_{\mathsf{in}},\infty)\times\mathbb{T}^2\,,\\ \theta(t_{\mathsf{in}},\cdot)=\theta_{\mathsf{in}} & \text{on } \mathbb{T}^2\end{cases} \tag{3.4}$$

satisfies

$$\kappa\|\nabla\theta\|^2_{L^2((t_{\mathsf{in}},t_{\mathsf{in}}+1)\times\mathbb{T}^2)} \geq c\left\|\theta_{\mathsf{in}}-(\theta_{\mathsf{in}})_{\mathbb{T}^2}\right\|^2_{L^2(\mathbb{T}^2)}, \tag{3.5}$$

and, for every $t \geq t_{\mathsf{in}}$,

$$\left\|\theta(t,\cdot)-(\theta_{\mathsf{in}})_{\mathbb{T}^2}\right\|_{L^2(\mathbb{T}^2)} \leq 2\exp(-c(t-t_{\mathsf{in}}))\left\|\theta_{\mathsf{in}}-(\theta_{\mathsf{in}})_{\mathbb{T}^2}\right\|_{L^2(\mathbb{T}^2)}. \tag{3.6}$$

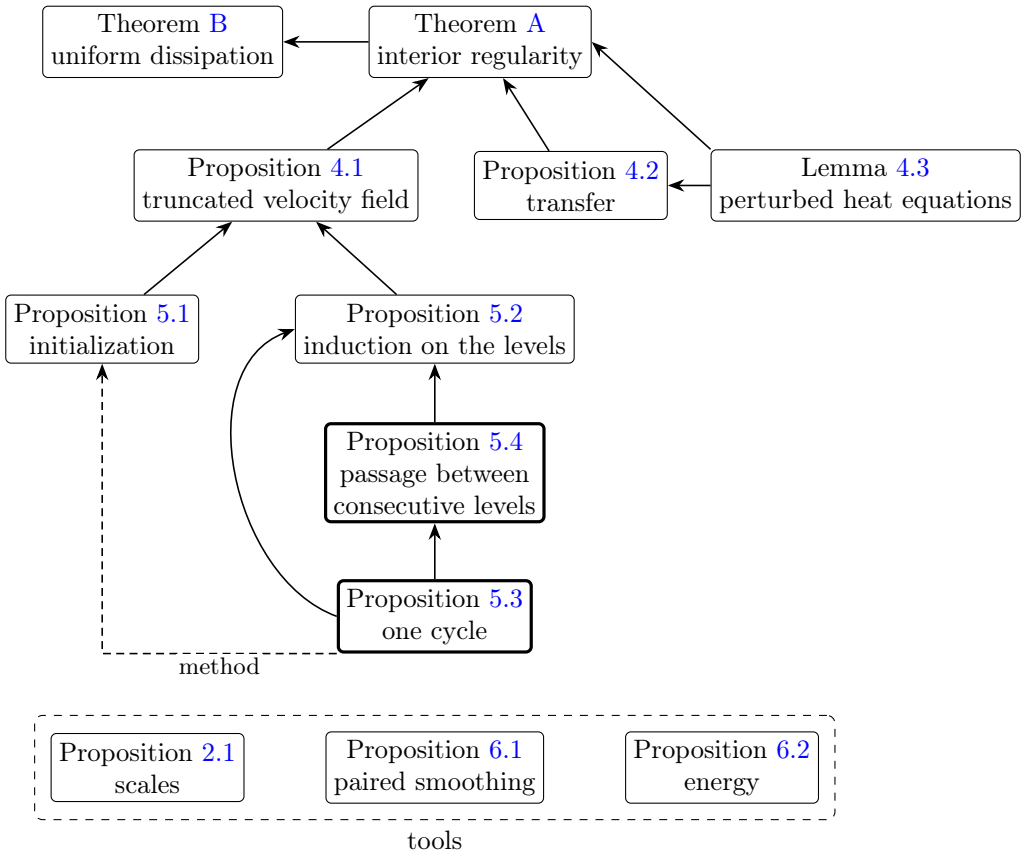


Figure 3.1: The statements and their uses. A solid arrow from one statement to another means that the proof of the second uses the first. The dashed arrow means that the entry repeats the method of the cycle at the terminal level. The two main steps are framed in bold. The dashed frame collects Proposition 2.1 and the two tools of Section 6, whose uses are not drawn.

We compare Theorem B with [AV25, Theorem 1.1], recalled in Subsection 5.1, which concerns the velocity field of [AV25] and gives a lower bound for $\limsup_{\kappa\to 0}\kappa\|\nabla\theta^\kappa\|^2_{L^2((0,1)\times\mathbb{T}^2)}$ for mean-zero data in H^1. The restriction to the bands and a lower bound for every κ in them are already in [AV25]: the discussion after [AV25, Theorem 1.1] states the lower bound for every κ in an explicit union of bands, the permissible diffusivities [AV25, (3.43)], with a constant depending on the datum. For the velocity field of this paper, Theorem B gains three things: the datum is only in $L^2(\mathbb{T}^2)$; the constant c is independent of the datum and of the initial time t_{in}, whereas the constant in [AV25] depends on the ratio $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}/\|\nabla\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$; and the variance decays exponentially, at a rate independent of κ. As a consequence, since the bands accumulate at 0, for every mean-zero $\theta_{\mathsf{in}}\in L^2(\mathbb{T}^2)$ the weak solutions θ^κ of (3.4) with $t_{\mathsf{in}}=0$, one for each $\kappa\in\mathcal{K}$, which are unique (Section 9), satisfy

$$\liminf_{\kappa\to 0,\ \kappa\in\mathcal{K}}\kappa\|\nabla\theta^\kappa\|^2_{L^2((0,1)\times\mathbb{T}^2)} \geq \inf_{\kappa\in\mathcal{K}}\kappa\|\nabla\theta^\kappa\|^2_{L^2((0,1)\times\mathbb{T}^2)} \geq c\,\|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)},$$

with the constant c of Theorem B: a lower bound on the limit inferior, where [AV25, Theorem 1.1] bounds the limit superior.

Idea of the proof of Theorem B. Fix t_{in} and κ. The solution operator $\mathcal{S}$, which maps θ_{in} to $\theta(t_{\mathsf{in}}+1,\cdot)$, is linear, positive by the maximum principle, and preserves constants and integrals. Theorem A on $(t_{\mathsf{in}}, t_{\mathsf{in}}+1)$, with $t_0 = \frac{1}{2}$, bounds it from $L^2(\mathbb{T}^2)$ into $C^{0,s}(\mathbb{T}^2)$ by a constant $C \geq 1$ independent of t_{in} and $\kappa \in \mathcal{K}$. With N the least integer such that $N^s \geq C$, Lemma 9.4 gives $\|\mathcal{S}\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)} \leq (1 - 1/(2N^4))\|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)}$ for every mean-zero $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$, and the energy identity turns this drop into (3.5), with $c = 1/(4N^4)$, and into a contraction by $1-2c$ of $\|\theta(t,\cdot) - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\|^2_{L^2(\mathbb{T}^2)}$ over each unit of time, which gives (3.6). The proof is in Subsection 9.6.

4. Reduction to a velocity field with finitely many scales

For $\kappa \in \mathcal{K}_M$, the levels of $\mathbf{b}$ much finer than level M are irrelevant to the solution: their stream functions are much smaller than κ, and molecular diffusion averages them out. In divergence form, a perturbation of the stream function of size $\kappa\varpi$ is a skew-symmetric perturbation of relative size ϖ of the diffusion matrix κI, and it moves the solution by at most $\varpi\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ in $L^2(\mathbb{T}^2)$ (Lemma 9.2, as in the comparison of θ with θ_M in [AV25, Section 5.4]). Proposition 4.2 turns this into a bound in $C^{0,s}(\mathbb{T}^2)$, at a cost which is bounded once the levels finer than $M+n$ are removed, for a fixed number n of extra levels. We therefore prove the estimate for the smooth velocity field $\mathbf{b}_{M+n}$, which has finitely many scales (Proposition 4.1, proved in Section 5), and transfer it to $\mathbf{b}$.

Proposition 4.1 (Regularity for the truncated velocity field)**.** *There exists $\Lambda_0(\alpha, s, q, \zeta) \geq 128$ such that, for every integer $\Lambda \geq \Lambda_0$, integer $n \geq 1$ and $0 < t_0 < T$, there exist $M_1(\alpha, s, q, \zeta, \Lambda, n, t_0, T) \in \mathbb{N}$ and $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ with the following property. For every $M \geq M_1$, $\kappa \in \mathcal{K}_M$, $t_{\mathsf{in}} \in \mathbb{R}$ and $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$, every weak solution θ of*

$$\begin{cases} \partial_t\theta + \mathbf{b}_{M+n}\cdot\nabla\theta - \kappa\Delta\theta = 0 & \text{in } (t_{\mathsf{in}}, t_{\mathsf{in}}+T)\times\mathbb{T}^2\,, \\ \theta(t_{\mathsf{in}},\cdot) = \theta_{\mathsf{in}} & \text{on } \mathbb{T}^2 \end{cases} \tag{4.1}$$

satisfies

$$\|\theta\|_{L^\infty([t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T];C^{0,s}(\mathbb{T}^2))} \leq C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,.$$

Proposition 4.2 (Transfer to the full velocity field)**.** *There exist $\varpi_0(s) > 0$ and $C(s) < \infty$ with the following property. Let $\Phi, \widetilde{\Phi}\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ be $\mathbb{Z}^2$-periodic in x, with Φ continuous and C^1 in x, $\nabla^\perp\Phi$ bounded and continuous, and $\widetilde{\Phi}$ smooth. Let $\kappa > 0$, $0 < r \leq 1/8$ and $\varpi \geq 0$ satisfy*

$$\|\Phi - \widetilde{\Phi}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \leq \kappa\varpi\,, \qquad \varpi + \frac{r^2}{\kappa}\|\nabla^2\widetilde{\Phi}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \leq \varpi_0\,.$$

Let $t_{\mathsf{in}}, t_1, t_2 \in \mathbb{R}$ with $t_{\mathsf{in}} \leq t_1 - r^2/\kappa$ and $t_1 < t_2$, let $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$, and let θ and $\widetilde{\theta}$ be weak solutions of

$$\begin{cases} \partial_t\theta + \nabla^\perp\Phi\cdot\nabla\theta - \kappa\Delta\theta = 0\,, \quad \partial_t\widetilde{\theta} + \nabla^\perp\widetilde{\Phi}\cdot\nabla\widetilde{\theta} - \kappa\Delta\widetilde{\theta} = 0 & \text{in } (t_{\mathsf{in}}, t_2 + r^2/\kappa)\times\mathbb{T}^2\,, \\ \theta(t_{\mathsf{in}},\cdot) = \widetilde{\theta}(t_{\mathsf{in}},\cdot) = \theta_{\mathsf{in}} & \text{on } \mathbb{T}^2\,. \end{cases}$$

Let $\Xi \geq 0$ be such that

$$\|\widetilde{\theta}\|_{L^\infty([t_1 - r^2/\kappa,\, t_2 + r^2/\kappa];C^{0,s}(\mathbb{T}^2))} \leq \Xi\,.$$

Then

$$\|\theta\|_{L^\infty([t_1,t_2];C^{0,s}(\mathbb{T}^2))} \leq C\big(\Xi + \varpi\, r^{-1-s}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\big)\,.$$

The power r^{-1-s} converts the $L^2(\mathbb{T}^2)$ deviation $\varpi\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ into a deviation in $C^{0,s}(\mathbb{T}^2)$. *Idea of the proof.* In a cylinder of radius r moving with $\nabla^\perp\widetilde{\Phi}$, after rescaling, θ solves the equation of Lemma 4.3: $\Phi-\widetilde{\Phi}$ gives the skew part, of size at most ϖ, and the variation of $\nabla^\perp\widetilde{\Phi}$ gives the drift, of size at most $r^2\|\nabla^2\widetilde{\Phi}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}/\kappa$. On the rescaled cylinder, the L^2 norm of $\theta-\widetilde{\theta}$ is at most $C\varpi r^{-1}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$, and $\widetilde{\theta}$ stays within $C\Xi r^s$ of a constant, by the bound on $\widetilde{\theta}$ and its equation tested with a fixed bump in the moving frame. So Lemma 4.3 places $\theta(t,\cdot)$ within $C(\varpi r^{-1}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}+\Xi r^s)$ of $\widetilde{\theta}(t,\cdot)$, and bounds its increments over distances $\rho<\frac{r}{4}$ by this quantity times $(\rho/r)^{(1+s)/2}$; since $\frac{1}{2}(1+s)>s$, the two bounds give the bound in $C^{0,s}(\mathbb{T}^2)$ at all distances (Step 4 of the proof in Subsection 9.2).

Lemma 4.3 (Hölder estimate for perturbed heat equations)**.** *There exist $\varpi_1(s)>0$ and $C(s)<\infty$ with the following property. Let $D_1:=\{x\in\mathbb{R}^2:|x|<1\}$ and $D_{1/2}:=\{x\in\mathbb{R}^2:|x|<\frac{1}{2}\}$, and let $[\cdot]_{C^{0,(1+s)/2}_{\mathrm{par}}((-\frac{1}{4},0]\times D_{1/2})}$ be the Hölder seminorm on $(-\frac{1}{4},0]\times D_{1/2}$ for the parabolic distance $|x-y|+|t-t'|^{1/2}$. Let $\mathbf{a}$ and $\mathbf{b}_*$ be measurable functions on $(-1,0)\times D_1$ with $\mathbf{a}-I$ skew-symmetric, $\mathbf{b}_*(t,\cdot)$ divergence free in the sense of distributions for almost every t, and $|\mathbf{a}-I|+|\mathbf{b}_*|\le\varpi_1$ almost everywhere, and let $U\in L^\infty((-1,0);L^2(D_1))$ with $\nabla U\in L^2((-1,0)\times D_1)$ be a weak solution of*

$$\partial_t U-\nabla\cdot(\mathbf{a}\nabla U)+\mathbf{b}_*\cdot\nabla U=0\quad\text{in }(-1,0)\times D_1\,.$$

Then $U(t,\cdot)$ converges in $L^2(D_{1/2})$ as $t\uparrow 0$, and, with this limit as $U(0,\cdot)$,

$$\begin{aligned}&[U]_{C^{0,(1+s)/2}_{\mathrm{par}}((-\frac{1}{4},0]\times D_{1/2})}+\big\|U-(U)_{(-1,0)\times D_1}\big\|_{L^\infty((-\frac{1}{4},0]\times D_{1/2})}\\&\qquad\le C\Big(\fint_{(-1,0)\times D_1}\big|U(t,x)-(U)_{(-1,0)\times D_1}\big|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}.\end{aligned}$$

Idea of the proof. Since $\mathbf{a}-I$ and $\mathbf{b}_*$ are small, U is close in L^2 to a caloric function, a solution of the heat equation, on every smaller cylinder, and Campanato's characterization of Hölder continuity by the decay of these L^2 distances gives the bound. The proof is in Subsection 9.1.

Idea of the proof of Theorem A. For $\kappa\in\mathcal{K}_M$ with M large, we continue θ to $(t_{\mathsf{in}},t_{\mathsf{in}}+T+1)$, as Proposition 4.2 uses the solutions after t_2, and compare it with the solution $\widetilde{\theta}$ of (4.1), with $\Phi=\phi$ and $\widetilde{\Phi}=\phi_{M+n}$; Proposition 4.1 bounds $\widetilde{\theta}$ in $L^\infty([t_{\mathsf{in}}+\frac{1}{2}t_0,t_{\mathsf{in}}+T+1];C^{0,s}(\mathbb{T}^2))$. By (2.14) and Proposition 7.3(ii), we may take ϖ a multiple of $\varepsilon^\beta_{M+n+1}/\kappa$ and r^2 a small multiple of $\kappa\varepsilon^{2-\beta}_{M+n}$, and a choice of n depending only on α,s,q makes ϖr^{-1-s} bounded (Step 2 of the proof in Subsection 9.3). The remaining finitely many bands lie in a compact subinterval of $(0,2]$; there the Hölder continuity of $\mathbf{b}$ makes the drift small at the scale $\kappa^{1/\beta}$ along its trajectories, and Lemma 4.3 applies in a frame moving along a trajectory of $\mathbf{b}$ (Lemma 9.1).

5. Iterated homogenization

This section reduces Proposition 4.1 to four propositions, proved in Part II, by an induction on the levels of $\mathbf{b}_{M+n}$, which are homogenized one at a time, from the finest to a fixed coarse level.

5.1 From L^2 to L^∞, scale by scale

This subsection is heuristic, except for the definition of the corrector amplitude and the inequality (5.2), which is proved here and used in the proof; the statements of the proof begin in Subsection 5.2.

Anomalous dissipation in [AV25]. For the velocity field of [AV25], that is, for the choice of q made there, [AV25, Theorem 1.1] states that the scalar variance is dissipated anomalously: for every mean-zero $\theta_{\mathsf{in}} \in H^1(\mathbb{T}^2)$ with $\theta_{\mathsf{in}} \neq 0$, the solutions θ^κ of

$$\begin{cases} \partial_t \theta^\kappa + \mathbf{b} \cdot \nabla \theta^\kappa - \kappa \Delta \theta^\kappa = 0 & \text{in } (0,1) \times \mathbb{T}^2 \,, \\ \theta^\kappa(0,\cdot) = \theta_{\mathsf{in}} & \text{on } \mathbb{T}^2 \end{cases}$$

satisfy

$$\limsup_{\kappa \to 0} \kappa \|\nabla \theta^\kappa\|^2_{L^2((0,1)\times\mathbb{T}^2)} > 0 \,,$$

with a lower bound depending only on $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ and the ratio $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}/\|\nabla\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. The proof compares, for $\kappa \in \mathcal{K}_M$, the solutions θ_m of the equations of level m, with velocity field $\mathbf{b}_m$, diffusivity κ_m and the same initial datum, and shows that the dissipation $\kappa_m \|\nabla \theta_m\|^2_{L^2((0,1)\times\mathbb{T}^2)}$ is multiplied only by $1 + O(\varepsilon_{m-1}^\delta)$ from one level to the next, at all levels finer than a threshold depending on the initial datum [AV25, Proposition 5.2].

The expected regularity. By the energy identity, the dissipation of each level satisfies $\kappa_m \|\nabla \theta_m\|^2_{L^2((0,\infty)\times\mathbb{T}^2)} \leq \frac{1}{2}\|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)}$. Heuristically, θ_m is the solution seen at the scale ε_m, with the oscillations of smaller wavelength averaged out, so at the scale ε_m the gradient of the solution is of size at most $\kappa_m^{-1/2} \asymp \varepsilon_m^{-(\beta+\gamma)/2}$ in $L^2((0,\infty)\times\mathbb{T}^2)$, for $1 \leq m < M$, by Proposition 2.1(b). If the gradient of θ_m were of this size pointwise in $(0,\infty)\times\mathbb{T}^2$, the solution would oscillate by at most $\varepsilon_m \cdot \varepsilon_m^{-(\beta+\gamma)/2}$ over a distance ε_m, at every level $1 \leq m < M$: it would be Hölder continuous with exponent $1 - \frac{1}{2}(\beta+\gamma)$. Since $\beta + \gamma = 2q\beta/(q+1)$,

$$1 - \frac{\beta+\gamma}{2} = \frac{1-q\alpha}{1+q} \nearrow \frac{1-\alpha}{2} \quad \text{as } q \downarrow 1 \,, \qquad \mu = (q-1)(1-s) - q\gamma = (q-1)\Big(1 - \frac{\beta+\gamma}{2} - s\Big). \tag{5.1}$$

The limit is the Obukhov–Corrsin exponent, and μ is $q-1$ times the margin by which s lies below the heuristic exponent. Theorem A makes this heuristic rigorous, for q close to 1. The cost of one level is read off from its correctors. Let a function be bounded in $C^{0,s}(\mathbb{T}^2)$ and smooth at the wavelength ε_{m-1}, so that its gradient is of size ε_{m-1}^{s-1}. The correctors of level m add to it a function that oscillates at the wavelength ε_m, of size the *corrector amplitude* $\bar\varepsilon_m := \varepsilon_m(\kappa_{m-1}/\kappa_m)^{1/2}$ times this gradient; for $2 \leq m < M$ the corrector amplitude is, up to a constant, ε_m times the Péclet number $\varepsilon_m^\beta/\kappa_m$ of level m, by Proposition 2.1(b) and $\varepsilon_m \asymp \varepsilon_{m-1}^q$. A function that oscillates at the wavelength ε_m has a $C^{0,s}(\mathbb{T}^2)$ norm of order ε_m^{-s} times its supremum. So, for $2 \leq m \leq M$, the correctors of level m cost, relative to the $C^{0,s}(\mathbb{T}^2)$ norm,

$$\varepsilon_m^{-s}\, \bar\varepsilon_m\, \varepsilon_{m-1}^{s-1} = \Big(\frac{\varepsilon_m}{\varepsilon_{m-1}}\Big)^{1-s} \Big(\frac{\kappa_{m-1}}{\kappa_m}\Big)^{1/2} \leq C\varepsilon_{m-1}^{(q-1)(1-s)-q\gamma} = C\varepsilon_{m-1}^{\mu} \tag{5.2}$$

by Proposition 2.1(c), the terminal level included, and by $\varepsilon_m \asymp \varepsilon_{m-1}^q$. Separation of scales gains $(\varepsilon_m/\varepsilon_{m-1})^{1-s}$, enhancement costs $(\kappa_{m-1}/\kappa_m)^{1/2}$, and their balance $\mu > 0$ is the condition that s lies below the exponent in (5.1).

Large-scale regularity. In the quantitative argument of large-scale regularity recalled in Subsection 1.3, the homogenization errors are summed over the scales in an iteration of Campanato type, in L^2 in [AKM19] and in L^∞ in [AS16]. Here the comparison is made once per level, on the whole torus. The seminorm $\lambda^{-s}[\,\cdot\,]_{C_\lambda^{N_{\mathrm{rep}}}(\mathbb{T}^2)}$ (see (5.12)) of a solution, bounded by the size of a representation (Subsection 5.6) at the wavelength λ, plays the role of the excess at the scale λ, and (5.2) is the cost of one level.

5.2 Overview: two dual cascades

Fix M, $\kappa \in \mathcal{K}_M$ and a weak solution θ of (4.1). For each level $m \le M$ the effective diffusivity κ_m was defined so that, once the pulses of level m are homogenized, the operator $\partial_t + \mathbf{b}_m \cdot \nabla - \kappa_m \Delta$ of level m becomes the operator of level $m-1$, with $\mathbf{b}_{m-1}$ and κ_{m-1}. The proof runs two dual cascades on the levels (Figure 5.1). The *homogenization cascade* goes *up the levels* one at a time, from M to a fixed top level m_0, toward coarser levels and forward in time; the *regularity cascade* then goes back down, toward finer levels, at one fixed time. The proof has four stages, and Proposition 5.2 runs both cascades between the seed and the top level.

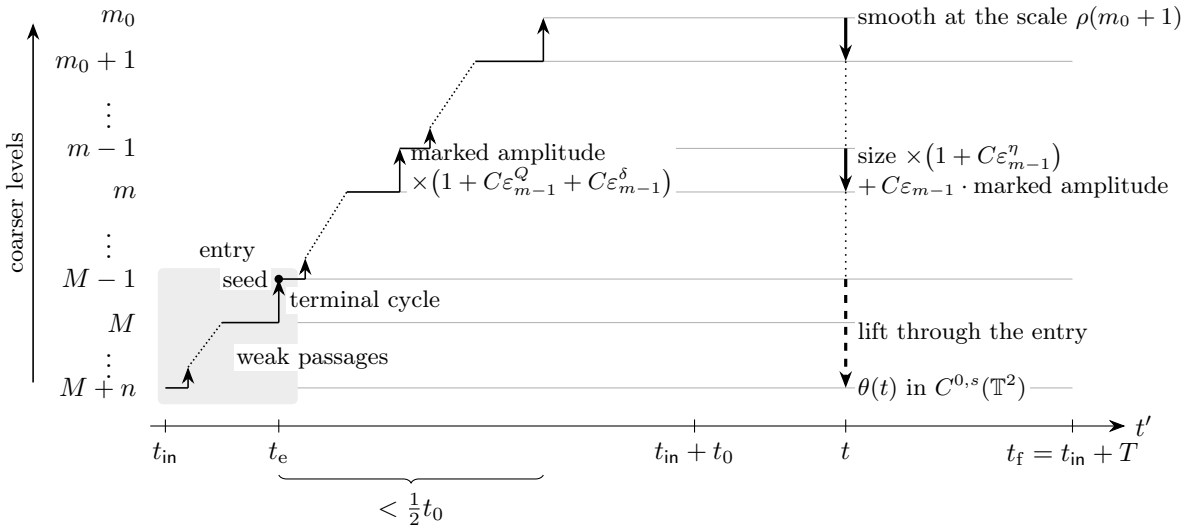


Figure 5.1: The induction of Subsection 5.2, levels against time t' (not to scale), coarser levels higher. The homogenization cascade, up the levels and forward in time: the entry (Proposition 5.1) takes θ, at level $M+n$ and time t_{in}, to the seed, at level $M-1$ and time t_{e}; each up arrow is then a cycle (Proposition 5.3), with the factor of (5.3), and the cycles end less than $\frac{1}{2}t_0$ after t_{e}. Grey: the solution U_m of each family, on $[t_{(m)}, t_{\mathsf{f}}]$. The regularity cascade, back down the levels at one time $t \in [t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T]$: each down arrow is a passage (Proposition 5.4), with the bound (5.4), and the dashed arrow is the lift through the entry. Dotted: levels not drawn.

1. *The entry* (Proposition 5.1) disposes of the levels $M+n, \dots, M$. At the levels finer than M, the Péclet number $\varepsilon_m^{\beta}/\kappa$ is small and molecular diffusion over the quiet margins smooths at the wavelength of the next coarser level, so these levels are crossed at the same diffusivity κ. At level M a full homogenization cycle passes to κ_{M-1}. The output is a system at level $M-1$, the *seed*, whose solution, lifted back to level $M+n$, reproduces θ up to a small error in $C^{0,s}(\mathbb{T}^2)$.

2. *The homogenization cascade* (Proposition 5.2) then homogenizes the levels $M-1, \dots, m_0+1$ one at a time. Each step up is a homogenization cycle (Proposition 5.3).

3. *At the top level* m_0, a fixed level independent of M and κ, the solution is smooth at a fixed scale.

4. *The regularity cascade* (Proposition 5.2) then goes back down the levels $m_0+1, \dots, M-1$ at one fixed time. Each step down is a passage (Proposition 5.4), and the lifts through the entry (Proposition 5.1) carry the Hölder bound from the seed to θ.

What the homogenization cascade carries up the levels. A cycle compares the solution at level m with a solution at level $m-1$, regenerating derivatives and keeping the marked amplitude, defined below, up to the factor in (5.3). The object passed from level m to level $m-1$ is a coupled system of equations with the operator of its level, a *family* (Subsection 5.4). Starting from the seed, the cycles produce families G_m, with solutions U_m on time windows $[t_{(m)}, t_{\mathrm{f}}]$ that end at a common final time t_{f}, where $t_{(M-1)} = t_{\mathrm{e}}$ and, for $m < M-1$, $t_{(m)}$ is the preparation time t_{pr} of the cycle at level $m+1$ (Proposition 5.3). The *marked amplitude* of U_m is its norm $\mathsf{A}_m := \|U_m\|_{L^\infty([t_{(m)},t_{\mathrm{f}}];L^2_{G_m}(\mathbb{T}^2))}$, where $\|\cdot\|_{L^2_{G_m}(\mathbb{T}^2)}$ is the marked norm of G_m (Subsection 5.4). One cycle costs

$$\mathsf{A}_{m-1} \le \big[1 + C\big(\varepsilon^Q_{m-1} + \varepsilon^\delta_{m-1}\big)\big]\mathsf{A}_m\,, \tag{5.3}$$

where ε^Q_{m-1} is the accuracy of the comparison and ε^δ_{m-1} is the cost of the retained trace, from which the solution of the successor starts (Subsection 5.5); Proposition 6.2 prevents growth along a family. The exponent Q, the order N_{ret} below and the other parameters of the proof are fixed in Subsection 5.3.

What the regularity cascade carries back down. The $C^{0,s}(\mathbb{T}^2)$ norm is carried by a *representation*: a smooth base plus oscillating profiles, each measured at its wavelength (Subsection 5.6). A representation is a Hölder bound together with bounds on derivatives at the scale of the current wavelength, which the correctors of the next finer level use. Read on representations, a cycle is a *passage*: it carries a representation of $U_{m-1}(t)$ at wavelength ε_{m-1} to a representation of $U_m(t)$ at wavelength ε_m, and the profiles of the representation of $U_{m-1}(t)$ are copied into that of $U_m(t)$ by a constant map (Proposition 5.4). With $R_m(t)$ a representation of $U_m(t)$, one passage costs

$$\mathrm{size}_{\varepsilon_m}(R_m(t)) \le \big(1 + C\varepsilon^\eta_{m-1}\big)\,\mathrm{size}_{\varepsilon_{m-1}}(R_{m-1}(t)) + C\varepsilon_{m-1}\mathsf{A}_m\,, \tag{5.4}$$

where η is the exponent of (2.3). The factor collects the readout (Subsection 5.5), the exponential part of the lift and the correctors of the passage, which cost (5.2), with $\eta \le \mu/4$; the additive term collects the comparison error of Proposition 5.3 and what the lift adds to the high frequencies of $U_{m-1}(t)$ (the idea of the proof of Proposition 5.4; Table 14.1). At the top level, the size of a representation of $U_{m_0}(t)$ is at most a constant, depending on m_0, times the marked amplitude A_{m_0+1}: the third bound of Proposition 5.3(b) at level m_0+1 gives N_{ret} derivatives of $U_{m_0}(t)$ at the scale $\rho(m_0+1)$, and these bound the size at the wavelength ε_{m_0}, by Sobolev embedding, since $N_{\mathrm{ret}} \ge N_{\mathrm{rep}} + 2$ by (14.3).

Why the products converge. Since $\varepsilon_0 = 1$ and $\varepsilon_{m-1}/\varepsilon_m \ge \Lambda \ge 128$ for $m \ge 1$, we have $\varepsilon_m \le \Lambda^{-m}$, so the sums $\sum_m \varepsilon^Q_{m-1}$, $\sum_m \varepsilon^\delta_{m-1}$ and $\sum_m \varepsilon^\eta_{m-1}$ are bounded independently of the number of levels, and so are the products $\prod_m(1 + C\varepsilon^Q_{m-1} + C\varepsilon^\delta_{m-1})$ and $\prod_m(1 + C\varepsilon^\eta_{m-1})$. Iterating (5.3) along the homogenization cascade and (5.4) along the regularity cascade therefore bounds the size of the representation at the finest level by a constant times the norm $\|g\|_{L^2_\omega(\mathbb{T}^2)}$ of the seed's datum g, uniformly in the number of levels, hence in M and κ (Proposition 5.2); this norm is the marked norm, since the markers of a seed are 1.

5.3 Parameters

Fix $\Lambda \ge \Lambda_0$, with Λ_0 as in Proposition 4.1, an integer $n \ge 1$ and $0 < t_0 < T$. The proof fixes the following parameters, all depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$:

- the corrector order N_{cor}, the largest degree of the correctors added by a lift (a corrector of degree j carries j derivatives, of the field or of the coefficients);

- the truncation order p of the exponential in a lift;
- the order $N_{\mathrm{ret}} \geq N_{\mathrm{cor}} + 2p$ up to which a cycle controls the derivatives of the solution of a successor, and the order $N_{\mathrm{out}} := N_{\mathrm{ret}} + 4p$ of the preparations;
- the order N_{rep} of the representations, which is fixed by the entry (Section 15);
- the order $N_{\mathrm{src}} \geq 2$ of the source terms of the families;
- an accuracy exponent $Q > 0$;
- the bounds on generation and word length of the successors (Subsection 5.4);
- the *top level* $m_0 \geq 2$, chosen after the others: the largest of the thresholds on the level in Sections 11–14 (item (4) of Subsection 11.1).

Table 11.1 lists them, with the parameters of Part II, in the order in which they are chosen. The preparations of level m are smooth at the effective diffusion length over one renewal time (Section 2),

$$\rho(m) = (\kappa_{m-1} h_m)^{1/2} . \tag{5.5}$$

The notions of family, admissible family, lift and representation below use these choices, families use the effective diffusivities of $\kappa \in \mathcal{K}_M$, and time windows have length at most T.

5.4 Families

A comparison between levels m and $m-1$ leaves error terms carrying derivatives of the coarse solution. Estimating them would cost derivatives at every level, which the induction cannot afford, so they become new unknowns, each solving the equation of its level forced by smoothed derivatives of earlier unknowns. These new sources are the higher-order terms of the averaged equation that the lift of Subsection 5.5 produces: the defect of the averaged flux, the means of the lift along the paths of sources of the family, and their conjugates by the truncated exponential. They, and the old sources carried over through their cell averages, are listed in Definition 11.3.

Families and their solutions. A family is the data of a system of linear equations of level m, one for each of its components, coupled through smoothed derivatives; a solution is a vector of functions on $[t_1, t_2] \times \mathbb{T}^2$, one for each component. Precisely, a *family* G of level m consists of

- a finite index set V, whose elements are the *components* (labels, which are words in the families of the proof, as described below), with *masses* $\omega_v > 0$ and *markers* $\Psi_v \geq 1$, $v \in V$;
- finitely many *sources* σ, each from a component v' to a component v (written $\sigma \colon v' \to v$; there may be several sources, or none, from v' to v), with a linear smoothing operator $S_\sigma(t)$ of width $r_\sigma > 0$, and coefficients $A_{\sigma,d}$, $2 \leq d \leq N_{\mathrm{src}}$;
- constants $c_j \geq 0$ such that the operators $S_\sigma(t)$, which commute with derivatives, satisfy, for every source σ, time t, $f \in L^2(\mathbb{T}^2)$ and $j \leq N_{\mathrm{src}} - 2$,

$$\|\nabla^j S_\sigma(t) f\|_{L^2(\mathbb{T}^2)} \leq c_j r_\sigma^{-j} \|f\|_{L^2(\mathbb{T}^2)} .$$

A *datum* of G is $g = (g_v)_{v\in V} \in L^2(\mathbb{T}^2)^V$, and the *solution* of G from g at time t_1, on a time window $[t_1, t_2]$ on which the hypotheses of Proposition 6.2 hold, is the solution $U = (U_v)_{v\in V}$ of

$$\begin{cases} \partial_t U_v + \mathbf{b}_m \cdot \nabla U_v - \kappa_m \Delta U_v \\ \qquad = \nabla \cdot \sum_{v'\in V} \sum_{\sigma\colon v'\to v} \sum_{d=2}^{N_{\mathrm{src}}} A_{\sigma,d} \nabla^{d-1}(S_\sigma U_{v'}) & \text{in } (t_1,t_2)\times\mathbb{T}^2\,, \qquad v\in V\,, \\ U_v(t_1,\cdot) = g_v & \text{on } \mathbb{T}^2\,, \end{cases} \tag{5.6}$$

in the class of Proposition 6.2, where the inner sum runs over the sources from v' to v, so that the right side collects all the sources into v; this is (6.3) with ϕ_m and κ_m in place of Φ and ν. The smoothing operators are products of two smoothings along ellipses (Proposition 6.1). For the sources that a cycle creates at level m the ellipses are thin. The averaged flux of a pulse is enhanced only along its direction of flow, and the two ellipses follow the averaged flux of a horizontal and of a vertical pulse, so that their axes are in a ratio of order $(\kappa_m/\kappa_{m-1})^{1/2}$, which is small below the terminal level M (Proposition 2.1(c)); together they smooth in every direction.

Norms. Fields of G are measured in the norms

$$\begin{aligned} \|g\|^2_{L^2_\omega(\mathbb{T}^2)} &\coloneqq \sum_{v\in V} \frac{\|g_v\|^2_{L^2(\mathbb{T}^2)}}{\omega_v}\,, \\ \|g\|^2_{L^2_G(\mathbb{T}^2)} &\coloneqq \sum_{v\in V} \frac{\Psi_v^2\,\|g_v\|^2_{L^2(\mathbb{T}^2)}}{\omega_v}\,, \\ \|g\|^2_{H^N_{\rho,G}(\mathbb{T}^2)} &\coloneqq \sum_{v\in V} \frac{\Psi_v^2\,\|g_v\|^2_{H^N_\rho(\mathbb{T}^2)}}{\omega_v}\,, \end{aligned} \tag{5.7}$$

where, for $\rho > 0$ and an integer $N \geq 0$,

$$\|f\|^2_{H^N_\rho(\mathbb{T}^2)} \coloneqq \sum_{j\leq N} \frac{\rho^{2j}}{(j!)^2}\,\|\nabla^j f\|^2_{L^2(\mathbb{T}^2)} \tag{5.8}$$

measures N derivatives at the length scale ρ. The norm $\|\cdot\|_{L^2_G(\mathbb{T}^2)}$ is the *marked norm*.

The coupling matrix. The coupling of the components is measured by the *coupling matrix* of G on a time window $[t_1, t_2]$,

$$B_{vv'} \coloneqq \Big(\frac{\omega_{v'}}{\omega_v}\Big)^{1/2} \sum_{\sigma\colon v'\to v} \sum_{d=2}^{N_{\mathrm{src}}} c_{d-2}\,\frac{\sup_{[t_1,t_2]\times\mathbb{T}^2} |A_{\sigma,d}|}{\kappa_m\, r_\sigma^{d-2}}\,, \qquad v, v'\in V\,, \tag{5.9}$$

with the constants c_j of G. By Schur's test, the operator norm of $(B_{vv'})$ on $\mathbb{R}^V$ with the Euclidean norm is at most the geometric mean of its largest row sum and its largest column sum. By Proposition 6.2, if all the row and column sums of the coupling matrix are at most $\frac{1}{8}$, then the energy $\|U(t)\|_{L^2_\omega(\mathbb{T}^2)}$ does not grow. Indeed, testing the equation for U_v with U_v/ω_v and writing $\nabla^{d-1} S_\sigma U_{v'} = \nabla^{d-2} S_\sigma \nabla U_{v'}$, we bound the contribution of the sources from v' to v by $\kappa_m B_{vv'} x_v x_{v'}$, where $x_v \coloneqq \omega_v^{-1/2}\|\nabla U_v\|_{L^2(\mathbb{T}^2)}$. By Schur's test the coupling is then at most one eighth of the dissipation $\kappa_m \sum_{v\in V} x_v^2$. The coupling matrix *for the masses* ω_v/Ψ_v^2 is (5.9) with ω_v/Ψ_v^2 in place of ω_v; with these masses the norm $\|\cdot\|_{L^2_\omega(\mathbb{T}^2)}$ is the marked norm.

Successors. A cycle turns a family G of level m into a *successor* G' of level $m-1$. Each component v of G is kept and becomes the *parent* of new components, its *children*, labelled by words whose letters name the source terms the comparison creates. A new component has mass ω_v times a positive power of ε_{m-1} and marker Ψ_v times a negative power of ε_{m-1}, so the markers grow with the generation (the number of creations). A successor is a family of Definition 11.3, which gives its letters, the words of its components and its sources; it keeps the components of bounded generation and bounded word length. The solution is controlled in the marked norm, so the components of high generation are small in $\|\cdot\|_{L^2_\omega(\mathbb{T}^2)}$ and the flux into the discarded ones is negligible (Lemma 13.9); components with longer words are small because each letter carries a positive power of ε_{m-1}.

Seeds and admissible families. A *seed* is a family whose components have generation 0 and marker 1. A family G of level m is *admissible* on a time window if it descends by cycles from an *admissible seed*, a seed whose coefficients obey the bounds of Part II (Definition 11.2), each cycle, at a level m', moving the start of the window forward by $3\tau_{m'}$ (Definition 11.4). Its coefficients then obey the bounds of Part II (Proposition 11.5), and the row and column sums of its coupling matrices, for the masses ω_v and ω_v/Ψ_v^2, are at most a constant times a fixed positive power of ε_{m-1}.

5.5 Lifts

Let G' be a successor of a family G of level m. The *lift* $Y^\sharp$ of a field Y of G' to the components of G is built in three steps:

$$(\Upsilon_m Y)_v := Y_v + \sum_{v' \text{ child of } v} \varepsilon_{m-1}^{\eta\,\#v'}\, Y_{v'}\,, \qquad Y^\sharp(t) := \mathrm{Corr}_m\big[\exp_p\big(\Gamma_m(t) : \nabla^2\big)\, \Upsilon_m Y(t)\big], \tag{5.10}$$

where the sum runs over the children of v (Subsection 5.4), $\#v'$ is the number of letters of the word of v', η is the exponent of (2.3), and Corr_m is the corrector operator of Definition 10.8, described below. First, the readout Υ_m adds each new component to its parent with a weight that is a positive power of ε_{m-1}. Second, the truncated exponential of $\Gamma_m(t) : \nabla^2 := \sum_{i,i'} (\Gamma_m)_{ii'}(t)\partial_i\partial_{i'}$ is applied, where

$$\exp_p\big(\pm\Gamma_m(t) : \nabla^2\big) := \sum_{l=0}^{p} \frac{1}{l!}\big(\pm\Gamma_m(t) : \nabla^2\big)^l\,, \qquad \Gamma_m(t) := \int_0^t \big(\mathbf{J}_m^{\kappa_m}(t') - \kappa_{m-1} I\big)\,\mathrm{d}t'\,. \tag{5.11}$$

The *covariance* $\Gamma_m(t)$ of the level-m pulses is the accumulated surplus of their averaged flux $\mathbf{J}_m^{\kappa_m}$ over its average $\kappa_{m-1}I$. Since $\partial_t\Gamma_m(t) = \mathbf{J}_m^{\kappa_m}(t) - \kappa_{m-1}I$ and operators with constant coefficients in space commute, the full exponential $\exp(\Gamma_m(t) : \nabla^2)$ would turn solutions of $\partial_t - \kappa_{m-1}\Delta$ into solutions of $\partial_t - \mathbf{J}_m^{\kappa_m}(t) : \nabla^2$; as $\Gamma_m(t)$ is symmetric and in general indefinite, this operator is unbounded on $L^2(\mathbb{T}^2)$, so the lift truncates it. The covariance is small at the scale $\rho(m)$: by Lemmas 10.3(b) and 13.7(iii), for every t,

$$\begin{aligned} |\Gamma_m(t)| &\le C\kappa_{m-1}\tau'_m \le C\varepsilon_{m-1}^{\delta}\rho(m)^2\,, \\ \big\|\big(\exp_p(\pm\Gamma_m(t) : \nabla^2) - I\big)F\big\|_{H^N_{\rho(m)}(\mathbb{T}^2)} &\le C\varepsilon_{m-1}^{\delta}\|F\|_{H^{N+2p}_{\rho(m)}(\mathbb{T}^2)}\,; \end{aligned}$$

this is the origin of the term $\varepsilon_{m-1}^{\delta}$ in the first bound of Proposition 5.3(b). Third, Corr_m (Definition 10.8) adds on each pulse the correctors of degrees up to N_{cor}, which oscillate at the wavelength ε_m in the Lagrangian coordinates of the pulse; they change a field by the corrector amplitude $\bar\varepsilon_m$ times its derivatives. The *retained trace* at time t_{pr} of a datum g of G is the datum of G' given by the truncated inverse of the second step, exact up to terms of degree at least $p+1$ in $\Gamma_m(t_{\mathrm{pr}}) : \nabla^2$: it is $\exp_p(-\Gamma_m(t_{\mathrm{pr}}) : \nabla^2)\, g_v$ on each component v of G and 0 on the new components.

5.6 Representations

Fields U of a family take values in $\mathbb{R}^V$, normed pointwise by $|U(x)|_\omega := (\sum_v \omega_v^{-1} U_v(x)^2)^{1/2}$, and $\|\cdot\|_{L^\infty(\mathbb{T}^2)}$ denotes the supremum of this pointwise norm; the Hölder norm and seminorm (3.1) of such fields use it in place of $|\cdot|$. For an integer $N \geq 1$, $0 < \lambda \leq 1$ and $F \in C^N(\mathbb{T}^2)^V$ we use the norm and seminorm at the scale λ

$$[F]_{C^N_\lambda(\mathbb{T}^2)} := \sum_{1 \leq |j| \leq N} \lambda^{|j|} \frac{\|\partial^j F\|_{L^\infty(\mathbb{T}^2)}}{j!}, \qquad \|F\|_{C^N_\lambda(\mathbb{T}^2)} := \|F\|_{L^\infty(\mathbb{T}^2)} + [F]_{C^N_\lambda(\mathbb{T}^2)}, \tag{5.12}$$

where $j = (j_1, j_2)$ runs over multi-indices, $|j| := j_1 + j_2$ and $j! := j_1!\, j_2!$. A *representation* R of U at wavelength λ_* is a decomposition into functions in $C^{N_{\mathrm{rep}}}(\mathbb{T}^2)^V$, a *base* E and a finite collection $\mathcal{P}$ of *profiles* P (repetitions allowed), each with a wavelength $\lambda_P \in [\lambda_*, 1]$, such that $U = E + \sum_{P \in \mathcal{P}} P$. Its *size* is

$$\mathrm{size}_{\lambda_*}(R) := \|E\|_{L^\infty(\mathbb{T}^2)} + [E]_{C^{0,s}(\mathbb{T}^2)} + \lambda_*^{-s} [E]_{C^{N_{\mathrm{rep}}}_{\lambda_*}(\mathbb{T}^2)} + \sum_{P \in \mathcal{P}} \lambda_P^{-s} \|P\|_{C^{N_{\mathrm{rep}}}_{\lambda_P}(\mathbb{T}^2)}. \tag{5.13}$$

Then $\|U\|_{C^{0,s}(\mathbb{T}^2)} \leq 3\,\mathrm{size}_{\lambda_*}(R)$, by Lemma 14.2(a). The readout Υ_m, like every constant linear map, maps a representation R of a field Y of G' to a representation of $\Upsilon_m Y$ of size at most $|\Upsilon_m|\,\mathrm{size}_{\lambda_*}(R)$, where $|\Upsilon_m|$ is its operator norm for the norms $|\cdot|_\omega$. Each profile keeps its wavelength: measured at $\lambda_* < \lambda_P$, its term $\lambda_P^{-s}\|P\|_{L^\infty(\mathbb{T}^2)}$ would grow by the factor $(\lambda_P/\lambda_*)^s$.

5.7 Initialization and induction

Proposition 5.1 (Initialization from rough data). *There exist $M_*(\alpha, s, q, \zeta, \Lambda, n, t_0, T) \in \mathbb{N}$ and $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ with the following property. For every $M \geq M_*$, $\kappa \in \mathcal{K}_M$ and $t_{\mathsf{in}} \in \mathbb{R}$, with $t_{\mathrm{f}} := t_{\mathsf{in}} + T$, there exist $t_{\mathrm{e}} \in [t_{\mathsf{in}}, t_{\mathsf{in}} + \frac{1}{4}t_0]$ and an admissible seed G of level $M-1$ on $[t_{\mathrm{e}}, t_{\mathrm{f}}]$ such that the following holds. For every $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$ and every weak solution θ of (4.1), there exists a datum g of G with $\|g\|_{L^2_\omega(\mathbb{T}^2)} \leq C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ such that the solution U of G from g at time t_{e} has the following property: for every interval $[t_1, t_2] \subset [t_{\mathsf{in}} + \frac{1}{4}t_0, t_{\mathrm{f}}]$ and every $\Xi \geq 0$, if, for almost every $t \in [t_1, t_2]$, there exists a representation $R(t)$ of $U(t)$ at wavelength ε_{M-1} with $\mathrm{size}_{\varepsilon_{M-1}}(R(t)) \leq \Xi$, then*

$$\|\theta\|_{L^\infty([t_1,t_2];C^{0,s}(\mathbb{T}^2))} \leq C\big(\Xi + \|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\big).$$

Idea of the proof. At the levels $m \in (M, M+n]$ the Péclet number $\varepsilon_m^\beta/\kappa$ is at most $C\varepsilon_{m-1}^{(q+2)\gamma}$, and molecular diffusion over a quiet margin reaches the wavelength ε_{m-1} of the next coarser level: $\varepsilon_{m-1}^2/(\kappa\tau'_m) \leq C\varepsilon_{m-1}^{\gamma-3\delta}$, with $\gamma > 3\delta$. So a *weak passage* compares the solution with a family of level $m-1$ at the same diffusivity, which keeps the enhancement of level m as a source instead of adding it to the diffusivity, and regains derivatives on the margins. After n of them the sources are smoothed once, by isotropic paired convolutions of width slightly below ε_M. At level M a margin is too short, $\kappa\tau'_M \asymp \varepsilon_{M-1}^{2+\gamma+3\delta} \ll \varepsilon_{M-1}^2$, and a *terminal cycle*, prepared over a time of order h_M by the method of Proposition 5.3, passes to $\kappa_{M-1} \in [\kappa, C\kappa]$ and yields the seed. For Proposition 4.1, θ is, up to $C\varepsilon_{M-1}^2\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ in $L^\infty([t_{\mathsf{in}} + \frac{1}{4}t_0, t_{\mathsf{in}} + T]; C^{0,s}(\mathbb{T}^2))$, the lift of U through the terminal cycle and the n weak passages (variants of (5.10), defined in Section 15), and these lifts carry a representation at ε_{M-1} to one at ε_{M+n} with fewer than N_{rep} derivatives, at a constant cost.

Proposition 5.2 (Induction on the levels). *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ with the following property. Let $m_0 < m < M$ and $\kappa \in \mathcal{K}_M$, let $t_{\mathrm{S}}, t_{\mathrm{f}} \in \mathbb{R}$ with $t_{\mathrm{S}} + \frac{1}{2}t_0 < t_{\mathrm{f}} \leq t_{\mathrm{S}} + T$, let G be an admissible seed of level m on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$, let g be a datum of G, and let U be the solution of G*

from g at time t_{S}. Then, for almost every $t \in [t_{\mathrm{S}} + \frac{1}{2}t_0, t_{\mathrm{f}}]$, there exists a representation $R(t)$ of $U(t)$ at wavelength ε_m such that

$$\mathrm{size}_{\varepsilon_m}(R(t)) \le C\|g\|_{L^2_\omega(\mathbb{T}^2)}\,.$$

The constant is independent of m, that is, of the number of cycles. *Idea of the proof.* This is the induction of Subsection 5.2: we apply Propositions 5.3 and 5.4 to G and its iterated successors, which gives (5.3) and (5.4), with $G = G_m$, $G' = G_{m-1}$, $U = U_m$, $Y = U_{m-1}$, $t_{\mathrm{pr}} = t_{(m-1)}$, $R = R_m(t)$ and $R' = R_{m-1}(t)$. The times of the cycles are chosen recursively (Step 2 of the proof in Subsection 14.4), and they are less than $t_{\mathrm{S}} + \frac{1}{2}t_0$, since m_0 is chosen so that $\sum_{j=m_0+1}^{m}(3\tau_j + 4h_j) < \frac{1}{2}t_0$.

Proof of Proposition 4.1. Let $\Lambda_0(\alpha, s, q, \zeta) \ge 128$ be at least the thresholds of Proposition 2.1 and of the statements of Part II; this choice is the first row of the table of the order of the parameters in Subsection 11.1. Let $\Lambda \ge \Lambda_0$, $n \ge 1$ and $0 < t_0 < T$ be given. Let M_* be as in Proposition 5.1, with m_0 the top level of Subsection 5.3, and let $M \ge M_1 := \max\{M_*, m_0 + 2\}$. Let G be the seed of Proposition 5.1 and g its datum. Proposition 5.2 applies to G and g at level $M-1$, with $t_{\mathrm{S}} = t_{\mathrm{e}}$ and $t_{\mathrm{f}} = t_{\mathsf{in}} + T$, since $m_0 < M-1 < M$, $t_{\mathrm{e}} + \frac{1}{2}t_0 \le t_{\mathsf{in}} + \frac{3}{4}t_0 < t_{\mathsf{in}} + T$ and $t_{\mathsf{in}} + T \le t_{\mathrm{e}} + T$; it gives for almost every $t \in [t_{\mathrm{e}} + \frac{1}{2}t_0, t_{\mathsf{in}} + T] \supset [t_{\mathsf{in}} + t_0, t_{\mathsf{in}} + T]$ a representation $R(t)$ of $U(t)$ at wavelength ε_{M-1} with $\mathrm{size}_{\varepsilon_{M-1}}(R(t)) \le C\|g\|_{L^2_\omega(\mathbb{T}^2)} \le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. Proposition 5.1 then applies with $[t_1, t_2] = [t_{\mathsf{in}} + t_0, t_{\mathsf{in}} + T]$ and $\Xi = C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$, and bounds $\|\theta\|_{L^\infty([t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T]; C^{0,s}(\mathbb{T}^2))}$ by $C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. □

5.8 One homogenization cycle and the passage to the next level

A cycle at level m has two parts (Subsection 5.2): the preparation, part (a) below, regenerates derivatives within one period $4h_m$, and the comparison, part (b), bounds the difference between the solution and the lift of the solution of a successor. Proposition 5.4 reads the cycle on representations.

Proposition 5.3 (One homogenization cycle)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ with the following property. Let $M \ge 1$, $\kappa \in \mathcal{K}_M$ and $m_0 < m < M$, let $t_{\mathrm{A}}, t_{\mathrm{c}}, t_{\mathrm{f}} \in \mathbb{R}$ with $t_{\mathrm{A}} + 3\tau_m \le t_{\mathrm{c}}$ and $t_{\mathrm{c}} + 4h_m \le t_{\mathrm{f}}$, let G be an admissible family of level m on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$, and let G' be a successor of G. There exist $t_{\mathrm{pr}} \in [t_{\mathrm{c}}, t_{\mathrm{c}} + 4h_m]$ and a linear map $\Pi\colon L^2(\mathbb{T}^2)^V \to H^{N_{\mathrm{out}}}(\mathbb{T}^2)^V$ such that, for every datum g of G, the solution U of G from g at time t_{c} and the solution Y of G' from the retained trace of Πg at time t_{pr} satisfy the following.*

(a) *The preparation Πg approximates $U(t_{\mathrm{pr}})$, is almost no larger than the datum, and has N_{out} derivatives at the scale $\rho(m)$:*

$$\|\Pi g - U(t_{\mathrm{pr}})\|_{L^2_G(\mathbb{T}^2)} \le C\varepsilon^Q_{m-1}\|g\|_{L^2_G(\mathbb{T}^2)}\,, \qquad \|\Pi g\|_{L^2_G(\mathbb{T}^2)} \le \big(1 + C\varepsilon^Q_{m-1}\big)\|g\|_{L^2_G(\mathbb{T}^2)}\,,$$
$$\|\Pi g\|_{H^{N_{\mathrm{out}}}_{\rho(m),G}(\mathbb{T}^2)} \le C\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

(b) *The successor G' is admissible on $[t_{\mathrm{A}} + 3\tau_m, t_{\mathrm{f}}]$, and*

$$\|Y\|_{L^\infty([t_{\mathrm{pr}}, t_{\mathrm{f}}]; L^2_{G'}(\mathbb{T}^2))} \le \big(1 + C(\varepsilon^Q_{m-1} + \varepsilon^\delta_{m-1})\big)\|g\|_{L^2_G(\mathbb{T}^2)}\,,$$
$$\|U - Y^\sharp\|_{L^\infty([t_{\mathrm{pr}}, t_{\mathrm{f}}]; L^2_\omega(\mathbb{T}^2))} + \kappa_m^{1/2}\|\nabla(U - Y^\sharp)\|_{L^2((t_{\mathrm{pr}}, t_{\mathrm{f}}); L^2_\omega(\mathbb{T}^2))} \le C\varepsilon^Q_{m-1}\|g\|_{L^2_G(\mathbb{T}^2)}\,,$$
$$\|Y\|_{L^\infty([t_{\mathrm{pr}}, t_{\mathrm{f}}]; H^{N_{\mathrm{ret}}}_{\rho(m),G'}(\mathbb{T}^2))} \le C\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

In (a), the preparation time t_{pr} is chosen so that the velocity added at level m vanishes near it: $\mathbf{b}_m = \mathbf{b}_{m-1}$ on $[t_{\mathrm{pr}} - \tau'_m, t_{\mathrm{pr}} + \tau'_m] \times \mathbb{T}^2$ (Proposition 12.1(i)), so that Corr_m adds no corrector

at t_{pr} and $Y^\sharp(t_{\mathrm{pr}}) = \exp_p(\Gamma_m(t_{\mathrm{pr}}) : \nabla^2)\exp_p(-\Gamma_m(t_{\mathrm{pr}}) : \nabla^2)\,\Pi g$ (Lemma 13.1). Since Πg has $N_{\mathrm{out}} = N_{\mathrm{ret}} + 4p$ derivatives and the retained trace uses $2p$ of them, $N_{\mathrm{ret}} + 2p$ remain; on $[t_{\mathrm{pr}}, t_{\mathrm{f}}]$ the third bound of (b) gives N_{ret} derivatives of Y, at least the $2p + N_{\mathrm{cor}}$ that its lift uses ($2p$ for the truncated exponential and N_{cor} for the correctors).

Idea of the proof. (a): in the Duhamel expansion of U in the sources, each term alternates free evolutions under the operator of level m with single applications of a source. The Bloch transform splits a free evolution, in the Lagrangian coordinates of a renewal block and with the conductivity frozen at a point, into evolutions on the period cell, one for each frequency. The principal branch of these cell evolutions dissipates over a complete renewal block inside $[t_{\mathrm{c}}, t_{\mathrm{c}} + 4h_m]$ at a rate proportional to the square of the frequency, in every direction and uniformly in the level, and the two orientations of the pulses damp the high frequencies. (b): the correctors cancel the leading terms of $(\partial_t + \mathbf{b}_m \cdot \nabla - \kappa_m \Delta)Y^\sharp$. The errors from switching the correctors on and off in the intervals between consecutive pulses are exponentially small, the errors of the smoothing operators are small by the estimates of Proposition 6.1, and the energy estimate with forcing at order zero (Proposition 8.3(a) at $N = 0$) bounds the error by the residual, which is controlled by the derivatives of (a). The coefficients of the successor are explicit combinations of those of G, each carrying a power of ε_{m-1}.

Proposition 5.4 (Passage between consecutive levels)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ with the following property. For every $M \geq 1$, $\kappa \in \mathcal{K}_M$, $m_0 < m < M$ and every G and G' as in Proposition 5.3 at level m, there exist t_{pr} and Π satisfying (a) and (b) of Proposition 5.3 with this C such that, for every datum g of G, with U and Y as there, and every $t \in [t_{\mathrm{pr}}, t_{\mathrm{f}}]$, the following hold: $U(t)$ and $Y(t)$ are of class $C^{N_{\mathrm{rep}}}$ on $\mathbb{T}^2$, and for every representation R' of $Y(t)$ at wavelength ε_{m-1} there exists a representation R of $U(t)$ at wavelength ε_m such that*

$$\mathrm{size}_{\varepsilon_m}(R) \leq \left(1 + C\varepsilon_{m-1}^{\eta}\right)\mathrm{size}_{\varepsilon_{m-1}}(R') + C\varepsilon_{m-1}\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

Proposition 5.3 is proved in Sections 12 and 13, and Proposition 5.4 in Section 14. *Idea of the proof.* Let S be a smoothing below the scale $\rho(m)$. Then U is the sum of $\Upsilon_m Y$ (old profiles copied by a constant map), the exponential and corrector parts of the lift of SY and of $(I - S)Y$, and the error $U - Y^\sharp$. On SY the corrector part costs (5.2) and the exponential part a power of ε_{m-1}. On $(I - S)Y$ the moments of S (Proposition 6.1) make both parts small, and $U - Y^\sharp$, which is small in $L^2_\omega(\mathbb{T}^2)$ by Proposition 5.3(b) and smooth at the molecular radius $\rho_{\mathrm{mol}}(m)$ of (11.1), is small at the wavelength ε_m by interpolation (Step 6 of the proof in Section 14).

6. Energy estimates and paired smoothing

Fix a smooth function $u \geq 0$ of integral one supported in the unit disc, and, for an integer $\ell \geq 1$, let W_ℓ be the kernel with Fourier transform

$$\widehat{W_\ell}(\xi) \coloneqq 1 - \left(1 - \left|\widehat{u}(\xi/(2\ell))\right|^2\right)^\ell, \qquad \xi \in \mathbb{R}^2\,. \tag{6.1}$$

It is smooth, supported in the unit disc, of integral one, with vanishing moments of orders 1 to $2\ell - 1$, and $\widehat{W_\ell}$ takes values in $[0, 1]$. For a linear map Ω of $\mathbb{R}^2$ and $r > 0$, the operator

$$S_{\ell,r,\Omega} g(y) \coloneqq \int_{\mathbb{R}^2} W_\ell(z)\, g(y - r\Omega z)\,\mathrm{d}z \tag{6.2}$$

takes the mean of g over the ellipse $\{y - r\Omega z : |z| < 1\}$ (a segment or a point if Ω is singular) with the signed weight W_ℓ.

Proposition 6.1 (Paired smoothing in eccentric geometry)**.** *Let $\ell \geq 1$ and $1 \leq N_1 \leq N_2$ be integers with $N_1 < 2\ell$. There exists $C(u, \ell, N_1, N_2) < \infty$ with the following property. Let $C_\Omega > 0$, and let Ω_1, Ω_2 be invertible linear maps of $\mathbb{R}^2$ with $|\Omega_i|_\infty \leq C_\Omega$ whose vectors $\Omega_1 e_1$, $\Omega_2 e_2$ are uniformly transverse, in the sense that the matrix with these columns has an inverse of $|\cdot|_\infty$-norm at most 2. Then, for every $\rho > 0$, $0 < r \leq \rho$ and $g \in H^{N_1}(\mathbb{T}^2)$, the operator $S := S_{\ell,r,\Omega_1} S_{\ell,r,\Omega_2}$ satisfies*

$$\|Sg\|_{H^{N_2}_\rho(\mathbb{T}^2)} \leq C\Big(\frac{\rho}{r}\Big)^{N_2-N_1} \|g\|_{H^{N_1}_\rho(\mathbb{T}^2)}\,, \qquad \|Sg\|_{H^{N_1}_\rho(\mathbb{T}^2)} \leq \|g\|_{H^{N_1}_\rho(\mathbb{T}^2)}\,,$$
$$\|g - Sg\|_{L^2(\mathbb{T}^2)} \leq C C_\Omega^{N_1} \Big(\frac{r}{\rho}\Big)^{N_1} \|g\|_{H^{N_1}_\rho(\mathbb{T}^2)}\,.$$

The third bound holds for all linear maps Ω_1, Ω_2 of $\mathbb{R}^2$ with $|\Omega_i|_\infty \leq C_\Omega$ and all $r, \rho > 0$. The constant of the third bound depends only on u and N_1.

Two ellipses whose vectors $\Omega_1 e_1$, $\Omega_2 e_2$ are transverse together smooth in every direction, whatever their eccentricity; in the applications these vectors are the long axes. For the sources of a seed the ellipses are round discs. *Idea of the proof.* Each ∂_i is a combination of the derivatives along $\Omega_1 e_1$ and $\Omega_2 e_2$, and along each of these vectors the corresponding smoothing gains one power of ρ/r per derivative, while its vanishing moments give the approximation.

Proposition 6.2 (Energy of coupled systems)**.** *Let $\nu > 0$ and $t_1 < t_2$, let $N_{\mathrm{src}} \geq 2$ be the order of Subsection 5.3, let Φ be $\mathbb{Z}^2$-periodic and smooth in x, with $\nabla^\perp \Phi$ bounded and Borel measurable in (t, x), and let V be a finite set with masses $\omega_v > 0$. Let finitely many sources $\sigma\colon v' \to v$, $v, v' \in V$, be given, with coefficients $A_{\sigma,d}$, $2 \leq d \leq N_{\mathrm{src}}$, that are Borel measurable in t and bounded, continuous and periodic in x, and linear operators $S_\sigma(t)$ on $L^2(\mathbb{T}^2)$ that commute with derivatives and such that $t \mapsto S_\sigma(t) g \in L^2(\mathbb{T}^2)$ is Borel measurable for every $g \in L^2(\mathbb{T}^2)$. Let constants $c_j \geq 0$ and widths $r_\sigma > 0$ be such that, for every σ, $t \in [t_1, t_2]$, $g \in L^2(\mathbb{T}^2)$ and $j \leq N_{\mathrm{src}} - 2$, $\|\nabla^j S_\sigma(t) g\|_{L^2(\mathbb{T}^2)} \leq c_j r_\sigma^{-j} \|g\|_{L^2(\mathbb{T}^2)}$. Consider the problem*

$$\begin{cases} \partial_t U_v + \nabla^\perp \Phi \cdot \nabla U_v - \nu \Delta U_v \\ \qquad = \nabla \cdot \displaystyle\sum_{v' \in V} \sum_{\sigma\colon v' \to v} \sum_{d=2}^{N_{\mathrm{src}}} A_{\sigma,d} \nabla^{d-1} (S_\sigma U_{v'}) & \text{in } (t_1, t_2) \times \mathbb{T}^2\,, \qquad v \in V\,. \\ U_v(t_1, \cdot) = g_v & \text{on } \mathbb{T}^2\,, \end{cases} \tag{6.3}$$

If all row and column sums of the coupling matrix

$$B_{vv'} := \Big(\frac{\omega_{v'}}{\omega_v}\Big)^{1/2} \sum_{\sigma\colon v' \to v} \sum_{d=2}^{N_{\mathrm{src}}} c_{d-2} \, \frac{\sup_{[t_1,t_2]\times\mathbb{T}^2} |A_{\sigma,d}|}{\nu \, r_\sigma^{d-2}}$$

are at most $\frac{1}{8}$, then, for every datum $g = (g_v)_{v \in V} \in L^2(\mathbb{T}^2)^V$, there exists a unique solution $U \in C([t_1, t_2]; L^2(\mathbb{T}^2)^V) \cap L^2((t_1, t_2); H^1(\mathbb{T}^2)^V)$ of (6.3) (in the integrated sense of (8.3), without forcing), and, for every $t', t \in [t_1, t_2]$ with $t' \leq t$,

$$\|U(t)\|^2_{L^2_\omega(\mathbb{T}^2)} + \nu \, \|\nabla U\|^2_{L^2((t',t);L^2_\omega(\mathbb{T}^2))} \leq \|U(t')\|^2_{L^2_\omega(\mathbb{T}^2)}\,.$$

Idea of the proof. We test the equation for U_v with U_v/ω_v and bound the coupling by Schur's test, as after (5.9). Its analogue at higher order, Proposition 8.3, gives the derivatives of Y used after Proposition 5.3 and the $C^{N_{\mathrm{rep}}}$ regularity in Proposition 5.4.

Part II. The proofs

7. Properties of the velocity field: flows and effective diffusivities

This section proves the properties of the construction that Section 2 states without proof, and Proposition 2.1. Other sections use the flows of Subsection 7.3: the short flows carry the tubes of Theorem C, and the renewal charts are the coordinates of Sections 10–13.

Throughout, $\Lambda \geq 128$ is an integer, and C and c denote positive constants depending only on α, q, ζ, unless another dependence is stated; they may change from line to line. Matrices and arrays are normed as in Section 2.

7.1 Scales and cutoffs

Lemma 7.1 (Scales and cutoffs)**.** *For every integer $m \geq 1$ the following hold.*

(a) Wavelengths. $\varepsilon_{m-1}/\varepsilon_m \geq \Lambda$ *and* $\varepsilon_m \leq \Lambda^{-m}$. *If $m \geq 2$, then* $\frac{1}{2}\varepsilon_{m-1}^q \leq \varepsilon_m \leq 2\varepsilon_{m-1}^q$.

(b) Times. $4\varepsilon_{m-1}^{-\delta} \leq 4\lceil \varepsilon_{m-1}^{-\delta} \rceil + 1 \leq 9\varepsilon_{m-1}^{-\delta}$; (2.5) *and* (2.13) *hold, and* $\varepsilon_m \leq 2^{35}\tau_m$.

(c) Exponents. $q(\beta-\gamma) = \beta+\gamma$, $q(2-\beta-\gamma) = 2-\beta+8\delta$, $(q-1)(\beta+\gamma) = 2q\gamma$, $3\delta < \gamma$, *and* $q-(2-\beta+4\delta) = \alpha + (q+\frac{1}{2})\gamma > 0$.

(d) Cutoffs. *Let $k, l \in \mathbb{Z}$ and let $j \geq 0$ be an integer. There exist $C(\zeta, j)$ and $C(j)$ such that* $\|\partial_t^j \zeta_{m,k}\|_{L^\infty(\mathbb{R})} \leq C(\zeta,j)\tau_m^{-j}$ *and* $\|\partial_t^j \widehat{\zeta}_{m,l}\|_{L^\infty(\mathbb{R})} \leq C(j)(\tau'_m)^{-j}$. *The fast cutoff $\zeta_{m,k}$ vanishes outside $[(k-\frac{2}{3})\tau_m, (k+\frac{2}{3})\tau_m]$. At each time at most two fast cutoffs are nonzero, and at most one with k odd. The slow cutoff satisfies $0 \leq \widehat{\zeta}_{m,l} \leq 1$ and*

$$\mathbf{1}_{[(l-\frac{1}{2})h_m + 2\tau'_m,\, (l+\frac{1}{2})h_m - 2\tau'_m]} \leq \widehat{\zeta}_{m,l} \leq \mathbf{1}_{((l-\frac{1}{2})h_m + \tau'_m,\, (l+\frac{1}{2})h_m - \tau'_m)}\,, \tag{7.1}$$

where $\mathbf{1}_E$ is the indicator of a set E. The slot $[(k-\frac{1}{2})\tau_m, (k+\frac{1}{2})\tau_m]$ lies in the block $[(l_{m,k}-\frac{1}{2})h_m, (l_{m,k}+\frac{1}{2})h_m]$.

Finally, for $\alpha, s > 0$ a number q satisfying (2.1) exists if and only if $\alpha < \frac{1}{3}$ and $s < \frac{1}{2}(1-\alpha)$; and for $q > 1$, $\delta > 0$ if and only if $q < \frac{1-\alpha}{2\alpha}$, while $\mu > 0$ if and only if $q < \frac{1-s}{\alpha+s}$.

Proof. (a) Let $m \geq 1$. Since $\Lambda^{q^m/(q-1)} \geq \Lambda^{1/(q-1)} \geq \log\Lambda/(q-1)$ and $\Lambda^{q^m-1} - 1 \geq (q-1)\log\Lambda$,

$$\Lambda^{q^{m+1}/(q-1)} - \Lambda \cdot \Lambda^{q^m/(q-1)} = \Lambda \cdot \Lambda^{q^m/(q-1)}\big(\Lambda^{q^m-1} - 1\big) \geq \Lambda(\log\Lambda)^2 \geq \Lambda\,.$$

So $\Lambda\varepsilon_m^{-1} \leq \Lambda(\Lambda^{q^m/(q-1)} + 1) \leq \varepsilon_{m+1}^{-1}$ by (2.4); with $\varepsilon_0 = 1$ and $\varepsilon_1^{-1} \geq \Lambda^{q/(q-1)} \geq \Lambda$, the first two bounds follow. For $m \geq 2$, $\Lambda^{q^{m-1}/(q-1)} \geq 128$, so its ceiling is at most $\frac{129}{128}$ times it, and $(\frac{129}{128})^q < 2$ gives $\varepsilon_{m-1}^{-q} \leq 2\Lambda^{q^m/(q-1)}$. Hence $\varepsilon_m \leq \Lambda^{-q^m/(q-1)} \leq 2\varepsilon_{m-1}^q$, and $\varepsilon_m \geq \frac{1}{2}\Lambda^{-q^m/(q-1)} \geq \frac{1}{2}\varepsilon_{m-1}^q$, since the ceiling of a number at least 1 is at most twice the number. (b) By (2.4), $h_m = 2^{-25}/\lceil \varepsilon_{m-1}^{-(2-\beta+2\delta)} \rceil$, and τ'_m and τ_m are h_m divided once and twice by $4\lceil \varepsilon_{m-1}^{-\delta} \rceil + 1$, so the bounds $y \leq \lceil y \rceil \leq 2y$ for $y \geq 1$ give the first bounds, (2.5), $h_m \leq 2^{-25}\varepsilon_{m-1}^{2-\beta+2\delta}$, which is (2.13), and $\tau_m \geq 2^{-26}\varepsilon_{m-1}^{2-\beta+4\delta}/81$. For $m \geq 2$, (a) and (c) then give $\varepsilon_m \leq 2\varepsilon_{m-1}^q \leq 2\varepsilon_{m-1}^{2-\beta+4\delta} \leq 2^{35}\tau_m$; for $m = 1$, $\varepsilon_1 \leq \Lambda^{-1} \leq 2^{35}\tau_1$. (c) and the equivalences are algebra on (2.1) and (2.3). (d) follows from (2.2), (2.7), (2.9) and the properties of H; the centres of odd slots are $2\tau_m$ apart, and $\zeta_{m,k}$ has support of length $\frac{4}{3}\tau_m$. Finally, h_m/τ_m is the odd integer $(4\lceil \varepsilon_{m-1}^{-\delta} \rceil + 1)^2$, so $k + \frac{1}{2}(h_m/\tau_m - 1)$ is an integer, and (2.8) gives $l_{m,k}h_m - \frac{1}{2}(h_m - \tau_m) \leq k\tau_m \leq l_{m,k}h_m + \frac{1}{2}(h_m - \tau_m)$. □

In particular, the envelope $\widetilde{\zeta}_{m,k} = \widehat{\zeta}_{m,l_{m,k}}\zeta_{m,k}$ of slot k vanishes outside the block of $l_{m,k}$ and within τ'_m of its ends. Cutoffs as in (2.2) exist: for $0 < w \le \frac{13}{80}$, the even function equal to 1 on $[0, \frac{1}{2} - w]$, to 0 on $[\frac{1}{2} + w, \infty)$ and affine in between, convolved with an even mollifier of width $\frac{1}{1000}$, satisfies the first two conditions and lies in $C_c^\infty((-\frac{2}{3}, \frac{2}{3}))$; its square integral is continuous in w, above $\frac{9}{10}$ at $w = \frac{1}{10}$ and at most $1 - \frac{2}{3}w < \frac{9}{10}$ at $w = \frac{13}{80}$, hence equal to $\frac{9}{10}$ in between.

Reliance on [AV25]. The proofs of [AV25, Proposition 2.2 and Corollary 2.4] use the scales and the cutoffs only through the facts of Lemma 7.1. These facts hold for every q satisfying (2.1), every integer $\Lambda \ge 128$ and every ζ as in Section 2, so those proofs apply to our construction, with $\varepsilon_m \le 2^{35}\tau_m$ of Lemma 7.1(b) in place of the exponent that [AV25] computes in [AV25, (2.61)] from its choice of q. We write out the arguments that differ.

7.2 The velocity field

The functions ϕ_m and $\mathbf{b}_m$ of Lemma 7.2 are well defined and smooth by Proposition 7.3(i), whose proof does not use Lemma 7.2.

Lemma 7.2 (Increments of the stream functions)**.** *There exists $C(\alpha, q, \zeta, \Lambda) < \infty$ such that, for every $m \ge 1$ and $i, i' \in \{1, 2\}$,*

$$\begin{gathered}\|\phi_m - \phi_{m-1}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le 10\,\varepsilon_m^\beta\,, \qquad \|\partial_i(\phi_m - \phi_{m-1})\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le 320\,\varepsilon_m^\alpha\,,\\ \|\partial_i\partial_{i'}(\phi_m - \phi_{m-1})\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le 36500\,\varepsilon_m^{\beta-2}\,,\end{gathered} \tag{7.2}$$

and, for every $t, t' \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$,

$$\big|(\mathbf{b}_m - \mathbf{b}_{m-1})(t,x) - (\mathbf{b}_m - \mathbf{b}_{m-1})(t',y)\big| \le C\big(|t-t'| + |x-y|\big)\varepsilon_m^{\beta-2}\,. \tag{7.3}$$

Proof. The bounds (7.2) are [AV25, Proposition 2.2, (2.39)]: the norm [AV25, (1.21)] at the radius $2^7/\varepsilon_m$ bounds each partial derivative of order j by $j!(2^7/\varepsilon_m)^j/(j+1)^2$ times the norm, so the bound $10\varepsilon_m^\beta$ gives $320\varepsilon_m^\alpha$ and $36500\varepsilon_m^{\beta-2}$. The bound (7.3) follows from them and from the bound $|\partial_t(\mathbf{b}_m - \mathbf{b}_{m-1})| \le C\varepsilon_m^{\beta-2}$ in the proof of [AV25, Proposition 2.2], after [AV25, (2.61)]; the first term of that bound, $C\varepsilon_m^{\beta-2}\varepsilon_m/\tau_m$, is at most $2^{35}C\varepsilon_m^{\beta-2}$ by Lemma 7.1(b). □

Proposition 7.3 (The velocity field)**.** *The following hold.*

(i) *For every $m \ge 0$, ϕ_m is well defined, smooth and $\mathbb{Z}\times\mathbb{Z}^2$-periodic, and $\mathbf{b}_m$ is divergence free.*

(ii) *For every $m \ge 0$,*

$$\|\nabla\mathbf{b}_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} = \|\nabla^2\phi_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le 78000\,\varepsilon_m^{\beta-2}\,. \tag{7.4}$$

(iii) *As $m \to \infty$, ϕ_m and $\mathbf{b}_m$ converge uniformly on $\mathbb{R}\times\mathbb{R}^2$ to functions ϕ and $\mathbf{b}$. The function ϕ is continuous, $\mathbb{Z}^2$-periodic in x and 1-periodic in t; for every $t \in \mathbb{R}$, $\phi(t,\cdot) \in C^1(\mathbb{R}^2)$ and $\nabla^\perp\phi(t,\cdot) = \mathbf{b}(t,\cdot)$; and (2.14) holds.*

(iv) *$\mathbf{b}$ is bounded, continuous, $\mathbb{Z}^2$-periodic in x and 1-periodic in t; for every $t \in \mathbb{R}$, the field $\mathbf{b}(t,\cdot)$ is divergence free in the sense of distributions, and (2.15) holds.*

The constants in (2.14) and (2.15) depend only on $\alpha, q, \zeta, \Lambda$.

Proof. (i) By induction on m: if ϕ_{m-1} is smooth and $\mathbb{Z}\times\mathbb{Z}^2$-periodic, its flow X_{m-1} exists, is smooth, and, by uniqueness, commutes with the translations by $\mathbb{Z}^2$ and satisfies $X_{m-1}(t+1, x, t'+1) =$

$X_{m-1}(t,x,t')$. By Lemma 7.1(d), finitely many terms of (2.12) are nonzero near each time, and each $\psi_{m,k}$ is $\mathbb{Z}^2$-periodic, since ε_m^{-1} is an integer; so ϕ_m is smooth and $\mathbb{Z}^2$-periodic in x. It is 1-periodic in t because $1/h_m$ and $1/\tau_m$ are integers divisible by 4, by (2.4): a shift of time by 1 shifts k by $1/\tau_m$ and $l_{m,k}$ by $1/h_m$, and keeps the class of k modulo 4. Finally, $\mathbf{b}_m = \nabla^\perp \phi_m$ is divergence free.

(ii) As $\phi_0 = 0$, Lemma 7.2 gives $|\partial_i \partial_{i'} \phi_m| \le 36500 \sum_{m'=1}^{m} \varepsilon_{m'}^{\beta-2}$, and $\varepsilon_{m'}^{\beta-2} \le \Lambda^{(2-\beta)(m'-m)} \varepsilon_m^{\beta-2} \le 16^{m'-m} \varepsilon_m^{\beta-2}$ by Lemma 7.1(a), since $2-\beta = 1-\alpha > \frac{2}{3}$ gives $\Lambda^{2-\beta} \ge 128^{2/3} > 16$. So every entry of $\nabla^2 \phi_m$ is at most $\frac{16}{15} \cdot 36500\, \varepsilon_m^{\beta-2} \le 39000\, \varepsilon_m^{\beta-2}$ in absolute value. The norm of a 2×2 matrix is at most twice its largest entry, and $\nabla \mathbf{b}_m$ is $\nabla^2 \phi_m$ composed with a rotation, which gives (7.4).

(iii) By Lemma 7.2 and $\varepsilon_{m'+1} \le \Lambda^{-1} \varepsilon_{m'}$, the sums $\sum_{m'>m} |\phi_{m'} - \phi_{m'-1}|$ and $\sum_{m'>m} |\mathbf{b}_{m'} - \mathbf{b}_{m'-1}|$ are at most $10 \varepsilon_{m+1}^{\beta} / (1-\Lambda^{-\beta})$ and $2 \cdot 320\, \varepsilon_{m+1}^{\alpha} / (1-\Lambda^{-\alpha})$, uniformly in (t,x). Hence ϕ_m and $\mathbf{b}_m = \nabla^\perp \phi_m$ converge uniformly, and (2.14) holds. The limit ϕ is continuous; since $\phi_m(t,\cdot)$ and its gradient converge uniformly, $\phi(t,\cdot)$ is C^1 with $\nabla^\perp \phi = \mathbf{b}$; and the periodicities persist.

(iv) By (iii), $\mathbf{b}$ is continuous and periodic, and $|\mathbf{b}| \le \sum_{m' \ge 1} 640\, \varepsilon_{m'}^{\alpha} \le C$. Each $\mathbf{b}_m(t,\cdot)$ is divergence free, and this passes to the uniform limit in the sense of distributions. For (2.15), let $t, t' \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$. If $(t,x) = (t',y)$ there is nothing to prove, and if $|t-t'| + |x-y| \ge 1$, (2.15) follows from the bound on $\mathbf{b}$. Otherwise, let $\overline{m} \ge 1$ be the least level with $\varepsilon_{\overline{m}} \le |t-t'| + |x-y|$. We write $\mathbf{b} = \sum_{m' \ge 1} (\mathbf{b}_{m'} - \mathbf{b}_{m'-1})$, bound the terms with $m' < \overline{m}$ by (7.3) and those with $m' \ge \overline{m}$ by (7.2) at each of the two points:

$$\begin{aligned} |\mathbf{b}(t,x) - \mathbf{b}(t',y)| &\le C\big(|t-t'| + |x-y|\big) \sum_{1 \le m' < \overline{m}} \varepsilon_{m'}^{\beta-2} + 1280 \sum_{m' \ge \overline{m}} \varepsilon_{m'}^{\alpha} \\ &\le C\big(|t-t'| + |x-y|\big) \varepsilon_{\overline{m}-1}^{\alpha-1} + C \varepsilon_{\overline{m}}^{\alpha} \le C\big(|t-t'| + |x-y|\big)^{\alpha}, \end{aligned}$$

since $\beta - 2 = \alpha - 1$, the two sums are geometric (Lemma 7.1(a)) and $\varepsilon_{\overline{m}} \le |t-t'| + |x-y| < \varepsilon_{\overline{m}-1}$; for $\overline{m} = 1$ the first sum is empty. This is (2.15). □

7.3 Flows and renewal charts

Lemma 7.4 (Short flows). *Let $m \ge 0$ and let $t, t_1 \in \mathbb{R}$ satisfy $|t - t_1|\, \varepsilon_m^{\beta-2} \le 2^{-22}$. Then $y \mapsto X_m(t,y,t_1)$ is a smooth, volume-preserving diffeomorphism of $\mathbb{R}^2$ commuting with the translations by $\mathbb{Z}^2$, with inverse $x \mapsto X_m(t_1,x,t)$, and, for every $x, y \in \mathbb{R}^2$,*

$$|\nabla_y X_m(t,y,t_1) - I| \le \tfrac{1}{2}, \qquad |\nabla_x X_m(t_1,x,t) - I| \le \tfrac{1}{2}; \tag{7.5}$$

in particular both maps are 2-Lipschitz. This applies when $|t - t_1| \le 8h_{m+1}$, by (2.13).

Proof. By Proposition 7.3(i), $\mathbf{b}_m$ is smooth, bounded, $\mathbb{Z}^2$-periodic and divergence free, so the two maps are smooth, mutually inverse by the group law of the flow, commute with the translations by $\mathbb{Z}^2$ by uniqueness, and preserve Lebesgue measure by Liouville's theorem. The derivative solves $\partial_t \nabla_y X_m(t,y,t_1) = \nabla \mathbf{b}_m(t, X_m(t,y,t_1))\, \nabla_y X_m(t,y,t_1)$, so Grönwall's inequality and (7.4) give

$$|\nabla_y X_m(t,y,t_1) - I| \le \exp\big(78000\, \varepsilon_m^{\beta-2} |t-t_1|\big) - 1 \le \exp\big(\tfrac{1}{50}\big) - 1 < \tfrac{1}{2},$$

and likewise for $\nabla_x X_m(t_1,x,t)$. □

Complex extensions. Section 11 bounds the derivatives of its families in their parameters by Cauchy's estimates, which needs holomorphic extensions of the flows to complex labels; they come from those of the velocity fields. For a vector of $\mathbb{C}^2$, Im is its imaginary part, taken coordinatewise, $|\cdot|_1$ is the sum of the moduli of its coordinates, and $|\cdot|_\infty$ is as in Section 2, on vectors and on

matrices. On the strips $\{|\operatorname{Im} x|_1 < c\,\varepsilon_m\}$, for a small fixed c, the shear streams $\varepsilon_m^\beta \sin(2\pi y_i/\varepsilon_m)$ stay below $2\varepsilon_m^\beta$, so Cauchy's estimates return the real Lipschitz scale $\varepsilon_m^{\beta-2}$; the imaginary part of a trajectory grows at most at this rate, so for $2^{21}\varepsilon_m^{\beta-2}|t-t'| \le \frac{3}{40}$ trajectories started in a slightly narrower strip stay in the strip. Complex trajectories $X_m(t,y,t')$ are defined by (2.11).

Proposition 7.5 (Analytic velocity fields)**.** *For every $m \ge 0$ and $t \in \mathbb{R}$, the field $\mathbf{b}_m(t,\cdot)$ extends to a holomorphic map on the strip $\{x \in \mathbb{C}^2 : |\operatorname{Im} x|_1 < \varepsilon_m/50\}$, which is continuous in (t,x), real at real points, and satisfies, for every x, x' in the strip,*

$$|\mathbf{b}_m(t,x) - \mathbf{b}_m(t,x')|_1 \le 2^{21}\varepsilon_m^{\beta-2}|x-x'|_1\,. \tag{7.6}$$

Lemma 7.6 (Complex flows)**.** *Let $m \ge 0$, $t_1 < t_2$ with $t_2 - t_1 \le \frac{6}{5}h_{m+1}$, and $t, t', t'' \in [t_1, t_2]$. For every $y \in \mathbb{C}^2$ with $|\operatorname{Im} y|_1 \le (1 - 2^{21}\varepsilon_m^{\beta-2}(t_2-t_1))\frac{19}{1000}\varepsilon_m$, the trajectory of $\mathbf{b}_m$ through y at time t' exists on $[t_1,t_2]$ and satisfies the first of the bounds*

$$\begin{aligned} |\operatorname{Im} X_m(t,y,t')|_1 &\le \frac{|\operatorname{Im} y|_1}{1 - 2^{21}\varepsilon_m^{\beta-2}(t_2-t_1)}\,, \\ |\nabla_y X_m(t,y,t') - I|_\infty &\le 10\cdot 2^{21}\varepsilon_m^{\beta-2}|t-t'| \le 2^{25}\varepsilon_m^{\beta-2}|t-t'|\,, \end{aligned} \tag{7.7}$$

and, if the inequality on $|\operatorname{Im} y|_1$ is strict, also the second. The map $(t,t',y) \mapsto X_m(t,y,t')$ is continuous for $t,t' \in [t_1,t_2]$ and these y, extends the real flow and, where the inequality on $|\operatorname{Im} y|_1$ is strict, is holomorphic in y. If $|\operatorname{Im} y|_1 \le (1-2^{21}\varepsilon_m^{\beta-2}(t_2-t_1))^2\frac{19}{1000}\varepsilon_m$, then $X_m(t, X_m(t'',y,t'),t'') = X_m(t,y,t')$. Since $2^{21}\varepsilon_m^{\beta-2}(t_2-t_1) \le \frac{3}{40}$ by (2.13), these domains contain $\{|\operatorname{Im} y|_1 < \frac{17}{1000}\varepsilon_m\}$ and $\{|\operatorname{Im} y|_1 \le \frac{15}{1000}\varepsilon_m\}$, respectively, and the factor in the first bound is at most $\frac{11}{10}$.

Proof of Proposition 7.5 and Lemma 7.6. Since $\mathbf{b}_0 = 0$, both follow from two implications: Proposition 7.5 at level m implies Lemma 7.6 at level m, and Proposition 7.5 and Lemma 7.6 at level $m-1$ imply Proposition 7.5 at level m.

Lemma 7.6 at level m. Since $\mathbf{b}_m$ is real at real points, (7.6), comparing $\mathbf{b}_m(t,x)$ with $\mathbf{b}_m(t,\operatorname{Re} x)$, gives $|\operatorname{Im}\mathbf{b}_m(t,x)|_1 \le 2^{21}\varepsilon_m^{\beta-2}|\operatorname{Im} x|_1$. By induction, every Picard iterate of the trajectory equation, started from the constant y, therefore has $|\operatorname{Im}\cdot|_1 \le |\operatorname{Im} y|_1/(1-2^{21}\varepsilon_m^{\beta-2}(t_2-t_1)) \le \frac{19}{1000}\varepsilon_m < \varepsilon_m/50$ on $[t_1,t_2]$. Since the first increment is bounded (as $\mathbf{b}_m$ is bounded at real points, hence on the strip by (7.6)), the iterates converge uniformly by (7.6), geometrically with ratio $2^{21}\varepsilon_m^{\beta-2}(t_2-t_1)$. Their limit is the trajectory, unique by (7.6); it satisfies the first bound in (7.7), and it is continuous in (t,t',y) and holomorphic in y, since the iterates are. Their derivatives in y solve the linearized iteration and converge; since $|\nabla\mathbf{b}_m|_\infty \le 2\cdot 2^{21}\varepsilon_m^{\beta-2}$ by (7.6) (each entry is at most $2^{21}\varepsilon_m^{\beta-2}$), Grönwall's inequality bounds them by $\exp(2\cdot\frac{3}{40}) < 5$, and their distance to I by $10\cdot 2^{21}\varepsilon_m^{\beta-2}|t-t'|$. At real points the trajectory is the real one, since $\mathbf{b}_m$ is real there. The group law holds by uniqueness, since, by the first bound, $X_m(t'',y,t')$ lies in the domain of existence.

Proposition 7.5 at level $m \ge 1$. For $|t - l_{m,k}h_m| \le \frac{1}{2}h_m$, Lemma 7.6 at level $m-1$ extends $x \mapsto X_{m-1}(l_{m,k}h_m, x, t)$ holomorphically, and continuously in (t,x), to $\{|\operatorname{Im} x|_1 < \varepsilon_m/25\}$, since $\varepsilon_m/25 < \frac{17}{1000}\varepsilon_{m-1}$, with values where $|\operatorname{Im}\cdot|_1 \le \frac{11}{10}\varepsilon_m/25$. The shear streams (2.6) are entire, and, as $|\sin z| \le \exp(|\operatorname{Im} z|)$, $|\psi_{m,k}| \le \varepsilon_m^\beta \exp(2\pi\cdot\frac{11}{250}) \le 2\varepsilon_m^\beta$ there. By Lemma 7.1(d), only the pulses with $l_{m,k}h_m$ within $\frac{1}{2}h_m$ of t contribute to (2.12), finitely many near each time, so $\phi_m - \phi_{m-1}$ extends to a function on $\mathbb{R}\times\{|\operatorname{Im} x|_1 < \varepsilon_m/25\}$, holomorphic in x, continuous in (t,x) (with its derivatives, by Cauchy's integral formula) and bounded by $2\varepsilon_m^\beta$, since $\sum_k \widetilde{\zeta}_{m,k} \le 1$. Cauchy's estimate on polydiscs of radius $\varepsilon_m/300$ about the points of $\{|\operatorname{Im} x|_1 < \varepsilon_m/50\}$, which lie in the strip of width $\varepsilon_m/25$, bounds the second derivatives of $\phi_m - \phi_{m-1}$ at these points by $2\cdot 2\varepsilon_m^\beta(300/\varepsilon_m)^2 = 360000\,\varepsilon_m^{\beta-2}$,

which gives the Lipschitz bound $720000\,\varepsilon_m^{\beta-2}$ for $\nabla^\perp(\phi_m-\phi_{m-1})$ on $\{|\operatorname{Im} x|_1<\varepsilon_m/50\}$. Since $\varepsilon_{m-1}^{\beta-2}\le\varepsilon_m^{\beta-2}/16$, the field $\mathbf{b}_m=\mathbf{b}_{m-1}+\nabla^\perp(\phi_m-\phi_{m-1})$ satisfies (7.6) with $2^{21}/16+720000\le 2^{21}$. At real points the extension is $\mathbf{b}_m$, so it is real there. □

Renewal charts. Let $m\ge 1$ and $l\in\mathbb{Z}$. In the block of l, the pulses of level m are written in the variable $y=X_{m-1}(lh_m,x,t)$ of (2.12). We call y the *Lagrangian label* of x at time t, the inverse map $y\mapsto X_{m-1}(t,y,lh_m)$ the *renewal chart* of the block at time t, and the chart of the block of $l_{m,k}$ the chart of pulse k. The *conductivity* of the chart is

$$\mathbf{K}_{m,l}(t,y)\coloneqq\big(\nabla_y X_{m-1}(t,y,lh_m)\big)^{-1}\big(\nabla_y X_{m-1}(t,y,lh_m)\big)^{-\mathsf{T}}\,, \tag{7.8}$$

the coefficient of the diffusion term in these coordinates (Lemma 7.9). For $m=1$ the chart is the identity, since $\mathbf{b}_0=0$, and $\mathbf{K}_{1,l}=I$.

Lemma 7.7 (Renewal charts)**.** *Let $m\ge 2$ and $l\in\mathbb{Z}$.*

(a) *Let $|t-lh_m|\le h_m$. Lemma 7.4 applies to X_{m-1} with $t_1=lh_m$, and, for every integer $j\ge 0$, there exists $C(j)$ such that, for every $x,y\in\mathbb{R}^2$,*

$$\begin{aligned}\varepsilon_{m-1}^j\big|\nabla_y^j\big(\nabla_y X_{m-1}(t,y,lh_m)-I\big)\big|+\varepsilon_{m-1}^j\big|\nabla_y^j\big(\mathbf{K}_{m,l}(t,y)-I\big)\big|&\le C(j)\,\varepsilon_{m-1}^{2\delta}\,,\\ \varepsilon_{m-1}^j\big|\nabla_x^{j+1}X_{m-1}(lh_m,x,t)\big|&\le C(j)\,.\end{aligned} \tag{7.9}$$

(b) *For every $t\in\mathbb{R}$ and $y\in\mathbb{R}^2$, with $y'\coloneqq X_{m-1}(lh_m,y,(l+1)h_m)$,*

$$\begin{aligned}X_{m-1}(t,y,(l+1)h_m)&=X_{m-1}(t,y',lh_m)\,,\\ \mathbf{K}_{m,l+1}(t,y)&=(\nabla_y y')^{-1}\,\mathbf{K}_{m,l}(t,y')\,(\nabla_y y')^{-\mathsf{T}}\,,\end{aligned} \tag{7.10}$$

and the map $y\mapsto y'$ preserves area.

(c) *Lemma 7.6 at level $m-1$ applies on the interval $[lh_m-\frac35 h_m,lh_m+\frac35 h_m]$ (of length $\frac65 h_m$) about the centre of the block, to $y\mapsto X_{m-1}(t,y,t')$, in particular to the chart ($t'=lh_m$) and to its inverse ($t=lh_m$).*

Proof. Lemmas 7.4 and 7.6 apply in (a) and (c), since $h_m\varepsilon_{m-1}^{\beta-2}\le 2^{-25}$ by (2.13). Next, [AV25, Proposition 2.2, (2.40)] bounds $\nabla_x X_{m-1}(t',x,t'')-I$, for every pair of times t',t'' with $|t'-t''|\le 2^{-25}\varepsilon_{m-1}^{2-\beta}$, in the norm [AV25, (1.21)] at the radius $2^{11}/\varepsilon_{m-1}$ by $2^{23}|t'-t''|\,\varepsilon_{m-1}^{\beta-2}$; read as in the proof of Lemma 7.2, this is $\varepsilon_{m-1}^j|\nabla_x^j(\nabla_x X_{m-1}(t',x,t'')-I)|\le C(j)\,|t'-t''|\,\varepsilon_{m-1}^{\beta-2}$. With $(t',t'')=(t,lh_m)$ and (lh_m,t), and $h_m\varepsilon_{m-1}^{\beta-2}\le 2^{-25}\varepsilon_{m-1}^{2\delta}$, this gives the bounds on $\nabla_y X_{m-1}(t,y,lh_m)-I$ and on $X_{m-1}(lh_m,x,t)$ in (7.9). Since $\det\nabla_y X_{m-1}=1$, the inverse of $\nabla_y X_{m-1}$ is its adjugate, so $\mathbf{K}_{m,l}-I$ is a quadratic polynomial without constant term in the entries of $\nabla_y X_{m-1}-I$, and the Leibniz rule bounds its jets. Finally, in (b), the first identity of (7.10) is the group law, the second follows from it by the chain rule, and $y\mapsto y'$ preserves area by Liouville's theorem. □

For $m\ge 1$, the *stream of the level-m pulses* [AV25, (3.3)] is

$$\psi_m(t,y)\coloneqq\sum_{k\in 2\mathbb{Z}+1}\widetilde{\zeta}_{m,k}(t)\,\psi_{m,k}(y)\,. \tag{7.11}$$

Lemma 7.8 (The level-m stream in a chart)**.** *Let $m\ge 1$ and $l\in\mathbb{Z}$.*

(i) *For every* $t \in [(l-\frac12)h_m, (l+\frac12)h_m]$ *and* $y \in \mathbb{R}^2$,

$$\begin{aligned}(\phi_m - \phi_{m-1})\big(t, X_{m-1}(t,y,lh_m)\big) &= \psi_m(t,y)\,,\\ \big(\nabla_y X_{m-1}(t,y,lh_m)\big)^{-1}(\mathbf{b}_m - \mathbf{b}_{m-1})\big(t, X_{m-1}(t,y,lh_m)\big) &= \nabla^\perp \psi_m(t,y)\,.\end{aligned} \tag{7.12}$$

(ii) *Let* k *be odd with* $l_{m,k} = l$, *and let* $t, t' \in \mathbb{R}$ *with* $|t - k\tau_m| < \frac43\tau_m$ *and* $|t' - k\tau_m| < \frac43\tau_m$. *Then, for every* $x \in \mathbb{R}^2$, *with* $y \coloneqq X_{m-1}(lh_m, x, t)$,

$$X_m(t', x, t) = X_{m-1}\Big(t',\, y + \Big(\int_t^{t'} \widetilde{\zeta}_{m,k}(t'')\,\mathrm{d}t''\Big)\nabla^\perp\psi_{m,k}(y),\, lh_m\Big)\,. \tag{7.13}$$

Proof. (i) By Lemma 7.1(d), at a time t of the block only the pulses with $l_{m,k} = l$ contribute to (2.12), and for them $X_{m-1}(lh_m, X_{m-1}(t,y,lh_m), t) = y$ by the group law. This is the first identity. The second follows from it, since $\det \nabla_y X_{m-1} = 1$ gives $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)(\nabla_y X_{m-1})^{\mathsf T} = (\nabla_y X_{m-1})^{-1}\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, so that $\nabla_y^\perp$ of a function composed with $X_{m-1}(t,\cdot,lh_m)$ is $(\nabla_y X_{m-1})^{-1}$ times its $\nabla^\perp$ at $X_{m-1}(t,y,lh_m)$.

(ii) Let t'' lie between t and t', and $x'' \in \mathbb{R}^2$. No pulse other than k is active at t'': for odd $k' \neq k$, the cutoff $\zeta_{m,k'}$ vanishes outside $[(k'-\frac23)\tau_m, (k'+\frac23)\tau_m]$, which is at distance at least $\frac43\tau_m$ from $k\tau_m$, while $|t'' - k\tau_m| < \frac43\tau_m$. If t'' lies in the block of l, (i) gives

$$(\mathbf{b}_m - \mathbf{b}_{m-1})(t'', x'') = \widetilde{\zeta}_{m,k}(t'')\,\nabla_y X_{m-1}(t'', y'', lh_m)\,\nabla^\perp\psi_{m,k}(y'')\,, \qquad y'' \coloneqq X_{m-1}(lh_m, x'', t'')\,.$$

Otherwise, by Lemma 7.1(d), t'' lies within $\frac56\tau_m < 5\tau_m \le \tau_m'$ of an end of the block, where all slow cutoffs vanish, and both sides vanish. Now consider the curve $t'' \mapsto X_{m-1}(t'', y + (\int_t^{t''}\widetilde{\zeta}_{m,k})\nabla^\perp\psi_{m,k}(y), lh_m)$. By (2.6), $\nabla^\perp\psi_{m,k}$ is a multiple of $e_{(k)}$ that depends only on the coordinate of its argument orthogonal to $e_{(k)}$, so it takes the value $\nabla^\perp\psi_{m,k}(y)$ at the displaced label. By the chain rule, the derivative of the curve is $\mathbf{b}_{m-1}$ at the curve plus $\widetilde{\zeta}_{m,k}(t'')\nabla_y X_{m-1}(t'', \cdot, lh_m)\nabla^\perp\psi_{m,k}$ at the displaced label, which is the label y'' of the point of the curve by the group law; by the previous display, the second term is $\mathbf{b}_m - \mathbf{b}_{m-1}$ at the curve. The curve passes through x at time t, so it is $t'' \mapsto X_m(t'', x, t)$. □

Shear displacements and phases. We call $y \mapsto y + (\int_t^{t'}\widetilde{\zeta}_{m,k}(t'')\,\mathrm{d}t'')\nabla^\perp\psi_{m,k}(y)$ in (7.13) the *shear displacement* of pulse k; it acts at the Lagrangian label y and depends on y only through y/ε_m modulo $\mathbb{Z}^2$, its *phase*. Section 11 composes the shear displacements across levels, with labels and phases as parameters.

Lemma 7.9 (The equation in a chart)**.** *Let* $m \ge 1$, $l \in \mathbb{Z}$, $(l-\frac12)h_m \le t_1 \le t_2 \le (l+\frac12)h_m$ *and* $\nu > 0$. *Let* $F \in L^2((t_1,t_2)\times\mathbb{T}^2;\mathbb{R}^2)$, $F^0 \in L^2((t_1,t_2)\times\mathbb{T}^2)$ *and* $g \in L^2(\mathbb{T}^2)$. *A function* θ *solves*

$$\begin{cases}\partial_t\theta + \mathbf{b}_m\cdot\nabla\theta - \nu\Delta\theta = \nabla\cdot F + F^0 & \text{in } (t_1,t_2)\times\mathbb{T}^2\,,\\ \theta(t_1,\cdot) = g & \text{on } \mathbb{T}^2\end{cases}$$

if and only if $U(t,y) \coloneqq \theta(t, X_{m-1}(t,y,lh_m))$ *solves*

$$\begin{cases}\partial_t U + \nabla^\perp\psi_m\cdot\nabla U - \nu\nabla\cdot(\mathbf{K}_{m,l}\nabla U)\\ \qquad = \nabla\cdot\big((\nabla_y X_{m-1})^{-1}F(t, X_{m-1})\big) + F^0(t, X_{m-1}) & \text{in } (t_1,t_2)\times\mathbb{T}^2\,,\\ U(t_1,\cdot) = g\big(X_{m-1}(t_1,\cdot,lh_m)\big) & \text{on } \mathbb{T}^2\,,\end{cases} \tag{7.14}$$

where X_{m-1} *and* $\nabla_y X_{m-1}$ *are evaluated at* (t,y,lh_m), *and* ∇ *acts in* y. *Solutions are energy solutions, as in* (8.3).

In the chart the drift $\mathbf{b}_{m-1}$ disappears, the identity in the diffusion term becomes the conductivity $\mathbf{K}_{m,l}$, which is close to I by Lemma 7.7, and the pulses become the explicit shears (7.11). Since, for $U \in H^1(\mathbb{T}^2)$, $-\nabla\cdot(\psi_m \nabla^\perp U) = \nabla^\perp\psi_m \cdot \nabla U$, the equation in (7.14) can equally be written in divergence form, with the matrix $\nu\mathbf{K}_{m,l} + \psi_m \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof. By Proposition 7.3(i) and as in the proof of Lemma 7.4, at each time the chart is a smooth, volume-preserving diffeomorphism commuting with the translations by $\mathbb{Z}^2$, so the change of variables preserves the class of solutions and the weak formulation: for $\varphi \in C^\infty(\mathbb{T}^2)$ and $\varphi_t := \varphi(X_{m-1}(lh_m, \cdot, t))$, $\langle U(t), \varphi\rangle_{L^2(\mathbb{T}^2)} = \langle \theta(t), \varphi_t\rangle_{L^2(\mathbb{T}^2)}$, so the equation for U tested against φ is the equation for θ tested against φ_t. As φ_t is transported by $\mathbf{b}_{m-1}$, $\partial_t\varphi_t + \mathbf{b}_{m-1}\cdot\nabla\varphi_t = 0$, so the drift $\mathbf{b}_{m-1}$ is absorbed into the time derivative of U. Since $\nabla_y U = (\nabla_y X_{m-1})^{\mathsf{T}} \nabla\theta(t, X_{m-1})$, the second identity of (7.12) turns $(\mathbf{b}_m - \mathbf{b}_{m-1})\cdot\nabla\theta$ into $\nabla^\perp\psi_m \cdot \nabla_y U$. Finally, since $\det \nabla_y X_{m-1} = 1$, the Piola identity (the rows of the cofactor matrix of a Jacobian matrix are divergence free) gives, for C^1 fields and, in the weak formulation, for fields in L^2, $(\nabla\cdot F)(t, X_{m-1}) = \nabla_y\cdot((\nabla_y X_{m-1})^{-1} F(t, X_{m-1}))$. Applied to F and to $\nu\nabla\theta = \nu(\nabla_y X_{m-1})^{-\mathsf{T}}\nabla_y U$, it gives the forcing and the term $\nu\nabla\cdot(\mathbf{K}_{m,l}\nabla U)$. □

7.4 The averaged flux

Lemma 7.10 (The averaged flux)**.** *Let $m \geq 1$ and $\nu > 0$.*

(i) *For every $t \in \mathbb{R}$, $\mathbf{J}^\nu_m(t)$ is diagonal, $\nu I \leq \mathbf{J}^\nu_m(t) \leq (\nu + \varepsilon_m^{2\beta}/(2\nu))I$, and, if $\widetilde{\zeta}_{m,k}(t) = 0$ for every odd k, then $\mathbf{J}^\nu_m(t) = \nu I$.*

(ii) *For every $t \in \mathbb{R}$, $\mathbf{J}^\nu_m(t+2h_m)$ is $\mathbf{J}^\nu_m(t)$ with its two diagonal entries exchanged, and $\mathbf{J}^\nu_m(t+4h_m) = \mathbf{J}^\nu_m(t)$.*

(iii) *(2.17) holds, and $\overline{K}_m(\nu) \geq \nu$.*

Proof. (i) The matrices $e_{(k)} \otimes e_{(k)}$ are diagonal, and $0 \leq \widetilde{\zeta}_{m,k} \leq 1$, so the time integral in (2.16) lies between 0 and $\int_{-\infty}^t \exp(-4\pi^2\nu(t-t')/\varepsilon_m^2)\,\mathrm{d}t' = \varepsilon_m^2/(4\pi^2\nu)$. At most one pulse is active at each time (Lemma 7.1(d)), so $0 \leq \mathbf{J}^\nu_m(t) - \nu I \leq 2\pi^2\varepsilon_m^{2\beta-2}\cdot\varepsilon_m^2/(4\pi^2\nu)\, I = \varepsilon_m^{2\beta}/(2\nu)\, I$.

(ii) Let $j \in 2\mathbb{Z}$. Since h_m/τ_m is an integer, (2.8), (2.7) and (2.9) give $l_{m,k+jh_m/\tau_m} = l_{m,k} + j$ and $\widetilde{\zeta}_{m,k+jh_m/\tau_m}(t + jh_m) = \widetilde{\zeta}_{m,k}(t)$, so the terms of $\mathbf{J}^\nu_m(t + jh_m)$ are those of $\mathbf{J}^\nu_m(t)$ with k replaced by $k + jh_m/\tau_m$. The integer h_m/τ_m is odd, so for $j = 2$ the shift keeps k odd and exchanges $4\mathbb{Z}+1$ and $4\mathbb{Z}+3$, hence $e_1\otimes e_1$ and $e_2\otimes e_2$, and for $j = 4$ it fixes the classes.

(iii) By (ii), $\mathbf{J}^\nu_m$ is $4h_m$-periodic and a shift by $2h_m$ exchanges its diagonal entries. So its averages over $[t, t+4h_m]$ and $[t+2h_m, t+6h_m]$ are equal and have exchanged entries; by (i) they are diagonal, hence half their trace times I, which is (2.17); and $\overline{K}_m(\nu) \geq \nu$. □

Section 10 identifies $\mathbf{J}^\nu_m$ with the spatial average of the flux of the correctors of the level-m pulses. The bands (2.18) are disjoint and accumulate at 0: by Lemma 7.1(a), $2\varepsilon_{M+1}^{\beta-\gamma} \leq 2\Lambda^{-(\beta-\gamma)}\varepsilon_M^{\beta-\gamma} < \frac{1}{2}\varepsilon_M^{\beta-\gamma}$, since $\beta - \gamma = 2\beta/(q+1) > \frac{32}{33}$ and $\Lambda \geq 128$.

7.5 Effective diffusivities

This subsection proves Proposition 2.1; its part (a) holds for every $\Lambda \geq 128$, with C depending only on ζ.

Proof of Proposition 2.1(a). Fix $m \geq 1$ and $\nu > 0$. By the recursion (2.19), we must show that $\overline{K}_m(\nu)$ is within $C(\varepsilon_m^{2\beta}/\nu)(\varepsilon_m^2/(\nu\tau_m) + \varepsilon_{m-1}^\delta)$ of $\nu + \frac{9}{80}\varepsilon_m^{2\beta}/\nu$.

Step 1: The time integral. Each pulse acts on the averaged flux as if it were held at its current value. Let k be odd. By Lemma 7.1(d) and $\tau_m \le \tau'_m$, $|\partial_t \widetilde{\zeta}_{m,k}| \le C/\tau_m$, so

$$\Big| \int_{-\infty}^{t} \exp\Big(-\frac{4\pi^2\nu(t-t')}{\varepsilon_m^2}\Big) \widetilde{\zeta}_{m,k}(t')\,\mathrm{d}t' - \frac{\varepsilon_m^2}{4\pi^2\nu}\,\widetilde{\zeta}_{m,k}(t) \Big| \le \frac{C}{\tau_m} \int_0^\infty t' \exp\Big(-\frac{4\pi^2\nu t'}{\varepsilon_m^2}\Big)\,\mathrm{d}t' \le \frac{C\varepsilon_m^4}{\nu^2\tau_m}\,.$$

As $\mathrm{tr}(e_{(k)} \otimes e_{(k)}) = 1$, $0 \le \widetilde{\zeta}_{m,k} \le 1$ and at most one pulse is active at a time, (2.16) gives, for every $t \in \mathbb{R}$,

$$\Big| \mathrm{tr}\, \mathbf{J}_m^\nu(t) - 2\nu - \frac{\varepsilon_m^{2\beta}}{2\nu} \sum_{k \in 2\mathbb{Z}+1} \widetilde{\zeta}_{m,k}(t)^2 \Big| \le C\,\frac{\varepsilon_m^{2\beta}}{\nu}\,\frac{\varepsilon_m^2}{\nu\tau_m}\,. \tag{7.15}$$

Step 2: The average of the squared envelopes. The function $\sum_{k\in 2\mathbb{Z}+1} \zeta_{m,k}^2$ is $2\tau_m$-periodic, and its integral over a period is $\tau_m \int_{\mathbb{R}} \zeta^2 = \frac{9}{10}\tau_m$ by (2.2). Since h_m/τ_m is an integer, its average over $[0, 4h_m]$ is exactly $\frac{9}{20}$: half of the slots are odd, and each carries $\frac{9}{10}$. The slow cutoffs change this sum only where $\widehat{\zeta}_{m,l_{m,k}}(t) < 1$ for an active pulse k. By Lemma 7.1(d), the slot of pulse k lies in its block and $\zeta_{m,k}$ vanishes outside the slot enlarged by $\frac{1}{6}\tau_m < 2\tau'_m$; so this happens only within $2\tau'_m$ of the ends of blocks, a fraction at most $4\tau'_m/h_m$ of $[0, 4h_m]$. As both sums lie in $[0,1]$, Lemma 7.1(b) gives

$$\Big| \fint_0^{4h_m} \sum_{k \in 2\mathbb{Z}+1} \widetilde{\zeta}_{m,k}(t)^2\,\mathrm{d}t - \frac{9}{20} \Big| \le \frac{4\tau'_m}{h_m} = \frac{4}{4\lceil \varepsilon_{m-1}^{-\delta} \rceil + 1} \le \varepsilon_{m-1}^{\delta}\,. \tag{7.16}$$

Step 3: Conclusion. By (2.17), $\overline{K}_m(\nu)$ is half the average of $\mathrm{tr}\,\mathbf{J}_m^\nu$ over $[0, 4h_m]$, so (7.15) and (7.16) give

$$\Big| \overline{K}_m(\nu) - \nu - \frac{\varepsilon_m^{2\beta}}{4\nu} \cdot \frac{9}{20} \Big| \le C\,\frac{\varepsilon_m^{2\beta}}{\nu} \Big(\frac{\varepsilon_m^2}{\nu\tau_m} + \varepsilon_{m-1}^{\delta} \Big)\,,$$

and $\frac{\varepsilon_m^{2\beta}}{4\nu} \cdot \frac{9}{20} = \frac{9}{80}\frac{\varepsilon_m^{2\beta}}{\nu}$. This is (a). □

Proof of Proposition 2.1(b). We may assume $M \ge 2$. Going up from M, we bound the logarithm of the ratio $\mathsf{s}_m := \kappa_m/((9/80)^{1/2}\varepsilon_m^{\beta+\gamma})$, uniformly in m. This normalization is the one a level reproduces: by (7.17) and (7.18) below, $\mathsf{s}_m = 1$ gives $\mathsf{s}_{m-1} \approx (\varepsilon_m/\varepsilon_{m-1}^q)^{\beta-\gamma}(1+\varepsilon_m^{2\gamma})$, a number close to 1.

Step 1: One level. Let $2 \le m \le M$. By (a), since $\kappa_m + \frac{9}{80}\varepsilon_m^{2\beta}/\kappa_m \ge \frac{9}{80}\varepsilon_m^{2\beta}/\kappa_m$,

$$\Big| \frac{\kappa_{m-1}}{\kappa_m + \frac{9}{80}\varepsilon_m^{2\beta}/\kappa_m} - 1 \Big| \le C\Big(\frac{\varepsilon_m^2}{\kappa_m\tau_m} + \varepsilon_{m-1}^{\delta} \Big)\,. \tag{7.17}$$

Since $q(\beta-\gamma) = \beta+\gamma$ (Lemma 7.1(c)), the comparison value satisfies the exact identity

$$\frac{\kappa_m + \frac{9}{80}\varepsilon_m^{2\beta}/\kappa_m}{(9/80)^{1/2}\varepsilon_{m-1}^{\beta+\gamma}} = \Big(\frac{\varepsilon_m}{\varepsilon_{m-1}^q}\Big)^{\beta-\gamma} \Big(\varepsilon_m^{2\gamma}\mathsf{s}_m + \frac{1}{\mathsf{s}_m} \Big)\,. \tag{7.18}$$

So, up to the factor $(\varepsilon_m/\varepsilon_{m-1}^q)^{\beta-\gamma}$ and the error in (7.17), a level maps s_m to $\varepsilon_m^{2\gamma}\mathsf{s}_m + 1/\mathsf{s}_m$, essentially $1/\mathsf{s}_m$, and inversion preserves $|\log|$: precisely, $|\log(\varepsilon_m^{2\gamma}\mathsf{s}_m + 1/\mathsf{s}_m)| \le |\log \mathsf{s}_m| + \varepsilon_m^{2\gamma}$, since $\varepsilon_m^{2\gamma}\mathsf{s}_m + 1/\mathsf{s}_m$ lies between $1/\mathsf{s}_m$ and $(1+\varepsilon_m^{2\gamma})\max\{\mathsf{s}_m, 1/\mathsf{s}_m\}$ and $\log(1+\varepsilon_m^{2\gamma}) \le \varepsilon_m^{2\gamma}$. Moreover:

- *The error of (a).* By Lemma 7.1(a)–(c), $\varepsilon_m^{2-\beta-\gamma} \le 2\varepsilon_{m-1}^{2-\beta+8\delta}$ and $\tau_m \ge c\varepsilon_{m-1}^{2-\beta+4\delta}$, so $\varepsilon_m^2/(\kappa_m\tau_m) = (\varepsilon_m^{2-\beta-\gamma}/\tau_m)(\varepsilon_m^{\beta+\gamma}/\kappa_m) \le C\varepsilon_{m-1}^{4\delta}/\mathsf{s}_m$.

- *The factor.* $|(\varepsilon_m/\varepsilon_{m-1}^q)^{\beta-\gamma}-1|\le C\varepsilon_{m-1}$: with $\lambda:=\Lambda^{-q^{m-1}/(q-1)}$, ε_{m-1}^{-1} and ε_m^{-1} are the ceilings of λ^{-1} and of λ^{-q}, so $\varepsilon_m/\varepsilon_{m-1}^q=\lceil\lambda^{-1}\rceil^q/\lceil\lambda^{-q}\rceil$ lies between $(1+\lambda^q)^{-1}$ and $(1+\lambda)^q$, and $\lambda\le 2\varepsilon_{m-1}$.

If the right side of (7.17) is at most $\frac12$, then, since the logarithm of a number in $[\frac12,\frac32]$ is at most twice its distance to 1 in absolute value, (7.18) gives

$$|\log \mathsf{s}_{m-1}|\le|\log \mathsf{s}_m|+C\varepsilon_{m-1}+\varepsilon_m^{2\gamma}+C\Big(1+\frac{1}{\mathsf{s}_m}\Big)\varepsilon_{m-1}^{\delta}\,. \tag{7.19}$$

Step 2: The start. Let $m=M$. Since $\frac12\varepsilon_M^{\beta-\gamma}\le\kappa_M\le 2\varepsilon_M^{\beta-\gamma}$, the two terms of the sum on the right side of (7.18) lie in $[\frac12(80/9)^{1/2},2(80/9)^{1/2}]\subset[1.49,5.97]$ and in $[0,2(9/80)^{1/2}\varepsilon_M^{2\gamma}]\subset[0,0.68]$, and the factor $(\varepsilon_M/\varepsilon_{M-1}^q)^{\beta-\gamma}$ lies in $[2^{-4/3},2^{4/3}]$, by Lemma 7.1(a) and $0<\beta-\gamma<\frac43$. By the first bullet, if $\Lambda\ge C$, the right side of (7.17) is at most $C\varepsilon_{M-1}^{\delta}\le\frac12$. So s_{M-1} lies in $[\frac12\cdot 2^{-4/3}\cdot 1.49,\frac32\cdot 2^{4/3}\cdot(5.97+0.68)]\subset[\exp(-4),\exp(4)]$.

Step 3: Up the levels. The constants C of Step 1 come from (a) and Lemma 7.1 and do not depend on Λ, and the start bound 4 is absolute; so we fix the bound $5=4+1$ first and Λ_0 last. We choose $\Lambda_0\ge 128$ so large that, for $\Lambda\ge\Lambda_0$: the requirement $\Lambda\ge C$ of Step 2 holds; for every $m\ge 2$, $C(1+\exp(5))(\varepsilon_{m-1}^{4\delta}+\varepsilon_{m-1}^{\delta})\le\frac12$; and

$$\sum_{m\ge 2}\Big(C\varepsilon_{m-1}+\varepsilon_m^{2\gamma}+C(1+\exp(5))\varepsilon_{m-1}^{\delta}\Big)\le 1\,,$$

which is possible since $\varepsilon_{m'}\le\Lambda^{-m'}$ by Lemma 7.1(a). We prove by induction on m, for $m=M-1,M-2,\dots,1$ in turn, that

$$|\log \mathsf{s}_m|\le 4+\sum_{m'=m+1}^{M-1}\Big(C\varepsilon_{m'-1}+\varepsilon_{m'}^{2\gamma}+C(1+\exp(5))\varepsilon_{m'-1}^{\delta}\Big)\,.$$

For $m=M-1$ this is Step 2. If it holds at some $m\ge 2$, then $|\log \mathsf{s}_m|\le 5$ by the display above, so the right side of (7.17) is at most $\frac12$, and (7.19) with $1/\mathsf{s}_m\le\exp(5)$ gives it at $m-1$. Since the sum is at most 1, $|\log \mathsf{s}_m|\le 5$ for $1\le m\le M-1$: this is (b). □

Proof of Proposition 2.1(c). Let $2\le m\le M$. First, $\kappa_m\le\overline{K}_m(\kappa_m)=\kappa_{m-1}$ by Lemma 7.10(iii). Since $1\le m-1<M$, (b) gives $\kappa_{m-1}\asymp\varepsilon_{m-1}^{\beta+\gamma}$, and (2.5) gives the bound on $(\kappa_{m-1}h_m)^{1/2}$. If $m<M$, then (b) gives $\kappa_m\asymp\varepsilon_m^{\beta+\gamma}$, and Lemma 7.1(a)–(c) give

$$\frac{\varepsilon_m^2}{\kappa_m\tau_m}\le C\,\frac{\varepsilon_m^{2-\beta-\gamma}}{\tau_m}\le C\varepsilon_{m-1}^{q(2-\beta-\gamma)-(2-\beta+4\delta)}=C\varepsilon_{m-1}^{4\delta}\,,\qquad \frac{\kappa_{m-1}}{\kappa_m}\asymp\frac{\varepsilon_{m-1}^{\beta+\gamma}}{\varepsilon_{m-1}^{q(\beta+\gamma)}}=\varepsilon_{m-1}^{-2q\gamma}\,.$$

If $m=M$, then $\kappa_M\ge\frac12\varepsilon_M^{\beta-\gamma}=\frac12\varepsilon_M^{\beta+\gamma}\varepsilon_M^{-2\gamma}$, so the same computation gives $\varepsilon_M^2/(\kappa_M\tau_M)\le C\varepsilon_{M-1}^{4\delta+2q\gamma}\le C\varepsilon_{M-1}^{4\delta}$. Moreover $\kappa_M\asymp\varepsilon_M^{\beta-\gamma}\asymp\varepsilon_{M-1}^{q(\beta-\gamma)}=\varepsilon_{M-1}^{\beta+\gamma}\asymp\kappa_{M-1}$, so $\kappa_{M-1}\le C\kappa_M$ and $(\kappa_{M-1}/\kappa_M)^{1/2}\le C\le C\varepsilon_{M-1}^{-q\gamma}$. □

8. Energy estimates for families and paired smoothing

In this section we prove Propositions 6.1 and 6.2. We prove Proposition 6.2 in the form at higher order announced after it, Proposition 8.3, together with its analogue for sources without smoothing, Proposition 8.6; Sections 12–15 use these two.

Arrays and coefficients. For a function g on $\mathbb{T}^2$ with values in a Euclidean space, $\nabla^j g$ is the array of its 2^j ordered partial derivatives of order j, with the Euclidean norm, as in (5.8). A *coefficient* is a function A on $\mathbb{T}^2$, or on $[t_1,t_2]\times\mathbb{T}^2$, whose values are linear maps between Euclidean spaces, normed as in Section 2. Its derivatives are measured by

$$|\nabla^j A(x)| := \sup\bigl\{|\nabla^j(A\mathsf{z})(x)| : |\mathsf{z}|\le 1\bigr\},$$

where $A\mathsf{z}$ is the function $x\mapsto A(x)\mathsf{z}$ for a fixed z. For the drift, the norm $|\nabla^j\nabla^\perp\Phi(x)|$ of the j-linear map of Section 2 equals this norm for the coefficient $\mathsf{z}\mapsto\nabla^\perp\Phi\cdot\mathsf{z}$, since the two matrices, acting on arrays, are transposes of each other. For functions g_v on $\mathbb{T}^2$, $v\in V$, masses $\omega_v>0$ and $g=(g_v)_{v\in V}$, we write

$$\|g\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} := \sum_{v\in V}\frac{\|g_v\|^2_{H^N_\rho(\mathbb{T}^2)}}{\omega_v}, \qquad \|g\|^2_{H^{(N_v)}_{\rho,\omega}(\mathbb{T}^2)} := \sum_{v\in V}\frac{\|g_v\|^2_{H^{N_v}_\rho(\mathbb{T}^2)}}{\omega_v}, \tag{8.1}$$

the second for integers $N_v\ge 0$; for $N=0$ the first is the norm $\|g\|_{L^2_\omega(\mathbb{T}^2)}$ of (5.7).

8.1 Products and derivatives in H^N_ρ

Lemma 8.1 (Products and derivatives at the scale ρ)**.** *Let $N\ge 0$ be an integer and $\rho>0$.*

(a) *For every coefficient A of class C^N on $\mathbb{T}^2$ and every $g\in H^N(\mathbb{T}^2)$ with values in the domain of A,*

$$\|Ag\|_{H^N_\rho(\mathbb{T}^2)} \le \Bigl(\sum_{j=0}^{N}\frac{\rho^j}{j!}\sup_{\mathbb{T}^2}|\nabla^j A|\Bigr)\|g\|_{H^N_\rho(\mathbb{T}^2)}.$$

(b) *For every integer $i\ge 0$ and every $g\in H^{N+i}(\mathbb{T}^2)$, $\|\nabla^i g\|_{H^N_\rho(\mathbb{T}^2)}\le\rho^{-i}((N+i)!/N!)\|g\|_{H^{N+i}_\rho(\mathbb{T}^2)}$.*

Proof. (a) The weights turn the Leibniz rule into a convolution: there are $\binom{j}{i}$ ways to let i of the j derivatives of $\nabla^j(Ag)$ fall on A, each giving an array of norm at most $|\nabla^i A|\,|\nabla^{j-i}g|$ at every point, and $\binom{j}{i}\rho^j/j! = (\rho^i/i!)(\rho^{j-i}/(j-i)!)$. So the sequence $(\rho^j/j!)\|\nabla^j(Ag)\|_{L^2(\mathbb{T}^2)}$ is bounded by the convolution of $(\rho^i/i!)\sup_{\mathbb{T}^2}|\nabla^i A|$ and $(\rho^i/i!)\|\nabla^i g\|_{L^2(\mathbb{T}^2)}$, and Young's inequality for a summable sequence convolved with a square-summable one gives (a). (b) For $j\le N$, $\rho^j/j! = \rho^{-i}((j+i)!/j!)\,\rho^{j+i}/(j+i)!$ and $(j+i)!/j!\le(N+i)!/N!$. □

8.2 Paired smoothing

The kernel. We prove the properties of W_ℓ stated before Proposition 6.1. With $u_\ell(z) := (2\ell)^2u(2\ell z)$, the autocorrelation $u_\ell * u_\ell(-\,\cdot\,)$ has Fourier transform $|\widehat{u}(\xi/(2\ell))|^2$, so expanding the power in (6.1) gives $W_\ell = \sum_{j=1}^{\ell}(-1)^{j+1}\binom{\ell}{j}(u_\ell * u_\ell(-\,\cdot\,))^{*j}$, where $(\cdot)^{*j}$ is the j-fold convolution. The autocorrelation is smooth with support in the disc of radius $1/\ell$, so W_ℓ is smooth with support in the unit disc, and $\int_{\mathbb{R}^2}W_\ell = \widehat{W}_\ell(0) = 1$. Since $u\ge 0$ has integral one, $0\le|\widehat{u}|^2\le 1$, so $\widehat{W}_\ell$ takes values in $[0,1]$. The smooth function $1-|\widehat{u}|^2$ vanishes at its minimum, the origin, so $1-\widehat{W}_\ell = (1-|\widehat{u}(\xi/(2\ell))|^2)^\ell = O(|\xi|^{2\ell})$, and the derivatives of $\widehat{W}_\ell$ of orders 1 to $2\ell-1$, which are constant multiples of the moments of W_ℓ, vanish at the origin. Consequently, each average $S_{\ell,r,\Omega}$ multiplies the Fourier coefficient of a function on $\mathbb{T}^2$ at the frequency ξ by $\widehat{W}_\ell(r\Omega^{\mathsf{T}}\xi)\in[0,1]$: it commutes with derivatives and, by Plancherel's theorem, contracts every $H^N_\rho(\mathbb{T}^2)$.

Proof of Proposition 6.1. Step 1: Derivatives along Ω_1e_1 and Ω_2e_2. Each of these derivatives falls on one average and costs r^{-1}, whatever the short axes of the ellipses (Figure 8.1). Let $\partial_{(1)} := \Omega_1e_1\cdot\nabla$

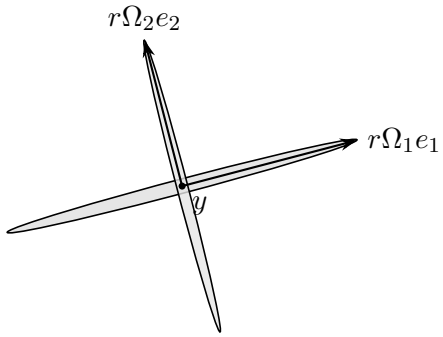


Figure 8.1: The two thin ellipses $\{y - r\Omega_1 z : |z| < 1\}$ and $\{y - r\Omega_2 z : |z| < 1\}$, centred at y, over which the two averages of $S_{\ell,r,\Omega_1} S_{\ell,r,\Omega_2}$ are taken, and the vectors $r\Omega_1 e_1$ and $r\Omega_2 e_2$.

and $\partial_{(2)} \coloneqq \Omega_2 e_2 \cdot \nabla$. On $y \mapsto g(y - r\Omega_1 z)$, $\partial_{(1)}$ is $-r^{-1}\partial_{z_1}$, and integration by parts in z_1 in (6.2) gives

$$\partial_{(1)} S_{\ell,r,\Omega_1} g(y) = r^{-1} \int_{\mathbb{R}^2} \partial_1 W_\ell(z)\, g(y - r\Omega_1 z)\, \mathrm{d}z\,,$$

and, by Minkowski's inequality, for every $j \geq 0$, $\|\partial_{(1)}^j S_{\ell,r,\Omega_1} g\|_{L^2(\mathbb{T}^2)} \leq r^{-j} \|\partial_1^j W_\ell\|_{L^1(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{T}^2)}$; similarly for $\partial_{(2)}$, S_{ℓ,r,Ω_2} and ∂_2. The pair $(\partial_{(1)}, \partial_{(2)})$ is the image of (∂_1, ∂_2) under the transpose of the matrix with columns $\Omega_1 e_1$, $\Omega_2 e_2$, so each ∂_i is a combination of $\partial_{(1)}$ and $\partial_{(2)}$ with coefficients that are entries of the inverse of that matrix, of absolute value at most its $|\cdot|_\infty$-norm, hence at most 2. An ordered derivative of order j is thus a sum of 2^j products of j derivatives $\partial_{(1)}$ or $\partial_{(2)}$, with coefficients of absolute value at most 2^j. The two averages commute with each other and with derivatives; letting $\partial_{(1)}$ act on S_{ℓ,r,Ω_1} and $\partial_{(2)}$ on S_{ℓ,r,Ω_2} gives, for every $j \geq 0$,

$$\|\nabla^j S g\|_{L^2(\mathbb{T}^2)} \leq c_j\, r^{-j} \|g\|_{L^2(\mathbb{T}^2)}\,,$$

where c_j depends only on u, ℓ and j. Proved for smooth g, this bound extends to every $g \in L^2(\mathbb{T}^2)$ by density, as the averages are bounded on $L^2(\mathbb{T}^2)$.

Step 2: The first two bounds. The averages contract $H^{N_1}_\rho(\mathbb{T}^2)$, which gives the second bound and the terms of order $j \leq N_1$ of the first. For $N_1 < j \leq N_2$, $\nabla^j S g = \nabla^{j-N_1} S \nabla^{N_1} g$, so Step 1 applied to the entries of $\nabla^{N_1} g$, with $r \leq \rho$ and $N_1! \leq j!$, gives the following bound, whose squares summed over $j \leq N_2$ give the first bound:

$$\frac{\rho^j}{j!} \|\nabla^j S g\|_{L^2(\mathbb{T}^2)} \leq c_{j-N_1} \Big(\frac{\rho}{r}\Big)^{j-N_1} \frac{\rho^{N_1}}{N_1!} \|\nabla^{N_1} g\|_{L^2(\mathbb{T}^2)} \leq c_{j-N_1} \Big(\frac{\rho}{r}\Big)^{N_2-N_1} \|g\|_{H^{N_1}_\rho(\mathbb{T}^2)}\,.$$

Step 3: The approximation. We use the vanishing moments in Fourier form. As $1 - |\widehat{u}|^2$ is smooth, with values in $[0,1]$, and vanishes at its minimum, the origin, $1 - |\widehat{u}|^2 \leq C_u |\cdot|^2$ with C_u depending only on u; so, by (6.1), $2\ell > N_1$ and $\sup_{\ell>0} (C_u/(4\ell^2))^\ell = \exp(\sqrt{C_u}/\mathrm{e})$, for every $\xi \in \mathbb{R}^2$,

$$0 \leq 1 - \widehat{W}_\ell(\xi) = \Big(1 - \Big|\widehat{u}\Big(\frac{\xi}{2\ell}\Big)\Big|^2\Big)^\ell \leq \min\Big\{1, \Big(\frac{C_u |\xi|^2}{4\ell^2}\Big)^\ell\Big\} \leq \exp\Big(\frac{\sqrt{C_u}}{\mathrm{e}}\Big) |\xi|^{N_1}\,.$$

Let $i \in \{1,2\}$. As S_{ℓ,r,Ω_i} multiplies the Fourier coefficient at ξ by $\widehat{W}_\ell(r\Omega_i^{\mathsf{T}}\xi)$, and $|\Omega_i^{\mathsf{T}}\xi| \leq \sqrt{2}\,|\Omega_i|_\infty |\xi|$, Plancherel's theorem gives

$$\|g - S_{\ell,r,\Omega_i} g\|_{L^2(\mathbb{T}^2)} \leq \exp\Big(\frac{\sqrt{C_u}}{\mathrm{e}}\Big) \big(\sqrt{2}\, r C_\Omega\big)^{N_1} \|\nabla^{N_1} g\|_{L^2(\mathbb{T}^2)} \leq C\, C_\Omega^{N_1} \Big(\frac{r}{\rho}\Big)^{N_1} \|g\|_{H^{N_1}_\rho(\mathbb{T}^2)}\,,$$

with C depending only on u and N_1. Since $g - Sg = (g - S_{\ell,r,\Omega_1} g) + S_{\ell,r,\Omega_1}(g - S_{\ell,r,\Omega_2} g)$ and $0 \le \widehat{W}_\ell \le 1$, this gives the third bound with the constant $2C$. This step uses neither the invertibility of Ω_i, nor the transversality of $\Omega_1 e_1$ and $\Omega_2 e_2$, nor $r \le \rho$, which proves the last sentence of the proposition. □

Lemma 8.2 (Time-dependent paired smoothing). *Let $\ell \ge 1$ and $r > 0$, and let $\Omega_1(t)$, $\Omega_2(t)$, $t \in \mathbb{R}$, be invertible linear maps of $\mathbb{R}^2$, Borel measurable in t, such that, for every $t \in \mathbb{R}$, the matrix with columns $\Omega_1(t)e_1$, $\Omega_2(t)e_2$ has an inverse of $|\cdot|_\infty$-norm at most 2. Then the operators $S(t) \coloneqq S_{\ell,r,\Omega_1(t)} S_{\ell,r,\Omega_2(t)}$ commute with derivatives; for every $g \in L^2(\mathbb{T}^2)$, $t \mapsto S(t)g \in L^2(\mathbb{T}^2)$ is Borel measurable; and there exist constants $c_j(u,\ell)$ such that, for every $t \in \mathbb{R}$, $j \ge 0$ and $g \in L^2(\mathbb{T}^2)$, $\|\nabla^j S(t) g\|_{L^2(\mathbb{T}^2)} \le c_j r^{-j} \|g\|_{L^2(\mathbb{T}^2)}$. Thus $S(t)$ is a smoothing operator of width r (Subsection 5.4 and the hypotheses of Proposition 6.2), with no bound on $|\Omega_i(t)|$.*

Proof. Step 1 of the proof of Proposition 6.1 at the time t gives the bound, and the multiplier form of $S(t)$ gives the commutation. The multiplier at each frequency depends continuously on the matrices, so $t \mapsto S(t)g$ is Borel measurable, by Plancherel's theorem and dominated convergence. □

8.3 Energy of coupled systems

We keep the setting of Proposition 6.2, without its condition on B, except that U_v, g_v and F_v^0 may take values in a finite-dimensional Euclidean space E_v ($E_v = \mathbb{R}$ for families), and F_v in E_v^2; $A_{\sigma,d}$ maps arrays of order $d-1$ over $E_{v'}$ to E_v^2, $S_\sigma(t)$ acts on each coordinate, and, in the component v, $L^2(\mathbb{T}^2)$ and $H^N(\mathbb{T}^2)$ stand for $L^2(\mathbb{T}^2; E_v)$ and $H^N(\mathbb{T}^2; E_v)$. We consider, for $v \in V$,

$$\begin{cases} \partial_t U_v + \nabla^\perp \Phi \cdot \nabla U_v - \nu \Delta U_v \\ \qquad = \nabla \cdot \Big(\displaystyle\sum_{v' \in V} \sum_{\sigma \colon v' \to v} \sum_{d=2}^{N_{\mathrm{src}}} A_{\sigma,d} \nabla^{d-1} (S_\sigma U_{v'}) + F_v \Big) + F_v^0 \quad \text{in } (t_1,t_2) \times \mathbb{T}^2 \,, \\ U_v(t_1,\cdot) = g_v \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \mathbb{T}^2 \,, \end{cases} \tag{8.2}$$

with a flux forcing F_v and a scalar forcing F_v^0. A *solution* of (8.2) is a function $U \in C([t_1,t_2]; L^2(\mathbb{T}^2)^V) \cap L^2((t_1,t_2); H^1(\mathbb{T}^2)^V)$ such that, for every $v \in V$ and $t \in [t_1,t_2]$,

$$U_v(t) - g_v = \int_{t_1}^t \Big(-\nabla^\perp \Phi \cdot \nabla U_v + \nu \Delta U_v + \nabla \cdot \Big(\sum_{v',\sigma,d} A_{\sigma,d} \nabla^{d-1} (S_\sigma U_{v'}) + F_v \Big) + F_v^0 \Big)(t'') \, \mathrm{d}t'' \,. \tag{8.3}$$

Here $\nabla^{d-1}(S_\sigma U_{v'}) = \nabla^{d-2} S_\sigma \nabla U_{v'}$ is measurable in time (by the linearity of $S_\sigma(t)$, its bound and the measurability of $t \mapsto S_\sigma(t)h$), with norm at most $c_{d-2} r_\sigma^{2-d} \|\nabla U_{v'}\|_{L^2((t_1,t_2);L^2(\mathbb{T}^2))}$ in $L^2((t_1,t_2); L^2(\mathbb{T}^2))$; (8.3) is an identity in $H^{-1}(\mathbb{T}^2)$.

For an integer $N \ge 0$ and $\rho > 0$, the *coupling matrix of order N at scale ρ* is

$$B_{vv'}^{N,\rho} \coloneqq \Big(\frac{\omega_{v'}}{\omega_v} \Big)^{1/2} \sum_{\sigma \colon v' \to v} \sum_{d=2}^{N_{\mathrm{src}}} \frac{c_{d-2}}{\nu \, r_\sigma^{d-2}} \sum_{j=0}^{N} \frac{\rho^j}{j!} \sup_{[t_1,t_2] \times \mathbb{T}^2} |\nabla^j A_{\sigma,d}| \,, \qquad v, v' \in V \,. \tag{8.4}$$

For $N = 0$ and every ρ, it is the coupling matrix B of Proposition 6.2.

Proposition 8.3 (Energy of coupled systems at higher order). *In the setting of (8.2), let $N \ge 0$ be an integer and $\rho > 0$, and assume the following.*

(i) *For every* $(t,x) \in [t_1,t_2] \times \mathbb{T}^2$ *and every integer* $1 \le i \le N$,

$$\rho^{i+1} \, |\nabla^i \nabla^\perp \Phi(t,x)| \le 2^{-i-2} \, i! \, \nu \, .$$

(ii) *For every* $t \in [t_1,t_2]$, *the coefficients* $A_{\sigma,d}(t,\cdot)$ *are of class* C^N *on* $\mathbb{T}^2$, *and, for every* $j \le N$, $\nabla^j A_{\sigma,d}$ *is bounded on* $[t_1,t_2] \times \mathbb{T}^2$.

(iii) *All row and column sums of* $B^{N,\rho}$ *are at most* $\frac{1}{8}$.

Then, for every datum g *and forcing* F, F^0 *such that, for every* $v \in V$, $g_v \in H^N(\mathbb{T}^2)$, $F_v \in L^2((t_1,t_2); H^N(\mathbb{T}^2)^2)$ *and* $F_v^0 \in L^1((t_1,t_2); H^N(\mathbb{T}^2))$, *there exists a unique solution* U *of* (8.2). *It belongs to* $C([t_1,t_2]; H^N(\mathbb{T}^2)^V) \cap L^2((t_1,t_2); H^{N+1}(\mathbb{T}^2)^V)$, *and the following hold.*

(a) *The solution satisfies*

$$\|U\|^2_{L^\infty([t_1,t_2];H^N_{\rho,\omega}(\mathbb{T}^2))} + \frac{\nu}{2} \, \|\nabla U\|^2_{L^2((t_1,t_2);H^N_{\rho,\omega}(\mathbb{T}^2))} \le 2\|g\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} + \frac{8}{\nu} \, \|F\|^2_{L^2((t_1,t_2);H^N_{\rho,\omega}(\mathbb{T}^2))} + 10 \, \|F^0\|^2_{L^1((t_1,t_2);H^N_{\rho,\omega}(\mathbb{T}^2))} \, .$$

(b) *Without forcing, for every* $t_1 \le t' \le t \le t_2$,

$$\|U(t)\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} + \nu \, \|\nabla U\|^2_{L^2((t',t);H^N_{\rho,\omega}(\mathbb{T}^2))} \le \|U(t')\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} \, .$$

(c) *Without forcing, the components satisfy, each with a separate supremum in time,*

$$\sum_{v \in V} \frac{1}{\omega_v} \Big(\|U_v\|^2_{L^\infty([t_1,t_2];H^N_\rho(\mathbb{T}^2))} + \nu \, \|\nabla U_v\|^2_{L^2((t_1,t_2);H^N_\rho(\mathbb{T}^2))} \Big) \le 2\|g\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} \, .$$

By (i), the velocity gradient times the diffusion time ρ^2/ν of the scale ρ is at most $\frac{1}{8}$. The sources cost at most an eighth of the dissipation $2\nu\|\nabla U\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)}$ (Lemma 8.4), and the drift a quarter (Lemma 8.5).

Lemma 8.4 (The source term). *Let* $N \ge 0$, $\rho > 0$ *and* $\vartheta \ge 0$, *assume (ii) of Proposition 8.3, and assume that all row and column sums of* $B^{N,\rho}$ *are at most* ϑ. *Then, for every* $t \in [t_1,t_2]$ *and every* $U = (U_v)_{v \in V}$ *with* $U_v \in H^{N+1}(\mathbb{T}^2)$,

$$\Big\| \Big(\sum_{v',\sigma,d} A_{\sigma,d}(t) \nabla^{d-1} (S_\sigma(t) U_{v'}) \Big)_{v \in V} \Big\|_{H^N_{\rho,\omega}(\mathbb{T}^2)} \le \vartheta\nu \, \|\nabla U\|_{H^N_{\rho,\omega}(\mathbb{T}^2)} \, .$$

Under (iii) of Proposition 8.3, this holds with $\vartheta = \frac{1}{8}$.

Proof. Fix $\sigma \colon v' \to v$ and d; we omit t. Since S_σ commutes with derivatives, its bound in $L^2(\mathbb{T}^2)$ holds in $H^N_\rho(\mathbb{T}^2)$, and with Lemma 8.1(a),

$$\big\| A_{\sigma,d} \nabla^{d-1} (S_\sigma U_{v'}) \big\|_{H^N_\rho(\mathbb{T}^2)} \le \frac{c_{d-2}}{r_\sigma^{d-2}} \Big(\sum_{j=0}^{N} \frac{\rho^j}{j!} \sup_{[t_1,t_2] \times \mathbb{T}^2} |\nabla^j A_{\sigma,d}| \Big) \|\nabla U_{v'}\|_{H^N_\rho(\mathbb{T}^2)} \, .$$

So the v-th component of the source term, divided by $\omega_v^{1/2}$, has norm in $H^N_\rho(\mathbb{T}^2)$ at most $\nu \sum_{v'} B^{N,\rho}_{vv'} \, \omega_{v'}^{-1/2} \|\nabla U_{v'}\|_{H^N_\rho(\mathbb{T}^2)}$, and Schur's test (Subsection 5.4) gives the lemma. □

Lemma 8.5 (The drift commutator). *Let $N \geq 1$, $\rho > 0$ and $t \in [t_1, t_2]$, and assume (i) of Proposition 8.3 at t. Then, for every smooth g on $\mathbb{T}^2$, with Φ taken at the time t,*

$$\sum_{j=1}^{N} \frac{\rho^{2j}}{(j!)^2} \left| \left\langle \nabla^j(\nabla^\perp \Phi \cdot \nabla g) - \nabla^\perp \Phi \cdot \nabla(\nabla^j g), \nabla^j g \right\rangle_{L^2(\mathbb{T}^2)} \right| \leq \frac{\nu}{4} \|\nabla g\|^2_{H^N_\rho(\mathbb{T}^2)} .$$

Proof. By the Leibniz rule, as in the proof of Lemma 8.1(a), the commutator is the sum over $1 \leq i \leq j$ of $\binom{j}{i}$ arrays in which i derivatives fall on $\nabla^\perp \Phi$ and $j - i$ on ∇g, each of norm at most $|\nabla^i \nabla^\perp \Phi|\, |\nabla^{j-i+1} g|$ at every point. Since $\binom{j}{i}\rho^{2j}/(j!)^2$ is the product of $\rho^{i+1}/(i!\, j)$, $\rho^{j-i}/(j-i)!$ and $\rho^{j-1}/(j-1)!$, the hypothesis and $1/j \leq 1$ bound the contribution of i to the term of j by

$$2^{-i-2}\nu \cdot \frac{\rho^{j-i}}{(j-i)!} \|\nabla^{j-i+1} g\|_{L^2(\mathbb{T}^2)} \cdot \frac{\rho^{j-1}}{(j-1)!} \|\nabla^j g\|_{L^2(\mathbb{T}^2)} .$$

For each i, the sum over j of these products is at most $2^{-i-2}\nu \|\nabla g\|^2_{H^N_\rho(\mathbb{T}^2)}$, by the Cauchy–Schwarz inequality, and $\sum_{i \geq 1} 2^{-i-2} = \frac{1}{4}$. □

Proof of Proposition 8.3. Step 1: The energy inequalities. Let U be a solution without forcing that is smooth in x, or one of the Galerkin system of Step 3 (the system projected onto trigonometric polynomials). We apply ∇^j, $j \leq N$, to the equation of U_v and take the inner product in $L^2(\mathbb{T}^2)$ with $(\rho^{2j}/(j!)^2)\nabla^j U_v$: the transport term $\langle \nabla^\perp \Phi \cdot \nabla(\nabla^j U_v), \nabla^j U_v \rangle_{L^2(\mathbb{T}^2)}$ vanishes because $\nabla^\perp \Phi$ is divergence free, the commutator is bounded by Lemma 8.5 applied to $U_v(t)$, and the source term pairs with ∇U_v after an integration by parts. Summing over j, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|U_v(t)\|^2_{H^N_\rho(\mathbb{T}^2)} + \frac{3}{2}\nu \|\nabla U_v(t)\|^2_{H^N_\rho(\mathbb{T}^2)} \leq 2 \left| \left\langle \sum_{v',\sigma,d} A_{\sigma,d} \nabla^{d-1}(S_\sigma U_{v'}), \nabla U_v \right\rangle_{H^N_\rho(\mathbb{T}^2)} \right| , \tag{8.5}$$

with the inner product of the norm (5.8). Divided by ω_v and summed over v, the right sides are at most $\frac{\nu}{4}\|\nabla U(t)\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)}$, by the Cauchy–Schwarz inequality and Lemma 8.4, so

$$\frac{\mathrm{d}}{\mathrm{d}t} \|U(t)\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} + \frac{5}{4}\nu \|\nabla U(t)\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} \leq 0 . \tag{8.6}$$

Integrating (8.6) over $[t', t]$ gives (b). For (c), we integrate (8.5) over $[t_1, t]$, drop its dissipation, take the supremum in t, divide by ω_v and sum. By the bound of the right sides above, and then by (8.6) integrated over $[t_1, t_2]$, the left side of (c) is at most

$$\|g\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} + \frac{5}{4}\nu \|\nabla U\|^2_{L^2((t_1,t_2);H^N_{\rho,\omega}(\mathbb{T}^2))} \leq 2\|g\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} .$$

Step 2: The forced estimate. With forcing, the computation of Step 1, summed over v, has the further terms $-2\langle F, \nabla U\rangle_{H^N_{\rho,\omega}(\mathbb{T}^2)}$ and $2\langle F^0, U\rangle_{H^N_{\rho,\omega}(\mathbb{T}^2)}$, with the inner product of the norm (8.1). Bounding the first by $\frac{\nu}{4}\|\nabla U\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} + \frac{4}{\nu}\|F\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)}$ and integrating gives, for every $t \in [t_1, t_2]$,

$$\begin{aligned} \|U(t)\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} + \nu \|\nabla U\|^2_{L^2((t_1,t);H^N_{\rho,\omega}(\mathbb{T}^2))} &\leq \|g\|^2_{H^N_{\rho,\omega}(\mathbb{T}^2)} + \frac{4}{\nu}\|F\|^2_{L^2((t_1,t_2);H^N_{\rho,\omega}(\mathbb{T}^2))} \\ &\quad + 2\|U\|_{L^\infty([t_1,t_2];H^N_{\rho,\omega}(\mathbb{T}^2))} \|F^0\|_{L^1((t_1,t_2);H^N_{\rho,\omega}(\mathbb{T}^2))} . \end{aligned} \tag{8.7}$$

Young's inequality absorbs the last term, in two stages. On the right side of (8.7), let Y be the sum of the first two terms and X and Z the two norms in the last, and let $D := \nu\|\nabla U\|^2_{L^2((t_1,t_2);H^N_{\rho,\omega}(\mathbb{T}^2))}$.

The supremum in t of (8.7) gives the first line below, and this supremum plus half of (8.7) at $t = t_2$ gives the second, which is (a), with the first line in its last step:

$$X^2 \le Y + 2XZ \le Y + \frac{1}{2}X^2 + 2Z^2\,, \qquad \text{so} \qquad X^2 \le 2Y + 4Z^2\,,$$
$$X^2 + \frac{1}{2}D \le \frac{3}{2}Y + 3XZ \le \frac{3}{2}Y + \frac{1}{4}X^2 + 9Z^2 \le 2Y + 10Z^2\,.$$

Step 3: Existence, uniqueness and regularity. By the standard Galerkin argument, with the Fourier truncations P_n, which are self-adjoint and commute with derivatives, Steps 1 and 2 hold for the Galerkin approximations uniformly in n, and a subsequence converges weakly-$*$ in $L^\infty([t_1,t_2];H^N(\mathbb{T}^2)^V) \cap L^2((t_1,t_2);H^{N+1}(\mathbb{T}^2)^V)$ to a solution U. For $j \le N$, $\partial_t \nabla^j U_v$ lies in $L^2((t_1,t_2);H^{-1}(\mathbb{T}^2)) + L^1((t_1,t_2);L^2(\mathbb{T}^2))$, by (i), Lemma 8.4 and the classes of F and F^0; testing with Fourier truncations then gives the energy identity for $\nabla^j U_v$, hence $U \in C([t_1,t_2];H^N(\mathbb{T}^2)^V)$ and Steps 1 and 2 at order N (Lemma 8.5 extends to $H^{N+1}(\mathbb{T}^2)$ by density), and so (a)–(c). At order 0 the same lines apply to every solution, with only the boundedness of $\nabla^\perp\Phi$, so the difference of two solutions vanishes by Step 1 with $N = 0$, since (i) is then void and $B^{0,\rho} \le B^{N,\rho}$. □

By the uniqueness at order 0 (Step 3), the restriction of a solution to $[t', t_2]$ is the solution from $U(t')$ with the restricted forcing, as it satisfies (8.3) there and Proposition 8.3 applies on $[t', t_2]$.

Proof of Proposition 6.2. Take $N = 0$ in Proposition 8.3, without forcing and with real components: (i) is void, (ii) is the boundedness of the coefficients, $B^{0,\rho} = B$, and (b) is the energy inequality. □

Proposition 8.6 (Systems without smoothing)**.** *In the setting of* (8.2)*, let* $\rho > 0$*, let* $N_v \ge 0$*,* $v \in V$*, be integers, and assume (i) of Proposition 8.3 at the order* $\max_v N_v$*. Let* $A_{vv',d}$*, for* $v, v' \in V$ *and finitely many* $d \ge 2$*, be coefficients as in Proposition 6.2 and in (ii) of Proposition 8.3 at the order* N_v*, vanishing unless* $N_{v'} \ge N_v + d - 2$*, such that all row and column sums of the matrix with entries*

$$\Big(\frac{\omega_{v'}}{\omega_v}\Big)^{1/2} \sum_d \frac{(N_v + d - 2)!}{N_v!\,\nu\,\rho^{d-2}} \sum_{j=0}^{N_v} \frac{\rho^j}{j!} \sup_{[t_1,t_2]\times\mathbb{T}^2} |\nabla^j A_{vv',d}|\,, \qquad v, v' \in V\,,$$

are at most $\frac{1}{8}$*. Then, for every* g*,* F*,* F^0 *such that, for every* $v \in V$*,* $g_v \in H^{N_v}(\mathbb{T}^2)$*,* $F_v \in L^2((t_1,t_2);H^{N_v}(\mathbb{T}^2)^2)$ *and* $F_v^0 \in L^1((t_1,t_2);H^{N_v}(\mathbb{T}^2))$*, the problem, for* $v \in V$*,*

$$\begin{cases} \partial_t U_v + \nabla^\perp\Phi\cdot\nabla U_v - \nu\Delta U_v = \nabla\cdot\Big(\displaystyle\sum_{v'\in V}\sum_d A_{vv',d}\nabla^{d-1}U_{v'} + F_v\Big) + F_v^0 & \text{in } (t_1,t_2)\times\mathbb{T}^2\,, \\ U_v(t_1,\cdot) = g_v & \text{on } \mathbb{T}^2\,, \end{cases}$$

has a unique solution U *(as in* (8.3)*) with* $U_v \in C([t_1,t_2];H^{N_v}(\mathbb{T}^2)) \cap L^2((t_1,t_2);H^{N_v+1}(\mathbb{T}^2))$*, and*

$$\|U\|^2_{L^\infty([t_1,t_2];H^{(N_v)}_{\rho,\omega}(\mathbb{T}^2))} + \frac{\nu}{2}\|\nabla U\|^2_{L^2((t_1,t_2);H^{(N_v)}_{\rho,\omega}(\mathbb{T}^2))} \le 2\|g\|^2_{H^{(N_v)}_{\rho,\omega}(\mathbb{T}^2)} + \frac{8}{\nu}\|F\|^2_{L^2((t_1,t_2);H^{(N_v)}_{\rho,\omega}(\mathbb{T}^2))} + 10\,\|F^0\|^2_{L^1((t_1,t_2);H^{(N_v)}_{\rho,\omega}(\mathbb{T}^2))}\,.$$

The system (8.2) with every S_σ replaced by the identity is of this form, with $A_{vv',d} = \sum_{\sigma\colon v'\to v} A_{\sigma,d}$.

Proof. The proof is that of Proposition 8.3, for each U_v at the order N_v and with uniqueness by (a) for differences, except for the source term: by Lemma 8.1, with $i = d - 2$,

$$\|A_{vv',d}\nabla^{d-1}U_{v'}\|_{H^{N_v}_\rho(\mathbb{T}^2)} \le \Big(\sum_{j=0}^{N_v}\frac{\rho^j}{j!}\sup_{[t_1,t_2]\times\mathbb{T}^2}|\nabla^j A_{vv',d}|\Big)\frac{(N_v+d-2)!}{N_v!\,\rho^{d-2}}\,\|\nabla U_{v'}\|_{H^{N_v+d-2}_\rho(\mathbb{T}^2)}\,,$$

and, when $A_{vv',d} \ne 0$, $\|\nabla U_{v'}\|_{H^{N_v+d-2}_\rho(\mathbb{T}^2)} \le \|\nabla U_{v'}\|_{H^{N_{v'}}_\rho(\mathbb{T}^2)}$; Schur's test concludes. □

9. Reduction and proofs of the main results

In this section we prove Lemma 4.3 and Proposition 4.2, deduce Theorem A from them and Proposition 4.1, state and prove Theorem C and Corollary D, and prove Theorem B.

Conventions. Throughout, C and c denote positive constants depending only on $\alpha, s, q, \zeta, \Lambda$ and on the quantities named in each statement. We write $D_r(x) \coloneqq \{y \in \mathbb{R}^2 : |y-x| < r\}$ and $D_r \coloneqq D_r(0)$; for $E \subset \mathbb{R}^2$, $\|g\|_{C^{0,s}(E)}$ and $[g]_{C^{0,s}(E)}$ are defined by (3.1), with E in place of $\mathbb{T}^2$ and $|x-y|$ in place of $|x-y|_{\mathbb{T}^2}$. We fix $C_0(\alpha, s, q, \zeta, \Lambda) \geq 1$ such that, for every $m \geq 0$,

$$\|\phi - \phi_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \leq C_0 \varepsilon_{m+1}^{\beta}\,, \quad \|\mathbf{b} - \mathbf{b}_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \leq C_0 \varepsilon_{m+1}^{\alpha}\,, \quad \|\nabla^2 \phi_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \leq C_0 \varepsilon_m^{\beta-2}\,, \tag{9.1}$$

by (2.14) and Proposition 7.3(ii). A disc of radius at most $\frac{1}{2}$ meets at most four translates of the unit cell, so, for every periodic g, $y \in \mathbb{R}^2$ and $r \leq \frac{1}{2}$, $\int_{D_r(y)} g(x)^2 \,\mathrm{d}x \leq 4\|g\|^2_{L^2(\mathbb{T}^2)}$.

Weak solutions are those of Section 3, and a weak solution on $(t_{\mathsf{in}}, \infty)$ is one on every $(t_{\mathsf{in}}, t_{\mathsf{in}}+T)$. For a bounded, measurable, $\mathbb{Z}^2$-periodic and divergence-free velocity field and $\kappa > 0$, the initial-value problem (3.2) has exactly one weak solution on every $(t_{\mathsf{in}}, t_{\mathsf{in}}+T)$; its restriction to $(t'_{\mathsf{in}}, t_{\mathsf{in}}+T)$ is the weak solution with datum $\theta(t'_{\mathsf{in}}, \cdot)$ at time t'_{in}, and, for every $t \in (t_{\mathsf{in}}, t_{\mathsf{in}}+T)$,

$$\|\theta(t,\cdot)\|^2_{L^2(\mathbb{T}^2)} + 2\kappa \|\nabla\theta\|^2_{L^2((t_{\mathsf{in}},t)\times\mathbb{T}^2)} = \|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)}\,. \tag{9.2}$$

These facts are standard (Galerkin approximation, Lions–Magenes), and so is the first assertion below; the second follows from it, since the balls of $C^{0,s}(\mathbb{T}^2)$ are weakly closed in $L^2(\mathbb{T}^2)$: for every $[t_1, t_2] \subset [t_{\mathsf{in}}, t_{\mathsf{in}}+T]$ and $t \in [t_1, t_2]$,

$$\theta \in C([t_{\mathsf{in}}, t_{\mathsf{in}}+T]; L^2(\mathbb{T}^2))\,, \qquad \|\theta(t,\cdot)\|_{C^{0,s}(\mathbb{T}^2)} \leq \|\theta\|_{L^\infty([t_1,t_2];C^{0,s}(\mathbb{T}^2))}\,. \tag{9.2$'$}$$

So the limit in L^2_{loc} of $\theta(t',\cdot)$ as $t' \uparrow t$, when it exists, is $\theta(t,\cdot)$.

For an open set $\mathcal{O} \subset \mathbb{R}\times\mathbb{R}^2$ with time sections $\mathcal{O}_t \coloneqq \{x : (t,x) \in \mathcal{O}\}$, a *local weak solution* of $\partial_t\theta + \mathbf{b}\cdot\nabla\theta - \kappa\Delta\theta = 0$ in $\mathcal{O}$ is a function $\theta \in L^2_{\mathrm{loc}}(\mathcal{O})$ such that, for almost every t, $\theta(t,\cdot)$ has a weak gradient on $\mathcal{O}_t$, with $\nabla\theta \in L^2_{\mathrm{loc}}(\mathcal{O})$, and, for every $g \in C_c^\infty(\mathcal{O})$,

$$\int_{\mathcal{O}} \big(-\theta\,\partial_t g + (\mathbf{b}\cdot\nabla\theta)g + \kappa\nabla\theta\cdot\nabla g\big)\,\mathrm{d}t\,\mathrm{d}x = 0\,. \tag{9.3}$$

A weak solution on $(t_{\mathsf{in}}, t_{\mathsf{in}}+T)$ is a local weak solution in $(t_{\mathsf{in}}, t_{\mathsf{in}}+T)\times\mathbb{R}^2$.

Finally, by Lemma 7.4 and (2.13), for every $m \geq 0$ and $|t - t_{\mathsf{f}}| \leq 8h_{m+1}$, the map $y \mapsto X_m(t, y, t_{\mathsf{f}})$ of (2.11) is a smooth, volume-preserving diffeomorphism of $\mathbb{R}^2$ commuting with the translations by $\mathbb{Z}^2$, with inverse $x \mapsto X_m(t_{\mathsf{f}}, x, t)$, and both maps are 2-Lipschitz, since

$$|\nabla_y X_m(t, y, t_{\mathsf{f}}) - I| \leq \tfrac{1}{2}\,, \qquad |\nabla_x X_m(t_{\mathsf{f}}, x, t) - I| \leq \tfrac{1}{2}\,. \tag{9.4}$$

9.1 Perturbed heat equations and moving cylinders

Proof of Lemma 4.3. We adapt Campanato's method of [Gia83, Chapter III, Section 2] to parabolic equations, with caloric comparison functions and the energy estimates of [LSU68]. In this proof, C depends only on s. We write $\mathcal{Q}_r(t,x) \coloneqq (t-r^2, t) \times D_r(x)$ and $\mathcal{Q}_r \coloneqq \mathcal{Q}_r(0,0)$, and take $\varpi_1 \leq 1$.

Step 1: Energy and a caloric comparison function. Testing the equation with $(U - (U)_{\mathcal{Q}_1})g^2$, for a cutoff g equal to 1 on $\mathcal{Q}_{3/4}$ and vanishing near the parabolic boundary of $\mathcal{Q}_1$, gives

$$\int_{\mathcal{Q}_{3/4}} |\nabla U|^2 \,\mathrm{d}t\,\mathrm{d}x \leq C \int_{\mathcal{Q}_1} \big|U - (U)_{\mathcal{Q}_1}\big|^2 \,\mathrm{d}t\,\mathrm{d}x\ : \tag{9.5}$$

the skew part has no principal term, since $\nabla U \cdot (\mathbf{a} - I)\nabla U = 0$, and the divergence-free drift contributes only $\frac{1}{2}\int (U - (U)_{\mathcal{Q}_1})^2 \mathbf{b}_* \cdot \nabla g^2 \, \mathrm{d}t \, \mathrm{d}x$. Since $\mathbf{b}_* \cdot \nabla U = \nabla \cdot ((U - (U)_{\mathcal{Q}_1})\mathbf{b}_*)$, U solves $\partial_t U - \Delta U = \nabla \cdot F$ with $F := (\mathbf{a} - I)\nabla U - (U - (U)_{\mathcal{Q}_1})\mathbf{b}_*$, and $|F| \le \varpi_1(|\nabla U| + |U - (U)_{\mathcal{Q}_1}|)$. Let W solve this forced heat equation in $\mathcal{Q}_{3/4}$ with zero data, and set $U_{\mathrm{cal}} := U - W$. Then $U_{\mathrm{cal}} - (U)_{\mathcal{Q}_1}$ is caloric in $\mathcal{Q}_{3/4}$, and by the energy estimate and (9.5)

$$\int_{\mathcal{Q}_{3/4}} |W|^2 \, \mathrm{d}t \, \mathrm{d}x \le C \int_{\mathcal{Q}_{3/4}} |F|^2 \, \mathrm{d}t \, \mathrm{d}x \le C\varpi_1^2 \int_{\mathcal{Q}_1} \left|U - (U)_{\mathcal{Q}_1}\right|^2 \mathrm{d}t \, \mathrm{d}x \, .$$

Step 2: One step of decay. By the heat kernel representation of a cut-off caloric function, for every $(t,x), (t',x') \in (-\frac{1}{4}, 0] \times D_{1/2}$,

$$|U_{\mathrm{cal}}(t,x) - U_{\mathrm{cal}}(t',x')| \le C(|t - t'| + |x - x'|) \int_{\mathcal{Q}_{3/4}} \left|U_{\mathrm{cal}}(\tau, z) - (U)_{\mathcal{Q}_1}\right| \mathrm{d}\tau \, \mathrm{d}z \, .$$

With Step 1 and $|\mathcal{Q}_1|/|\mathcal{Q}_r| = r^{-4}$, this gives, for every $0 < r \le \frac{1}{2}$,

$$\fint_{\mathcal{Q}_r} \left|U - (U)_{\mathcal{Q}_r}\right|^2 \mathrm{d}t \, \mathrm{d}x \le C\big(r^2 + \varpi_1^2 r^{-4}\big) \fint_{\mathcal{Q}_1} \left|U - (U)_{\mathcal{Q}_1}\right|^2 \mathrm{d}t \, \mathrm{d}x \, . \tag{9.6}$$

Step 3: Iteration. Since $1 + s < 2$, we may fix $r_0 \in (0, \frac{1}{2}]$ with $Cr_0^2 \le \frac{1}{2} r_0^{1+s}$ and then ϖ_1 with $C\varpi_1^2 r_0^{-4} \le \frac{1}{2} r_0^{1+s}$; then, at $r = r_0$, the constant on the right side of (9.6) is at most r_0^{1+s}. For $(t,x) \in \mathbb{R} \times \mathbb{R}^2$ and $0 < r \le 1$, the rescaled function $(\tau, z) \mapsto U(t + r^2\tau, x + rz)$ solves an equation of the same form, with the drift multiplied by $r \le 1$. We iterate with the radii $\frac{1}{2} r_0^j$, $j \ge 1$, starting from $\mathcal{Q}_{1/2}(t,x) \subset \mathcal{Q}_1$, with a constant depending only on r_0 for the first step and for the radii between two consecutive ones, and obtain, for every $(t,x) \in (-\frac{1}{4}, 0] \times D_{1/2}$ and $0 < r \le \frac{1}{2}$,

$$\fint_{\mathcal{Q}_r(t,x)} \left|U(\tau, z) - (U)_{\mathcal{Q}_r(t,x)}\right|^2 \mathrm{d}\tau \, \mathrm{d}z \le C r^{1+s} \fint_{\mathcal{Q}_1} \left|U(\tau, z) - (U)_{\mathcal{Q}_1}\right|^2 \mathrm{d}\tau \, \mathrm{d}z \, . \tag{9.7}$$

Step 4: Hölder continuity. By (9.7), the averages of U over $\mathcal{Q}_r(t,x)$ and $\mathcal{Q}_{r/2}(t,x)$ differ by at most $Cr^{(1+s)/2}(\fint_{\mathcal{Q}_1} |U - (U)_{\mathcal{Q}_1}|^2)^{1/2}$. So the averages over $\mathcal{Q}_{2^{-i}}(t,x)$ converge uniformly in (t,x) to U, which is within C times this bound of $(U)_{\mathcal{Q}_r(t,x)}$. One comparison of $(U)_{\mathcal{Q}_{1/2}(t,x)}$ with $(U)_{\mathcal{Q}_1}$ gives the L^∞ term of Lemma 4.3. Comparing averages over $\mathcal{Q}_{2r}(t,x) \supset \mathcal{Q}_r(t',x')$, at the radius $r = |x - x'| + |t - t'|^{1/2}$ for the points (t,x) and (t',x') with $t' \le t$, gives the seminorm for $r \le \frac{1}{4}$, where (9.7) applies at the radius $2r$. For $r > \frac{1}{4}$ the seminorm follows from the L^∞ term, since $|U(t,x) - U(t',x')| \le 2\|U - (U)_{\mathcal{Q}_1}\|_{L^\infty((-\frac{1}{4},0] \times D_{1/2})}$ and $1 < (4r)^{(1+s)/2}$. The uniform continuity of U on $(-\frac{1}{4}, 0] \times D_{1/2}$ gives the convergence of $U(t,\cdot)$ as $t \uparrow 0$. □

A trajectory of $\mathbf{b}$ on a time interval is a C^1 curve X with $\partial_{t'} X(t') = \mathbf{b}(t', X(t'))$; since $\mathbf{b}$ is bounded and continuous, a trajectory ends at every point at every time.

Lemma 9.1 (Hölder bounds along moving cylinders)**.** *There exist $c_{\mathrm{cyl}} > 0$ and $C < \infty$, depending only on $\alpha, s, q, \zeta, \Lambda$, with the following property. Let $\kappa > 0$, let $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^2$ be open, let θ be a local weak solution of $\partial_t \theta + \mathbf{b} \cdot \nabla \theta - \kappa \Delta \theta = 0$ in $\mathcal{O}$, and let $\Xi \ge 0$, $t \in \mathbb{R}$, $E \subset \mathbb{R}^2$ and $0 < r \le \min\{\frac{1}{8}, c_{\mathrm{cyl}} \kappa^{1/\beta}\}$. Assume that, for almost every $t' \in \mathbb{R}$ and every disc $D_r(y) \subset \mathcal{O}_{t'}$, $\int_{D_r(y)} \theta(t', x')^2 \, \mathrm{d}x' \le \Xi^2$, and that, for every $x \in E$ and every trajectory X of $\mathbf{b}$ on $[t - r^2/\kappa, t]$ with $X(t) = x$, the moving cylinder $\{(t', y) : t - r^2/\kappa \le t' \le t, \ |y - X(t')| \le r\}$ lies in $\mathcal{O}$. Then, for every $x \in E$, $\theta(t', \cdot)$ converges in $L^2(D_{r/4}(x))$ as $t' \uparrow t$; the limit $\theta(t, \cdot)$ is continuous on $\bigcup_{x \in E} D_{r/4}(x)$, and*

$$\|\theta(t, \cdot)\|_{C^{0,s}(E)} \le C r^{-1-s} \, \Xi \, . \tag{9.8}$$

Proof. Fix $x \in E$ and a trajectory X of $\mathbf{b}$ on $[t - r^2/\kappa, t]$ with $X(t) = x$, and set

$$U(t', y) := \theta\Big(t + \frac{r^2}{\kappa}t', X\Big(t + \frac{r^2}{\kappa}t'\Big) + ry\Big), \qquad (t', y) \in (-1, 0) \times D_1 .$$

Then U solves the equation of Lemma 4.3 with $\mathbf{a} = I$ and $\mathbf{b}_*(t', y) = \frac{r}{\kappa}[\mathbf{b}(t'', X(t'') + ry) - \mathbf{b}(t'', X(t''))]$, $t'' = t + r^2 t'/\kappa$: the frame moves with $\mathbf{b}$ at its centre. This drift is divergence free, and by (2.15) $|\mathbf{b}_*| \le Cr^{1+\alpha}/\kappa \le \varpi_1$ once c_{cyl} is small. Since the discs $D_r(X(t''))$ lie in $\mathcal{O}_{t''}$, $\fint_{(-1,0)\times D_1} U(\tau, z)^2 \,\mathrm{d}\tau \,\mathrm{d}z \le \Xi^2/(\pi r^2)$. Lemma 4.3 bounds U on $(-\frac{1}{4}, 0] \times D_{1/2}$; since $X(t'') \to x$ as $t'' \uparrow t$, its Hölder bound gives the convergence of $\theta(t'', \cdot)$ to $U(0, (\cdot - x)/r)$, uniformly on $D_{r/4}(x)$ and locally uniformly on $D_{r/2}(x)$, and, on $D_{r/2}(x)$, the limit $\theta(t, \cdot)$ satisfies the bounds $|\theta(t, \cdot)| \le C\Xi/r$ and $|\theta(t, z) - \theta(t, z')| \le C\Xi r^{-1}(|z - z'|/r)^{(1+s)/2}$; the limits from different cylinders agree on overlaps. For $z, z' \in E$ with $|z - z'| < \frac{r}{4}$, the cylinder at z gives $|\theta(t, z) - \theta(t, z')| \le C\Xi r^{-1-s}|z - z'|^s$, since $s < \frac{1}{2}(1 + s)$; for $|z - z'| \ge \frac{r}{4}$ the bound on $|\theta(t, \cdot)|$ gives the same. □

9.2 Transfer to the full velocity field

Lemma 9.2 (Stream comparison)**.** *Let Φ and $\widetilde{\Phi}$ be as in Proposition 4.2, and let $\kappa > 0$ and $\varpi \ge 0$ with $\|\Phi - \widetilde{\Phi}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le \kappa\varpi$. Let $t_{\mathsf{in}} \in \mathbb{R}$, $T > 0$, $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$, and let θ and $\widetilde{\theta}$ be weak solutions on $(t_{\mathsf{in}}, t_{\mathsf{in}} + T)$ of the two equations of Proposition 4.2, with the datum θ_{in} at time t_{in}. Then*

$$\|\theta - \widetilde{\theta}\|_{L^\infty([t_{\mathsf{in}}, t_{\mathsf{in}}+T]; L^2(\mathbb{T}^2))} \le \varpi \, \|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)} .$$

Proof. Since $\nabla^\perp(\Phi - \widetilde{\Phi}) \cdot \nabla\widetilde{\theta} = -\nabla \cdot ((\Phi - \widetilde{\Phi})\nabla^\perp\widetilde{\theta})$, the difference $\theta - \widetilde{\theta}$ solves

$$\partial_t(\theta - \widetilde{\theta}) + \nabla^\perp\Phi \cdot \nabla(\theta - \widetilde{\theta}) - \kappa\Delta(\theta - \widetilde{\theta}) = \nabla \cdot \big((\Phi - \widetilde{\Phi})\nabla^\perp\widetilde{\theta}\big)$$

with zero datum. Its energy identity, Young's inequality and (9.2) for $\widetilde{\theta}$ give, for $t_{\mathsf{in}} < t < t_{\mathsf{in}} + T$, $\|(\theta - \widetilde{\theta})(t, \cdot)\|^2_{L^2(\mathbb{T}^2)} \le \kappa\varpi^2\|\nabla\widetilde{\theta}\|^2_{L^2((t_{\mathsf{in}},t)\times\mathbb{T}^2)} \le \varpi^2\|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)}$. □

Proof of Proposition 4.2. Let $\varpi_0 := \varpi_1$, the constant of Lemma 4.3, and fix $t \in [t_1, t_2]$.

Step 1: The inputs. By Lemma 9.2 with $t_{\mathsf{in}} + T = t_2 + r^2/\kappa$, $\|\theta - \widetilde{\theta}\|_{L^\infty([t_{\mathsf{in}}, t_2 + r^2/\kappa]; L^2(\mathbb{T}^2))} \le \varpi\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$; by (9.2′), the bound Ξ on $\widetilde{\theta}$ holds at every time of $[t_1 - r^2/\kappa, t_2 + r^2/\kappa]$.

Step 2: A cylinder moving with $\nabla^\perp\widetilde{\Phi}$. Fix $x \in \mathbb{R}^2$ and the trajectory $\widetilde{X}$ of $\nabla^\perp\widetilde{\Phi}$ with $\widetilde{X}(t) = x$, and set, for $(t', y) \in (-1, 0) \times D_1$ and $t'' := t + r^2 t'/(4\kappa)$,

$$U(t', y) := \theta\big(t'', \widetilde{X}(t'') + \tfrac{r}{2}y\big), \qquad \widetilde{U}(t', y) := \widetilde{\theta}\big(t'', \widetilde{X}(t'') + \tfrac{r}{2}y\big) .$$

By the identity in the proof of Lemma 9.2, applied to θ, U solves the equation of Lemma 4.3 with

$$\mathbf{a} - I = \frac{\Phi - \widetilde{\Phi}}{\kappa}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{b}_*(t', y) = \frac{r}{2\kappa}\big[\nabla^\perp\widetilde{\Phi}\big(t'', \widetilde{X}(t'') + \tfrac{r}{2}y\big) - \nabla^\perp\widetilde{\Phi}\big(t'', \widetilde{X}(t'')\big)\big],$$

the first evaluated at $(t'', \widetilde{X}(t'') + \frac{r}{2}y)$; $\mathbf{a} - I$ is skew with $|\mathbf{a} - I| \le \varpi$, and $\mathbf{b}_*$ is divergence free with $|\mathbf{b}_*| \le \frac{r^2}{4\kappa}\|\nabla^2\widetilde{\Phi}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}$, so $|\mathbf{a} - I| + |\mathbf{b}_*| \le \varpi_0$; $\widetilde{U}$ solves it with $\mathbf{a} = I$; and $t'' \in (t_1 - r^2/\kappa, t_2]$.

Step 3: Closeness to a constant. By Step 1, $\widetilde{U}(t', \cdot)$ oscillates on D_1 by at most Ξr^s. The function $u_1 := 4u(2\cdot)$, with u fixed at the start of Section 6, is smooth, nonnegative, of integral one and supported in $\overline{D_{1/2}}$. Testing the equation of $\widetilde{U}$ with it gives, in the sense of distributions in t',

$$\frac{\mathrm{d}}{\mathrm{d}t'}\big\langle\widetilde{U}(t', \cdot), u_1\big\rangle_{L^2(D_1)} = \int_{D_1}\big(\widetilde{U}(t', y) - \widetilde{U}(t', 0)\big)\big(\Delta + \mathbf{b}_* \cdot \nabla\big)u_1(y) \,\mathrm{d}y ,$$

because $(\Delta+\mathbf{b}_*\cdot\nabla)u_1$ has integral zero ($\nabla\cdot\mathbf{b}_*=0$). Hence the pairing varies by at most $C\Xi r^s$ over $(-1,0)$, and $\widetilde{U}$ stays within $C\Xi r^s$ of $\langle\widetilde{U}(0,\cdot),u_1\rangle_{L^2(D_1)}$. By Step 1 and the four translates of the unit cell, $\int_{D_1}|U-\widetilde{U}|^2(t',y)\,\mathrm{d}y\le 16r^{-2}\varpi^2\|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)}$: normalizing the L^2 deviation to the cylinder costs r^{-1}. Altogether,

$$\Big(\fint_{(-1,0)\times D_1}\Big|U(t',y)-\big\langle\widetilde{U}(0,\cdot),u_1\big\rangle_{L^2(D_1)}\Big|^2\,\mathrm{d}t'\,\mathrm{d}y\Big)^{1/2}\le C\big(\varpi r^{-1}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}+\Xi r^s\big)\,. \tag{9.9}$$

Step 4: All distances. Since $\widetilde{X}(t'')\to x$ as $t''\uparrow t$, Lemma 4.3 and (9.9) give that $\theta(t'',\cdot)$ converges to $U(0,2(\cdot-x)/r)$ in $L^2_{\mathrm{loc}}(D_{r/4}(x))$, and, by (9.2′), this limit is $\theta(t,\cdot)$. It is within $C(\varpi r^{-1}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}+\Xi r^s)$ of the constant in (9.9), hence of $\widetilde{\theta}(t,x)=\widetilde{U}(0,0)$, and its increments over $|z-z'|<\frac{r}{4}$ are at most $C(\varpi r^{-1}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}+\Xi r^s)(|z-z'|/r)^{(1+s)/2}$. Since x is arbitrary,

$$\|\theta(t,\cdot)-\widetilde{\theta}(t,\cdot)\|_{L^\infty(\mathbb{T}^2)}\le C\big(\varpi r^{-1}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}+\Xi r^s\big)\,.$$

For $|z-z'|_{\mathbb{T}^2}<\frac{r}{4}$ the increment bound gives $|\theta(t,z)-\theta(t,z')|\le C(\varpi r^{-1-s}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}+\Xi)|z-z'|^s_{\mathbb{T}^2}$; for $|z-z'|_{\mathbb{T}^2}\ge\frac{r}{4}$, the comparison with $\widetilde{\theta}(t,\cdot)$ and the bound on $\widetilde{\theta}$ give the same, since $1\le 4^s r^{-s}|z-z'|^s_{\mathbb{T}^2}$. Hence $\|\theta(t,\cdot)\|_{C^{0,s}(\mathbb{T}^2)}\le C(\Xi+\varpi r^{-1-s}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)})$, since $r\le 1$; and $t\in[t_1,t_2]$ is arbitrary. □

9.3 Proof of Theorem A

Proof of Theorem A. We choose the number n of extra levels (Step 1); for the bands far out, Proposition 4.1 bounds the solution for the truncated velocity field $\mathbf{b}_{M+n}$, and Proposition 4.2 transfers this bound to θ (Step 2); the finitely many remaining bands follow from Lemma 9.1 (Step 3). Let Λ_0 be the threshold of Proposition 4.1, and fix $\Lambda\ge\Lambda_0$ and $0<t_0<T$; the constants below are independent of t_{in}, κ, θ_{in} and θ. We continue θ to the weak solution on $(t_{\mathsf{in}},t_{\mathsf{in}}+T+1)$ with datum θ_{in}, which agrees with it on $(t_{\mathsf{in}},t_{\mathsf{in}}+T)$.

Step 1: The number of extra levels. The bracket $\beta q-\frac{1}{2}(1+s)(2-\beta)$ is positive, since $\beta q>\beta$ and, as $\beta=1+\alpha$ and $s<\frac{1}{2}(1-\alpha)$, $(1+s)(2-\beta)<\frac{1}{2}(3-\alpha)(1-\alpha)<2\beta$. Fix an integer $n\ge 1$ with

$$q^n\Big(\beta q-\frac{(1+s)(2-\beta)}{2}\Big)-\frac{(\beta-\gamma)(3+s)}{2}\ge 0\,; \tag{9.10}$$

it depends only on α,s,q.

Step 2: The bands far out. Let $\kappa\in\mathcal{K}_M$, so that $\frac{1}{2}\varepsilon_M^{\beta-\gamma}\le\kappa\le 2\varepsilon_M^{\beta-\gamma}$. We apply Proposition 4.2, whose regularity hypotheses hold by Proposition 7.3(i), (iii) and (iv), with $\Phi=\phi$, $\widetilde{\Phi}=\phi_{M+n}$ and

$$r:=\Big(\frac{\varpi_0\kappa}{2C_0}\Big)^{1/2}\varepsilon_{M+n}^{(2-\beta)/2}\,,\qquad \varpi:=\frac{C_0\varepsilon_{M+n+1}^{\beta}}{\kappa}\,,$$

so that, by (9.1), $\|\phi-\phi_{M+n}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\le\kappa\varpi$ and $\frac{r^2}{\kappa}\|\nabla^2\phi_{M+n}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\le\frac{1}{2}\varpi_0$. Since $\varepsilon_m\asymp\varepsilon_{m-1}^q$, we have $\varepsilon_{M+n+1}\le C\varepsilon_M^{q^{n+1}}$ and $\varepsilon_{M+n}\ge c\varepsilon_M^{q^n}$, with constants depending on n, and, as $\kappa\ge\frac{1}{2}\varepsilon_M^{\beta-\gamma}$,

$$\varpi\, r^{-1-s}\le C\varepsilon_{M+n+1}^{\beta}\kappa^{-(3+s)/2}\varepsilon_{M+n}^{-(2-\beta)(1+s)/2}\le C\varepsilon_M^{q^n(\beta q-(1+s)(2-\beta)/2)-(\beta-\gamma)(3+s)/2}\le C$$

by (9.10). Moreover $r\le C\varepsilon_M^{(\beta-\gamma)/2}$, $r^2/\kappa\le C\varepsilon_M^{2-\beta}$ and $\varpi\le C\varepsilon_M^{\beta q^{n+1}-(\beta-\gamma)}$ tend to zero as $M\to\infty$, uniformly in $\kappa\in\mathcal{K}_M$; we fix M_{A}, depending only on $\alpha,s,q,\zeta,\Lambda,t_0,T$, such that, for $M\ge M_{\mathrm{A}}$, $r\le\frac{1}{8}$, $r^2/\kappa\le\min\{\frac{1}{2}t_0,1\}$, $\varpi\le\frac{1}{2}\varpi_0$, and M is at least the threshold M_1 of Proposition 4.1 for the

times $\frac{1}{2}t_0 < T+1$. For such M, Proposition 4.1 bounds the solution $\widetilde{\theta}$ of (4.1) on $(t_{\mathsf{in}}, t_{\mathsf{in}}+T+1)$ in $L^\infty([t_{\mathsf{in}}+\frac{1}{2}t_0, t_{\mathsf{in}}+T+1]; C^{0,s}(\mathbb{T}^2))$ by $C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$, and, by Proposition 4.2 with $t_1 = t_{\mathsf{in}}+t_0$ and $t_2 = t_{\mathsf{in}}+T$, we obtain $\|\theta\|_{L^\infty([t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T]; C^{0,s}(\mathbb{T}^2))} \le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$.

Step 3: The remaining bands. If $\kappa \in \mathcal{K}_M$ with $M < M_{\mathrm{A}}$, then $\kappa \ge \frac{1}{2}\varepsilon_{M_{\mathrm{A}}-1}^{\beta-\gamma}$. Let $t \in [t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T]$ and

$$r := \min\Big\{\frac{1}{8},\ c_{\mathrm{cyl}}\Big(\frac{1}{2}\varepsilon_{M_{\mathrm{A}}-1}^{\beta-\gamma}\Big)^{1/\beta},\ \Big(\frac{1}{4}\varepsilon_{M_{\mathrm{A}}-1}^{\beta-\gamma}t_0\Big)^{1/2}\Big\}.$$

Then $r \le c_{\mathrm{cyl}}\kappa^{1/\beta}$ and $r^2/\kappa \le \frac{1}{2}t_0$, so every moving cylinder ending at time t lies in $\mathcal{O} := (t_{\mathsf{in}}, t_{\mathsf{in}}+T+1)\times\mathbb{R}^2$. By (9.2) and the four translates of the unit cell, the disc bound of Lemma 9.1 holds with $\Xi = 2\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. By the lemma with $E = \mathbb{R}^2$, we obtain $\|\theta(t,\cdot)\|_{C^{0,s}(\mathbb{T}^2)} \le Cr^{-1-s}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ (for periodic functions the norms on $\mathbb{R}^2$ and $\mathbb{T}^2$ agree), and r depends only on $\alpha, s, q, \zeta, \Lambda, t_0, T$. □

9.4 Local regularity in tubes: Theorem C

For an integer $m \ge 0$, a radius $r_{\mathrm{tube}} > 0$ and a final time $t_{\mathrm{f}} \in \mathbb{R}$, the *tube* is

$$\mathcal{T} := \big\{(t, X_m(t, y, t_{\mathrm{f}})) : t_{\mathrm{f}} - 8h_{m+1} < t < t_{\mathrm{f}},\ |y| < r_{\mathrm{tube}}\big\}, \tag{9.11}$$

the disc $D_{r_{\mathrm{tube}}}$ carried by the trajectories (2.11) of $\mathbf{b}_m$ over the time $8h_{m+1}$. For $\kappa > 0$, a *local energy solution* of $\partial_t\theta + \mathbf{b}\cdot\nabla\theta - \kappa\Delta\theta = 0$ in $\mathcal{T}$ is a function $\theta \in L^2(\mathcal{T})$ such that, for every $g \in C_c^\infty(\mathcal{T})$,

$$\int_{\mathcal{T}} \theta\,(\partial_t g + \mathbf{b}\cdot\nabla g + \kappa\Delta g)\,\mathrm{d}t\,\mathrm{d}x = 0, \tag{9.12}$$

which is the equation, written as $\partial_t\theta + \nabla\cdot(\mathbf{b}\theta) - \kappa\Delta\theta = 0$ since $\nabla\cdot\mathbf{b} = 0$, and such that, in the coordinates y, $U(t,y) := \theta(t, X_m(t,y,t_{\mathrm{f}}))$ satisfies $U \in L^\infty((t_{\mathrm{f}}-8h_{m+1}, t_{\mathrm{f}}); L^2(D_{r_{\mathrm{tube}}}))$ and $\nabla_y U \in L^2((t_{\mathrm{f}}-8h_{m+1}, t_{\mathrm{f}})\times D_{r_{\mathrm{tube}}})$. No lateral data are prescribed.

Theorem C (Local interior regularity)**.** *In the setting of Theorem A, with Λ_0 as there, let $\Lambda \ge \Lambda_0$ be an integer. There exists $C_1(\alpha, s, q, \zeta, \Lambda) \ge 1$ with the following property. Let $m \ge 0$ be an integer and let r_{tube} satisfy*

$$C_1 \max\big\{\varepsilon_{m+1}^{\alpha}h_{m+1},\ (h_{m+1}\sup\mathcal{K})^{1/2}\big\} \le r_{\mathrm{tube}} \le \frac{1}{8}. \tag{9.13}$$

For every $0 < r_{\mathrm{obs}} < \frac{1}{8}r_{\mathrm{tube}}$ and $4h_{m+1} < t_0 < 8h_{m+1}$ there exists $C(\alpha, s, q, \zeta, \Lambda, m, r_{\mathrm{tube}}, r_{\mathrm{obs}}, t_0) < \infty$ such that, for every $t_{\mathrm{f}} \in \mathbb{R}$, every $\kappa \in \mathcal{K}$ and every local energy solution θ of $\partial_t\theta + \mathbf{b}\cdot\nabla\theta - \kappa\Delta\theta = 0$ in the tube $\mathcal{T}$ of (9.11), with $U(t,y) := \theta(t, X_m(t,y,t_{\mathrm{f}}))$, the following hold. For every $t \in [t_{\mathrm{f}}-8h_{m+1}+t_0, t_{\mathrm{f}}]$, $U(t',\cdot)$ converges in $L^2(D_{r_{\mathrm{obs}}})$ as $t' \uparrow t$ to a continuous function $U(t,\cdot)$, and

$$\|U\|_{L^\infty([t_{\mathrm{f}}-8h_{m+1}+t_0,\, t_{\mathrm{f}}]; C^{0,s}(D_{r_{\mathrm{obs}}}))} \le C\Big(\fint_{\mathcal{T}} \theta(t,x)^2\,\mathrm{d}t\,\mathrm{d}x\Big)^{1/2}.$$

The bound holds from a time after the middle of the tube up to its end. At the end, where $X_m(t_{\mathrm{f}}, y, t_{\mathrm{f}}) = y$, the limit of $U(t',\cdot)$ defines $\theta(t_{\mathrm{f}},\cdot) := U(t_{\mathrm{f}},\cdot)$ on $D_{r_{\mathrm{tube}}/8}$. Since $\mathcal{K}$ is bounded and $h_{m+1} \to 0$ and $\varepsilon_{m+1} \to 0$ as $m \to \infty$, the range (9.13) is nonempty for all large m. Since $X_m(t,\cdot,t_{\mathrm{f}})$ and its inverse are 2-Lipschitz by (9.4), Theorem C bounds $\theta(t,\cdot)$ in $C^{0,s}$ on the moving disc $X_m(t, D_{r_{\mathrm{obs}}}, t_{\mathrm{f}})$, with $2C$ in place of C. The proof compares θ with the solution on the whole torus from a cut-off datum, to which Theorem A applies; the localization error is exponentially small in $r_{\mathrm{tube}}^2/(\kappa h_{m+1})$, and this offsets the smoothing at the radius $r \le c_{\mathrm{cyl}}\kappa^{1/\beta}$ of Lemma 9.1.

Lemma 9.3 (Localization). *In the setting of Theorem A, with Λ_0 as there, let $\Lambda \geq \Lambda_0$ be an integer. There exist $C_1 \geq 1$, $C < \infty$ and $c > 0$, depending only on $\alpha, s, q, \zeta, \Lambda$, with the following property. Let $m \geq 0$, let $r_{\mathrm{tube}} \leq \frac{1}{8}$ satisfy $C_1 \varepsilon_{m+1}^{\alpha} h_{m+1} \leq r_{\mathrm{tube}}$, let $t_{\mathrm{f}} \in \mathbb{R}$, let $\kappa > 0$ satisfy $C_1(\kappa h_{m+1})^{1/2} \leq r_{\mathrm{tube}}$, let θ be a local energy solution in $\mathcal{T}$, and let $t_{\mathsf{in}} \coloneqq t_{\mathrm{f}} - \frac{9}{2} h_{m+1}$. There exists $\theta_{\mathsf{in}} \in L^2(\mathbb{T}^2)$ with $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)} \leq C r_{\mathrm{tube}} (⨍_{\mathcal{T}} \theta(t,x)^2 \,\mathrm{d}t \,\mathrm{d}x)^{1/2}$ such that the weak solution θ^{per} of (3.2) on $(t_{\mathsf{in}}, t_{\mathsf{in}} + 5h_{m+1})$ with datum θ_{in} satisfies, with $U^{\mathrm{per}}(t,y) \coloneqq \theta^{\mathrm{per}}(t, X_m(t,y,t_{\mathrm{f}}))$,*

$$\|U - U^{\mathrm{per}}\|_{L^\infty((t_{\mathsf{in}}, t_{\mathrm{f}}); L^2(D_{r_{\mathrm{tube}}/8}))} \leq C r_{\mathrm{tube}} \Big(⨍_{\mathcal{T}} \theta(t,x)^2 \,\mathrm{d}t \,\mathrm{d}x \Big)^{1/2} \exp\Big(- \frac{c\, r_{\mathrm{tube}}^2}{\kappa h_{m+1}} \Big) . \tag{9.14}$$

Proof. Step 1: The equation in the coordinates y. For $t_{\mathrm{f}} - 8h_{m+1} \leq t \leq t_{\mathrm{f}}$, changing variables by the volume-preserving map $y \mapsto X_m(t,y,t_{\mathrm{f}})$, whose time derivative is $\mathbf{b}_m$, turns (9.12) into

$$\begin{aligned} \partial_t U &= \nabla_y \cdot \big(\kappa \mathbf{M} \nabla_y U - U \mathbf{b}_* \big) , \\ \mathbf{M} \coloneqq (\nabla_y X_m)^{-1} (\nabla_y X_m)^{-\mathsf{T}}, &\qquad \mathbf{b}_* \coloneqq (\nabla_y X_m)^{-1} (\mathbf{b} - \mathbf{b}_m)(t, X_m(t,y,t_{\mathrm{f}})) , \end{aligned} \tag{9.15}$$

in $(t_{\mathrm{f}} - 8h_{m+1}, t_{\mathrm{f}}) \times D_{r_{\mathrm{tube}}}$, where $\nabla_y X_m$ is evaluated at (t,y,t_{f}), and U^{per} solves it in $(t_{\mathsf{in}}, t_{\mathrm{f}}) \times \mathbb{R}^2$. Since $\det \nabla_y X_m = 1$, the Piola identity (the cofactor matrix of a smooth map is divergence free) gives $\nabla_y \cdot \mathbf{b}_* = (\nabla \cdot (\mathbf{b} - \mathbf{b}_m)) \circ X_m = 0$, and, applied to $\nabla\theta$ in place of $\mathbf{b} - \mathbf{b}_m$, it puts the equation in this divergence form. By (9.1) and (9.4), $|\mathbf{b}_*| \leq 2C_0 \varepsilon_{m+1}^{\alpha}$, and, as $\det \nabla_y X_m = 1$, $\mathbf{M}$ has the eigenvalues of $(\nabla_y X_m)^{\mathsf{T}} \nabla_y X_m$, which lie in $[\frac{1}{4}, \frac{9}{4}]$.

Step 2: Energy after the start of the tube. Let $g \in C_c^\infty(D_{r_{\mathrm{tube}}})$ with $0 \leq g \leq 1$ and $|\nabla_y g| \leq C / r_{\mathrm{tube}}$. Testing (9.15) with $g^2 U$ times a time cutoff that rises on $(t_{\mathrm{f}} - 8h_{m+1}, t_{\mathsf{in}})$ shows that gU is continuous in time with values in $L^2(D_{r_{\mathrm{tube}}})$ on $[t_{\mathsf{in}}, t_{\mathrm{f}}]$ (Lions–Magenes) and, since $\int_{(t_{\mathrm{f}} - 8h_{m+1}, t_{\mathrm{f}}) \times D_{r_{\mathrm{tube}}}} U^2 = 8\pi h_{m+1} r_{\mathrm{tube}}^2 ⨍_{\mathcal{T}} \theta^2$ by volume preservation,

$$\|gU\|^2_{L^\infty([t_{\mathsf{in}}, t_{\mathrm{f}}]; L^2(D_{r_{\mathrm{tube}}}))} \leq C \Big(\frac{1}{h_{m+1}} + \frac{\kappa}{r_{\mathrm{tube}}^2} + \frac{\varepsilon_{m+1}^{\alpha}}{r_{\mathrm{tube}}} \Big) \int_{\mathcal{T}} \theta^2 \,\mathrm{d}t \,\mathrm{d}x \leq C r_{\mathrm{tube}}^2 ⨍_{\mathcal{T}} \theta^2 \,\mathrm{d}t \,\mathrm{d}x , \tag{9.16}$$

by the hypotheses on r_{tube} and κ, with $C_1 \geq 1$. We apply (9.16) with $g = g_1$ and with $g = g_2$, where $g_1(y) \coloneqq \mathsf{H}(3 - 4|y|/r_{\mathrm{tube}})$ equals 1 on $D_{r_{\mathrm{tube}}/2}$ and 0 outside $D_{3r_{\mathrm{tube}}/4}$, and $g_2(y) \coloneqq \mathsf{H}(7 - 8|y|/r_{\mathrm{tube}})$ equals 1 on $D_{3r_{\mathrm{tube}}/4}$ and 0 outside $D_{7r_{\mathrm{tube}}/8}$.

Step 3: The datum. The function $g_1 U(t_{\mathsf{in}}, \cdot)$ is supported in $D_{3r_{\mathrm{tube}}/4} \subset D_{1/8}$, so its translates by $\mathbb{Z}^2$ have disjoint supports. Let θ_{in} be their sum, composed with $x \mapsto X_m(t_{\mathrm{f}}, x, t_{\mathsf{in}})$. It is $\mathbb{Z}^2$-periodic, $U^{\mathrm{per}}(t_{\mathsf{in}}, \cdot)$ equals $g_1 U(t_{\mathsf{in}}, \cdot)$ on $D_{r_{\mathrm{tube}}}$, and $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)} \leq C r_{\mathrm{tube}} (⨍_{\mathcal{T}} \theta^2)^{1/2}$ by (9.16) with $g = g_1$. By (9.2) and the four translates of the unit cell, since $X_m(t, D_{r_{\mathrm{tube}}}, t_{\mathrm{f}})$ lies in a disc of radius $2r_{\mathrm{tube}} < \frac{1}{2}$ by (9.4), $\|U^{\mathrm{per}}\|^2_{L^\infty((t_{\mathsf{in}}, t_{\mathrm{f}}); L^2(D_{r_{\mathrm{tube}}}))} \leq 4 \|\theta_{\mathsf{in}}\|^2_{L^2(\mathbb{T}^2)} \leq C r_{\mathrm{tube}}^2 ⨍_{\mathcal{T}} \theta^2$.

Step 4: Exponential separation. We use energy estimates with an exponential weight of the form used in [Dav87, Section 2]. For a length $r' > 0$, the weight

$$\exp\Big(- \frac{r_{\mathrm{tube}}}{8r'} \mathsf{H}\Big(\frac{4|y|}{r_{\mathrm{tube}}} - \frac{1}{2} \Big) \Big)$$

equals 1 on $D_{r_{\mathrm{tube}}/8}$, equals $\exp(-r_{\mathrm{tube}}/(8r'))$ outside $D_{3r_{\mathrm{tube}}/8}$, and the gradient of its logarithm is at most C/r'. Testing the equation of $U - U^{\mathrm{per}}$ shows that the integral over $D_{r_{\mathrm{tube}}}$ of the square of this weight times $g_2(y)^2 (U - U^{\mathrm{per}})(t,y)^2$ grows at most at the rate $C(\kappa / r'^2 + \varepsilon_{m+1}^{\alpha} / r')$: the derivatives of the weight cost $1/r'$ against the diffusion and the divergence-free drift, and Young's inequality absorbs the rest into the dissipation. The other terms contain derivatives of g_2, which vanish on $D_{3r_{\mathrm{tube}}/4}$, so they carry the factor $\exp(-r_{\mathrm{tube}}/(4r'))$, the square of the weight outside

$D_{3r_{\mathrm{tube}}/8}$. After Young's inequality, which keeps g_2 in front of the gradient of $U-U^{\mathrm{per}}$, their integral in time is at most $C(\kappa/r_{\mathrm{tube}}^2+\varepsilon_{m+1}^{\alpha}/r_{\mathrm{tube}})$ times the integral of $U^2+(U^{\mathrm{per}})^2$ over $(t_{\mathsf{in}},t_{\mathrm{f}})\times D_{r_{\mathrm{tube}}}$, which is at most $Ch_{m+1}r_{\mathrm{tube}}^2\fint_{\mathcal{T}}\theta^2$ by Steps 2 and 3; and $(\kappa/r_{\mathrm{tube}}^2+\varepsilon_{m+1}^{\alpha}/r_{\mathrm{tube}})h_{m+1}\le 2$ by the hypotheses. The initial value is bounded by (9.16) with $g=g_2$, since at time t_{in} the difference $U-U^{\mathrm{per}}=(1-g_1)U(t_{\mathsf{in}},\cdot)$ vanishes on $D_{r_{\mathrm{tube}}/2}$. Since $t_{\mathrm{f}}-t_{\mathsf{in}}=\frac{9}{2}h_{m+1}$, Grönwall's inequality gives, for $t_{\mathsf{in}}<t<t_{\mathrm{f}}$,

$$\int_{D_{r_{\mathrm{tube}}/8}}(U-U^{\mathrm{per}})(t,y)^2\,\mathrm{d}y\le Cr_{\mathrm{tube}}^2\fint_{\mathcal{T}}\theta(t',x)^2\,\mathrm{d}t'\,\mathrm{d}x\cdot\exp\Bigl(C(\kappa/r'^2+\varepsilon_{m+1}^{\alpha}/r')h_{m+1}-\frac{r_{\mathrm{tube}}}{4r'}\Bigr).$$

With $r'\coloneqq C'\kappa h_{m+1}/r_{\mathrm{tube}}$, and C' and then C_1 large, the exponent is at most $-cr_{\mathrm{tube}}^2/(\kappa h_{m+1})$, using $C_1\varepsilon_{m+1}^{\alpha}h_{m+1}\le r_{\mathrm{tube}}$ for the drift term. This proves (9.14). □

Proof of Theorem C. Let C_1 be that of Lemma 9.3; for $\kappa\in\mathcal{K}$, (9.13) gives its hypotheses. Fix r_{obs}, t_0, t_{f}, κ and θ, and let t_{in}, θ_{in} and θ^{per} be as in Lemma 9.3.

Step 1: The comparison solution. By Theorem A on $(t_{\mathsf{in}},t_{\mathsf{in}}+5h_{m+1})$, with the waiting time $t_0-4h_{m+1}\in(0,4h_{m+1})$, we obtain

$$\|\theta^{\mathrm{per}}\|_{L^\infty([t_{\mathsf{in}}+t_0-4h_{m+1},\,t_{\mathsf{in}}+5h_{m+1}];C^{0,s}(\mathbb{T}^2))}\le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\le Cr_{\mathrm{tube}}\Bigl(\fint_{\mathcal{T}}\theta(t,x)^2\,\mathrm{d}t\,\mathrm{d}x\Bigr)^{1/2},$$

at every time of this interval, by (9.2′), which contains $[t_{\mathrm{f}}-8h_{m+1}+t_0,t_{\mathrm{f}}]$.

Step 2: The difference. In the open set $\mathcal{O}\coloneqq\{(t,X_m(t,y,t_{\mathrm{f}}))\,:\,t_{\mathsf{in}}<t<t_{\mathrm{f}},\ |y|<\frac{1}{8}r_{\mathrm{tube}}\}$, $\theta-\theta^{\mathrm{per}}$ is a local weak solution (the weak gradients are those of U and U^{per}, in the coordinates y). By volume preservation and (9.14), $\int_{\mathcal{O}_{t'}}(\theta-\theta^{\mathrm{per}})(t',\cdot)^2\le\Xi^2$ for almost every $t'\in(t_{\mathsf{in}},t_{\mathrm{f}})$, with $\Xi\coloneqq Cr_{\mathrm{tube}}(\fint_{\mathcal{T}}\theta^2)^{1/2}\exp(-cr_{\mathrm{tube}}^2/(\kappa h_{m+1}))$.

Step 3: Moving cylinders stay in $\mathcal{O}$. Let

$$r\coloneqq\min\Bigl\{\frac{1}{8},\ \frac{\frac{1}{8}r_{\mathrm{tube}}-r_{\mathrm{obs}}}{8},\ c_{\mathrm{cyl}}\kappa^{1/\beta},\ \Bigl(\kappa\min\Bigl\{\frac{t_0-4h_{m+1}}{2},\ \frac{\frac{1}{8}r_{\mathrm{tube}}-r_{\mathrm{obs}}}{24(1+C_0\varepsilon_{m+1}^{\alpha})}\Bigr\}\Bigr)^{1/2}\Bigr\},$$

so that

$$2r+6C_0\varepsilon_{m+1}^{\alpha}\frac{r^2}{\kappa}\le\frac{1}{2}\Bigl(\frac{1}{8}r_{\mathrm{tube}}-r_{\mathrm{obs}}\Bigr),\qquad\frac{r^2}{\kappa}\le\frac{t_0-4h_{m+1}}{2}.\tag{9.17}$$

Let $t\in[t_{\mathrm{f}}-8h_{m+1}+t_0,t_{\mathrm{f}})$, $E_t\coloneqq X_m(t,D_{r_{\mathrm{obs}}},t_{\mathrm{f}})$, and let X be a trajectory of $\mathbf{b}$ on $[t-r^2/\kappa,t]$ ending in E_t. The second bound in (9.17) gives $t-r^2/\kappa\ge t_{\mathrm{f}}-6h_{m+1}+\frac{1}{2}t_0>t_{\mathsf{in}}$. In the coordinates y, the point $X(t')$ moves with the velocity $\mathbf{b}_*$ of (9.15), which equals $\nabla_xX_m(t_{\mathrm{f}},x,t')(\mathbf{b}-\mathbf{b}_m)(t',x)$ at $x=X(t')$ and has size at most $\frac{3}{2}C_0\varepsilon_{m+1}^{\alpha}$ by (9.1) and (9.4), so it stays within $\frac{3}{2}C_0\varepsilon_{m+1}^{\alpha}r^2/\kappa$ of $D_{r_{\mathrm{obs}}}$, a quarter of the term $6C_0\varepsilon_{m+1}^{\alpha}r^2/\kappa$ allowed by (9.17); and $x\mapsto X_m(t_{\mathrm{f}},x,t')$ is 2-Lipschitz. By the first bound in (9.17), the moving cylinder of radius r around X lies in $\mathcal{O}$.

Step 4: The bound before the end of the tube. Lemma 9.1 applies to $\theta-\theta^{\mathrm{per}}$ in $\mathcal{O}$ at every $t\in[t_{\mathrm{f}}-8h_{m+1}+t_0,t_{\mathrm{f}})$, with $E=E_t$ and Ξ of Step 2, and gives $\|(\theta-\theta^{\mathrm{per}})(t,\cdot)\|_{C^{0,s}(E_t)}\le Cr^{-1-s}\Xi$. The power of κ in r^{-1-s} is absorbed by the exponential in Ξ, since $\kappa\mapsto\kappa^{-(1+s)/\beta}\exp(-cr_{\mathrm{tube}}^2/(\kappa h_{m+1}))$ and $\kappa\mapsto\kappa^{-(1+s)/2}\exp(-cr_{\mathrm{tube}}^2/(\kappa h_{m+1}))$ are bounded on $(0,\infty)$: $r^{-1-s}\Xi\le C(\fint_{\mathcal{T}}\theta^2)^{1/2}$, with C depending on m, r_{tube}, r_{obs} and t_0. With Step 1, $\theta(t,\cdot)$ is continuous on E_t and $\|\theta(t,\cdot)\|_{C^{0,s}(E_t)}\le C(\fint_{\mathcal{T}}\theta^2)^{1/2}$. Since $y\mapsto X_m(t,y,t_{\mathrm{f}})$ is 2-Lipschitz by (9.4), $U(t,\cdot)$ is continuous and $\|U(t,\cdot)\|_{C^{0,s}(D_{r_{\mathrm{obs}}})}\le 2^s\|\theta(t,\cdot)\|_{C^{0,s}(E_t)}$; the convergence of $U(t',\cdot)$ to $U(t,\cdot)$ in $L^2(D_{r_{\mathrm{obs}}})$ as $t'\uparrow t$ follows from that of Lemma 9.1 for $\theta-\theta^{\mathrm{per}}$, the continuity in time of θ^{per} and the continuity of X_m.

Step 5: The end of the tube. By Step 2 of the proof of Lemma 9.3, $U(t',\cdot)$ converges in $L^2(D_{r_{\mathrm{obs}}})$ as $t'\uparrow t_{\mathrm{f}}$; by the uniform bound of Step 4, the limit is continuous and satisfies the same bound. □

9.5 Vanishing diffusivity: Corollary D

As $\nabla \cdot \mathbf{b} = 0$, the transport equation reads $\partial_t \theta + \nabla \cdot (\mathbf{b}\theta) = 0$, meaningful for continuous θ.

Corollary D (Vanishing-diffusivity limits)**.** *In the setting of Theorem A, with Λ_0 as there, let $\Lambda \geq \Lambda_0$ be an integer.*

(a) *There exist constants $C(\alpha, s, q, \zeta, \Lambda, t_0, T)$, for $0 < t_0 < T$, with the following property. Let $t_{\mathsf{in}} \in \mathbb{R}$, let $\mathcal{K}' \subset \mathcal{K}$ be an infinite subset with 0 as its only accumulation point, and, for $\kappa \in \mathcal{K}'$, let $\theta^\kappa_{\mathsf{in}} \in L^2(\mathbb{T}^2)$ with $A_0 := \sup_{\kappa \in \mathcal{K}'} \|\theta^\kappa_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)} < \infty$, let θ^κ be the weak solution of*

$$\begin{cases} \partial_t \theta^\kappa + \mathbf{b} \cdot \nabla \theta^\kappa - \kappa \Delta \theta^\kappa = 0 & \text{in } (t_{\mathsf{in}}, \infty) \times \mathbb{T}^2 , \\ \theta^\kappa(t_{\mathsf{in}}, \cdot) = \theta^\kappa_{\mathsf{in}} & \text{on } \mathbb{T}^2 , \end{cases}$$

and, with M such that $\kappa \in \mathcal{K}_M$, let $\widetilde{\theta}^\kappa$ be the weak solution of the same problem with $\mathbf{b}_M$ in place of $\mathbf{b}$. Then every θ^κ is continuous on $(t_{\mathsf{in}}, \infty) \times \mathbb{T}^2$, and there exist an infinite subset $\mathcal{K}'' \subset \mathcal{K}'$, $\theta_{\mathsf{in},*} \in L^2(\mathbb{T}^2)$ and a continuous function θ_* on $(t_{\mathsf{in}}, \infty) \times \mathbb{T}^2$ such that:

(i) *$\theta^\kappa_{\mathsf{in}} \rightharpoonup \theta_{\mathsf{in},*}$ weakly in $L^2(\mathbb{T}^2)$, and $\theta^\kappa \to \theta_*$ locally uniformly on $(t_{\mathsf{in}}, \infty) \times \mathbb{T}^2$, as $\kappa \to 0$ in $\mathcal{K}''$;*

(ii) *$\partial_t \theta_* + \nabla \cdot (\mathbf{b}\theta_*) = 0$ in the sense of distributions on $(t_{\mathsf{in}}, \infty) \times \mathbb{R}^2$, and $\theta_*(t, \cdot) \rightharpoonup \theta_{\mathsf{in},*}$ weakly in $L^2(\mathbb{T}^2)$ as $t \downarrow t_{\mathsf{in}}$;*

(iii) *for all such t_0 and T, $\|\theta_*\|_{L^\infty([t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T]; C^{0,s}(\mathbb{T}^2))} \leq C(t_0, T) A_0$, and θ_* is Hölder continuous in time on $[t_{\mathsf{in}} + t_0, t_{\mathsf{in}} + T] \times \mathbb{T}^2$, uniformly in space, with exponent $s/(1+s)$;*

(iv) *$\theta^\kappa - \widetilde{\theta}^\kappa \to 0$ in the sense of distributions on $(t_{\mathsf{in}}, \infty) \times \mathbb{R}^2$ as $\kappa \to 0$ in $\mathcal{K}'$.*

(b) *Let C_1, m, r_{tube}, t_{f} and the tube $\mathcal{T}$ be as in Theorem C, and let $\mathcal{T}' := \{(t, X_m(t, y, t_{\mathrm{f}})) : t_{\mathrm{f}} - 4h_{m+1} < t \leq t_{\mathrm{f}},\ |y| < \frac{1}{8} r_{\mathrm{tube}}\}$. There exist constants $C(\alpha, s, q, \zeta, \Lambda, m, r_{\mathrm{tube}}, r_{\mathrm{obs}}, t_0)$, for $0 < r_{\mathrm{obs}} < \frac{1}{8} r_{\mathrm{tube}}$ and $4h_{m+1} < t_0 < 8h_{m+1}$, with the following property. Let $\mathcal{K}' \subset \mathcal{K}$ be an infinite subset with 0 as its only accumulation point, and let θ^κ, $\kappa \in \mathcal{K}'$, be local energy solutions in $\mathcal{T}$ with $\sup_{\kappa \in \mathcal{K}'} \fint_{\mathcal{T}} \theta^\kappa(t, x)^2 \, \mathrm{d}t \, \mathrm{d}x < \infty$. Then every θ^κ is continuous on $\mathcal{T}'$, and there exist an infinite subset $\mathcal{K}'' \subset \mathcal{K}'$ and a continuous function θ_* on $\mathcal{T}'$ such that $\theta^\kappa \to \theta_*$ uniformly on the compact subsets of $\mathcal{T}'$ as $\kappa \to 0$ in $\mathcal{K}''$; the function θ_* satisfies $\partial_t \theta_* + \nabla \cdot (\mathbf{b}\theta_*) = 0$ in the sense of distributions in the interior of $\mathcal{T}'$ and, for every such r_{obs} and t_0, with $U_*(t, y) := \theta_*(t, X_m(t, y, t_{\mathrm{f}}))$,*

$$\|U_*\|_{L^\infty([t_{\mathrm{f}} - 8h_{m+1} + t_0, t_{\mathrm{f}}]; C^{0,s}(D_{r_{\mathrm{obs}}}))} \leq C \sup_{\kappa \in \mathcal{K}'} \Big(\fint_{\mathcal{T}} \theta^\kappa(t, x)^2 \, \mathrm{d}t \, \mathrm{d}x \Big)^{1/2} .$$

Proof. (a), compactness. For $0 < t_0 < T$ and every $t \in [t_{\mathsf{in}} + t_0, t_{\mathsf{in}} + T]$, Theorem A, with (9.2′), bounds $\theta^\kappa(t, \cdot)$ in $C^{0,s}(\mathbb{T}^2)$ by CA_0. Let $x \in \mathbb{T}^2$, $t, t' \in [t_{\mathsf{in}} + t_0, t_{\mathsf{in}} + T]$ and $0 < r \leq 1$. We average $\theta^\kappa(t, \cdot)$ against $r^{-2} u(r^{-1}(x - \cdot))$, with u the function fixed at the start of Section 6. The average differs from $\theta^\kappa(t, x)$ by at most $C r^s A_0$; testing the equation with this function and using the bound of θ^κ in L^∞ at the times between t' and t, we obtain

$$\Big| \int_{\mathbb{R}^2} \big(\theta^\kappa(t, x') - \theta^\kappa(t', x') \big) \, r^{-2} u\big(r^{-1}(x - x') \big) \, \mathrm{d}x' \Big| \leq C |t - t'| \big(\|\mathbf{b}\|_{L^\infty(\mathbb{R} \times \mathbb{T}^2)} r^{-1} + \kappa r^{-2} \big) A_0 .$$

Since $\mathcal{K}'$ is bounded, for $|t - t'| \leq 1$ the choice $r = |t - t'|^{1/(s+2)}$ gives a modulus $C|t - t'|^{s/(s+2)}$ in time, uniform in κ; for $|t - t'| > 1$ the bound CA_0 on the slices gives it. So every θ^κ is continuous,

and the θ^κ are bounded and equicontinuous on every $[t_{\mathsf{in}}+t_0, t_{\mathsf{in}}+T]\times\mathbb{T}^2$; the theorem of Arzelà and Ascoli with a diagonal argument, and the weak compactness of the balls of $L^2(\mathbb{T}^2)$, give (i).

(a), the limit. For $g\in C_c^\infty((t_{\mathsf{in}},\infty)\times\mathbb{R}^2)$, $\int\theta^\kappa(\partial_t g+\mathbf{b}\cdot\nabla g+\kappa\Delta g)=0$; since $\mathbf{b}\theta^\kappa\to\mathbf{b}\theta_*$ uniformly on the support of g as $\kappa\to0$ in $\mathcal{K}''$, the limit is (ii). The bound of (iii) passes to the limit pointwise, and the computation above with $\kappa=0$, for θ_*, and $r=|t-t'|^{1/(1+s)}$ for $|t-t'|\le1$ give the exponent $s/(1+s)$. For the initial trace, let g be smooth and $\mathbb{Z}^2$-periodic. By the weak form of the equation, the Cauchy–Schwarz inequality and (9.2), for $t>t_{\mathsf{in}}$,

$$\Big|\int_{\mathbb{T}^2}\big(\theta^\kappa(t,x)-\theta^\kappa_{\mathsf{in}}(x)\big)g(x)\,\mathrm{d}x\Big|\le(t-t_{\mathsf{in}})A_0\big\|\mathbf{b}\cdot\nabla g+\kappa\Delta g\big\|_{L^\infty((t_{\mathsf{in}},t)\times\mathbb{T}^2)}.$$

Letting first $\kappa\to0$ in $\mathcal{K}''$ and then $t\downarrow t_{\mathsf{in}}$ gives $\int\theta_*(t,\cdot)g\to\int\theta_{\mathsf{in},*}g$; since $\|\theta_*(t,\cdot)\|_{L^2(\mathbb{T}^2)}\le A_0$, this extends to every $g\in L^2(\mathbb{T}^2)$.

(a), part (iv). Each finite union of bands is bounded away from zero, so $M\to\infty$ as $\kappa\to0$ in $\mathcal{K}'$, where $\kappa\in\mathcal{K}_M$. Lemma 9.2, with $\Phi=\phi$, $\widetilde{\Phi}=\phi_M$ and, by (9.1) and $\kappa\ge\frac12\varepsilon_M^{\beta-\gamma}$,

$$\varpi:=\frac{C_0\varepsilon_{M+1}^{\beta}}{\kappa}\le C\varepsilon_M^{q\beta-\beta+\gamma}=C\varepsilon_M^{(q+2)\gamma}\to0,$$

where $\beta(q-1)=(q+1)\gamma$, gives, for every $T>0$, $\|\theta^\kappa-\widetilde{\theta}^\kappa\|_{L^\infty([t_{\mathsf{in}},t_{\mathsf{in}}+T];L^2(\mathbb{T}^2))}\le\varpi A_0$, hence (iv).

(b). This is the same argument in the tube, with Theorem C in place of Theorem A. The sets $\{(t,X_m(t,y,t_{\mathrm{f}})):t_{\mathrm{f}}-8h_{m+1}+t_0\le t\le t_{\mathrm{f}},\ |y|<r_{\mathrm{obs}}\}$, for $r_{\mathrm{obs}}\uparrow\frac18 r_{\mathrm{tube}}$ and $t_0\downarrow4h_{m+1}$, exhaust $\mathcal{T}'$, and Theorem C bounds the θ^κ in $C^{0,s}$ on each of them, uniformly in κ and in time, up to t_{f}. The averages of (a), centred at a point $x=X_m(t,y,t_{\mathrm{f}})$ of such a set and with r and $t-t'>0$ small, depending on m, $\frac18 r_{\mathrm{tube}}-r_{\mathrm{obs}}$ and t_0-4h_{m+1}, are supported in a slightly larger such set at every time $\tau\in[t',t]$, since, for $z\in D_r(x)$, $|X_m(t_{\mathrm{f}},z,\tau)-y|\le2(r+\|\mathbf{b}_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}(t-\tau))$ by (9.4). They give the same modulus in time for $t<t_{\mathrm{f}}$, and, by the limit of Theorem C at the end of the tube, for $t=t_{\mathrm{f}}$. □

9.6 Uniform anomalous dissipation: Theorem B

Lemma 9.4 (Positive Hölder-smoothing operators contract)**.** *Let $N\ge1$ be an integer, and let $\mathcal{S}$ be a linear map from $L^2(\mathbb{T}^2)$ to the continuous functions on $\mathbb{T}^2$ such that $\mathcal{S}1=1$ and, for every $f\in L^2(\mathbb{T}^2)$, $\int_{\mathbb{T}^2}\mathcal{S}f(x)\,\mathrm{d}x=\int_{\mathbb{T}^2}f(x)\,\mathrm{d}x$, $\|\mathcal{S}f\|_{C^{0,s}(\mathbb{T}^2)}\le N^s\|f\|_{L^2(\mathbb{T}^2)}$ and, if $f\ge0$, $\mathcal{S}f\ge0$. Then, for every $f\in L^2(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2}f(x)\,\mathrm{d}x=0$,*

$$\|\mathcal{S}f\|^2_{L^2(\mathbb{T}^2)}\le\Big(1-\frac{1}{2N^4}\Big)\|f\|^2_{L^2(\mathbb{T}^2)}.$$

Proof. Step 1: Rows. For every $x\in\mathbb{T}^2$, $f\mapsto\mathcal{S}f(x)$ is bounded on $L^2(\mathbb{T}^2)$, so, by the Riesz representation theorem, $\mathcal{S}f(x)=\langle\mathcal{S}_x,f\rangle_{L^2(\mathbb{T}^2)}$ for some $\mathcal{S}_x\in L^2(\mathbb{T}^2)$, the row of $\mathcal{S}$ at x. The Hölder bound gives $\|\mathcal{S}_x-\mathcal{S}_y\|_{L^2(\mathbb{T}^2)}\le N^s|x-y|^s_{\mathbb{T}^2}$. Positivity gives $\mathcal{S}_x\ge0$, since $0\le\mathcal{S}(\mathcal{S}_x^-)(x)=-\|\mathcal{S}_x^-\|^2_{L^2(\mathbb{T}^2)}$ for $\mathcal{S}_x^-:=\max\{-\mathcal{S}_x,0\}$; $\mathcal{S}1=1$ gives $\int_{\mathbb{T}^2}\mathcal{S}_x(z)\,\mathrm{d}z=1$, hence $\|\mathcal{S}_x\|_{L^2(\mathbb{T}^2)}\ge1$; and the preservation of integrals gives $\int_{\mathbb{T}^2}\mathcal{S}_x\,\mathrm{d}x=1$.

Step 2: The kernel of $\mathcal{S}\mathcal{S}^*$. The function $\langle\mathcal{S}_x,\mathcal{S}_y\rangle_{L^2(\mathbb{T}^2)}$ is continuous, symmetric and nonnegative, for every x its integral in y is 1, and, by polarization, it equals $\frac12(\|\mathcal{S}_x\|^2_{L^2(\mathbb{T}^2)}+\|\mathcal{S}_y\|^2_{L^2(\mathbb{T}^2)}-\|\mathcal{S}_x-\mathcal{S}_y\|^2_{L^2(\mathbb{T}^2)})\ge1-\frac12N^{2s}|x-y|^{2s}_{\mathbb{T}^2}$, which, if $|x-y|_{\mathbb{T}^2}\le1/N$, is at least $\frac12$. The adjoint is $\mathcal{S}^*g=\int_{\mathbb{T}^2}g(x)\mathcal{S}_x\,\mathrm{d}x$, and expanding $\|\mathcal{S}^*g\|^2_{L^2(\mathbb{T}^2)}$ gives, for continuous g,

$$\|g\|^2_{L^2(\mathbb{T}^2)}-\|\mathcal{S}^*g\|^2_{L^2(\mathbb{T}^2)}=\frac12\int_{\mathbb{T}^2\times\mathbb{T}^2}\langle\mathcal{S}_x,\mathcal{S}_y\rangle_{L^2(\mathbb{T}^2)}\,|g(x)-g(y)|^2\,\mathrm{d}x\,\mathrm{d}y.\tag{9.18}$$

Step 3: A Poincaré inequality by translations. Let g be continuous, periodic and of mean zero. Then the average over $z \in [-\frac{1}{2}, \frac{1}{2})^2$ of $\int_{\mathbb{T}^2} |g(x+z) - g(x)|^2 \, \mathrm{d}x$ is $2\|g\|^2_{L^2(\mathbb{T}^2)}$. Writing $g(\cdot + Nz) - g$ as a sum of N increments by z, Cauchy–Schwarz bounds the integral at Nz by N^2 times the integral at z, and the change of variables from Nz to z gives

$$2\|g\|^2_{L^2(\mathbb{T}^2)} \le N^4 \int_{[-\frac{1}{2N}, \frac{1}{2N})^2 \times \mathbb{T}^2} |g(x+z) - g(x)|^2 \, \mathrm{d}z \, \mathrm{d}x \,.$$

For $z \in [-\frac{1}{2N}, \frac{1}{2N})^2$, $|z| \le 1/N$, so, by Step 2 and the change of variables $y = x + z$ in (9.18), the right side of (9.18) is at least $\frac{1}{4} \int_{[-\frac{1}{2N}, \frac{1}{2N})^2 \times \mathbb{T}^2} |g(x+z) - g(x)|^2 \, \mathrm{d}z \, \mathrm{d}x \ge \frac{1}{2N^4} \|g\|^2_{L^2(\mathbb{T}^2)}$.

Step 4: Conclusion. Let $\int_{\mathbb{T}^2} f = 0$ and $g := \mathcal{S}f$, which is continuous and of mean zero. Then $\|g\|^2_{L^2(\mathbb{T}^2)} = \langle f, \mathcal{S}^* g \rangle_{L^2(\mathbb{T}^2)} \le \|f\|_{L^2(\mathbb{T}^2)} \|\mathcal{S}^* g\|_{L^2(\mathbb{T}^2)} \le (1 - \frac{1}{2N^4})^{1/2} \|f\|_{L^2(\mathbb{T}^2)} \|g\|_{L^2(\mathbb{T}^2)}$ by Step 3. □

Proof of Theorem B. Fix $t_{\mathsf{in}} \in \mathbb{R}$ and $\kappa \in \mathcal{K}$, and let $\mathcal{S}\theta_{\mathsf{in}} := \theta(t_{\mathsf{in}} + 1, \cdot)$, with θ the weak solution with datum θ_{in}. By uniqueness, $\mathcal{S}$ is linear; constants are solutions, so $\mathcal{S}1 = 1$; testing the equation with a function of time only gives $\int_{\mathbb{T}^2} \mathcal{S}\theta_{\mathsf{in}} = \int_{\mathbb{T}^2} \theta_{\mathsf{in}}$, since $\nabla \cdot \mathbf{b} = 0$. The operator $\mathcal{S}$ is positive by the maximum principle: if $\theta_{\mathsf{in}} \ge 0$, testing the equation with $\min\{\theta, 0\}$ (Stampacchia's truncation) gives $\int_{\mathbb{T}^2} \min\{\theta(t,\cdot), 0\}^2 \, \mathrm{d}x \le \int_{\mathbb{T}^2} \min\{\theta_{\mathsf{in}}, 0\}^2 \, \mathrm{d}x = 0$ for every $t > t_{\mathsf{in}}$: the diffusion term has a sign, and the transport term vanishes, as $\nabla \cdot \mathbf{b} = 0$.

By Theorem A on $(t_{\mathsf{in}}, t_{\mathsf{in}} + 1)$ with $t_0 = \frac{1}{2}$ and (9.2′), $\mathcal{S}$ is bounded from $L^2(\mathbb{T}^2)$ into $C^{0,s}(\mathbb{T}^2)$, and, by Lemma 9.4 with N the least integer such that N^s is at least the constant of Theorem A, we obtain, with $c := 1/(4N^4)$,

$$\|\mathcal{S}g\|^2_{L^2(\mathbb{T}^2)} \le (1 - 2c)\|g\|^2_{L^2(\mathbb{T}^2)} \qquad \text{for every } g \in L^2(\mathbb{T}^2) \text{ with } \int_{\mathbb{T}^2} g(x) \, \mathrm{d}x = 0 \,.$$

Since the constant of Theorem A is independent of t_{in} and of $\kappa \in \mathcal{K}$, so are N and c. Since constants are solutions, $\theta - (\theta_{\mathsf{in}})_{\mathbb{T}^2}$ is the weak solution with datum $\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2}$, and $\theta(t_{\mathsf{in}} + 1, \cdot) - (\theta_{\mathsf{in}})_{\mathbb{T}^2} = \mathcal{S}(\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2})$. The energy identity (9.2) for $\theta - (\theta_{\mathsf{in}})_{\mathbb{T}^2}$ gives

$$\begin{aligned} 2\kappa \|\nabla\theta\|^2_{L^2((t_{\mathsf{in}}, t_{\mathsf{in}}+1)\times\mathbb{T}^2)} &= \big\|\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\big\|^2_{L^2(\mathbb{T}^2)} - \big\|\mathcal{S}(\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2})\big\|^2_{L^2(\mathbb{T}^2)} \\ &\ge 2c \big\|\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\big\|^2_{L^2(\mathbb{T}^2)} \,, \end{aligned}$$

which is (3.5). For $t'_{\mathsf{in}} > t_{\mathsf{in}}$, the restriction of θ to $(t'_{\mathsf{in}}, \infty)$ is the weak solution with datum $\theta(t'_{\mathsf{in}}, \cdot)$, whose mean is $(\theta_{\mathsf{in}})_{\mathbb{T}^2}$. So the contraction, with $t_{\mathsf{in}} + k$ in place of t_{in}, gives $\|\theta(t_{\mathsf{in}} + k + 1, \cdot) - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\|^2_{L^2(\mathbb{T}^2)} \le (1 - 2c)\|\theta(t_{\mathsf{in}} + k, \cdot) - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\|^2_{L^2(\mathbb{T}^2)}$ for every integer $k \ge 0$, and, by (9.2), $\|\theta(t, \cdot) - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\|_{L^2(\mathbb{T}^2)}$ does not increase in t. Hence, for every $t \ge t_{\mathsf{in}}$, as $1 - 2c \le \exp(-2c)$ and $c \le \frac{1}{2}$, we obtain (3.6):

$$\big\|\theta(t,\cdot) - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\big\|^2_{L^2(\mathbb{T}^2)} \le (1-2c)^{\lfloor t - t_{\mathsf{in}} \rfloor} \big\|\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\big\|^2_{L^2(\mathbb{T}^2)} \le 4\exp(-2c(t - t_{\mathsf{in}})) \big\|\theta_{\mathsf{in}} - (\theta_{\mathsf{in}})_{\mathbb{T}^2}\big\|^2_{L^2(\mathbb{T}^2)} \,.$$

□

10. Cell problems and the averaged flux

In this section we construct, on each pulse, the finite lift of a system with sources: correctors of every degree up to N_{cor}, which oscillate at the wavelength ε_m, and the means they leave behind, the sources of the successor. In the chart of its block (Section 7), a pulse of level m is a single shear, whose cell problems are one-dimensional and explicitly solvable. The first correctors are those of [AV25,

Section 3.2]. Throughout, $\Lambda \geq 128$ is an integer, and C, c denote positive constants depending only on α, s, q, ζ, unless another dependence is stated. For $M \geq 1$, $\kappa \in \mathcal{K}_M$ and $1 \leq m \leq M$, recall that $\bar{\varepsilon}_m = \varepsilon_m(\kappa_{m-1}/\kappa_m)^{1/2}$ is the corrector amplitude of Subsection 5.1, and $\rho(m) = (\kappa_{m-1}h_m)^{1/2}$ is the effective diffusion length (5.5).

The window of a pulse. Let $m \geq 1$ and k be odd, and let $e_{(k)}^{\perp} := e_1$ for $k \in 4\mathbb{Z}+1$ and $e_{(k)}^{\perp} := e_2$ for $k \in 4\mathbb{Z}+3$, the coordinate vector other than $e_{(k)}$. By (2.6), $\psi_{m,k}(y) = \varepsilon_m^{\beta} \sin(2\pi y \cdot e_{(k)}^{\perp}/\varepsilon_m)$ and

$$\nabla^{\perp}\psi_{m,k}(y) = \pm 2\pi\varepsilon_m^{\beta-1} \cos\bigl(2\pi y \cdot e_{(k)}^{\perp}/\varepsilon_m\bigr)\, e_{(k)}\,, \tag{10.1}$$

with $+$ for $k \in 4\mathbb{Z}+1$ and $-$ for $k \in 4\mathbb{Z}+3$. The *window of pulse* k and the *time cutoff* of k are

$$|t - k\tau_m| \leq \frac{61}{48}\tau_m\,, \qquad \mathsf{H}_{m,k}(t) := \mathsf{H}\Bigl(\frac{5}{2} - \frac{2|t-k\tau_m|}{\tau_m}\Bigr), \tag{10.2}$$

with H of (2.9), which is 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$. As the envelope $\widetilde{\zeta}_{m,k}$ satisfies $0 \leq \widetilde{\zeta}_{m,k} \leq \zeta_{m,k}$ (by (2.2), Lemma 7.1(d)), it vanishes for $|t-k\tau_m| \geq \frac{2}{3}\tau_m$; as the centres of odd slots are $2\tau_m$ apart, no other pulse is active on the window of pulse k, where therefore $\psi_m(t,y) = \widetilde{\zeta}_{m,k}(t)\psi_{m,k}(y)$ (see (7.11)). The cutoff $\mathsf{H}_{m,k}$ is 1 for $|t-k\tau_m| \leq \frac{3}{4}\tau_m$, hence where $\widetilde{\zeta}_{m,k} \neq 0$; it vanishes for $|t-k\tau_m| \geq \frac{5}{4}\tau_m$; and $\sum_{k \in 2\mathbb{Z}+1} \mathsf{H}_{m,k} = 1$, since $\mathsf{H}(x) + \mathsf{H}(1-x) = 1$. By Lemma 7.8(i), in the chart $y \mapsto X_{m-1}(t, y, l_{m,k}h_m)$ of k the velocity added at level m is $\nabla^{\perp}\psi_m(t,y)$ in the block of k, and it vanishes on the rest of the window; the chart and $\mathbf{K}_{m,l_{m,k}}$ are data on the whole window.

A *two-scale form* on the window of pulse k is a function $\mathcal{A}(t,y,z)$ of t in the window and $y, z \in \mathbb{R}^2$, 1-periodic in the *fast variable* z, read at $z = y/\varepsilon_m$; it is *transverse* if it depends on z only through $z \cdot e_{(k)}^{\perp}$. We write $(\mathcal{A})_{\mathbb{T}^2}(t,y) := \fint_{[0,1]^2} \mathcal{A}(t,y,z)\,\mathrm{d}z$ for its average in z, and, for a transverse form, $\widehat{\mathcal{A}}(t,y;\xi)$ for its ξ-th Fourier coefficient in $z \cdot e_{(k)}^{\perp}$, $\xi \in \mathbb{Z}$, as in Section 11.

10.1 The corrector of a pulse and the averaged flux

Lemma 10.1 (The corrector of a pulse)**.** *Let $m \geq 1$, $\nu > 0$ and k odd, and let*

$$\chi_{m,k}^{\nu}(t,y) := -\Bigl(\int_0^{\infty} \exp\Bigl(-\frac{4\pi^2\nu t'}{\varepsilon_m^2}\Bigr)\widetilde{\zeta}_{m,k}(t-t')\,\mathrm{d}t'\Bigr)\nabla^{\perp}\psi_{m,k}(y)\,. \tag{10.3}$$

(a) *$\chi_{m,k}^{\nu}$ is $\mathbb{Z}^2$-periodic and smooth in y, continuously differentiable in t, of mean zero over $[0,1]^2$, parallel to $e_{(k)}$, a function of $y \cdot e_{(k)}^{\perp}$, and it solves, componentwise,*

$$\begin{cases} \partial_t\chi_{m,k}^{\nu} + \widetilde{\zeta}_{m,k}\nabla^{\perp}\psi_{m,k}\cdot\nabla\chi_{m,k}^{\nu} - \nu\Delta\chi_{m,k}^{\nu} = -\widetilde{\zeta}_{m,k}\nabla^{\perp}\psi_{m,k} & \text{in } \mathbb{R}\times\mathbb{R}^2\,, \\ \chi_{m,k}^{\nu} = 0 & \text{on } (-\infty, (k-\frac{2}{3})\tau_m]\times\mathbb{R}^2\,, \end{cases} \tag{10.4}$$

whose transport term vanishes identically.

(b) $\|\chi_{m,k}^{\nu}\|_{L^{\infty}(\mathbb{R}\times\mathbb{T}^2)} \leq \varepsilon_m^{\beta+1}/(2\pi\nu)$.

(c) *If $m \geq 2$, there exist $\Lambda_0(\alpha,s,q,\zeta) \geq 128$ and $C(\alpha,s,q,\zeta) < \infty$ such that, for every $\Lambda \geq \Lambda_0$, $M \geq m$ and $\kappa \in \mathcal{K}_M$,*

$$\frac{\varepsilon_m^{2\beta}}{\kappa_m} \leq 20\kappa_{m-1}\,, \qquad \|\chi_{m,k}^{\kappa_m}\|_{L^{\infty}(\mathbb{R}\times\mathbb{T}^2)} \leq C\bar{\varepsilon}_m \leq C\varepsilon_{m-1}^{3\delta}\rho(m)\,. \tag{10.5}$$

This is the corrector of [AV25, (3.6)]; (b) sharpens [AV25, (3.10)], and (c) bounds $\chi^{\kappa_m}_{m,k}$ by a constant times the corrector amplitude $\bar\varepsilon_m$ of Part I.

Proof. (a) The integral in (10.3) is $\int_{-\infty}^t \exp(-4\pi^2\nu(t-t')/\varepsilon_m^2)\widetilde\zeta_{m,k}(t')\,\mathrm{d}t'$; it vanishes before the pulse, and its derivative is $\widetilde\zeta_{m,k}(t)$ minus $4\pi^2\nu/\varepsilon_m^2$ times itself. By (10.1), $-\nu\Delta\nabla^\perp\psi_{m,k} = (4\pi^2\nu/\varepsilon_m^2)\nabla^\perp\psi_{m,k}$, which gives the equation; the transport term vanishes because the gradient of a function of $y\cdot e_{(k)}^\perp$ is orthogonal to $e_{(k)}$. (b) The integral is at most $\varepsilon_m^2/(4\pi^2\nu)$, and $|\nabla^\perp\psi_{m,k}| \le 2\pi\varepsilon_m^{\beta-1}$. (c) By Proposition 2.1(a) and (c), and as $\varepsilon_{m-1}\le\Lambda^{-1}$ (Lemma 7.1(a)), for Λ_0 large,

$$\kappa_{m-1} \ge \Big(\frac{9}{80} - C\varepsilon_{m-1}^{4\delta} - C\varepsilon_{m-1}^{\delta}\Big)\frac{\varepsilon_m^{2\beta}}{\kappa_m} \ge \frac{1}{20}\,\frac{\varepsilon_m^{2\beta}}{\kappa_m}\,,$$
$$\frac{\varepsilon_m^{\beta+1}}{\kappa_m} = \varepsilon_m\Big(\frac{\varepsilon_m^{2\beta}}{\kappa_m}\Big)^{1/2}\kappa_m^{-1/2} \le C\varepsilon_m\Big(\frac{\kappa_{m-1}}{\kappa_m}\Big)^{1/2} = C\bar\varepsilon_m\ :$$

the first line is the first bound of (10.5), and the second, with (b), gives the second. The third follows from $(\kappa_{m-1}/\kappa_m)^{1/2}\le C\varepsilon_{m-1}^{-q\gamma}$, $\rho(m)\ge c\varepsilon_{m-1}^{1+\gamma/2+\delta}$ (Proposition 2.1(c)), $\varepsilon_m\le 2\varepsilon_{m-1}^q$ (Lemma 7.1(a)) and $q-q\gamma-1-\frac{\gamma}{2}-\delta = 3\delta$, by (2.3). □

Lemma 10.2 (The averaged flux is the flux of the correctors)**.** *Let $m\ge 1$ and $\nu>0$.*

(a) *For every odd k, $t\in\mathbb{R}$ and $e\in\mathbb{R}^2$, with the term k of (2.16) on the right,*

$$\fint_{[0,1]^2}\Big(\nu I + \widetilde\zeta_{m,k}(t)\psi_{m,k}(y)\begin{pmatrix}0&-1\\1&0\end{pmatrix}\Big)\big(e+\nabla(e\cdot\chi^\nu_{m,k}(t,y))\big)\,\mathrm{d}y$$
$$= \nu e + 2\pi^2\varepsilon_m^{2\beta-2}\,\widetilde\zeta_{m,k}(t)\Big(\int_0^\infty \exp\Big(-\frac{4\pi^2\nu t'}{\varepsilon_m^2}\Big)\widetilde\zeta_{m,k}(t-t')\,\mathrm{d}t'\Big)(e_{(k)}\otimes e_{(k)})\,e\,.$$

(b) *The glued corrector $\chi^\nu_m := \sum_{k\in 2\mathbb{Z}+1}\mathsf{H}_{m,k}\chi^\nu_{m,k}$ satisfies, for every $t\in\mathbb{R}$ and $e\in\mathbb{R}^2$,*

$$\fint_{[0,1]^2}\Big(\nu I + \psi_m(t,y)\begin{pmatrix}0&-1\\1&0\end{pmatrix}\Big)\big(e+\nabla(e\cdot\chi^\nu_m(t,y))\big)\,\mathrm{d}y = \mathbf{J}^\nu_m(t)e\,, \tag{10.6}$$

and there exists $C(\alpha,q,\zeta)<\infty$ such that

$$\partial_t\chi^\nu_m + \nabla^\perp\psi_m\cdot\nabla\chi^\nu_m - \nu\Delta\chi^\nu_m + \nabla^\perp\psi_m = \sum\nolimits_{k\in 2\mathbb{Z}+1}\mathsf{H}'_{m,k}\chi^\nu_{m,k}\,,$$
$$\Big\|\sum\nolimits_{k\in 2\mathbb{Z}+1}\mathsf{H}'_{m,k}\chi^\nu_{m,k}\Big\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le \frac{C\varepsilon_m^{\beta+1}}{\nu\tau_m}\exp\Big(-\frac{\pi^2\nu\tau_m}{3\varepsilon_m^2}\Big). \tag{10.7}$$

The integrand of (10.6) is the flux of the corrected coordinate $y\cdot e + e\cdot\chi^\nu_m$ (the divergence form after Lemma 7.9, with I in place of $\mathbf{K}_{m,l}$; [AV25, (3.15), (3.18)]), and the glued correctors solve the cell problem up to (10.7) ([AV25, (3.11)–(3.13)]).

Proof. (a) Let $k\in 4\mathbb{Z}+1$ (the other case is symmetric). Then $\psi_{m,k} = \varepsilon_m^\beta\sin(2\pi y_1/\varepsilon_m)$, and $\chi^\nu_{m,k}$, $\nabla(e\cdot\chi^\nu_{m,k})$ are $-2\pi\varepsilon_m^{\beta-1}\cos(2\pi y_1/\varepsilon_m)e_2$ and $4\pi^2\varepsilon_m^{\beta-2}\sin(2\pi y_1/\varepsilon_m)(e\cdot e_2)e_1$ times the integral in (10.3). Only the product of $\psi_{m,k}$ and $\nabla(e\cdot\chi^\nu_{m,k})$ has nonzero mean; since $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)e_1 = e_2$ and $\fint\sin^2 = \frac12$, its mean is $2\pi^2\varepsilon_m^{2\beta-2}\widetilde\zeta_{m,k}(t)$ times the integral times $(e\cdot e_2)e_2$, and $e_{(k)} = e_2$.

(b) If $\psi_m(t,\cdot)\neq 0$, exactly one pulse k is active, $\mathsf{H}_{m,k}(t)=1$ and the other cutoffs vanish at t, as $|t-k'\tau_m|\ge\frac43\tau_m$ for odd $k'\neq k$; so $\chi^\nu_m(t) = \chi^\nu_{m,k}(t)$, and (a) gives (10.6). Otherwise both sides

are νe (Lemma 7.10(i)). The product rule and (10.4) give the equation, since $\mathsf{H}_{m,k} = 1$ where $\widetilde{\zeta}_{m,k} \neq 0$ and the transport term vanishes. Finally, $\mathsf{H}'_{m,k} \neq 0$ only for $\frac{3}{4}\tau_m \le |t - k\tau_m| \le \frac{5}{4}\tau_m$, where $|\mathsf{H}'_{m,k}| \le C/\tau_m$ and either $\chi^\nu_{m,k} = 0$ or the pulse ended at least $\frac{1}{12}\tau_m$ earlier; after the pulse the integral in (10.3) decays at the rate $4\pi^2\nu/\varepsilon_m^2$, so $|\chi^\nu_{m,k}| \le (\varepsilon_m^{\beta+1}/2\pi\nu)\exp(-\pi^2\nu\tau_m/(3\varepsilon_m^2))$ there. □

10.2 The covariance

Lemma 10.3 (The covariance). (a) *There exists $C(\zeta) < \infty$ such that, for every $m \ge 1$, $\nu > 0$ and $t, t' \in \mathbb{R}$,*

$$\Big| \int_{t'}^{t} \big(\mathbf{J}^\nu_m(t'') - \overline{K}_m(\nu) I \big)\, \mathrm{d}t'' \Big| \le C\, \frac{\varepsilon_m^{2\beta}}{\nu} \Big(\tau'_m + \frac{\varepsilon_m^2 h_m}{\nu \tau_m} \Big). \tag{10.8}$$

(b) *There exist $\Lambda_0(\alpha, s, q, \zeta) \ge 128$ and $C(\alpha, s, q, \zeta) < \infty$ such that, for every $\Lambda \ge \Lambda_0$, $M \ge 2$, $\kappa \in \mathcal{K}_M$, $2 \le m \le M$ and $t, t' \in \mathbb{R}$, the covariance (5.11) satisfies*

$$|\Gamma_m(t)| + |\Gamma_m(t) - \Gamma_m(t')| \le C \kappa_{m-1} \tau'_m \le C \varepsilon_{m-1}^{\delta} \rho(m)^2 . \tag{10.9}$$

The matrices are diagonal (Lemma 7.10(i)); Section 15 uses the two-time form (b).

Proof. (a) We compare $\mathbf{J}^\nu_m$ with the flux $\mathbf{J}^{\nu,\mathrm{fast}}_m := \nu I + (\varepsilon_m^{2\beta}/2\nu) \sum_{k \in 2\mathbb{Z}+1} \zeta^2_{m,k} e_{(k)} \otimes e_{(k)}$ of pulses held at their current value and without slow cutoffs. By Step 1 of the proof of Proposition 2.1(a), holding the pulses changes $\mathbf{J}^\nu_m$ by at most $C(\varepsilon_m^{2\beta}/\nu)\varepsilon_m^2/(\nu\tau_m)$ at each time; removing the slow cutoffs changes it only within $2\tau'_m$ of the ends of the blocks, where the slow cutoffs differ from 1 by (7.1), as in (7.16). So $\int |\mathbf{J}^\nu_m - \mathbf{J}^{\nu,\mathrm{fast}}_m| \le C(\varepsilon_m^{2\beta}/\nu)(\tau'_m + \varepsilon_m^2 h_m/(\nu\tau_m))$ over every interval of length $4h_m$. The fast flux is $4\tau_m$-periodic with mean $(\nu + \frac{9}{80}\varepsilon_m^{2\beta}/\nu)I$, since each period holds two odd slots, one for each direction $e_{(k)}$, and $\int_{\mathbb{R}} \zeta^2_{m,k} = \frac{9}{10}\tau_m$, so its centred primitive is at most $C(\varepsilon_m^{2\beta}/\nu)\tau_m$. The flux $\mathbf{J}^\nu_m$ is $4h_m$-periodic with mean $\overline{K}_m(\nu)I$ (Lemma 7.10), and $4h_m$ is a multiple of $4\tau_m$, so the two means differ by at most $(4h_m)^{-1}$ times the integral above. The left side of (10.8) is unchanged when t or t' moves by $4h_m$, since the integral of $\mathbf{J}^\nu_m - \overline{K}_m(\nu)I$ over a period vanishes; so it suffices to bound the centred primitive over one period. Adding the errors, with $\tau_m \le \tau'_m$, gives (10.8).

(b) By (5.11), $\Gamma_m(0) = 0$ and $\Gamma_m(t) - \Gamma_m(t') = \int_{t'}^{t} (\mathbf{J}^{\kappa_m}_m(t'') - \kappa_{m-1} I)\, \mathrm{d}t''$. We take $\nu = \kappa_m$ in (a), so that $\overline{K}_m(\kappa_m) = \kappa_{m-1}$ by (2.19), and it suffices to bound the right side of (10.8) by $C\kappa_{m-1}\tau'_m$. By (10.5), Proposition 2.1(c) and Lemma 7.1(b), $\varepsilon_m^{2\beta}/\kappa_m \le C\kappa_{m-1}$ and $\varepsilon_m^2 h_m/(\kappa_m\tau_m) \le C\varepsilon_{m-1}^{4\delta} h_m \le C\tau'_m$, because $h_m/\tau'_m = 4\lceil \varepsilon_{m-1}^{-\delta} \rceil + 1 \le 9\varepsilon_{m-1}^{-\delta}$; and $\kappa_{m-1}\tau'_m \le \frac{1}{4}\varepsilon_{m-1}^{\delta}\kappa_{m-1}h_m = \frac{1}{4}\varepsilon_{m-1}^{\delta}\rho(m)^2$, as $\rho(m) = (\kappa_{m-1}h_m)^{1/2}$. □

10.3 The cell inverse, the finite lift and the means

In this subsection and the next one, $M \ge 2$, $\kappa \in \mathcal{K}_M$, $2 \le m \le M$, k is odd, $N_{\mathrm{cor}} \ge 1$, and $\mathbf{K}(t, y)$ is a *conductivity* on the window of pulse k: a symmetric matrix, measurable in t, smooth in y, with $\frac{3}{4}I \le \mathbf{K} \le \frac{5}{4}I$, for instance $\mathbf{K}_{m,l_{m,k}}$ (Lemma 7.7(a), Λ large). The *cell operator* is the part of $\partial_t + \nabla^\perp\psi_m \cdot \nabla - \kappa_m \nabla \cdot (\mathbf{K}\nabla)$, applied to a form $\mathcal{A}$ read at $z = y/\varepsilon_m$, without derivative in y:

$$\mathcal{L}_0 \mathcal{A} := \partial_t \mathcal{A} + \varepsilon_m^{-1} (\nabla^\perp \psi_m)(t, \varepsilon_m z) \cdot \nabla_z \mathcal{A} - \kappa_m \varepsilon_m^{-2} \nabla_z \cdot (\mathbf{K} \nabla_z \mathcal{A}) . \tag{10.10}$$

On transverse forms its shear term vanishes by (10.1), and $\mathcal{L}_0 = \partial_t - \kappa_m \varepsilon_m^{-2} (e^\perp_{(k)} \cdot \mathbf{K} e^\perp_{(k)})(e^\perp_{(k)} \cdot \nabla_z)^2$: the cell problems across the pulse are one-dimensional heat equations.

Lemma 10.4 (The cell inverse). *Let F^0 and F be transverse forms of zero average in z, measurable in t, of class C^2 and C^3 in z respectively, whose derivatives in z up to these orders are bounded on $[(k-61/48)\tau_m,(k+61/48)\tau_m]\times\mathbb{R}^2\times\mathbb{R}^2$. There exists exactly one* energy solution *$\mathcal{A}$, a transverse form of zero average in z with, for every $y\in\mathbb{R}^2$, $\nabla_z\mathcal{A}(\cdot,y,\cdot)\in L^2([(k-61/48)\tau_m,(k+61/48)\tau_m]\times[0,1]^2)$ and $\mathcal{A}(\cdot,y,\cdot)\in C([(k-61/48)\tau_m,(k+61/48)\tau_m];L^2([0,1]^2))$, of the following problem, in the integrated form in t tested against smooth periodic functions of z, for every y:*

$$\begin{cases}\mathcal{L}_0\mathcal{A}=F^0+e^\perp_{(k)}\cdot\nabla_z F & \text{in } \big((k-\frac{61}{48})\tau_m,(k+\frac{61}{48})\tau_m\big)\times\mathbb{R}^2\times\mathbb{R}^2\,,\\ \mathcal{A}=0 & \text{on } \{(k-\frac{61}{48})\tau_m\}\times\mathbb{R}^2\times\mathbb{R}^2\,;\end{cases}\tag{10.11}$$

it is the cell inverse *of $F^0+e^\perp_{(k)}\cdot\nabla_z F$. For every $\xi\in\mathbb{Z}\setminus\{0\}$, $y\in\mathbb{R}^2$ and $t\in[(k-61/48)\tau_m,(k+61/48)\tau_m]$,*

$$\widehat{\mathcal{A}}(t,y;\xi)=\int_{(k-61/48)\tau_m}^{t}\exp\Big(-\frac{4\pi^2\xi^2\kappa_m}{\varepsilon_m^2}\int_{t'}^{t}e^\perp_{(k)}\cdot\mathbf{K}(t'',y)e^\perp_{(k)}\,\mathrm{d}t''\Big)\big(\widehat{F^0}+2\pi\mathrm{i}\xi\widehat{F}\big)(t',y;\xi)\,\mathrm{d}t'\,,\tag{10.12}$$

$$\begin{aligned}\|\widehat{\mathcal{A}}(\cdot,y;\xi)\|_{L^\infty([(k-61/48)\tau_m,(k+61/48)\tau_m])}&\le\frac{\varepsilon_m^2}{3\pi^2\xi^2\kappa_m}\|\widehat{F^0}(\cdot,y;\xi)\|_{L^\infty([(k-61/48)\tau_m,(k+61/48)\tau_m])}\\&\quad+\frac{2\varepsilon_m^2}{3\pi|\xi|\kappa_m}\|\widehat{F}(\cdot,y;\xi)\|_{L^\infty([(k-61/48)\tau_m,(k+61/48)\tau_m])}\,.\end{aligned}$$

If $\mathbf{K}$ is instead a complex symmetric matrix, measurable in t, whose entry $e^\perp_{(k)}\cdot\mathbf{K}e^\perp_{(k)}$ is bounded with real part at least $\frac34$ (as in Section 11), the right side of (10.12) defines a transverse form $\mathcal{A}$ of zero average in z, the cell inverse *at $\mathbf{K}$, with the same bound. The cell inverse of $-(\nabla^\perp\psi_m)(t,\varepsilon_m z)$ is the* corrector in the conductivity $\mathbf{K}$, *written $\chi_{\mathbf{K}}(t,y,z)$; $\chi_I(t,y,z)=\chi^{\kappa_m}_{m,k}(t,\varepsilon_m z)$.*

Proof. On the mode ξ, $\mathcal{L}_0$ is $\partial_t+(4\pi^2\xi^2\kappa_m/\varepsilon_m^2)e^\perp_{(k)}\cdot\mathbf{K}e^\perp_{(k)}$: Duhamel's formula gives (10.12); for a conductivity, the energy identity in z gives uniqueness. As $\operatorname{Re}e^\perp_{(k)}\cdot\mathbf{K}e^\perp_{(k)}\ge\frac34$, the exponential is at most $\exp(-3\pi^2\xi^2\kappa_m(t-t')/\varepsilon_m^2)$, of integral in t' at most $\varepsilon_m^2/(3\pi^2\xi^2\kappa_m)$, which gives the bound. The source $F^0+e^\perp_{(k)}\cdot\nabla_z F$ is of class C^2 in z with bounded derivatives, so its modes are $O(\xi^{-2})$ and, by (10.12), those of $\mathcal{A}$ are $O(\xi^{-4})$; the series of $\mathcal{A}$ converges with two derivatives in z, continuously in t with values in $L^2([0,1]^2)$, and is an energy solution. For $\mathbf{K}=I$, (10.12) on the modes $\xi=\pm1$ is (10.3), since the pulse starts at $(k-\frac23)\tau_m$, inside the window. □

A *system on the window* of pulse k has finitely many components v, with masses $\omega_v>0$, and finitely many sources $\sigma\colon v'\to v$. A source has a birth order $2\le d_\sigma\le N_{\mathrm{src}}$, its order when created (Subsection 11.1), a factor $\mathrm{fac}(\sigma)>0$, a majorant $\mathrm{maj}(\sigma)\ge0$, and transverse two-scale forms $\mathcal{A}_{\sigma,d}$, $2\le d\le d_\sigma$, whose values are coefficients from arrays of order $d-1$ to $\mathbb{R}^2$ (Section 8); we set $\mathcal{A}_{\sigma,d}:=0$ for $d>d_\sigma$. The system acts on fields $Y=(Y_v)_v$ by

$$\begin{aligned}&\partial_t Y_v+\nabla^\perp\psi_m\cdot\nabla Y_v-\kappa_m\nabla\cdot(\mathbf{K}\nabla Y_v)\\&\quad-\sum\nolimits_{\sigma\colon v'\to v}\sum\nolimits_{d=2}^{d_\sigma}\nabla\cdot\big(\kappa_m\varepsilon_m^{d-2}\,\mathrm{fac}(\sigma)\mathcal{A}_{\sigma,d}(t,y,y/\varepsilon_m)\nabla^{d-1}Y_{v'}\big);\end{aligned}\tag{10.13}$$

for the families of Section 11 these are the coefficients (11.2) in the chart of k. A *path* $\sigma_1\cdots\sigma_i$ is a sequence of sources, each starting where the previous one ends; the empty path $\emptyset$ has $i=0$. We assume that every path has at most N_{path} sources, as for the families of Section 11. On a form $\mathcal{A}$

read at $z = y/\varepsilon_m$, the chain rule $\nabla = \nabla_y + \varepsilon_m^{-1}\nabla_z$ splits (10.13) by the number of derivatives in y into $\mathcal{L}_0$,

$$\begin{aligned}\mathcal{L}_1\mathcal{A} &\coloneqq (\nabla^\perp\psi_m)(t,\varepsilon_m z)\cdot\nabla_y\mathcal{A} - \kappa_m\varepsilon_m^{-1}\big(\nabla_z\cdot(\mathbf{K}\nabla_y\mathcal{A}) + \nabla_y\cdot(\mathbf{K}\nabla_z\mathcal{A})\big),\\ \mathcal{L}_2\mathcal{A} &\coloneqq -\kappa_m\nabla_y\cdot(\mathbf{K}\nabla_y\mathcal{A}),\end{aligned} \tag{10.14}$$

and the terms $-\mathcal{L}_j^{\mathrm{src}}[\sigma]$, where the *source parts* $\mathcal{L}_j^{\mathrm{src}}[\sigma]$ are the parts with exactly j derivatives in y of the term of σ in (10.13) without its minus sign, those that fall on $\mathcal{A}_{\sigma,d}$ included.

The finite lift below collects by degree the terms of the operator (10.13) applied to the sum of $\mathcal{V}_j[\sigma_1\cdots\sigma_i]Y$ over the degrees and the paths, for a field Y that solves the averaged equation stated after Proposition 10.6. The time derivative falls on the coefficients of the correctors, which is the term ∂_t of $\mathcal{L}_0$, and on Y, where the averaged equation replaces it; this gives the last sum of (10.16), in which a path splits at the component where the mean ends and the corrector begins. The proof of Lemma 13.2 carries out this computation. In $\mathcal{M}_{j,d}$, as in $\mathcal{A}_{\sigma,d}$ and in the coefficients of (5.6), the order d counts the derivatives on the field together with the outer divergence, while in $\mathcal{V}_{j,d}$ it counts the derivatives on the field. The means start at $d = 2$: in (10.15) the field Y appears only differentiated, since $\mathcal{V}_0[\emptyset]Y = Y$ appears only under a derivative and the correctors of positive degree annihilate constants (Proposition 10.6(a)).

Definition 10.5 (The finite lift on a pulse)**.** The *finite lift* of a system on the window of pulse k assigns to every path $\sigma_1\cdots\sigma_i$ and every *degree* $0 \le j \le N_{\mathrm{cor}}$, the number of derivatives in y it carries (on the field or the coefficients), a *corrector* $\mathcal{V}_j[\sigma_1\cdots\sigma_i]$, a linear map from smooth fields $Y(y)$ to transverse two-scale forms, $\mathcal{V}_j[\sigma_1\cdots\sigma_i]Y = \sum_{d=0}^{j}\mathcal{V}_{j,d}[\sigma_1\cdots\sigma_i]\nabla^d Y$, and to every degree $2 \le j \le N_{\mathrm{cor}} + N_{\mathrm{src}}$ the *means* $\mathcal{M}_{j,d}[\sigma_1\cdots\sigma_i](t,y)$, $2 \le d \le j$, arrays symmetric in the order of the derivatives. With $\mathcal{V}_0[\emptyset] \coloneqq I$, $\mathcal{V}_0[\sigma_1\cdots\sigma_i] \coloneqq 0$ for $i \ge 1$, and $\mathcal{V}_j[\sigma_1\cdots\sigma_i] \coloneqq 0$ for $j < 0$ and $j > N_{\mathrm{cor}}$, they are defined by induction on i, then on j. The means are the coefficients in

$$\begin{aligned}\sum_{d=2}^{j}\mathcal{M}_{j,d}[\sigma_1\cdots\sigma_i]\nabla^{d-1}Y \coloneqq{}& \mathbf{1}_{\{j=2,\,i=0\}}\kappa_m\mathbf{K}\nabla Y - \big((\nabla^\perp\psi_m)(t,\varepsilon_m z)\,\mathcal{V}_{j-1}[\sigma_1\cdots\sigma_i]Y\big)_{\mathbb{T}^2}\\ &+\Big(\text{the part with } j-1 \text{ derivatives in } y \text{ of } \sum_{d'=2}^{d_{\sigma_i}}\kappa_m\varepsilon_m^{d'-2}\,\mathrm{fac}(\sigma_i)\mathcal{A}_{\sigma_i,d'}(\nabla_y+\varepsilon_m^{-1}\nabla_z)^{d'-1}\\ &\sum_{j''=0}^{N_{\mathrm{cor}}}\mathcal{V}_{j''}[\sigma_1\cdots\sigma_{i-1}]Y\Big)_{\mathbb{T}^2},\end{aligned} \tag{10.15}$$

where, for $i = 0$, the last term is absent, and $\mathcal{V}_{j''}$ counts as j'' derivatives in y. For every $1 \le j \le N_{\mathrm{cor}}$, $\mathcal{V}_j[\sigma_1\cdots\sigma_i]$ is the cell inverse, coefficient by coefficient, of the part of zero average in z of

$$\begin{aligned}-\Big(\mathcal{L}_1\mathcal{V}_{j-1}[\sigma_1\cdots\sigma_i]Y + \mathcal{L}_2\mathcal{V}_{j-2}[\sigma_1\cdots\sigma_i]Y - \sum_{j'=0}^{\min\{j,N_{\mathrm{src}}\}}\mathcal{L}_{j'}^{\mathrm{src}}[\sigma_i]\mathcal{V}_{j-j'}[\sigma_1\cdots\sigma_{i-1}]Y\Big)\\ -\sum_{j''=0}^{j-2}\sum_{i'=0}^{i}\mathcal{V}_{j''}[\sigma_{i'+1}\cdots\sigma_i]\nabla\cdot\Big(\sum_{d=2}^{j-j''}\mathcal{M}_{j-j'',d}[\sigma_1\cdots\sigma_{i'}]\nabla^{d-1}Y\Big),\end{aligned} \tag{10.16}$$

with, for $i = 0$, no source sum; the parenthesis, for any $j \ge 1$, is the *bracket of degree* j. The finite lift of the system (10.13) itself is (10.15)–(10.16) read for matrices over the components, with the source sum over the sources into each component and matrix products in the last sum.

As $\mathcal{L}_0^{\mathrm{src}}$ acts along a shorter path and the means of degree j use only lower degrees or shorter paths, the induction is well founded. The definition extends to any cell operator with an inverse as in Lemma 10.4; the statements below are for transverse forms.

Proposition 10.6 (The finite lift and its means)**.** *There exists $\Lambda_0(\alpha, s, q, \zeta) \geq 128$ with the following property. Let $\Lambda \geq \Lambda_0$ and $N \geq 0$, and consider a system on the window of pulse k whose* majorant matrix*, with the entries $(\omega_{v'}/\omega_v)^{1/2} \sum_{\sigma:\, v' \to v} \mathrm{fac}(\sigma)\, \mathrm{maj}(\sigma)$ as in (11.3), has row and column sums at most 1, and whose conductivity and forms are measurable in t and satisfy, with constants $C(j_1)$, $C(j_1, j_2)$, for every t in the window of pulse k, $y, z \in \mathbb{R}^2$, $2 \leq d \leq N_{\mathrm{src}}$, $j_1 \leq N + 4(N_{\mathrm{path}} + 1)(N_{\mathrm{cor}} + N_{\mathrm{src}})^2$ and $j_2 \leq N + 2 + N_{\mathrm{path}} N_{\mathrm{src}} + 2N_{\mathrm{cor}} + 2N_{\mathrm{src}}$,*

$$\varepsilon_{m-1}^{j_1} |\nabla_y^{j_1} \mathbf{K}(t, y)| \leq C(j_1)\,, \qquad \varepsilon_{m-1}^{j_1} |\nabla_y^{j_1} \nabla_z^{j_2} \mathcal{A}_{\sigma,d}(t, y, z)| \leq C(j_1, j_2)\, \mathrm{maj}(\sigma)\,. \tag{10.17}$$

Then:

(a) Existence and means. *The correctors of positive degree exist, are unique, transverse, of zero average in z, zero at the start of the window of pulse k, and annihilate constants, and the means are $N+1$ times differentiable in y. For every path $\sigma_1 \cdots \sigma_i$ and $1 \leq j \leq N_{\mathrm{cor}} + N_{\mathrm{src}}$, the average in z of the bracket of degree j along it is $-\nabla \cdot \big(\sum_{d=2}^{j} \mathcal{M}_{j,d}[\sigma_1 \cdots \sigma_i] \nabla^{d-1} Y\big)$, and, for $j = 1$, it vanishes.*

(b) The cell equations. *For every $1 \leq j \leq N_{\mathrm{cor}}$, (10.16) has zero average in z, and $\mathcal{V}_j[\sigma_1 \cdots \sigma_i] Y$ solves (10.11) with right side (10.16). The means of degree $j \geq N_{\mathrm{cor}} + 2$ along $\emptyset$ vanish.*

(c) The averaged flux. *If $\mathbf{K} = I$, then $\mathcal{M}_{2,2}[\emptyset] = \mathbf{J}_m^{\kappa_m}(t)$. If, for every t in the window of pulse k, $y \in \mathbb{R}^2$ and j_1 as in (10.17), $\varepsilon_{m-1}^{j_1} |\nabla_y^{j_1} (\mathbf{K} - I)| \leq C(j_1) \varepsilon_{m-1}^{2\delta}$, with $C(j_1)$ of (10.17), then there exists C, depending only on $\alpha, s, q, \zeta, N_{\mathrm{cor}}, N_{\mathrm{src}}, N$ and the constants of (10.17), such that, for every such t and y and every $j_1 \leq N$, $\varepsilon_{m-1}^{j_1} |\nabla_y^{j_1} (\mathcal{M}_{2,2}[\emptyset] - \mathbf{J}_m^{\kappa_m}(t))| \leq C \kappa_{m-1} \varepsilon_{m-1}^{2\delta}$.*

(d) Quiet times. *Let $N \geq N_{\mathrm{src}} - 2$, and let t be a time of the window of pulse k at which $\widetilde{\zeta}_{m,k} = 0$ and the forms do not depend on z. Then, at t: $\mathcal{M}_{2,2}[\emptyset] = \kappa_m \mathbf{K}$; for every path σ of one source, every degree $2 \leq j \leq N_{\mathrm{cor}} + N_{\mathrm{src}}$ and every smooth field Y, $\mathcal{M}_{j,j}[\sigma] \nabla^{j-1} Y = \kappa_m \varepsilon_m^{j-2}\, \mathrm{fac}(\sigma) \mathcal{A}_{\sigma,j} \nabla^{j-1} Y$; and all other coefficients of the means vanish.*

(e) Quiet vanishing. *Let t be in the window of pulse k. If, at all times of the window up to t, $\widetilde{\zeta}_{m,k} = 0$ and the forms do not depend on z, then, for every $1 \leq j \leq N_{\mathrm{cor}}$ and every path $\sigma_1 \cdots \sigma_i$, $\mathcal{V}_j[\sigma_1 \cdots \sigma_i] = 0$ at t.*

(f) The system. *The finite lift of the system (10.13) has from v' to v the correctors and means $\sum \mathcal{V}_j[\sigma_1 \cdots \sigma_i]$ and $\sum \mathcal{M}_{j,d}[\sigma_1 \cdots \sigma_i]$, summed over the paths $\sigma_1 \cdots \sigma_i$ from v' to v.*

So, up to the cell tail (Section 13), the averaged equation of the finite lift is $\partial_t Y_v = \sum \nabla \cdot (\mathcal{M}_{j,d}[\sigma_1 \cdots \sigma_i] \nabla^{d-1} Y_{v'})$, over j, d and the paths, with the sign of (5.6).

Proof. Centring. The average in z of a derivative in z vanishes, and $(\nabla^\perp \psi_m)(t, \varepsilon_m z)$ does not depend on y. So, for a transverse form $\mathcal{A}$ of zero average in z, $(\mathcal{L}_1 \mathcal{A})_{\mathbb{T}^2} = \nabla \cdot ((\nabla^\perp \psi_m)(t, \varepsilon_m z) \mathcal{A})_{\mathbb{T}^2}$ and $(\mathcal{L}_2 \mathcal{A})_{\mathbb{T}^2} = 0$, while $(\mathcal{L}_1 Y)_{\mathbb{T}^2} = 0$ and $(\mathcal{L}_2 Y)_{\mathbb{T}^2} = -\kappa_m \nabla \cdot (\mathbf{K} \nabla Y)$. A source part is $\nabla_y + \varepsilon_m^{-1} \nabla_z$ applied to a flux, with average $\nabla \cdot$ of the averaged flux.

(a), (b). By induction on i and then on j. The centring and (10.15) give the average of the bracket of degree j. At $j = 1$ every term vanishes: along $\emptyset$ since the shear has mean zero; along a path of positive length since $\mathcal{L}_0^{\mathrm{src}}$ is a divergence in z and, on $\mathcal{V}_0$, which does not depend on z, the part of a source flux with no derivative in y vanishes, as $d' \geq 2$. So in (10.16) the term $j'' = 0$,

to which only $i' = i$ contributes, cancels the average of the bracket, and the terms $j'' \geq 1$ have zero average: (10.16) has zero average in z. It is transverse, since $\mathbf{K}$ does not depend on z, and Lemma 10.4 gives $\mathcal{V}_j[\sigma_1 \cdots \sigma_i]$ and (b). On constants, $\mathcal{L}_1$, $\mathcal{L}_2$ and every $\mathcal{L}^{\mathrm{src}}_{j'}[\sigma]$ vanish, hence so do the correctors of positive degree. The regularity of the means is that of Proposition 10.7, which also gives the derivatives in y of the lower correctors and means used to form (10.16). Existence and these bounds are proved together, by the induction of its proof.

(c). Along $\emptyset$, at $j = 1$ the bracket is $(\nabla^\perp \psi_m)(t, \varepsilon_m z) \cdot \nabla Y$, so $\mathcal{V}_1[\emptyset]Y = \chi_{\mathbf{K}} \cdot \nabla Y$ and, by (10.15), $\mathcal{M}_{2,2}[\emptyset] = \kappa_m \mathbf{K} - ((\nabla^\perp \psi_m)(t, \varepsilon_m z) \otimes \chi_{\mathbf{K}})_{\mathbb{T}^2}$. For $\mathbf{K} = I$, $\chi_I = \chi^{\kappa_m}_{m,k}(t, \varepsilon_m z)$, and the mean of $\cos^2$ is $\frac{1}{2}$, so $\mathcal{M}_{2,2}[\emptyset]$ is $\kappa_m I$ plus the term k of (2.16), that is, $\mathbf{J}^{\kappa_m}_m(t)$. For general $\mathbf{K}$, by (10.12) the modes $\xi = \pm 1$ of $\chi_{\mathbf{K}}$ and χ_I differ by the integral of $\widetilde{\zeta}_{m,k}$ against the difference of two exponentials, at most $(4\pi^2 \kappa_m / \varepsilon_m^2)(t - t') \sup |\mathbf{K} - I| \exp(-3\pi^2 \kappa_m (t - t')/\varepsilon_m^2)$, of integral in t' at most $(4\varepsilon_m^2 / 9\pi^2 \kappa_m) \sup |\mathbf{K} - I|$. With $|\nabla^\perp \psi_{m,k}| \leq 2\pi \varepsilon_m^{\beta - 1}$ and (10.5), $|\mathcal{M}_{2,2}[\emptyset] - \mathbf{J}^{\kappa_m}_m(t)| \leq C(\kappa_m + \varepsilon_m^{2\beta}/\kappa_m) \sup |\mathbf{K} - I| \leq C \kappa_{m-1} \varepsilon^{2\delta}_{m-1}$; derivatives in y fall on $\mathbf{K}$ and cost $C\varepsilon^{-1}_{m-1}$ each.

(d), (e). At a time as in (d), the shear term of (10.15) vanishes, and so does the average of the source flux of a corrector of positive degree; what remains is $\kappa_m \mathbf{K} \nabla Y$ and the sources applied to $\mathcal{V}_0[\emptyset]Y = Y$, of degree d for the order d. For (e), by induction: if the lower correctors of positive degree vanish up to t, then, up to t, the bracket is $\mathcal{L}_2 Y$ (for $i = 0$ and $j = 2$), a source part applied to Y (for $i = 1$), or zero, and it does not depend on z. The terms $j'' \geq 1$ of the last sum of (10.16) vanish, and its term $j'' = 0$ does not depend on z. So (10.16), of zero average by (b), vanishes up to t, and (10.12) gives $\mathcal{V}_j[\sigma_1 \cdots \sigma_i] = 0$ at t.

(f). The definition is linear and the cell inverse unique. In the finite lift of the system, a term carrying i sources is a product of the forms along one path, in order; its source parts extend a shorter path by its last source, and its products split a path at one point. So the sums over paths satisfy the definition for the system. Its equations are triangular in the length of the longest path into v, which is at most N_{path}. At degree j, the source part $\mathcal{L}^{\mathrm{src}}_0[\sigma]$ of a source $\sigma \colon w \to v$ couples the corrector from v' to v to the corrector from v' to w at the same degree, and the longest path into w is shorter than that into v. So the definition for the system has exactly one solution, and uniqueness gives (f).

Paths and the threshold. The system along a path (its components, one source per step) satisfies the same hypotheses, and (a) along the path is (a) for it. The threshold Λ_0 is the larger of that of Lemma 10.1(c) and the one beyond which $\bar{\varepsilon}_m \leq \rho(m) \leq \varepsilon_{m-1}$, by (10.5) and Proposition 2.1(c). □

10.4 Bounds on the correctors and the means

For a transverse form $\mathcal{A}$ and integers $N, j_1 \geq 0$, the *Wiener norm*

$$\|\mathcal{A}\|_{N,j_1} \coloneqq \sum_{\xi \in \mathbb{Z}} (1 + |\xi|)^N \big\| \nabla_y^{j_1} \widehat{\mathcal{A}}(\cdot, \cdot\,; \xi) \big\|_{L^\infty([(k-61/48)\tau_m, (k+61/48)\tau_m] \times \mathbb{R}^2)} \tag{10.18}$$

is summed over the entries for arrays. For every $j_2 \leq N$, it bounds $(2\pi)^{-j_2}$ times the norm of $\nabla_y^{j_1} \nabla_z^{j_2} \mathcal{A}$ in $L^\infty([(k - 61/48)\tau_m, (k + 61/48)\tau_m] \times \mathbb{R}^2 \times \mathbb{R}^2)$, and obeys the Leibniz rule, as $1 + |\xi + \xi'| \leq (1 + |\xi|)(1 + |\xi'|)$. A matrix over the components is bounded *in row and column sums* if the sum of its entries over each row and over each column is.

Normalized variables. In the time $\kappa_m t/\varepsilon_m^2$ and the slow variable $y/\rho(m)$, $(\varepsilon_m^2/\kappa_m)\mathcal{L}_0$ has the diffusivity $e^\perp_{(k)} \cdot \mathbf{K} e^\perp_{(k)} \in [\frac{3}{4}, \frac{5}{4}]$ in the fast variable. Relative to $(\varepsilon_m^2/\kappa_m)\mathcal{L}_0$ and to the derivatives in the slow variable, $(\varepsilon_m^2/\kappa_m)\mathcal{L}_1$, $(\varepsilon_m^2/\kappa_m)\mathcal{L}_2$, $(\varepsilon_m^2/\kappa_m)\mathcal{L}^{\mathrm{src}}_{j'}[\sigma]$ have the sizes of the powers $1, 2, j'$ of $\bar{\varepsilon}_m/\rho(m) \leq C\varepsilon^{3\delta}_{m-1}$: the drift $\varepsilon_m(\kappa_m \kappa_{m-1})^{-1/2} \nabla^\perp \psi_m$ of $\mathcal{L}_1$ is at most C by (10.5), and the sources carry $(\varepsilon_m/\rho(m))^{j'}$, smaller as $\kappa_m \leq \kappa_{m-1}$.

Proposition 10.7 (Sizes). *In the setting of Proposition 10.6, there exists C, depending only on $\alpha, s, q, \zeta, N_{\mathrm{cor}}, N_{\mathrm{src}}, N_{\mathrm{path}}, N$ and the constants of (10.17), such that, with sums over the paths $\sigma_1 \cdots \sigma_i$ from v' to v:*

(a) *for every $1 \le d \le j \le N_{\mathrm{cor}}$ and $j_1 \le N$, in row and column sums over the components v, v',*

$$\Big(\frac{\omega_{v'}}{\omega_v}\Big)^{1/2} \varepsilon_{m-1}^{j_1} \Big\| \sum \mathcal{V}_{j,d}[\sigma_1 \cdots \sigma_i] \Big\|_{N,j_1} \le C \bar{\varepsilon}_m^{j} \varepsilon_{m-1}^{d-j} ; \tag{10.19}$$

(b) *for every $2 \le d \le j \le N_{\mathrm{cor}} + N_{\mathrm{src}}$ and $j_1 \le N$, in row and column sums as in (a),*

$$\begin{aligned} \Big(\frac{\omega_{v'}}{\omega_v}\Big)^{1/2} \varepsilon_{m-1}^{j_1} \Big\| \nabla_y^{j_1} \sum \mathcal{M}_{j,d}[\sigma_1 \cdots \sigma_i] \Big\|_{L^\infty([(k-61/48)\tau_m, (k+61/48)\tau_m] \times \mathbb{R}^2)} \\ \le C \kappa_{m-1} \bar{\varepsilon}_m^{j-2} \varepsilon_{m-1}^{d-j} ; \end{aligned} \tag{10.20}$$

(c) *for every path $\sigma_1 \cdots \sigma_i$, $2 \le d \le j \le N_{\mathrm{cor}} + N_{\mathrm{src}}$ and $j_1 \le N+1$,*

$$\begin{aligned} \varepsilon_{m-1}^{j_1} \| \nabla_y^{j_1} \mathcal{M}_{j,d}[\sigma_1 \cdots \sigma_i] \|_{L^\infty([(k-61/48)\tau_m, (k+61/48)\tau_m] \times \mathbb{R}^2)} \\ \le C \operatorname{fac}(\sigma_1) \operatorname{maj}(\sigma_1) \cdots \operatorname{fac}(\sigma_i) \operatorname{maj}(\sigma_i) \, \kappa_{m-1} \bar{\varepsilon}_m^{j-2} \varepsilon_{m-1}^{d-j} . \end{aligned} \tag{10.21}$$

Proof. By induction on the length of the paths and then on the degree; for (c), the majorant matrix is replaced by the entries $\operatorname{fac}(\sigma) \operatorname{maj}(\sigma)$ of the sources of the path.

The inverse and the four products. By Lemma 10.4, in the norms (10.18) with the weights $\varepsilon_{m-1}^{j_1}$, the cell inverse costs $C\varepsilon_m^2/\kappa_m$ and returns two derivatives in z on F^0 and one on F of (10.11). The cross terms of $\mathcal{L}_1$ on a transverse form $\mathcal{B}$ have the form $e_{(k)}^\perp \cdot \nabla_z F$, as $\nabla_z \cdot (\mathbf{K} \nabla_y \mathcal{B}) = e_{(k)}^\perp \cdot \nabla_z (e_{(k)}^\perp \cdot \mathbf{K} \nabla_y \mathcal{B})$ and $\nabla_y \cdot (\mathbf{K} \nabla_z \mathcal{B}) = e_{(k)}^\perp \cdot \nabla_z \nabla_y \cdot (\mathbf{K} e_{(k)}^\perp \mathcal{B})$. A derivative in y of the exponential of (10.12) brings down $-(4\pi^2 \xi^2 \kappa_m / \varepsilon_m^2) \int_{t'}^t e_{(k)}^\perp \cdot \nabla_y \mathbf{K}(t'', y) e_{(k)}^\perp \, \mathrm{d}t''$, of size at most $(4\pi^2 \xi^2 \kappa_m / \varepsilon_m^2)(t - t') \sup |\nabla_y \mathbf{K}|$. So j_1 derivatives bring down products of at most j_1 such terms, with j_1 derivatives on $\mathbf{K}$ in all. After the integration in t' each derivative costs $C\varepsilon_{m-1}^{-1}$, as $\int_0^\infty (a\tau)^l \mathrm{e}^{-3a\tau/4} \, \mathrm{d}\tau = l! (\frac{4}{3})^{l+1} / a$ for $a > 0$. Each term of (10.16) is a lower corrector or mean times one of four coefficients, which the inverse turns into powers of the corrector amplitude; by (10.5), the definition of $\bar{\varepsilon}_m$ and $\varepsilon_m \le \bar{\varepsilon}_m$ (Proposition 2.1(c)), the four products satisfy

$$\frac{\varepsilon_m^2}{\kappa_m} \varepsilon_m^{\beta-1} \le C \bar{\varepsilon}_m, \quad \frac{\varepsilon_m^2}{\kappa_m} \cdot \frac{\kappa_m}{\varepsilon_m} = \varepsilon_m , \quad \frac{\varepsilon_m^2}{\kappa_m} \kappa_m \varepsilon_m^{j'-2} = \varepsilon_m^{j'}, \quad \frac{\varepsilon_m^2}{\kappa_m} \kappa_{m-1} = \varepsilon_m^2 \frac{\kappa_{m-1}}{\kappa_m} = \bar{\varepsilon}_m^2 . \tag{10.22}$$

The step. Assume the bounds along shorter paths, and along the same path at lower degrees. In each term, a derivative in y raises the order of the derivatives of Y or falls on a coefficient, at the cost ε_{m-1}^{-1}, which keeps the power ε_{m-1}^{d-j}. The shear term of $\mathcal{L}_1 \mathcal{V}_{j-1}$, of size $2\pi \varepsilon_m^{\beta-1}$ with two modes, gets the first product; the cross terms of $\mathcal{L}_1 \mathcal{V}_{j-1}$, with one derivative in z that the inverse returns, the second; $\mathcal{L}_2 \mathcal{V}_{j-2}$ the third, with $j' = 2$. A term $\mathcal{L}_{j'}^{\mathrm{src}}[\sigma_i] \mathcal{V}_{j-j'}[\sigma_1 \cdots \sigma_{i-1}]$ has the coefficient $\kappa_m \varepsilon_m^{d'-2} \operatorname{fac}(\sigma_i) \mathcal{A}_{\sigma_i, d'}$ and $d' - j'$ derivatives in z, each costing ε_m^{-1}, so it gets the third product times $C \operatorname{fac}(\sigma_i) \operatorname{maj}(\sigma_i)$. A product term of (10.16) with $j'' \ge 1$ carries κ_{m-1} and $j - 2$ powers of the corrector amplitude, and the fourth product turns κ_{m-1} into two more. This gives (a) at j.

The means. By (10.15), the means are $\kappa_m \mathbf{K}$, at most $\frac{5}{4} \kappa_{m-1}$, at the degree 2, plus two averages. For the shear term, $\varepsilon_m^{\beta-1} \bar{\varepsilon}_m = (\kappa_{m-1} \varepsilon_m^{2\beta} / \kappa_m)^{1/2} \le C \kappa_{m-1}$ by (10.5), so the bounds of the induction on $\mathcal{V}_{j-1}$, those of (a) with the $N+1$ derivatives in y of its reserve and, for (c), along the path,

give (b) and (c). For the source term, the part with j' derivatives in y of a source applied to $\mathcal{V}_{j''}$, $j''+j'=j-1$, is $\kappa_m\varepsilon_m^{j'-1}\operatorname{fac}(\sigma_i)\operatorname{maj}(\sigma_i)$ times the bound of the induction for $\mathcal{V}_{j''}$, and this is at most the bound (b): by $\varepsilon_m\le\bar\varepsilon_m$ if $j'\ge1$, and by $(\kappa_m/\varepsilon_m)\bar\varepsilon_m=(\kappa_m\kappa_{m-1})^{1/2}\le\kappa_{m-1}$ if $j'=0$.

Components and orders. Over the components, each term is a product of the majorant matrix and of lower correctors and means; row and column sums are submultiplicative, and at most 1 for the majorant matrix. The derivatives are counted by reserves: the induction at path length i and degree j carries bounds on the correctors with

$$N+1+(N_{\mathrm{cor}}+N_{\mathrm{src}}-j)\text{ derivatives in }y\quad\text{and}\quad N+2+(N_{\mathrm{path}}-i)N_{\mathrm{src}}+2(N_{\mathrm{cor}}+N_{\mathrm{src}}-j)\text{ in }z\,,$$

its *reserves* at (i,j), and on the means, which do not depend on z, with $N+2+(N_{\mathrm{cor}}+N_{\mathrm{src}}-j)$ derivatives in y, one more than a corrector of the same degree. A reserve r in z is carried in the Wiener norm (10.18) of order $r-2$, as the cell inverse acts mode by mode. A form with r bounded derivatives in z has this norm, since $(1+|\xi|)^r|\widehat{\mathcal{A}}(\xi)|\le 2^r\big(\|\mathcal{A}\|_{L^\infty}+(2\pi)^{-r}\|(e_{(k)}^\perp\cdot\nabla_z)^r\mathcal{A}\|_{L^\infty}\big)$ and $\sum_{\xi\in\mathbb{Z}}(1+|\xi|)^{-2}<\infty$. So the bases $N+1$ and $N+2$ give the $N+1$ derivatives in y of (c) and the Wiener norms (10.18) of order N. Lowering the degree by j' raises the reserve in y by j' and that in z by $2j'$, and shortening the path by one raises the reserve in z by N_{src}. A source part $\mathcal{L}^{\mathrm{src}}_{j'}[\sigma_i]$, $i\ge1$, $0\le j'\le\min\{j,N_{\mathrm{src}}\}$, puts at most j' derivatives in y and $N_{\mathrm{src}}-j'$ in z on its predecessor, of path length $i-1$ and degree $j-j'$, whose reserves exceed those at (i,j) by j' in y and by $N_{\mathrm{src}}+2j'\ge N_{\mathrm{src}}-j'$ in z. The terms with $\mathcal{L}_1$ and $\mathcal{L}_2$ put at most j' derivatives in y and $2j'$ in z on $\mathcal{V}_{j-j'}$, $j'=1,2$, whose reserves exceed those at (i,j) by exactly these numbers. By (10.15), a mean of degree j asks of $\mathcal{V}_{j-1}$ the reserve in y of the means at j, which is that of $\mathcal{V}_{j-1}$, and of a source part with j' derivatives in y on $\mathcal{V}_{j''}$, $j''+j'=j-1$, that reserve plus j', which is that of $\mathcal{V}_{j''}$. The term $\kappa_m\mathbf{K}$ of (10.15) asks at most $N+N_{\mathrm{cor}}+N_{\mathrm{src}}$ derivatives of $\mathbf{K}$. In a product term with $j''\ge1$ (the term $j''=0$ does not depend on z, and the part of zero average removes it), the path splits into two paths, of total length i, with the degrees j'' and $j-j''$, both below j. By Leibniz's rule, the mean, of degree $j-j''$, receives at most $j''+1$ derivatives in y beyond those of the target, and its reserve in y exceeds that of the correctors at (i,j) by exactly $j''+1$. The corrector, of degree j'', receives at most those of the target, within its reserve, and both have reserves in z at least those at (i,j). So the induction closes. This induction runs together with that of Proposition 10.6(a): at (i,j), the bounds at the lower stages bound (10.16) in the Wiener norm of order $r-4$, with $r\ge N+2+2N_{\mathrm{src}}\ge6$ the reserve in z at (i,j), so that (10.16), as a scalar source, has the two bounded derivatives in z that Lemma 10.4 assumes. The largest reserves, at $(0,0)$, are $N+2+N_{\mathrm{path}}N_{\mathrm{src}}+2N_{\mathrm{cor}}+2N_{\mathrm{src}}$ in z, which is the order in z of (10.17), and $N+1+N_{\mathrm{cor}}+N_{\mathrm{src}}\le N+4(N_{\mathrm{path}}+1)(N_{\mathrm{cor}}+N_{\mathrm{src}})^2$ in y, since $N_{\mathrm{src}}\ge2$, which is within its order in y. □

10.5 The corrector operator and the mean flux

Let G be a family of level m, $2\le m\le M$ (Subsection 5.4), whose sources have coefficients (11.2) satisfying the hypotheses of Proposition 10.6 on the window of every pulse k, in its chart, with $\mathbf{K}=\mathbf{K}_{m,l_{m,k}}$. Let $\mathcal{V}_j[\sigma_1\cdots\sigma_i]$ and $\mathcal{M}_{j,d}[\sigma_1\cdots\sigma_i]$ be the correctors and means of its finite lift on the window of pulse k, and, for a field Y on the components of G, let $\widetilde{Y}_k(t,y)\coloneqq Y(t,X_{m-1}(t,y,l_{m,k}h_m))$.

Definition 10.8 (The corrector operator and the mean flux)**.** For a field Y as above, a component v, $t\in\mathbb{R}$ and $x\in\mathbb{R}^2$, with $y\coloneqq X_{m-1}(l_{m,k}h_m,x,t)$, ∇_yX_{m-1} taken at $(t,y,l_{m,k}h_m)$ and $\mathcal{V}_j[\sigma_1\cdots\sigma_i]$, $\mathcal{M}_{j,d}[\sigma_1\cdots\sigma_i]$ those of the window of pulse k, in the term of k, with the sums over the paths $\sigma_1\cdots\sigma_i$

ending at v, from the component v',

$$(\mathrm{Corr}_{m,k}\, Y)_v(t,x) \coloneqq \sum_{j=0}^{N_{\mathrm{cor}}} \sum \big(\mathcal{V}_j[\sigma_1 \cdots \sigma_i]\widetilde{Y}_{k,v'}\big)(t,y,y/\varepsilon_m)\,, \quad \mathrm{Corr}_m\, Y \coloneqq \sum_{k\in 2\mathbb{Z}+1} \mathsf{H}_{m,k}\, \mathrm{Corr}_{m,k}\, Y\,,$$

$$(\mathcal{M}_m Y)_v(t,x) \coloneqq \sum_{k\in 2\mathbb{Z}+1} \mathsf{H}_{m,k}(t)\, \nabla_y X_{m-1} \sum_{j=2}^{N_{\mathrm{cor}}+N_{\mathrm{src}}} \sum \sum_{d=2}^{j} \big(\mathcal{M}_{j,d}[\sigma_1 \cdots \sigma_i]\nabla^{d-1}\widetilde{Y}_{k,v'}\big)(t,y)\,. \quad (10.23)$$

The operator Corr_m is that of (5.10). In (10.23), the term of k vanishes outside the window of k, where $\mathsf{H}_{m,k} = 0$, as $\frac{5}{4} < \frac{61}{48}$, and $\nabla^{d-1}\widetilde{Y}_{k,v'}$ is taken in the label y. The *mean flux* $\mathcal{M}_m$ glues the means, carried to Eulerian coordinates by the Piola identity of Lemma 7.9, under which a flux in the chart is multiplied by $\nabla_y X_{m-1}$. By Propositions 10.6(c) and 10.7(b) and Lemma 7.7(a), the part along the empty paths is close to $\mathbf{J}_m^{\kappa_m}(t)\nabla$; the difference, of all orders, is the defect of the averaged flux, a source of the successor (Section 13).

11. Admissible families

In this section we define the admissible families, those that descend from an admissible seed by cycles (Subsection 5.4), and prove by induction along the cycles that they satisfy the invariant of Definition 11.1 (Proposition 11.5), whose clauses (a2)–(a4), (a6), (b)–(d) are used in Sections 12–14.

11.1 Parameters and sources

Parameters. The following parameters depend only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. With those of Subsection 5.3 and Section 14, they are fixed in the order of Table 11.1, which omits only the absolute constant c_{an} of Subsection 11.3 and computed orders such as N_{rec} and N_{word}. The orders of the entry, among them $N_{\mathrm{src,e}}$ and $N_{\mathrm{out,e}}$, and the bound B on the paths of the seed are defined in Subsection 15.1, in (15.3), item (3) there and (15.7).

(1) The orders of the entry, which fix Part I's N_{rep}, and the seed's bounds (Section 15): orders at most $N_{\mathrm{src,e}}$, and at most B sources on a path. Then, with the other parameters of Subsection 5.3 and the orders of Section 14: N_{gen}, the number of generations kept; N_{src}; the width exponent Θ of (14.1); N_{len}, the length of the longest word kept, and N_{path}; N_{mom}, the moment order of the smoothings, with $2\ell - 1 \geq N_{\mathrm{mom}}$ and the conditions of (13.5); N_{coef}, the order of the derivative bounds, at least N_{ret} and the orders at which the preparation uses these bounds (Section 12). A generation adds at most $2N_{\mathrm{cor}} + 4p + 4$ to the orders and multiplies the number of components on a path by at most $N_{\mathrm{len}} + 2$; this gives the orders N_{src} at generation $N_{\mathrm{gen}} + 1$ and the *path bound* N_{path} at generation N_{gen} (the paragraph after Definition 11.4), and Θ is small:

$$N_{\mathrm{src}} \coloneqq N_{\mathrm{src,e}} + (N_{\mathrm{gen}}+1)(2N_{\mathrm{cor}}+4p+4), \quad N_{\mathrm{path}} \coloneqq (B+1)(N_{\mathrm{len}}+2)^{N_{\mathrm{gen}}} - 1, \quad (N_{\mathrm{src}}-2)\Theta \leq \frac{\delta}{8}\,.$$

(2) The constant $c_{\mathrm{mol}} \in (0,1]$ of the *molecular radius*

$$\rho_{\mathrm{mol}}(m) \coloneqq c_{\mathrm{mol}} \min\big\{\varepsilon_m,\, (\kappa_m \varepsilon_m^{2-\beta})^{1/2}\big\}\,, \quad (11.1)$$

the scale at which a solution of a family of level m has derivatives after the time $\rho_{\mathrm{mol}}(m)^2/\kappa_m$; c_{mol} is fixed in Section 12, before the orders of item (1). Then the seed's orders and constants (Section 15): its derivative order, at least $\max\{N_{\mathrm{coef}}, N_{\mathrm{out,e}}\}$ and large enough in terms of the

orders of item (1); its weight order (the exponent of $1+|\xi|$ in (11.9)), at least its derivative order; the offset $N_{\mathrm{off}} \geq n$ of the analytic norm (Subsection 11.3; used in Definition 15.3); $C_{\mathrm{seed}} \geq 1$ (its mass, coefficients and majorant sums); and its constants of the kinds in item (3).

(3) The invariant's constants, weaker than the seed's (constants larger; orders smaller): $C_\Omega \geq 3$ ($|\cdot|_\infty$-norms of the matrices of the smoothings and of the inverses of their long axes); c_j (the smoothings); C_{wid} (the widths); C_{der}, $C_{\mathrm{jet}}(j)$ (the derivative bounds); the weight order $N_{\mathrm{an}} \geq N_{\mathrm{coef}}$ of the analytic norm. They are so large that Steps 6 and 7 of the proof of Lemma 11.7 close; in particular, the c_j are at least the $c_j(u,\ell)$ of Lemma 8.2 for the kernel orders in use.

(4) $C_{\mathrm{adm}} \geq C_{\mathrm{seed}}$, the constant of the majorant and coupling sums, and C_{maj}, the factor that a cycle applies to majorants, both large enough given item (3) (Lemma 11.6; Step 5 of the proof of Lemma 11.7); the *carrier level* m_{c}, the coarsest level of the chart parameters (11.19); the top level $m_0 \geq \max\{m_{\mathrm{c}}, 2\}$ of Subsection 5.3, the largest of the thresholds on the level in Sections 11–14, in particular so large that $C_{\mathrm{adm}}\varepsilon_{m-1}^{q\delta/4} \leq \frac{1}{8}$ for every $m \geq m_0$, that $\sum_{i>m_0}(3\tau_i + 4h_i) < \frac{1}{2}t_0$ (Lemma 11.8(iv)), and that the *fresh smoothings* (those of the sources a cycle creates, Definition 11.3(iv)) created at the levels finer than m_0 have matrices of long axes with inverses of $|\cdot|_\infty$-norm ≤ 2. Also, C_{maj} is large enough given the constants of Lemma 15.4(b) and of Steps 4 and 5 of the proof of Lemma 11.7 for the terminal cycle (Proposition 15.10).

Sources. A source $\sigma\colon v' \to v$ carries a *birth order* d_σ, its order when created, and a *current order*, beyond which its coefficients vanish; transports keep both, and only for a seed's source may the current order be below d_σ. It also carries a *birth radius* ρ_σ; a *factor* $\mathrm{fac}(\sigma) > 0$, the constant in front of its two-scale forms in (11.2); and a *majorant* $\mathrm{maj}(\sigma) \geq 0$, which bounds its two-scale forms, so that $\mathrm{fac}(\sigma)\,\mathrm{maj}(\sigma)$ bounds the source. The *generation* $\mathrm{gen}(v)$ counts the cycles that created v.

Two-scale forms. On the window $|t - k\tau_m| \leq \frac{61}{48}\tau_m$ of a pulse k of level m (Section 10), σ acts in the chart of k (Section 7), at the label y, through the coefficients

$$\kappa_m \varepsilon_m^{d-2}\,\mathrm{fac}(\sigma)\,\mathcal{A}_{\sigma,d}\Big(t, y, \frac{y}{\varepsilon_m}\Big), \qquad 2 \leq d \leq d_\sigma\,, \tag{11.2}$$

with *two-scale forms* $\mathcal{A}_{\sigma,d}(t,y,z)$ in the sense of Section 10, periodic in the fast variable z, with values indexed by the two coordinates of the flux and the entries of ∇^{d-1}. They also depend on *chart parameters*, the shears of the coarser levels (precise form (11.7)), and (11.2) uses those of the chart of k. The $A_{\sigma,d}$ of (5.6) are the sums over k, with time cutoffs, of the *transports* of the coefficients (11.2) from y to x, order d giving orders $2,\dots,d$ (precise form (11.8)).

A two-scale form $\mathcal{A}$ has an *analytic norm* $\|\mathcal{A}\|_{\mathrm{an},v}$, a weighted sum over its Fourier modes in z of its suprema over a complex neighbourhood of the chart parameters that shrinks by a fixed step per generation of v (precise form (11.9)). Cauchy's estimate bounds the derivatives of a cycle at the cost of this step, for at most N_{gen} generations. The *majorant matrix* and its *priced* form are

$$\begin{aligned}
&(\omega_{v'}/\omega_v)^{1/2} \sum\nolimits_{\sigma\colon v'\to v} \mathrm{fac}(\sigma)\,\mathrm{maj}(\sigma),\\
&(\omega_{v'}/\omega_v)^{1/2} \sum\nolimits_{\sigma\colon v'\to v} \mathrm{fac}(\sigma)\,\max\{1, (\rho_\sigma/r_\sigma)^{d_\sigma-2}\}\,\mathrm{maj}(\sigma);
\end{aligned} \tag{11.3}$$

the price compensates for the $d-2$ derivatives that (5.9) puts on the kernel of width r_σ; for the masses ω_v/Ψ_v^2, with the markers Ψ_v of Subsection 5.4, the entries are multiplied by $\Psi_v/\Psi_{v'}$.

chosen, in this order	uses	conditions, where stated
$\Lambda_0(\alpha, s, q, \zeta) \geq 128$	α, s, q, ζ	at least the threshold $\Lambda_0(\alpha, s, q, \zeta)$ of every statement that declares one (Theorem A)
$\Lambda \geq \Lambda_0$, $n \geq 1$, $0 < t_0 < T$		given (Subsection 5.3); the proof of Theorem A takes n depending only on α, s, q, in (9.10)
$c_{\mathrm{mol}} \in (0, 1]$	$\alpha, s, q, \zeta, \Lambda$	(12.1)
$N_{\mathrm{cor,w}}$, $N_{\mathrm{len,w}}$		(15.1)
Q_{e}; then $N_{\mathrm{cor,e}}$, p_{e}, $N_{\mathrm{len,e}}$, $N_{\mathrm{cl,e}}$; then $N_{\mathrm{src,e}}$, Θ_{e}	$N_{\mathrm{cor,w}}$	(15.2), (15.3)
N_{rep}; the seed's bounds on its orders and path lengths	$N_{\mathrm{cor,w}}$, $N_{\mathrm{cor,e}}$, p_{e}, $N_{\mathrm{src,e}}$, $N_{\mathrm{len,e}}$, $N_{\mathrm{len,w}}$	(15.4); Subsection 15.1, item (3)
Z; then Q	N_{rep}	(14.2); $Q \geq 4 + 2qZ$ in (14.3)
N_{gen}, N_{cor}, p, N_{cl}	Q	their conditions in (13.5)
N_{src}; then Θ	the seed's bounds, N_{gen}, N_{cor}, p, N_{rep}	item (1): N_{src}, and $(N_{\mathrm{src}} - 2)\Theta \leq \delta/8$; (14.4)
N_{len}; then the path bound N_{path}	Q, the seed's bounds, N_{gen}	(13.5); item (1)
N_{mom}	Q, Θ	(13.5), which gives the first condition of (12.2); the kernels of the smoothings have $2\ell - 1 \geq N_{\mathrm{mom}}$ (clause (c1), (14.9))
N_{dep}; then N_{buf}	Q, N_{src}, N_{mom}	(12.2)
N_{ret}; then N_{out}	the orders above	the floors (13.6) and (14.3); $N_{\mathrm{out}} \coloneqq N_{\mathrm{ret}} + 4p$ (Section 13)
the internal orders of Lemmas 12.8–12.15 and Proposition 12.7; then N_{coef}	the orders above	N_{coef} at least N_{ret}, N_{buf} and these orders (item (1)), among them the numbers of fast derivatives of the insertions of Lemmas 12.12 and 12.15 (Subsection 12.6, derivative counts)
$N_{\mathrm{mom,w}}$, $N_{\mathrm{mom,e}}$	the orders above	(15.5)
the kernel orders ℓ of S_{w} and of the smoothing of the new sources of Section 15	$N_{\mathrm{mom,w}}$, $N_{\mathrm{mom,e}}$, N_{mom}	$2\ell - 1 \geq \max\{N_{\mathrm{mom,w}}, N_{\mathrm{mom}}\}$ (Subsection 15.3) and $2\ell - 1 \geq \max\{N_{\mathrm{mom,e}}, N_{\mathrm{mom}}\}$ (Definition 15.9(iii))
$N_{\mathrm{rd,e}}$; then $N_{\mathrm{out,e}}$; then $N_{\mathrm{tr,w}}$	the orders above	$N_{\mathrm{rd,e}} \geq nN_{\mathrm{cor,w}} + 5$ and (15.6); then (15.7); $N_{\mathrm{tr,w}}$ also at least the order that Section 12 reads at the terminal level to produce Π at $N_{\mathrm{out,e}}$ and $2N_{\mathrm{rd,e}} + 2$ (Subsection 15.1, item (4))
the seed's constants	all the orders	item (2): weight order $\geq$ derivative order $\geq \max\{N_{\mathrm{coef}}, N_{\mathrm{out,e}}\}$; $N_{\mathrm{off}} \geq n$; $C_{\mathrm{seed}} \geq 1$
the invariant's constants; then C_{adm}, C_{maj}	the seed's constants	items (3)–(4): $C_\Omega \geq 3$, c_j, C_{wid}, C_{der}, $C_{\mathrm{jet}}(j)$, $N_{\mathrm{an}} \geq N_{\mathrm{coef}}$; $C_{\mathrm{adm}} \geq C_{\mathrm{seed}}$
the carrier level m_{c}		(11.19)
m_0	everything above the double line, and m_{c}	item (4), with $m_0 \geq m_{\mathrm{c}}$; every level finer than m_0 satisfies the inequalities used in Sections 11–14, (14.19) included, and $m_0 \geq 2$
M_*; then $M_1 \coloneqq \max\{M_*, m_0 + 2\}$	everything above	Proposition 5.1, last; the proof of Proposition 4.1

Table 11.1: The order in which the parameters of Parts I and II are chosen. Each row may use the rows above it; the middle column names the earlier choices it uses (in the first three rows, the parameters it uses). The local paragraphs (Subsection 5.3, the items above, Sections 12–15) say why each condition is needed. Everything above the double line depends only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$ and is fixed before the levels M, m and the diffusivity κ; below it are thresholds on the levels, chosen after everything above and changing none of it, each so large that the inequalities of the proofs hold beyond it. The constants c_{br}, c_{lo}, c_{dis}, c_{hi} of Section 12 are absolute.

11.2 Definitions and the main proposition

Each clause is stated here in words; its precise form, given in Subsection 11.3, is the definition.

Definition 11.1 (The invariant)**.** A family of level m satisfies the *invariant* on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$ if, with the constants of items (1), (3) and (4) of Subsection 11.1 (for a seed: its constants, its derivative order (item (2)) in place of N_{coef} in (b2), (b2′), (c3) and (d), and C_{seed} for C_{adm}):

(a) *Cell structure* (cycles solve one-dimensional cell problems; charts exact; precise forms (11.10)–(11.13)):

- (a1) the two-scale forms are built from the construction's functions and the family's *cell solutions* (built from the solutions of its two-dimensional cell problems) by sums, products, derivatives, averages and primitives in z, the cell inverse across the pulse and changes of chart;
- (a2) on the window of pulse k, at all chart parameters, the two-scale forms of k are transverse (Section 10), that is, independent of the coordinate of z along $e_{(k)}$;
- (a3) at every $t \in [t_{\mathrm{A}}, t_{\mathrm{f}}]$ with $|t - k\tau_m| \le \frac{3}{4}\tau_m$, the forms of all pulses whose windows contain t are invariant along $e_{(k)}$;
- (a4) at every $t \in [t_{\mathrm{A}}, t_{\mathrm{f}}]$ such that the velocity added at level m vanishes on $[t - \frac{\tau_m}{24}, t + \frac{\tau_m}{24}]$, the forms of every pulse whose window contains t do not depend on z;
- (a5) an integer shift of the base label does not change the forms;
- (a6) where the velocity added at level m vanishes, consecutive charts describe one coefficient: the change of chart maps (11.2) of one to the next's, with z fixed;
- (a7) near each chart parameter, a pulse's forms depend only on the flow over its *history*.

(b) *Majorants* (the derivative bounds of Sections 12–13). Here $\|\cdot\|_{L^\infty_m([t_{\mathrm{A}},t_{\mathrm{f}}])}$ is the norm of (11.14), and (b2)–(b3) hold for both masses.

- (b1) for every source $\sigma\colon v' \to v$ and $2 \le d \le d_\sigma$, the entries of $\mathcal{A}_{\sigma,d}$ extend as in (11.9) and $(\varepsilon_m/\rho_\sigma)^{d-2}\|\mathcal{A}_{\sigma,d}\|_{\mathrm{an},v} \le \mathrm{maj}(\sigma)$;
- (b2) for every v, v', in each pulse's chart, the sum

$$(\omega_{v'}/\omega_v)^{1/2} \sum\nolimits_{\sigma\colon v'\to v} \mathrm{fac}(\sigma) \sum\nolimits_{d=2}^{d_\sigma} \sum\nolimits_{j,j'\le N_{\mathrm{coef}}} \varepsilon_{m-1}^{j} \big\|\nabla_y^j \nabla_z^{j'} \mathcal{A}_{\sigma,d}\big\|_{L^\infty_m([t_{\mathrm{A}},t_{\mathrm{f}}])} \tag{11.4}$$

  is at most C_{der} times the entry (v, v') of the majorant matrix;
- (b2′) for every $N, j \le N_{\mathrm{coef}}$, the sums of (8.4) of order N at scale $\rho_{\mathrm{mol}}(m)$ for (11.2), after j more slow derivatives (in y), are at most $C_{\mathrm{jet}}(j)\,\varepsilon_{m-1}^{-j}$ times the majorant matrix (precise form (11.15));
- (b3) the priced majorant matrix has row and column sums at most $C_{\mathrm{adm}}\varepsilon_m^{\delta/4}$.

(c) *Smoothing* (the c_j of Subsection 5.4; moments; small widths), for every source σ:

- (c1) $S_\sigma(t) = S_{\ell,r_\sigma,\Omega_1(t)} S_{\ell,r_\sigma,\Omega_2(t)}$ as in (6.2) (ℓ, Ω_i depending on σ), $2\ell - 1 \ge N_{\mathrm{mom}}$, Ω_i Borel measurable in t, $|\Omega_i(t)|_\infty \le C_\Omega$, and the matrix of long axes $\Omega_1(t)e_1$, $\Omega_2(t)e_2$ is invertible, with inverse of $|\cdot|_\infty$-norm $\le C_\Omega$;
- (c2) $r_\sigma \le C_{\mathrm{wid}}\rho_{\mathrm{mol}}(m)\,\varepsilon_m^{\delta}$ and $r_\sigma \le C_{\mathrm{wid}}\rho(m)\,\varepsilon_{m-1}^{4\delta}$;
- (c3) for every $t \in [t_{\mathrm{A}}, t_{\mathrm{f}}]$, $\rho > 0$, $N \le N_{\mathrm{coef}}$ and $j \le N_{\mathrm{src}} - 2$, $\|\nabla^j S_\sigma(t) g\|_{H^N_\rho(\mathbb{T}^2)} \le c_j r_\sigma^{-j}\|g\|_{H^N_\rho(\mathbb{T}^2)}$ (precise form (11.16)).

(d) *Coupling* (for Proposition 8.3): for every $N \le N_{\mathrm{coef}}$ and both masses, the coupling matrices (8.4) on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$ of order N at scale $\rho_{\mathrm{mol}}(m)$ have row and column sums at most $C_{\mathrm{adm}}\varepsilon_{m-1}^{q\delta/4}$.

Definition 11.2 (Admissible seed)**.** An *admissible seed* of level $m_{\mathrm{S}} < M$ on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$, $t_{\mathrm{f}} - t_{\mathrm{S}} \le T$, is a family of level m_{S} whose components have generation 0 and marker 1, such that:

(i) its total mass is at most C_{seed}; its paths of sources and current orders obey the seed's bounds; its widths are at most $\varepsilon_{m_{\mathrm{S}}}$, and so are the birth radii of its sources of birth order at least 3;

(ii) for every source σ, $0 < \rho \le \varepsilon_{m_{\mathrm{S}}}$, N at most the seed's derivative order (item (2)) and $2 \le d \le d_\sigma$, the sum of (8.4) satisfies $\sum_{j \le N} \frac{\rho^j}{j!} \sup_{[t_{\mathrm{S}}, t_{\mathrm{f}}] \times \mathbb{T}^2} |\nabla^j A_{\sigma,d}| \le C_{\mathrm{seed}} \operatorname{maj}(\sigma) \kappa_{m_{\mathrm{S}}} \rho_\sigma^{d-2}$;

(iii) its cell solutions are built from finitely many two-dimensional cell problems, each solved once on the window of its pulse, and are unchanged when the phases move along the pulses (precise form (11.17));

(iv) it satisfies the invariant on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$, also with the supremum in time in (b2′) (precise form (11.17)).

Definition 11.3 (Successor)**.** Let G be a family of level $m > m_0$ with finite lifts along its paths of sources (Section 10). A *successor* G' of G, of level $m - 1$, is a family that the comparison of Section 13 constructs with the lift (5.10), solving no new two-dimensional cell problem; its data are:

(i) *Letters:* the *defect letter* (the defect of the averaged flux at level m; birth order $N_{\mathrm{cor}}+1+4p$); for every $3 \le d \le N_{\mathrm{src}} - N_{\mathrm{cor}} - 4p - 4$ and every nonzero path length up to the path bound N_{path}, a *mean letter* (birth order d) for the means of degree d of the lift along paths of that length (for a path of one source, less the cell average of that source in the order d, which its transport carries); for every $2 \le d \le N_{\mathrm{src}} - N_{\mathrm{cor}} - 4p - 4$, every nonzero path length up to N_{path} and every $j, j' \le p$ with $j + j' \ge 1$, a *covariance letter* (birth order $d + 2(j + j')$) for the means of degree d along paths of that length, with no subtraction, conjugated by $(-\Gamma_m : \nabla^2)^j/j!$ on the left and $(\Gamma_m : \nabla^2)^{j'}/j'!$ on the right (summed over $j, j' \le p$, the term $j = j' = 0$ included, these give the mean times the truncated exponentials (5.11) on both sides, as in the flux (13.20)).

(ii) *Components:* those of G, with their masses and markers, and their children v', labelled by nonempty words of letters, with $\mathrm{gen}(v') = \mathrm{gen}(v) + 1 \le N_{\mathrm{gen}}$, $\#v' \le N_{\mathrm{len}}$ and

$$\omega_{v'} \coloneqq \frac{1 - 128^{-\eta/2}}{2N_{\mathrm{word}}} \varepsilon_{m-1}^{\eta/2} \omega_v \,, \qquad \Psi_{v'} \coloneqq \varepsilon_{m-1}^{-\eta/8} \Psi_v \,, \tag{11.5}$$

where N_{word} counts the nonempty words of length $\le N_{\mathrm{len}} + 1$.

(iii) *Old sources,* in each word: the transport of the cell average (the average in z) of every source, in its orders $d \ge 3$ (the same factor, width, smoothing, birth radius, birth order and current order; majorant times C_{maj}); the mean of degree 2 along each nonempty path of sources (birth order 2; width, smoothing and birth radius of its widest source; product of factors; $C_{\mathrm{maj}}\times$ product of majorants).

(iv) *New sources:* for each letter and path it labels (each component, for the defect letter), a source from the first component's word to the last one's word plus the letter: coefficients multiplied by $\varepsilon_{m-1}^{-\eta}$, the fresh smoothing (14.9) with the ellipses of (11.18), birth radius $\rho(m)$, the letter's birth order, factor the path's product of factors (1 for the defect letter), majorant $C_{\mathrm{maj}}\varepsilon_{m-1}^{\delta-\eta}$ times the product of the majorants along the path.

The components of G' are thus the pairs (v, w) of a component v of G and a word w of length at most N_{len}, with $\mathrm{gen}(v) < N_{\mathrm{gen}}$ if w is nonempty (the empty word gives v, a nonempty one a child

of v). In each word w, a source of (iii) along a path from v' to v (one source, for a transport) goes from (v', w) to (v, w), and a source of (iv) of a letter ℓ along such a path (with $v' = v$ for the defect letter) from (v', w) to $(v, w\ell)$. The children of word length $N_{\mathrm{len}}+1$ or generation $N_{\mathrm{gen}}+1$ are created with the masses and markers of (11.5), then dropped with the sources at them (Subsection 13.3).

Definition 11.4 (Admissible family)**.** A family G of level $m \geq m_0$ is *admissible* on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$ if there exist an admissible seed $G_{m_{\mathrm{S}}}$ of level $m_{\mathrm{S}} \geq m$, $m_{\mathrm{S}} > m_0$, on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$ with $t_{\mathrm{S}} + \frac{1}{2}t_0 < t_{\mathrm{f}}$, and families $G_{m_{\mathrm{S}}-1}, \dots, G_m = G$, each G_{i-1} a successor of G_i, such that $t_{\mathrm{A}} = t_{\mathrm{A}}^{(m)}$, where $t_{\mathrm{A}}^{(m_{\mathrm{S}})} \coloneqq t_{\mathrm{S}}$ and, for every $m < i \leq m_{\mathrm{S}}$,

$$t_{\mathrm{A}}^{(i-1)} \coloneqq t_{\mathrm{A}}^{(i)} + 3\tau_i \leq t_{\mathrm{f}} - 4h_i\,. \tag{11.6}$$

The path bound. For a family G and $g \geq 0$, let $P_g(G)$ be the largest number of sources on a path of G whose components have generation at most g. We claim that in the families G_i of Definition 11.4 no source goes to a smaller generation and $P_g(G_i) \leq B + gN_{\mathrm{len}}$ for every g; so every path of G_i has at most $B + N_{\mathrm{gen}}N_{\mathrm{len}} \leq N_{\mathrm{path}}$ sources. This holds for the seed (generation 0) and passes to a successor G' of G. A component (u, w) of G', with u a component of G and w a word, has generation $\mathrm{gen}(u)+1$ if w is nonempty and $\mathrm{gen}(u)$ otherwise; an old source goes from (u, w) to (u'', w), and a new one from (u, w) to $(u'', w\ell)$, along a nonempty path of G from u to u'', or with $u'' = u$ for the defect letter (Definition 11.3). So an old source adds the same 0 or 1 to both generations, and a new one ends at generation $\mathrm{gen}(u'') + 1 \geq \mathrm{gen}(u) + 1$, by the claim for G. Along a path of G' the word only grows, by a letter at each new source, so the path has at most N_{len} new sources; it projects onto a path of G, each old source onto at least one source. If it ends at generation at most g with an empty word, its sources are all old, at most $P_g(G)$; with a nonempty word, the projected path ends at generation at most $g-1$, so lies in the generations at most $g-1$ by the monotonicity in G, and the path has at most $P_{g-1}(G) + N_{\mathrm{len}}$ sources. Thus $P_g(G') \leq \max\{P_g(G), P_{g-1}(G) + N_{\mathrm{len}}\}$.

Proposition 11.5 (Admissible families)**.** *An admissible family of level m on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$ satisfies the invariant with the constants of items (1), (3) and (4) of Subsection 11.1, which depend only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. Moreover, for its sources with a fresh smoothing the inverse in (c1) has $|\cdot|_\infty$-norm ≤ 2, so that the long axes are uniformly transverse in the sense of Proposition 6.1, and (c3) holds with the constants $c_j(u, \ell)$ of Lemma 8.2; and the sums in (d) are at most $C_{\mathrm{adm}}\varepsilon_{m-1}^{q\delta/4} \leq \frac{1}{8}$.*

In the main estimate of the proof, the recursion for (b3), a new source gains $\varepsilon_{m-1}^{\delta-\eta}$, loses $\varepsilon_{m-1}^{-3\eta/8-(N_{\mathrm{src}}-2)\Theta}$ to its masses, markers and price, and keeps $\varepsilon_{m-1}^{45\delta/64} \leq \varepsilon_{m-1}^{\delta/4}$, as $\eta, (N_{\mathrm{src}}-2)\Theta \leq \delta/8$.

11.3 Precise form of the admissibility conditions

Pulses and chains. The chart of a pulse k of level m is *used* where its time cutoff $\mathsf{H}_{m,k}$ (10.2) is positive; *quiet times* are those at which the velocity added at level m vanishes; $c_{\mathrm{an}} \coloneqq 1/20000$ is the strip constant.

(11.7) *Chart parameters.* A *chain* $\mathcal{C}$ of k consists of a level $m' \leq m$, pulses k_i of the levels $m' \leq i \leq m$, $k_m = k$, each k_{i+1} with its chart used in the window of k_i, a *base label* $y^{(m'-1)} \in \mathbb{R}^2$, and, for each $m' \leq i < m$, a *centre* $y^{(i)} \in \mathbb{R}^2$ and a *phase* $z^{(i)} \in \mathbb{R}^2$. With $\bar{t}_{i,k} \coloneqq l_{i,k}h_i$, the centre of the block of a pulse k of level i, and $\bar{t}_i \coloneqq \bar{t}_{i,k_i}$, its flows $\mathcal{X}_i$ and centre points $\bar{x}_i$ are $\mathcal{X}_{m'-1} \coloneqq X_{m'-1}$, $\bar{x}_{m'-1} \coloneqq y^{(m'-1)}$ and, for $m' \leq i < m$, $\bar{x}_i \coloneqq \mathcal{X}_{i-1}(\bar{t}_{i+1}, \bar{x}_{i-1} + y^{(i)}, \bar{t}_i)$ and

$$\mathcal{X}_i(t', \cdot, t) \coloneqq \mathcal{X}_{i-1}(t', \cdot, \bar{t}_i) \circ \Big[y \mapsto y + \Big(\int_t^{t'} \widetilde{\zeta}_{i,k_i}(t'')\,\mathrm{d}t''\Big)\nabla^\perp\psi_{i,k_i}\big(y + \varepsilon_i z^{(i)} - \bar{x}_{i-1} - y^{(i)}\big)\Big]$$
$$\circ\, \mathcal{X}_{i-1}(\bar{t}_i, \cdot, t)\,;$$

at the matched phase $\varepsilon_i z^{(i)} = \bar{x}_{i-1} + y^{(i)}$ the bracket is the shear displacement of Section 7. The chain is *allowed* if $m' \geq m_{\mathrm{c}}$, the carrier level, and, for every $m' \leq i < m$, $|y^{(i)}|_1 < 3c_{\mathrm{an}}\varepsilon_i/4$. The chain $\mathcal{C}^{[y]}$ *recentred* by $y \in \mathbb{R}^2$ has the centre point $\bar{x}_{m-1} + y$: it moves the base label if $m' = m$, and otherwise the last centre, keeping $\varepsilon_{m-1}z^{(m-1)} - y^{(m-1)}$ fixed. For $m' = m$ the chain is the chart of k with base label $y^{(m-1)}$. The two-scale forms of k depend also on the chain: each is a function $\mathcal{A}(t, \mathcal{C}, z)$ of t in the window of k, an allowed chain $\mathcal{C}$ of k and z, whose value at the chain with $m' = m$ and base label y (the *chain without shear*) is the two-scale form $\mathcal{A}(t, y, z)$ of Section 10 in (11.2). The *history* of k is the smallest interval containing its window, $\bar{t}_m$ and the histories of the pulses of level $m+1$ whose charts are used in the window of k (at the finest level $M+n$ of $\mathbf{b}_{M+n}$: its window and $\bar{t}_m$). For t in the window of k, a chain $\mathcal{C}$ of k, a pulse k' of level $m+1$ with $\mathsf{H}_{m+1,k'}(t) > 0$ and $z \in \mathbb{C}^2$ in the strip of the phases of (11.9), the chain *extended at* (t, z) has the data of $\mathcal{C}$, $k_{m+1} = k'$, the phase $z^{(m)} = z$ and the centre $y^{(m)} = (\int_t^{\bar{t}_{m+1}} \widetilde{\zeta}_{m,k}(t'')\,\mathrm{d}t'')\nabla^{\perp}\psi_{m,k}(\varepsilon_m z)$. Its centre point at the time t is that of $\mathcal{C}$, as the shear over $[\bar{t}_{m+1}, t]$ moves $\bar{x}_{m-1} + y^{(m)}$ to $\bar{x}_{m-1}$; the centre is along $e_{(k)}$, depends on z only through $z \cdot e_{(k)}^{\perp}$, and has $|y^{(m)}|_1 \leq 4 \cdot 10^{-6} c_{\mathrm{an}}\varepsilon_m$ by (11.19), as $|t - \bar{t}_{m+1}| \leq \frac{1}{2}h_{m+1} + \frac{3}{4}\tau_{m+1}$. At $\varepsilon_m z = \bar{x}_{m-1}$ its map on the history of k' is the shear displacement of (7.13).

(11.8) *Transport.* Let $x \mapsto y$ be an area-preserving change of variables. The Piola identity writes the divergence in y of a vector field F as $\nabla_x \cdot \big((\nabla_x y)^{-1} F\big)$, and the chain rule, applied one derivative at a time in the order of an entry of ∇_y^{d-1}, writes that entry as a combination of the entries of $\nabla_x, \dots, \nabla_x^{d-1}$. Substituting both into $\nabla_y \cdot (\text{coefficient} \cdot \nabla_y^{d-1} g)$ for the orders $2 \leq d \leq d_\sigma$ and collecting the terms in each entry of each $\nabla_x^{d'-1} g$ gives the transported coefficients, of orders $2 \leq d' \leq d_\sigma$, indexed by the entries of $\nabla_x^{d'-1}$ (not symmetrized).

(11.9) *Analytic norm.* For a component v, the *reserve* $\mathrm{res}(v) \coloneqq (N_{\mathrm{gen}} + 2 - \mathrm{gen}(v))/(N_{\mathrm{gen}} + 2 + N_{\mathrm{off}})$, with N_{off} the offset of item (2). Let $\mathcal{D}_v$ be the set of complex chart parameters, with the discrete data of an allowed chain, such that $|\operatorname{Im} y^{(m'-1)}|_1 < (1 + \mathrm{res}(v)/4)\,\varepsilon_{m'-1}/100$ and, for every $m' \leq i < m$,

$$|y^{(i)}|_1 < \frac{1 + \mathrm{res}(v)}{4}\, c_{\mathrm{an}}\varepsilon_i\,, \qquad |\operatorname{Im} z^{(i)} \cdot e_{(k_i)}^{\perp}| < \Big(3 + \frac{\mathrm{res}(v)}{4} + \frac{1}{64}\Big) c_{\mathrm{an}}\,,$$

with $e_{(k)}^{\perp}$ of Section 10; the *outer domain* is defined with 2 in place of $\mathrm{res}(v)$, and its real points are the allowed chains. If the entries of a two-scale form $\mathcal{A}$ of k extend holomorphically to the outer domain (for each t; Borel measurable in t, locally bounded uniformly in t; periodic in the phases and independent of their coordinates along the pulses), then

$$\|\mathcal{A}\|_{\mathrm{an},v} \coloneqq \sum_{\text{entries}} \sum_{\xi \in \mathbb{Z}} \exp(3\pi c_{\mathrm{an}}|\xi|)(1 + |\xi|)^{N_{\mathrm{an}}} \sup |\widehat{\mathcal{A}}(t, \mathcal{C}; \xi)|\,,$$

with $\widehat{\mathcal{A}}(t, \mathcal{C}; \xi)$ the ξ-th Fourier coefficient in $z \cdot e_{(k)}^{\perp}$, N_{an} the weight order of item (3), and the supremum over the pulses k of level m, t in the window of k and $\mathcal{C} \in \mathcal{D}_v$.

(11.10) *Clause (a1).* Every entry of a two-scale form of k is built from: the Jacobian of $\mathcal{X}_{m-1}(t, \cdot, \bar{t}_m)$ at $\bar{x}_{m-1}$, its inverse, the conductivity (as for the renewal charts of Section 7) and the strain (the time derivative of the Jacobian times its inverse); the same for the chain extended at (t, z) (end of (11.7)) toward a pulse k' of level $m+1$ whose chart is used in the window of k, where the Jacobian, its inverse and the conductivity take their identity values outside the support of the time cutoff of k'; the drift of the cell problem of k; the time cutoffs; the matrices $\mathbf{J}_i^{\kappa_i}$ of (2.16) and Γ_i of (5.11); constants; and the family's cell solutions. The operations are sums, products, derivatives in z across the pulse, in the phases and along recentring, averages and transverse primitives in z, the inverse of the one-dimensional cell operator across the pulse (zero at the start of the window), and the

evaluation of an entry built in this way for a pulse k' as above, at the level $m+1$ and independent of the fast variable of k', at the chain extended at (t,z) toward k', where the time cutoff of k' is positive, times that cutoff.

(11.11) *Clauses (a2)–(a5).* They hold at all allowed chains; in (a5), the base label moves in $\mathbb{Z}^2$.

(11.12) *Clause (a6).* If $k' \neq k$ is in the block of k or the next one and $t \in [t_{\mathrm{A}}, t_{\mathrm{f}}]$ is quiet in both windows, then at t the forms of k, k' at chains without shear do not depend on z, and, as coefficients of symmetric arrays ∇^{d-1}, (11.2) of k' at y' is the transport of that of k by $y = X_{m-1}(\bar t_{m,k}, y', \bar t_{m,k'})$.

(11.13) *Clause (a7).* If allowed chains $\mathcal{C}$, $\mathcal{C}'$ of k have the same map $y \mapsto \mathcal{X}_{m-1}(t, \bar x_{m-1} + y, \bar t_m)$ on the history of k, then, for y near 0 and every t, z, their two-scale forms at $\mathcal{C}^{[y]}$ and $\mathcal{C}'^{[y]}$ agree.

(11.14) *Clause (b2).* In (11.4), the norm is $\|\cdot\|_{L^\infty_m([t_{\mathrm{A}},t_{\mathrm{f}}])}$, the essential supremum over $t \in [t_{\mathrm{A}}, t_{\mathrm{f}}]$ of the supremum over the pulses k whose window contains t, the allowed chains $\mathcal{C}$ of k with all $|y^{(i)}|_1 < c_{\mathrm{an}}\varepsilon_i/4$ and $z \in \mathbb{R}^2$, summed over the entries; and ∇_y differentiates $\mathcal{A}_{\sigma,d}(t, \mathcal{C}^{[y]}, z)$ at $y = 0$ (for a chain without shear, this is the ∇_y of (11.4)).

(11.15) *Clause (b2′).* For both masses, every v, v' and every $N, j \le N_{\mathrm{coef}}$, the sum (11.4), with $\|\cdot\|_{L^\infty_m([t_{\mathrm{A}},t_{\mathrm{f}}])}$ and ∇_y as in (11.14) and with the inner double sum of (11.4) replaced by

$$\sum_{j_1+j_2\le N} \frac{(N+d-2)!\,\rho_{\mathrm{mol}}(m)^{j_1+j_2+2-d}}{N!\,j_1!\,j_2!\,\varepsilon_m^{j_2+2-d}}\, \|\nabla_y^{j_1+j}\nabla_z^{j_2}\mathcal{A}_{\sigma,d}\|_{L^\infty_m([t_{\mathrm{A}},t_{\mathrm{f}}])}\,,$$

is at most $C_{\mathrm{jet}}(j)\,\varepsilon_{m-1}^{-j}$ times the entry (v,v') of the majorant matrix.

(11.16) *Clause (c3).* The operators $S_\sigma(t)$ commute with derivatives, $t \mapsto S_\sigma(t)g$ is Borel measurable for every $g \in L^2(\mathbb{T}^2)$, and, for every $g \in H^N_\rho(\mathbb{T}^2)$, the bound holds.

(11.17) *Seeds, (iii)–(iv).* (iii) Each cell solution of the seed is given by an expression built as in (11.10) from finitely many two-dimensional cell problems, each solved once on the window of its pulse, in the cell time $t\kappa_{\min\{i,M\}}/\varepsilon_i^2$ of its level i, from its initial value, and depending, among the pulses of level $i+1$, only on those whose charts are used in that window. It is unchanged when the phases of an allowed chain move along their pulses, and any two functions given by the same expression agree on the window and the allowed chains. (iv) In (11.15) at $m = m_{\mathrm{S}}$, the essential supremum in time is replaced by the supremum.

(11.18) *Fresh smoothings (Definition 11.3(iv)).* The averaged flux $\mathbf{J}_m^{\kappa_m}$ is diagonal and $4h_m$-periodic, with its entries exchanged by a shift of $2h_m$ (Lemma 7.10). For $l \in \mathbb{Z}$, let $t_{\mathrm{hor}} \in [(l+\frac12)h_m, (l+\frac92)h_m)$ be a time at which the first entry of $\mathbf{J}_m^{\kappa_m}$ is largest, t_{ver} the time of this interval at distance $2h_m$ from it, where the second entry is largest, and l_{hor}, l_{ver} the blocks containing them. The fresh smoothing of the successor of a family of level m is (14.9) with, for $(l+\frac12)h_m \le t < (l+\frac32)h_m$,

$$\Omega_1(t) := \nabla_y X_{m-1}(t, y_{\mathrm{hor}}, l_{\mathrm{hor}}h_m)\big(\kappa_{m-1}^{-1}\mathbf{J}_m^{\kappa_m}(t_{\mathrm{hor}})\big)^{1/2}, \qquad y_{\mathrm{hor}} := X_{m-1}\big(l_{\mathrm{hor}}h_m, 0, (l+\tfrac12)h_m\big),$$
$$\Omega_2(t) := \nabla_y X_{m-1}(t, y_{\mathrm{ver}}, l_{\mathrm{ver}}h_m)\big(\kappa_{m-1}^{-1}\mathbf{J}_m^{\kappa_m}(t_{\mathrm{ver}})\big)^{1/2}, \qquad y_{\mathrm{ver}} := X_{m-1}\big(l_{\mathrm{ver}}h_m, 0, (l+\tfrac12)h_m\big),$$

with entrywise square roots. These are the ellipses of the averaged flux at its horizontal and vertical maxima, in the label of their block, carried to the time t by the coarse flow.

11.4 The induction

Proposition 11.5 is proved by induction along the cycles of Definition 11.4: Lemma 11.6 starts it at the seed, and Lemma 11.7, proved in Subsection 11.6 with the complex chains of Subsection 11.5, passes the invariant to the successor, at the cost of the time $3\tau_m$. Lemma 11.8 records what Sections 13 and 14 use about successors. In Subsections 11.4–11.6, C and c denote positive constants depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$, fixed before $m_0 \ge m_{\mathrm{c}}$, beyond which the inequalities below hold.

Lemma 11.6 (Seeds). *An admissible seed of level $m_{\mathrm{S}} > m_0$ on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$ with $t_{\mathrm{S}} + \frac{1}{2}t_0 < t_{\mathrm{f}}$ satisfies the invariant on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$ with the constants of items (1), (3) and (4) of Subsection 11.1.*

Proof. By Definition 11.2(iv), the seed satisfies the invariant with its constants. Each clause of Definition 11.1 other than (d) weakens when a constant grows or an order decreases (in (b1), a smaller weight order decreases the analytic norm, and the reserves are the same, since the offset is), and (d) is increasing in the c_j. As the constants of families are weaker than the seed's (items (1)–(3)), this gives the invariant except for the sums of (d); for the sums of (b3), $C_{\mathrm{seed}}\varepsilon_{m_{\mathrm{S}}}^{\delta/4} \le C_{\mathrm{adm}}\varepsilon_{m_{\mathrm{S}}}^{\delta/4}$. By (b1) for the seed, the coefficient (11.2) of order d of a source σ is at most $\kappa_{m_{\mathrm{S}}}\rho_\sigma^{d-2}\,\mathrm{fac}(\sigma)\,\mathrm{maj}(\sigma)$. Its transport to x gives the orders $d' \le d$ at the cost $\varepsilon_{m_{\mathrm{S}}-1}^{d'-d} \le \rho_\sigma^{d'-d}$: for $d' < d$ the birth order of σ is at least 3, so $\rho_\sigma \le \varepsilon_{m_{\mathrm{S}}}$ by Definition 11.2(i). Hence the argument of Step 8 of Subsection 11.6, at the level m_{S} in place of $m-1$ and with the Cauchy estimates of Step 6, shows that the coupling matrices of (d), formed with the c_j of families, are at most C times the priced majorant matrix, entry by entry. By Lemma 7.1(a) and as C_{adm} is large given item (3), their sums are at most

$$CC_{\mathrm{seed}}\varepsilon_{m_{\mathrm{S}}}^{\delta/4} \le 2^{\delta/4}CC_{\mathrm{seed}}\varepsilon_{m_{\mathrm{S}}-1}^{q\delta/4} \le C_{\mathrm{adm}}\varepsilon_{m_{\mathrm{S}}-1}^{q\delta/4}\,. \qquad \square$$

Lemma 11.7 (One cycle). *Let $m > m_0$ and $t_{\mathrm{A}} + 3\tau_m + 4h_m \le t_{\mathrm{f}}$. Let G be a family of level m obtained by successors from an admissible seed on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$, with $t_{\mathrm{S}} \le t_{\mathrm{A}}$, that satisfies the invariant on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$ with the constants of items (1), (3) and (4) of Subsection 11.1, and let G' be a successor of G. Then G' satisfies the invariant on $[t_{\mathrm{A}} + 3\tau_m, t_{\mathrm{f}}]$ with the same constants.*

Proof of Proposition 11.5. Let G be admissible of level m on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$, with the seed $G_{m_{\mathrm{S}}}$ and the families G_i of Definition 11.4. By Lemma 11.6, $G_{m_{\mathrm{S}}}$ satisfies the invariant on $[t_{\mathrm{A}}^{(m_{\mathrm{S}})}, t_{\mathrm{f}}] = [t_{\mathrm{S}}, t_{\mathrm{f}}]$, and for $i = m_{\mathrm{S}}, \dots, m+1$ in turn Lemma 11.7 passes it from G_i on $[t_{\mathrm{A}}^{(i)}, t_{\mathrm{f}}]$ to G_{i-1} on $[t_{\mathrm{A}}^{(i-1)}, t_{\mathrm{f}}]$, since $i > m_0$, $t_{\mathrm{S}} \le t_{\mathrm{A}}^{(i)}$ and $t_{\mathrm{A}}^{(i)} + 3\tau_i + 4h_i \le t_{\mathrm{f}}$ by (11.6). The fresh smoothings of the sources of G were created above $m \ge m_0$, so Step 7 of Subsection 11.6 gives the inverse bound 2 and (c3) with the constants of Lemma 8.2; the bound $\frac{1}{8}$ holds by item (4). $\square$

Lemma 11.8 (Successors). *Let G be a family of level $m > m_0$ obtained from an admissible seed by successors, and let G' be a successor of G.*

(i) *Each component of G is one of G', with the same masses and markers.*

(ii) *The total mass of G' is at most twice that of the seed.*

(iii) *The readout (5.10) satisfies $|\Upsilon_m| \le 1 + N_{\mathrm{word}}^{1/2}\varepsilon_{m-1}^{5\eta/4}$.*

(iv) *For every admissible seed $G_{m_{\mathrm{S}}}$ of level $m_{\mathrm{S}} > m_0$ on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$ with $t_{\mathrm{S}} + \frac{1}{2}t_0 < t_{\mathrm{f}}$, there exist families $G_{m_{\mathrm{S}}-1}, \dots, G_{m_0}$, each G_{i-1} a successor of G_i, for which (11.6) holds with $m = m_0$. In particular, each G_i is an admissible family of level i on $[t_{\mathrm{A}}^{(i)}, t_{\mathrm{f}}]$.*

Proof. (i) A component of G is labelled in G' by the empty word and has generation at most N_{gen}, so Definition 11.3(ii) keeps it with its mass and marker. (ii) By (11.5), a cycle at level i gives each component v at most N_{word} children, of total mass at most $\frac{1}{2}(1 - 128^{-\eta/2})\varepsilon_{i-1}^{\eta/2}\omega_v$, and $\varepsilon_{i-1} \le 128^{-(i-1)}$ by Lemma 7.1(a). So the cycles multiply the total mass by at most $\exp(\frac{1}{2}(1 - 128^{-\eta/2})\sum_{i\ge 2} 128^{-(i-1)\eta/2}) \le \exp(1/2) < 2$. (iii) In the coordinates scaled by the masses to the power $-\frac{1}{2}$, the readout (5.10) is the identity on the components of G plus a matrix with one entry $\varepsilon_{m-1}^{\eta\#v'}(\omega_{v'}/\omega_v)^{1/2} \le \varepsilon_{m-1}^{5\eta/4}$, by (11.5), for each component v and child v' of v. A row has at most N_{word} entries and a column one, so Schur's test (Subsection 5.4) bounds this matrix by $N_{\mathrm{word}}^{1/2}\varepsilon_{m-1}^{5\eta/4}$. (iv)

By Proposition 11.5 and the setting of the lift (Section 13), where (b1), (b3), Cauchy's estimate and (7.9) give the hypotheses of Proposition 10.6, an admissible family of a level finer than m_0 has finite lifts up to t_{f} on the windows that start at or after t_{A}, as the cell inverse is solved forward from the start of the window (11.10); these include every window that meets $[t_{\mathrm{A}}+3\tau_m, t_{\mathrm{f}}]$, of length $\frac{61}{24}\tau_m < 3\tau_m$ by (10.2). So Section 13 constructs its successor. As Proposition 11.5 for G_i uses only $G_{m_{\mathrm{S}}}, \dots, G_i$, the families are constructed one level at a time, up to G_{m_0} as long as (11.6) holds, and it holds: as $t_{\mathrm{A}}^{(i)} = t_{\mathrm{S}} + \sum_{i'=i+1}^{m_{\mathrm{S}}} 3\tau_{i'}$, item (4) gives, for every $m_0 < i \le m_{\mathrm{S}}$,

$$t_{\mathrm{A}}^{(i)} + 3\tau_i + 4h_i \le t_{\mathrm{S}} + \sum_{i'>m_0} (3\tau_{i'} + 4h_{i'}) < t_{\mathrm{S}} + \tfrac{1}{2}t_0 < t_{\mathrm{f}}\,. \qquad \square$$

11.5 Holomorphic extension of the flows

The analytic norm takes suprema over complex chart parameters, so the flows $\mathcal{X}_i$ of a chain (11.7) must extend holomorphically to the outer domain. They are compositions of a coarse flow, extended by Lemma 7.6, with entire shear displacements. The domains close because the renewal times of a chain are short, which the carrier level provides: m_{c} satisfies, for every level $i \ge m_{\mathrm{c}} - 1$,

$$\varepsilon_i^{\beta-2} h_{i+1} \le 10^{-6} c_{\mathrm{an}}\,, \qquad 48\, h_{i+2} < \tau_{i+1}\,, \tag{11.19}$$

and such a level exists, by (2.13) and, for the second bound, by (2.5) and $\varepsilon_{i+1} \le 2\varepsilon_i^q$, as $q(2-\beta+2\delta) - (2-\beta+4\delta) = (q-1)(1-\frac{\beta}{2q+2}) + 2q\delta > 0$ by (2.3). For a chain of a pulse k of level m and $m'-1 \le i \le m-1$, the flow $\mathcal{X}_i$ is anchored at the centre $\bar t_{i+1}$ of the block of k_{i+1} ($k_m = k$); we call $|t - \bar t_{i+1}| \le \frac{3}{5}h_{i+1}$ the *interval* of $\mathcal{X}_i$, and write $\bar x_i(t) \coloneqq \mathcal{X}_i(t, \bar x_i, \bar t_{i+1})$ for the centre point at time t.

The intervals of the flows and the histories of the pulses are nested. Let $i \ge m_{\mathrm{c}}$, and let a pulse k_{i+1} of level $i+1$ have its chart used at a time s of the window of a pulse k_i of level i, so that $|s - k_{i+1}\tau_{i+1}| < \frac{5}{4}\tau_{i+1}$ and $|s - k_i\tau_i| \le \frac{61}{48}\tau_i$ by (10.2). The time of a pulse of level j is within $\frac{1}{2}h_j$ of the centre of its block (Lemma 7.1(d)), and $\tau_j \le h_j/25$ by (2.4). Hence, by the second bound of (11.19) at the level $i-1 \ge m_{\mathrm{c}}-1$ and induction on the level, starting from the finest level (where a history is the window and $\bar t_{M+n}$), the interval $|t - \bar t_{i+1,k_{i+1}}| \le \frac{3}{5}h_{i+1}$ and the history of k_{i+1} lie within

$$\Big(\frac{3}{5}+\frac{1}{2}\Big)h_{i+1} + \frac{5}{4}\tau_{i+1} \le \frac{6}{5}h_{i+1} < \frac{\tau_i}{40} \ \text{ of } s, \text{ hence within } \ \frac{1}{2}h_i + \frac{61}{48}\tau_i + \frac{\tau_i}{40} < \frac{3}{5}h_i \ \text{ of } \bar t_{i,k_i},$$

as $\frac{1}{2} + \frac{1}{25}(\frac{61}{48} + \frac{1}{40}) < \frac{3}{5}$; so does the history of k_i.

Lemma 11.9 (Complex chains)**.** *Fix the levels and pulses of a chain of a pulse k of level m, with $m' \ge m_{\mathrm{c}}$. For every $m'-1 \le i \le m-1$, the maps $\mathcal{X}_i(t', y, t)$, for t, t' in the interval of $\mathcal{X}_i$, and the centre points $\bar x_i(t)$ defined above extend from the allowed chains to the parameters of the outer domain of (11.9) and to the points y with $|y - \bar x_i(t)|_\infty < 8c_{\mathrm{an}}\varepsilon_i$, holomorphically in the parameters and in y, such that, for every t, t', t'' in the interval of $\mathcal{X}_i$ and every y with $|y - \bar x_i(t)|_\infty < 8c_{\mathrm{an}}\varepsilon_i$:*

(i) *$\mathcal{X}_i(t, y, t) = y$ and $|\nabla_y \mathcal{X}_i(t', y, t) - I|_\infty \le 2^{25}\varepsilon_i^{\beta-2}|t'-t|$;*

(ii) *$\mathcal{X}_i(t', \bar x_i(t), t) = \bar x_i(t')$, and, if $|y - \bar x_i(t)|_\infty < 2c_{\mathrm{an}}\varepsilon_i$, then $\mathcal{X}_i(t'', \mathcal{X}_i(t', y, t), t') = \mathcal{X}_i(t'', y, t)$.*

By (i), (ii) and $2^{25}\varepsilon_i^{\beta-2} \cdot \frac{6}{5}h_{i+1} \le \frac{1}{100}$, from (11.19), the map $\mathcal{X}_i(t', \cdot, t)$ sends every ball about $\bar x_i(t)$ of radius at most $8c_{\mathrm{an}}\varepsilon_i$ into the ball about $\bar x_i(t')$ of $\frac{101}{100}$ times that radius, and, by Cauchy's estimate, its derivative is Lipschitz on the ball of radius $4c_{\mathrm{an}}\varepsilon_i$, with constant at most $1/(4c_{\mathrm{an}}\varepsilon_i)$.

Proof. By induction on i. *The first level.* We apply Lemma 7.6 at the level $m'-1$ on the interval of $\mathcal{X}_{m'-1} = X_{m'-1}$, of length $t_2 - t_1 = \frac{6}{5}h_{m'}$, where $\bar{x}_{m'-1}(t) = X_{m'-1}(t, y^{(m'-1)}, \bar{t}_{m'})$. By (11.19), the number $1 - 2^{21}\varepsilon_{m'-1}^{\beta-2}(t_2 - t_1)$ of that lemma satisfies

$$1 - 2^{21}\varepsilon_{m'-1}^{\beta-2} \cdot \tfrac{6}{5}h_{m'} \geq 1 - \tfrac{6}{5} \cdot 2^{21} \cdot 10^{-6} c_{\mathrm{an}} \geq 0.9998\,.$$

In units of $\varepsilon_{m'-1}/1000$, in which $c_{\mathrm{an}}\varepsilon_{m'-1} = \frac{1}{20}$, Lemma 7.6 with this bound gives holomorphy and (7.7) where $|\operatorname{Im} y|_1 < 0.9998 \cdot 19$, and the group law where $|\operatorname{Im} y|_1 \leq 0.9998^2 \cdot 19$. In the outer domain $|\operatorname{Im} y^{(m'-1)}|_1 < 15$, so $|\operatorname{Im} \bar{x}_{m'-1}(t)|_1 < 15/0.9998 < 15.2$ by (7.7). As $|\cdot|_1 \leq 2|\cdot|_\infty$, the balls about $\bar{x}_{m'-1}(t)$ of radii $8c_{\mathrm{an}}\varepsilon_{m'-1}$ and $2c_{\mathrm{an}}\varepsilon_{m'-1}$, that is, $\frac{2}{5}$ and $\frac{1}{10}$, lie in $|\operatorname{Im} y|_1 < 16$ and $|\operatorname{Im} y|_1 < 15.4$. This gives (i), as $10 \cdot 2^{21} \leq 2^{25}$, and, with the group law at the base label, (ii).

From $i-1$ to i. As the chart of k_{i+1} is used in the window of k_i, the interval of $\mathcal{X}_i$ lies within $\frac{3}{5}h_i$ of $\bar{t}_i$, inside that of $\mathcal{X}_{i-1}$, by the nesting above ($i \geq m' \geq m_{\mathrm{c}}$). For t, t' in it, (11.7) composes $\mathcal{X}_{i-1}(\bar{t}_i, \cdot, t)$, the *shear map* (the bracket in (11.7)) and $\mathcal{X}_{i-1}(t', \cdot, \bar{t}_i)$. The shear $\nabla^\perp \psi_{i,k_i}$ is entire and depends on its argument only through the coordinate along $e^\perp_{(k_i)}$, which the shear map therefore does not change. Where the imaginary part of this coordinate is less than $15c_{\mathrm{an}}\varepsilon_i$, the shear is at most $2\pi\varepsilon_i^{\beta-1}\exp(30\pi c_{\mathrm{an}})$ and its derivative at most $4\pi^2\varepsilon_i^{\beta-2}\exp(30\pi c_{\mathrm{an}})$, by (2.6). As $0 \leq \widetilde{\zeta}_{i,k_i} \leq 1$ by (2.2) and (2.10), (11.19) shows that the shear map over an interval of length at most $\frac{6}{5}h_{i+1}$ moves each point of the strip by at most $\frac{6}{5}h_{i+1} \cdot 2\pi\varepsilon_i^{\beta-1}\exp(30\pi c_{\mathrm{an}}) \leq 10^{-5}c_{\mathrm{an}}\varepsilon_i$. The point $\bar{x}_{i-1} + y^{(i)}$, and its image under the shear map over $[\bar{t}_{i+1}, t]$, which $\mathcal{X}_{i-1}(t, \cdot, \bar{t}_i)$ maps to $\bar{x}_i(t)$ by (11.7) and (ii) for $\mathcal{X}_{i-1}$, lie within $(\frac{3}{4} + 10^{-5})c_{\mathrm{an}}\varepsilon_i < 2c_{\mathrm{an}}\varepsilon_{i-1}$ of $\bar{x}_{i-1} = \bar{x}_{i-1}(\bar{t}_i)$, as $|y^{(i)}|_1 < \frac{3}{4}c_{\mathrm{an}}\varepsilon_i$. So $|\bar{x}_i(t) - \bar{x}_{i-1}(t)|_\infty \leq \frac{101}{100}(\frac{3}{4} + 10^{-5})c_{\mathrm{an}}\varepsilon_i \leq c_{\mathrm{an}}\varepsilon_i$, by (i) for $\mathcal{X}_{i-1}$, and all points below lie within $10c_{\mathrm{an}}\varepsilon_i \leq \frac{10}{128}c_{\mathrm{an}}\varepsilon_{i-1}$ of $\bar{x}_{i-1}(t)$ or $\bar{x}_{i-1}$, in the balls of (i), (ii) and of the Lipschitz bound for $\mathcal{X}_{i-1}$. Hence $\mathcal{X}_{i-1}(\bar{t}_i, \cdot, t)$ sends the ball $|y - \bar{x}_i(t)|_\infty < 8c_{\mathrm{an}}\varepsilon_i$ into the ball of radius $\frac{101}{100} \cdot 8c_{\mathrm{an}}\varepsilon_i$ about the image of $\bar{x}_i(t)$, which is $\bar{x}_{i-1} + y^{(i)}$ moved along $e_{(k_i)}$ by a shear map. For y' in the image ball, the bound on $\operatorname{Im} z^{(i)} \cdot e^\perp_{(k_i)}$ of the outer domain gives

$$\bigl|\operatorname{Im}\bigl((y' + \varepsilon_i z^{(i)} - \bar{x}_{i-1} - y^{(i)}) \cdot e^\perp_{(k_i)}\bigr)\bigr| < \frac{101}{100} \cdot 8c_{\mathrm{an}}\varepsilon_i + \Bigl(3 + \frac{1}{2} + \frac{1}{64}\Bigr)c_{\mathrm{an}}\varepsilon_i < 12c_{\mathrm{an}}\varepsilon_i\ :$$

the argument of the shear stays in the strip, and $\mathcal{X}_i$ and $\bar{x}_i(t)$ are holomorphic. At $t' = t$ the shear map is the identity, and the shear maps over $[t, t']$ and $[t', t'']$ compose to the one over $[t, t'']$, since the integrals of $\widetilde{\zeta}_{i,k_i}$ add and the shear map does not change the coordinate along $e^\perp_{(k_i)}$; with (i), (ii) for $\mathcal{X}_{i-1}$, this gives the identities of (i), (ii). By the chain rule and the group law (ii) for $\mathcal{X}_{i-1}$, $\nabla_y \mathcal{X}_i(t', y, t) - I$ is the sum of three terms: $\nabla_y \mathcal{X}_{i-1}(t', y, t) - I$, bounded by (i) for $\mathcal{X}_{i-1}$; the change of $\nabla_y \mathcal{X}_{i-1}(t', \cdot, \bar{t}_i)$ over the shear displacement (at most $|t' - t|\,2\pi\varepsilon_i^{\beta-1}\exp(30\pi c_{\mathrm{an}})$), bounded by the Lipschitz bound above, times $\nabla_y \mathcal{X}_{i-1}(\bar{t}_i, y, t)$, of norm at most $\frac{101}{100}$; and $\int_t^{t'} \widetilde{\zeta}_{i,k_i}(t'')\,\mathrm{d}t''$, at most $|t' - t|$, times the derivative of the shear between two matrices of norm at most $\frac{101}{100}$. As $\varepsilon_{i-1}^{\beta-2} \leq 128^{\beta-2}\varepsilon_i^{\beta-2} \leq \frac{1}{2}\varepsilon_i^{\beta-2}$, since $2 - \beta = 1 - \alpha > \frac{2}{3}$,

$$\begin{aligned}\frac{|\nabla_y \mathcal{X}_i(t', y, t) - I|_\infty}{|t' - t|} &\leq 2^{25}\varepsilon_{i-1}^{\beta-2} + \frac{101}{100} \cdot \frac{2\pi\varepsilon_i^{\beta-1}\exp(30\pi c_{\mathrm{an}})}{4c_{\mathrm{an}}\varepsilon_{i-1}} + \Bigl(\frac{101}{100}\Bigr)^2 4\pi^2\varepsilon_i^{\beta-2}\exp(30\pi c_{\mathrm{an}}) \\ &\leq (2^{24} + 10^3)\varepsilon_i^{\beta-2}\,,\end{aligned}$$

which is at most $2^{25}\varepsilon_i^{\beta-2}$. At real parameters the extensions are given by the formulas (11.7). □

11.6 Proof of Lemma 11.7

Proof of Lemma 11.7. Each clause for G', of level $m-1$, is read off its explicit data (Definition 11.3), those of G and the invariant of G, on $[t_{\mathrm{A}} + 3\tau_m, t_{\mathrm{f}}]$, the time window of Proposition 5.3(b). At a

time t, the forms of G' read those of G at the pulses k of level m with $\mathsf{H}_{m,k}(t) > 0$ (Step 1) at times of the window of k up to t, the lift being solved forward on it (11.10), and only the flows of the chains, functions of the construction, reach back along the histories; for $t \geq t_{\mathrm{A}} + 3\tau_m$ these times lie in $[t_{\mathrm{A}}, t_{\mathrm{f}}]$, where G satisfies the invariant, as in the proof of Lemma 11.8(iv). Step 3, the recursion for (b3), is the main estimate. It comes before (b1) because the majorants of G' are numbers fixed by Definition 11.3, so (b3) is a recursion on numbers, while (b1) says that the forms obey them, which needs the analytic calculus of Step 4. Step 5 proves (b1), and (b2), (b2′), (d) follow. Components of G are written v, v'; a component $\tilde{v}$ of G' *over* v is v or a child of v.

Step 1: (a1)–(a4). By Definition 11.3 and Section 13, the forms of a source of G' at a pulse k' of level $m-1$ are of four kinds: transports (11.8) of cell averages of forms of G; means of the lift along paths of sources of G; these means conjugated by powers of $\Gamma_m : \nabla^2$; and the defect of the averaged flux, with the commutator of the covariance with $\mathbf{b}_{m-1}$, of flux (13.17). At a time t of the window of k', they read the construction's functions and the forms of G at the pulses k of level m with $\mathsf{H}_{m,k}(t) > 0$, at the chains of k extended at (t, z) from those of k' (end of (11.7), one level lower), whose phase z is the fast variable of k'; as the forms of G may contain evaluations toward the level $m+1$, these evaluations are nested, which (11.10) allows. The lift of Section 10 is built from these forms by the cell inverse across the pulse, derivatives and averages in the fast variable of level m, products and slow derivatives. So the forms of G' are built as in (11.10) from the construction's functions and the cell solutions of the seed, and no two-dimensional cell problem is solved: this is (a1). They depend on z only through the coordinate along $e^{\perp}_{(k')}$ of the phase of the shear of k': so do the shear map and the centre of the extended chain, and so do the forms of G, since forms of finite analytic norm do not depend on the coordinates of the phases along the pulses (11.9). So the forms of G' are transverse: this is (a2). The same argument, with (b1) for G, shows that they are periodic in the phases and independent of their coordinates along the pulses, as (11.9) requires.

For (a4), let t, in the window of k', be as in (a4) at the level $m-1$. The forms of k' at t read those of G at the pulses k of level m with $\mathsf{H}_{m,k}(t) > 0$ on their histories, within $\frac{6}{5}h_m < \frac{\tau_{m-1}}{40}$ of t by the nesting after (11.19) at the level $m - 1 \geq m_{\mathrm{c}}$. There the shear of k' vanishes, so the extended chains have centre 0 and their maps on the history of k and centre points do not depend on z; by (a7) for G, neither do the forms of G and the correctors at these chains, nor the forms of k' at t. For (a3), let $|t - k'\tau_{m-1}| \leq \frac{3}{4}\tau_{m-1}$: the forms of k' are transverse by (a2), and the window of another pulse meets this interval only where $|t - k'\tau_{m-1}| \geq \frac{35}{48}\tau_{m-1}$, and there no pulse is active within $\frac{\tau_{m-1}}{24}$ of t, as $\zeta_{m-1,k''}$ vanishes outside $|t - k''\tau_{m-1}| \leq \frac{2}{3}\tau_{m-1}$ (Lemma 7.1(d)), so that (a4) applies. The commutator term of the defect letter reads no form of G: in its flux (13.17), $\nabla\mathbf{b}_{m-1}$ read in the chart of k' is the strain of $\mathcal{X}_{m-2}$ plus $\nabla_y(\nabla_y\mathcal{X}_{m-2}\, w)\,(\nabla_y\mathcal{X}_{m-2})^{-1}$, with w the drift of k', which depends only on $z \cdot e^{\perp}_{(k')}$ and vanishes at the times of (a4) and (a6). So this term is built as in (11.10) and is transverse; at those times it is the strain, the transport of the one Eulerian coefficient $\nabla\mathbf{b}_{m-1}$, which gives (a4) and (a6); and, a function of the flow, it satisfies (a5) and (a7).

Step 2: (a5)–(a7). Moving the base label by an integer vector commutes with every operation of (11.10): the flows commute with the translations by $\mathbb{Z}^2$ (Lemma 7.4), the arguments of the shear streams and the centres of the extended chains do not change, the cutoffs do not depend on the label, and the forms of G are invariant by (a5) for G: this gives (a5). Clauses (a6) and (a7) follow from (a7) for G, as a form of G' at k' is built from forms of G over the histories of the pulses of level m whose charts are used in the window of k', which lie in the history of k'. For (a7), two allowed chains of k' with the same map over this history, extended at the same (t, z), have the same centre, which depends only on k', k, t, z, and the same maps over these histories; so the forms of G agree there near the chain, by (a7) for G, and so do the correctors, solved from zero, and the forms of G'. For (a6), the windows of two pulses k', k'' of level $m-1$ meet only if their indices differ by 2, and their overlap lies at distance at least $(\frac{35}{48} - \frac{2}{3})\tau_{m-1} = \frac{\tau_{m-1}}{16}$ from the supports of both fast

cutoffs, so both shears vanish within $\frac{\tau_{m-1}}{16}$ of a time t of the overlap, and, as in Step 1 and by (a7) for G, the extended chains reduce to the charts of k' and k'', related by Lemma 7.7(b); the forms of G' at k', k'' are then transports of the same forms of G through the two charts, and transport composes exactly, as the chain rule taken one derivative at a time is associative: this is (11.12).

Step 3: the recursion for (b3). Fix one of the two masses. By (b3) for G, every row and column sum of the priced majorant matrix (11.3) of G is at most $C_{\mathrm{adm}}\varepsilon_m^{\delta/4}$, at most $\frac{1}{2}$ for $m > m_0$ by item (4) and Lemma 7.1(a). For a path of sources of G from v' to v, the product of the corresponding entries of this matrix is at least $(\omega_{v'}/\omega_v)^{1/2}\prod_{\sigma\in\mathrm{path}}\mathrm{fac}(\sigma)\,\mathrm{maj}(\sigma)$: the mass ratios telescope, and every price is at least 1. Their sums over the paths of a given length into or out of a component are at most the row or column sums of a power of the matrix, so over all nonzero lengths at most $2C_{\mathrm{adm}}\varepsilon_m^{\delta/4}$.

Old sources. Each word contains a copy of G (Definition 11.3(iii)), its sources joining the components of the word over v' and v, whose masses and markers are those of v' and v times numbers depending only on the word (Lemma 11.8(i), (11.5)); so the masses in the entries are those of G. A transported source keeps its factor, birth radius, birth order and width, and its majorant is multiplied by C_{maj}; a second-order mean has birth order 2, hence price 1, factor the product of the factors and majorant C_{maj} times the product of the majorants. Counting transports as paths of one source, the copies into or out of a component of a word contribute at most $2C_{\mathrm{maj}}C_{\mathrm{adm}}\varepsilon_m^{\delta/4}$ per kind.

New sources. A letter labelling a path from v' to v ($v' = v$ for the defect letter) gives a source from the component of a word over v' to the child over v of the extended word. By (11.5), the square root of its mass ratio is at most $C\varepsilon_{m-1}^{-\eta/4}(\omega_{v'}/\omega_v)^{1/2}$, and its marker ratio at most $\varepsilon_{m-1}^{-\eta/8}\Psi_v/\Psi_{v'}$. Its birth radius is $\rho(m)$, its width $r_m = \rho(m)\varepsilon_{m-1}^{\Theta}$ by (14.1), and its birth order at most N_{src}, so its price is at most $\varepsilon_{m-1}^{-(N_{\mathrm{src}}-2)\Theta}$. Its entry is thus at most

$$C C_{\mathrm{maj}}\,\varepsilon_{m-1}^{\delta-\eta-\frac{\eta}{4}-\frac{\eta}{8}-(N_{\mathrm{src}}-2)\Theta}$$

times the product along its path. Into or out of a component, the entries of one letter add up to at most twice this, by the bound on path sums.

The recursion. Adding over the at most $N_{\mathrm{word}}+1$ words and over the letters, every row and column sum of the priced majorant matrix of G' is at most

$$C\varepsilon_{m-1}^{\delta-11\eta/8-(N_{\mathrm{src}}-2)\Theta} + 4(N_{\mathrm{word}}+1)(C_{\mathrm{maj}}+1)C_{\mathrm{adm}}\varepsilon_m^{\delta/4}\,, \tag{11.20}$$

with C depending on η, N_{word}, the number of letters and C_{maj}, and

$$\delta - \tfrac{11}{8}\eta - (N_{\mathrm{src}}-2)\Theta \geq \tfrac{45}{64}\delta > \tfrac{\delta}{4}\,,$$

as $\eta, (N_{\mathrm{src}}-2)\Theta \leq \delta/8$ by (2.3) and item (1). In the second term, $\varepsilon_m^{\delta/4} \leq 2\varepsilon_{m-1}^{(q-1)\delta/4}\varepsilon_{m-1}^{\delta/4}$ by Lemma 7.1(a): the old sums gain one step of the scales. So, for $m > m_0$, both terms of (11.20) are at most $\frac{1}{2}C_{\mathrm{adm}}\varepsilon_{m-1}^{\delta/4}$, m_0 being chosen after C_{adm}, C_{maj} and the exponents: this is (b3). For the masses ω_v/Ψ_v^2, the markers appear only through the ratio $\varepsilon_{m-1}^{-\eta/8}$ of a first letter, counted above.

Step 4: analyticity. The construction's functions extend to the outer domain: the flows of the chains and their centre points by Lemma 11.9; the shear streams are entire; the cutoffs, $\mathbf{J}_m^{\kappa_m}$ and Γ_m depend on time only; and the Jacobians, their inverses, the conductivity and the strain are built from the flows. The strain is defined on the complex domain, as $\partial_t\nabla_y X_i = \nabla\mathbf{b}_i(t, X_i)\nabla_y X_i$ is continuous in t and holomorphic in y (Proposition 7.5, Lemma 7.6), and so are the compositions of (11.7), with C^1 time weights $\int\widetilde{\zeta}_{i,k_i}$; near $\bar{x}_i(t)$, the strain of $\mathcal{X}_i$ is the limit of $h^{-1}(\nabla_y\mathcal{X}_i(t+h,\cdot,t) - I)$ as $h\to 0$, so its $|\cdot|_\infty$-norm is at most $2^{25}\varepsilon_i^{\beta-2}$, by Lemma 11.9. As $2^{25}\varepsilon_{m-1}^{\beta-2}\cdot\frac{6}{5}h_m \leq \frac{1}{100}$ by (11.19), on the outer

domain and in the window of k the Jacobian of $\mathcal{X}_{m-1}(t,\cdot,\bar t_m)$ is within $\frac{1}{100}$ of I, by Lemma 11.9(i), and its inverse within $\frac{1}{99}$. The conductivity of the chain is $(\nabla_y\mathcal{X}_{m-1})^{-1}(\nabla_y\mathcal{X}_{m-1})^{-\mathsf{T}}$, as in (7.8), with the transpose and no complex conjugation, so that it is holomorphic; its entry across the pulse, the only one the cell inverse uses, is the sum of the squares of the entries of the row of the inverse along $e^\perp_{(k)}$, within $\frac{1}{99}$ of $e^\perp_{(k)}$ in $|\cdot|_1$, so it is within $\frac{2}{99}+\frac{1}{99^2}\le\frac{1}{10}$ of 1 and has real part at least $\frac{9}{10}$. Let v be a component of G and $\tilde v$ one of G' over v, so that $\mathrm{res}(\tilde v)\ge\mathrm{res}(v)-1/(N_{\mathrm{gen}}+2+N_{\mathrm{off}})$. As sources never go to a smaller generation (Subsection 11.1), the forms of the sources on a path into v are bounded on $\mathcal{D}_v$ by their analytic norms. We split the step $1/(N_{\mathrm{gen}}+2+N_{\mathrm{off}})$ of the reserve into N_{src} equal parts, one for each slow derivative a cycle takes; $\mathcal{D}$ and $\|\cdot\|_{\mathrm{an}}$ are those of (11.9) at the level of the form, with a number r in place of the reserve where written. The extension at (t,z) of a chain of $\mathcal{D}_{\tilde v}$ at the level $m-1$, with $|\operatorname{Im} z\cdot e^\perp_{(k')}|<(3+\frac{1}{64})c_{\mathrm{an}}$, keeps the lower parameters and has a centre of $|\cdot|_1$-norm at most $4\cdot10^{-6}c_{\mathrm{an}}\varepsilon_{m-1}$, below all radii of centres, so it lies in $\mathcal{D}_{\tilde v}$ at the level m, hence in $\mathcal{D}_v$.

A slow derivative costs one part. Let $\mathcal{A}$ have finite analytic norm at the reserve $\mathrm{res}(v)$, and let $\mathrm{res}(\tilde v)\le\mathrm{res}(v)-1/(N_{\mathrm{src}}(N_{\mathrm{gen}}+2+N_{\mathrm{off}}))$. For a chain $\mathcal{C}$ in $\mathcal{D}_{\tilde v}$, the recentred chains $\mathcal{C}^{[y]}$ with $|y|_\infty<c_{\mathrm{an}}\varepsilon_{m-1}/(9N_{\mathrm{src}}(N_{\mathrm{gen}}+2+N_{\mathrm{off}}))$ lie in $\mathcal{D}_v$. Indeed, recentring moves the base label by y if $m'=m$; otherwise, by the group law of Lemma 11.9(ii), it moves the last centre by $w:=\mathcal{X}_{m-2}(\bar t_{m-1},\bar x_{m-1}+y,\bar t_m)-\mathcal{X}_{m-2}(\bar t_{m-1},\bar x_{m-1},\bar t_m)$ and the last phase by w/ε_{m-1}, with $|w|_\infty\le(1+a)|y|_\infty$, $a:=2^{25}\cdot\frac{3}{5}\cdot10^{-6}c_{\mathrm{an}}<2\cdot10^{-3}$, by Lemma 11.9(i) and (11.19) at the level $m-2$, as $|\bar t_m-\bar t_{m-1}|\le\frac{3}{5}h_{m-1}$ by the nesting after (11.19) and the segment from $\bar x_{m-1}$ to $\bar x_{m-1}+y$ lies within $c_{\mathrm{an}}\varepsilon_{m-1}$ of $\bar x_{m-2}(\bar t_m)$ (proof of Lemma 11.9). By (11.9), the radii of $\mathcal{D}_v$ exceed those of $\mathcal{D}_{\tilde v}$ by $c_{\mathrm{an}}\varepsilon_{m-1}$, c_{an} and $\varepsilon_{m-1}/100$ times $1/(4N_{\mathrm{src}}(N_{\mathrm{gen}}+2+N_{\mathrm{off}}))$, and $2(1+a)/9<\frac{1}{4}$. Cauchy's estimate in y_1, mode by mode, gives

$$\varepsilon_{m-1}\big|\partial_{y_1}\widehat{\mathcal{A}}(t,\mathcal{C}^{[y]};\xi)\big|_{y=0}\le\frac{9N_{\mathrm{src}}(N_{\mathrm{gen}}+2+N_{\mathrm{off}})}{c_{\mathrm{an}}}\sup_{\mathcal{D}_v}\big|\widehat{\mathcal{A}}(t,\cdot\,;\xi)\big|,$$

and likewise in y_2, so that $\varepsilon_{m-1}\|\nabla_y\mathcal{A}\|_{\mathrm{an},\tilde v}\le18N_{\mathrm{src}}(N_{\mathrm{gen}}+2+N_{\mathrm{off}})c_{\mathrm{an}}^{-1}\|\mathcal{A}\|_{\mathrm{an},v}$.

A transverse primitive costs nothing. On a transverse form with zero average in z, it multiplies the mode $\xi\ne0$ of $z\cdot e^\perp_{(k)}$ by $(2\pi\mathrm{i}\xi)^{-1}$, of modulus at most $\frac{1}{2\pi}$, and keeps the mode 0 at zero. *The other operations* of (11.10) cost nothing, a constant or one part, by the same two mechanisms. Products cost a constant, the Fourier weights $\exp(3\pi c_{\mathrm{an}}|\xi|)(1+|\xi|)^{N_{\mathrm{an}}}$ of the analytic norm being submultiplicative in ξ; the average in z keeps the mode 0; the cell inverse of Section 10 maps the mode ξ of a flux written $(\kappa_m/\varepsilon_m^2)e^\perp_{(k)}\cdot\nabla_zF$ to that of the gradient of the solution by a time integral of $4\pi^2\xi^2\kappa_m/\varepsilon_m^2$ times exponentials whose exponents have real parts at most $-3\pi^2\xi^2\kappa_m/\varepsilon_m^2$ times the elapsed time, as the real part of the entry across the pulse is at least $\frac{9}{10}\ge\frac{3}{4}$, so with a constant at most $\frac{4}{3}$; on a scalar source it costs the power $\varepsilon_m^2/(3\pi^2\xi^2\kappa_m)$ of Lemma 10.4, which the proof of Proposition 10.7 pairs with the coefficients of (10.16). A derivative in the fast variable of level m multiplies its mode by $2\pi\mathrm{i}\xi$, which the polynomial weight absorbs: a cycle takes at most $2+N_{\mathrm{path}}N_{\mathrm{src}}+2N_{\mathrm{cor}}+2N_{\mathrm{src}}$ of them, the order in z of (10.17) at $N=0$, and $N_{\mathrm{an}}\ge N_{\mathrm{coef}}\ge N_{\mathrm{ret}}$ (items (1) and (3)) exceeds this number by (13.6). A derivative in a centre costs one part. After the averages in the fast variable of level m, the Fourier weights of the analytic norm of G' are restored: its fast variable is the phase of level $m-1$ of the chains of G, in which the forms of G are holomorphic on $|\operatorname{Im} z^{(m-1)}\cdot e^\perp_{(k')}|<(3+\frac{1}{64})c_{\mathrm{an}}$, so their modes in it decay faster than $\exp(-6\pi c_{\mathrm{an}}|\xi|)$. Transports and second-order means use only operations that cost nothing or a constant. A letter takes at most $N_{\mathrm{src}}-2$ slow derivatives and goes to a child, so its forms have analytic norms at $\mathrm{res}(\tilde v)$ bounded by constants times those of forms of G at $\mathrm{res}(v)$. With $P:=N_{\mathrm{src}}(N_{\mathrm{gen}}+2+N_{\mathrm{off}})$ and $\mathcal{D}(r)$ the set $\mathcal{D}$ of the level m with r in place of the reserve, these costs give, for a path $\sigma_1\cdots\sigma_i$ of sources of G

into v, $r = \mathrm{res}(v)$ and $2 \le d \le j \le N_{\mathrm{cor}} + N_{\mathrm{src}}$ with $j - d \le N_{\mathrm{src}} - 2$,

$$\sup_{\mathcal{D}(r-(j-d)/P)} \big|\mathcal{M}_{j,d}[\sigma_1 \cdots \sigma_i]\big| \le C \,\mathrm{fac}(\sigma_1)\,\mathrm{maj}(\sigma_1) \cdots \mathrm{fac}(\sigma_i)\,\mathrm{maj}(\sigma_i)\,\kappa_{m-1}\bar{\varepsilon}_m^{j-2}\varepsilon_{m-1}^{d-j}\,.$$

This is the proof of Proposition 10.7(c), with Lemma 10.4 in its complex form (it uses only $\mathrm{Re}\, e_{(k)}^{\perp} \cdot \mathbf{K} e_{(k)}^{\perp} \ge \frac{3}{4}$), the costs above for derivatives, and the monotone generations along the path; Proposition 10.6(c) passes in the same way to complex conductivities, as $|e^{-a} - e^{-b}| \le |a-b| \max\{|e^{-a}|, |e^{-b}|\}$ for complex a, b. Their hypotheses (10.17) hold on every window by (b1) for G, as in Step 6.

Step 5: (b1). For a source σ' of G' into $\tilde{v}$ we check, kind by kind, that the forms satisfy $(\varepsilon_{m-1}/\rho_{\sigma'})^{d-2}\|\mathcal{A}_{\sigma',d}\|_{\mathrm{an},\tilde{v}} \le \mathrm{maj}(\sigma')$; the constants produced by Step 4 are at most C_{maj}, which is large given item (3). Each form of G' at k' is, at a time t, a sum over the at most two pulses k of level m with $\mathsf{H}_{m,k}(t) > 0$ of the transport of a form at k by the maps of Lemma 11.9, with the Piola factor of (10.23); this costs a constant (Lemma 11.9, Cauchy's estimate on its balls), the coefficient of order d' from the order d carrying $\varepsilon_{m-1}^{d'-d}$, so we bound the forms at k. The birth radius of a source of G of birth order at least 3 is at most ε_m: for one of the seed by Definition 11.2(i), and for one created at a level $i > m$ since $\rho(i) \le C\varepsilon_{i-1}^{1+\gamma/2+\delta} \le \varepsilon_{i-1}$ by (14.7), as $i > m_0$.

Transports. Let σ be a source of G. Its coefficients (11.2) of orders $d \ge 3$, averaged in z, are transported to the chart of k'; the order d gives the orders $d' \le d$, with the $d - d'$ derivatives of the change of variables at the scale ε_{m-1} (Lemma 11.9). Writing the result of order d' as $\kappa_{m-1}\varepsilon_{m-1}^{d'-2}\,\mathrm{fac}(\sigma)\mathcal{A}_{\sigma',d'}$, we obtain, by (b1) for G, $\kappa_m \le \kappa_{m-1}$ (Proposition 2.1(c)) and $\rho_\sigma \le \varepsilon_m$,

$$\Big(\frac{\varepsilon_{m-1}}{\rho_\sigma}\Big)^{d'-2}\|\mathcal{A}_{\sigma',d'}\|_{\mathrm{an},\tilde{v}} \le C\frac{\kappa_m}{\kappa_{m-1}}\sum_{d=\max\{3,d'\}}^{d_\sigma}\Big(\frac{\rho_\sigma}{\varepsilon_{m-1}}\Big)^{d-d'}\Big(\frac{\varepsilon_m}{\rho_\sigma}\Big)^{d-2}\|\mathcal{A}_{\sigma,d}\|_{\mathrm{an},v} \le C\,\mathrm{maj}(\sigma)\,.$$

Second-order means are at most C times the product of the $\mathrm{fac}(\sigma)\,\mathrm{maj}(\sigma)$ along their path (Section 10), in the analytic norm with no loss of reserve, as averages of products of forms and correctors (Step 4).

Mean letters. Let σ' be the source of a mean letter of birth order $d_{\sigma'} \ge 3$ along a path of G into v. By Proposition 10.7(c), the coefficient of order $d \le d_{\sigma'}$ of the mean of degree $d_{\sigma'}$ along the path is $\kappa_{m-1}\bar{\varepsilon}_m^{d_{\sigma'}-2}\varepsilon_{m-1}^{d-d_{\sigma'}}$ times a form bounded by $C\prod_{\sigma\in\mathrm{path}}\mathrm{fac}(\sigma)\,\mathrm{maj}(\sigma)$. Its recursion takes $d_{\sigma'} - d$ slow derivatives and otherwise the operations of Step 4, so the bound holds in the analytic norm at the reserve of $\tilde{v}$. By Definition 11.3(iv), $\mathcal{A}_{\sigma',d}$ is $\varepsilon_{m-1}^{-\eta}$ times this coefficient divided by $\kappa_{m-1}\varepsilon_{m-1}^{d-2}\prod_{\sigma\in\mathrm{path}}\mathrm{fac}(\sigma)$, and $\rho_{\sigma'} = \rho(m) \le \varepsilon_{m-1}$, by (14.7) as $m > m_0$, so

$$\Big(\frac{\varepsilon_{m-1}}{\rho_{\sigma'}}\Big)^{d-2}\|\mathcal{A}_{\sigma',d}\|_{\mathrm{an},\tilde{v}} \le C\varepsilon_{m-1}^{-\eta}\Big(\frac{\bar{\varepsilon}_m}{\rho(m)}\Big)^{d_{\sigma'}-2}\prod_{\sigma\in\mathrm{path}}\mathrm{maj}(\sigma) \le C\varepsilon_{m-1}^{\delta-\eta}\prod_{\sigma\in\mathrm{path}}\mathrm{maj}(\sigma)\,.$$

The last step is the smallness of the correctors: by (10.5), $\bar{\varepsilon}_m \le C\varepsilon_{m-1}^{3\delta}\rho(m)$, so the ratio in parentheses is at most $C\varepsilon_{m-1}^{3\delta}$, and $3\delta(d_{\sigma'} - 2) \ge \delta$. For a path of one source σ, the letter subtracts its cell average, at most $C\varepsilon_{m-1}^{-\eta}(\kappa_m/\kappa_{m-1})\,\mathrm{maj}(\sigma)$ by the transport bound and $\rho_\sigma \le \varepsilon_m \le \rho_{\sigma'}$; and $\kappa_m/\kappa_{m-1} \le C\varepsilon_{m-1}^{2q\gamma} \le C\varepsilon_{m-1}^{\delta}$, by Proposition 2.1(c) ($m \le m_{\mathrm{S}} < M$) and $3\delta < \gamma$.

Covariance letters ($j + j' \ge 1$) are identical, with the mean conjugated: each operator $\Gamma_m : \nabla^2$ adds two orders and a coefficient $|\Gamma_m(t)| \le C\varepsilon_{m-1}^{\delta}\rho(m)^2$ (Lemma 10.3(b)), so the gain $(|\Gamma_m|/\rho_{\sigma'}^2)^{j+j'} \le C\varepsilon_{m-1}^{\delta}$ replaces that of the correctors, and derivatives on a coefficient cost $\rho_{\sigma'}/\varepsilon_{m-1} \le 1$ each.

The defect letter has no path. Its forms are $\varepsilon_{m-1}^{-\eta}$ times the defect of the averaged flux at level m (Section 13), normalized by $\kappa_{m-1}\varepsilon_{m-1}^{d-2}$: the diagonal means of the lift less $\mathbf{J}_m^{\kappa_m}\nabla$, and the commutator

of the covariance with the coarse velocity, conjugated by the truncated exponentials. Each part is small by a separation of scales. With the conductivity I, the second-order diagonal mean is exactly $\mathbf{J}_m^{\kappa_m}$ (Proposition 10.6(c)); in a chain the conductivity differs from I by at most $C\varepsilon_{m-1}^{\beta-2}h_m \le C\varepsilon_{m-1}^{2\delta}$, by Step 4 and (2.13), which changes the mean by at most $C\kappa_{m-1}\varepsilon_{m-1}^{2\delta}$ (Proposition 10.6(c)). The Piola factor $\nabla_y X_{m-1}$ of (10.23), on the left and, through the gradient of $\widetilde{Y}_k$, transposed on the right, changes it by at most $C|\nabla_y X_{m-1} - I|\,|\mathbf{J}_m^{\kappa_m}| \le C\kappa_{m-1}\varepsilon_{m-1}^{2\delta}$, by Lemma 7.10(i) and (10.5). The diagonal means of degrees 3 to $N_{\mathrm{cor}}+1$ gain $C\varepsilon_{m-1}^{3\delta}$, as above. The commutator is at most $|\Gamma_m(t)|$ times the strain of the chain, which is $\nabla\mathbf{b}_{m-1}$ at a real chain with matched phases and has $|\cdot|_\infty$-norm at most $2^{25}\varepsilon_{m-1}^{\beta-2}$ on the outer domain (Step 4), its derivatives costing $C\varepsilon_{m-1}^{-1}$ each (Cauchy's estimate). So the commutator is at most $C\varepsilon_{m-1}^{\beta-2}\kappa_{m-1}\tau_m' \le C\kappa_{m-1}\varepsilon_{m-1}^{2\delta}$, by Lemma 10.3(b) and (2.13), and the truncated exponentials cost at most $(1+C|\Gamma_m|/\rho(m)^2)^{2p} \le C$, derivatives on a coefficient costing at most 1 each. So the coefficient of order d of the defect is at most $C\varepsilon_{m-1}^{\delta}\kappa_{m-1}\rho(m)^{d-2}$ in the analytic norm, and with the birth radius $\rho(m)$ this is (b1) with the majorant $C_{\mathrm{maj}}\varepsilon_{m-1}^{\delta-\eta}$.

Step 6: (b2) and (b2′), in their precise forms (11.14) and (11.15) at the level $m-1$. Let $\mathcal{C}$ be an allowed chain with all $|y^{(i)}|_1 < \frac14 c_{\mathrm{an}}\varepsilon_i$. Every component has reserve at least $2/(N_{\mathrm{gen}}+2+N_{\mathrm{off}})$, so $\mathcal{D}_{\tilde v}$ contains a complex neighbourhood of $\mathcal{C}$ and of its recentrings by $|y|_\infty$ at most a fixed multiple of ε_{m-2}. Cauchy's estimate in y, and the weight $\exp(3\pi c_{\mathrm{an}}|\xi|)$ in z, give, for all j and j', with C depending also on j and j', by (b1) and $\rho_{\sigma'} \le \varepsilon_{m-1}$ when $d \ge 3$,

$$\varepsilon_{m-2}^{j}\big\|\nabla_y^j\nabla_z^{j'}\mathcal{A}_{\sigma',d}\big\|_{L^\infty_{m-1}([t_{\mathrm{A}}+3\tau_m,t_{\mathrm{f}}])} \le C\|\mathcal{A}_{\sigma',d}\|_{\mathrm{an},\tilde v} \le C\Big(\frac{\rho_{\sigma'}}{\varepsilon_{m-1}}\Big)^{d-2}\mathrm{maj}(\sigma') \le C\,\mathrm{maj}(\sigma')\,.$$

Summing with the weights of (11.4) gives (b2), as $C_{\mathrm{der}} \ge C$. In (11.15) at level $m-1$, slow and fast derivatives cost $(\rho_{\mathrm{mol}}(m-1)/\varepsilon_{m-2})^{j_1} \le 1$ and $(\rho_{\mathrm{mol}}(m-1)/\varepsilon_{m-1})^{j_2} \le 1$, and the power $(\varepsilon_{m-1}/\rho_{\mathrm{mol}}(m-1))^{d-2}$ is absorbed by (b1) if every source of G' of current order at least 3 has birth radius at most $\rho_{\mathrm{mol}}(m-1)$. A transported source keeps its birth radius $\le \varepsilon_m$; a new one has $\rho(m)$. By Lemma 7.1(a), (c), Proposition 2.1(b) ($m-1<M$) and (14.7),

$$\varepsilon_m \le 2\varepsilon_{m-1}^{q}\,,\qquad \rho(m) \le C\varepsilon_{m-1}^{1+\gamma/2+\delta}\,,\qquad \rho_{\mathrm{mol}}(m-1) \ge c\varepsilon_{m-1}^{1+\gamma/2}\,,$$

and $q-1-\frac{\gamma}{2} = 4\delta+q\gamma$ (Lemma 7.1(c)), so, for $m > m_0$, both birth radii are at most $\rho_{\mathrm{mol}}(m-1)$. Hence (b2′); for the masses ω_v/Ψ_v^2, both sides gain the factor $\Psi_v/\Psi_{v'}$.

Step 7: (c). A transported source keeps its smoothing, and a second-order mean takes that of its widest source, so (c1) and (c3) are inherited, and (c2) follows from (c2) for G, since $\rho_{\mathrm{mol}}(m)\varepsilon_m^{\delta} \le \rho_{\mathrm{mol}}(m-1)\varepsilon_{m-1}^{\delta}$ and $\rho(m)\varepsilon_{m-1}^{4\delta} \le \rho(m-1)\varepsilon_{m-2}^{4\delta}$ ($\varepsilon_m \le \varepsilon_{m-1}$, $\kappa_m \le \kappa_{m-1} \le \kappa_{m-2}$, $h_m \le h_{m-1}$).

The new sources have the fresh smoothing (14.9), of width r_m, with the ellipses of (11.18). Then Ω_1, Ω_2 are piecewise continuous in t, and $|\Omega_i(t)|_\infty \le C \le C_\Omega$, by Lemma 7.4 ($|t-l_{\mathrm{hor}}h_m|, |t-l_{\mathrm{ver}}h_m| \le \frac92 h_m$), Lemma 7.10(i) and Proposition 2.1(a), (c). Both are invertible, as $\det\nabla_y X_{m-1} = 1$ (Lemma 7.4) and $\mathbf{J}_m^{\kappa_m} \ge \kappa_m I$ (Lemma 7.10(i)). The long axes $\Omega_1(t)e_1$, $\Omega_2(t)e_2$ are the columns e_1, e_2 of Jacobians within $C\varepsilon_{m-1}^{2\delta}$ of I (proof of Lemma 7.4, (2.13)), times square roots of maxima of the diagonal entries of $\kappa_{m-1}^{-1}\mathbf{J}_m^{\kappa_m}$, at least 1 as these entries have average 1 over a period by (2.17) and (2.19). So the matrix of long axes is within $C\varepsilon_{m-1}^{2\delta}$ of I times a diagonal matrix with entries ≥ 1, and, for $m > m_0$, its inverse has $|\cdot|_\infty$-norm at most $(1-C\varepsilon_{m-1}^{2\delta})^{-1} \le 2 \le C_\Omega$.

This gives (c1), with $2\ell-1 \ge N_{\mathrm{mom}}$ by item (1). Lemma 8.2 gives the bound of (c3) with its constants, which the c_j of item (3) dominate; the bound passes to $H_\rho^N(\mathbb{T}^2)$ since $S(t)$ commutes with derivatives. The widths follow from (14.1), (14.7), $\varepsilon_{m-1} \le 2\varepsilon_{m-2}^q$ and $4\delta < q-1$ (Step 6):

$$r_m \le C\varepsilon_{m-1}^{1+\frac{\gamma}{2}+\delta+\Theta} \le C\rho_{\mathrm{mol}}(m-1)\,\varepsilon_{m-1}^{\delta+\Theta} \le C_{\mathrm{wid}}\rho_{\mathrm{mol}}(m-1)\,\varepsilon_{m-1}^{\delta}\,,$$
$$r_m \le C\rho(m-1)\,\varepsilon_{m-2}^{(q-1)(1+\frac{\gamma}{2}+\delta)+q\Theta} \le C_{\mathrm{wid}}\rho(m-1)\,\varepsilon_{m-2}^{4\delta}\,.$$

Step 8: (d). The coupling matrix (8.4) of G' of order N at scale $\rho_{\mathrm{mol}}(m-1)$ is at most C times the priced majorant matrix, entry by entry, for both masses. Indeed, by (b1), the coefficient (11.2) of order d of a source σ' is at most $\kappa_{m-1}\rho_{\sigma'}^{d-2}\operatorname{fac}(\sigma')\operatorname{maj}(\sigma')$. Its transport to x gives the orders $d' \le d$ at the cost $\varepsilon_{m-2}^{d'-d} \le \rho_{\sigma'}^{d'-d}$; if $d' < d$, then $d \ge 3$, so σ' has current order at least 3 and $\rho_{\sigma'} \le \varepsilon_{m-2}$, by Step 6. Its derivatives at the scale $\rho_{\mathrm{mol}}(m-1) \le \varepsilon_{m-1}$ cost constants, as in Step 6: by the chain rule, ∇_x^j of the transported coefficient, at $z = y/\varepsilon_{m-1}$, combines $\nabla_y^{j_1}\nabla_z^{j_2}$ of the form, $j_1 + j_2 \le j$, at the cost $\varepsilon_{m-2}^{-j_1}\varepsilon_{m-1}^{-j_2}$, as the derivatives of order $j+1$ of $\mathcal{X}_{m-2}(t,\cdot,\bar t_{m-1})$ and of its inverse are at most $C\varepsilon_{m-2}^{-j}$ (Lemma 11.9(i), Cauchy's estimate), and at most two time cutoffs are positive at once. So

$$\frac{c_{d'-2}}{\kappa_{m-1}r_{\sigma'}^{d'-2}}\sum_{j\le N}\frac{\rho_{\mathrm{mol}}(m-1)^j}{j!}\sup_{[t_{\mathrm{A}}+3\tau_m,t_{\mathrm{f}}]\times\mathbb{T}^2}|\nabla^j A_{\sigma',d'}| \le C\operatorname{fac}(\sigma')\operatorname{maj}(\sigma')\Big(\frac{\rho_{\sigma'}}{r_{\sigma'}}\Big)^{d'-2},$$

and the last power is at most the price $\max\{1,(\rho_{\sigma'}/r_{\sigma'})^{d_{\sigma'}-2}\}$, which compensates for the coefficient $c_{d'-2}r_{\sigma'}^{2-d'}$ of the kernel. So the row and column sums of the coupling matrix are at most C times (11.20). As in Step 3, with one more step of the scales ($\varepsilon_{m-1}^{\delta/4} \le 2\varepsilon_{m-2}^{q\delta/4}$), for $m > m_0$, m_0 being chosen after C, C_{adm} and C_{maj}, both terms are at most $\frac12 C_{\mathrm{adm}}\varepsilon_{m-2}^{q\delta/4}$: this is (d). □

12. Smoothing within one period: Bloch wave analysis

This section proves part (a) of Proposition 5.3, the *preparation*, by Bloch wave analysis; part (b) is proved in Section 13, for the time t_{pr} and the map Π constructed here. Part (a) says that within one period $4h_m$ of the pulses the solution of an admissible family of level m becomes, up to a small error, smooth at the scale $\rho(m)$, with bounds by the L^2 norm of the datum. This scale is the distance over which the coarse diffusivity κ_{m-1} spreads in one renewal time; below the terminal level it is far larger than the distance that molecular diffusion reaches in that time. So the smoothing must come from the homogenization of the pulses of level m, and it must hold to the accuracy ε_{m-1}^{Q} of the cycle, uniformly in the contrast, and without any derivative of the datum.

We obtain it from an exact decomposition of the evolution. In the chart of a renewal block, with the conductivity frozen at a point, Bloch waves split the evolution into evolutions on the period cell, one for each frequency (Subsection 12.2). At small frequencies, the cell mean of the principal branch of these evolutions evolves at a scalar rate, the symbol (Subsection 12.3). To leading order the symbol is the homogenized operator of the pulses; as a block carries pulses of both orientations, over one block the principal branch is damped by a Gaussian in the frequency at the scale $1/\rho(m)$, which is the smoothing we need (Subsection 12.4). Everything else decays at the *cell rate*, the inverse of the time that molecular diffusion takes to cross a period cell, and over a block its norm is multiplied by a number smaller than any power of ε_{m-1} (Subsection 12.5). Subsections 12.6–12.9 add the sources of the family, which take derivatives of the solution. We expand the solution in its sources by Duhamel's formula, freeze the coefficients on patches of size $\rho(m)$, and estimate each term by duality, against solutions of the adjoint cell evolutions, which carry the derivatives that the sources take, except near the ends of a block (Lemma 12.12(iii)). On the principal branch a source then costs, over a block, at most a power of $1+\rho(m)|\xi|$, which the Gaussian absorbs; everything else is damped. Subsection 12.10 assembles the proof.

Conventions. In Subsections 12.2–12.5 every constant of a statement is named in it, and C is absolute in the proofs of Subsections 12.2 and 12.3; Subsection 12.6 states the conventions of Subsections 12.6–12.9. Elsewhere, C and c denote positive constants depending only on α, s, q, ζ, Λ, n, t_0, T and on the orders and constants fixed in Subsection 11.1 and below, which may change

from line to line; $C(j)$ depends also on j, and $C(J)$ and $c(J)$ on J. The corrector amplitude $\bar\varepsilon_m = \varepsilon_m(\kappa_{m-1}/\kappa_m)^{1/2}$ of (5.2), with the *contrast* κ_{m-1}/κ_m, is at most $C\varepsilon_{m-1}^{3\delta}\rho(m)$, by (10.5). The cell rate is κ_m/ε_m^2. Two identities, from the definition of $\bar\varepsilon_m$ and (5.5), translate between cell units and the coarse scale:

$$\frac{\kappa_m}{\varepsilon_m^2}\,\bar\varepsilon_m^2 = \kappa_{m-1}\,, \qquad \frac{\rho(m)^2}{\bar\varepsilon_m^2} = \frac{\kappa_m h_m}{\varepsilon_m^2}\ :$$

in cell units the homogenized equation damps the frequency ξ at the rate $(\bar\varepsilon_m|\xi|)^2$, and a renewal block lasts $(\rho(m)/\bar\varepsilon_m)^2$ cell diffusion times.

The constant c_{mol}. The constant $c_{\mathrm{mol}} \in (0,1]$ of the molecular radius (11.1) is fixed here, after Λ: it is at most a threshold depending only on $\alpha, s, q, \zeta, \Lambda$, below which, for every M, $\kappa \in \mathcal{K}_M$ and $2 \le m \le M$,

$$\begin{gathered}\rho_{\mathrm{mol}}(m)^{j+1}\big\|\nabla^j \mathbf{b}_m\big\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le 2^{-j-2}\, j!\,\kappa_m \quad (j \ge 1)\,,\\ \frac{\rho_{\mathrm{mol}}(m)}{\varepsilon_m} \le 2^{-11}\,, \qquad \varepsilon_m^{\beta-2}\,\frac{\rho_{\mathrm{mol}}(m)^2}{\kappa_m} \le 2^{-20}\,.\end{gathered} \tag{12.1}$$

The first condition is (i) of Proposition 8.3 for the velocity $\mathbf{b}_m$ at the scale $\rho_{\mathrm{mol}}(m)$ and every order. It holds for small c_{mol}: Proposition 7.5 and Cauchy's estimate on polydiscs of radius $\varepsilon_m/100$ give $\|\nabla^j \mathbf{b}_m\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le C^j(j-1)!\,\varepsilon_m^{\beta-1-j}$ with an absolute C, and (11.1) bounds the left side by $c_{\mathrm{mol}}^{j+1}C^j(j-1)!\,\kappa_m$. The last condition makes the molecular time $\rho_{\mathrm{mol}}(m)^2/\kappa_m$ short compared with the strain time $\varepsilon_m^{2-\beta}$ of $\mathbf{b}_m$ (7.4).

The order of choices. The *depth* N_{dep} of the Duhamel expansion (Subsection 12.6) is the least integer with the second condition below, N_{buf} is the order of the molecular smoothing, and the moment order N_{mom} of the smoothings satisfies the first condition (item (1) of Subsection 11.1, Table 11.1):

$$q\delta N_{\mathrm{mom}} > Q+3\,, \qquad \frac{q\delta}{4}(N_{\mathrm{dep}}+1) > Q+3\,, \qquad N_{\mathrm{buf}} \coloneqq (N_{\mathrm{dep}}+1)(N_{\mathrm{src}}-2) + N_{\mathrm{mom}}\,. \tag{12.2}$$

The integers introduced in Subsections 12.6–12.9 depend only on these orders, on N_{src}, Q and N_{out}, and N_{coef} is at least each of them and N_{buf}. The top level m_0 is fixed after c_{mol}.

12.1 Statements

The preparation runs over the interval of Figure 12.1. From t_{c} the solution runs for the molecular time, which gives it derivatives at the scale $\rho_{\mathrm{mol}}(m)$ (Lemma 12.17), and from the end t_1 of the molecular interval it is expanded in its sources. The rest of the block of t_1 may carry pulses of one orientation only; the next, complete, block smooths, and t_{pr} is its end, inside a quiet margin. Proposition 12.1 gives four further properties of t_{pr} and Π. By (i) the comparison of Section 13 starts at t_{pr}, where the lift adds no corrector (Lemma 13.1); (iii) gives the derivatives that the passage of Section 14 spends; and (iv) bounds the datum of the successor, the retained trace of Πg.

Proposition 12.1 (Further properties of the preparation)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ such that, for every M, κ, m, t_{A}, t_{c}, t_{f}, G and G' as in Proposition 5.3, the time t_{pr} and the map Π of Proposition 5.3 can be chosen, satisfying* (a) *and* (b) *with this C, such that also the following hold.*

(i) *The time t_{pr} comes after the molecular time, and the velocity added at level m vanishes near it:*

$$t_{\mathrm{c}} + \frac{\rho_{\mathrm{mol}}(m)^2}{\kappa_m} \le t_{\mathrm{pr}}\,, \qquad \mathbf{b}_m = \mathbf{b}_{m-1} \quad \text{on } [t_{\mathrm{pr}} - \tau_m', t_{\mathrm{pr}} + \tau_m'] \times \mathbb{T}^2\,.$$

(ii) Π *maps every field of* G *whose components are constant functions to itself.*

(iii) *For every datum* g*, the solution* U *of* G *from* g *at time* t_{c} *satisfies*

$$\|U\|_{L^\infty([t_{\mathrm{pr}},t_{\mathrm{f}}];H^{2N_{\mathrm{rep}}+4}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2))} \le C\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

(iv) *For every datum* g*,*

$$\|\exp_p(-\Gamma_m(t_{\mathrm{pr}}):\nabla^2)\,\Pi g\|_{H^{N_{\mathrm{ret}}}_{\rho(m),G}(\mathbb{T}^2)} \le C\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

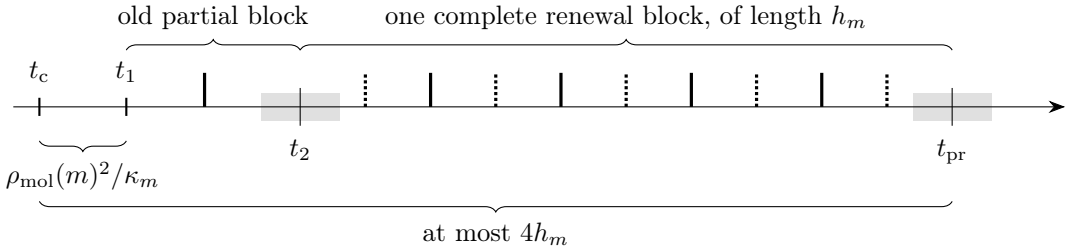


Figure 12.1: The preparation of a cycle, in time (not to scale): the molecular interval from t_{c} to t_1, the rest of the block containing t_1 (the old partial block), and one complete renewal block, from the block boundary t_2 to t_{pr}. Vertical strokes: pulses, solid and dotted for the two orientations. Shaded: the quiet margins around t_2 and t_{pr}, of half-width τ'_m.

12.2 The frozen operator and its Bloch waves

In the chart of a renewal block (Lemma 7.9) the coarse velocity is gone: what remains is the periodic shear of the pulses and a conductivity close to the identity that varies only on the scale ε_{m-1}, much larger than $\rho(m)$. As a point of the datum influences, over one block, a neighbourhood of size about $\rho(m)$, we freeze the conductivity at a point y_0; Subsection 12.9 bounds the error. The frozen operator has periodic coefficients, and Bloch waves reduce it to one evolution on the period cell for each frequency, the fibre equation (12.5).

Standing hypotheses. Throughout Subsections 12.2–12.5, let $\Lambda \ge \Lambda_0$, where $\Lambda_0(\alpha,s,q,\zeta) \ge 128$ is at least the thresholds of Proposition 2.1 and Lemma 10.1 and is so large that

$$\frac{4\pi^2\kappa_m\tau_m}{\varepsilon_m^2} \ge 6\,, \qquad \varepsilon_{m-1}^{\delta} \le \frac{1}{16} \qquad \text{for every } M\ge 2,\ \kappa\in\mathcal{K}_M,\ 2\le m\le M \tag{12.3}$$

(by Proposition 2.1(c)). Fix $M\ge 2$, $\kappa\in\mathcal{K}_M$, $2\le m\le M$, $l\in\mathbb{Z}$, a point $y_0\in\mathbb{R}^2$ and a measurable map $\mathbf{K}$ from $\mathbb{R}$ to the symmetric 2×2 matrices with $\|\mathbf{K}-I\|_{L^\infty(\mathbb{R})}\le\varepsilon_{m-1}^{\delta}$. In the applications $\mathbf{K}(t)=\mathbf{K}_{m,l}(t,y_0)$ for $|t-lh_m|\le h_m$, the conductivity of the chart of the block of l frozen at the label y_0, and $\mathbf{K}(t)=I$ otherwise; by (7.9), $\|\mathbf{K}-I\|_{L^\infty(\mathbb{R})}\le C\varepsilon_{m-1}^{2\delta}\le\varepsilon_{m-1}^{\delta}$ for Λ_0 large. By Lemma 7.9, the level-m operator in the chart of the block is $\partial_t+\nabla^\perp\psi_m\cdot\nabla-\kappa_m\nabla\cdot(\mathbf{K}_{m,l}\nabla\cdot)$, with ψ_m of (7.11). With its conductivity frozen at y_0, it becomes the *frozen operator*

$$\partial_t+\nabla^\perp\psi_m(t,y)\cdot\nabla-\kappa_m\nabla\cdot\big(\mathbf{K}(t)\nabla\cdot\big) \qquad \text{on } \mathbb{R}\times\mathbb{R}^2\,, \tag{12.4}$$

whose coefficients are ε_m-periodic in y (Section 2).

Fibres. We write $z \coloneqq y/\varepsilon_m$ for the fast variable, as in Section 10. The velocity $(\nabla^\perp\psi_m)(t,\varepsilon_m z)$ is 1-periodic in z, divergence free and of mean zero over the cell. On $\mathbb{C}^2$ the dot is bilinear, $\xi\cdot\eta \coloneqq \xi_1\eta_1+\xi_2\eta_2$, so that $|\xi|^2=\xi\cdot\bar\xi$. For $\xi\in\mathbb{C}^2$, the function $\exp(\mathrm{i}\xi\cdot(y-y_0))U(t,y/\varepsilon_m)$, with U periodic in z, is annihilated by (12.4) if and only if U solves the *fibre equation at* ξ,

$$\partial_t U = \frac{\kappa_m}{\varepsilon_m^2}(\nabla+\mathrm{i}\varepsilon_m\xi)\cdot\mathbf{K}(t)(\nabla+\mathrm{i}\varepsilon_m\xi)U - \frac{1}{\varepsilon_m}(\nabla^\perp\psi_m)(t,\varepsilon_m z)\cdot(\nabla+\mathrm{i}\varepsilon_m\xi)U \tag{12.5}$$

on $\mathbb{T}^2$, with ∇ in z and $\nabla^\perp\psi_m$ the velocity of (12.4), in y. Solutions are energy solutions, one for each datum. The real frequencies with $\varepsilon_m|\xi|_\infty\le\pi$ form the (Brillouin) *zone*. For real ξ the transport term of (12.5) is skew-adjoint, as the velocity is divergence free, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2_{L^2(\mathbb{T}^2)} = -\frac{2\kappa_m}{\varepsilon_m^2}\big\langle\mathbf{K}(t)(\nabla+\mathrm{i}\varepsilon_m\xi)U,(\nabla+\mathrm{i}\varepsilon_m\xi)U\big\rangle_{L^2(\mathbb{T}^2)}\le 0\,, \tag{12.6}$$

and, by Fourier series, for real ξ in the zone and every $U\in H^1(\mathbb{T}^2)$,

$$\|(\nabla+\mathrm{i}\varepsilon_m\xi)U\|_{L^2(\mathbb{T}^2)} \ge \max\big\{\varepsilon_m|\xi|\,\|U\|_{L^2(\mathbb{T}^2)},\ \pi\|U-(U)_{\mathbb{T}^2}\|_{L^2(\mathbb{T}^2)}\big\}\,, \tag{12.7}$$

since $|2\pi j+\varepsilon_m\xi|\ge\varepsilon_m|\xi|$ for every $j\in\mathbb{Z}^2$, and $\ge\pi$ for $j\neq 0$.

Bloch sums. Fields $g=(g_v)_{v\in V}$ on $\mathbb{R}^2$ carry the masses ω_v of (5.7), and their moments about y_0 at the scale $\rho(m)$ are measured by

$$\begin{aligned}\|g\|_{L^2_{\omega,N'}(\mathbb{R}^2)} &\coloneqq \Big\|\Big(1+\frac{|y-y_0|^2}{\rho(m)^2}\Big)^{N'/2}g\Big\|_{L^2_\omega(\mathbb{R}^2)}\,,\\ \|g\|^2_{H^N_{\rho,\omega,N'}(\mathbb{R}^2)} &\coloneqq \sum_{j\le N}\frac{\rho^{2j}}{(j!)^2}\|\nabla^j g\|^2_{L^2_{\omega,N'}(\mathbb{R}^2)}\,;\end{aligned} \tag{12.8}$$

the centre y_0 is always the point of the frozen operator. For a measurable field g on $\mathbb{R}^2$ with $\|g\|_{L^2_{\omega,2}(\mathbb{R}^2)}<\infty$, its *Bloch sums* about y_0 are the sums

$$\mathcal{B}g(\xi,z) \coloneqq \sum_{j\in\mathbb{Z}^2}\exp(-\mathrm{i}\xi\cdot(\varepsilon_m(z+j)-y_0))g\big(\varepsilon_m(z+j)\big)\,, \tag{12.9}$$

periodic in z. At every real ξ they converge absolutely for almost every z, and in $L^2(\mathbb{T}^2)$, by the Cauchy–Schwarz inequality against the weight $(1+|y-y_0|^2/\rho(m)^2)^{-2}$, which is summable over the points $y=\varepsilon_m(z+j)$, $j\in\mathbb{Z}^2$. Their mean over $\mathbb{T}^2$ is $\varepsilon_m^{-2}\widehat g(\xi)$, where $\widehat g(\xi)\coloneqq\int_{\mathbb{R}^2}\exp(-\mathrm{i}\xi\cdot(y-y_0))g(y)\,\mathrm{d}y$ is the Fourier transform about y_0. They satisfy $\mathcal{B}(\nabla g)=\varepsilon_m^{-1}(\nabla+\mathrm{i}\varepsilon_m\xi)\mathcal{B}g$, with ∇ in y on the left and in z on the right, and $\mathcal{B}(a(\cdot/\varepsilon_m)g)=a\mathcal{B}g$ for every 1-periodic function a. The following lemma, Plancherel's identity for Bloch sums, returns bounds proved at each frequency to the plane.

Lemma 12.2 (Bloch sums)**.** *Let g be a measurable field on $\mathbb{R}^2$ with $\|g\|_{L^2_{\omega,2}(\mathbb{R}^2)}<\infty$. Then the Bloch sums of g satisfy*

$$\int_{zone}\|\mathcal{B}g(\xi,\cdot)\|^2_{L^2_\omega(\mathbb{T}^2)}\,\mathrm{d}\xi = (2\pi)^2\varepsilon_m^{-4}\|g\|^2_{L^2_\omega(\mathbb{R}^2)}\,.$$

Proof. For almost every z, $\mathcal{B}g(\xi,z)$ is a complex number of modulus one times the trigonometric series $\sum_{j\in\mathbb{Z}^2}\exp(-\mathrm{i}\varepsilon_m\xi\cdot j)g(\varepsilon_m(z+j))$ of the cell frequency $\varepsilon_m\xi\in[-\pi,\pi]^2$, and Parseval's identity gives $\int_{\mathrm{zone}}|\mathcal{B}g(\xi,z)|^2\,\mathrm{d}\xi=(2\pi)^2\varepsilon_m^{-2}\sum_j|g(\varepsilon_m(z+j))|^2$; integrating in z gives the identity. □

We write $\mathcal{L}_0$ for the cell operator (10.10) with the conductivity $\mathbf{K}(t)$ of the standing hypotheses, at every $t \in \mathbb{R}$. The next subsection constructs the principal branch as a fixed point of the cell problem forced at all times, which Lemma 12.3 solves with a constant independent of the velocity: the transport is skew and drops out of the energy, and the diffusion damps functions of mean zero at the cell rate. For a Banach space X, forcings and solutions with values in X are measured over intervals of one cell diffusion time ε_m^2/κ_m,

$$\|F\|_{L^2_{\mathrm{ul}}(\mathbb{R};X)} := \sup_{t\in\mathbb{R}} \Big(\frac{\kappa_m}{\varepsilon_m^2} \int_t^{t+\varepsilon_m^2/\kappa_m} \|F(t')\|_X^2 \,\mathrm{d}t' \Big)^{1/2},$$

and the space $L^\infty(\mathbb{R};L^2(\mathbb{T}^2)) \cap L^2_{\mathrm{ul}}(\mathbb{R};H^1(\mathbb{T}^2))$ carries the norm $\|w\|_{L^\infty(\mathbb{R};L^2(\mathbb{T}^2))} + \|\nabla w\|_{L^2_{\mathrm{ul}}(\mathbb{R};L^2(\mathbb{T}^2))}$.

Lemma 12.3 (Bounded solutions at every time)**.** *Let F^0 and F be measurable on $\mathbb{R}\times\mathbb{T}^2$, with values in $\mathbb{C}$ and $\mathbb{C}^2$, with finite $L^2_{\mathrm{ul}}(\mathbb{R};L^2(\mathbb{T}^2))$ norms. Then the problem*

$$\mathcal{L}_0 U = \frac{\kappa_m}{\varepsilon_m^2}\big(F^0 - (F^0)_{\mathbb{T}^2} + \nabla\cdot F\big) \qquad \text{in } \mathbb{R}\times\mathbb{T}^2$$

has exactly one energy solution $U \in L^\infty(\mathbb{R};L^2(\mathbb{T}^2))$ with $(U(t))_{\mathbb{T}^2} = 0$ for every t, and

$$\|U\|_{L^\infty(\mathbb{R};L^2(\mathbb{T}^2))} + \|\nabla U\|_{L^2_{\mathrm{ul}}(\mathbb{R};L^2(\mathbb{T}^2))} \le 100\big(\|F^0\|_{L^2_{\mathrm{ul}}(\mathbb{R};L^2(\mathbb{T}^2))} + \|F\|_{L^2_{\mathrm{ul}}(\mathbb{R};L^2(\mathbb{T}^2))}\big).$$

Proof. We measure time in units of ε_m^2/κ_m, so that the coefficients κ_m/ε_m^2 disappear. Testing with $\bar U$ and using $\mathbf{K} \ge \frac12 I$ and Poincaré's inequality $\|\nabla U\|_{L^2(\mathbb{T}^2)} \ge 2\pi\|U\|_{L^2(\mathbb{T}^2)}$ for mean-zero U, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2_{L^2(\mathbb{T}^2)} + \pi^2\|U\|^2_{L^2(\mathbb{T}^2)} + \tfrac14\|\nabla U\|^2_{L^2(\mathbb{T}^2)} \le 4\big(\|F^0\|^2_{L^2(\mathbb{T}^2)} + \|F\|^2_{L^2(\mathbb{T}^2)}\big).$$

Summing over intervals of unit length, we find that every energy solution on $[t',\infty)$ satisfies

$$\|U(t)\|_{L^2(\mathbb{T}^2)} \le e^{-\pi^2(t-t')/2}\|U(t')\|_{L^2(\mathbb{T}^2)} + 2.01\big(\|F^0\|^2_{L^2_{\mathrm{ul}}(\mathbb{R};L^2(\mathbb{T}^2))} + \|F\|^2_{L^2_{\mathrm{ul}}(\mathbb{R};L^2(\mathbb{T}^2))}\big)^{1/2},$$

and the first display, integrated over an interval of unit length, then bounds ∇U. The solutions from zero at the times $t' = -1, -2, \dots$ converge to one with the bound, and two bounded solutions coincide (let $t' \to -\infty$). □

12.3 The principal branch at small frequencies

At $\xi = 0$ the constants solve the fibre equation (12.5), and every other solution approaches its cell mean at the cell rate. We show that this splitting persists at small frequencies. As the coefficients depend on time, there is no eigenvalue to continue. We seek instead a cell function $\mathcal{R}_\xi(t)$ of mean one and a scalar rate $\mathcal{Z}_\xi(t)$ such that $\mathcal{R}_\xi$, multiplied by a scalar function of time that grows at the rate $\mathcal{Z}_\xi$, solves (12.5); in the plane this is a plane wave corrected at the cell scale. Averaging the fibre equation over $\mathbb{T}^2$, with the velocity of mean zero, gives the *symbol*

$$\mathcal{Z}_\xi = -\kappa_m\,\xi\cdot\mathbf{K}\xi - \mathrm{i}\,\xi\cdot\big((\nabla^\perp\psi_m)(t,\varepsilon_m\cdot)\,(\mathcal{R}_\xi - 1)\big)_{\mathbb{T}^2}\,, \tag{12.10}$$

and its part of mean zero is the *graph equation*

$$\begin{aligned}\mathcal{L}_0(\mathcal{R}_\xi - 1) = {}& \frac{2\mathrm{i}\kappa_m}{\varepsilon_m}\nabla\cdot\big(\mathbf{K}\xi\,(\mathcal{R}_\xi - 1)\big)\\ & - \mathrm{i}\big(\xi\cdot(\nabla^\perp\psi_m)(t,\varepsilon_m\cdot)\,\mathcal{R}_\xi - (\xi\cdot(\nabla^\perp\psi_m)(t,\varepsilon_m\cdot)\,(\mathcal{R}_\xi - 1))_{\mathbb{T}^2}\big)\\ & + \mathrm{i}(\mathcal{R}_\xi - 1)\big(\xi\cdot(\nabla^\perp\psi_m)(t,\varepsilon_m\cdot)\,(\mathcal{R}_\xi - 1)\big)_{\mathbb{T}^2}\,.\end{aligned} \tag{12.11}$$

This is the cell problem of Lemma 12.3 for $\mathcal{R}_\xi - 1$, with a forcing that is small in cell units when $\bar{\varepsilon}_m|\xi|$ is small: whatever the Péclet number (the velocity in cell units), the transport term on a mode of frequency ξ has relative size

$$\frac{\varepsilon_m^2}{\kappa_m}\,\|\nabla^\perp\psi_m\|_{L^\infty(\mathbb{R}\times\mathbb{R}^2)}\,|\xi| \le \frac{2\pi\varepsilon_m^{\beta+1}|\xi|}{\kappa_m} \le C\bar{\varepsilon}_m|\xi| \tag{12.12}$$

with an absolute C (Step 1 of the proof of Proposition 12.4). So the graph equation has exactly one small solution of mean zero, and $\mathcal{R}_\xi$ is the *principal branch*, the branch that continues the constants. By (a) below it differs from 1 by the relative size of the first corrector of a plane wave, and by (b) the symbol is at most of the size of the rate of the homogenized equation at the frequency ξ.

Proposition 12.4 (The principal branch)**.** *There exist absolute constants $c_{\mathrm{br}} \in (0,1]$ and C_{br} such that, under the standing hypotheses, there exist $\mathcal{R}_\xi(t) \in L^2(\mathbb{T}^2)$ and $\mathcal{Z}_\xi(t) \in \mathbb{C}$, for $t \in \mathbb{R}$ and $\xi \in \mathbb{C}^2$ with $\bar{\varepsilon}_m|\xi| < c_{\mathrm{br}}$, power series in ξ, such that for every such ξ the following hold.*

(a) *For every $t \in \mathbb{R}$, $(\mathcal{R}_\xi(t))_{\mathbb{T}^2} = 1$, and*

$$\|\mathcal{R}_\xi - 1\|_{L^\infty(\mathbb{R};L^2(\mathbb{T}^2))} \le C_{\mathrm{br}}\bar{\varepsilon}_m|\xi|\,.$$

(b) *$\mathcal{Z}_\xi = \sum_{j\ge2}\mathcal{Z}_{\xi,j}$, where $\mathcal{Z}_{\xi,j}$ is homogeneous of degree j in ξ, and for every $j \ge 2$*

$$\|\mathcal{Z}_{\xi,j}\|_{L^\infty(\mathbb{R})} \le C_{\mathrm{br}}\frac{\kappa_m}{\varepsilon_m^2}\Big(\frac{2\bar{\varepsilon}_m|\xi|}{c_{\mathrm{br}}}\Big)^j\,.$$

(c) *For every $t' \in \mathbb{R}$, the function $t \mapsto \mathcal{R}_\xi(t)\exp(\int_{t'}^t \mathcal{Z}_\xi(t'')\,\mathrm{d}t'')$ solves the fibre equation (12.5) on $[t',\infty)$.*

(d) *For every $\xi \in \mathbb{R}^2$ with $\bar{\varepsilon}_m|\xi| < c_{\mathrm{br}}$ and every $t' \le t$,*

$$\Big|\exp\Big(\int_{t'}^t \mathcal{Z}_\xi(t'')\,\mathrm{d}t''\Big)\Big| \le C_{\mathrm{br}}\,.$$

(e) *$\mathcal{R}_\xi - 1$ is the only solution of the graph equation (12.11) that has mean zero at every time and norm at most one in $L^\infty(\mathbb{R};L^2(\mathbb{T}^2)) \cap L^2_{\mathrm{ul}}(\mathbb{R};H^1(\mathbb{T}^2))$, and $\mathcal{Z}_\xi$ is given by (12.10).*

Proof. Step 1: The branch and the symbol. As $\|\nabla^\perp\psi_m\|_{L^\infty(\mathbb{R}\times\mathbb{R}^2)} \le 2\pi\varepsilon_m^{\beta-1}$ and $\varepsilon_m^{2\beta}/\kappa_m \le 20\kappa_{m-1}$ (Lemma 10.1(c)), the constant of (12.12) is absolute. The right side of the graph equation (12.11) is the cell rate times $F^0 - (F^0)_{\mathbb{T}^2} + \nabla\cdot F$, where $F = 2\mathrm{i}\varepsilon_m\mathbf{K}\xi\,(\mathcal{R}_\xi - 1)$ and F^0 is ε_m^2/κ_m times the transport terms, with $\|F\|_{L^2(\mathbb{T}^2)} \le 4\bar{\varepsilon}_m|\xi|\,\|\mathcal{R}_\xi - 1\|_{L^2(\mathbb{T}^2)}$ and $\|F^0\|_{L^2(\mathbb{T}^2)} \le C\bar{\varepsilon}_m|\xi|(1 + \|\mathcal{R}_\xi - 1\|_{L^2(\mathbb{T}^2)})^2$, by (12.12) and $\varepsilon_m|\xi| \le \bar{\varepsilon}_m|\xi|$. So Lemma 12.3 makes the map that sends a mean-zero function, put in place of $\mathcal{R}_\xi - 1$ on the right side, to the bounded mean-zero solution of (12.11) a contraction of the unit ball of $L^\infty(\mathbb{R};L^2(\mathbb{T}^2)) \cap L^2_{\mathrm{ul}}(\mathbb{R};H^1(\mathbb{T}^2))$ when $\bar{\varepsilon}_m|\xi| < c_{\mathrm{br}}$, with an absolute $c_{\mathrm{br}} \le \frac12$; its fixed point has norm at most $C\bar{\varepsilon}_m|\xi|$. The map is a polynomial in ξ and in the function, so its Picard iterates are polynomials in ξ and converge uniformly on the complex ball $\bar{\varepsilon}_m|\xi| < c_{\mathrm{br}}$: $\mathcal{R}_\xi$ and $\mathcal{Z}_\xi$ are power series in ξ. For $\bar{\varepsilon}_m|\xi| \le c_{\mathrm{br}}/2$, $|\mathcal{Z}_\xi|$ is at most an absolute constant times κ_m/ε_m^2, by (12.10) and (12.12); we take $C_{\mathrm{br}} \ge 2$, at least this constant and the constant of the bound on the fixed point, which gives the bound of (a). Cauchy's estimate on the ball of radius $c_{\mathrm{br}}/2$ bounds the term of degree j by $(2\bar{\varepsilon}_m|\xi|/c_{\mathrm{br}})^j$ times the supremum, which gives (b); the terms of degree 0 and 1 vanish, as $\mathcal{R}_\xi - 1$ is of degree at least one in (12.10). A mean-zero solution of (12.11) in that unit

ball is a fixed point of the map, by the uniqueness in Lemma 12.3; this is (e), and (c) and the mean in (a) hold by construction.

Step 2: Real frequencies. By (12.6), and by (a) with $c_{\mathrm{br}} \le \frac{1}{2}$ and $C_{\mathrm{br}} \ge 2$, respectively,

$$\|\mathcal{R}_\xi(t)\|_{L^2(\mathbb{T}^2)} \Big| \exp\Big(\int_{t'}^{t} \mathcal{Z}_\xi(t'')\,\mathrm{d}t'' \Big) \Big| \le \|\mathcal{R}_\xi(t')\|_{L^2(\mathbb{T}^2)}\,, \qquad 1 \le \|\mathcal{R}_\xi\|_{L^2(\mathbb{T}^2)} \le 1 + C_{\mathrm{br}}\bar{\varepsilon}_m|\xi| \le C_{\mathrm{br}}\,,$$

which gives (d). □

12.4 Dissipation of the branch: the averaged flux

The quadratic part of the symbol is given by the averaged flux of Part I, up to an error small relative to $\kappa_{m-1}|\xi|^2$, from the conductivity and from switching the pulses on and off, because the first-order part of the branch is the corrector of Section 10. At each time the averaged flux enhances diffusion only along the flow of the active pulse; a block, however, carries pulses of both orientations, and over a block the averaged flux integrates to at least a constant times $\rho(m)^2$ in every direction. So over one block the branch is damped by the Gaussian $\exp(-c_{\mathrm{dis}}\rho(m)^2|\xi|^2)$, by (b) below: this is the homogenized equation of level $m-1$ seen over the time h_m. Item (c) also covers intervals that are not complete blocks: the symbol is at most $\pi^2/16$ times the cell rate, so the branch changes more slowly than diffusion damps functions of mean zero on the cell, and frequency derivatives of its integral, at the scale $1/\rho(m)$, cost only powers of $1+\rho(m)|\xi|$. Below, the *core* of a pulse k is $[(k-\frac{1}{3})\tau_m, (k+\frac{1}{3})\tau_m]$; it is *complete* if the slow cutoff of its block is 1 on it.

Proposition 12.5 (Dissipation of the branch)**.** *There exist $\Lambda_0(\alpha,s,q,\zeta) \ge 128$, $C(\alpha,s,q,\zeta)$ and absolute constants $c_{\mathrm{lo}} \in (0, c_{\mathrm{br}}/8]$ and $c_{\mathrm{dis}} > 0$ such that, under the standing hypotheses with this Λ_0, the following hold.*

(a) *For every $\xi \in \mathbb{C}^2$,*

$$\big\| \mathcal{Z}_{\xi,2} + \xi \cdot \mathbf{J}_m^{\kappa_m} \xi \big\|_{L^\infty(\mathbb{R})} \le C \varepsilon_{m-1}^{\delta} \kappa_{m-1} |\xi|^2\,.$$

(b) *For every $l' \in \mathbb{Z}$, every $\xi \in \mathbb{R}^2$ with $\bar{\varepsilon}_m|\xi| \le 2c_{\mathrm{lo}}$ and every integer $N \ge 2$, on the renewal block $[t_-, t_+] := [(l'-\frac{1}{2})h_m, (l'+\frac{1}{2})h_m]$,*

$$\mathrm{Re} \int_{t_-}^{t_+} \mathcal{Z}_\xi(t)\,\mathrm{d}t \le -c_{\mathrm{dis}}\,\rho(m)^2|\xi|^2\,, \qquad \mathrm{Re} \int_{t_-}^{t_+} \sum_{j=2}^{N} \mathcal{Z}_{\xi,j}(t)\,\mathrm{d}t \le -c_{\mathrm{dis}}\,\rho(m)^2|\xi|^2\,.$$

(c) *For every $\xi \in \mathbb{C}^2$ with $\bar{\varepsilon}_m|\xi| \le 2c_{\mathrm{lo}}$,*

$$C_{\mathrm{br}}\bar{\varepsilon}_m|\xi| \le \frac{1}{3}\,, \qquad \|\mathcal{Z}_\xi\|_{L^\infty(\mathbb{R})} \le \frac{\pi^2 \kappa_m}{16\varepsilon_m^2}\,.$$

For every interval $[t', t'']$ of length at most $2h_m$ and every $\xi \in \mathbb{R}^2$ with $\bar{\varepsilon}_m|\xi| \le 2c_{\mathrm{lo}}$, with ∇_ξ^j the array of the derivatives of order j and $C(j)$ depending only on j, the following hold for every integer $j \ge 1$:

$$\Big| \rho(m)^{-j} \nabla_\xi^j \int_{t'}^{t''} \mathcal{Z}_\xi(t)\,\mathrm{d}t \Big| \le \begin{cases} C(1)\rho(m)|\xi|\,, & j = 1\,, \\ C(j)\,, & j \ge 2\,, \end{cases}$$

and

$$\Big| \rho(m)^{-j} \nabla_\xi^j \exp\Big(\pm \int_{t'}^{t''} \mathcal{Z}_\xi(t)\,\mathrm{d}t \Big) \Big| \le C(j)(1+\rho(m)|\xi|)^j \Big| \exp\Big(\pm \int_{t'}^{t''} \mathcal{Z}_\xi(t)\,\mathrm{d}t \Big) \Big|\,.$$

Proof. Step 1: The quadratic part. The solution operator of Lemma 12.3 is linear and independent of ξ. So the term $\mathcal{R}_{\xi,1}$ of degree one in ξ of the branch is the bounded mean-zero solution of (12.11) with $\mathcal{R}_{\xi,1}$ in place of $\mathcal{R}_\xi - 1$ on the left side and with the part of degree one of the right side, $-\mathrm{i}\xi\cdot(\nabla^\perp\psi_m)(t,\varepsilon_m z)$, on the right. By (12.10),

$$\mathcal{Z}_{\xi,2} = -\kappa_m\,\xi\cdot\mathbf{K}(t)\xi - \mathrm{i}\big(\xi\cdot(\nabla^\perp\psi_m)(t,\varepsilon_m\cdot)\,\mathcal{R}_{\xi,1}\big)_{\mathbb{T}^2}\,. \tag{12.13}$$

Step 2: Comparison with the averaged flux. In this step C depends only on α, s, q, ζ, as in (a). We use the glued corrector $\chi_m^{\kappa_m}$ of Lemma 10.2(b). For $e\in\mathbb{R}^2$, since $(\psi_m)_{\mathbb{T}^2}=0$, $(\nabla(e\cdot\chi_m^{\kappa_m}))_{\mathbb{T}^2}=0$ and $\nabla^\perp\psi_m = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\nabla\psi_m$, an integration by parts in (10.6) gives the exact identity

$$\mathbf{J}_m^{\kappa_m}(t)e = \kappa_m e - \fint_{\mathbb{T}^2}\nabla^\perp\psi_m(t,y)\,\big(e\cdot\chi_m^{\kappa_m}(t,y)\big)\,\mathrm{d}y\,, \tag{12.14}$$

so that $-\xi\cdot\mathbf{J}_m^{\kappa_m}\xi = -\kappa_m\,\xi\cdot\xi + \fint_{\mathbb{T}^2}(\xi\cdot\nabla^\perp\psi_m)(\xi\cdot\chi_m^{\kappa_m})\,\mathrm{d}y$ for every $\xi\in\mathbb{C}^2$, by bilinearity. By (10.7), read in the fast variable, $\mathrm{i}\xi\cdot\chi_m^{\kappa_m}(t,\varepsilon_m z)$ solves the equation of $\mathcal{R}_{\xi,1}$ at $\mathbf{K}=I$ up to the switching residual. Hence $\mathcal{R}_{\xi,1} - \mathrm{i}\xi\cdot\chi_m^{\kappa_m}(t,\varepsilon_m z)$ is the bounded mean-zero solution of Lemma 12.3 with $F^0 = -(\varepsilon_m^2/\kappa_m)\,\mathrm{i}\xi\cdot\sum_k \mathsf{H}'_{m,k}\chi_{m,k}^{\kappa_m}(t,\varepsilon_m z)$, with $\mathsf{H}'_{m,k}$ the derivative of $\mathsf{H}_{m,k}$ in t, and $F = (\mathbf{K}-I)\nabla_z\big(\mathrm{i}\xi\cdot\chi_m^{\kappa_m}(t,\varepsilon_m z)\big)$. The gradient in z of $\chi_m^{\kappa_m}(t,\varepsilon_m z)$ is at most $\varepsilon_m^{\beta+1}/\kappa_m$, by (10.3), (10.1) and $\sum_k \mathsf{H}_{m,k}=1$; with (10.7) and Lemma 12.3,

$$\big\|\mathcal{R}_{\xi,1} - \mathrm{i}\xi\cdot\chi_m^{\kappa_m}(\cdot,\varepsilon_m\cdot)\big\|_{L^\infty(\mathbb{R};L^2(\mathbb{T}^2))} \le C|\xi|\,\frac{\varepsilon_m^{\beta+1}}{\kappa_m}\Big(\frac{\varepsilon_m^2}{\kappa_m\tau_m} + \varepsilon_{m-1}^{\delta}\Big).$$

The averages over $\mathbb{T}^2$ in z are those over $\mathbb{T}^2$ in y, as the functions are ε_m-periodic; so (12.13) and (12.14) give

$$\mathcal{Z}_{\xi,2} + \xi\cdot\mathbf{J}_m^{\kappa_m}\xi = -\kappa_m\,\xi\cdot(\mathbf{K}(t)-I)\xi - \mathrm{i}\Big(\xi\cdot(\nabla^\perp\psi_m)(t,\varepsilon_m\cdot)\,\big(\mathcal{R}_{\xi,1} - \mathrm{i}\xi\cdot\chi_m^{\kappa_m}(t,\varepsilon_m\cdot)\big)\Big)_{\mathbb{T}^2}\,,$$

$$\big|\mathcal{Z}_{\xi,2} + \xi\cdot\mathbf{J}_m^{\kappa_m}\xi\big| \le \kappa_m\varepsilon_{m-1}^{\delta}|\xi|^2 + C\frac{\varepsilon_m^{2\beta}}{\kappa_m}\Big(\frac{\varepsilon_m^2}{\kappa_m\tau_m} + \varepsilon_{m-1}^{\delta}\Big)|\xi|^2 \le C\varepsilon_{m-1}^{\delta}\kappa_{m-1}|\xi|^2\,,$$

the last by $\kappa_m\le\kappa_{m-1}$, $\varepsilon_m^{2\beta}/\kappa_m \le 20\kappa_{m-1}$ and $\varepsilon_m^2/(\kappa_m\tau_m)\le C\varepsilon_{m-1}^{4\delta}$ (Proposition 2.1(c) and (10.5)). This is (a).

Step 3: Coercivity over a block. We show, with $c_{\mathrm{dis}} := (1-\exp(-2))/192$,

$$\int_{t_-}^{t_+}\xi\cdot\mathbf{J}_m^{\kappa_m}(t)\xi\,\mathrm{d}t \ge 2c_{\mathrm{dis}}\,\kappa_{m-1}h_m|\xi|^2 \qquad (\xi\in\mathbb{R}^2)\,. \tag{12.15}$$

On a complete core $\widetilde{\zeta}_{m,k}=1$, since $\zeta(\cdot-k)$ is the only term of the partition (2.2) that is nonzero there. So for $t\in[k\tau_m,(k+\frac13)\tau_m]$ the time integral of (2.16) is at least $\int_0^{\tau_m/3}\exp(-4\pi^2\kappa_m\tilde t/\varepsilon_m^2)\,\mathrm{d}\tilde t \ge (1-\exp(-2))\varepsilon_m^2/(4\pi^2\kappa_m)$, by (12.3), and (2.16) gives

$$e_{(k)}\cdot\mathbf{J}_m^{\kappa_m}(t)e_{(k)} \ge \kappa_m + \big(1-\exp(-2)\big)\frac{\varepsilon_m^{2\beta}}{2\kappa_m}\,.$$

A block has $h_m/\tau_m = (4\lceil\varepsilon_{m-1}^{-\delta}\rceil+1)^2$ slots, and only those within $2\tau'_m+\tau_m$ of its ends fail to have complete cores (Lemma 7.1(d)); so the complete cores are those of $h_m/\tau_m - 4\tau'_m/\tau_m$ consecutive integers, with $\tau'_m/\tau_m = 4\lceil\varepsilon_{m-1}^{-\delta}\rceil+1\ge 65$ by (12.3), and it has at least $h_m/(32\tau_m)$ complete cores of each orientation. Since $\mathbf{J}_m^{\kappa_m}$ is diagonal and at least $\kappa_m I$ (Lemma 7.10), for $e\in\{e_1,e_2\}$

$$\int_{t_-}^{t_+} e\cdot\mathbf{J}_m^{\kappa_m}e\,\mathrm{d}t \ge \kappa_m h_m + \frac{h_m}{32\tau_m}\cdot\frac{\tau_m}{3}\cdot\big(1-\exp(-2)\big)\frac{\varepsilon_m^{2\beta}}{2\kappa_m} \ge 2c_{\mathrm{dis}}h_m\Big(\kappa_m + \frac{\varepsilon_m^{2\beta}}{2\kappa_m}\Big) \ge 2c_{\mathrm{dis}}h_m\kappa_{m-1}\,,$$

because $\kappa_{m-1} = \overline{K}_m(\kappa_m) \le \kappa_m + \varepsilon_m^{2\beta}/(2\kappa_m)$ (Lemma 7.10(i) and (2.17)). This is (12.15).

Step 4: The block integral. For real ξ with $\bar\varepsilon_m|\xi| \le c_{\mathrm{br}}/4$, by (a), (12.15), Proposition 12.4(b) and $\frac{\kappa_m}{\varepsilon_m^2}(\bar\varepsilon_m|\xi|)^2 = \kappa_{m-1}|\xi|^2$,

$$\mathrm{Re}\int_{t_-}^{t_+} \mathcal{Z}_\xi(t)\,\mathrm{d}t \le -\big(2c_{\mathrm{dis}} - C\varepsilon_{m-1}^{\delta}\big)\kappa_{m-1}h_m|\xi|^2 + C_{\mathrm{br}}h_m\frac{\kappa_m}{\varepsilon_m^2}\sum_{j\ge 3}\Big(\frac{2\bar\varepsilon_m|\xi|}{c_{\mathrm{br}}}\Big)^j$$
$$\le -\Big(2c_{\mathrm{dis}} - C\varepsilon_{m-1}^{\delta} - \frac{16C_{\mathrm{br}}}{c_{\mathrm{br}}^3}\bar\varepsilon_m|\xi|\Big)\rho(m)^2|\xi|^2\,.$$

We choose $c_{\mathrm{lo}} := \min\{c_{\mathrm{br}}/8, c_{\mathrm{br}}^3c_{\mathrm{dis}}/(64C_{\mathrm{br}}), \pi c_{\mathrm{br}}/(32C_{\mathrm{br}}^{1/2}), 1/(6C_{\mathrm{br}})\}$, and Λ_0 so large that $C\varepsilon_{m-1}^{\delta} \le c_{\mathrm{dis}}/2$. Then the right side, and that of every truncation, is at most $-c_{\mathrm{dis}}\rho(m)^2|\xi|^2$ for $\bar\varepsilon_m|\xi| \le 2c_{\mathrm{lo}}$.

Step 5: Frequency derivatives. By Proposition 12.4(b), summing a geometric series of ratio at most $\frac34$, $\|\mathcal{Z}_{\xi'}\|_{L^\infty(\mathbb{R})} \le 16C_{\mathrm{br}}c_{\mathrm{br}}^{-2}\frac{\kappa_m}{\varepsilon_m^2}(\bar\varepsilon_m|\xi'|)^2$ for complex ξ' with $\bar\varepsilon_m|\xi'| \le \frac38 c_{\mathrm{br}}$; as $2c_{\mathrm{lo}} \le \frac14 c_{\mathrm{br}}$, the last two entries of the minimum defining c_{lo} give the first sentence of (c). There the integral over $[t', t'']$ is holomorphic and at most $32C_{\mathrm{br}}c_{\mathrm{br}}^{-2}\rho(m)^2|\xi'|^2$, as $t''-t' \le 2h_m$ and $\bar\varepsilon_m^2\kappa_m h_m/\varepsilon_m^2 = \rho(m)^2$. Cauchy's estimates on the polydiscs about a real ξ of radii $c_{\mathrm{br}}/(16\bar\varepsilon_m)$ and $|\xi|/4$ give the bounds of (c) on this integral for $j \ge 2$ (as $\rho(m)^2/\bar\varepsilon_m^2 = \kappa_m h_m/\varepsilon_m^2 \ge 1$) and $j = 1$ (at $\xi = 0$ the gradient vanishes, as $\mathcal{Z}_\xi$ has no term of degree one). Faà di Bruno's formula gives the last bound. □

12.5 The other frequencies: two orientations

At the frequencies with $\bar\varepsilon_m|\xi| \ge c_{\mathrm{lo}}$ the whole fibre must decay at the cell rate. If the contrast is bounded, as only at the terminal level, the inequality (12.7) suffices. If it is large, then at the frequencies with $\varepsilon_m|\xi| \ll 1$ molecular diffusion alone damps the cell mean, the plane wave of frequency ξ, far more slowly than at the cell rate. The damping comes instead from the shear: a pulse flowing along e_2 moves the cell mean into the first cell harmonic, at a rate at least comparable to the cell rate once $\bar\varepsilon_m|\xi_2| \ge c_{\mathrm{lo}}/2$, and diffusion damps the harmonic. Such a pulse does not see the frequencies along e_1, as it moves functions of y_1 along their level lines; so both orientations are needed, and a block has many pulses of each.

Proposition 12.6 (Damping of the other frequencies)**.** *There exist absolute constants $c_{\mathrm{hi}}, C_{\mathrm{hi}} > 0$ such that the following holds under the standing hypotheses, with m above a level depending only on $\alpha, s, q, \zeta, \Lambda$. For every $l' \in \mathbb{Z}$, every $\xi \in \mathbb{R}^2$ in the zone with $\bar\varepsilon_m|\xi| \ge c_{\mathrm{lo}}$ and every solution U of the fibre equation (12.5) on the renewal block $[t_-, t_+] := [(l'-\frac12)h_m, (l'+\frac12)h_m]$,*

$$\|U(t_+)\|_{L^2(\mathbb{T}^2)} \le \exp\Big(-c_{\mathrm{hi}}\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)\|U(t_-)\|_{L^2(\mathbb{T}^2)}\,.$$

If moreover $\kappa_m \le \frac12\kappa_{m-1}$, then, for every interval $[t', t''] \subset [t_-, t_+]$ of length at least $4C_{\mathrm{hi}}\varepsilon_m^2/\kappa_m$ that lies in a complete core of a pulse k with $\bar\varepsilon_m|\xi\cdot e_{(k)}| \ge c_{\mathrm{lo}}/2$,

$$\|U(t'')\|_{L^2(\mathbb{T}^2)} \le \tfrac12\|U(t')\|_{L^2(\mathbb{T}^2)}\,.$$

Proof. By (12.6) the norm never grows. We distinguish two cases.

Bounded contrast, $\kappa_m \ge \frac12\kappa_{m-1}$. Then $\varepsilon_m|\xi| \ge 2^{-1/2}\bar\varepsilon_m|\xi| \ge c_{\mathrm{lo}}/2$, and (12.6), (12.7) and $\mathbf{K} \ge \frac12 I$ give $\frac{\mathrm{d}}{\mathrm{d}t}\|U\|_{L^2(\mathbb{T}^2)}^2 \le -\frac{\kappa_m}{\varepsilon_m^2}\cdot\frac{c_{\mathrm{lo}}^2}{4}\|U\|_{L^2(\mathbb{T}^2)}^2$, whatever the velocity. For m large, this case is $m = M$ (Proposition 2.1).

Large contrast, $\kappa_m \le \frac12\kappa_{m-1}$. Then $\kappa_{m-1} = \overline{K}_m(\kappa_m) \le \kappa_m + \varepsilon_m^{2\beta}/(2\kappa_m)$ (Lemma 7.10(i) and (2.17)) forces $\varepsilon_m^{2\beta}/\kappa_m \ge \kappa_{m-1}$. One of the coordinates of ξ, say ξ_2, has $\bar\varepsilon_m|\xi_2| \ge c_{\mathrm{lo}}/2$.

Let $k \in 4\mathbb{Z}+1$ be a pulse, whose flow direction is e_2, with a complete core. On the core only pulse k is active and $\widetilde{\zeta}_{m,k} = 1$, so by (10.1) the velocity of (12.5) is the stationary shear $2\pi\varepsilon_m^{\beta-1}\cos(2\pi z_1)e_2$. The shear couples the cell mean $(U)_{\mathbb{T}^2}$ to the first harmonic $(\sqrt{2}\cos(2\pi z_1)U)_{\mathbb{T}^2}$: testing (12.5) with 1 and with $\sqrt{2}\cos(2\pi z_1)$, and integrating by parts in z_2 and, where derivatives fall on the test function, in z_1, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(U)_{\mathbb{T}^2} &= -\kappa_m\,\xi\cdot\mathbf{K}(t)\xi\,(U)_{\mathbb{T}^2} - \mathrm{i}\sqrt{2}\pi\varepsilon_m^{\beta-1}\xi_2\big(\sqrt{2}\cos(2\pi z_1)U\big)_{\mathbb{T}^2}\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\big(\sqrt{2}\cos(2\pi z_1)U\big)_{\mathbb{T}^2} &= -\mathrm{i}\sqrt{2}\pi\varepsilon_m^{\beta-1}\xi_2\,(U)_{\mathbb{T}^2} + \text{a remainder}\,, \end{aligned} \tag{12.16}$$

with the remainder at most $C(\varepsilon_m^{\beta-1}|\xi_2|+\kappa_m/\varepsilon_m^2)\|U-(U)_{\mathbb{T}^2}\|_{L^2(\mathbb{T}^2)}$ for an absolute C, as $\mathbf{K}$ does not depend on z and $\varepsilon_m|\xi| \le \sqrt{2}\pi$; here $|(\sqrt{2}\cos(2\pi z_1)U)_{\mathbb{T}^2}| \le \|U-(U)_{\mathbb{T}^2}\|_{L^2(\mathbb{T}^2)}$, since the harmonic has mean zero. Hence the imaginary part of $\overline{(U)_{\mathbb{T}^2}}\,(\sqrt{2}\cos(2\pi z_1)U)_{\mathbb{T}^2}$ has the derivative $\sqrt{2}\pi\varepsilon_m^{\beta-1}\xi_2$ $(|(\sqrt{2}\cos(2\pi z_1)U)_{\mathbb{T}^2}|^2 - |(U)_{\mathbb{T}^2}|^2)$ up to $C(\varepsilon_m^{\beta-1}|\xi_2|+\kappa_m/\varepsilon_m^2)|(U)_{\mathbb{T}^2}|\,\|U-(U)_{\mathbb{T}^2}\|_{L^2(\mathbb{T}^2)}$, the damping of the first line included, as $\varepsilon_m|\xi| \le \sqrt{2}\pi$. We integrate over an interval $[t',t'']$ of the core; at its ends the imaginary part is at most $\frac{1}{2}\|U(t')\|^2_{L^2(\mathbb{T}^2)}$, as the norm does not increase, and Young's inequality on the error gives

$$\varepsilon_m^{\beta-1}|\xi_2|\int_{t'}^{t''}|(U)_{\mathbb{T}^2}|^2\,\mathrm{d}t \le C\|U(t')\|^2_{L^2(\mathbb{T}^2)} + C\varepsilon_m^{\beta-1}|\xi_2|\Big(1+\frac{\kappa_m}{\varepsilon_m^{\beta+1}|\xi_2|}\Big)^2\int_{t'}^{t''}\|U-(U)_{\mathbb{T}^2}\|^2_{L^2(\mathbb{T}^2)}\,\mathrm{d}t\,.$$

By (12.6) and (12.7), the mean-free part satisfies $\pi^2(\kappa_m/\varepsilon_m^2)\int_{t'}^{t''}\|U-(U)_{\mathbb{T}^2}\|^2_{L^2(\mathbb{T}^2)}\,\mathrm{d}t \le \|U(t')\|^2_{L^2(\mathbb{T}^2)}$. Now $\varepsilon_m^{\beta+1}|\xi_2|/\kappa_m \ge \bar{\varepsilon}_m|\xi_2| \ge c_{\mathrm{lo}}/2$, by $\varepsilon_m^{2\beta}/\kappa_m \ge \kappa_{m-1}$, so that

$$(\varepsilon_m^{\beta-1}|\xi_2|)^{-1} = (\kappa_m/(\varepsilon_m^{\beta+1}|\xi_2|))\,\varepsilon_m^2/\kappa_m \le (2/c_{\mathrm{lo}})\,\varepsilon_m^2/\kappa_m\,.$$

Dividing the first bound by $\varepsilon_m^{\beta-1}|\xi_2|$ and adding the two bounds, we obtain $\int_{t'}^{t''}\|U\|^2_{L^2(\mathbb{T}^2)}\,\mathrm{d}t \le C_{\mathrm{hi}}(\varepsilon_m^2/\kappa_m)\|U(t')\|^2_{L^2(\mathbb{T}^2)}$ with an absolute C_{hi}, as c_{lo} is absolute; and since the norm does not increase, $\|U(t'')\|_{L^2(\mathbb{T}^2)} \le \frac{1}{2}\|U(t')\|_{L^2(\mathbb{T}^2)}$ as soon as $t''-t' \ge 4C_{\mathrm{hi}}\varepsilon_m^2/\kappa_m$. This is the last assertion of the proposition for $k \in 4\mathbb{Z}+1$; for $k \in 4\mathbb{Z}+3$ we exchange the coordinates.

A core has length $\frac{2}{3}\tau_m$, and $\kappa_m\tau_m/\varepsilon_m^2 \to \infty$ with m (Proposition 2.1(c)); for m large, each complete core of the right orientation halves the norm at least $\kappa_m\tau_m/(12C_{\mathrm{hi}}\varepsilon_m^2)$ times. The block has at least $h_m/(32\tau_m)$ such cores (Step 3 of the proof of Proposition 12.5), so the norm decreases by the factor $\exp(-c_{\mathrm{hi}}\kappa_m h_m/\varepsilon_m^2)$, with $c_{\mathrm{hi}} \coloneqq \min\{\log 2/(384C_{\mathrm{hi}}), c_{\mathrm{lo}}^2/8\}$. □

12.6 The Duhamel expansion and the adjoint equation

In a family, sources couple the components, and a source of order d takes d derivatives. The preparation expands the solution in its sources by Duhamel's formula, and as the sources are small, finitely many terms suffice (Subsection 12.10). Proposition 12.7, the main result of Subsections 12.6–12.9, prepares their sum. A direct estimate of each term would lose derivatives of the datum and, through the transport, powers of the contrast; the map $\widetilde{\Pi}$ of that proposition loses neither, and only its error, of size ε_{m-1}^{Q+3}, uses derivatives of the datum, at the molecular scale.

The *Duhamel expansion* of a family G of level m from a datum g at a time t_1 is the sequence $U^{\{0\}}, U^{\{1\}}, \dots$ of solutions of

$$\big(\partial_t + \mathbf{b}_m\cdot\nabla - \kappa_m\Delta\big)U_v^{\{j\}} = \begin{cases} 0\,, & j=0\,, \\ \nabla\cdot\sum_{v'}\sum_{\sigma\colon v'\to v}\sum_{d=2}^{N_{\mathrm{src}}} A_{\sigma,d}\nabla^{d-1}U_{v'}^{\{j-1\}}\,, & j\ge 1\,, \end{cases}$$

with $U^{\{0\}}(t_1) = g$ and $U^{\{j\}}(t_1) = 0$ for $j \geq 1$: the system (5.6) with its sources applied one at a time and *without the smoothing operators* S_σ. The terms exist in $C([t_1, t_\mathrm{f}]; H^{N_\mathrm{buf} - j(N_\mathrm{src}-2)}(\mathbb{T}^2)^V)$ for $j \leq N_\mathrm{dep} + 1$ when $g \in H^{N_\mathrm{buf}}(\mathbb{T}^2)^V$, by Proposition 8.6, whose condition at the scale $\rho_\mathrm{mol}(m)$ is (b2′) of Definition 11.1 with majorant sums at most $C_*^2 \varepsilon_m^{\delta/4}$.

Proposition 12.7 (Preparation of the Duhamel expansion)**.** *Let $C_* \geq 1$. There exist C and a level, both depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$ and C_*, such that the following holds for every M, $\kappa \in \mathcal{K}_M$ and every $m \leq M$ above this level, the terminal level included. Let G be a family of level m that satisfies the invariant (Definition 11.1) on $[t_\mathrm{A}, t_\mathrm{f}]$, with C_* in place of the constants C_Ω, c_j ($j \leq N_\mathrm{src} - 2$), C_wid, C_der, $C_\mathrm{jet}(j)$ ($j \leq N_\mathrm{coef}$) and C_adm of items (3)–(4) of Subsection 11.1, and whose majorant matrix has nonzero entries that, as arrows between components, form no path of more than N_path arrows, let $t_1 \in [t_\mathrm{A}, t_\mathrm{f} - 2h_m]$, and let t_pr be the end of the block that follows the block containing t_1: with $(l - \frac{1}{2})h_m \leq t_1 < (l + \frac{1}{2})h_m$ and $t_2 := (l + \frac{1}{2})h_m$,*

$$t_\mathrm{pr} := t_2 + h_m\,, \qquad h_m < t_\mathrm{pr} - t_1 \leq 2h_m\,. \tag{12.17}$$

Then the velocity added at level m vanishes near t_pr,

$$\mathbf{b}_m = \mathbf{b}_{m-1} \quad \text{on } [t_\mathrm{pr} - \tau_m', t_\mathrm{pr} + \tau_m'] \times \mathbb{T}^2\,,$$

and there exists a linear map $\widetilde{\Pi}\colon L^2(\mathbb{T}^2)^V \to H^{N_\mathrm{out}}(\mathbb{T}^2)^V$ that maps fields with constant components to themselves, such that, for both choices ω_v and ω_v/Ψ_v^2 of the masses and for every $g \in H^{N_\mathrm{buf}}(\mathbb{T}^2)^V$, with $U^{\{j\}}$ the Duhamel expansion of G from g at t_1,

$$\|\widetilde{\Pi} g\|_{H^{N_\mathrm{out}}_{\rho(m),\omega}(\mathbb{T}^2)} \leq C \|g\|_{L^2_\omega(\mathbb{T}^2)}\,,$$

$$\Big\| \sum_{j=0}^{N_\mathrm{dep}} U^{\{j\}}(t_\mathrm{pr}) - \widetilde{\Pi} g \Big\|_{L^2_\omega(\mathbb{T}^2)} \leq C \varepsilon_{m-1}^{Q+3} \|g\|_{H^{N_\mathrm{buf}}_{\rho_\mathrm{mol}(m),\omega}(\mathbb{T}^2)}\,.$$

With the masses ω_v/Ψ_v^2, these norms are those of L^2_G and $H^N_{\rho,G}$ in (5.7).

The plan. The proof (end of Subsection 12.9) cuts the datum into *pieces* within $\rho(m)/4$ of their centres y_0 and freezes the conductivity and the forms of the sources at y_0 as Taylor polynomials, which become *insertions*, operators inserted between free evolutions. Each term of the expansion becomes a finite sum of *Duhamel terms* of the frozen operator (12.4), across the old partial block, the change of chart at t_2, and one complete block (Figure 12.1); Lemma 12.15 gives the model of each and its error.

Sources moved onto tests. In the bounds that determine the model, the derivatives that the sources take are never taken of a term of the expansion: we pair the term with a *test*, a solution of the adjoint fibre equation run backward from the final time, and move each source onto the test, which carries all of them. Against the principal branch $\mathcal{R}_\xi$, a source in divergence form acts at most at a constant times the sum of the rate $\kappa_{m-1}|\xi|^2$ of the homogenized equation and one over the length h_m of a block (Lemma 12.10), and over a block it costs at most a power of $1 + \rho(m)|\xi|$, which the Gaussian of Proposition 12.5(b) absorbs. These bounds lose no power of the contrast. Everything else decays over a block by a factor smaller than any power of ε_{m-1}; it is estimated with losses polynomial in the contrast factor and in the number of cell times ε_m^2/κ_m in a block, and it goes into the error.

Conventions for Subsections 12.6–12.9. The standing hypotheses hold with $m \leq M$, the terminal level included, and m above a level that is at least that of Proposition 12.6 and that each

lemma may enlarge; $l \in \mathbb{Z}$ is fixed, and *the block* is $[(l-\frac{1}{2})h_m, (l+\frac{1}{2})h_m]$. Unless a statement says otherwise, $t' < t''$ lie in the block, ξ is real with $\bar{\varepsilon}_m|\xi| \le 2c_{\mathrm{lo}}$, and insertions have strength at most one; for insertions of any strengths, every bound on a Duhamel term or its moment terms holds with its constant multiplied by the product of the strengths of its insertions, as these terms are linear in each insertion. The constants C, $C(N)$, $c > 0$, the integers N_0, N_2, N_3 and the level of a statement depend only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$, on N_{path} and on the integers that the statement fixes: the lengths of its Duhamel terms, the orders and Taylor degrees of their insertions, the moment order N_1, and its Sobolev orders and weight exponents. Proofs measure time in cell units $\kappa_m t/\varepsilon_m^2$. Quiet times are those of Section 11, at which no pulse of level m is active; no pulse is active within $\tau_m' \ge 5\tau_m$ of the ends of the block, its quiet margins. Windows of pulses are those of (10.2), and complete cores those of Subsection 12.4. We call $\big(1+\bar{\varepsilon}_m/\varepsilon_m\big) = 1 + (\kappa_{m-1}/\kappa_m)^{1/2}$ the *contrast factor*, and we write $\|f\|_{H^J(\mathbb{T}^2)} := \|f\|_{H_1^J(\mathbb{T}^2)}$ for the norm (5.8) at $\rho = 1$.

Insertions. After freezing, the sources become insertions; by (12.18), in cell units an insertion of order d is the cell rate times an operator of order d in z, with coefficients bounded in z and polynomial in $y - y_0$. The frozen operator (12.4) acts componentwise on fields of a finite set V of components with masses ω_v; we write $\langle f, f'\rangle_{L^2(\mathbb{T}^2)} := (f\,\overline{f'})_{\mathbb{T}^2}$ and $\langle U, U'\rangle_{L^2_\omega(\mathbb{T}^2)} := \sum_v \omega_v^{-1}\langle U_v, U_v'\rangle_{L^2(\mathbb{T}^2)}$. An *insertion* is an operator $\mathcal{I}U := \nabla \cdot \sum_{d=2}^{N_{\mathrm{src}}} A_d \nabla^{d-1} U$ between such fields with coefficients

$$A_d(t,y) = \kappa_m \varepsilon_m^{d-2}\, \mathcal{A}_d\Big(t, y, \frac{y}{\varepsilon_m}\Big), \tag{12.18}$$

where $\mathcal{A}_d(t,y,z) = \sum_a ((y-y_0)/\rho(m))^a \mathcal{A}_{d,a}(t,z)$, the sum over the multi-indices $a \in \mathbb{N}^2$ with $|a|$ at most the *Taylor degree*, has coefficients $\mathcal{A}_{d,a}$ that are two-scale forms (Section 10) independent of y, measurable in t. On the block they satisfy:

(i) the structure clauses (a2)–(a4) of Definition 11.1: transverse (independent of $z \cdot e_{(k)}$) on the window of each pulse k, invariant along $e_{(k)}$ where $|t - k\tau_m| \le \frac{3}{4}\tau_m$, and independent of z at times with no pulse active within $\tau_m/24$;

(ii) the derivative bound: with $M_{vv'} := \sum_d \sum_a \sum_{j'} \|\nabla_z^{j'} (\mathcal{A}_{d,a})_{vv'}\|_{L^\infty}$, the sum over j' up to the derivative counts below, the matrix of entries $(\omega_{v'}/\omega_v)^{1/2} M_{vv'}$ has row and column sums at most the *strength*;

(iii) either this matrix is diagonal and $\mathcal{A}_d$ has no term of degree 0 in $y - y_0$, or its nonzero entries, as arrows between components, form no path of more than N_{path} arrows. The sources of Section 11 satisfy the second alternative, and the conductivity insertions the first.

The *order* is the largest d with $A_d \neq 0$. We sum the forms of a source over the pulses, with their time cutoffs, each replaced by its Taylor polynomial at y_0 and written in the chart of the block by clause (a6) of Definition 11.1. On the block, extended by zero outside the interval of the Duhamel term, this sum is an insertion of strength C times its majorant sums. The Taylor polynomial at y_0 of the conductivity minus its value gives diagonal insertions of order 2 and strength $C\varepsilon_{m-1}^{2\delta}\rho(m)/\varepsilon_{m-1}$ on fields supported near y_0, each power of $(y-y_0)/\varepsilon_{m-1}$ costing one moment at the scale $\rho(m)$.

Duhamel terms. A *Duhamel term* of length N on an interval $[t_1, t_2]$ of the block from a datum g is the sequence $U^{[0]}, \dots, U^{[N]}$ of energy solutions of (12.4) on $[t_1, t_2]$ with $U^{[0]}(t_1) = g$ and, for $1 \le j \le N$, the forcing $\mathcal{I}_j U^{[j-1]}$ and $U^{[j]}(t_1) = 0$, where $\mathcal{I}_1, \dots, \mathcal{I}_N$ are insertions of orders $d_1, \dots, d_N$. It is *centred at* y_0 when the conductivity of (12.4) is frozen at y_0 and the insertions are polynomials in $y - y_0$. For an integer $K \ge 0$, let $\mathcal{D}_K$ be the class of fields g on $\mathbb{R}^2$ with

$\nabla^b g \in L^2_{\omega,K}(\mathbb{R}^2)^V$ for every $b \le d_1 + \cdots + d_N - 2N$, with the moments of (12.8). For data in $\mathcal{D}_K$, with K the sum of the Taylor degrees of the insertions, the Duhamel term exists and is unique, by the energy of Proposition 8.6 with the moment weights (the estimate of Lemma 8.5 is pointwise). Along the sequence the orders decrease by $d_j - 2$ and the weights by the Taylor degree of $\mathcal{I}_j$ at step j.

Tests. For real ξ, a *test* on $[t', t'']$ is an energy solution $U^\dagger$ of the *adjoint fibre equation*

$$-\partial_t U^\dagger = \frac{\kappa_m}{\varepsilon_m^2}(\nabla + \mathrm{i}\varepsilon_m\xi)\cdot\mathbf{K}(t)(\nabla + \mathrm{i}\varepsilon_m\xi)U^\dagger + \frac{1}{\varepsilon_m}\nabla^\perp\psi_m(t, \varepsilon_m z)\cdot(\nabla + \mathrm{i}\varepsilon_m\xi)U^\dagger \tag{12.19}$$

on $[t', t''] \times \mathbb{T}^2$ with a datum at t'' (componentwise for fields): the fibre equation (12.5) run backward in time with the velocity reversed. The transport term is skew-adjoint and the diffusion self-adjoint, so for every solution U of (12.5) with a forcing F,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle U(t), U^\dagger(t)\big\rangle_{L^2_\omega(\mathbb{T}^2)} = \big\langle F(t), U^\dagger(t)\big\rangle_{L^2_\omega(\mathbb{T}^2)}. \tag{12.20}$$

As $\mathcal{R}_\xi(t)\exp(\int_{t'}^t \mathcal{Z}_\xi(\tilde t)\,\mathrm{d}\tilde t)$ solves (12.5) (Proposition 12.4(c)), $\exp(\int_{t'}^t \mathcal{Z}_\xi(\tilde t)\,\mathrm{d}\tilde t)\langle\mathcal{R}_\xi(t), U_v^\dagger(t)\rangle_{L^2(\mathbb{T}^2)}$ is constant. A test is *off-branch* at t if $\langle\mathcal{R}_\xi(t), U_v^\dagger(t)\rangle_{L^2(\mathbb{T}^2)} = 0$ for every v; it is then off-branch at every earlier time, and it reads only the part of a solution off the branch. For a time t'' of the block with no pulse of level m active within $\tau_m/24$ of it, the *branch test* $U^\dagger_{\mathrm{br}}$ is the test with $U^\dagger_{\mathrm{br}}(t'') = 1$; paired with a Bloch sum at t'', it reads the cell mean, that is, by (12.22) below, the Fourier transform.

Moment terms. The coefficients of insertions are polynomials in $y - y_0$, which do not act fibre by fibre; so we carry the moments of the Duhamel terms as further unknowns. A power of $y - y_0$ then maps moment terms to higher ones, every coefficient becomes periodic in the fast variable, and Bloch sums decouple the fibres. The moments are also the frequency derivatives of the Fourier transform (12.22), which localize the model near y_0. Let $U^{[0]}, \dots, U^{[N]}$ be a Duhamel term on $[t_1, t_2]$, let $N_1 \ge 0$, let N' be N_1 plus the sum of the Taylor degrees of its insertions, and let its datum lie in $\mathcal{D}_{N'+2}$. For a multi-index $i \in \mathbb{N}^2$ with $|i|$ at most N_1 plus the sum of the Taylor degrees of $\mathcal{I}_{j+1}, \dots, \mathcal{I}_N$, the *moment term* (j, i) is $U^{[j,i]} := ((y - y_0)/\rho(m))^i U^{[j]}$. Its *degree* is $|i|$ plus the sum of the Taylor degrees of $\mathcal{I}_1, \dots, \mathcal{I}_j$, at most N'; a moment term is *lower* than (j, i) if its degree is smaller, or equal with a smaller j, and the moment terms with given j and $|i|$ form a *group*. Commuting the weight with the frozen operator, we obtain

$$\begin{aligned}\big(\partial_t + \nabla^\perp\psi_m\cdot\nabla - \kappa_m\nabla\cdot(\mathbf{K}(t)\nabla\cdot)\big)U^{[j,i]} \\ = \Big(\frac{y - y_0}{\rho(m)}\Big)^i \mathcal{I}_j U^{[j-1]} + \sum_e \frac{i\cdot e}{\rho(m)}\big(\nabla^\perp\psi_m\cdot e - 2\kappa_m\mathbf{K}(t)e\cdot\nabla\big)U^{[j,i-e]} \\ + \kappa_m\sum_{e,e'}\frac{(i\cdot e)(i\cdot e' - e\cdot e')}{\rho(m)^2}\, e\cdot\mathbf{K}(t)e'\, U^{[j,i-e-e']}\end{aligned} \tag{12.21}$$

(the sums over $e, e' \in \{e_1, e_2\}$; without the first term for $j = 0$). Moving the weight through $\mathcal{I}_j$, whose coefficients are polynomials in $(y - y_0)/\varepsilon_{m-1} = (\rho(m)/\varepsilon_{m-1})(y - y_0)/\rho(m)$, makes the first term a sum of operators of order at most d_j on the moment terms $(j - 1, i')$, $|i'|$ at most $|i|$ plus the Taylor degree of $\mathcal{I}_j$, with coefficients bounded as those of $\mathcal{I}_j$ and $\rho(m)^{-1}$ per derivative on the weight. All moment terms on the right are lower than (j, i).

By the rules for Bloch sums after (12.9), $\mathcal{B}U^{[j,i]}(t)$, short for $\mathcal{B}(U^{[j,i]}(t))(\xi, \cdot)$, solves (12.5) forced by the sum over the lower (j', i') of $\mathcal{I}_{(j,i),(j',i')}(t)\mathcal{B}U^{[j',i']}(t)$, the fibre forms of the terms of (12.21);

$\mathcal{I}^*_{(j,i),(j',i')}(t)$ is the adjoint for $\langle\cdot,\cdot\rangle_{L^2_\omega(\mathbb{T}^2)}$. As the mean of a Bloch sum is ε_m^{-2} times the Fourier transform (Subsection 12.2), with $\nabla^i_\xi := \partial^{i_1}_{\xi_1}\partial^{i_2}_{\xi_2}$,

$$\rho(m)^{-|i|}\nabla^i_\xi\widehat{U^{[j]}(t)}(\xi) = (-\mathrm{i})^{|i|}\widehat{U^{[j,i]}(t)}(\xi) = (-\mathrm{i})^{|i|}\varepsilon_m^2\big(\mathcal{B}U^{[j,i]}(t)\big)_{\mathbb{T}^2}\,. \tag{12.22}$$

The moment terms have two further moments: at step j the weight available, $N'+2$ minus the Taylor degrees of $\mathcal{I}_1,\dots,\mathcal{I}_j$, exceeds the largest $|i|$ by two. So, by the moment-weighted energy of the Duhamel term, each $U^{[j,i]}$, its gradient (in L^2 in time) and the flux of its forcing lie in $L^2_{\omega,2}(\mathbb{R}^2)$, their Bloch sums converge (paragraph of (12.9)), and Lemma 12.2 applies. For an array of tests $U^\dagger = (U^\dagger_{(j,i)})$ indexed by a group, and for a lower group, $\mathcal{I}^*U^\dagger$ is the array indexed by the lower group with entries $\sum_{(j,i)}\mathcal{I}^*_{(j,i),(j',i')}U^\dagger_{(j,i)}$. Arrays are normed by the square sums of the norms of their entries; for a function f on $\mathbb{T}^2$ and a field or array U, $|\langle U,f\rangle| = |\langle f,U\rangle|$ is the square root of the sum over the entries and components of $\omega_v^{-1}|\langle U_v,f\rangle_{L^2(\mathbb{T}^2)}|^2$.

Derivative counts. The proofs below differentiate the coefficients of the insertions of a Duhamel term at most $2(N_1+N'+d_1+\cdots+d_N+2)+\max\{d_1,\dots,d_N\}+2$ times in the fast variable, and, for a Duhamel term across the renewal (Lemma 12.15), at most $2(3N_1+d_1+\cdots+d_{N_-+N_+}+4)+\max\{d_1,\dots,d_{N_-+N_+}\}+2$ plus twice the sum of the Taylor degrees; N_{coef} is at least these counts.

12.7 Estimates for the adjoint equation

Tests carry the derivatives that the sources take, and in cell units the velocity of the fibre equation has the size of the Péclet number, at most a constant times the contrast factor, by (12.12) with $|\xi| = 1/\varepsilon_m$; so a derivative of a test may cost a power of the contrast factor. Lemma 12.8 gives the bounds that accept this loss, for the parts that decay at the cell rate. The bounds without loss, Lemma 12.9, rest on a feature of shears: on the window of a pulse, a function independent of the coordinate along its flow is not transported, and on such functions the adjoint fibre equation is a heat equation with a potential of the size of $\bar\varepsilon_m|\xi|$, whatever the contrast. Lemma 12.10 says how insertions act on tests.

Halving intervals. For real ξ in the zone with $\bar\varepsilon_m|\xi| \ge c_{\mathrm{lo}}$, a *halving interval* is an interval of the block of length $\max\{4C_{\mathrm{hi}}, 16c_{\mathrm{lo}}^{-2}\}\varepsilon_m^2/\kappa_m$, contained, if $\kappa_m \le \frac12\kappa_{m-1}$, in a complete core of a pulse k with $\bar\varepsilon_m|\xi\cdot e_{(k)}| \ge c_{\mathrm{lo}}/2$. Solutions of (12.5) lose at least half of their $L^2(\mathbb{T}^2)$ norm across each (Proposition 12.6, and for $\kappa_m \ge \frac12\kappa_{m-1}$ the first case of its proof), and so do tests, as by (12.20) their map from the right end to the left end is the adjoint of the solution map. The block contains $c\kappa_m h_m/\varepsilon_m^2$ disjoint ones (the count at the end of that proof). If $\#[t',t'']$ is the largest number of disjoint halving intervals in $[t',t'']$ (and 0 if $\bar\varepsilon_m|\xi| < c_{\mathrm{lo}}$), then $\#[t',t]+\#[t,t''] \ge \#[t',t''] - 1$, and shortening $[t',t'']$ by a cell time ε_m^2/κ_m at either end lowers it by at most 2.

In Lemma 12.8, (a) and (b) say that the fibre evolution propagates and creates derivatives at the cell scale ((b) over one cell time, from a datum of negative order and with a forcing), at the cost of a power of the contrast factor, which the decay of Proposition 12.6 over a block absorbs; by (c), without pulses each cell mode evolves separately, and the part of mean zero decays with constant one. The negative norm of (b) is $\|f\|_{H^{-J'}(\mathbb{T}^2)} := \sup\{|\langle f,\vartheta\rangle_{L^2(\mathbb{T}^2)}| : \|\vartheta\|_{H^{J'}(\mathbb{T}^2)} \le 1\}$.

Lemma 12.8 (The fibre evolution smooths)**.** *For every choice of integers $J, J' \ge 0$ there exists C such that, for every real ξ in the zone, every solution U of (12.5) on $[t',t'']$ satisfies the following. Tests satisfy the same bounds with $U^\dagger(t')$ on the left side and $U^\dagger(t'')$ on the right side.*

(a) Long times.

$$\|U(t'')\|_{H^J(\mathbb{T}^2)} \le C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C 2^{-\#[t',t'']} \\ \times \min\big\{\|U(t')\|_{H^J(\mathbb{T}^2)}, (\min\{1,\kappa_m(t''-t')/\varepsilon_m^2\})^{-J/2}\|U(t')\|_{L^2(\mathbb{T}^2)}\big\}\,.$$

(b) One cell time. *Let $\kappa_m(t''-t') \le \varepsilon_m^2$, and let U have a forcing F^0, possibly zero. Then*

$$\|U(t'')\|_{H^J(\mathbb{T}^2)} \le C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C \Big(\min\Big\{\|U(t')\|_{H^J(\mathbb{T}^2)}, \Big(\frac{\kappa_m(t''-t')}{\varepsilon_m^2}\Big)^{-\frac{J+J'}{2}}\|U(t')\|_{H^{-J'}(\mathbb{T}^2)}\Big\} \\ + \int_{t'}^{t''}\|F^0(t)\|_{H^J(\mathbb{T}^2)}\,\mathrm{d}t\Big)\,.$$

(c) Quiet stretches. *If no pulse of level m is active on $[t',t'']$, then, with constant* 1,

$$(U(t''))_{\mathbb{T}^2} = \exp\Big(-\kappa_m\int_{t'}^{t''}\xi\cdot\mathbf{K}(t)\xi\,\mathrm{d}t\Big)(U(t'))_{\mathbb{T}^2}\,, \\ \big\|U(t'')-(U(t''))_{\mathbb{T}^2}\big\|_{H^J(\mathbb{T}^2)} \le e^{-\pi^2\kappa_m(t''-t')/(2\varepsilon_m^2)}\big\|U(t')-(U(t'))_{\mathbb{T}^2}\big\|_{H^J(\mathbb{T}^2)}\,.$$

Proof. (a) The $L^2(\mathbb{T}^2)$ norm does not increase, by (12.6), and halves across halving intervals. When we differentiate (12.5), only the terms in which derivatives fall on the velocity, of size $C(J)\big(1+\bar\varepsilon_m/\varepsilon_m\big)$, are neither skew nor dissipative. In cell time and in the zone, as $|2\pi\mathbf{n}+\varepsilon_m\xi| \ge \frac12|2\pi\mathbf{n}|$ for $\mathbf{n}\in\mathbb{Z}^2$ and $\mathbf{K}\ge\frac{15}{16}I$,

$$\tfrac14\|\nabla^{j+1}U\|^2_{L^2(\mathbb{T}^2)} \le \|(\nabla+\mathrm{i}\varepsilon_m\xi)\nabla^jU\|^2_{L^2(\mathbb{T}^2)} \le 2\|\nabla^{j+1}U\|^2_{L^2(\mathbb{T}^2)} + 4\pi^2\|\nabla^jU\|^2_{L^2(\mathbb{T}^2)}\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^jU\|^2_{L^2(\mathbb{T}^2)} \le -\tfrac{15}{8}\|(\nabla+\mathrm{i}\varepsilon_m\xi)\nabla^jU\|^2_{L^2(\mathbb{T}^2)} \\ + C(j)\big(1+\bar\varepsilon_m/\varepsilon_m\big)\sum_{i<j}\|(\nabla+\mathrm{i}\varepsilon_m\xi)\nabla^iU\|_{L^2(\mathbb{T}^2)}\|\nabla^jU\|_{L^2(\mathbb{T}^2)}\,.$$

So, for $t-t' \le c(J)/\big(1+\bar\varepsilon_m/\varepsilon_m\big)^2$, the weighted energy $\sum_{j\le J}8^{-j}(j!)^{-1}(t-t')^j\|\nabla^jU(t)\|^2_{L^2(\mathbb{T}^2)}$ does not increase, and the unweighted one at most doubles, by Young's inequality on the cross terms:

$$(t-t')^J\|U(t)\|^2_{H^J(\mathbb{T}^2)} \le C(J)\|U(t')\|^2_{L^2(\mathbb{T}^2)}\,, \qquad \|U(t)\|_{H^J(\mathbb{T}^2)} \le C(J)\|U(t')\|_{H^J(\mathbb{T}^2)}\,.$$

Longer intervals use the L^2 bound up to $t''-c(J)/\big(1+\bar\varepsilon_m/\varepsilon_m\big)^2$, which loses at most two halving intervals and so gives a constant multiple of $2^{-\#[t',t'']}$, and then the first of these bounds, which costs $\big(1+\bar\varepsilon_m/\varepsilon_m\big)^J$.

(b) Forcings are added by Duhamel's formula. For a datum of negative order, we pair U at the midpoint with the tests from $L^2(\mathbb{T}^2)$, by (12.20), and smooth after it.

(c) The coefficients do not depend on z: the mode $\exp(2\pi\mathrm{i}\mathbf{n}\cdot z)$, $\mathbf{n}\in\mathbb{Z}^2$, is multiplied by $\exp(-\frac{\kappa_m}{\varepsilon_m^2}\int_{t'}^{t''}(2\pi\mathbf{n}+\varepsilon_m\xi)\cdot\mathbf{K}(t)(2\pi\mathbf{n}+\varepsilon_m\xi)\,\mathrm{d}t)$, and $|2\pi\mathbf{n}+\varepsilon_m\xi|\ge\pi$ for $\mathbf{n}\ne0$, with $\mathbf{K}\ge\frac12 I$. □

Activity. A test *transverse* to a pulse, that is, independent of the coordinate $z\cdot e_{(k)}$ along its flow, has derivatives bounded without loss on the window of the pulse. The rest of a test does not: the shear turns its dependence along the flow into gradients across it, larger by at most a power of the Péclet number, which diffusion damps after about $C\log\big(1+\bar\varepsilon_m/\varepsilon_m\big)$ cell times. The activity weight

carries a power of the contrast factor on this part over that time. For t in the block, $t_{\mathrm{act}}(t)$ is the supremum of the times of the block up to t at which a pulse of level m is active, $e_{(k)}$ is the flow direction of the pulse k with $|t_{\mathrm{act}}(t) - k\tau_m| \le \frac{2}{3}\tau_m$ (any direction if there is none), and $(U)_{e_{(k)}}$ is the average of U along the lines parallel to $e_{(k)}$. For integers $J, N_{\mathrm{act}} \ge 0$, the *activity weight* and the weighted norm are

$$\begin{aligned} w_{N_{\mathrm{act}}}(t) &:= \max\Big\{1, \big(1 + \bar\varepsilon_m/\varepsilon_m\big)^{N_{\mathrm{act}}} \exp\Big(-\frac{3\pi^2\kappa_m(t - t_{\mathrm{act}}(t))}{8\varepsilon_m^2}\Big)\Big\}, \\ \|U^\dagger\|_{H^J(\mathbb{T}^2);N_{\mathrm{act}},t} &:= \big\|(U^\dagger)_{e_{(k)}}\big\|_{H^J(\mathbb{T}^2)} + w_{N_{\mathrm{act}}}(t)\big\|U^\dagger - (U^\dagger)_{e_{(k)}}\big\|_{H^J(\mathbb{T}^2)} \end{aligned} \tag{12.23}$$

(masses for fields, square sums for arrays), with $w_{N_{\mathrm{act}}}(t) := 1$ if no pulse is active in the block before t. The norm is at least $\|U^\dagger\|_{H^J(\mathbb{T}^2)}$, as $(\cdot)_{e_{(k)}}$ is an orthogonal projection in $H^J(\mathbb{T}^2)$.

In Lemma 12.9 no constant depends on the contrast. By (a), a transverse test has the bounds of a heat equation over two cell times, and from 1 it stays within a constant times $\bar\varepsilon_m|\xi|$ of 1. By (b), an off-branch test has a small cell mean and decays backward at a rate comparable to the cell rate, and in the norm (12.23) it does so with N_0 fewer powers of the contrast factor in the weight at the earlier time, which compensates for crossing a pulse. By (c), the branch test is, up to a relative error of the size of $\bar\varepsilon_m|\xi|$, the complex conjugate of the scalar evolution of the branch, run backward.

Lemma 12.9 (Three kinds of tests)**.** *For every integer $J \ge 0$ there exists an integer N_0 and, for every integer $N_{\mathrm{act}} \ge 0$, a constant C such that the following hold.*

(a) Transverse tests. *Let k be a pulse such that $\nabla^\perp\psi_m(t,\cdot)$ is a multiple of $\nabla^\perp\psi_{m,k}$ for every $t \in [t', t'']$, as it is on the window of k, and let the test $U^\dagger$ be transverse at t''. Then it is transverse on $[t', t'']$, and*

$$\|U^\dagger(t')\|_{H^J(\mathbb{T}^2)} \le \begin{cases} C\|U^\dagger(t'')\|_{H^J(\mathbb{T}^2)} & \text{if } \kappa_m(t'' - t') \le \varepsilon_m^2\,, \\ C\|U^\dagger(t'')\|_{L^2(\mathbb{T}^2)} & \text{if } \varepsilon_m^2 \le \kappa_m(t'' - t') \le 2\varepsilon_m^2\,. \end{cases}$$

If moreover $U^\dagger(t'') = 1$ and $\kappa_m(t'' - t') \le 2\varepsilon_m^2$, then

$$\|U^\dagger(t') - 1\|_{H^J(\mathbb{T}^2)} \le C\bar\varepsilon_m|\xi|\,.$$

With a transverse forcing F^0, the first two bounds hold with $C\int_{t'}^{t''} \|F^0(t)\|_{H^J(\mathbb{T}^2)}\,\mathrm{d}t$ added to their right sides.

(b) Off-branch tests. *Let $U^\dagger$ be off-branch at t'', and $N_{\mathrm{act}} \ge N''_{\mathrm{act}} \ge N'_{\mathrm{act}} + N_0$, $N'_{\mathrm{act}} \ge 0$. Then*

$$\begin{aligned} \|U^\dagger(t')\|_{L^2(\mathbb{T}^2)} &\le e^{-\pi^2\kappa_m(t''-t')/(8\varepsilon_m^2)}\|U^\dagger(t'')\|_{L^2(\mathbb{T}^2)}\,, \\ \|U^\dagger(t')\|_{H^J(\mathbb{T}^2);N'_{\mathrm{act}},t'} &\le Ce^{-\pi^2\kappa_m(t''-t')/(8\varepsilon_m^2)}\|U^\dagger(t'')\|_{H^J(\mathbb{T}^2);N''_{\mathrm{act}},t''}\,. \end{aligned}$$

(c) The branch test. *For every time t'' of the block with no pulse of level m active within $\tau_m/24$ of it, every $0 \le N'_{\mathrm{act}} \le N_{\mathrm{act}}$ and every $t \le t''$ of the block,*

$$\begin{aligned} \big\langle \mathcal{R}_\xi(t), U^\dagger_{\mathrm{br}}(t)\big\rangle_{L^2(\mathbb{T}^2)} &= \exp\Big(\int_t^{t''} \mathcal{Z}_\xi(\tilde t)\,\mathrm{d}\tilde t\Big)\,, \\ \Big\|U^\dagger_{\mathrm{br}}(t) - \overline{\exp\Big(\int_t^{t''} \mathcal{Z}_\xi(\tilde t)\,\mathrm{d}\tilde t\Big)}\Big\|_{H^J(\mathbb{T}^2);N'_{\mathrm{act}},t} &\le C\bar\varepsilon_m|\xi|\Big|\exp\Big(\int_t^{t''} \mathcal{Z}_\xi(\tilde t)\,\mathrm{d}\tilde t\Big)\Big|\,. \end{aligned}$$

Proof. Time is in cell units, so that τ_m stands for $\kappa_m\tau_m/\varepsilon_m^2$ and $\mathcal{Z}_\xi$ for $\varepsilon_m^2\mathcal{Z}_\xi/\kappa_m$; tests run backward, and "before" and "after" refer to t. On the window of k the velocity is a shear parallel to $e_{(k)}$ depending on $z\cdot e_{(k)}^\perp$ (paragraph of (10.2)); it vanishes outside the active intervals $|t-k\tau_m|\le\frac{2}{3}\tau_m$.

(a) If $U^\dagger$ is transverse, $e_{(k)}\cdot\nabla U^\dagger=0$, so on transverse functions (12.19) is the heat equation with a potential

$$-\partial_t U^\dagger=(\nabla+\mathrm{i}\varepsilon_m\xi)\cdot\mathbf{K}(\nabla+\mathrm{i}\varepsilon_m\xi)U^\dagger+\mathrm{i}\frac{\varepsilon_m^2}{\kappa_m}\xi\cdot(\nabla^\perp\psi_m)(t,\varepsilon_m z)\,U^\dagger\,,$$
$$\Big\|\nabla_z^j\Big(\frac{\varepsilon_m^2}{\kappa_m}\xi\cdot\nabla^\perp\psi_m\Big)\Big\|_{L^\infty(\mathbb{T}^2)}\le(2\pi)^j C\bar\varepsilon_m|\xi|\quad(j\ge0)\,,$$

by (10.1) and (12.12); it maps transverse functions to transverse ones, and its energy estimates over two cell times (with the standard weight $(t''-t)^J$ for the second bound) do not see the contrast; from 1, $U^\dagger-1$ is forced by $-\varepsilon_m^2\xi\cdot\mathbf{K}\xi$ plus the potential, of size $C\bar\varepsilon_m|\xi|$. With a transverse forcing F^0, Duhamel's formula gives the forced bounds: for $t'\le\tilde t\le t''$ the test from the datum $F^0(\tilde t)$ at $\tilde t$ is at most $C\|F^0(\tilde t)\|_{H^J(\mathbb{T}^2)}$ in $H^J(\mathbb{T}^2)$ at t', by the first bound if $\tilde t-t'\le1$ and by the second if $1\le\tilde t-t'\le2$.

(b), Step 1: the L^2 rate. By Propositions 12.4(a) and 12.5(c), then (12.6), (12.7) and $\mathbf{K}\ge\frac{15}{16}I$, for each component,

$$\big|(U_v^\dagger)_{\mathbb{T}^2}\big|=\big|\langle U_v^\dagger,1-\mathcal{R}_\xi\rangle_{L^2(\mathbb{T}^2)}\big|\le\frac{1}{3}\|U_v^\dagger\|_{L^2(\mathbb{T}^2)}\,,\qquad\frac{\mathrm{d}}{\mathrm{d}t}\|U_v^\dagger\|_{L^2(\mathbb{T}^2)}^2\ge\frac{15\pi^2}{8}\cdot\frac{8}{9}\|U_v^\dagger\|_{L^2(\mathbb{T}^2)}^2\,.$$

Step 2: two bounds. Let N_0 be the least integer at least the exponent C of $\big(1+\bar\varepsilon_m/\varepsilon_m\big)$ in Lemma 12.8(a) for J. For $U^\dagger$ off-branch at t'',

$$\|U^\dagger(t')\|_{H^J(\mathbb{T}^2)}\le\begin{cases}C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_0}e^{-\pi^2(t''-t')/8}\|U^\dagger(t'')\|_{H^J(\mathbb{T}^2)} & (\text{crude})\,,\\ \frac{3}{2}e^{-\pi^2(t''-t')/2}\|U^\dagger(t'')\|_{H^J(\mathbb{T}^2)} & (\text{quiet})\,,\end{cases}\tag{12.24}$$

the second if no pulse is active on $[t',t'']$: by Lemma 12.8(a) on $[t',\min\{t'+1,t''\}]$ and Step 1, and by Lemma 12.8(c), the mean being at most half the mean-free part (Step 1). An off-branch datum U at a time t_* splits along $e_{(k)}$ into off-branch parts:

$$U^{\mathrm{tr}}:=(U)_{e_{(k)}}-\big\langle(U)_{e_{(k)}},\mathcal{R}_\xi(t_*)\big\rangle_{L^2(\mathbb{T}^2)}\,,\qquad\|U-U^{\mathrm{tr}}\|_{H^J(\mathbb{T}^2)}\le\frac{7}{3}\big\|U-(U)_{e_{(k)}}\big\|_{H^J(\mathbb{T}^2)}\,,\tag{12.25}$$

componentwise, since $\langle(U)_{e_{(k)}},\mathcal{R}_\xi\rangle_{L^2(\mathbb{T}^2)}=-\langle U-(U)_{e_{(k)}},\mathcal{R}_\xi\rangle_{L^2(\mathbb{T}^2)}$ and $\|\mathcal{R}_\xi\|_{L^2(\mathbb{T}^2)}\le\frac{4}{3}$.

Step 3: the windows. Let $\hat k_0$ be the pulse of $t_{\mathrm{act}}(t'')$ (if there is none, (12.24) suffices), and for $i\ge1$ let $\hat k_i:=\hat k_0-2i$ and $\hat t_i:=(\hat k_i+\frac{11}{12})\tau_m$, a quarter of τ_m after the activity of $\hat k_i$; $\hat t_0:=t''$. On $[\hat t_{i+1},\hat t_i]$ the velocity is a multiple of $\nabla^\perp\psi_{m,\hat k_i}$, for $i\ge1$ no pulse is active on $[\hat t_i,\hat t_i+\frac{5}{12}\tau_m]$, and, as $\tau_m\ge c\varepsilon_{m-1}^{-4\delta}$ and $\big(1+\bar\varepsilon_m/\varepsilon_m\big)\le C\varepsilon_{m-1}^{-q\gamma}$ (Proposition 2.1(c)), weights of exponent at most $N_{\mathrm{act}}+N_0$ are 1 for m large where the direction in (12.23) is not $e_{(\hat k_i)}$. We normalize the right side of (b) to 1. With $U_{-1}^\dagger:=U^\dagger$, let $U_i^\dagger:=(U_{i-1}^\dagger(\hat t_i))^{\mathrm{tr}}$ along $e_{(\hat k_i)}$, by (12.25), $i\ge0$. With the same letters for the tests from these data,

$$U^\dagger(t)=U_i^\dagger(t)+\sum_{i'=0}^{i}\big(U_{i'-1}^\dagger-U_{i'}^\dagger\big)(t)\quad(t\le\hat t_i)\,,\qquad U_i^\dagger\text{ transverse to }e_{(\hat k_i)}\text{ on }[\hat t_{i+1},\hat t_i]\,.$$

Three mechanisms bound the parts at the rate $\pi^2/8$, without compounding constants.

- *The part $U^\dagger - U_0^\dagger$ not transverse at t''.* After $t_{\mathrm{act}}(t'')$ the weight grows backward at the rate $3\pi^2/8$ and the quiet bound gains $\pi^2/2 = 3\pi^2/8 + \pi^2/8$; before it, the crude loss $\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_0}$ is absorbed by $N''_{\mathrm{act}} \ge N'_{\mathrm{act}} + N_0$:

$$\begin{aligned} w_{N'_{\mathrm{act}}}(t)\, e^{-3\pi^2(t''-t)/8} &\le w_{N''_{\mathrm{act}}}(t'') \quad (t \ge t_{\mathrm{act}}(t''))\,, \\ \big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N'_{\mathrm{act}}+N_0} e^{-3\pi^2(t''-t_{\mathrm{act}}(t''))/8} &\le w_{N''_{\mathrm{act}}}(t'')\,, \end{aligned}$$

  so $w_{N'_{\mathrm{act}}}(t)\|(U^\dagger - U_0^\dagger)(t)\|_{H^J(\mathbb{T}^2)} \le Ce^{-\pi^2(t''-t)/8}$.

- *The parts $U_{i-1}^\dagger - U_i^\dagger$, $i \ge 1$,* by induction on i (the third item for $U_{i-1}^\dagger$ bounds it at the end of the stretch; the C below is fixed before the level). At $\hat t_i$, after the quiet stretch, the part of $U_{i-1}^\dagger$ not transverse to $e_{(\hat k_i)}$ is mean-free (at $\hat t_i$, $U_{i-1}^\dagger$ is transverse to $e_{(\hat k_{i-1})} \perp e_{(\hat k_i)}$, so its average along $e_{(\hat k_i)}$ is its mean) and damped by $e^{-5\pi^2\tau_m/24}$ (Lemma 12.8(c)); with the crude bound,

$$\|(U_{i-1}^\dagger - U_i^\dagger)(t)\|_{H^J(\mathbb{T}^2)} \le C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_0} e^{-5\pi^2\tau_m/32} e^{-\pi^2(t''-t)/8}\,.$$

  There are at most $C\varepsilon_{m-1}^{-2\delta}$ of them, so their sum times $\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_{\mathrm{act}}}$ is at most $e^{-\pi^2(t''-t)/8}$ for m large.

- *The transverse parts.* By (a), $\|U_i^\dagger(t)\|_{H^J(\mathbb{T}^2)}$ is at most C times $\|U_i^\dagger(t+1)\|_{L^2(\mathbb{T}^2)}$ (or times $\|U_i^\dagger(\hat t_i)\|_{H^J(\mathbb{T}^2)}$ near $\hat t_i$; by (12.25) and $\|\mathcal{R}_\xi\|_{L^2(\mathbb{T}^2)} \le \frac43$ this is at most $\frac73$ for $i=0$, and at most $\frac73\|U_{i-1}^\dagger(\hat t_i)\|_{H^J(\mathbb{T}^2)} \le C\|U_{i-1}^\dagger(\hat t_i+1)\|_{L^2(\mathbb{T}^2)}$ for $i \ge 1$, by (a), as $\hat t_{i-1} - \hat t_i \ge 1$ for m large), and $\|U_i^\dagger\|_{L^2(\mathbb{T}^2)} \le \|U^\dagger\|_{L^2(\mathbb{T}^2)} + \sum_{i'\le i}\|U_{i'-1}^\dagger - U_{i'}^\dagger\|_{H^J(\mathbb{T}^2)} \le Ce^{-\pi^2(t''-t)/8}$ by Step 1 and the two items above: every step returns to the L^2 bound. With (12.23) this gives (b).

(c), Step 1: the identity and the L^2 bound. The first line below, for $t' \le t \le t''$, follows from Proposition 12.4(c) and (12.20); we call it the identity. Let $\mathring U_{\mathrm{br}}^\dagger := U_{\mathrm{br}}^\dagger - (U_{\mathrm{br}}^\dagger)_{\mathbb{T}^2}$; it vanishes at t'' and solves the mean-free part of (12.19), forced by the mean and damped at the rate $\frac{15}{16}\pi^2$. The identity and Proposition 12.4(a) give the other two lines:

$$\begin{aligned} \exp\Big(\int_{t'}^{t} \mathcal{Z}_\xi\Big)\big\langle \mathcal{R}_\xi(t), U_{\mathrm{br}}^\dagger(t)\big\rangle_{L^2(\mathbb{T}^2)} &= \exp\Big(\int_{t'}^{t''} \mathcal{Z}_\xi\Big)\big\langle \mathcal{R}_\xi(t''), 1\big\rangle_{L^2(\mathbb{T}^2)} = \exp\Big(\int_{t'}^{t''} \mathcal{Z}_\xi\Big)\,, \\ \Big|(U_{\mathrm{br}}^\dagger(t))_{\mathbb{T}^2} - \overline{\exp\Big(\int_{t}^{t''} \mathcal{Z}_\xi\Big)}\Big| &\le C_{\mathrm{br}}\bar\varepsilon_m|\xi|\,\|\mathring U_{\mathrm{br}}^\dagger(t)\|_{L^2(\mathbb{T}^2)}\,, \\ \|\mathring U_{\mathrm{br}}^\dagger(t)\|_{L^2(\mathbb{T}^2)} &\le C\bar\varepsilon_m|\xi| \int_t^{t''} e^{-15\pi^2(\tilde t - t)/16}\big|(U_{\mathrm{br}}^\dagger(\tilde t))_{\mathbb{T}^2}\big|\,\mathrm{d}\tilde t\,. \end{aligned}$$

As the scalar factor changes at most at the rate $\pi^2/16$ (Proposition 12.5(c)), a bootstrap in the ratio of $\|\mathring U_{\mathrm{br}}^\dagger\|_{L^2(\mathbb{T}^2)}$ to it (closing since $\bar\varepsilon_m|\xi| \le \frac14$, $C_{\mathrm{br}}\bar\varepsilon_m|\xi| \le \frac13$ and the constant of (12.12) is at most $2\pi\sqrt{20}$) gives (c) in $L^2(\mathbb{T}^2)$, for every such t''.

Step 2: the pulses. If no pulse is active in the block before t'', then the velocity vanishes on the block up to t'', $U_{\mathrm{br}}^\dagger(t)$ is the constant $\overline{\exp\int_t^{t''}\mathcal{Z}_\xi}$, by the identity, and the left side of the second line of (c) vanishes. Otherwise, with $\hat k_i$, $\hat t_i$ as in (b), Step 3, let $U_{\mathrm{br},i}^\dagger$ be the branch test from the quiet time $\hat t_i$; it is transverse on $[\hat t_{i+1}, \hat t_i]$. Its difference from the conjugate scalar factor, zero at $\hat t_i$, is forced transversally with size $C\bar\varepsilon_m|\xi|$ times the scalar factor, as $|\mathcal{Z}_\xi| \le C(\bar\varepsilon_m|\xi|)^2$ by Proposition 12.4(b). By Step 1 at $t+1$ and the forced bounds of (a) on $[t, t+1]$ (from the datum 0 at $\hat t_i$ when $t+1 > \hat t_i$),

over which the scalar factor changes in modulus by at most $e^{\pi^2/16}$ (Proposition 12.5(c)), with C_{tr} independent of the contrast,

$$\Big\| U^\dagger_{\mathrm{br},i}(t) - \overline{\exp\Big(\int_t^{\hat t_i} \mathcal{Z}_\xi\Big)} \Big\|_{H^J(\mathbb{T}^2)} \le C_{\mathrm{tr}} \bar\varepsilon_m |\xi| \Big| \exp\Big(\int_t^{\hat t_i} \mathcal{Z}_\xi\Big)\Big| \qquad (\hat t_{i+1} \le t \le \hat t_i)\,.$$

The remainder $U^\dagger_{\mathrm{off},i}$ below is off-branch at $\hat t_i$, by the identity, and $U^\dagger_{\mathrm{off},0} = 0$:

$$U^\dagger_{\mathrm{off},i} \coloneqq U^\dagger_{\mathrm{br}} - \overline{\exp\Big(\int_{\hat t_i}^{t''} \mathcal{Z}_\xi\Big)}\, U^\dagger_{\mathrm{br},i}\,,$$

$$U^\dagger_{\mathrm{br}}(t) - \overline{\exp\Big(\int_t^{t''} \mathcal{Z}_\xi\Big)} = U^\dagger_{\mathrm{off},i}(t) + \overline{\exp\Big(\int_{\hat t_i}^{t''} \mathcal{Z}_\xi\Big)} \Big(U^\dagger_{\mathrm{br},i}(t) - \overline{\exp\Big(\int_t^{\hat t_i} \mathcal{Z}_\xi\Big)} \Big)\,.$$

By (b), with its constant C_{off} (exponents $N'_{\mathrm{act}} + N_0$ at $\hat t_i$, where the weight is 1, and N'_{act} at t; at most $N_{\mathrm{act}} + N_0$, for which the level is taken), and Proposition 12.5(c),

$$\|U^\dagger_{\mathrm{off},i}(t)\|_{H^J(\mathbb{T}^2);N'_{\mathrm{act}},t} \le 2C_{\mathrm{off}} e^{-\pi^2(\hat t_i - t)/16}\, \|U^\dagger_{\mathrm{off},i}(\hat t_i)\|_{H^J(\mathbb{T}^2)}\, \frac{|\exp\int_t^{t''} \mathcal{Z}_\xi|}{|\exp\int_{\hat t_i}^{t''} \mathcal{Z}_\xi|}\,.$$

Step 3: the induction. As $\hat t_i - \hat t_{i+1} \ge \frac{5}{12}\tau_m$, for m large:

$$\begin{aligned}
&\|U^\dagger_{\mathrm{off},i}(\hat t_i)\|_{H^J(\mathbb{T}^2)} \le 4C_{\mathrm{tr}} \bar\varepsilon_m |\xi| \Big| \exp\Big(\int_{\hat t_i}^{t''} \mathcal{Z}_\xi\Big)\Big| && \text{(true for } i = 0) \\
\Longrightarrow\ &\text{(c) at } t \text{ holds with } \big(8C_{\mathrm{off}} C_{\mathrm{tr}} e^{-\pi^2(\hat t_i - t)/16} + 2C_{\mathrm{tr}}\big) \text{ for } C && \text{on } [\hat t_{i+1}, \hat t_i], \text{ by Step 2} \\
\Longrightarrow\ &\|U^\dagger_{\mathrm{off},i+1}(\hat t_{i+1})\|_{H^J(\mathbb{T}^2)} \le 4C_{\mathrm{tr}} \bar\varepsilon_m |\xi| \Big| \exp\Big(\int_{\hat t_{i+1}}^{t''} \mathcal{Z}_\xi\Big)\Big| && \text{at } t = \hat t_{i+1}, \text{ as } 4C_{\mathrm{off}} e^{-5\pi^2\tau_m/192} \le 1.
\end{aligned}$$

□

Insertions against tests. In a Duhamel term an insertion acts at each time on the previous term, and the result is evolved by the frozen operator; by (12.20), the pairing of the next term with a test is therefore an integral in time of the pairing of the previous term with the adjoint insertion $\mathcal{I}^*$ applied to the test. So what matters is what $\mathcal{I}^*$ does to a test at one time: how many derivatives it takes, and how large a multiple of the branch test its output contains. The pairing with $\mathcal{R}_\xi$ measures that multiple; the rest of the output is orthogonal to $\mathcal{R}_\xi$, and the test it starts decays at the cell rate (Lemma 12.9(b)). The insertion acts at the cell rate, and its pairing with $\mathcal{R}_\xi$ is at most $C\kappa_m/\varepsilon_m^2$ times the factor $\bar\varepsilon_m(|\xi| + \rho(m)^{-1})$, the corrector amplitude divided by the length on which the field varies. Indeed, paired with 1, the outer divergence of the insertion leaves only $\varepsilon_m\xi$ times a cell mean, $\mathcal{R}_\xi$ differs from 1 by the size of the corrector, and the commutators of (12.21) carry the corrector amplitude, or ε_m, divided by $\rho(m)$. If the test is itself nearly constant, as the branch test is, the innermost derivative of the insertion gives a second factor, at most the same one. The factor, and the rate at which an insertion acts when it carries the factor twice, satisfy

$$\bar\varepsilon_m\Big(|\xi| + \frac{1}{\rho(m)}\Big) \le 1\,, \qquad \kappa_{m-1}|\xi|^2 + \frac{1}{h_m} \le \frac{\kappa_m}{\varepsilon_m^2}\,\bar\varepsilon_m^2\Big(|\xi| + \frac{1}{\rho(m)}\Big)^2 \le 2\Big(\kappa_{m-1}|\xi|^2 + \frac{1}{h_m}\Big) \tag{12.26}$$

by (10.5) for m large, which gives $\bar\varepsilon_m/\rho(m) \le \frac34$, by $\bar\varepsilon_m|\xi| \le 2c_{\mathrm{lo}} \le \frac14$, and, for the bounds on the rate, by the identities of the Conventions. So over a block the rate (12.26) integrates to at most

$2(1+\rho(m)|\xi|)^2$. A single factor is the cost of passing between the branch test and the tests orthogonal to $\mathcal{R}_\xi$. These decay faster than the scalar factor of the branch changes (Proposition 12.5(c)), so leaving the branch and returning cost the factor twice over about one cell time, at most a constant times the cost of one cell time on the branch.

Lemma 12.10(a),(b) makes these bounds precise, and (c) is the forward bound used in Lemma 12.12(iii). The map $\mathcal{I}^*$ keeps the norm (12.23), as, near a pulse, insertions do not depend on the coordinate along its flow.

Lemma 12.10 (Insertions against tests)**.** *Consider the moment terms of a Duhamel term on $[t_1,t_2]$, with moment order N_1. Let a group and a lower group be given, let d be the order of the terms of (12.21) that connect them (the order of the insertion that connects them, or 1 for the commutator terms), and let $J, N_{\mathrm{act}} \geq 0$ be integers, with $J+d-1$ at most the derivative counts of Subsection 12.6. There exists C such that, for almost every $t \in [t_1,t_2]$, every real ξ in the zone, every array $U^\dagger$ of fields in $H^{J+d}(\mathbb{T}^2)^V$ indexed by the group and every $0 \leq N'_{\mathrm{act}} \leq N_{\mathrm{act}}$, the following hold.*

(a) Mapping, and the pairing with $\mathcal{R}_\xi$. *The first bound below holds; if moreover $\bar\varepsilon_m|\xi| \leq 2c_{\mathrm{lo}}$ and $J+d \geq \max\{d_1,\dots,d_N\}+3$, so does the second:*

$$\|\mathcal{I}^*U^\dagger\|_{H^J(\mathbb{T}^2);N'_{\mathrm{act}},t} \leq C\frac{\kappa_m}{\varepsilon_m^2}\|U^\dagger\|_{H^{J+d}(\mathbb{T}^2);N'_{\mathrm{act}},t}\,,$$
$$\big|\langle \mathcal{R}_\xi(t), \mathcal{I}^*U^\dagger\rangle\big| \leq C\frac{\kappa_m}{\varepsilon_m^2}\bar\varepsilon_m\Big(|\xi|+\frac{1}{\rho(m)}\Big)\|U^\dagger\|_{H^{J+d}(\mathbb{T}^2)}\,.$$

(b) The branch test. *Let $\bar\varepsilon_m|\xi| \leq 2c_{\mathrm{lo}}$, let $t'' \geq t$ be a time as in Lemma 12.9(c), and let every entry of $U^\dagger$ be the branch test of that lemma at t times a constant vector of $\mathbb{C}^V$, these vectors having square sum one in $|\cdot|_\omega$. Then*

$$\begin{aligned}\big|\langle \mathcal{R}_\xi(t), \mathcal{I}^*U^\dagger\rangle\big| &\leq C\Big(\kappa_{m-1}|\xi|^2+\frac{1}{h_m}\Big)\Big|\exp\Big(\int_t^{t''}\mathcal{Z}_\xi(\tilde t)\,\mathrm{d}\tilde t\Big)\Big|\,,\\ \|\mathcal{I}^*U^\dagger\|_{H^J(\mathbb{T}^2);N'_{\mathrm{act}},t} &\leq C\frac{\kappa_m}{\varepsilon_m^2}\bar\varepsilon_m\Big(|\xi|+\frac{1}{\rho(m)}\Big)\Big|\exp\Big(\int_t^{t''}\mathcal{Z}_\xi(\tilde t)\,\mathrm{d}\tilde t\Big)\Big|\,.\end{aligned} \tag{12.27}$$

(c) Forward. *For every array U' in $H^{J+d}(\mathbb{T}^2)^V$ indexed by the lower group and every (j,i) in the group,*

$$\Big\|\sum_{(j',i')}\mathcal{I}_{(j,i),(j',i')}(t)U'_{(j',i')}\Big\|_{H^J(\mathbb{T}^2)} \leq C\frac{\kappa_m}{\varepsilon_m^2}\|U'\|_{H^{J+d}(\mathbb{T}^2)}\,.$$

Proof. We bound one entry and sum by Schur's test in the components with the weighted row and column sums of the matrices of the strengths, so that C does not depend on the number of components. Recall $\varepsilon_m \leq \bar\varepsilon_m$, by Proposition 2.1(c), and $\bar\varepsilon_m/\rho(m) \leq \frac34$ (paragraph of (12.26)).

(a), the mapping, and (c). In cell time the insertion part of an entry of $\mathcal{I}^*$ is, up to sign and terms with $\varepsilon_m/\rho(m)$ in place of a derivative,

$$\sum_{d'\leq d}(\nabla+\mathrm{i}\varepsilon_m\xi)^{d'-1}\cdot\big(\mathcal{A}^*_{d'}(\nabla+\mathrm{i}\varepsilon_m\xi)\cdot\big)\,,$$

with coefficients bounded with their z-derivatives by clause (ii) of the insertions and $\rho(m) \leq C\varepsilon_{m-1}$; the commutator part has coefficients $\varepsilon_m/\rho(m)$, $(\varepsilon_m/\rho(m))^2$ and at most $C\bar\varepsilon_m/\rho(m)$, with their z-derivatives, by (12.12) and the display in the proof of Lemma 12.9(a). The forward operator of (c) is the adjoint of this one, with the same conjugated matrix and at most $J+1 \leq J+d-1$

derivatives on the coefficients of an insertion. For the weights, let k be the pulse of (12.23) at t, with $|t_{\mathrm{act}}(t) - k\tau_m| \le \frac{2}{3}\tau_m$. Where the activity weight exceeds 1, t lies within $C\log\big(1+\bar\varepsilon_m/\varepsilon_m\big)$ cell times after the activity of k, so $|t-k\tau_m| \le \frac{3}{4}\tau_m$ for m large, and there the coefficients do not depend on $z\cdot e_{(k)}$, by item (i) of the insertions, the velocity being parallel to $e_{(k)}$; hence

$$\big\|\mathcal{I}^*U^\dagger - (\mathcal{I}^*U^\dagger)_{e_{(k)}}\big\|_{H^J(\mathbb{T}^2)} \le C\frac{\kappa_m}{\varepsilon_m^2}\big\|U^\dagger - (U^\dagger)_{e_{(k)}}\big\|_{H^{J+d}(\mathbb{T}^2)}\,.$$

Where the weight is 1, the bound is the plain one, by the last sentence after (12.23); elsewhere $\mathcal{I}^*$ commutes with the average along $e_{(k)}$.

(a), the pairing with $\mathcal{R}_\xi$. For one entry, the mean of a divergence is that of its zeroth-order part:

$$\big\langle \mathcal{R}_\xi, \mathcal{I}^*U^\dagger\big\rangle = \overline{(\mathcal{I}^*U^\dagger)_{\mathbb{T}^2}} + \big\langle \mathcal{R}_\xi - 1, \mathcal{I}^*U^\dagger\big\rangle\,, \qquad \big((\nabla + \mathrm{i}\varepsilon_m\xi)\cdot F\big)_{\mathbb{T}^2} = \mathrm{i}\varepsilon_m\xi\cdot(F)_{\mathbb{T}^2}\,,$$

for a vector field F on $\mathbb{T}^2$, and the other terms carry $\varepsilon_m/\rho(m)$, $\bar\varepsilon_m/\rho(m)$ or $C_{\mathrm{br}}\bar\varepsilon_m|\xi|$. The first bound with $J=0$ and $N'_{\mathrm{act}}=0$ gives $\|\mathcal{I}^*U^\dagger\|_{L^2(\mathbb{T}^2)} \le C\frac{\kappa_m}{\varepsilon_m^2}\|U^\dagger\|_{H^d(\mathbb{T}^2)}$, so only $\|U^\dagger\|_{H^d(\mathbb{T}^2)}$ appears.

(b) Let a be the array of the constant vectors, so that

$$U^\dagger = a\Big(U^\dagger_{\mathrm{br}}(t) - \overline{\exp\Big(\int_t^{t''}\mathcal{Z}_\xi\Big)}\Big) + a\,\overline{\exp\Big(\int_t^{t''}\mathcal{Z}_\xi\Big)}\,.$$

By the first bound of (a) and Lemma 12.9(c) at the order $J+d$ with the exponent N'_{act}, the image under $\mathcal{I}^*$ of the first part is at most $C\frac{\kappa_m}{\varepsilon_m^2}\bar\varepsilon_m|\xi|\,|\exp\int_t^{t''}\mathcal{Z}_\xi|$ in the norm $\|\cdot\|_{H^J(\mathbb{T}^2);N'_{\mathrm{act}},t}$. On the second part, the innermost derivative of an insertion term gives $\mathrm{i}\varepsilon_m\xi$, or $\varepsilon_m/\rho(m)$ stands in its place; the commutator terms carry $(\varepsilon_m/\rho(m))\varepsilon_m|\xi|$, $C\bar\varepsilon_m/\rho(m)$ and $(\varepsilon_m/\rho(m))^2$. So the image of the second part is at most $C\frac{\kappa_m}{\varepsilon_m^2}\bar\varepsilon_m(|\xi| + \frac{1}{\rho(m)})|\exp\int_t^{t''}\mathcal{Z}_\xi|$ in $H^J(\mathbb{T}^2)$; where the weight exceeds 1 this field is invariant along $e_{(k)}$, so its weighted norm is at most $\sqrt{2}$ times this in every case. This is the second line of (12.27). For the first, in the mean a term of order d' carries $d'\ge 2$ quantities among $\varepsilon_m|\xi|$, $\varepsilon_m/\rho(m)$ and the bound $C\bar\varepsilon_m|\xi|\,|\exp\int_t^{t''}\mathcal{Z}_\xi|$ of $(\nabla+\mathrm{i}\varepsilon_m\xi)U^\dagger_{\mathrm{br}}(t)$ given by Lemma 12.9(c). The commutators carry $(\varepsilon_m/\rho(m))\varepsilon_m|\xi|$ and $(\varepsilon_m/\rho(m))^2$, and the drift, of mean zero against a constant, carries $(\bar\varepsilon_m/\rho(m))\bar\varepsilon_m|\xi|$; against $\mathcal{R}_\xi - 1$ the second line gains $C_{\mathrm{br}}\bar\varepsilon_m|\xi|$. We conclude with (12.26).

The derivative count. In $H^J(\mathbb{T}^2)$, the product rule puts at most $J+d-1$ derivatives on the coefficients of an insertion of order d, as $d-1$ of the derivatives stand outside them. □

12.8 Duhamel terms over a renewal block

We follow one Duhamel term of the frozen operator, centred at y_0, over a segment that ends at a block boundary, and look at the Fourier transform of its last term at the end of the segment, at the low frequencies. There, by (12.22), it is ε_m^2 times the pairing with the branch test, which is 1 at the end, and Lemma 12.11 follows this pairing across the segment.

Over a complete block, Lemma 12.12 splits the term into a *model* $\Pi_{\mathrm{blk}}g$, smooth at the scale $\rho(m)$, and an error smaller than any power of ε_{m-1}; the model is the last term at the end of the block, cut off smoothly to low frequencies, which is its part in the principal branch. The low-frequency cutoff is $\mathsf{H}^{\mathrm{lo}}_m(\xi) := \mathsf{H}(2 - \bar\varepsilon_m|\xi|/c_{\mathrm{lo}})$, where H is the smooth step of (2.9) and c_{lo} the absolute constant of Proposition 12.5: the cutoff is 1 where $\bar\varepsilon_m|\xi| \le c_{\mathrm{lo}}$ and 0 where $\bar\varepsilon_m|\xi| \ge 2c_{\mathrm{lo}}$. For a field F on $\mathbb{R}^2$ (or any function of ξ with values in $\mathbb{C}^V$ in place of $\widehat F$) and a datum g, we write

$$|\widehat F|_{\rho(m),\omega,N_1;\xi} := \sum_{j\le N_1}\frac{|\nabla_\xi^j\widehat F(\xi)|_\omega}{\rho(m)^j\,j!}\,, \qquad \|g\|_{\rho(m),\omega,N';\xi} := \sum_{|i|\le N'}\Big\|\mathcal{B}\Big(\Big(\frac{y-y_0}{\rho(m)}\Big)^i g\Big)(\xi,\cdot)\Big\|_{L^2_\omega(\mathbb{T}^2)}\,,$$

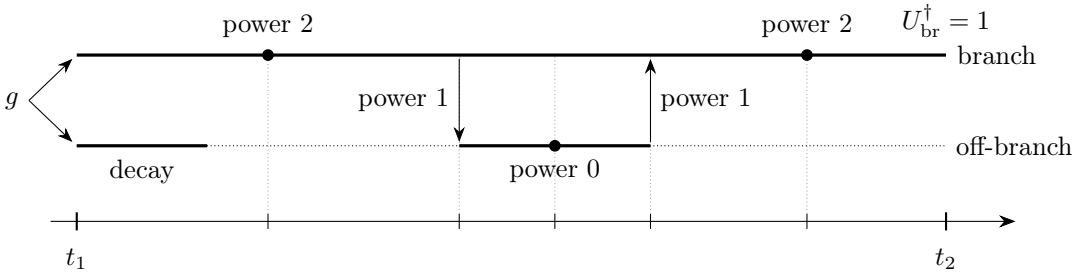


Figure 12.2: The proof of Lemma 12.11 in cell time. Tracks: pairings with the branch test (1 at t_2) and with off-branch tests; arrows from g: the split of the datum; ticks: insertion times; dots: insertions within a track; vertical arrows: passages between the tracks; labels: the power of $\bar{\varepsilon}_m(|\xi| + \rho(m)^{-1})$ each costs; thick lower segments: decay at the cell rate.

with ∇^j_ξ the array of the derivatives of order j in ξ, and $|c|^2_\omega := \sum_v \omega_v^{-1}|c_v|^2$ for $c \in \mathbb{C}^V$; the second is the *moment data* of g at ξ.

Without insertions, the left side of (12.28) is, up to the corrector, the Fourier transform of the datum, whose frequency derivatives are its moments. By Lemma 12.11, insertions of any orders change this only by a polynomial factor in $\rho(m)|\xi|$, with no power of the contrast, on a segment that may be a whole block or arbitrarily short. In the proof (Figure 12.2) we pair the moment terms with the branch test and with off-branch tests: insertions act on the branch with the square of the factor $\bar{\varepsilon}_m(|\xi| + \rho(m)^{-1})$ of (12.26), a passage between the branch and the off-branch part costs the factor once, and off-branch tests decay at the cell rate, so a triangular Grönwall inequality closes with a polynomial loss in $\rho(m)|\xi|$. The part of the datum off the branch is read by tests of norm at most one, which is why no derivative of the datum enters.

Lemma 12.11 (The Fourier transform on a segment)**.** *Let $t_2 := (l + \frac{1}{2})h_m$, $t_1 \in [t_2 - h_m, t_2]$ and $N_1 \geq 0$ an integer, and let $U^{[0]}, \dots, U^{[N]}$ be a Duhamel term on $[t_1, t_2]$, centred at y_0, from a datum $g \in H^{d_1+\cdots+d_N-2N}(\mathbb{R}^2)^V$ with compact support, with N' as for the moment terms. There exist C and an integer N_2 such that the function of ξ inside the norm on the left of (12.28) is N_1 times differentiable and*

$$\Big|\exp\Big(-\int_{t_1}^{t_2} \mathcal{Z}_\xi(t)\,\mathrm{d}t\Big)\mathsf{H}^{\mathrm{lo}}_m \widehat{U^{[N]}(t_2)}\Big|_{\rho(m),\omega,N_1;\xi} \leq C(1 + \rho(m)|\xi|)^{N_2}\varepsilon_m^2 \|g\|_{\rho(m),\omega,N';\xi}\,. \tag{12.28}$$

Proof. Fix ξ, and let $\mathcal{B}U^{[j,i]}(t)$ be the Bloch sums at ξ of the moment terms with N_1.

Step 1: Coordinates. Let $U^\dagger_{\mathrm{br}}$ be the branch test of Lemma 12.9(c) with $t'' = t_2$, which lies in the quiet margin at the end of the block. Each moment term (j, i) gets a test order $N_{j,i}$ and an exponent of the weight, fixed in Step 7; the norm (12.23) of this order and exponent is the *weighted norm* of (j, i). The *branch coordinate* and the *off-branch seminorm* of (j, i) at t are, with the supremum over the $U^\dagger$ off-branch at t of weighted norm at most one,

$$|\mathcal{B}U^{[j,i]}(t)|_{\mathrm{br},t} := \frac{|\langle \mathcal{B}U^{[j,i]}(t), U^\dagger_{\mathrm{br}}(t)\rangle|}{|\exp\int_{t_1}^{t_2}\mathcal{Z}_\xi|}\,, \quad \|\mathcal{B}U^{[j,i]}(t)\|_{\mathrm{off},t} := \frac{\sup_{U^\dagger}|\langle \mathcal{B}U^{[j,i]}(t), U^\dagger\rangle_{L^2_\omega(\mathbb{T}^2)}|}{|\exp\int_{t_1}^{t}\mathcal{Z}_\xi|}\,;$$

the scalar factors normalize the branch: on the solution $\mathcal{R}_\xi(t)\exp(\int_{t_1}^t \mathcal{Z}_\xi)$ the branch coordinate is 1 and the off-branch seminorm 0.

Step 2: The branch inequality. Fix a constant vector of $\mathbb{C}^V$ of norm one in $|\cdot|_\omega$. By (12.20), the derivative of the pairing of $\mathcal{B}U^{[j,i]}$ with $U^\dagger_{\rm br}$ times this vector is the sum over the lower (j',i') of the pairings of $\mathcal{B}U^{[j',i']}$ with $\mathcal{I}^*_{(j,i),(j',i')}$ applied to it. We split the latter, component by component, into a multiple of $U^\dagger_{\rm br}(t)$ and a rest off-branch at t: by Lemmas 12.9(c) and 12.10(b) and the chain in (12.26), the multiples are at most C times the square of the factor $\bar\varepsilon_m(|\xi|+\rho(m)^{-1})$, and the rest at most C times the factor times $|\exp\int_t^{t_2}\mathcal{Z}_\xi|$ in the weighted norm of (j',i'). As the scalar factors multiply, integrating gives, with the sum over the terms on the right of (12.21),

$$|\mathcal{B}U^{[j,i]}(t)|_{{\rm br},t} \le |\mathcal{B}U^{[j,i]}(t_1)|_{{\rm br},t_1} + C\int_{t_1}^t \sum_{(j',i')} \Big(\bar\varepsilon_m^2\Big(|\xi|+\frac{1}{\rho(m)}\Big)^2 |\mathcal{B}U^{[j',i']}(t')|_{{\rm br},t'} + \bar\varepsilon_m\Big(|\xi|+\frac{1}{\rho(m)}\Big)\|\mathcal{B}U^{[j',i']}(t')\|_{{\rm off},t'}\Big)\,{\rm d}t'\,.$$

Step 3: The off-branch inequality. Let $U^\dagger$ be the test on $[t_1,t]$ from a field off-branch at t of weighted norm at most one for (j,i). By Lemma 12.9(b) it stays off-branch, and, for every term (j',i') on the right of (12.21), its norm of order $N_{j,i}$ and exponent $N_{j',i'}N_0$ at t' is at most $Ce^{-\pi^2(t-t')/8}$. By (12.20) its pairing with $\mathcal{B}U^{[j,i]}(t)$ is that at t_1 plus the integral of the pairings of the lower terms with $\mathcal{I}^*U^\dagger$. We split $\mathcal{I}^*U^\dagger(t')$ as in Step 2: by Lemma 12.10(a), the multiples of $U^\dagger_{\rm br}(t')$ are at most C times the factor $\bar\varepsilon_m(|\xi|+\rho(m)^{-1})$ times $e^{-\pi^2(t-t')/8}/|\exp\int_{t'}^{t_2}\mathcal{Z}_\xi|$, and the rest is at most $Ce^{-\pi^2(t-t')/8}$ in the weighted norm. With $|\exp(-\int_{t'}^t\mathcal{Z}_\xi)|\le e^{\pi^2(t-t')/16}$ (Proposition 12.5(c)),

$$\|\mathcal{B}U^{[j,i]}(t)\|_{{\rm off},t} \le Ce^{-\pi^2(t-t_1)/16}\|\mathcal{B}U^{[j,i]}(t_1)\|_{L^2_\omega(\mathbb{T}^2)} + C\int_{t_1}^t e^{-\pi^2(t-t')/16}\sum_{(j',i')}\Big(\bar\varepsilon_m\Big(|\xi|+\frac{1}{\rho(m)}\Big)|\mathcal{B}U^{[j',i']}(t')|_{{\rm br},t'} + \|\mathcal{B}U^{[j',i']}(t')\|_{{\rm off},t'}\Big)\,{\rm d}t'\ :$$

the coupling between off-branch parts has no small coefficient.

Step 4: Data. For $j\ge 1$ the moment terms vanish at t_1. For $j=0$, $\|U^\dagger_{\rm br}(t_1)\|_{L^2(\mathbb{T}^2)}\le C|\exp\int_{t_1}^{t_2}\mathcal{Z}_\xi|$ by Lemma 12.9(c), and the norm (12.23) is at least the L^2 norm; so

$$|\mathcal{B}U^{[0,i]}(t_1)|_{{\rm br},t_1} + \|\mathcal{B}U^{[0,i]}(t_1)\|_{{\rm off},t_1} \le C\|g\|_{\rho(m),\omega,N';\xi}\,.$$

Step 5: The triangular Grönwall inequality. We claim that, at t, the branch coordinate and the off-branch seminorm of (j,i) are at most $\|g\|_{\rho(m),\omega,N';\xi}$ times, respectively,

$$C_{j,i}\Big(1+\bar\varepsilon_m^2\Big(|\xi|+\frac{1}{\rho(m)}\Big)^2(t-t_1)\Big)^{N_{j,i}}, \qquad \text{the same times } \Big(\bar\varepsilon_m\Big(|\xi|+\frac{1}{\rho(m)}\Big)+e^{-\pi^2(t-t_1)/32}\Big),$$

$C_{j,i}$ depending only on $N_{j,i}$ and the fixed integers. The off-branch seminorm, a supremum over tests that change with t, is not integrated in t: Steps 2–3 hold with any continuous functions above the lower quantities. We prove the claim by induction on $N_{j,i}$, inserting the majorants of the lower terms, which are the terms on the right of (12.21), of test orders lower by at least one (Step 7). In Step 2 the square of the factor times the first majorant and the factor times the second integrate to at most the first majorant, as $\int_0^\infty e^{-\pi^2t'/32}\,{\rm d}t'\le 4$ and the factor is at most 1. In Step 3 the integrand is the same power times twice the factor plus $e^{-\pi^2(t'-t_1)/32}$, which against $e^{-\pi^2(t-t')/16}$ gives at most the second majorant. Over the segment $t-t_1\le\kappa_m h_m/\varepsilon_m^2$, and $t-t_1$ times the square of the factor is at most $2(1+\rho(m)|\xi|)^2$ by (12.26).

Step 6: Output. The function of ξ in (12.28) is N_1 times differentiable, as $U^{[N]}(t_2)$ has N_1+2 moments and $\int_{t_1}^{t_2}\mathcal{Z}_\xi$ is analytic on the support of $\mathsf{H}^{\rm lo}_m$ (Proposition 12.4). At t_2 the branch test is 1,

so, by (12.22) and Step 5, for $|i| \le N_1$,

$$\rho(m)^{-|i|}\big|\nabla_\xi^i \widehat{U^{[N]}(t_2)}(\xi)\big|_\omega = \varepsilon_m^2 \big|\langle \mathcal{B}U^{[N,i]}(t_2), 1\rangle\big| \le C\varepsilon_m^2 \Big|\exp\Big(\int_{t_1}^{t_2} \mathcal{Z}_\xi\Big)\Big| (1+\rho(m)|\xi|)^C \|g\|_{\rho(m),\omega,N';\xi}\,.$$

By Leibniz's rule, the derivatives in (12.28) are sums of products of these, of derivatives of $\mathsf{H}_m^{\mathrm{lo}}$, bounded at the scale $\rho(m)$ since $\bar\varepsilon_m \le \rho(m)$, and of derivatives of $\exp(-\int_{t_1}^{t_2} \mathcal{Z}_\xi)$, normalized as in (12.28), at most $C(1+\rho(m)|\xi|)^{N_1} |\exp(-\int_{t_1}^{t_2} \mathcal{Z}_\xi)|$ by Proposition 12.5(c); the factors $|\exp(\pm\int_{t_1}^{t_2} \mathcal{Z}_\xi)|$ cancel, and N_2 is the power above plus N_1.

Step 7: Test orders. The moment term (j,i) gets the *test order* $N_{j,i} := \max\{d_1,\dots,d_N\}+3+d_1+\cdots+d_j$ plus its degree, and the exponent of the weight $N_{j,i}N_0$, with N_0 the largest of the integers of Lemma 12.9(b) for these orders (the statement of (b) holds with any larger integer); Lemmas 12.9 and 12.10 are applied with N_{act} the largest of these exponents. Along (12.21) the test order drops by at least the order d of the connecting terms: by at least d_j from (j,i) to the terms $(j-1,i')$, as $|i'|$ is at most $|i|$ plus the Taylor degree of $\mathcal{I}_j$, and by at least 1 to the commutator terms. So Lemma 12.10 applies to tests of order $N_{j,i}$ with $J = N_{j',i'}$, the test order of the lower term, and the exponents drop by at least N_0, as Step 3 requires. Lemma 12.10 differentiates the coefficients at most $J+d-1 \le N_{j,i}-1 \le \max\{d_1,\dots,d_N\}+2+d_1+\cdots+d_N+N'$ times, within the derivative counts of Subsection 12.6; the constants $C_{j,i}$ are fixed in the order of the test orders. □

Over a complete block the scalar factor $|\exp\int_{t_1}^{t_2} \mathcal{Z}_\xi|$ is at most the Gaussian of Proposition 12.5(b), and Lemma 12.11 gives (ii): the model $\Pi_{\mathrm{blk}}g$ is smooth at the scale $\rho(m)$ and localized near y_0 by its moments. Item (iii) bounds the rest, which the block damps at the cell rate, with losses in powers of $\big(1+\bar\varepsilon_m/\varepsilon_m\big)$ and of the number of cell times, and with derivatives taken forward over the last cell time; the damping, smaller than any power of ε_{m-1}, absorbs these losses.

Lemma 12.12 (Duhamel terms over a renewal block)**.** *Let $N_1 \ge 0$ be an integer, and let $U^{[0]},\dots,U^{[N]}$ be a Duhamel term over the block $[t_1,t_2] = [(l-\frac12)h_m,(l+\frac12)h_m]$, centred at y_0, with N' and the class $\mathcal{D}_{N'+2}$ of its data as for the moment terms. There exist C, $c>0$, an integer N_2 and a linear map Π_{blk}, independent of the datum g, such that:*

(i) *for every $g \in \mathcal{D}_{N'+2}$, $\Pi_{\mathrm{blk}}g$ is smooth, and $\widehat{\Pi_{\mathrm{blk}}g}(\xi) = 0$ for every ξ with $\bar\varepsilon_m|\xi| > 2c_{\mathrm{lo}}$;*

(ii) *for every $g \in \mathcal{D}_{N'+2}$ and every real ξ,*

$$|\widehat{\Pi_{\mathrm{blk}}g}|_{\rho(m),\omega,N_1;\xi} \le C(1+\rho(m)|\xi|)^{N_2} e^{-c\rho(m)^2|\xi|^2} \varepsilon_m^2 \|g\|_{\rho(m),\omega,N';\xi}\,;$$

(iii) *for every $g \in \mathcal{D}_{N'+2} \cap H^{N_1+d_1+\cdots+d_N-2N}(\mathbb{R}^2)^V$,*

$$\begin{aligned}&\|U^{[N]}(t_2) - \Pi_{\mathrm{blk}}g\|_{H^{N_1}_{\rho_{\mathrm{mol}}(m),\omega,N_1}(\mathbb{R}^2)}\\ &\quad\le C\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^{N_2}\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_2}\exp\Big(-\frac{c\kappa_m\tau_m}{\varepsilon_m^2}\Big)\|g\|_{L^2_{\omega,N'}(\mathbb{R}^2)} \le C\varepsilon_{m-1}^{Q+3}\|g\|_{L^2_{\omega,N'}(\mathbb{R}^2)}\,.\end{aligned}$$

Proof. The map. For $g \in \mathcal{D}_{N'+2}$ let $\widehat{\Pi_{\mathrm{blk}}g}(\xi)$ be $\mathsf{H}_m^{\mathrm{lo}}(\xi)^{N'+1}$ times $\widehat{U^{[N]}(t_2)}(\xi)$. It is linear, with the Fourier support of (i), and smooth since its Fourier transform has compact support.

(ii). Lemma 12.11 over the block, times $\mathsf{H}_m^{\mathrm{lo}}(\xi)^{N'}$, bounds the derivatives of $\exp(-\int_{t_1}^{t_2} \mathcal{Z}_\xi)\widehat{\Pi_{\mathrm{blk}}g}(\xi)$ for compactly supported $g \in H^{d_1+\cdots+d_N-2N}(\mathbb{R}^2)^V$; those of $\exp(\int_{t_1}^{t_2} \mathcal{Z}_\xi)$ of order j, times $\rho(m)^{-j}$, are at most $C(j)(1+\rho(m)|\xi|)^j e^{-c_{\mathrm{dis}}\rho(m)^2|\xi|^2}$ by Proposition 12.5(b),(c), and Leibniz's rule gives

the bound with $c = c_{\mathrm{dis}}$, the power of $1+\rho(m)|\xi|$ rising by N_1. It extends to $g \in \mathcal{D}_{N'+2}$ by the limit $g^{(r)} := \chi((y-y_0)/r)g \to g$ in $\mathcal{D}_{N'+2}$ as $r \to \infty$, χ a smooth cutoff equal to 1 near 0: by the moment-weighted energy, $U^{[N]}_{g^{(r)}}(t_2) \to U^{[N]}_g(t_2)$ in $L^2_{\omega,N_1+2}(\mathbb{R}^2)$, so the Fourier transforms converge in $C^{N_1}(\mathbb{R}^2)$; and the moment data of $g^{(r)}$ converge to those of g at every ξ (paragraph of (12.9), with two extra moments).

(iii), Step 1: Against tests. Let ξ be in the zone and $N_{j,i}$ the test orders of the proof of Lemma 12.11. For every moment term, $t \in [t_1,t_2]$ and $U^\dagger \in H^{N_{j,i}}(\mathbb{T}^2)^V$, no orthogonality required,

$$\big|\big\langle \mathcal{B}U^{[j,i]}(t), U^\dagger\big\rangle_{L^2_\omega(\mathbb{T}^2)}\big| \le C\Big(1+\frac{\kappa_m(t-t_1)}{\varepsilon_m^2}\Big)^C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C 2^{-\#[t_1,t]}\|U^\dagger\|_{H^{N_{j,i}}(\mathbb{T}^2)}\|g\|_{\rho(m),\omega,N';\xi}\,. \tag{12.29}$$

Indeed, by (12.20) the pairing is that at t_1 with the test from $U^\dagger$, plus the integral of the pairings of the lower moment terms with $\mathcal{I}^*$ of the test, which Lemmas 12.8(a) and 12.10(a) bound by $C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C 2^{-\#[t',t]}$ in $H^{N_{j',i'}}(\mathbb{T}^2)$. By induction, with $2^{-\#[t_1,t']}2^{-\#[t',t]} \le 2\cdot 2^{-\#[t_1,t]}$, each pairing is a nested integral over a simplex of insertion times, of volume polynomial in the cell duration, with every derivative on a smooth test: there is no short-time singularity.

Step 2: The full field. On $N+N'+1$ equal parts of the last cell time of the block, Lemma 12.8(b) smooths the moment terms one after the other, from the negative order of (12.29) for the part without forcing and with the forcing for the Duhamel part, to the *forward order* of (j,i): $N_1+d_{j+1}+\cdots+d_N$ plus N_1 plus the Taylor degrees of $\mathcal{I}_{j+1},\dots,\mathcal{I}_N$ minus $|i|$. The rank of (j,i), j plus its degree, lies in $\{0,\dots,N+N'\}$ and drops by at least one along every arrow of (12.21), as the degree does not increase from (j,i) to the terms $(j-1,i')$ and drops along the commutator terms; so the terms of rank r are smoothed on the part $r+1$, their forcings being smooth from its start. The forward order drops along (12.21) by at least the order of the connecting terms, so the forcing lies in the next forward order by Lemma 12.10(c), which differentiates the coefficients at most $N_1+N'+d_1+\cdots+d_N+\max\{d_1,\dots,d_N\}-1$ times here, within the derivative counts. So every (N,i) is at most $C(1+\kappa_m h_m/\varepsilon_m^2)^C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C\|g\|_{\rho(m),\omega,N';\xi}$ in $H^{N_1}(\mathbb{T}^2)$ at t_2.

Step 3: The discarded frequencies. The Fourier transform of $U^{[N]}(t_2) - \Pi_{\mathrm{blk}}g$ is $(1-\mathsf{H}^{\mathrm{lo}}_m(\xi)^{N'+1})\widehat{U^{[N]}(t_2)}(\xi)$, supported where $\bar\varepsilon_m|\xi| \ge c_{\mathrm{lo}}$; we split it by the smooth step $\mathsf{H}(2-4\varepsilon_m|\xi|/\pi)$, which is 1 for $\varepsilon_m|\xi| \le \pi/4$ and 0 for $\varepsilon_m|\xi| \ge \pi/2$. At *moderate* frequencies, $\varepsilon_m|\xi| \le \pi/2$, (12.22) makes $\rho(m)^{-|i|}|\nabla^i_\xi \widehat{U^{[N]}(t_2)}(\xi)|_\omega$ equal to $\varepsilon_m^2|\langle \mathcal{B}U^{[N,i]}(t_2),1\rangle|$, which (12.29), with the test 1 and the $c\kappa_m h_m/\varepsilon_m^2$ halving intervals of the block, bounds by

$$C\varepsilon_m^2\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C e^{-c\kappa_m h_m/\varepsilon_m^2}\|g\|_{\rho(m),\omega,N';\xi}\,.$$

As $\rho_{\mathrm{mol}}(m)|\xi| \le \pi/2$ on this band, Plancherel's identity, Leibniz's rule on the two cutoffs and Lemma 12.2 give the bound of (iii) for this part. At *high* frequencies, $\varepsilon_m|\xi| \ge \pi/4$, we use the modes of Bloch sums: for h with $\|h\|_{L^2_{\omega,2}(\mathbb{R}^2)} < \infty$, ξ' in the zone and $\mathbf{n} \in \mathbb{Z}^2$, by Poisson's summation formula,

$$\int_{\mathbb{T}^2} \mathcal{B}h(\xi',z)\,e^{-2\pi\mathrm{i}\mathbf{n}\cdot z}\,\mathrm{d}z = \frac{e^{-2\pi\mathrm{i}\mathbf{n}\cdot y_0/\varepsilon_m}}{\varepsilon_m^2}\,\widehat{h}\Big(\xi'+\frac{2\pi\mathbf{n}}{\varepsilon_m}\Big)\,, \tag{12.30}$$

and the translates of the zone by $2\pi\mathbb{Z}^2/\varepsilon_m$ tile $\mathbb{R}^2$. So, with ξ' the frequency of the zone with $\varepsilon_m(\xi-\xi') \in 2\pi\mathbb{Z}^2$, $\widehat{U^{[N,i]}(t_2)}(\xi)$ is ε_m^2 times a unimodular number times the mode $e^{\mathrm{i}\varepsilon_m(\xi-\xi')\cdot z}$ of the Bloch sum at ξ'. Let t be the start of the quiet stretch, of length at least $\tau'_m - \tau_m/24$, that ends the block. By (12.29) with this test,

$$\big|\big\langle \mathcal{B}U^{[j,i]}(t)(\xi',\cdot), e^{\mathrm{i}\varepsilon_m(\xi-\xi')\cdot z}\big\rangle\big| \le C\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C(1+\varepsilon_m|\xi|)^{N_{j,i}}\|g\|_{\rho(m),\omega,N';\xi'}\,.$$

Across the stretch the velocity vanishes and the insertions do not depend on z, by clause (i) of the definition of insertions, so each mode evolves separately and is damped at the rate $\frac{1}{2}\varepsilon_m^2|\xi|^2 \geq \pi^2/32$ (the proof of Lemma 12.8(c)). The triangular forcing costs a polynomial in the length of the stretch, and, summed over the modes, the Gaussian damping in $\varepsilon_m|\xi|$ absorbs the growth $(1+\varepsilon_m|\xi|)^{N_{j,i}}$ and leaves the factor $e^{-c\kappa_m\tau'_m/\varepsilon_m^2}$ in $H^{N_1}(\mathbb{T}^2)$ at t_2. We return to the plane: by (12.30), $\nabla + \mathrm{i}\varepsilon_m\xi'$ acts on the mode $\mathbf{n}$ of $\mathcal{B}h(\xi',\cdot)$ as $\mathrm{i}\varepsilon_m\xi$, with $\xi = \xi' + 2\pi\mathbf{n}/\varepsilon_m$, so, by Parseval's identity,

$$\int_{\text{zone}} \big\|(\nabla + \mathrm{i}\varepsilon_m\xi')^j \mathcal{B}h(\xi',\cdot)\big\|^2_{L^2_\omega(\mathbb{T}^2)}\,\mathrm{d}\xi' = (2\pi)^2\varepsilon_m^{2j-4}\|\nabla^j h\|^2_{L^2_\omega(\mathbb{R}^2)}$$

whenever the left side is finite, and $\nabla^j h$ then lies in $L^2_\omega(\mathbb{R}^2)$. We apply this with $h = ((y-y_0)/\rho(m))^a U^{[N]}(t_2)$, $|a| \leq N_1$, and commute the weight with $\rho_{\mathrm{mol}}(m)\nabla$ at the cost $\rho_{\mathrm{mol}}(m)/\rho(m) \leq 1$, the derivatives at the scale $\rho_{\mathrm{mol}}(m) \leq \varepsilon_m$ costing a constant. As $h_m \geq \tau'_m \geq \tau_m$, this is the first bound of (iii) for $g \in \mathcal{D}_{N'+2}$.

Step 4: The power of ε_{m-1}. By Proposition 2.1(c) and (2.5), $\kappa_m\tau_m/\varepsilon_m^2 \geq c\varepsilon_{m-1}^{-4\delta}$, $\bar{\varepsilon}_m/\varepsilon_m \leq C\varepsilon_{m-1}^{-q\gamma}$, and $\kappa_m h_m/\varepsilon_m^2 \leq \rho(m)^2/\varepsilon_m^2 \leq C\varepsilon_{m-1}^{-2q}$ as $\kappa_m \leq \kappa_{m-1}$, $\rho(m) \leq 1$ and $\varepsilon_m \geq \frac{1}{2}\varepsilon_{m-1}^q$ (Lemma 7.1(a)). So the first bound of (iii) is at most $C\varepsilon_{m-1}^{-C}\exp(-c\varepsilon_{m-1}^{-4\delta}) \leq C\varepsilon_{m-1}^{Q+3}$ for m large. □

12.9 The change of Lagrangian coordinates, and the proof of Proposition 12.7

The old partial block, from t_1 to the block boundary t_2, may be short and carry pulses of one orientation only (Figure 12.1): it need not damp the frequencies away from the branch, and its scalar factor has no Gaussian. At t_2 the chart changes, by an area-preserving map that does not keep the support of the Fourier transform. So the old term is carried across t_2 at low frequency, where it is bounded only in negative norms, and the next block, the complete *future block*, smooths it (Figure 12.3): its model sees its datum only through the zone part below, and its Gaussian absorbs any power of $1+\rho(m)|\xi|$.

Let $t_2 \coloneqq (l+\frac{1}{2})h_m$ and $t_{\mathrm{pr}} \coloneqq t_2 + h_m$, as in (12.17), let y'_0 be a label of the chart of the block of $l+1$, and let $y_0 \coloneqq X_{m-1}(lh_m, y'_0, (l+1)h_m)$ be the label of the same point in the chart of the block of l (Lemma 7.7(b)). A *Duhamel term across the renewal* of lengths N_-, N_+, from a datum g at a time $t_1 \in [t_2 - h_m, t_2]$, is the sequence $U^{[0]}, \dots, U^{[N_-+N_+]}$ in which $U^{[0]}, \dots, U^{[N_-]}$ is a Duhamel term on $[t_1, t_2]$ for the block of l, centred at y_0, from g, with the *old* insertions $\mathcal{I}_1, \dots, \mathcal{I}_{N_-}$, of orders $d_1, \dots, d_{N_-}$; $U^{[N_-]}$ continues on $[t_2, t_{\mathrm{pr}}]$ by the frozen operator of the block of $l+1$, centred at y'_0, from $U^{[N_-]}(t_2) \circ X_{m-1}(lh_m, \cdot, (l+1)h_m)$; and $U^{[N_-+1]}, \dots, U^{[N_-+N_+]}$ start from zero at t_2, forced by N_+ *future* insertions $\mathcal{I}_{N_-+1}, \dots, \mathcal{I}_{N_-+N_+}$ of the block of $l+1$, of orders $d_{N_-+1}, \dots, d_{N_-+N_+}$. Here and below, $U^{[N_-]}(t_2)$ always denotes the value at t_2 in the chart of the block of l.

The *low-frequency part* and the *zone part* of a field F on $\mathbb{R}^2$ are given, with F a function of the labels of a chart and the Fourier transform about its centre, by

$$\widehat{F^{\mathrm{lo}}}(\xi) \coloneqq \mathsf{H}^{\mathrm{lo}}_m(\xi)\widehat{F}(\xi)\,, \qquad \widehat{F^{\mathrm{zone}}}(\xi) \coloneqq \mathsf{H}\Big(4 - \frac{8\varepsilon_m|\xi|}{\pi}\Big)\widehat{F}(\xi)\,;$$

the second multiplier, the *zone multiplier*, is 1 for $\varepsilon_m|\xi| \leq 3\pi/8$ and 0 for $\varepsilon_m|\xi| \geq \pi/2$, so the Fourier transform of F^{zone} is supported in the zone $\varepsilon_m|\xi|_\infty \leq \pi$. The zone part is taken after the change of chart.

Negative norms. Fix an integer $N_1 \geq 0$, the moment order of the output, and let N' be that of the future block: N_1 plus the sum of the Taylor degrees of the future insertions. For an integer $N \geq 0$ and $\rho \in \{\varepsilon_m, \rho(m)\}$, over Schwartz fields ϑ,

$$\|F\|_{H^{-N}_{\rho,\omega,N'+2}(\mathbb{R}^2)} \coloneqq \sup\Big\{\big|\langle F, \vartheta\rangle_{L^2_\omega(\mathbb{R}^2)}\big| : \|\vartheta\|_{H^N_{\rho,\omega,-(N'+2)}(\mathbb{R}^2)} \leq 1\Big\}\,,$$

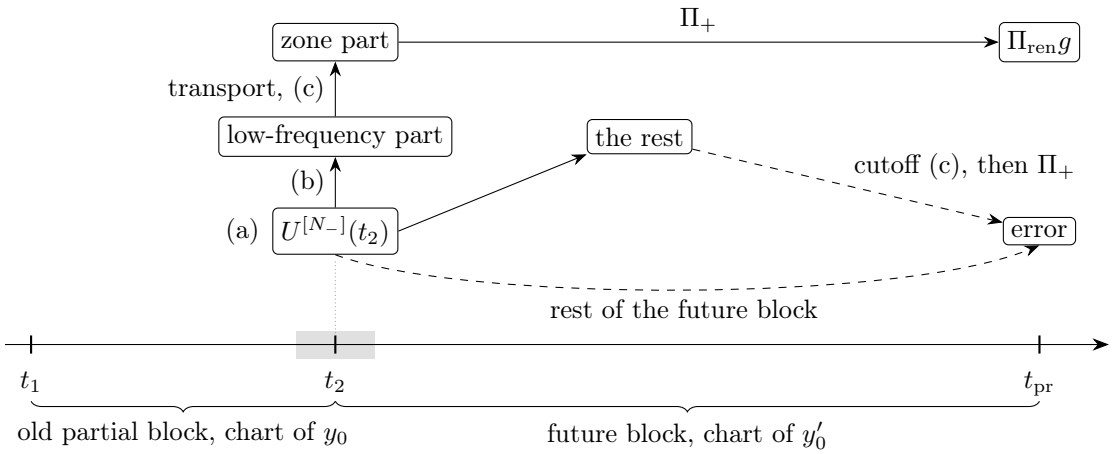


Figure 12.3: The first renewal (Lemmas 12.13–12.15). At t_2, in the quiet margin (shaded), $U^{[N_-]}(t_2)$ splits into its low-frequency part, transported to the chart of the future block and cut to its zone part, and the rest. Solid: the model $\Pi_{\mathrm{ren}}g$; dashed: its two errors; (a)–(c): the items of Lemma 12.13; Π_+: the model of the future block.

with $\langle F,\vartheta\rangle_{L^2_\omega(\mathbb{R}^2)} := \sum_v \omega_v^{-1}\int_{\mathbb{R}^2} F_v\overline{\vartheta_v}\,\mathrm{d}y$, and with the weight $(1+|y-y_*|^2/\rho(m)^2)^{-(N'+2)/2}$ in (12.8), y_* the centre of the chart of F, about which the moments are taken. A larger N or the scale $\rho(m)$ in place of ε_m gives a smaller norm. The *negative order* $N_{\mathrm{neg}} := N_1+d_1+\cdots+d_{N_-}+2$ exceeds the number of derivatives that the old insertions put on a test field.

In Lemma 12.13, (a) bounds the old term with losses, and (b) its low-frequency part without loss but, with no Gaussian to absorb the polynomial factor of Lemma 12.11, only in a negative norm at the scale $\rho(m)$. By (c) the change of chart preserves these norms, and the high frequencies, far above $1/\rho(m)$, gain a power of $\bar\varepsilon_m/\rho(m)$ at that scale, which absorbs the losses of (a).

Lemma 12.13 (The old block, carried across)**.** *In the setting above (a Duhamel term across the renewal from g at t_1), there exist C, an integer N_3 and, for every integer $N\ge 0$, a constant $C(N)$ such that:*

(a) In a negative norm. *For every $g\in H^{d_1+\cdots+d_{N_-}-2N_-}(\mathbb{R}^2)^V$ with compact support,*

$$\|U^{[N_-]}(t_2)\|_{H^{-N_{\mathrm{neg}}}_{\varepsilon_m,\omega,N'+2}(\mathbb{R}^2)} \le C\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^{N_3}\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_3}\|g\|_{L^2_{\omega,N_3}(\mathbb{R}^2)}\,.$$

(b) At low frequency, *for the same g, with no loss in the contrast or the number of cell times:*

$$\big\|(U^{[N_-]}(t_2))^{\mathrm{lo}}\big\|_{H^{-N_3}_{\rho(m),\omega,N'+2}(\mathbb{R}^2)} \le C\|g\|_{L^2_{\omega,N_3}(\mathbb{R}^2)}\,.$$

(c) Transport and the cutoff. *For every $F\in L^2(\mathbb{R}^2)^V$, every N and $\rho\in\{\varepsilon_m,\rho(m)\}$, with F in the chart of the block of l,*

$$\big\|F\circ X_{m-1}(lh_m,\cdot,(l+1)h_m)\big\|_{H^{-N}_{\rho,\omega,N'+2}(\mathbb{R}^2)} \le C(N)\|F\|_{H^{-N}_{\rho,\omega,N'+2}(\mathbb{R}^2)}\,,$$

$$\big\|F-F^{\mathrm{lo}}\big\|_{H^{-(N+N_{\mathrm{neg}})}_{\rho(m),\omega,N'+2}(\mathbb{R}^2)} \le C(N)\Big(\frac{\bar\varepsilon_m}{\rho(m)}\Big)^N\|F\|_{H^{-N_{\mathrm{neg}}}_{\varepsilon_m,\omega,N'+2}(\mathbb{R}^2)}\,.$$

Proof. (a) By (12.20), the pairing of $U^{[N_-]}(t_2)$ with a Schwartz field of norm one in $H^{N_{\mathrm{neg}}}_{\varepsilon_m,\omega,-(N'+2)}(\mathbb{R}^2)$ is a sum of nested integrals, over a simplex of insertion times, of the pairing of g with this field

run backward by the adjoint frozen operator on $\mathbb{R}^2$ and acted on by the adjoint old insertions, each costing its order in derivatives and its Taylor degree in the weight. The simplex volume, the cost of the adjoint evolution in these weighted norms, and the rate from the transport of the moment weight w, by $\nabla^\perp\psi_m\cdot\nabla w=-\nabla\cdot(\psi_m\nabla^\perp w)$ and the Cauchy–Schwarz inequality against the dissipation (a constant over $t_2-t_1\le h_m$, by Grönwall's inequality), are

$$\frac{1}{N_-!}\Big(\frac{\kappa_m(t_2-t_1)}{\varepsilon_m^2}\Big)^{N_-},\qquad C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C\Big(1+\frac{\kappa_m(t_2-t_1)}{\varepsilon_m^2}\Big)^C,\qquad \frac{C|\psi_m|^2}{\kappa_m\rho(m)^2}\le\frac{C}{h_m},$$

the last as $|\psi_m|^2\le C\varepsilon_m^{2\beta}$ and $\varepsilon_m^{2\beta}/\kappa_m\le 20\kappa_{m-1}$; there is no short-time singularity, as at the scale $\varepsilon_m/(C\big(1+\bar\varepsilon_m/\varepsilon_m\big))$ the field $\nabla^\perp\psi_m$ satisfies (i) of Proposition 8.3, whose energy holds with the weight on $\mathbb{R}^2$, its commutator estimate being pointwise.

(b) Let N_3 be even, at least $N_2+N'+2$, at least the exponents of $1+\kappa_m h_m/\varepsilon_m^2$ and of $\big(1+\bar\varepsilon_m/\varepsilon_m\big)$ in the proof of (a), and at least $N'+2$ plus the sum of the Taylor degrees of the old insertions, the weight used in (a), with N_2 the integer of Lemma 12.11 with $N'+2$ in place of N_1. That lemma on $[t_1,t_2]$, and Propositions 12.5(c) and 12.4(d) for the scalar factor (no Gaussian, but no growth either), give

$$\big|\widehat{(U^{[N_-]}(t_2))^{\mathrm{lo}}}\big|_{\rho(m),\omega,N'+2;\xi}\le C(1+\rho(m)|\xi|)^{N_2+N'+2}\varepsilon_m^2\|g\|_{\rho(m),\omega,N_3;\xi}\,.$$

As N_3 is even, $(1-\rho(m)^2\Delta)^{N_3/2}$ moves onto the test fields of the negative norm at the cost C, and the factor $(1+\rho(m)^2|\xi|^2)^{-N_3/2}$ it leaves absorbs the power of $1+\rho(m)|\xi|$; by Plancherel, moments becoming ξ-derivatives, and Lemma 12.2 summing over ξ,

$$\big\|(U^{[N_-]}(t_2))^{\mathrm{lo}}\big\|_{H^{-N_3}_{\rho(m),\omega,N'+2}(\mathbb{R}^2)}\le C\varepsilon_m^2\Big(\int_{\mathrm{zone}}\|g\|^2_{\rho(m),\omega,N_3;\xi}\,\mathrm{d}\xi\Big)^{1/2}\le C\|g\|_{L^2_{\omega,N_3}(\mathbb{R}^2)}\,.$$

(c) The map $X_{m-1}(lh_m,\cdot,(l+1)h_m)$ preserves area, maps y_0' to y_0, and has a gradient within $C\varepsilon_{m-1}^{2\delta}$ of the identity, with derivatives at the scale $\varepsilon_{m-1}\ge\rho(m)$ ((7.9), Lemma 7.7(b)); by duality, the first bound follows from the chain rule for the inverse map, as the moment weights $1+|X_{m-1}(lh_m,y,(l+1)h_m)-y_0|^2/\rho(m)^2$ and $1+|y-y_0'|^2/\rho(m)^2$ are comparable. The Fourier transform of $F-F^{\mathrm{lo}}$ is supported in $\bar\varepsilon_m|\xi|\ge c_{\mathrm{lo}}$; as $\rho(m)^2=\kappa_{m-1}h_m$,

$$\frac{\rho(m)}{\bar\varepsilon_m}=\Big(\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^{1/2}.\tag{12.31}$$

By duality it suffices that $1-\mathsf{H}^{\mathrm{lo}}_m(-\mathrm{i}\nabla)$ maps $H^{N+N_{\mathrm{neg}}}_{\rho(m),\omega,-(N'+2)}(\mathbb{R}^2)$ to $H^{N_{\mathrm{neg}}}_{\rho(m),\omega,-(N'+2)}(\mathbb{R}^2)$, whose norm exceeds that at the scale ε_m, with norm $C(N)(\bar\varepsilon_m/\rho(m))^N$. The kernel of $\mathsf{H}^{\mathrm{lo}}_m(-\mathrm{i}\nabla)$ is a Schwartz function at the scale $\bar\varepsilon_m/c_{\mathrm{lo}}$ with integral 1 and vanishing moments, as $\mathsf{H}^{\mathrm{lo}}_m\equiv 1$ near 0; so $(1-\mathsf{H}^{\mathrm{lo}}_m(-\mathrm{i}\nabla))\vartheta$ is its integral against the Taylor remainder of order N of ϑ, of size $|\tilde y|^N|\nabla^N\vartheta|$. Peetre's inequality moves the moment weight through the shift by $\tilde y$ at the cost $(1+|\tilde y|/\rho(m))^{N'+2}$, and Minkowski's inequality gives the coefficient $C(N)(\bar\varepsilon_m/\rho(m))^N$. □

By Lemma 12.14(a), the future model maps the zone part from negative norms at the scale $\rho(m)$ to positive ones, the Gaussian of Lemma 12.12(ii) absorbing every power of $1+\rho(m)|\xi|$; by (b), the future block differs from this model by losses times a coefficient smaller than any power of ε_{m-1}.

Lemma 12.14 (The future block)**.** *In the setting above, let Π_+ be the map Π_{blk} of Lemma 12.12 for the future block, centred at y_0', with the future insertions and N_1. There exist C, $c>0$, an integer N_3 and, for every integer $N\ge 0$, a constant $C(N)$ such that:*

(a) The future model on negative norms. *For every $F \in L^2(\mathbb{R}^2)^V$ with finite right side,*

$$\big\|\Pi_+(F^{\mathrm{zone}})\big\|_{H^N_{\rho(m),\omega,N_1}(\mathbb{R}^2)} \le C(N)\|F\|_{H^{-N}_{\rho(m),\omega,N'+2}(\mathbb{R}^2)}\,.$$

(b) The rest of the future block. *Let F satisfy $\nabla^b F \in L^2_{\omega,N'}(\mathbb{R}^2)^V$ for $b \le N_1 + d_{N_-+1} + \cdots + d_{N_-+N_+} - 2N_+$ and $\|F\|_{L^2_{\omega,N'+2}(\mathbb{R}^2)} < \infty$, and let $U_F^{[N_+]}$ be the last term of its Duhamel term over the block of $l+1$ from t_2, with the future insertions. Then*

$$\begin{aligned}&\big\|U_F^{[N_+]}(t_{\mathrm{pr}}) - \Pi_+(F^{\mathrm{zone}})\big\|_{H^{N_1}_{\rho_{\mathrm{mol}}(m),\omega,N_1}(\mathbb{R}^2)}\\ &\qquad\le C\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^{N_3}\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_3} e^{-c\kappa_m\tau_m/\varepsilon_m^2}\|F\|_{H^{-N_{\mathrm{neg}}}_{\varepsilon_m,\omega,N'+2}(\mathbb{R}^2)}\,.\end{aligned}$$

Proof. (a) As the zone multiplier has a Schwartz kernel at the scale ε_m and the right side is finite, F^{zone} and its derivatives have finite moments of order $N'+2$, by Peetre's inequality; so F^{zone} lies in the class $\mathcal{D}_{N'+2}$ of the future block, and Lemma 12.12(ii) applies. At ξ in the zone its Bloch sums are constants, by (12.30), as $|\varepsilon_m\xi + 2\pi\mathbf{n}| \ge \pi$ for $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$:

$$\mathcal{B}\Big(\Big(\frac{y-y_0'}{\rho(m)}\Big)^i F^{\mathrm{zone}}\Big)(\xi,\cdot) = \frac{\mathrm{i}^{|i|}}{\varepsilon_m^2\rho(m)^{|i|}}\nabla_\xi^i\widehat{F^{\mathrm{zone}}}(\xi)\,.$$

So Plancherel's identity and Lemma 12.12(ii), with its integer N_2, the Gaussian absorbing $(1+\rho(m)|\xi|)^{4N+2N_2}$, give

$$\big\|\Pi_+(F^{\mathrm{zone}})\big\|^2_{H^N_{\rho(m),\omega,N_1}(\mathbb{R}^2)} \le C(N)\sum_{|i|\le N'}\int_{\mathbb{R}^2}\big(1+\rho(m)^2|\xi|^2\big)^{-N}\rho(m)^{-2|i|}\big|\nabla_\xi^i\widehat{F^{\mathrm{zone}}}(\xi)\big|^2_\omega\,\mathrm{d}\xi\,.$$

The zone multiplier has derivatives bounded at the scale $\varepsilon_m \le \rho(m)$, so F may replace F^{zone} on the right; by Plancherel this is at most $C(N)$ times the sum over $|i| \le N'$ of the squared $H^{-N}_{\rho(m),\omega,0}(\mathbb{R}^2)$ norms of $((y-y_0')/\rho(m))^i F$, and, as multiplication by $((y-y_0')/\rho(m))^i$ maps $H^N_{\rho(m),\omega,0}(\mathbb{R}^2)$ into $H^N_{\rho(m),\omega,-(N'+2)}(\mathbb{R}^2)$, duality gives (a).

(b) The Duhamel terms from F^{zone} and $F^{\mathrm{hi}} := F - F^{\mathrm{zone}}$ exist, as both lie in the class of (b) (F^{zone} by Peetre's inequality, as in (a)), and by linearity $U_F^{[N_+]} - \Pi_+(F^{\mathrm{zone}})$ is the sum of $U_{F^{\mathrm{zone}}}^{[N_+]} - \Pi_+(F^{\mathrm{zone}})$ and $U_{F^{\mathrm{hi}}}^{[N_+]}$. As in (a), at the scale ε_m, $\|F^{\mathrm{zone}}\|_{L^2_{\omega,N'}(\mathbb{R}^2)} \le C\|F\|_{H^{-N_{\mathrm{neg}}}_{\varepsilon_m,\omega,N'+2}(\mathbb{R}^2)}$, so Lemma 12.12(iii) bounds the first term at t_{pr}. For the second we use the *quiet stretch* $[t_2, t_{\mathrm{q}}]$, $t_{\mathrm{q}} := t_2 + \tau_m' - \tau_m/24$, the first part of the quiet margin of the block of $l+1$. As $\tau_m' \ge 5\tau_m$, its length in cell units lies between $\kappa_m\tau_m'/(2\varepsilon_m^2)$, which is at least 1 for m large, and $\kappa_m h_m/\varepsilon_m^2$. Let $U_{F^{\mathrm{hi}}}^{[j,i]}$ be the moment terms of the Duhamel term of F^{hi} over the block of $l+1$, centred at y_0'.

The quiet stretch. On $[t_2, t_{\mathrm{q}}]$ no pulse is active within $\tau_m/24$, so $\psi_m = 0$ and the coefficients of the insertions do not depend on z, by clause (i) of the definition of insertions; once the weight is moved through as in the sentence after (12.21), they do not depend on y either. So at each frequency η the system (12.21) for the Fourier transforms of the $U_{F^{\mathrm{hi}}}^{[j,i]}$ is triangular, with the common diagonal $-\kappa_m\eta\cdot\mathbf{K}(t)\eta$ and off-diagonal entries κ_m/ε_m^2 times polynomials in $\varepsilon_m\eta$ with bounded coefficients (the commutator terms carry $\varepsilon_m/\rho(m)$ and $(\varepsilon_m/\rho(m))^2$, and $\varepsilon_m \le \rho(m) \le \varepsilon_{m-1}$). As the terms with $j \ge 1$ vanish at t_2,

$$\widehat{U_{F^{\mathrm{hi}}}^{[j,i]}}(t_{\mathrm{q}},\eta) = \exp\Big(-\kappa_m\int_{t_2}^{t_{\mathrm{q}}}\eta\cdot\mathbf{K}(t)\eta\,\mathrm{d}t\Big)\sum_{|i'|\le N'}P_{j,i,i'}(\varepsilon_m\eta)\widehat{U_{F^{\mathrm{hi}}}^{[0,i']}}(t_2,\eta)\,,$$

with $P_{j,i,i'}(x)$ a linear map on $\mathbb{C}^V$, polynomial in the length of the stretch, whose norm on $(\mathbb{C}^V, |\cdot|_\omega)$ is at most $C(1+\kappa_m h_m/\varepsilon_m^2)^C(1+|x|)^C$. By (12.22), $\widehat{U_{F^{\mathrm{hi}}}^{[0,i']}}(t_2,\eta) = \mathrm{i}^{|i'|}\rho(m)^{-|i'|}\nabla_\eta^{i'}\widehat{F^{\mathrm{hi}}}(\eta)$, which vanishes for $\varepsilon_m|\eta| \le 3\pi/8$. As $\mathbf{K} \ge \frac{15}{16}I$, the exponential is at most $\exp(-\frac{15}{32}\kappa_m\tau'_m|\eta|^2)$: half of the exponent absorbs $(1+\varepsilon_m|\eta|)^C$, as $\kappa_m\tau'_m/\varepsilon_m^2 \ge 2$, and the other half is at most $\exp(-\kappa_m\tau'_m/(4\varepsilon_m^2))$ where $\varepsilon_m|\eta| \ge 3\pi/8$. Each ∇_η that falls on the zone multiplier costs $\varepsilon_m \le \rho(m)$, and multiplication by $((y-y'_0)/\rho(m))^{i'}$, $|i'| \le N'$, maps $H^{N_{\mathrm{neg}}}_{\varepsilon_m,\omega,0}(\mathbb{R}^2)$ into $H^{N_{\mathrm{neg}}}_{\varepsilon_m,\omega,-(N'+2)}(\mathbb{R}^2)$, as in (a) at the scale $\varepsilon_m \le \rho(m)$. So

$$\sum_{(j,i)}\int_{\mathbb{R}^2}\big(1+\varepsilon_m^2|\eta|^2\big)^{-N_{\mathrm{neg}}}\big|\widehat{U_{F^{\mathrm{hi}}}^{[j,i]}}(t_{\mathrm{q}},\eta)\big|_\omega^2\,\mathrm{d}\eta \le C\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^C\exp\Big(-\frac{\kappa_m\tau'_m}{2\varepsilon_m^2}\Big)\|F\|^2_{H^{-N_{\mathrm{neg}}}_{\varepsilon_m,\omega,N'+2}(\mathbb{R}^2)}\,.$$

The same formula, with up to two further η-derivatives, shows that the moment terms have at t_{q} every derivative with two moments, so their Bloch sums are defined; the moments persist for $t \ge t_{\mathrm{q}}$ by the moment-weighted energy (paragraph *Duhamel terms*), and the Bloch sums solve the fibre system, by the paragraph of (12.9).

From the plane to the fibres. Let h be a field on $\mathbb{R}^2$ with two moments and ξ in the zone. The modes of $\mathcal{B}h(\xi,\cdot)$ are given by (12.30), and, as $|\varepsilon_m\xi|^2 \le 2\pi^2$, the weights $1+|2\pi\mathbf{n}|^2$ and $1+|2\pi\mathbf{n}+\varepsilon_m\xi|^2$ are comparable with the constant $2+4\pi^2$. So, componentwise and then with the masses, for every integer $K \ge 0$, with $\|\cdot\|_{H^{-K}(\mathbb{T}^2)}$ the norm dual to $H^K(\mathbb{T}^2)$ for $\langle\cdot,\cdot\rangle_{L^2_\omega(\mathbb{T}^2)}$,

$$\int_{\mathrm{zone}}\|\mathcal{B}h(\xi,\cdot)\|^2_{H^{-K}(\mathbb{T}^2)}\,\mathrm{d}\xi \le C(K)\varepsilon_m^{-4}\int_{\mathbb{R}^2}\big(1+\varepsilon_m^2|\eta|^2\big)^{-K}\big|\widehat{h}(\eta)\big|_\omega^2\,\mathrm{d}\eta\,.$$

The rest of the block. Let $t \in [t_{\mathrm{q}}, t_{\mathrm{pr}}]$, let ξ be in the zone, and let $U^\dagger \in H^{N_{j,i}+N_{\mathrm{neg}}}(\mathbb{T}^2)^V$. The induction of Step 1 of the proof of Lemma 12.12(iii), started at t_{q} with a data term for each moment term and at the test orders raised by N_{neg}, gives

$$\begin{aligned}\big|\big\langle\mathcal{B}U_{F^{\mathrm{hi}}}^{[j,i]}(t),U^\dagger\big\rangle_{L^2_\omega(\mathbb{T}^2)}\big| \le{}& C\Big(1+\frac{\kappa_m(t-t_{\mathrm{q}})}{\varepsilon_m^2}\Big)^C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C\|U^\dagger\|_{H^{N_{j,i}+N_{\mathrm{neg}}}(\mathbb{T}^2)}\\ &\times\sum_{(j',i')}\big\|\mathcal{B}U_{F^{\mathrm{hi}}}^{[j',i']}(t_{\mathrm{q}})\big\|_{H^{-N_{\mathrm{neg}}}(\mathbb{T}^2)}\,,\end{aligned}$$

the sum over the moment terms (j',i') equal to or lower than (j,i), with the coefficients differentiated within the derivative counts for a Duhamel term across the renewal. Step 2 of that proof, on $N_+ + N' + 1$ equal parts of the last cell time of the block, which lies in $[t_{\mathrm{q}}, t_{\mathrm{pr}}]$, from the negative order raised by N_{neg}, bounds every (N_+,i), $|i| \le N_1$, in $H^{N_1}(\mathbb{T}^2)$ at t_{pr} by $C(1+\kappa_m h_m/\varepsilon_m^2)^C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C$ times the same sum. We square and integrate over the zone: the Parseval identity of Step 3 of that proof on the left side, and the bound from the plane to the fibres with $K = N_{\mathrm{neg}}$ and the bound at t_{q} on the right side, give, the powers ε_m^4 and ε_m^{-4} cancelling and as $\tau'_m \ge \tau_m$,

$$\big\|U_{F^{\mathrm{hi}}}^{[N_+]}(t_{\mathrm{pr}})\big\|_{H^{N_1}_{\rho_{\mathrm{mol}}(m),\omega,N_1}(\mathbb{R}^2)} \le C\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^C\big(1+\bar\varepsilon_m/\varepsilon_m\big)^C\exp\Big(-\frac{\kappa_m\tau_m}{4\varepsilon_m^2}\Big)\|F\|_{H^{-N_{\mathrm{neg}}}_{\varepsilon_m,\omega,N'+2}(\mathbb{R}^2)}\,.$$

This is (b) for the second term, and Lemma 12.12(iii) for the future block gives (b) for the first, for c at most $\frac14$ and the c of that lemma, and N_3 at least the exponents above and its N_2. □

Lemma 12.15 assembles the two lemmas along Figure 12.3: its model is bounded without loss, and the losses in its two errors, the future block less its model and the high frequencies of the old term, are offset by the damping of the future block and the gain of Lemma 12.13(c).

Lemma 12.15 (The first renewal)**.** *Let $J \geq 0$ be an integer, and consider a Duhamel term across the renewal from a datum $g \in H^{N_1+d_1+\cdots+d_{N_-+N_+}-2(N_-+N_+)}(\mathbb{R}^2)^V$ supported within distance $\rho(m)$ of y_0. There exist C and a linear map Π_{ren}, independent of g, such that, with the moments about y_0' on the left sides,*

$$\|\Pi_{\mathrm{ren}} g\|_{H^J_{\rho(m),\omega,N_1}(\mathbb{R}^2)} \leq C\|g\|_{L^2_\omega(\mathbb{R}^2)}\,,$$
$$\big\|U^{[N_-+N_+]}(t_{\mathrm{pr}}) - \Pi_{\mathrm{ren}} g\big\|_{H^{N_1}_{\rho_{\mathrm{mol}}(m),\omega,N_1}(\mathbb{R}^2)} \leq C\varepsilon_{m-1}^{Q+3}\|g\|_{L^2_\omega(\mathbb{R}^2)}\,.$$

Proof. Let N_3 be the larger of the integers of Lemmas 12.13 and 12.14; both hold with it. The model is the solid path of Figure 12.3,

$$\Pi_{\mathrm{ren}} g := \Pi_+\Big(\big((U^{[N_-]}(t_2))^{\mathrm{lo}} \circ X_{m-1}(lh_m, \cdot, (l+1)h_m)\big)^{\mathrm{zone}}\Big)\,.$$

It is linear, and $\|g\|_{L^2_{\omega,N_3}(\mathbb{R}^2)} \leq 2^{N_3/2}\|g\|_{L^2_\omega(\mathbb{R}^2)}$ by the support of g. By Lemma 12.14(a) at the order $\max\{J, N_3\}$, and Lemma 12.13(c) at the scale $\rho(m)$ and Lemma 12.13(b), $\|\Pi_{\mathrm{ren}} g\|_{H^J_{\rho(m),\omega,N_1}(\mathbb{R}^2)} \leq C\|g\|_{L^2_\omega(\mathbb{R}^2)}$.

The error. $U^{[N_-+N_+]}(t_{\mathrm{pr}})$ is the last term at t_{pr} of the Duhamel term over the block of $l+1$ from $F := U^{[N_-]}(t_2) \circ X_{m-1}(lh_m, \cdot, (l+1)h_m)$, which lies in the class of Lemma 12.14(b): as g has compact support, the weighted energy of Proposition 8.6 with the moment weights gives finite moments to $U^{[N_-]}(t_2)$ and to its derivatives of order at most $N_1 + d_{N_-+1} + \cdots + d_{N_-+N_+} - 2N_+$, the j-th old insertion lowering the order by $d_j - 2$, and the change of variables, area preserving with derivatives bounded at the scale $\varepsilon_{m-1} \geq \rho(m)$, keeps the class. So $U^{[N_-+N_+]}(t_{\mathrm{pr}}) - \Pi_{\mathrm{ren}} g$ is the sum of $U^{[N_-+N_+]}(t_{\mathrm{pr}}) - \Pi_+(F^{\mathrm{zone}})$ and of Π_+ applied to the zone part of $(U^{[N_-]}(t_2) - (U^{[N_-]}(t_2))^{\mathrm{lo}}) \circ X_{m-1}(lh_m, \cdot, (l+1)h_m)$. By Lemmas 12.14(b), 12.13(c) at the scale ε_m and 12.13(a) for the first, and by Lemmas 12.14(a) with $N_{\mathrm{err}} + N_{\mathrm{neg}}$ in place of N ($N_{\mathrm{err}} \geq N_1$ is chosen below; $\rho_{\mathrm{mol}}(m) \leq \rho(m)$), 12.13(c) with (12.31), and 12.13(a) for the second, they are at most, times $\|g\|_{L^2_\omega(\mathbb{R}^2)}$,

$$C\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^{2N_3}\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{2N_3} e^{-\frac{c\kappa_m\tau_m}{\varepsilon_m^2}}\,, \quad C(N_{\mathrm{err}})\Big(\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^{-\frac{N_{\mathrm{err}}}{2}}\Big(1+\frac{\kappa_m h_m}{\varepsilon_m^2}\Big)^{N_3}\big(1+\bar\varepsilon_m/\varepsilon_m\big)^{N_3}\,.$$

The powers of ε_{m-1}. With the bounds of Step 4 of the proof of Lemma 12.12, which with $h_m/\tau_m \geq \varepsilon_{m-1}^{-2\delta}$ (by (2.4)) give $\kappa_m h_m/\varepsilon_m^2 \geq c\varepsilon_{m-1}^{-6\delta}$, the first term is at most $C\varepsilon_{m-1}^{Q+3}\|g\|_{L^2_\omega(\mathbb{R}^2)}$ for m large, and the second, as $N_{\mathrm{err}} \geq 2N_3$ by the choice below, at most $C(N_{\mathrm{err}})\varepsilon_{m-1}^{6\delta(N_{\mathrm{err}}/2-N_3)-q\gamma N_3}\|g\|_{L^2_\omega(\mathbb{R}^2)}$; we choose N_{err}, after N_3, with $6\delta(N_{\mathrm{err}}/2 - N_3) - q\gamma N_3 \geq Q+3$. □

In the proof of Proposition 12.7, moments control the freezing: the coefficients vary on the scale ε_{m-1}, much larger than $\rho(m)$, and the moments of a piece are multiplied by at most a constant over each block (Step 3); the models of the pieces, each localized near its centre, are almost orthogonal (Step 5).

Proof of Proposition 12.7. Step 1: The time interval. With t_2 and t_{pr} as in (12.17), $t_1 < t_2 \leq t_1 + h_m$ gives the bounds there, and the slow cutoff (2.9) vanishes within τ_m' of every block boundary, which gives the margin.

Step 2: Patches. We cover the labels at time t_1, in the chart of the block of t_1, by a smooth partition of unity at the scale $\rho(m)/8$, with bounded overlap, whose functions are supported within $\rho(m)/4$ of their centres. Their products with g, the *pieces*, have squared $L^2_\omega(\mathbb{T}^2)$ norms of sum at most $\|g\|^2_{L^2_\omega(\mathbb{T}^2)}$, and the same in $H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)$ up to a constant, as the partition varies on the scale

$\rho(m) \geq \rho_{\mathrm{mol}}(m)$ and the chart of the block of t_1 preserves area and has derivatives bounded at the scale $\varepsilon_{m-1} \geq \rho(m)$, by (7.9). Each piece is a datum on $\mathbb{R}^2$. Fix a piece, and let y_0 be its centre.

Step 3: Freezing (the decisive estimate). On the patch of y_0, we replace the conductivity and the two-scale forms of the sources by their Taylor polynomials of degree N_{Tay} at y_0, on the block of $l+1$ at y_0', the label of the same point in its chart ($X_{m-1}(lh_m, y_0', (l+1)h_m) = y_0$). We expand the conductivity polynomial as insertions (Subsection 12.6), up to N_{Tay} of them on the two blocks together. Near the ends of a block, the forms given in the chart of the neighbouring block are insertions of the block, of C times the same strength (paragraph *Insertions*; (a4), (a6), (7.9); the items (a1)–(d) cited in this proof are those of Definition 11.1). Moments bound the remainders: by (7.9), the conductivity differs from its Taylor polynomial by at most

$$C\varepsilon_{m-1}^{2\delta}\Big(\frac{|y-y_0|}{\varepsilon_{m-1}}\Big)^{N_{\mathrm{Tay}}+1} = C\varepsilon_{m-1}^{2\delta}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{N_{\mathrm{Tay}}+1}\Big(\frac{|y-y_0|}{\rho(m)}\Big)^{N_{\mathrm{Tay}}+1},$$

and the same holds for the forms in the sums of (b2′) with $j = N_{\mathrm{Tay}}+1$, with C_* times the entry of the majorant matrix in place of $\varepsilon_{m-1}^{2\delta}$; here $N_{\mathrm{Tay}}+1$ is an internal order of Proposition 12.7, so at most N_{coef} (the table of Subsection 11.1). By Proposition 2.1(c), and as each power of $(y-y_0)/\varepsilon_{m-1}$ costs one moment and the strength below,

$$\frac{\rho(m)}{\varepsilon_{m-1}} \leq C\varepsilon_{m-1}^{\gamma/2+\delta}, \qquad C\varepsilon_{m-1}^{2\delta}\frac{\rho(m)}{\varepsilon_{m-1}} \leq C\varepsilon_{m-1}^{\gamma/2+3\delta} \quad \text{(the strength of a conductivity insertion)}.$$

Telescoping the frozen evolution with its insertions against the actual one leaves, as forcing of the difference, the Taylor remainders and the term with $N_{\mathrm{Tay}}+1$ conductivity insertions. These forcings take at most $N_\sharp := N_{\mathrm{Tay}}(N_{\mathrm{Tay}}+N_{\mathrm{dep}}+1)+1$ moments from the datum, as at most $N_{\mathrm{Tay}}+N_{\mathrm{dep}}$ insertions, of Taylor degrees at most N_{Tay}, precede a forcing of degree at most $N_{\mathrm{Tay}}+1$. We use an energy with the moment weights $w := (1+|y-y_0|^2/\rho(m)^2)^{N_{\mathrm{w}}}$, about y_0' in place of y_0 on the block of $l+1$, as in Proposition 8.6, at moment orders N_{w} that decrease along the chain from $N_\sharp+3$ for the datum to 3 for the difference, as the Sobolev orders do. Each term is bounded through the previous ones, and the moments of the previous term absorb the polynomial growth of the frozen coefficients. The difference at t_{pr} is therefore at most, in $L^2_{\omega,3}(\mathbb{R}^2)$ about y_0', with g_{y_0} the piece and the norm (12.8),

$$C\Big[\big(C\varepsilon_{m-1}^{\gamma/2+\delta}\big)^{N_{\mathrm{Tay}}+1} + \big(C\varepsilon_{m-1}^{\gamma/2+3\delta}\big)^{N_{\mathrm{Tay}}+1}\Big]\,\|g_{y_0}\|_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega,N_\sharp+3}(\mathbb{R}^2)}.$$

The shears of the chart move labels by far more than $\rho(m)$ over a block, but they enter the weighted energy only through their stream function ψ_m, which is small. For a solution D of the equation of the block with a forcing f,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} wD^2\,\mathrm{d}y + 2\kappa_m\int_{\mathbb{R}^2} w\mathbf{K}\nabla D\cdot\nabla D\,\mathrm{d}y = \int_{\mathbb{R}^2} D^2\,\nabla^\perp\psi_m\cdot\nabla w\,\mathrm{d}y + \kappa_m\int_{\mathbb{R}^2} D^2\Delta w\,\mathrm{d}y$$
$$- 2\kappa_m\int_{\mathbb{R}^2} D\,\nabla w\cdot(\mathbf{K}-I)\nabla D\,\mathrm{d}y + 2\int_{\mathbb{R}^2} wDf\,\mathrm{d}y.$$

As in the proof of Lemma 12.13(a), the first term on the right is at most a quarter of the dissipation plus $C|\psi_m|^2/(\kappa_m\rho(m)^2) \leq C/h_m$ times the weighted energy, and the second at most $C\kappa_m/\rho(m)^2 \leq C/h_m$ times it; as $\kappa_m|\nabla w|^2/w \leq Cw/h_m$, the Cauchy–Schwarz inequality bounds the third by half of the dissipation plus C/h_m times the weighted energy. At the orders i of derivatives at the scale $\rho_{\mathrm{mol}}(m)$, the sources add $2\int\nabla w\cdot\nabla^i D\,\nabla^i F\,\mathrm{d}y$, with F the flux of the forcing of D, which Young's inequality bounds by $Ch_m^{-1}\int w|\nabla^i D|^2\,\mathrm{d}y + \kappa_m^{-1}\int w|\nabla^i F|^2\,\mathrm{d}y$, the size of the forcing in Proposition 8.6. With

condition (i) of Proposition 8.3 for $\nabla^\perp\psi_m$ at the scale $\rho_{\mathrm{mol}}(m)$, by the second line of (12.1), the energy runs on $\mathbb{R}^2$ with the weight w, and Grönwall's inequality at the rate C/h_m multiplies it by at most a constant on each of the two blocks, the weights about y_0 and y_0' being matched at t_2 as in the proof of Lemma 12.13(c). The degree N_{Tay} is chosen so that, for m large, the difference is at most $\varepsilon_{m-1}^{Q+4}\|g_{y_0}\|_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{R}^2)}$; the norm of the display is at most a constant times this norm, as g_{y_0} is supported within $\rho(m)/4$ of y_0.

Step 4: The frozen terms. Each frozen patch evolution is a finite sum of Duhamel terms across the renewal at t_2, of lengths at most $N_{\mathrm{dep}}+N_{\mathrm{Tay}}$. Lemma 12.15, with $J\coloneqq N_{\mathrm{out}}$ and $N_1\coloneqq 3$, gives their models and errors $C\varepsilon_{m-1}^{Q+3}$, multiplied by the product of the strengths of the insertions, which for n_{src} source insertions, by (b2) and (b3), and n_{con} conductivity insertions is at most

$$\big(CC_*^2\varepsilon_m^{\delta/4}\big)^{n_{\mathrm{src}}}\Big(C\varepsilon_{m-1}^{2\delta}\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{n_{\mathrm{con}}}.$$

A piece of $g\in H^{N_{\mathrm{buf}}}(\mathbb{T}^2)^V$ has the regularity that the lemma asks, as the conductivity insertions have order 2: $N_1+d_1+\cdots-2(N_-+N_+)\le 3+N_{\mathrm{dep}}(N_{\mathrm{src}}-2)\le N_{\mathrm{buf}}$.

Step 5: Sums. Let $\Pi^{\mathrm{sum}}g$ be the sum, over the pieces and over the lattice images, of the models of all these Duhamel terms, returned to $\mathbb{T}^2$ by the chart at t_{pr}, and for the piece g_{y_0} centred at y_0 let $\Pi^{\mathrm{sum}}_{y_0}g$ be the sum of the models of its Duhamel terms. By (a5), the periodicity of $\mathbf{K}$ and $X_{m-1}(t,y+\mathbf{n},t')=X_{m-1}(t,y,t')+\mathbf{n}$ for $\mathbf{n}\in\mathbb{Z}^2$, the model of a shifted piece is the shifted model; the chart at t_{pr} changes the norm of $H^{N_{\mathrm{out}}}_{\rho(m),\omega}$ by at most a constant, as in Step 2. For two centres y_0 and $\tilde y_0$, let y_0' and $\tilde y_0'$ be their labels in the chart of the block of $l+1$, and $r\coloneqq|y_0'-\tilde y_0'|$. By the moments of order $N_1=3$ about y_0' in Lemma 12.15 and Peetre's inequality, for $j\le N_{\mathrm{out}}$,

$$\big|\big\langle\nabla^j\Pi^{\mathrm{sum}}_{y_0}g,\nabla^j\Pi^{\mathrm{sum}}_{\tilde y_0}g\big\rangle_{L^2_\omega(\mathbb{R}^2)}\big|\le C\rho(m)^{-2j}\Big(1+\frac{r}{\rho(m)}\Big)^{-3}\|g_{y_0}\|_{L^2_\omega(\mathbb{R}^2)}\|g_{\tilde y_0}\|_{L^2_\omega(\mathbb{R}^2)},$$

and the matrix $(1+r/\rho(m))^{-3}$, indexed by the centres, has bounded row sums: the labels of the centres in the chart of the block of $l+1$ form a net of mesh comparable to $\rho(m)/8$ in $\mathbb{R}^2$, as the transition $X_{m-1}(lh_m,\cdot,(l+1)h_m)$ has gradient within $C\varepsilon_{m-1}^{2\delta}$ of the identity, by (7.9). So Schur's test in the patch index gives, by (5.8),

$$\|\Pi^{\mathrm{sum}}g\|_{H^{N_{\mathrm{out}}}_{\rho(m),\omega}(\mathbb{T}^2)}\le C\|g\|_{L^2_\omega(\mathbb{T}^2)}.$$

By density, Π^{sum} extends to a bounded linear map on $L^2(\mathbb{T}^2)^V$.

The errors add in the same way. For a piece g_{y_0}, let E_{y_0} be its error at t_{pr} in the chart of the block of $l+1$, the sum of the difference of Step 3 and of the errors of Step 4. By Step 3, and by Lemma 12.15 with $N_1=3$ and the strengths of Step 4, at most 1 for m large,

$$\|E_{y_0}\|_{L^2_{\omega,3}(\mathbb{R}^2)}\le Ca_{y_0},\qquad a_{y_0}\coloneqq\varepsilon_{m-1}^{Q+3}\|g_{y_0}\|_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{R}^2)},$$

with the moments about y_0'. By the moments, (a5), the periodicity of the coefficients and the uniqueness of the energy solutions on $\mathbb{T}^2$, the lattice sum over the pieces of the terms on $\mathbb{R}^2$ is $U^{\{j\}}$, for each j, and the charts preserve area. So $\sum_{j=0}^{N_{\mathrm{dep}}}U^{\{j\}}(t_{\mathrm{pr}})-\Pi^{\mathrm{sum}}g$ is the sum of the errors E_{y_0} over the pieces and their lattice images. At each y, one of $|y-y_0'|$ and $|y-\tilde y_0'|$ is at least $r/2$, so $|\langle E_{y_0},E_{\tilde y_0}\rangle_{L^2_\omega(\mathbb{R}^2)}|\le C(1+r/\rho(m))^{-3}a_{y_0}a_{\tilde y_0}$, and Schur's test over the centres and their lattice images, with Step 2, gives

$$\Big\|\sum_{j=0}^{N_{\mathrm{dep}}}U^{\{j\}}(t_{\mathrm{pr}})-\Pi^{\mathrm{sum}}g\Big\|^2_{L^2_\omega(\mathbb{T}^2)}\le C\sum_{y_0}a_{y_0}^2\le C\varepsilon_{m-1}^{2(Q+3)}\|g\|^2_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)}.$$

Step 6: Constants. A field with constant components solves the system without smoothing, whose sources carry at least one derivative, so its Duhamel expansion is the field itself in $U^{\{0\}}$ and zero in $U^{\{j\}}$, $j \geq 1$. With $(g)_{\mathbb{T}^2}$ the field of the means of the components, Steps 3–5 therefore give the first bound below, and we define $\widetilde{\Pi}$ by the formula on the right:

$$\big\|\Pi^{\mathrm{sum}}[(g)_{\mathbb{T}^2}] - (g)_{\mathbb{T}^2}\big\|_{L^2_\omega(\mathbb{T}^2)} \leq C\varepsilon_{m-1}^{Q+3}\|g\|_{L^2_\omega(\mathbb{T}^2)}\,, \qquad \widetilde{\Pi}g \coloneqq \Pi^{\mathrm{sum}}g - \Pi^{\mathrm{sum}}[(g)_{\mathbb{T}^2}] + (g)_{\mathbb{T}^2}\,.$$

The map $\widetilde{\Pi}$ fixes the fields with constant components and satisfies both bounds of the proposition, since $\|(g)_{\mathbb{T}^2}\|_{L^2_\omega(\mathbb{T}^2)} \leq \|g\|_{L^2_\omega(\mathbb{T}^2)}$.

Both choices of the masses. The construction of t_{pr} and $\widetilde{\Pi}$ does not use the masses, and the hypotheses hold for both choices (Definition 11.1), so one $\widetilde{\Pi}$ serves both. □

Remark 12.16 (The terminal level). Propositions 12.4–12.7 and Lemmas 12.8–12.15 hold at the terminal level $m = M$; Section 15 uses them, with Lemma 15.8 in place of Lemma 12.17, which is stated for $m < M$.

12.10 Proof of Proposition 5.3(a) and of Proposition 12.1

The Duhamel expansion applies the sources without their smoothing operators, and each source it applies takes derivatives of the datum. Lemma 12.17 supplies them, at the molecular scale; the truncated expansion is within $C\varepsilon_{m-1}^{Q+3}$ of the solution (Step 4 below), and Proposition 12.7 prepares it.

Lemma 12.17 (Molecular smoothing). *For every integer $0 \leq N \leq N_{\mathrm{coef}}$ there exists a constant $C(N)$, depending only on N, such that the following holds. Let $\kappa \in \mathcal{K}_M$ and $m_0 < m < M$, let G be an admissible family of level m on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$, and let $t_{\mathrm{c}} \in [t_{\mathrm{A}}, t_{\mathrm{f}} - \rho_{\mathrm{mol}}(m)^2/\kappa_m]$. For both choices ω_v and ω_v/Ψ_v^2 of the masses and for every datum g, the solution U of G from g at t_{c} satisfies*

$$\|U\|_{L^\infty([t_{\mathrm{c}}+\rho_{\mathrm{mol}}(m)^2/\kappa_m, t_{\mathrm{f}}]; H^N_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2))} \leq C(N)\|g\|_{L^2_\omega(\mathbb{T}^2)}\,.$$

Proof. Proposition 8.3 applies at every order $j \leq N$ at the scale $\rho_{\mathrm{mol}}(m)$, with its diffusivity equal to κ_m: its hypothesis (i) is (12.1); (ii) follows from (b1), (b2) of Definition 11.1 and (7.9); and (iii) is its (d), with sums at most $\frac{1}{8}$ for $m > m_0$ (Proposition 11.5); by its part (b), $\|U(t)\|_{H^j_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)}$ does not increase in t. The $(t-t_{\mathrm{c}})^j$-weighted energy estimate, by induction on j with $\|U\|^2_{H^j_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)} \leq \|U\|^2_{L^2_\omega(\mathbb{T}^2)} + \rho_{\mathrm{mol}}(m)^2\|\nabla U\|^2_{H^{j-1}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)}$ (by (5.8)), gives for $0 < t - t_{\mathrm{c}} \leq \rho_{\mathrm{mol}}(m)^2/\kappa_m$

$$\begin{aligned}&(t-t_{\mathrm{c}})^j\|U(t)\|^2_{H^j_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)} + \kappa_m\int_{t_{\mathrm{c}}}^{t}(t'-t_{\mathrm{c}})^j\|\nabla U(t')\|^2_{H^j_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)}\,\mathrm{d}t'\\ &\qquad\leq (j+1)!\left(\frac{\rho_{\mathrm{mol}}(m)^2}{\kappa_m}\right)^j\|g\|^2_{L^2_\omega(\mathbb{T}^2)}\,,\end{aligned}$$

as $1 + j \cdot j! \leq (j+1)!$: the weight vanishes at t_{c}, whatever the derivatives of the datum. At $t = t_{\mathrm{c}} + \rho_{\mathrm{mol}}(m)^2/\kappa_m$ this is the bound, with $C(N) \coloneqq ((N+1)!)^{1/2}$, which persists. □

Proof of Proposition 5.3(a). The top level m_0 is at least the levels of Propositions 12.5–12.7 and Lemmas 12.8–12.15, and so large that $\rho_{\mathrm{mol}}(m)^2/\kappa_m \leq C\varepsilon_{m-1}^{q(2-\beta)} \leq 2h_m$, by (12.1), Lemma 7.1(a), (2.3) and (2.5). Fix g, let U be the solution of G from g at t_{c}, and put $t_1 \coloneqq t_{\mathrm{c}} + \rho_{\mathrm{mol}}(m)^2/\kappa_m$. Then $t_1 \leq t_{\mathrm{c}} + 2h_m \leq t_{\mathrm{f}} - 2h_m$, as $t_{\mathrm{c}} + 4h_m \leq t_{\mathrm{f}}$ (Proposition 5.3).

Step 1: Smoothing. By Lemma 12.17 with $N = N_{\mathrm{buf}} \le N_{\mathrm{coef}}$,

$$\|U(t_1)\|_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)} \le C\|g\|_{L^2_\omega(\mathbb{T}^2)}\,.$$

Step 2: The remainder of the expansion. Let $U^{\{j\}}$ be the Duhamel expansion of G from $U(t_1)$ at t_1. Applying the operator of level m to each term, the remainder $U - \sum_{j=0}^{N_{\mathrm{dep}}} U^{\{j\}}$ solves on $[t_1, t_{\mathrm{f}}]$ the system (8.2) of G, with its smoothings, from the datum 0 at t_1, with the flux forcing $F := F^{(1)} + F^{(2)}$,

$$F^{(1)}_v := \sum_{v',\sigma,d} A_{\sigma,d}\nabla^{d-1}\big(S_\sigma U^{\{N_{\mathrm{dep}}\}}_{v'}\big)\,, \qquad F^{(2)}_v := \sum_{v',\sigma,d} A_{\sigma,d}\nabla^{d-1}\Big((S_\sigma - I)\sum_{j<N_{\mathrm{dep}}} U^{\{j\}}_{v'}\Big)\,:$$

the last term of the expansion fed back through the sources, and the error of applying the sources without their smoothing. By Proposition 8.3(a) at the order 0, whose (iii) is clause (d) of Definition 11.1,

$$\Big\|U - \sum_{j=0}^{N_{\mathrm{dep}}} U^{\{j\}}\Big\|_{L^\infty([t_1,t_{\mathrm{f}}];L^2_\omega(\mathbb{T}^2))} \le \Big(\frac{8}{\kappa_m}\Big)^{1/2}\|F\|_{L^2((t_1,t_{\mathrm{f}});L^2_\omega(\mathbb{T}^2))}\,.$$

Step 3: The terms of the expansion. The coefficients of the sources are summed by Schur's test: by (b2′) and the transport to x (as in Step 8 of the proof of Lemma 11.7, at the cost of a constant), for every $N \le N_{\mathrm{buf}}$ the matrix with entries

$$\Big(\frac{\omega_{v'}}{\omega_v}\Big)^{1/2}\sum_{\sigma\colon v'\to v}\sum_{d=2}^{N_{\mathrm{src}}}\frac{(N+d-2)!}{N!\,\kappa_m\rho_{\mathrm{mol}}(m)^{d-2}}\sum_{i\le N}\frac{\rho_{\mathrm{mol}}(m)^i}{i!}\|\nabla^i A_{\sigma,d}\|_{L^\infty((t_1,t_{\mathrm{f}})\times\mathbb{T}^2)}$$

is at most C times the majorant matrix, so its row and column sums are at most $C\varepsilon_m^{\delta/4}$, by (b3). Proposition 8.3(a) at the scale $\rho_{\mathrm{mol}}(m)$ and the order $N_{\mathrm{buf}} - j(N_{\mathrm{src}}-2)$, with (12.1) for its (i) and with Lemma 8.1 and these sums for the forcing as in the proof of Proposition 8.6, gives by induction on j, with a constant C,

$$\kappa_m^{1/2}\|\nabla U^{\{j\}}\|_{L^2((t_1,t_{\mathrm{f}});H^{N_{\mathrm{buf}}-j(N_{\mathrm{src}}-2)}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2))} \le C^{j+1}\varepsilon_m^{j\delta/4}\|U(t_1)\|_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)}\,.$$

Step 4: The forcing. We use the order and width conditions, with ℓ the order of the kernels W_ℓ of S_σ in (c1), which depends on σ,

$$N_{\mathrm{mom}} \le 2\ell-1\,, \quad N_{\mathrm{buf}} - j(N_{\mathrm{src}}-2) \ge N_{\mathrm{mom}} + N_{\mathrm{src}} - 2\ (j < N_{\mathrm{dep}})\,, \quad (r_\sigma/\rho_{\mathrm{mol}}(m))^{N_{\mathrm{mom}}} \le C\varepsilon_{m-1}^{q\delta N_{\mathrm{mom}}}\,,$$

the first by (c1), the second by the definition (12.2) of N_{buf}, the third by (c2) and $\varepsilon_m \le 2\varepsilon_{m-1}^q$. In $L^2((t_1,t_{\mathrm{f}});L^2_\omega(\mathbb{T}^2))$, the two parts of the forcing satisfy

$$\begin{aligned}
&\|F^{(1)}\| \le C_{\mathrm{adm}}\varepsilon_{m-1}^{q\delta/4}\kappa_m\|\nabla U^{\{N_{\mathrm{dep}}\}}\|_{L^2((t_1,t_{\mathrm{f}});L^2_\omega(\mathbb{T}^2))} \qquad \text{(Lemma 8.4 and (d))}\,,\\
&\|F^{(2)}\| \le C\kappa_m^{1/2}\varepsilon_{m-1}^{q\delta N_{\mathrm{mom}}}\|U(t_1)\|_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)} \qquad \text{(Proposition 6.1, (b2′), Step 3)}\,;
\end{aligned}$$

in the first line, the sums $C_{\mathrm{adm}}\varepsilon_{m-1}^{q\delta/4}$ of (d) replace the $\frac{1}{8}$ of Lemma 8.4; in the second line, the third bound of Proposition 6.1, which holds for every $r, \rho > 0$ and $|\Omega_i|_\infty \le C_\Omega$, with a constant independent of ℓ as $N_{\mathrm{mom}} \le 2\ell-1$, at the scale $\rho_{\mathrm{mol}}(m)$ with $N_1 = N_{\mathrm{mom}}$, bounds $(I - S_\sigma)\nabla^{d-1}U^{\{j\}}$

by $C(r_\sigma/\rho_{\mathrm{mol}}(m))^{N_{\mathrm{mom}}}\|\nabla^{d-1}U^{\{j\}}\|_{H^{N_{\mathrm{mom}}}_{\rho_{\mathrm{mol}}(m)}(\mathbb{T}^2)}$, and the coefficients $A_{\sigma,d}$ are summed by the matrix of Step 3 at $N=0$. By Steps 2 and 3, $\varepsilon_m^{\delta/4}\le 2\varepsilon_{m-1}^{q\delta/4}$, the choice of N_{dep}, (12.2) and Step 1,

$$\Big\|U-\sum_{j=0}^{N_{\mathrm{dep}}}U^{\{j\}}\Big\|_{L^\infty([t_1,t_{\mathrm{f}}];L^2_\omega(\mathbb{T}^2))}\le C\big(\varepsilon_{m-1}^{q\delta(N_{\mathrm{dep}}+1)/4}+\varepsilon_{m-1}^{q\delta N_{\mathrm{mom}}}\big)\|g\|_{L^2_\omega(\mathbb{T}^2)}\le C\varepsilon_{m-1}^{Q+3}\|g\|_{L^2_\omega(\mathbb{T}^2)}\,,$$

and the same holds with the masses ω_v/Ψ_v^2, for which (b2′), (b3) and (d) also hold.

Step 5: The preparation. Proposition 12.7, with the constants of Definition 11.1 (Proposition 11.5) and with its path hypothesis (an admissible family has paths of at most N_{path} sources, item (1) of Subsection 11.1), gives $t_{\mathrm{pr}}\in(t_1,t_1+2h_m]\subset[t_{\mathrm{c}},t_{\mathrm{c}}+4h_m]$ and $\widetilde\Pi$; we put $\Pi g:=\widetilde\Pi\,U(t_1)$, which is linear in g. Step 4, Proposition 12.7 and Step 1, for the masses ω_v/Ψ_v^2, give the first bound of (a),

$$\|\Pi g-U(t_{\mathrm{pr}})\|_{L^2_G(\mathbb{T}^2)}\le C\varepsilon_{m-1}^{Q+3}\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

Step 6: The other two bounds. The solution U does not grow in $L^2_G(\mathbb{T}^2)$, by Proposition 6.2 for the masses ω_v/Ψ_v^2, whose coupling sums are at most $\frac18$ by (d); the near-unit norm of Π comes from this energy, not from the model. With Step 5 and the first bound of Proposition 12.7 for these weights,

$$\begin{aligned}\|\Pi g\|_{L^2_G(\mathbb{T}^2)}&\le\|U(t_{\mathrm{pr}})\|_{L^2_G(\mathbb{T}^2)}+C\varepsilon_{m-1}^{Q}\|g\|_{L^2_G(\mathbb{T}^2)}\le\big(1+C\varepsilon_{m-1}^{Q}\big)\|g\|_{L^2_G(\mathbb{T}^2)}\,,\\ \|\Pi g\|_{H^{N_{\mathrm{out}}}_{\rho(m),G}(\mathbb{T}^2)}&\le C\|U(t_1)\|_{L^2_G(\mathbb{T}^2)}\le C\|g\|_{L^2_G(\mathbb{T}^2)}\,.\end{aligned}$$ □

Proof of Proposition 12.1. We take t_{pr} and Π as just constructed. (i) $t_{\mathrm{pr}}>t_1=t_{\mathrm{c}}+\rho_{\mathrm{mol}}(m)^2/\kappa_m$, and the margin is that of Proposition 12.7. (ii) A field with constant components is a solution of G at every time, so $U(t_1)=g$, and $\widetilde\Pi$ fixes it. (iii) Lemma 12.17 with $N=2N_{\mathrm{rep}}+4\le N_{\mathrm{ret}}\le N_{\mathrm{coef}}$, as $t_{\mathrm{pr}}\ge t_{\mathrm{c}}+\rho_{\mathrm{mol}}(m)^2/\kappa_m$, and $\|g\|_{L^2_\omega(\mathbb{T}^2)}\le\|g\|_{L^2_G(\mathbb{T}^2)}$, as $\Psi_v\ge1$. (iv) We expand $\exp_p(-\Gamma_m(t_{\mathrm{pr}}):\nabla^2)$: by Lemma 8.1(b) each application of $\Gamma_m(t_{\mathrm{pr}}):\nabla^2$ costs two orders and at most $C(N_{\mathrm{ret}},p)|\Gamma_m(t_{\mathrm{pr}})|/\rho(m)^2\le C\varepsilon_{m-1}^{\delta}$ in $H^{N_{\mathrm{ret}}}_{\rho(m),G}(\mathbb{T}^2)$ (Lemma 10.3(b)), and $N_{\mathrm{ret}}+2p\le N_{\mathrm{out}}$; the third bound of Proposition 5.3(a) concludes. □

13. Two-scale expansion and the homogenization error

In this section we prove Proposition 5.3(b), the comparison of the solution U of G with its two-scale expansion, the lift $Y^\sharp$ of (5.10): the successor G' of Definition 11.3 is admissible, the solution Y of G' keeps the amplitude of the datum, and the homogenization error $U-Y^\sharp$ is small. We also give the formulas for the sources of the successor, whose data Definition 11.3 lists. Section 12 proves part (a) and constructs the preparation time t_{pr} and the map Π used below.

The route. Each of the three steps of the lift (5.10) turns one equation into another, exactly, up to explicit error terms (Figure 13.1).

- *The correctors* (Subsection 13.1). The correctors of Section 10 cancel every term of degree at most N_{cor} of the operator of G applied to a lifted field. There remain the averaged operator $\partial_t+\mathbf{b}_{m-1}\cdot\nabla-\nabla\cdot\mathcal{M}_m$, with the mean flux $\mathcal{M}_m$ of the lift, whose part of order 2 without sources is $\mathbf{J}_m^{\kappa_m}\nabla$ up to a defect, and a tail of mean zero in the fast variable. Each degree of the tail costs the square root $(\varepsilon_m^2/(\kappa_m h_m))^{1/2}$ of the ratio of the cell time to the renewal time. The correctors are switched from pulse to pulse in the quiet intervals, where they decay exponentially.

- *The truncated exponential* (Subsection 13.2). Conjugation by $\exp_p(\Gamma_m : \nabla^2)$ turns $\kappa_{m-1}\Delta$ into $\mathbf{J}_m^{\kappa_m}(t) : \nabla^2$, up to a term of degree p in Γ_m times $\partial_t\Gamma_m = \mathbf{J}_m^{\kappa_m} - \kappa_{m-1}I$, and commutes with $\mathbf{b}_{m-1}\cdot\nabla$ up to the divergence of an explicit flux.
- *The readout* (Subsection 13.3). The part of $\mathcal{M}_m$ beyond $\mathbf{J}_m^{\kappa_m}\nabla$ and the commutator, conjugated by the truncated exponential, are not small. They are kept as the sources of G': this is the construction of the successor. The readout weight ε_{m-1}^{η} that each letter (Definition 11.3(i)) adds cancels the power $\varepsilon_{m-1}^{-\eta}$ of the coefficients of the new source that carries it (Lemma 13.8), so the letters telescope.

The residual of $Y^\sharp$ in the system of G is then a forcing of size $C\varepsilon_{m-1}^{Q}\|g\|_{L^2_G(\mathbb{T}^2)}$ (Subsection 13.4), and Proposition 8.3(a) bounds $U - Y^\sharp$ by it (Subsection 13.5).

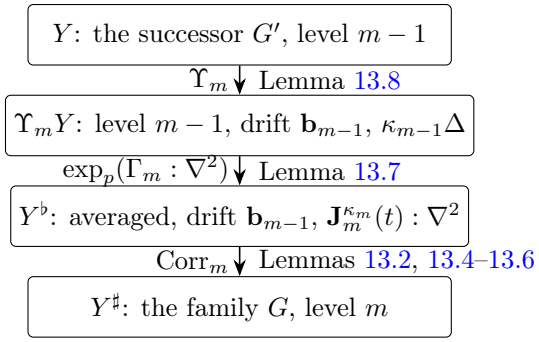


Figure 13.1: The three steps of the lift (5.10), $Y^\sharp = \mathrm{Corr}_m[\exp_p(\Gamma_m : \nabla^2)\Upsilon_m Y]$, from top to bottom. Each box names a field and the equation it solves, up to error terms; $\mathbf{J}_m^{\kappa_m}$ is the averaged flux (2.16).

Setting. Throughout, C and c denote positive constants depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. In Subsection 13.1, *the setting of the lift* is: $M \geq 1$, $\kappa \in \mathcal{K}_M$, $2 \leq m \leq M$, and a family G of level m that satisfies the invariant (Definition 11.1) on an interval $[t_{\mathrm{A}}, t_{\mathrm{f}}]$, whose sources have current order at most N_{src} (the sums of (10.13) stop at the current order), whose paths of sources have at most N_{path} sources, and whose sources of birth order at least 3 have birth radius at most ε_m. An admissible family has these properties (item (1) of Subsection 11.1, Definition 11.2(i), and Step 5 of the proof of Lemma 11.7). Its constants are those of Section 11, or other constants fixed depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$ (for the terminal family of Section 15, with the terminal orders and path bound). Above a level that depends only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$ and is at least m_{c}, such a family has the finite lifts of Section 10 at every order N, with constants depending only on N and $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. Indeed, the majorant matrix is at most the priced one, so its row and column sums are at most 1 by (b3). By (b1) and the bound on the birth radii, $\|\mathcal{A}_{\sigma,d}\|_{\mathrm{an},v}$ is at most a constant times $\mathrm{maj}(\sigma)$. Cauchy's estimate in the base label on $\mathcal{D}_v$, whose supremum runs over the whole window of k, and the exponential weight in z give (10.17) for the forms, at every j_1, j_2; and (7.9) gives it for the conductivity, as the window of k lies in $|t - l_{m,k}h_m| \leq h_m$. Below, Proposition 10.7 is used at the order $N = N_{\mathrm{ret}}$. Clauses (a1)–(d) are those of Definition 11.1.

For a field F of a family of level m we write

$$(\mathbf{A}_m F)_v := \sum_{v'\in V}\sum_{\sigma\colon v'\to v}\sum_{d=2}^{N_{\mathrm{src}}} A_{\sigma,d}\nabla^{d-1}F_{v'}\,, \qquad (\mathbf{A}_m^S F)_v := \sum_{v'\in V}\sum_{\sigma\colon v'\to v}\sum_{d=2}^{N_{\mathrm{src}}} A_{\sigma,d}\nabla^{d-1}(S_\sigma F_{v'}) \quad (13.1)$$

for the flux of its sources without and with the smoothing, so that the system (5.6) of the family reads $\partial_t U + \mathbf{b}_m\cdot\nabla U - \kappa_m\Delta U = \nabla\cdot\mathbf{A}_m^S U$.

Subsections 13.3–13.5 work in the setting of Proposition 5.3: $m_0 < m < M$, an admissible family G of level m on $[t_{\mathrm{A}}, t_{\mathrm{f}}]$ and its successor G', with set of components V', a time t_{c} with $t_{\mathrm{A}} + 3\tau_m \le t_{\mathrm{c}} \le t_{\mathrm{f}} - 4h_m$, the time t_{pr} and the map Π of Section 12, a datum g of G, the solution U of G from g at time t_{c}, and the solution Y of G' from the retained trace of Πg at time t_{pr}. Recall from Subsection 5.5 that the readout is $(\Upsilon_m Y)_v = Y_v + \sum_{v'} \varepsilon_{m-1}^{\eta \#v'} Y_{v'}$, over the children v' of v in G', where $\#v'$ is the number of letters of the word of v'. The retained trace of Πg at time t_{pr} is $\exp_p(-\Gamma_m(t_{\mathrm{pr}}) : \nabla^2)(\Pi g)_v$ on each component v of G and 0 on the new components. Here $t_{\mathrm{f}} - t_{\mathrm{pr}} \le T$ (Definitions 11.2 and 11.4). The family G' is admissible on $[t_{\mathrm{A}} + 3\tau_m, t_{\mathrm{f}}]$ (Subsection 13.5, which uses only Definition 11.4), so, by Proposition 11.5, G and G' satisfy the invariant, with sums at most $\frac{1}{8}$ in clause (d) for both masses, and below $\frac{1}{8}$ at the order 0 when m is above a level depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. Being admissible, G is in the setting of the lift. We write

$$Y^\flat(t) \coloneqq \exp_p(\Gamma_m(t) : \nabla^2)\, \Upsilon_m Y(t)\,, \qquad \text{so that} \qquad Y^{\sharp\sharp}(t) = \mathrm{Corr}_m\, Y^\flat(t)\,. \tag{13.2}$$

Two norms. For a field F of a family with masses ω_v on $[t_1, t_2]$, an integer $N \ge 0$, $\rho > 0$ and $\nu > 0$, the *component energy* of F of order N at the scale ρ, with the diffusivity ν, on $[t_1, t_2]$ is

$$\Big(\sum_{v \in V} \frac{1}{\omega_v} \Big(\|F_v\|^2_{L^\infty([t_1,t_2];H^N_\rho(\mathbb{T}^2))} + \nu \, \|\nabla F_v\|^2_{L^2((t_1,t_2);H^N_\rho(\mathbb{T}^2))} \Big)\Big)^{1/2}, \tag{13.3}$$

the square root of the left side of Proposition 8.3(c), with the supremum in time taken in each component. In this section, *the energy of F of order N* is its component energy (13.3) of order N at the scale $\rho(m)$, with the diffusivity κ_{m-1}, on the interval at hand. For a forcing $(F^{\mathrm{f}}, F^{\mathrm{f},0})$ of the system (8.2) of a family of level m on $[t_1, t_2]$, its *norm* is

$$\kappa_m^{-1/2} \|F^{\mathrm{f}}\|_{L^2((t_1,t_2);L^2_\omega(\mathbb{T}^2))} + \|F^{\mathrm{f},0}\|_{L^1((t_1,t_2);L^2_\omega(\mathbb{T}^2))}\,. \tag{13.4}$$

Orders. Besides the parameters of Subsection 5.3 and Sections 11 and 14, the comparison uses the number N_{cl} of antidivergence steps in the reconstruction of the cell tail (Lemma 13.4; "cl" for cleanup), and fixes the moment order N_{mom} for the smoothings of the sources of G' at the scale $\rho(m)$ and of G at the scale $\rho_{\mathrm{mol}}(m)$ (Step 3 of the proof of Proposition 13.11). They are fixed in the order of item (1) of Subsection 11.1 (Table 11.1), as follows. Given Q (Section 14) and the seed's bounds on its orders and path lengths (Section 15), we choose N_{gen}, N_{cor}, p and N_{cl}; then N_{src} and Θ, with $(N_{\mathrm{src}} - 2)\Theta \le \delta/8$; then N_{len}; then N_{mom}, such that

$$\begin{gathered} \frac{\eta(N_{\mathrm{gen}} + 1)}{8}, \quad \frac{\delta(N_{\mathrm{len}} + 1)}{2}, \quad 3\delta N_{\mathrm{cor}}, \quad \delta p, \quad \Theta N_{\mathrm{mom}} \;>\; Q + q\gamma + 4\,, \\ (q\gamma + 3\delta) N_{\mathrm{cl}} \ge 2 - \beta + 8\delta\,; \end{gathered} \tag{13.5}$$

these give $q\delta N_{\mathrm{mom}} > 12\Theta N_{\mathrm{mom}} > Q + 4$, since $q\delta > \delta > 12\Theta$ by (14.4); the kernels then have vanishing moments up to N_{mom} (clause (c1)), and the third bound of Proposition 6.1 applies with $N_1 = N_{\mathrm{mom}}$. Last, we choose N_{ret} and set $N_{\mathrm{out}} \coloneqq N_{\mathrm{ret}} + 4p$. With $N_{\mathrm{rec}} \coloneqq 4(N_{\mathrm{cor}} + N_{\mathrm{src}}) + 2N_{\mathrm{cl}} + 16$, the order of the reconstruction, and the path bound N_{path} of Section 11, N_{ret} is at least the passage floor of (14.3) and

$$\begin{aligned} &N_{\mathrm{ret}} \ge \max\big\{ N_{\mathrm{rec}} + N_{\mathrm{src}} + N_{\mathrm{mom}} + 6p,\; N_{\mathrm{src}} - 2 + N_{\mathrm{mom}} + N_{\mathrm{cor}} + 2p + 1 \big\}\,, \\ &N_{\mathrm{ret}} \ge N_{\mathrm{rec}} + 2N_{\mathrm{src}} N_{\mathrm{path}} \\ &\qquad + \max\big\{ N_{\mathrm{src}} + N_{\mathrm{mom}} + N_{\mathrm{cor}} + 6p + 4,\; (N_{\mathrm{path}} + 3)(N_{\mathrm{cor}} + N_{\mathrm{src}} + 12) + 2p \big\}\,. \end{aligned} \tag{13.6}$$

The second floor bounds the number of derivatives in the fast variable that a cycle takes along the paths of sources, which have at most N_{path} sources (proof of Lemma 11.7); it depends on N_{len} through N_{path}.

13.1 The two-scale expansion

For a pulse k of level m, a field F of G is read in the chart of its block as $\widetilde{F}_k(t,y)$, with the label $y = X_{m-1}(l_{m,k}h_m, x, t)$ and the fast variable $z = y/\varepsilon_m$. The parts $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}^{\mathrm{src}}_{j'}[\sigma]$ of the operator of the system are those of (10.10) and (10.14). Written without a path, $\mathcal{V}_j$ and $\mathcal{M}_{j,d}$ are the correctors and means of the system, the sums over the paths of Proposition 10.6(f), with $\mathcal{V}_0 = I$. The operators $\mathrm{Corr}_{m,k}$, Corr_m and the mean flux $\mathcal{M}_m$ are those of Definition 10.8, the part of $\mathcal{M}_m$ of degree d along the paths of i sources being the sum of the means of degree d along them.

The cost of a degree. Since $\rho(m)^2 = \kappa_{m-1}h_m$, the corrector amplitude $\bar{\varepsilon}_m$ of Subsection 5.1, divided by $\rho(m)$, is the square root of the ratio of the cell time ε_m^2/κ_m to the renewal time:

$$\frac{\bar{\varepsilon}_m}{\rho(m)} = \Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{1/2} \le C\varepsilon_{m-1}^{3\delta}\,, \qquad \frac{\varepsilon_m}{\rho(m)} = \Big(\frac{\kappa_m}{\kappa_{m-1}}\Big)^{1/2}\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{1/2} \le C\varepsilon_{m-1}^{q\gamma+3\delta}\,. \tag{13.7}$$

The first bound is Lemma 10.1(c); the second holds at every $2 \le m \le M$, since $\varepsilon_m \le 2\varepsilon_{m-1}^q$ (Lemma 7.1(a)), $\rho(m) = (\kappa_{m-1}h_m)^{1/2} \ge c\varepsilon_{m-1}^{1+\gamma/2+\delta}$ (Proposition 2.1(c)), and $q - 1 - \frac{\gamma}{2} - \delta = q\gamma + 3\delta$ by (2.3). The degree is the number of derivatives in y, and every derivative that falls on the field is one of them, so the order never exceeds the degree. By Proposition 10.7, the coefficients of order d of a corrector of degree j are bounded, with their derivatives (each slow derivative costing ε_{m-1}^{-1}, each fast one a constant), by

$$C\bar{\varepsilon}_m^j\varepsilon_{m-1}^{d-j} = C\rho(m)^d\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{j/2}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{j-d}\,, \tag{13.8}$$

in row and column sums of the matrices with the entries multiplied by $(\omega_{v'}/\omega_v)^{1/2}$, as in (11.3). The excess of the degree over the order costs $\rho(m)/\varepsilon_{m-1} \le C\varepsilon_{m-1}^{\gamma/2+\delta}$. The coefficients of the means of degree j satisfy the same bound times κ_m/ε_m^2. We also use $c\varepsilon_m \le \rho(m) \le \varepsilon_{m-1}$, so that $(\rho(m)/\varepsilon_{m-1})^{j-d} \le 1$.

The correctors of a pulse start from zero when its window opens, and stay zero while the shear is off and the forms do not depend on the fast variable (Proposition 10.6(e)). So at a time of $[t_{\mathrm{A}} + 3\tau_m, t_{\mathrm{f}}]$ with no pulse of level m active within τ'_m, such as t_{pr} (Proposition 12.1(i)), the lift adds no corrector.

Lemma 13.1 (Quiet entrance)**.** *The following holds in the setting of the lift, with m above a level depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. Let $t \in [t_{\mathrm{A}} + 3\tau_m, t_{\mathrm{f}}]$ be such that the velocity added at level m vanishes on $[t - \tau'_m, t + \tau'_m]$. Then, for every pulse k with $\mathsf{H}_{m,k}(t) \ne 0$, every corrector $\mathcal{V}_j$ of k with $j \ge 1$ vanishes at t. In particular, for every field F of G, $\mathrm{Corr}_m F = F$ at the time t.*

Proof. Let $\mathsf{H}_{m,k}(t) \ne 0$, so $|t - k\tau_m| < \frac{5}{4}\tau_m$. The window of k starts at $(k - \frac{61}{48})\tau_m > t - \frac{121}{48}\tau_m$. Since $\tau'_m/\tau_m = 4\lceil\varepsilon_{m-1}^{-\delta}\rceil + 1 \ge 5$, every t' between the start of the window and t has $[t' - \frac{\tau_m}{24}, t' + \frac{\tau_m}{24}] \subset [t - \tau'_m, t + \tau'_m]$, and $t' \ge t - 3\tau_m \ge t_{\mathrm{A}}$. So the shear of k vanishes there, and, by (a4), the two-scale forms of G do not depend on z. For m large, the conductivity of the chart lies between $\frac{3}{4}I$ and $\frac{5}{4}I$, by (7.9). So, by Proposition 10.6(e), the correctors of positive degree vanish at t. □

When we apply the operator of the system to the field with its correctors and sort the terms by degree, each corrector of degree $j \le N_{\mathrm{cor}}$ removes the part of mean zero in z of the terms of degree j, and their means move into the averaged equation of F (Definition 10.5). The cell tail is what is left, of degrees above N_{cor}, each degree carrying the small factor $\bar{\varepsilon}_m/\rho(m)$ of (13.7); in Lemma 13.4 one of them pays for the cell rate κ_m/ε_m^2 that multiplies its coefficients.

Lemma 13.2 (The correctors cancel the leading terms)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ such that the following holds in the setting of the lift, with m above a level depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. Let k be a pulse, let I be an interval in the window of k, and let $\widetilde{F}(t, y)$ be a field on I, in the chart of k, with $\widetilde{F}(t, \cdot) \in C^{2N_{\mathrm{cor}}+N_{\mathrm{src}}+1}(\mathbb{R}^2)$ with bounded derivatives, and with $\nabla_y^a \widetilde{F}$, $|a| \le N_{\mathrm{cor}} + 1$, Lipschitz in t, uniformly in y. Then there exists a differential operator $\mathcal{T}_k$ in y, the* cell tail *on the pulse k, depending on no field, of orders 1 to $2N_{\mathrm{cor}} + N_{\mathrm{src}}$, whose coefficients have mean zero in z and whose terms have degrees $N_{\mathrm{cor}} + 1$ to $2N_{\mathrm{cor}} + N_{\mathrm{src}}$, such that, off a null set of t independent of (y, z),*

$$\Big(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 - \sum_{\sigma} \sum_{j' \le N_{\mathrm{src}}} \mathcal{L}_{j'}^{\mathrm{src}}[\sigma]\Big) \sum_{j''=0}^{N_{\mathrm{cor}}} \mathcal{V}_{j''} \widetilde{F} \\ - \sum_{j''=0}^{N_{\mathrm{cor}}} \mathcal{V}_{j''} \Big(\partial_t \widetilde{F} - \sum_{j=2}^{N_{\mathrm{cor}}+N_{\mathrm{src}}} \sum_{d=2}^{j} \nabla_y \cdot \big(\mathcal{M}_{j,d} \nabla_y^{d-1} \widetilde{F}\big)\Big) = \mathcal{T}_k \widetilde{F}\,. \tag{13.9}$$

Its coefficients of order d and degree j satisfy (13.8) times κ_m/ε_m^2, in row and column sums as there, each slow derivative up to the order N_{cl} costing ε_{m-1}^{-1}.

Proof. Off a null set of t, the derivatives in t of the $\nabla_y^a \widetilde{F}$, $|a| \le N_{\mathrm{cor}}$, and of the coefficients $\mathcal{V}_{j,d}$ exist at every (y, z): their difference quotients are Lipschitz in (y, z), uniformly in the increment (by the Lipschitz bound one derivative higher, and by (10.12) with the Wiener norm of Proposition 10.7), and they converge on a countable dense set, at the Lebesgue points of the data of the cell equations. So the limits for the $\nabla_y^a \widetilde{F}$ commute with ∇_y, and the limit for $\mathcal{V}_{j,d}$ is the right side of its cell equation at every (y, z), by continuity. Thus, off this null set, the product rule holds for $\mathcal{V}_{j,d} \nabla_y^d \widetilde{F}$, Proposition 10.6(b) holds at every (y, z), and the left side of (13.9) is computed at the time t as an identity between differential operators applied to $\widetilde{F}$. We sort its terms by degree. The cell operator $\mathcal{L}_0$ contains ∂_t, so on $\mathcal{V}_j \widetilde{F}$ it acts on the coefficients and gives $\mathcal{V}_j \partial_t \widetilde{F}$, which cancels. The remaining terms act on $\widetilde{F}$ through its derivatives in y, with coefficients built from the correctors, the means and the parts of the operator, which depend on no field. At the degree 0 only $\sum_\sigma \mathcal{L}_0^{\mathrm{src}}[\sigma] \widetilde{F}$ remains, and it vanishes, as $\widetilde{F}$ does not depend on z. The terms of degree $j \ge 1$ are then $\mathcal{L}_0$ acting on the coefficients of $\mathcal{V}_j \widetilde{F}$, plus the bracket of degree j of Definition 10.5 (for the system, by Proposition 10.6(f)), plus $\sum_{j''=0}^{j-2} \mathcal{V}_{j''} \sum_d \nabla_y \cdot (\mathcal{M}_{j-j'',d} \nabla_y^{d-1} \widetilde{F})$, with $\mathcal{V}_{j''} = 0$ for $j'' > N_{\mathrm{cor}}$ and no term for $j - j'' > N_{\mathrm{cor}} + N_{\mathrm{src}}$. For $1 \le j \le N_{\mathrm{cor}}$ this is $\mathcal{L}_0 \mathcal{V}_j \widetilde{F}$ minus (10.16), which vanishes by Proposition 10.6(b).

For $j > N_{\mathrm{cor}}$ no corrector is added, and the terms of degree j are the bracket plus the sum. The term $j'' = 0$ of the sum cancels the mean of the bracket (Proposition 10.6(a); the bracket and the means of Definition 10.5 are both formed with $\mathcal{V}_j = 0$ for $j > N_{\mathrm{cor}}$), and the terms $j'' \ge 1$ have mean zero, as the correctors do. For $j > N_{\mathrm{cor}} + N_{\mathrm{src}}$ the bracket and the term $j'' = 0$ vanish, since a source has at most N_{src} slow derivatives. So the degrees are $N_{\mathrm{cor}} + 1$ to $2N_{\mathrm{cor}} + N_{\mathrm{src}}$, and every term has order at least 1, since the correctors of positive degree annihilate constants and $\mathcal{L}_1$, $\mathcal{L}_2$, the source parts and the means differentiate.

Each term of degree j is a lower corrector times one of the four products (10.22) divided by ε_m^2/κ_m, or a corrector times a mean. By Proposition 10.7, its coefficients obey (13.8) times κ_m/ε_m^2, in row and column sums, which are submultiplicative, those of the majorant matrix being at most 1 by (b3). For the slow derivatives up to the order N_{cl}, we use Proposition 10.7 at an order $N \ge N_{\mathrm{cl}} + N_{\mathrm{cor}} + N_{\mathrm{src}}$ (such as N_{ret}, by (13.6)). In a coefficient of the tail, a corrector carries at most N_{src} slow derivatives and a mean at most $N_{\mathrm{cor}} + 1$ (one from the divergence and at most N_{cor} from the corrector applied

to it), so that after $N_{\rm cl}$ further slow derivatives each carries at most N. Each slow derivative of a corrector, a mean, the conductivity or a form costs ε_{m-1}^{-1} (Proposition 10.7, (10.17)). □

At one time, the correctors change a field by at most a constant times $\bar\varepsilon_m/\rho(m)$ times its derivatives at the scale $\rho(m)$, and, as they oscillate at the wavelength ε_m, each derivative of the change costs ε_m^{-1} (Lemma 13.3). On a field smooth at the wavelength ε_{m-1}, this gives the cost (5.2) of the correctors in Step 4 of the proof of Proposition 5.4, where the condition $s<\frac12(1-\alpha)$ is used.

Lemma 13.3 (The correctors on a field)**.** *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)<\infty$ such that, in the setting of the lift with $m_0<m<M$, for every $t\in[t_{\rm A}+3\tau_m,t_{\rm f}]$, every field F of G of class $C^{N_{\rm rep}+N_{\rm cor}+2}$ on $\mathbb{T}^2$, every multi-index a with $|a|\le N_{\rm rep}$ and every $x\in\mathbb{T}^2$,*

$$\varepsilon_m^{|a|}\left|\partial^a(\mathrm{Corr}_m F-F)(x)\right|_\omega \le C\,\frac{\bar\varepsilon_m}{\rho(m)}\sum_{1\le|a'|\le N_{\rm rep}+N_{\rm cor}}\rho(m)^{|a'|}\,|\partial^{a'}F(x)|_\omega\,, \tag{13.10}$$

where Corr_m *acts at the time* t*, as in* (5.10)*.*

Proof. At each time at most two time cutoffs are positive, so it suffices to bound $\mathrm{Corr}_{m,k}F-F=\sum_{j=1}^{N_{\rm cor}}\mathcal{V}_j\widetilde F_k$, read at $y=X_{m-1}(l_{m,k}h_m,x,t)$, $z=y/\varepsilon_m$. The coefficients of the correctors of positive degree act on derivatives of order $d\ge1$, since these correctors annihilate constants. A derivative in x falls on the fast variable (cost ε_m^{-1}), on a slow coefficient or on the chart (cost $C\varepsilon_{m-1}^{-1}$, by (7.9)), or on F. So, by (13.8) and the chain rule, $\varepsilon_m^{|a|}|\partial^a\mathcal{V}_j\widetilde F_k|$ is at most a sum, over $1\le d'\le d\le j$ and $d'\le|a'|\le d'+|a|$, of $C\rho(m)^{d'}(\varepsilon_m^2/(\kappa_m h_m))^{j/2}(\rho(m)/\varepsilon_{m-1})^{j-d'}\varepsilon_m^{|a'|-d'}|\partial^{a'}F|$. Indeed, $\nabla_y^d\widetilde F_k$ contains the derivatives of F of every order $1\le d'\le d$, the other $d-d'$ derivatives falling on the chart at the cost $C\varepsilon_{m-1}^{d'-d}$, which turns (13.8) at the order d into (13.8) at the order d'. Moreover

$$\rho(m)^{d'}\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{\frac j2}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{j-d'}\varepsilon_m^{|a'|-d'}=\frac{\bar\varepsilon_m}{\rho(m)}\,\rho(m)^{|a'|}\cdot\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{\frac{j-1}2}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{j-d'}\Big(\frac{\varepsilon_m}{\rho(m)}\Big)^{|a'|-d'},$$

and the last three powers are bounded: the first and the third by (13.7), as $j\ge1$ and $|a'|\ge d'$, and the second by $\rho(m)\le\varepsilon_{m-1}$, as $j\ge d'$. Here $1\le|a'|\le N_{\rm cor}+N_{\rm rep}$. By Proposition 10.7(a), whose hypothesis holds by (b3), and Schur's test, the coefficients $\sum\mathcal{V}_{j,d}$, summed over the paths, act on $(\mathbb{R}^V,|\cdot|_\omega)$ with norm at most $C\bar\varepsilon_m^j\varepsilon_{m-1}^{d-j}$, so these bounds hold in the norm $|\cdot|_\omega$. □

The tail in physical space. The cell tail of (13.9) has mean zero in the fast variable, and is read at $z=y/\varepsilon_m$. A function of mean zero in z is a derivative in z, and a derivative in z read at $z=y/\varepsilon_m$ is ε_m times a derivative in y, up to a new slow derivative. So each antidivergence step gains the factor $\varepsilon_m/\rho(m)$, and after $N_{\rm cl}$ steps only a scalar remainder of that size is left. In (13.11) the factor before the parenthesis is the power $N_{\rm cor}$ of $(\varepsilon_m^2/(\kappa_m h_m))^{1/2}=\bar\varepsilon_m/\rho(m)$ (by (13.7)), one degree paying for the cell rate against the dissipation. In the second term, the remainder, the power $N_{\rm cl}$ of $\varepsilon_m/\rho(m)$ beats, by the choice of $N_{\rm cl}$ in (13.5), the square root of the number of cell diffusion times in $[t_1,t_2]$ when $t_2-t_1\le T$.

Lemma 13.4 (The cell tail)**.** *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)<\infty$ such that the following holds in the setting of the lift, with m above a level depending only on $\alpha,s,q,\zeta,\Lambda,n,t_0,T$. Let $[t_1,t_2]\subset[t_{\rm A}+3\tau_m,t_{\rm f}]$, and let F be a field of G on $[t_1,t_2]$, of class $C^{N_{\rm rec}+1}$ in x at each time, with derivatives measurable in (t,x). Then there exists a forcing $(F^{\rm tl},F^{\rm tl,0})$ such that the fields $\mathcal{T}_k\widetilde F_k$, with $\mathcal{T}_k$ the cell tail of Lemma 13.2, read at $z=y/\varepsilon_m$, carried to x by the charts and glued by the*

cutoffs $\mathsf{H}_{m,k}(t)$*, are* $\nabla\cdot F^{\mathrm{tl}}+F^{\mathrm{tl},0}$*, and such that its norm* (13.4) *is at most the energy of* F *of order* N_{rec} *on* $[t_1,t_2]$ *times*

$$C\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{N_{\mathrm{cor}}/2}\Big(1+(t_2-t_1)^{1/2}\frac{\kappa_m^{1/2}}{\varepsilon_m}\Big(\frac{\varepsilon_m}{\rho(m)}\Big)^{N_{\mathrm{cl}}}\Big). \tag{13.11}$$

Proof. Step 1: Antidivergence. The forms are transverse, by clause (a2), so each coefficient of the cell tail depends on z only through $z\cdot e_{(k)}^{\perp}$. A function of mean zero in $z\cdot e_{(k)}^{\perp}$ has a primitive $(e_{(k)}^{\perp}\cdot\nabla_z)^{-1}$ of mean zero, bounded by its supremum, with the same slow derivatives. Let F_0^{cl} be one term of the cell tail of $\widetilde{F}_k$, a function of (t,y,z) of mean zero in z, and let the cleanup iterates be $F_{i+1}^{\mathrm{cl}}:=-e_{(k)}^{\perp}\cdot\nabla_y(e_{(k)}^{\perp}\cdot\nabla_z)^{-1}F_i^{\mathrm{cl}}$, again of mean zero. The chain rule gives

$$F_0^{\mathrm{cl}}\Big(t,y,\frac{y}{\varepsilon_m}\Big)=\nabla_y\cdot\Big(\sum_{i=0}^{N_{\mathrm{cl}}-1}\varepsilon_m^{i+1}e_{(k)}^{\perp}\big((e_{(k)}^{\perp}\cdot\nabla_z)^{-1}F_i^{\mathrm{cl}}\big)\Big(t,y,\frac{y}{\varepsilon_m}\Big)\Big)+\varepsilon_m^{N_{\mathrm{cl}}}F_{N_{\mathrm{cl}}}^{\mathrm{cl}}\Big(t,y,\frac{y}{\varepsilon_m}\Big). \tag{13.12}$$

Each slow derivative in F_{i+1}^{cl} falls on a coefficient (cost $\varepsilon_{m-1}^{-1}\le\rho(m)^{-1}$) or on $\widetilde{F}_k$.

Step 2: Size of the flux. Let the term F_0^{cl} have degree $j\ge N_{\mathrm{cor}}+1$ and order d. Lemma 13.2 bounds the flux of (13.12) with $i=0$ by $C\varepsilon_m\kappa_m/\varepsilon_m^2$ times the bound (13.8) times $|\nabla_y^d\widetilde{F}_k|$. By the chain rule, the first line of (7.9), area preservation and $\rho(m)\le\varepsilon_{m-1}$, and by (5.8) and (13.3), for $d\ge1$, $\|\nabla_y^d\widetilde{F}_k\|_{L^2((t_1,t_2);L^2_\omega(\mathbb{T}^2))}$ is at most $C\rho(m)^{1-d}\kappa_{m-1}^{-1/2}$ times the energy of F of order d. Also $\kappa_m^{-1/2}\kappa_m\varepsilon_m^{-1}\kappa_{m-1}^{-1/2}\rho(m)=(\kappa_m h_m/\varepsilon_m^2)^{1/2}$. So the flux contributes to (13.4) at most

$$C\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{\frac{j-1}{2}}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{j-d}\le C\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{\frac{N_{\mathrm{cor}}}{2}}$$

times the energy, by (13.7). The steps $i\ge1$ add the factors $(\varepsilon_m/\rho(m))^i\le C$.

Step 3: The remainder. The remainder $\varepsilon_m^{N_{\mathrm{cl}}}F_{N_{\mathrm{cl}}}^{\mathrm{cl}}$ is at most $(\varepsilon_m/\rho(m))^{N_{\mathrm{cl}}}$ times the bound of Step 2 divided by ε_m, on derivatives $\nabla_y^{d'}\widetilde{F}_k$ with $d\le d'\le d+N_{\mathrm{cl}}$. Its norm in $L^1((t_1,t_2);L^2_\omega(\mathbb{T}^2))$ is at most $(t_2-t_1)^{1/2}$ times its norm in L^2. As in Step 2, with the relation $\kappa_m\varepsilon_m^{-2}\kappa_{m-1}^{-1/2}\rho(m)=(\kappa_m^{1/2}/\varepsilon_m)(\kappa_m h_m/\varepsilon_m^2)^{1/2}$, this gives the second term of (13.11).

Step 4: Back to x. The charts preserve area and turn the divergence in y into that in x (Piola, Lemma 7.9, with $|\nabla_y X_{m-1}|\le2$), and derivatives in y of order d' into derivatives in x of orders $d''\le d'$ at the cost $C\varepsilon_{m-1}^{d''-d'}$ (Lemma 7.7); at most two cutoffs are positive at each time, and the orders used are at most $2N_{\mathrm{cor}}+N_{\mathrm{src}}+N_{\mathrm{cl}}\le N_{\mathrm{rec}}$, by Step 3 and Lemma 13.2. On the paths with sources, the sums over the paths are bounded by (b3). □

Switching. Lemma 13.5 glues the identities of Lemma 13.2 in the coordinates x, using the lift of each pulse where its time cutoff is positive. The gluing produces the derivatives $\mathsf{H}'_{m,k}=\partial_t\mathsf{H}_{m,k}$ of the cutoffs, supported in the *quiet intervals* $[(k+\frac23)\tau_m,(k+\frac43)\tau_m]$, $k\in2\mathbb{Z}+1$, between consecutive pulses, where the correctors of the pulse that has just switched off decay at the rate $c\kappa_m/\varepsilon_m^2$, and those of the next pulse have not started.

Lemma 13.5 (Switching between pulses)**.** *There exist* $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)<\infty$ *and* $c(\alpha,s,q,\zeta,\Lambda,n,t_0,T)>0$ *such that the following holds in the setting of the lift, with* m *above a level depending only on* $\alpha,s,q,\zeta,\Lambda,n,t_0,T$*. Let* $[t_1,t_2]\subset[t_{\mathrm{A}}+3\tau_m,t_{\mathrm{f}}]$ *with* $t_2-t_1\le T$*, and let* F *be a field of* G *on* $[t_1,t_2]$*, of class* $C^{N_{\mathrm{rec}}+1}$ *in* x *at each time, with derivatives measurable in* (t,x)*, with* $\nabla_x^a F$*,*

$|a| \le N_{\mathrm{cor}}+2$, bounded on $[t_1,t_2]\times\mathbb{T}^2$, and with $\nabla_x^a F$, $|a|\le N_{\mathrm{cor}}+1$, Lipschitz in t, uniformly in x. Then, with a forcing $(F^{\mathrm{tl}},F^{\mathrm{tl},0})$ as in Lemma 13.4, at almost every $t\in(t_1,t_2)$,

$$\begin{aligned}&(\partial_t+\mathbf{b}_m\cdot\nabla-\kappa_m\Delta)\operatorname{Corr}_m F-\nabla\cdot(\mathbf{A}_m\operatorname{Corr}_m F)-\operatorname{Corr}_m\big(\partial_t F+\mathbf{b}_{m-1}\cdot\nabla F-\nabla\cdot(\mathcal{M}_m F)\big)\\&\qquad=\nabla\cdot F^{\mathrm{tl}}+F^{\mathrm{tl},0}+\sum_{k\in2\mathbb{Z}+1}\mathsf{H}'_{m,k}(t)\,(\operatorname{Corr}_{m,k}F-F)\,,\end{aligned}\tag{13.13}$$

and the norm of the last sum in $L^1((t_1,t_2);L^2_\omega(\mathbb{T}^2))$ is at most the energy of F of order N_{rec} on $[t_1,t_2]$ times

$$C\Big(\frac{\varepsilon_m^2}{\kappa_m}\Big)^{1/2}\Big(\frac{1+(t_2-t_1)/\tau_m}{\tau_m}\Big)^{1/2}\exp\Big(-c\,\frac{\kappa_m\tau_m}{\varepsilon_m^2}\Big).\tag{13.14}$$

Moreover $\kappa_m\tau_m/\varepsilon_m^2\ge c\varepsilon_{m-1}^{-4\delta}$, so the exponential is smaller than any power of ε_{m-1}. The field $\operatorname{Corr}_m F$ is Lipschitz in time in $L^2(\mathbb{T}^2)$, so (13.13) holds in the form (8.3).

Proof. Step 1: One pulse. Where $\mathsf{H}_{m,k}>0$, the velocity added at level m is the shear of k, and $\mathbf{A}_m$ is the transport of the coefficients of k. Where $\mathsf{H}_{m,k}=1$, this holds as the other cutoffs vanish; where a neighbouring cutoff is also positive, the velocity added at level m vanishes, and by (a6) the transports of the coefficients of the two pulses agree. By Lemma 7.9, the chart turns $\partial_t+\mathbf{b}_m\cdot\nabla-\kappa_m\Delta-\nabla\cdot\mathbf{A}_m$ into the operator on the left of (13.9) in the labels, and ∂_t at fixed label into $\partial_t+\mathbf{b}_{m-1}\cdot\nabla$ at fixed x. On the part of the window outside the block of $l_{m,k}$, the velocity added at level m vanishes, and the proof of Lemma 7.9 applies verbatim. Lemma 13.2 applies to $\widetilde{F}_k$ on the interval I where $[t_1,t_2]$ meets the window of k: its class follows from that of F, by $\partial_t X_{m-1}=\mathbf{b}_{m-1}(t,X_{m-1})$ and (7.9). So Lemma 13.2, carried to x, is (13.13) with $\operatorname{Corr}_{m,k}$ in place of Corr_m, the means of the pulse k carried to x in place of $\mathcal{M}_m$, the cell tail of k, and without the last sum.

Step 2: Gluing. The cutoffs depend only on t, $\sum_k\mathsf{H}_{m,k}=1$ and $\sum_k\mathsf{H}'_{m,k}=0$. So the operator of G without smoothing, applied to $\operatorname{Corr}_m F=\sum_k\mathsf{H}_{m,k}\operatorname{Corr}_{m,k}F$, is $\sum_k\mathsf{H}_{m,k}$ times its application to $\operatorname{Corr}_{m,k}F$, plus $\sum_k\mathsf{H}'_{m,k}(\operatorname{Corr}_{m,k}F-F)$. By Definition 10.8, $\mathcal{M}_m$ is the sum over k of $\mathsf{H}_{m,k}$ times the means of the pulse k carried to x. By Step 1, the first sum gives (13.13) when, in the term of k, the means of k are replaced by $\mathcal{M}_m$. Their difference is nonzero only where the cutoffs of k and of a neighbour k' overlap, where $\frac34\tau_m\le|t-k\tau_m|\le\frac54\tau_m$, inside the quiet intervals. There the means of each pulse are the operator of G itself, read in its chart (Proposition 10.6(d)). Their part $\kappa_m\mathbf{K}$ carries to the flux $\kappa_m\nabla F$ in every chart, as $\nabla_y X_{m-1}\,\mathbf{K}_{m,l}\,(\nabla_y X_{m-1})^{\mathsf{T}}=I$ by (7.8), and consecutive charts describe the same coefficients of the sources, by (a6); so the replacement costs nothing. The field $\operatorname{Corr}_m F$ is a finite sum of products of $\mathsf{H}_{m,k}(t)$, the coefficients $\mathcal{V}_{j,d}$, derivatives of the chart and $\nabla_y^d\widetilde{F}_k$, each Lipschitz in t uniformly in x, which proves the last assertion of the lemma.

Step 3: The switching term. The derivative $\mathsf{H}'_{m,k}$ is supported where $\frac34\tau_m\le|t-k\tau_m|\le\frac54\tau_m$, and $|\mathsf{H}'_{m,k}|\le C/\tau_m$. There the shear of k vanishes, since its envelope is supported where $|t-k\tau_m|<\frac23\tau_m$. On the side before k, the positive correctors of k vanish, as in Lemma 13.1, since the stretch from the start $(k-\frac{61}{48})\tau_m$ of the window to $(k-\frac34)\tau_m$, enlarged by $\frac{\tau_m}{24}$, lies in the quiet interval between $k-2$ and k. On the side after k, for $t'\ge(k+\frac{17}{24})\tau_m$, $[t'-\frac{\tau_m}{24},t'+\frac{\tau_m}{24}]$ lies in the quiet interval $[(k+\frac23)\tau_m,(k+\frac43)\tau_m]$, on which the velocity added at level m vanishes, as ζ has compact support in $(-\frac23,\frac23)$. So, at these times, the shear of k vanishes, the forms do not depend on z, by (a4), and neither do the conductivity and the means. As in the proof of Proposition 10.6(e), the terms of the brackets (10.16) with $\mathcal{V}_0[\emptyset]=I$ then do not depend on z and give nothing, and the positive correctors solve a triangular system of homogeneous cell equations, with no datum except the lower correctors, the mode ξ in $z\cdot e_{(k)}^{\perp}$ of each forced only by the modes ξ of the lower ones. On every transverse Fourier mode the cell inverse decays at a rate at least $c\kappa_m/\varepsilon_m^2$, by the ellipticity of

the conductivity (Lemma 10.4). After the time $\frac{\tau_m}{24}$ the positive correctors are therefore bounded by $C\exp(-c\kappa_m\tau_m/\varepsilon_m^2)$ times their sizes (13.8); the polynomial factors of the triangular system are absorbed by lowering c. The supports of the $\mathsf{H}'_{m,k}$ in (t_1,t_2) are at most $C(1+(t_2-t_1)/\tau_m)$ intervals of length at most τ_m. So, by the Cauchy–Schwarz inequality in time, the norm of the last sum of (13.13) in $L^1((t_1,t_2);L^2_\omega(\mathbb{T}^2))$ is at most $C((1+(t_2-t_1)/\tau_m)/\tau_m)^{1/2}$ times the norm of the fields $\mathrm{Corr}_{m,k}F-F$ in $L^2((t_1,t_2);L^2_\omega(\mathbb{T}^2))$ over these supports. As in Lemma 13.4, for $j\geq 1$, (13.8) times $\rho(m)^{1-d}\kappa_{m-1}^{-1/2}$ is at most $\rho(m)\kappa_{m-1}^{-1/2}(\varepsilon_m^2/(\kappa_m h_m))^{1/2}=(\varepsilon_m^2/\kappa_m)^{1/2}$, which gives (13.14). Last, $\kappa_m\tau_m/\varepsilon_m^2\geq c\varepsilon_{m-1}^{-4\delta}$ by Proposition 2.1(c). □

In Proposition 13.11, the field $Y^\flat$ of (13.2) solves the averaged equation up to the divergence of a small flux F^{av}, which the corrector operator carries into the system of G. The correctors of positive degree give the second, small term of (13.15), as in Lemma 13.4. The first term is the cost of the identity part: the flux is bounded against the dissipation of level $m-1$, but the norm (13.4) uses κ_m. For $m<M$ this cost is at most $C\varepsilon_{m-1}^{-q\gamma}$, which the term $q\gamma$ in (13.5) absorbs, and at $m=M$ at most C, by Proposition 2.1(c).

Lemma 13.6 (A coarse flux under the lift)**.** *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)<\infty$ such that the following holds in the setting of the lift, with m above a level depending only on $\alpha,s,q,\zeta,\Lambda,n,t_0,T$. Let $[t_1,t_2]\subset[t_{\mathrm{A}}+3\tau_m,t_{\mathrm{f}}]$, and let F^{av} be a flux, a vector field for each component of G, on $[t_1,t_2]$, of class $C^{N_{\mathrm{rec}}+1}$ in x at each time, with derivatives measurable in (t,x). Then there exists a forcing $(F^{\mathrm{co}},F^{\mathrm{co},0})$ such that $\mathrm{Corr}_m(\nabla\cdot F^{\mathrm{av}})=\nabla\cdot F^{\mathrm{co}}+F^{\mathrm{co},0}$ and such that its norm (13.4) is at most*

$$C\Big[\Big(\frac{\kappa_{m-1}}{\kappa_m}\Big)^{1/2}+\frac{\varepsilon_m^2}{\kappa_m h_m}\Big(1+(t_2-t_1)^{1/2}\frac{\kappa_m^{1/2}}{\varepsilon_m}\Big(\frac{\varepsilon_m}{\rho(m)}\Big)^{N_{\mathrm{cl}}}\Big)\Big]\kappa_{m-1}^{-1/2}\|F^{\mathrm{av}}\|_{L^2((t_1,t_2);H^{N_{\mathrm{rec}}}_{\rho(m),\omega}(\mathbb{T}^2))}\,. \tag{13.15}$$

Proof. The identity part of Corr_m gives $\nabla\cdot F^{\mathrm{av}}$, whose flux contributes $\kappa_m^{-1/2}\|F^{\mathrm{av}}\|_{L^2((t_1,t_2);L^2_\omega(\mathbb{T}^2))}=(\kappa_{m-1}/\kappa_m)^{1/2}\kappa_{m-1}^{-1/2}\|F^{\mathrm{av}}\|_{L^2((t_1,t_2);L^2_\omega(\mathbb{T}^2))}$. A corrector of positive degree j has mean zero in z, so its action on $\nabla\cdot F^{\mathrm{av}}$ is reconstructed by (13.12), as in Lemma 13.4. The first flux is at most ε_m times (13.8) times $|\nabla_y^d((\nabla\cdot F^{\mathrm{av}})\circ X_{m-1})|$, whose norm in $L^2((t_1,t_2);L^2_\omega(\mathbb{T}^2))$ is at most $C\rho(m)^{-d-1}\|F^{\mathrm{av}}\|_{L^2((t_1,t_2);H^{d+1}_{\rho(m),\omega}(\mathbb{T}^2))}$, by the chain rule and (7.9). As $\kappa_m^{-1/2}\varepsilon_m\rho(m)^{-1}=(\varepsilon_m^2/(\kappa_m h_m))^{1/2}\kappa_{m-1}^{-1/2}$, it contributes at most

$$C\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{\frac{j+1}{2}}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{j-d}\leq C\,\frac{\varepsilon_m^2}{\kappa_m h_m}$$

times $\kappa_{m-1}^{-1/2}\|F^{\mathrm{av}}\|_{L^2((t_1,t_2);H^{d+1}_{\rho(m),\omega}(\mathbb{T}^2))}$, since $j\geq 1$. The remainder after N_{cl} steps is bounded as in Step 3 of Lemma 13.4. □

13.2 Conjugation by the exponential of the covariance

The averaged operator has the time-dependent averaged flux $\mathbf{J}_m^{\kappa_m}(t)$; the operator of level $m-1$ has the constant diffusivity κ_{m-1}, the average of $\mathbf{J}_m^{\kappa_m}$ over a period (2.17). The covariance $\Gamma_m(t)$ of (5.11) is the primitive of their difference, and the full exponential $\exp(\Gamma_m:\nabla^2)$ would conjugate one into the other if the drift were absent. We write $[\cdot,\cdot]$ for the commutator of two operators, and $A:\nabla^2:=\sum_{i,i'}A_{ii'}\partial_i\partial_{i'}$ for a 2×2 matrix A, as for Γ_m before (5.11). In Lemma 13.7, (i) writes the commutator of the coarse drift with the truncated exponential as the divergence of an explicit flux, as $\Gamma_m(t)$ is a symmetric matrix independent of x and the drift is divergence free; (ii) is the exact conjugation identity, with this commutator and the truncation at the order p as error terms; by (iii),

as the covariance is a small multiple of $\rho(m)^2$ (Lemma 10.3(b)), each power of $\Gamma_m : \nabla^2$ gains $\varepsilon_{m-1}^{\delta}$ on fields smooth at the scale $\rho(m)$, at the cost of two derivatives.

Lemma 13.7 (Temporal conjugation)**.** *Let $M \geq 1$, $\kappa \in \mathcal{K}_M$ and $2 \leq m \leq M$.*

(i) *For every $t \in \mathbb{R}$, $\Gamma_m = \Gamma_m(t)$ is a symmetric matrix independent of x, and, for every function F of class C^{2p+2} on $\mathbb{T}^2$,*

$$[\mathbf{b}_{m-1} \cdot \nabla, \Gamma_m : \nabla^2]F = \nabla \cdot \big(-(\nabla \mathbf{b}_{m-1}\, \Gamma_m + \Gamma_m\, (\nabla \mathbf{b}_{m-1})^{\mathsf{T}})\nabla F\big)\,, \tag{13.16}$$

where $(\nabla \mathbf{b}_{m-1})_{ii'} \coloneqq \partial_{i'}(\mathbf{b}_{m-1})_i$. Consequently the commutator with the truncated exponential is a divergence,

$$\begin{aligned}&[\mathbf{b}_{m-1} \cdot \nabla, \exp_p(\Gamma_m : \nabla^2)]F \\ &\quad = \nabla \cdot \Big(\sum_{j=1}^{p} \frac{1}{j!} \sum_{j'+j''=j-1} (\Gamma_m : \nabla^2)^{j'} \Big(-\big(\nabla \mathbf{b}_{m-1}\, \Gamma_m + \Gamma_m (\nabla \mathbf{b}_{m-1})^{\mathsf{T}}\big)\nabla (\Gamma_m : \nabla^2)^{j''} F\Big)\Big),\end{aligned} \tag{13.17}$$

and we call the field in the divergence the flux of the commutator *at F.*

(ii) *For every $t_1 < t_2$ and every field F on $[t_1, t_2]$ of class C^{2p+2} in x at each time, with $\nabla_x^a F$, $|a| \leq 2p+1$, Lipschitz in t, uniformly in x, there exists a null set of t, independent of x, off which*

$$\begin{aligned}&\big(\partial_t + \mathbf{b}_{m-1} \cdot \nabla - \mathbf{J}_m^{\kappa_m}(t) : \nabla^2\big) \exp_p(\Gamma_m : \nabla^2)F \\ &\quad = \exp_p(\Gamma_m : \nabla^2)\big(\partial_t + \mathbf{b}_{m-1} \cdot \nabla - \kappa_{m-1}\Delta\big)F \\ &\qquad + [\mathbf{b}_{m-1} \cdot \nabla, \exp_p(\Gamma_m : \nabla^2)]F - \big(\mathbf{J}_m^{\kappa_m}(t) - \kappa_{m-1} I\big) : \nabla^2 \frac{(\Gamma_m : \nabla^2)^p}{p!} F\,.\end{aligned} \tag{13.18}$$

(iii) *There exist $\Lambda_0(\alpha, s, q, \zeta) \geq 128$ and, for every integer $N \geq 0$, a constant $C(\alpha, s, q, \zeta, p, N) < \infty$ such that, for every $\Lambda \geq \Lambda_0$, $t \in \mathbb{R}$ and $F \in H^{N+4p}(\mathbb{T}^2)$ (in the second bound, $F \in H^{N+2p}(\mathbb{T}^2)$), with $\Gamma_m = \Gamma_m(t)$,*

$$\begin{aligned}\big\|\big(\exp_p(\Gamma_m : \nabla^2) \exp_p(-\Gamma_m : \nabla^2) - I\big)F\big\|_{H^N_{\rho(m)}(\mathbb{T}^2)} &\leq C\varepsilon_{m-1}^{\delta(p+1)} \|F\|_{H^{N+4p}_{\rho(m)}(\mathbb{T}^2)}\,, \\ \big\|\big(\exp_p(\pm\Gamma_m : \nabla^2) - I\big)F\big\|_{H^N_{\rho(m)}(\mathbb{T}^2)} &\leq C\varepsilon_{m-1}^{\delta} \|F\|_{H^{N+2p}_{\rho(m)}(\mathbb{T}^2)}\,.\end{aligned} \tag{13.19}$$

Proof. (i) The matrix $\Gamma_m(t)$ is symmetric by (5.11). With summation over the repeated coordinate indices i, i', i'',

$$[\mathbf{b}_{m-1} \cdot \nabla, \Gamma_m : \nabla^2]F = -(\Gamma_m)_{ii'}\, \partial_i \partial_{i'}(\mathbf{b}_{m-1})_{i''}\, \partial_{i''} F - 2(\Gamma_m)_{ii'}\, \partial_i (\mathbf{b}_{m-1})_{i''}\, \partial_{i'} \partial_{i''} F\,,$$

and expanding the divergence on the right side of (13.16) gives the same two terms, as $\nabla \cdot \mathbf{b}_{m-1} = 0$ and Γ_m is symmetric. The operators $\Gamma_m : \nabla^2$ and $\nabla\cdot$ commute, and $[\mathbf{b}_{m-1} \cdot \nabla, (\Gamma_m : \nabla^2)^j] = \sum_{j'+j''=j-1} (\Gamma_m : \nabla^2)^{j'} [\mathbf{b}_{m-1} \cdot \nabla, \Gamma_m : \nabla^2] (\Gamma_m : \nabla^2)^{j''}$, which gives (13.17).

(ii) By (5.11), Γ_m is Lipschitz in time with $\partial_t \Gamma_m = \mathbf{J}_m^{\kappa_m} - \kappa_{m-1} I$, and constant-coefficient operators commute (off a null set of t independent of x, the derivatives $\partial_t \nabla^a F$, $|a| \leq 2p$, exist at every x and equal $\nabla^a \partial_t F$, as in the proof of Lemma 13.2). So

$$\partial_t \exp_p(\Gamma_m : \nabla^2)F = \exp_p(\Gamma_m : \nabla^2)\partial_t F + (\mathbf{J}_m^{\kappa_m} - \kappa_{m-1} I) : \nabla^2 \Big(\exp_p(\Gamma_m : \nabla^2) - \frac{(\Gamma_m : \nabla^2)^p}{p!}\Big)F\,.$$

Adding $\mathbf{b}_{m-1}\cdot\nabla$, which commutes with $\exp_p(\Gamma_m:\nabla^2)$ up to (i), and subtracting $\mathbf{J}_m^{\kappa_m}:\nabla^2\exp_p(\Gamma_m:\nabla^2)F=\exp_p(\Gamma_m:\nabla^2)(\kappa_{m-1}\Delta F)+(\mathbf{J}_m^{\kappa_m}-\kappa_{m-1}I):\nabla^2\exp_p(\Gamma_m:\nabla^2)F$ gives (13.18).

(iii) The operator $(\Gamma_m:\nabla^2)^j$ is a sum of 4^j derivatives of order $2j$ with coefficients of absolute value at most $|\Gamma_m|^j$, so Lemma 8.1(b) gives, for $\rho>0$,

$$\|(\Gamma_m:\nabla^2)^jF\|_{H_\rho^N(\mathbb{T}^2)}\le\Big(\frac{4|\Gamma_m|}{\rho^2}\Big)^j\frac{(N+2j)!}{N!}\|F\|_{H_\rho^{N+2j}(\mathbb{T}^2)}\,.$$

In $\exp_p(\Gamma_m:\nabla^2)\exp_p(-\Gamma_m:\nabla^2)-I$ the terms of powers 1 to p of $\Gamma_m:\nabla^2$ cancel, since they are those of $\exp(\Gamma_m:\nabla^2)\exp(-\Gamma_m:\nabla^2)$, which is the identity, and the powers $p+1$ to $2p$ remain. Lemma 10.3(b) gives $|\Gamma_m(t)|\le C\varepsilon_{m-1}^{\delta}\rho(m)^2$. □

13.3 The coupled system at the coarser level

The sources of the successor. Let G be as in Definition 11.3. Its successor G' has the components and weights of Definition 11.3(ii). Its sources carry the pieces of the operator

$$\begin{aligned}F\mapsto\nabla\cdot\Big(&\exp_p(-\Gamma_m:\nabla^2)\big((\mathcal{M}_m-\mathbf{J}_m^{\kappa_m}\nabla)\exp_p(\Gamma_m:\nabla^2)F\big)\Big)\\&-\exp_p(-\Gamma_m:\nabla^2)\,[\mathbf{b}_{m-1}\cdot\nabla,\exp_p(\Gamma_m:\nabla^2)]F\end{aligned}\tag{13.20}$$

on fields F of G, which is in divergence form by (13.17). The pieces are the following.

- *The defect letter:* the part of $\mathcal{M}_m$ on the paths without sources, of all orders, less $\mathbf{J}_m^{\kappa_m}\nabla$ (the defect of the averaged flux), conjugated as in (13.20), together with the commutator term of (13.20), whose flux is $-\exp_p(-\Gamma_m:\nabla^2)$ applied to the flux of the commutator at F.
- *Old sources:* the part of $\mathcal{M}_m$ of degree 2 along the nonempty paths of sources (Definition 11.3(iii)); and the cell averages of the sources of G in their orders $d\ge3$, carried to the charts of level $m-1$, which are the part of $\mathcal{M}_m$ along the paths of one source that involves no corrector.
- *Mean letters:* for $d\ge3$ and $i\ge1$, the part of $\mathcal{M}_m$ of degree d along the paths of i sources; for $i=1$, less the cell average of the source in the order d, which the old source carries.
- *Covariance letters:* for $d\ge2$, $i\ge1$, and $j,j'\le p$ with $j+j'\ge1$, the part of $\mathcal{M}_m$ of degree d along the paths of i sources, times $(-\Gamma_m:\nabla^2)^j/j!$ on the left and $(\Gamma_m:\nabla^2)^{j'}/j'!$ on the right.

Their sum is (13.20), since the sum over $j,j'\le p$ of a part multiplied in this way is that part multiplied by $\exp_p(-\Gamma_m:\nabla^2)$ on the left and by $\exp_p(\Gamma_m:\nabla^2)$ on the right. By item (1) of Subsection 11.1, the sources of G have current orders at most $N_{\mathrm{src}}-2N_{\mathrm{cor}}-4p-4$, so by (10.15) the means along their paths vanish above the degree $N_{\mathrm{src}}-N_{\mathrm{cor}}-4p-4$; as these paths have at most N_{path} sources, every piece above has its letter in Definition 11.3(i). They are carried by the sources of Definition 11.3(iii)–(iv), with its words, smoothings and the power $\varepsilon_{m-1}^{-\eta}$ of the coefficients of the new sources; the components of generation above N_{gen} or word length above N_{len} are omitted, with the sources into them. In the charts of level $m-1$ these fluxes are the transports (11.8) of their two-scale forms. So the successor exists for every such G: its sources are explicit in the correctors and the means of Section 10, and no further cell problem is solved.

The successor at the scale $\rho(m)$. In the setting of Proposition 5.3, for m above a level that depends only on $\alpha,s,q,\zeta,\Lambda,n,t_0,T$, $\rho(m)\le\rho_{\mathrm{mol}}(m-1)$, as $\rho(m)$ is below c_{mol} times each entry of the minimum (11.1):

$$\rho(m)\le C\varepsilon_{m-1}^{1+\gamma/2+\delta}\le c_{\mathrm{mol}}\varepsilon_{m-1}\,,\qquad\rho(m)^2=\kappa_{m-1}h_m\le2^{-25}\varepsilon_{m-1}^{2\delta}\kappa_{m-1}\varepsilon_{m-1}^{2-\beta}\le c_{\mathrm{mol}}^2\kappa_{m-1}\varepsilon_{m-1}^{2-\beta}\,,$$

by Proposition 2.1(c) and (2.13), as c_{mol} is fixed before this level (Section 12). So Proposition 8.3 applies to G' on $[t_{\mathrm{pr}}, t_{\mathrm{f}}]$, at every order $N \le N_{\mathrm{ret}}$ and the scale $\rho(m)$, with the masses ω_v/Ψ_v^2 (the marked norm) and with the masses ω_v: its hypothesis (iii) holds by clause (d) of G' ($N_{\mathrm{ret}} \le N_{\mathrm{coef}}$), since the coupling matrix (8.4) increases with the scale, and (i) by (12.1) at the level $m-1$; hypothesis (ii) holds by clause (b1), whose analytic norm runs over every time of the window, through Cauchy's estimate on the polydiscs of Step 6 of Subsection 11.6, the chains being smooth (Lemma 11.9). The datum of Y is the retained trace, with the weights of G on the components of G (Lemma 11.8(i)), so Proposition 12.1(iv) gives $\|Y(t_{\mathrm{pr}})\|_{H^{N_{\mathrm{ret}}}_{\rho(m),G'}(\mathbb{T}^2)} \le C\|g\|_{L^2_G(\mathbb{T}^2)}$. Lemma 13.9 and Proposition 13.10 use these facts, as m_0 is at least this level.

Lemma 13.8 checks the construction: summed with the readout weights, the equations of the components of G' give the equation of level $m-1$ for $\Upsilon_m Y$, forced by (13.20), up to the fluxes of the smoothings and of the omitted components.

Lemma 13.8 (The successor read out)**.** *In the setting of Proposition 5.3, let Y be a solution of the system of G' on $[t_{\mathrm{pr}}, t_{\mathrm{f}}]$ in $C([t_{\mathrm{pr}}, t_{\mathrm{f}}]; H^{N_{\mathrm{ret}}+3}(\mathbb{T}^2)^{V'})$. Then*

$$\begin{aligned}(\partial_t + \mathbf{b}_{m-1}\cdot\nabla - \kappa_{m-1}\Delta)\Upsilon_m Y = \nabla\cdot\Big(\exp_p(-\Gamma_m:\nabla^2)\big((\mathcal{M}_m - \mathbf{J}_m^{\kappa_m}\nabla)\exp_p(\Gamma_m:\nabla^2)\Upsilon_m Y\big)\Big)\\ - \exp_p(-\Gamma_m:\nabla^2)\,[\mathbf{b}_{m-1}\cdot\nabla, \exp_p(\Gamma_m:\nabla^2)]\Upsilon_m Y\\ - \nabla\cdot(F^{\mathrm{sm}} + F^{\mathrm{fr}})\end{aligned} \tag{13.21}$$

in the weak sense of (8.3). Here F^{sm} is the smoothing defect*: the sum, over the pieces of (13.20), of the flux of each piece acting on $(I - S_\sigma)\Upsilon_m Y$, with S_σ the smoothing of the source of the construction that carries it, before the omission. And F^{fr} is the* frontier flux*: the readout of the flux of the construction into the omitted components, from the retained ones.*

Proof. By the readout formula, the readout weight of a component v' over a component v of G is $\varepsilon_{m-1}^{\eta\#v'}$ ($\#v' = 0$ for $v' = v$). We multiply the equation of each component over v by its readout weight and sum. Old sources act between components with the same word, with the same coefficient and smoothing, so they sum to the old pieces acting on $S_\sigma\Upsilon_m Y$. A letter source (a new source of Definition 11.3(iv)), with the same coefficients and smoothing for every word of its origin, runs from a component over a component v'' of G to the component v' over v whose word is that of the first followed by the letter, so the word of its origin has length $\#v' - 1$. Its coefficients carry $\varepsilon_{m-1}^{-\eta}$, so

$$\varepsilon_{m-1}^{\eta\#v'}\,\varepsilon_{m-1}^{-\eta} = \varepsilon_{m-1}^{\eta(\#v'-1)}\,,$$

the readout weight of its origin, and the letter sources sum to the letter pieces acting on $S_\sigma\Upsilon_m Y$, as long as their targets are retained. The sources into omitted components are missing from the sum, and their readout is F^{fr}. Writing $S_\sigma\Upsilon_m Y = \Upsilon_m Y - (I - S_\sigma)\Upsilon_m Y$ and summing the pieces into (13.20) gives (13.21). As $Y \in C([t_{\mathrm{pr}}, t_{\mathrm{f}}]; H^{N_{\mathrm{ret}}+3}(\mathbb{T}^2)^{V'})$, the identities (8.3) of the components, combined with the readout weights, give (13.21) in that sense. □

Lemma 13.9 bounds the flux into the omitted components, in the norm in which Lemma 13.6 reads a coarse flux. Those of generation $N_{\mathrm{gen}}+1$ have markers at least $\varepsilon_{m-1}^{-\eta(N_{\mathrm{gen}}+1)/8}$, and the solution is bounded in the marked norm: the first term of (13.22). Those with words of length $N_{\mathrm{len}}+1$ are reached through $N_{\mathrm{len}}+1$ letters, each gaining $\varepsilon_{m-1}^{\delta/2}$ in the coupling: the second.

Lemma 13.9 (The omitted components)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ such that, in the setting of Proposition 5.3, for every datum g and every integer $0 \le N \le N_{\mathrm{ret}}$,*

$$\kappa_{m-1}^{-1/2}\|F^{\mathrm{fr}}\|_{L^2((t_{\mathrm{pr}},t_{\mathrm{f}});H^N_{\rho(m),\omega}(\mathbb{T}^2))} \le C\big(\varepsilon_{m-1}^{\eta(N_{\mathrm{gen}}+1)/8} + \varepsilon_{m-1}^{\delta(N_{\mathrm{len}}+1)/2}\big)\|g\|_{L^2_G(\mathbb{T}^2)}\,. \tag{13.22}$$

Proof. The datum of Y satisfies $\|Y(t_{\mathrm{pr}})\|_{H^N_{\rho(m),G'}(\mathbb{T}^2)} \le C\|g\|_{L^2_G(\mathbb{T}^2)}$ for $0 \le N \le N_{\mathrm{ret}}$, by the paragraph on the successor at the scale $\rho(m)$, as the norm increases with N.

Omitted generation. A child v' of generation $N_{\mathrm{gen}}+1$, an omitted component, was created by $N_{\mathrm{gen}}+1$ cycles, at levels $m' \ge m$. By (11.5), applied before the omission, its marker is that of its root, at least 1, times $\varepsilon_{m'-1}^{-\eta/8} \ge \varepsilon_{m-1}^{-\eta/8}$ for each of these cycles, so at least $\varepsilon_{m-1}^{-\eta(N_{\mathrm{gen}}+1)/8}$. The sources of the construction into v', before the omission, have row and column sums at most C, at the order N and the scale $\rho(m)$, in the coupling matrix for the masses ω_v/Ψ_v^2. Indeed, Steps 3 and 8 of the proof of Lemma 11.7 use Definition 11.3 only through the masses and markers of the copies and the bound of a new source by a fixed number times the product along its path, which hold before the omission. So do Steps 4 and 5, as a child of generation $N_{\mathrm{gen}}+1$ has reserve $1/(N_{\mathrm{gen}}+2+N_{\mathrm{off}}) > 0$; so (b1) holds for the sources into it, and Step 8 reads their derivatives by Cauchy's estimate on the margin of this reserve. Lemma 8.4, with $\vartheta = C$, bounds their flux, measured with the mass and the marker of v', by $C\kappa_{m-1}$ times the norm of ∇Y in $L^2((t_{\mathrm{pr}},t_{\mathrm{f}});H^N_{\rho(m),G'}(\mathbb{T}^2))$. Dividing by the marker of v' gives the first term of (13.22); the readout weight and the ratio of the masses of v' and of the component of G whose child it is cost at most C. Proposition 6.2 for G' with the masses ω_v/Ψ_v^2 bounds the norm of ∇Y in the marked norm by $\kappa_{m-1}^{-1/2}\|Y(t_{\mathrm{pr}})\|_{L^2_{G'}(\mathbb{T}^2)}$, and at order N Proposition 8.3(b) does so, by the paragraph on the successor at the scale $\rho(m)$.

Omitted length. The components of G' whose words have length j form a family, the *layer of length* j, whose coupling matrix is a principal submatrix of that of G', with sums at most $\frac{1}{8}$; the sources within a word are old sources, and those from the layer of length $j-1$ letter sources. For $j \ge 1$ its datum is zero, and its forcing is the flux of the letter sources from the layer of length $j-1$. By Steps 3 and 8 of the proof of Lemma 11.7, for each letter the entries of the coupling matrix of these sources into a component, or out of one, add up to at most $C\varepsilon_{m-1}^{\delta-11\eta/8-(N_{\mathrm{src}}-2)\Theta} \le C\varepsilon_{m-1}^{45\delta/64} \le C\varepsilon_{m-1}^{\delta/2}$ (as $\eta \le \delta/8$ by (2.3), and $(N_{\mathrm{src}}-2)\Theta \le \delta/8$), the factor $\varepsilon_{m-1}^{-\eta}$ and the mass and marker ratios included. So Proposition 8.3(a), with Lemma 8.4 at $\vartheta = C\varepsilon_{m-1}^{\delta/2}$, bounds the dissipation $\kappa_{m-1}^{1/2}\|\nabla Y\|_{L^2((t_{\mathrm{pr}},t_{\mathrm{f}});H^N_{\rho(m),G'}(\mathbb{T}^2))}$ of the components of Y in the layer of length $j \ge 1$ by $C\varepsilon_{m-1}^{\delta/2}$ times that of the layer of length $j-1$, with the marked norms (5.7). One more letter, from the layer of length N_{len}, bounds the flux into the omitted layer of length $N_{\mathrm{len}}+1$: in all, $N_{\mathrm{len}}+1$ letters. The readout weight costs at most C. □

13.4 Bounds on the residual of the two-scale expansion

Lemmas 13.4–13.6 bound the residual of the lift by energies of the fields they act on; Proposition 13.10, with Lemma 13.7(iii), bounds these and the energy of the lift. By item (i) of Proposition 13.10, the solution of the successor keeps the N_{ret} derivatives at the scale $\rho(m)$ of its datum. By item (ii), the lift has derivatives at the scale $\rho_{\mathrm{mol}}(m)$, with the diffusivity κ_m: a fast derivative of a corrector costs ε_m^{-1}, against $\kappa_m^{1/2}$ in the dissipation, and one degree offsets it.

Proposition 13.10 (Energy of the successor and of the lift)**.** *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T) < \infty$ such that, in the setting of Proposition 5.3, for every datum g of G the following hold.*

(i) *The solution Y lies in $C([t_{\mathrm{pr}},t_{\mathrm{f}}];H^{N_{\mathrm{ret}}+3}(\mathbb{T}^2)^{V'})$, and*

$$\|Y\|_{L^\infty([t_{\mathrm{pr}},t_{\mathrm{f}}];H^{N_{\mathrm{ret}}}_{\rho(m),G'}(\mathbb{T}^2))} \le C\|g\|_{L^2_G(\mathbb{T}^2)};$$

and the energy of $\Upsilon_m Y$ of order N_{ret} on $[t_{\mathrm{pr}},t_{\mathrm{f}}]$ is at most $C\|g\|_{L^2_G(\mathbb{T}^2)}$.

(ii) *For every integer $0 \le N \le N_{\mathrm{ret}} - N_{\mathrm{cor}} - 2p$, the component energy of $Y^\sharp$ of order N at the*

scale $\rho_{\mathrm{mol}}(m)$*, with the diffusivity* κ_m*, on* $[t_{\mathrm{pr}}, t_{\mathrm{f}}]$*, is at most* $C\|g\|_{L^2_G(\mathbb{T}^2)}$*. In particular, for every* $t \in [t_{\mathrm{pr}}, t_{\mathrm{f}}]$*,* $\|Y^\sharp(t)\|_{H^{2N_{\mathrm{rep}}+4}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)} \le C\|g\|_{L^2_G(\mathbb{T}^2)}$*.*

Proof. (i) By the paragraph on the successor at the scale $\rho(m)$, Proposition 8.3(b) with the marked masses gives the bound on Y and the continuity in $H^{N_{\mathrm{ret}}}(\mathbb{T}^2)^{V'}$, and (c) with the masses ω_v gives the component energy of Y, since the markers are at least 1. By Lemma 11.8(iii), the readout has norm at most $1 + N_{\mathrm{word}}^{1/2}\varepsilon_{m-1}^{5\eta/4} \le 2$ in the coordinates scaled by $\omega_v^{-1/2}$, with nonnegative entries; applied to the component norms, it bounds the component energy of $\Upsilon_m Y$ by twice that of Y.

The solution Y lies in $C([t_{\mathrm{pr}}, t_{\mathrm{f}}]; H^{N_{\mathrm{ret}}+3}(\mathbb{T}^2)^{V'})$, by Proposition 8.3 at the order $N_{\mathrm{ret}} + 3$ and a scale $\rho' \le \rho(m)$ small enough, and the uniqueness at the order 0. Indeed, the datum lies in $H^{N_{\mathrm{out}}-2p}(\mathbb{T}^2)^{V'}$, with $N_{\mathrm{out}} - 2p = N_{\mathrm{ret}} + 2p \ge N_{\mathrm{ret}} + 3$ ($p \ge 2$ by (13.5)); hypothesis (i) holds at ρ' as at $\rho(m)$; (ii) holds as the coefficients of G' have bounded derivatives of every order, by Cauchy's estimate on its forms at positive reserve (Steps 4 and 6 of Subsection 11.6) and Lemma 11.9; and (iii) holds as, when $\rho' \to 0$, the coupling sums of order $N_{\mathrm{ret}} + 3$ tend to those at the order 0, below $\frac{1}{8}$.

(ii) By Lemma 13.7(iii), the energy of $Y^\flat$ of order $N + N_{\mathrm{cor}}$ is at most C times that of $\Upsilon_m Y$ of order $N + N_{\mathrm{cor}} + 2p$. In $Y^\sharp = \mathrm{Corr}_m Y^\flat$, we consider the part of a corrector of degree $j \ge 1$ on the paths without sources, acting on derivatives of order d of $Y^\flat$, with $1 \le d \le j$ (the order 0 vanishes, as these correctors annihilate constants). In the dissipation, the costly derivative is the fast one, ε_m^{-1}, against $\kappa_m^{1/2}$. Its contribution to $\kappa_m^{1/2}\|\nabla Y^\sharp\|_{L^2((t_{\mathrm{pr}},t_{\mathrm{f}});H^N_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2))}$ is at most

$$C\kappa_m^{1/2}\varepsilon_m^{-1}\rho(m)\kappa_{m-1}^{-1/2}\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{\frac{j}{2}}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{j-d} = C\Big(\frac{\varepsilon_m^2}{\kappa_m h_m}\Big)^{\frac{j-1}{2}}\Big(\frac{\rho(m)}{\varepsilon_{m-1}}\Big)^{j-d} \le C$$

times the energy of $Y^\flat$, by (13.8) and the exact relation $\kappa_m^{1/2}\varepsilon_m^{-1}\rho(m)\kappa_{m-1}^{-1/2} = \rho(m)/\bar{\varepsilon}_m = (\kappa_m h_m/\varepsilon_m^2)^{1/2}$ of (13.7). A derivative that falls on $\nabla^d Y^\flat$ in place of the fast variable costs instead $(\kappa_m/\kappa_{m-1})^{1/2}(\varepsilon_m^2/(\kappa_m h_m))^{j/2}(\rho(m)/\varepsilon_{m-1})^{j-d} \le C$ times the energy; the transport of the coefficients to x by (7.9) lowers the order d at the cost ε_{m-1}^{-1} per order, which keeps the form of (13.8); and $\rho_{\mathrm{mol}}(m) \le c_{\mathrm{mol}}\varepsilon_m \le C\rho(m)$. The supremum in time is at most $C(\varepsilon_m^2/(\kappa_m h_m))^{j/2}$ times the energy. Derivatives at the scale $\rho_{\mathrm{mol}}(m)$ cost at most C, and the identity part of Corr_m costs $(\kappa_m/\kappa_{m-1})^{1/2} \le 1$ in the dissipation. The paths with sources are bounded in the same way, the sums over the paths by (b3). With $2N_{\mathrm{rep}} + 4 \le N_{\mathrm{ret}} - N_{\mathrm{cor}} - 2p$ by (14.3), this gives the last bound. □

In Proposition 13.11, the lift starts from the preparation, up to the two truncated exponentials (13.23), since it adds no corrector at the preparation time (Lemma 13.1). Its residual has four terms: the coarse flux (Lemma 13.6), the cell tail, the switching term, and the fine smoothing defect (the error of the smoothings of G); Table 13.1 lists their bounds and the origin of their smallness.

Proposition 13.11 (The residual of the lift)**.** *There exists* $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ *such that, in the setting of Proposition 5.3, for every datum* g *of* G*, the lift* $Y^\sharp$ *solves the system* (8.2) *of* G *on* $[t_{\mathrm{pr}}, t_{\mathrm{f}}]$ *from*

$$Y^\sharp(t_{\mathrm{pr}}) = \exp_p(\Gamma_m(t_{\mathrm{pr}}) : \nabla^2)\exp_p(-\Gamma_m(t_{\mathrm{pr}}) : \nabla^2)\,\Pi g\,, \tag{13.23}$$

with a forcing (F, F^0) *such that*

$$\kappa_m^{-1/2}\|F\|_{L^2((t_{\mathrm{pr}},t_{\mathrm{f}});L^2_\omega(\mathbb{T}^2))} + \|F^0\|_{L^1((t_{\mathrm{pr}},t_{\mathrm{f}});L^2_\omega(\mathbb{T}^2))} \le C\varepsilon_{m-1}^Q\|g\|_{L^2_G(\mathbb{T}^2)}\,. \tag{13.24}$$

Proof. The top level m_0 is at least the levels of the lemmas above, and so large that the finitely many inequalities below between powers of ε_{m-1} hold for $m > m_0$.

Step 1: The initial value. The velocity added at level m vanishes on $[t_{\rm pr}-\tau'_m, t_{\rm pr}+\tau'_m]$ (Proposition 12.1(i)), and $t_{\rm pr}\ge t_{\rm c}\ge t_{\rm A}+3\tau_m$, so Lemma 13.1 gives $Y^\sharp(t_{\rm pr})=Y^\flat(t_{\rm pr})$. As $Y(t_{\rm pr})$ vanishes on the new components, $\Upsilon_m Y(t_{\rm pr})=\exp_p(-\Gamma_m(t_{\rm pr}):\nabla^2)\Pi g$, and (13.23) follows.

Step 2: The exact identity. By Proposition 13.10(i) and (8.3), $\Upsilon_m Y$ is Lipschitz in t with values in $H^{N_{\rm ret}+3-N_{\rm src}}(\mathbb{T}^2)$: the drift costs one order, $\kappa_{m-1}\Delta$ two, a source at most $N_{\rm src}$, and the coefficients of G' have bounded derivatives of every order. As Γ_m is Lipschitz in t, $Y^\flat$ of (13.2) is Lipschitz in t with values in $H^{N_{\rm ret}+3-N_{\rm src}-2p}(\mathbb{T}^2)$ and bounded in $H^{N_{\rm ret}+3-2p}(\mathbb{T}^2)$, and by (13.6) and Sobolev's embedding these fields have the classes of Lemmas 13.7(ii) and 13.5. So (13.21) holds at almost every t, and, with Lemma 13.7(ii), it gives the averaged equation of $Y^\flat$, with the *flux of the averaged equation* $F^{\rm av}$:

$$\begin{aligned}&\partial_t Y^\flat+\mathbf{b}_{m-1}\cdot\nabla Y^\flat-\nabla\cdot(\mathcal{M}_m Y^\flat)=\nabla\cdot F^{\rm av}\,,\qquad\text{where}\\ &F^{\rm av}:=\big(\exp_p(\Gamma_m:\nabla^2)\exp_p(-\Gamma_m:\nabla^2)-I\big)\big((\mathcal{M}_m-\mathbf{J}_m^{\kappa_m}\nabla)Y^\flat-F^{\rm com}\big)\\ &\qquad-\exp_p(\Gamma_m:\nabla^2)(F^{\rm sm}+F^{\rm fr})-(\mathbf{J}_m^{\kappa_m}-\kappa_{m-1}I)\nabla\frac{(\Gamma_m:\nabla^2)^p}{p!}\Upsilon_m Y\,.\end{aligned}\tag{13.25}$$

Here $F^{\rm com}$ is the flux of the commutator at $\Upsilon_m Y$; the first line of $F^{\rm av}$ is the *truncation part*, and its last term the *Taylor part*. Indeed, $[\mathbf{b}_{m-1}\cdot\nabla,\exp_p(\Gamma_m:\nabla^2)]\Upsilon_m Y=\nabla\cdot F^{\rm com}$ by (13.17), $\mathbf{J}_m^{\kappa_m}:\nabla^2=\nabla\cdot(\mathbf{J}_m^{\kappa_m}\nabla)$, and $\exp_p(\pm\Gamma_m:\nabla^2)$ commute with $\nabla\cdot$. Lemma 13.5, applied to the field $Y^\flat$, then gives, with $\mathbf{A}_m=\mathbf{A}_m^S+(\mathbf{A}_m-\mathbf{A}_m^S)$,

$$\begin{aligned}&(\partial_t+\mathbf{b}_m\cdot\nabla-\kappa_m\Delta)Y^\sharp-\nabla\cdot(\mathbf{A}_m^S Y^\sharp)\\ &\quad=\mathrm{Corr}_m(\nabla\cdot F^{\rm av})+\nabla\cdot F^{\rm tl}+F^{\rm tl,0}+\sum_k \mathsf{H}'_{m,k}(\mathrm{Corr}_{m,k}\,Y^\flat-Y^\flat)+\nabla\cdot F^{\rm fine}\,,\end{aligned}$$

with the *fine smoothing defect* $F^{\rm fine}:=(\mathbf{A}_m-\mathbf{A}_m^S)Y^\sharp$, the sources of G acting on $(I-S_\sigma)Y^\sharp$. The forcing (F,F^0) of (13.24) collects the four terms on the right side, the coarse flux, the cell tail, the switching term and the fine smoothing defect: with the forcing $(F^{\rm co},F^{\rm co,0})$ of Lemma 13.6 for $F^{\rm av}$, it is $F=F^{\rm co}+F^{\rm tl}+F^{\rm fine}$ and $F^0=F^{\rm co,0}+F^{\rm tl,0}+\sum_k \mathsf{H}'_{m,k}(\mathrm{Corr}_{m,k}\,Y^\flat-Y^\flat)$. And $Y^\sharp$ lies in the class of solutions: by Lemma 13.5 it is Lipschitz with values in $L^2(\mathbb{T}^2)^V$, the display holds in the form (8.3), and Proposition 13.10(ii) gives the rest.

Step 3: The four terms. By Proposition 13.10(i), the energy of $\Upsilon_m Y$ of order $N_{\rm ret}$ is at most $C\|g\|^2_{L^2_G(\mathbb{T}^2)}$, and so, by Lemma 13.7(iii), is that of $Y^\flat$ of order $N_{\rm ret}-2p$. The truncation part reads the energy of $\Upsilon_m Y$ of order at most $N_{\rm rec}+6p+N_{\rm src}-2$, and the smoothing defect that of order at most $N_{\rm rec}+2p+N_{\rm src}+N_{\rm mom}-2$, both at most $N_{\rm ret}$ by the first line of (13.6). The fine smoothing defect reads the component energy of $Y^\sharp$ of Proposition 13.10(ii) at the order $N_{\rm mom}+N_{\rm src}-2\le N_{\rm ret}-N_{\rm cor}-2p$, by the same line. Moreover, at each time $Y^\flat$ and $F^{\rm av}$ have $N_{\rm rec}+3$ derivatives in L^2, by Proposition 13.10(i) and (13.6), hence $N_{\rm rec}+1$ continuous ones (measurable in (t,x), being continuous in x and measurable in t, the smoothings by clause (c3) and Lemma 8.2), as Lemmas 13.4–13.6 require. Table 13.1 lists the terms.

- *The coarse flux*, by Lemma 13.6: its cost is $C(\kappa_{m-1}/\kappa_m)^{1/2}\le C\varepsilon_{m-1}^{-q\gamma}$, since its second term is at most C, by (13.7) and the condition on $N_{\rm cl}$ in (13.5) (see the cell tail below). Then $\kappa_{m-1}^{-1/2}\|F^{\rm av}\|_{L^2((t_{\rm pr},t_{\rm f});H^{N_{\rm rec}}_{\rho(m),\omega}(\mathbb{T}^2))}$ has four parts. The *truncation part* is at most $C\varepsilon_{m-1}^{\delta(p+1)}\|g\|_{L^2_G(\mathbb{T}^2)}$, by (13.19): the coefficients of $\mathcal{M}_m-\mathbf{J}_m^{\kappa_m}\nabla$ and of the flux of the commutator are at most $C\kappa_{m-1}$ at the scale $\rho(m)$, by Proposition 10.7 and, for the flux of the commutator, by

$$\rho(m)^l\|\nabla^{l+1}\mathbf{b}_{m-1}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\,|\Gamma_m(t)|\le C(l)\,\kappa_{m-1}\varepsilon_{m-1}^{\delta}\,,\qquad l\ge 0\,,$$

which holds by (10.9), with (7.4) and (2.13) at $l = 0$, and with the first line of (12.1) at the level $m-1$ and $\rho(m) \le \rho_{\mathrm{mol}}(m-1)$ at $l \ge 1$. The *Taylor part* is at most $C\varepsilon_{m-1}^{\delta p}\|g\|_{L^2_G(\mathbb{T}^2)}$, since $|\mathbf{J}_m^{\kappa_m}(t)| \le \kappa_m + \varepsilon_m^{2\beta}/(2\kappa_m) \le C\kappa_{m-1}$. The *smoothing defect* F^{sm} appears under the truncated exponential, so by the second bound of (13.19) we bound it in $H^{N_{\mathrm{rec}}+2p}_{\rho(m),\omega}(\mathbb{T}^2)$. It is at most $C\varepsilon_{m-1}^{\Theta N_{\mathrm{mom}}}\|g\|_{L^2_G(\mathbb{T}^2)}$: by the third bound of Proposition 6.1 at each time (which needs no transversality and has a constant independent of ℓ), with $N_1 = N_{\mathrm{mom}}$ and the bound C_Ω of (c1), each piece acting on $(I - S_\sigma)\Upsilon_m Y$ costs $(r_\sigma/\rho(m))^{N_{\mathrm{mom}}}$, with $r_\sigma \le C\rho(m)\varepsilon_{m-1}^{\Theta}$ by (c2) for G, whose old sources G' copies, by (14.1) for the letters, and $4\delta > \Theta$ (by (14.4)). The coefficients are bounded by clauses (b1)–(b3) of the admissible family G'; a letter piece into a component of G of generation N_{gen} has no source in G', and its coefficients obey the same bounds before the omission, as in the proof of Lemma 13.9. The *frontier part* is Lemma 13.9 at the order $N_{\mathrm{rec}} + 2p$, under the truncated exponential as for F^{sm}.

- *The cell tail*, by Lemma 13.4: at most $C\varepsilon_{m-1}^{3\delta N_{\mathrm{cor}}}$ by (13.7). Its term in parentheses is bounded:

$$(t_{\mathrm{f}} - t_{\mathrm{pr}})^{1/2}\,\frac{\kappa_m^{1/2}}{\varepsilon_m}\Big(\frac{\varepsilon_m}{\rho(m)}\Big)^{N_{\mathrm{cl}}} \le CT^{1/2}\varepsilon_{m-1}^{-(2-\beta+8\delta)/2+(q\gamma+3\delta)N_{\mathrm{cl}}} \le C\,,$$

  by (13.7), the condition on N_{cl} in (13.5), and $\kappa_m/\varepsilon_m^2 \le C\varepsilon_m^{\beta+\gamma-2} \le C\varepsilon_{m-1}^{-(2-\beta+8\delta)}$ (Proposition 2.1(b) and Lemma 7.1(c)).

- *Switching*, by Lemma 13.5: at most $C\varepsilon_{m-1}^{-C}\exp(-c\varepsilon_{m-1}^{-4\delta}) \le C\varepsilon_{m-1}^{Q}$, since $t_{\mathrm{f}} - t_{\mathrm{pr}} \le T$, $\tau_m \asymp \varepsilon_{m-1}^{2-\beta+4\delta}$ by (2.5), and $\varepsilon_m^2/\kappa_m \le C\tau_m$ by Proposition 2.1(c).

- *The fine smoothing defect.* The sources of G at the scale $\rho_{\mathrm{mol}}(m)$ are bounded by clause (b2′). By the third bound of Proposition 6.1 at this scale, at each time, as for F^{sm}, each costs $(r_\sigma/\rho_{\mathrm{mol}}(m))^{N_{\mathrm{mom}}} \le C\varepsilon_m^{\delta N_{\mathrm{mom}}} \le C\varepsilon_{m-1}^{q\delta N_{\mathrm{mom}}}$, by (c2). The remaining norm of $Y^\sharp$, of order $N_{\mathrm{mom}} + N_{\mathrm{src}} - 2$ at the scale $\rho_{\mathrm{mol}}(m)$, is bounded by Proposition 13.10(ii).

Step 4: The orders. By (13.5), whose margin $q\gamma$ absorbs the cost $\varepsilon_{m-1}^{-q\gamma}$ of the coarse flux, the norm (13.4) of each term is at most $C\varepsilon_{m-1}^{Q+4}\|g\|_{L^2_G(\mathbb{T}^2)}$, and (13.24) follows. □

term	bound (times $\|g\|_{L^2_G(\mathbb{T}^2)}$)	smallness from	statement
cell tail	$C\varepsilon_{m-1}^{3\delta N_{\mathrm{cor}}}$	correctors up to degree N_{cor}	13.2, 13.4
switching	$C\varepsilon_{m-1}^{-C}\exp(-c\varepsilon_{m-1}^{-4\delta})$	decay in the quiet intervals	13.5
coarse smoothing	$C\varepsilon_{m-1}^{\Theta N_{\mathrm{mom}}-q\gamma}$	moments of the smoothings of G'	6.1, 13.6
fine smoothing	$C\varepsilon_{m-1}^{q\delta N_{\mathrm{mom}}}$	moments of the smoothings of G	6.1, 13.10
truncation, Taylor	$C\varepsilon_{m-1}^{\delta p-q\gamma}$	truncation of the exponential	13.7, 13.6
frontier, generation	$C\varepsilon_{m-1}^{\eta(N_{\mathrm{gen}}+1)/8-q\gamma}$	markers	13.9, 13.6
frontier, length	$C\varepsilon_{m-1}^{\delta(N_{\mathrm{len}}+1)/2-q\gamma}$	one letter per word length	13.9, 13.6

Table 13.1: The residual of the lift (Proposition 13.11) in the norm (13.4), the coarse flux split as in Step 3, each part with its cost $\varepsilon_{m-1}^{-q\gamma}$ (Lemma 13.6). Each bound is at most $C\varepsilon_{m-1}^{Q}$ by (13.5).

13.5 Proof of Proposition 5.3(b)

Here m_0 is the top level of item (4) of Subsection 11.1, at least every threshold below (Table 11.1).

Admissibility. By Definition 11.4, G descends from an admissible seed by successors, with (11.6) and $t_{\mathrm{A}} = t_{\mathrm{A}}^{(m)}$; we append G' to it. The new condition (11.6) is $t_{\mathrm{A}}^{(m-1)} = t_{\mathrm{A}} + 3\tau_m \le t_{\mathrm{f}} - 4h_m$, which holds since $t_{\mathrm{A}} + 3\tau_m \le t_{\mathrm{c}} \le t_{\mathrm{f}} - 4h_m$, and $m - 1 \ge m_0$. So G' is admissible on $[t_{\mathrm{A}} + 3\tau_m, t_{\mathrm{f}}]$.

The marked amplitude. So G' satisfies clause (d) on $[t_{\mathrm{pr}}, t_{\mathrm{f}}]$ (Proposition 11.5), whose matrix at the order 0 is the coupling matrix of Proposition 6.2; with the masses ω_v/Ψ_v^2, this proposition gives $\|Y\|_{L^\infty([t_{\mathrm{pr}},t_{\mathrm{f}}];L^2_{G'}(\mathbb{T}^2))} \le \|Y(t_{\mathrm{pr}})\|_{L^2_{G'}(\mathbb{T}^2)}$. The datum, the retained trace, has the masses and markers of G on the components of G (Lemma 11.8(i)), so Lemma 13.7(iii) and Proposition 5.3(a), with $N_{\mathrm{out}} \ge 2p$, give the first bound of Proposition 5.3(b):

$$\begin{aligned}\|Y(t_{\mathrm{pr}})\|_{L^2_{G'}(\mathbb{T}^2)} &\le \|\Pi g\|_{L^2_G(\mathbb{T}^2)} + C\varepsilon_{m-1}^{\delta}\|\Pi g\|_{H^{N_{\mathrm{out}}}_{\rho(m),G}(\mathbb{T}^2)} \\ &\le \big(1 + C\varepsilon_{m-1}^{Q}\big)\|g\|_{L^2_G(\mathbb{T}^2)} + C\varepsilon_{m-1}^{\delta}\|g\|_{L^2_G(\mathbb{T}^2)}\,.\end{aligned}$$

The comparison. The restriction of U to $[t_{\mathrm{pr}}, t_{\mathrm{f}}]$ solves the system of G from $U(t_{\mathrm{pr}})$ (Section 8). By Proposition 13.11, $U - Y^\sharp$ solves (8.2) with the forcing $-(F, F^0)$ from

$$U(t_{\mathrm{pr}}) - Y^\sharp(t_{\mathrm{pr}}) = \big(U(t_{\mathrm{pr}}) - \Pi g\big) - \big(\exp_p(\Gamma_m(t_{\mathrm{pr}}) : \nabla^2)\exp_p(-\Gamma_m(t_{\mathrm{pr}}) : \nabla^2) - I\big)\Pi g\,.$$

We bound both terms in $L^2_\omega(\mathbb{T}^2)$ by their norms with the markers, which are at least 1. The first term is at most $C\varepsilon_{m-1}^{Q}\|g\|_{L^2_G(\mathbb{T}^2)}$ by the first bound of Proposition 5.3(a). By (13.19) at $N = 0$, the third bound of Proposition 5.3(a) with $N_{\mathrm{out}} \ge 4p$, and $\delta p > Q$, the second term is at most

$$C\varepsilon_{m-1}^{\delta(p+1)}\|\Pi g\|_{H^{4p}_{\rho(m),G}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^{Q}\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

As G satisfies (d) with sums at most $\frac{1}{8}$ for the masses ω_v (Proposition 11.5), by linearity and uniqueness (Proposition 8.3, $N = 0$), $U - Y^\sharp$ is the solution given there for this datum and the forcing $-(F, F^0)$, and Proposition 8.3(a) bounds $\|U - Y^\sharp\|_{L^\infty([t_{\mathrm{pr}},t_{\mathrm{f}}];L^2_\omega(\mathbb{T}^2))} + \kappa_m^{1/2}\|\nabla(U - Y^\sharp)\|_{L^2((t_{\mathrm{pr}},t_{\mathrm{f}});L^2_\omega(\mathbb{T}^2))}$ by C times the norms of this datum in $L^2_\omega(\mathbb{T}^2)$ and of the forcing in (13.4), hence by $C\varepsilon_{m-1}^{Q}\|g\|_{L^2_G(\mathbb{T}^2)}$ by (13.24): the second bound of Proposition 5.3(b). The third is Proposition 13.10(i). □

The terminal cycle. Section 15 compares the terminal cycle, at level M, in the same way: Lemmas 13.1, 13.2 and 13.4–13.6 hold at $m = M$, where the cost $(\kappa_{M-1}/\kappa_M)^{1/2}$ in Lemma 13.6 is bounded; Lemma 13.3 is for $m < M$. Lemma 13.7(i)–(ii) hold at every level, and (13.19) holds at $m = M$, with its proof, for the anchored covariance $\Gamma_M(t) - \Gamma_M(t_{\mathrm{pr}}^{(M)})$, which Lemma 10.3(b) bounds.

14. Passage between consecutive levels and induction on the levels

In this section we prove Proposition 5.4, one passage: the cycle from level m to the coarser level $m-1$, read on representations; and Proposition 5.2, the induction on the levels, which runs the two dual cascades.

The five profiles. In the notation of Proposition 5.3, a passage writes $U(t)$ as $\Upsilon_m Y(t)$ plus the five profiles of Table 14.1: the exponential and corrector parts (14.10) of the lift, applied to $SY(t)$, where $S = S(t)$ is the fresh smoothing (14.9), and to $(I - S)Y(t)$, and the comparison error

$U(t) - Y^\sharp(t)$. The first two profiles are bounded by a small multiple of the size of a representation R' of $Y(t)$ at wavelength ε_{m-1} (Lemma 14.2(a)), the last three by $C\varepsilon_{m-1}^2 \|g\|_{L^2_G(\mathbb{T}^2)}$. This absolute smallness absorbs the power ε_{m-1}^{-Z} of (14.2) lost when a Sobolev bound is read as a profile.

Parameters. Throughout, C and c denote positive constants depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0$, T, which may change from line to line. Besides the parameters of Subsection 5.3, among them the order N_{ret}, fixed in Section 13 with the floors (13.6) and (14.3), the proofs use the following ones, depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$ (Table 11.1):

- the width exponent $\Theta > 0$ of the width
$$r_m := \rho(m)\, \varepsilon_{m-1}^{\Theta} \tag{14.1}$$
of the fresh smoothing (14.9) of the new sources of a successor created at level m, and the order N_{mom} of its vanishing moments (Section 11);
- the constant $c_{\mathrm{mol}} \in (0, 1]$ of the molecular radius $\rho_{\mathrm{mol}}(m)$ of (11.1), the scale at which a solution of a family of level m has derivatives after the time $\rho_{\mathrm{mol}}(m)^2/\kappa_m$ (Section 12);
- the power-loss exponent
$$Z := 1 + s + \Big(\frac{\gamma}{2} + \delta\Big)(N_{\mathrm{rep}} + 1)\,, \tag{14.2}$$
lost when a Sobolev bound is read as a profile.

These parameters and N_{ret} satisfy
$$N_{\mathrm{ret}} \geq 2N_{\mathrm{rep}} + 4 + N_{\mathrm{cor}} + 2p + N_{\mathrm{mom}}\,, \qquad Q \geq 4 + 2qZ\,, \tag{14.3}$$
and the width exponent is so small that
$$12\Theta < \delta\,, \qquad \Theta N_{\mathrm{cor}} \leq \Big(\frac{\gamma}{2} + \delta\Big)(N_{\mathrm{rep}} - 1)\,. \tag{14.4}$$

14.1 Estimates from the homogenization cycle

For a field F of a family G and $\rho > 0$, the norm (8.1) of $H^N_{\rho,\omega}(\mathbb{T}^2)$ is the norm (5.7) of $H^N_{\rho,G}(\mathbb{T}^2)$ with all markers replaced by 1. Since the markers are at least 1,
$$\|F\|_{H^N_{\rho,\omega}(\mathbb{T}^2)} \leq \|F\|_{H^N_{\rho,G}(\mathbb{T}^2)}\,. \tag{14.5}$$

profile	wavelength	profile norm at most	smallness from
$(SY(t))^{\mathrm{exp}}$	ε_{m-1}	$C\varepsilon_{m-1}^{\delta-2\Theta} \cdot \varepsilon_{m-1}^{-s}[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}$	covariance against r_m^2
$(SY(t))^{\mathrm{cor}}$	ε_m	$C\varepsilon_{m-1}^{\mu} \cdot \varepsilon_{m-1}^{-s}[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}$	the balance (5.2)
$((I-S)Y(t))^{\mathrm{exp}}$	ε_{m-1}	$C\varepsilon_{m-1}^2 \|g\|_{L^2_G(\mathbb{T}^2)}$	moments of S
$((I-S)Y(t))^{\mathrm{cor}}$	ε_m	$C\varepsilon_{m-1}^2 \|g\|_{L^2_G(\mathbb{T}^2)}$	moments of S
$U(t) - Y^\sharp(t)$	ε_m	$C\varepsilon_{m-1}^2 \|g\|_{L^2_G(\mathbb{T}^2)}$	Proposition 5.3

Table 14.1: The five profiles that a passage at level m appends to the readout of a representation R' of $Y(t)$: the wavelength λ_P of each profile P, the bound on $\lambda_P^{-s}\|P\|_{C^{N_{\mathrm{rep}}}_{\lambda_P}(\mathbb{T}^2)}$ proved in Steps 3–6 of the proof of Proposition 5.4, and the origin of its smallness. With the readout of Proposition 14.1(i), the first two give the coefficient $1 + C\varepsilon_{m-1}^{\eta}$ in Proposition 5.4; the last three give its additive term.

A passage works at one time. Proposition 14.1 bounds the factor by which the readout multiplies the size of a representation (Lemma 14.2(b)), states the comparison at every time, and gives $U(t)$ the $2N_{\mathrm{rep}}+4$ derivatives at the scale $\rho_{\mathrm{mol}}(m)$ that the lift has (Proposition 13.10(ii)); Step 6 of the proof of Proposition 5.4 interpolates between the comparison and these derivatives.

Proposition 14.1 (Further properties of a cycle)**.** *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)<\infty$ such that, for every M, κ, m, t_{A}, t_{c}, t_{f}, G and G' as in Proposition 5.3, there exist t_{pr} and Π satisfying* (a) *and* (b) *of Proposition 5.3 with this C, such that the following holds for every datum g of G, with U and Y as there.*

(i) *The readout satisfies $|\Upsilon_m|\le 1+C\varepsilon_{m-1}^{5\eta/4}$.*

(ii) *For every $t\in[t_{\mathrm{pr}},t_{\mathrm{f}}]$,*

$$\|U(t)-Y^\sharp(t)\|_{L^2_\omega(\mathbb{T}^2)}\le C\varepsilon_{m-1}^{Q}\|g\|_{L^2_G(\mathbb{T}^2)}\,,$$
$$\|U(t)\|_{H^{2N_{\mathrm{rep}}+4}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)}\le C\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

Proposition 14.1 holds for the time t_{pr} and the map Π constructed in Section 12: part (i) is Lemma 11.8(iii). The first bound of (ii) is Proposition 5.3(b), and the second is Proposition 12.1(iii), each at every time, as U (Proposition 6.2) and $Y^\sharp$ (Proposition 13.11) are continuous in time with values in $L^2(\mathbb{T}^2)^V$. For the same t_{pr} and Π, Proposition 13.10 bounds Y and $Y^\sharp(t)$. Proposition 5.4 is proved below with these t_{pr} and Π. Lemma 13.3 applies to the family G of Proposition 5.3 at every time $t\in[t_{\mathrm{c}},t_{\mathrm{f}}]$, since G is admissible on $[t_{\mathrm{A}},t_{\mathrm{f}}]$, hence in the setting of the lift of Section 13 (Proposition 11.5 gives the invariant), and $t_{\mathrm{c}}\ge t_{\mathrm{A}}+3\tau_m$.

By Lemma 10.3(b) and (14.1), the covariance is small against the square of the width:

$$|\Gamma_m(t)|\le C\varepsilon_{m-1}^{\delta-2\Theta}\,r_m^2\,. \tag{14.6}$$

By Proposition 2.1(c), for $m\ge 2$, $\rho(m)\asymp\varepsilon_{m-1}^{1+\gamma/2+\delta}$, so that

$$c\,\varepsilon_{m-1}^{1+\gamma/2+\delta}\le\rho(m)\le C\varepsilon_{m-1}^{1+\gamma/2+\delta}\le C\varepsilon_{m-1}\,,\qquad r_m\le\rho(m)\,. \tag{14.7}$$

14.2 Representations and the smoothing

By Lemma 14.2(a), the size of a representation bounds the field, its derivatives at the wavelength and its Hölder norm. By (b), the readout carries a representation to a finer wavelength at the cost of the norm of the readout, as each profile keeps its wavelength and the base, read at a finer wavelength, does not grow; added profiles add their norms.

Lemma 14.2 (Bounds from a representation, and copies)**.** *Let $0<\lambda\le\lambda'\le 1$.*

(a) *For every field U of a family and every representation R of U at wavelength λ,*

$$\|U\|_{L^\infty(\mathbb{T}^2)}+\lambda^{-s}[U]_{C^{N_{\mathrm{rep}}}_\lambda(\mathbb{T}^2)}\le\mathrm{size}_\lambda(R)\,,\qquad \|U\|_{C^{0,s}(\mathbb{T}^2)}\le 3\,\mathrm{size}_\lambda(R)\,.$$

(b) *Let R' be a representation, with base E and profiles $\mathcal{P}$, of a field Y of a successor G' of a family G of level m at wavelength λ'. Let finitely many fields $P_i\in C^{N_{\mathrm{rep}}}(\mathbb{T}^2)^V$ of G be given, with wavelengths $\lambda_i\in[\lambda,1]$. Then the base $\Upsilon_m E$, the profiles $\Upsilon_m P$, $P\in\mathcal{P}$, with the wavelengths λ_P, and the profiles P_i with the wavelengths λ_i form a representation R of $\Upsilon_m Y+\sum_i P_i$ at wavelength λ, and*

$$\mathrm{size}_\lambda(R)\le|\Upsilon_m|\,\mathrm{size}_{\lambda'}(R')+\sum_i\lambda_i^{-s}\|P_i\|_{C^{N_{\mathrm{rep}}}_{\lambda_i}(\mathbb{T}^2)}\,.$$

Proof. (a) For a profile P and $|j| \geq 1 > s$, $\lambda \leq \lambda_P \leq 1$ gives $\lambda^{|j|-s} \leq \lambda_P^{|j|-s}$ and $1 \leq \lambda_P^{-s}$; adding the definitions (5.12) over the base and the profiles gives the first bound. For the second, for $x, y \in \mathbb{T}^2$,

$$\begin{aligned} |P(x) - P(y)|_\omega &\leq \min\big\{2\|P\|_{L^\infty(\mathbb{T}^2)},\ \big(\|\partial_1 P\|_{L^\infty(\mathbb{T}^2)} + \|\partial_2 P\|_{L^\infty(\mathbb{T}^2)}\big)|x-y|_{\mathbb{T}^2}\big\} \\ &\leq 2\lambda_P^{-s}\|P\|_{C^{N_{\mathrm{rep}}}_{\lambda_P}(\mathbb{T}^2)}|x-y|_{\mathbb{T}^2}^{s}\,, \end{aligned}$$

since $\lambda_P(\|\partial_1 P\|_{L^\infty(\mathbb{T}^2)} + \|\partial_2 P\|_{L^\infty(\mathbb{T}^2)}) \leq [P]_{C^{N_{\mathrm{rep}}}_{\lambda_P}(\mathbb{T}^2)}$ and $s < 1$, and adding $[E]_{C^{0,s}(\mathbb{T}^2)}$ and the first bound gives $\|U\|_{C^{0,s}(\mathbb{T}^2)} \leq 3\,\mathrm{size}_\lambda(R)$.

(b) The readout, a constant linear map, commutes with derivatives and at every point costs at most its operator norm $|\Upsilon_m|$ for $|\cdot|_\omega$, hence at most $|\Upsilon_m|$ in each term of (5.13). The base, measured at $\lambda \leq \lambda'$, only decreases by the monotonicity of (a), and the profiles P_i add their terms. □

Lemma 14.3 turns a Sobolev bound into a profile: $N_{\mathrm{rep}} + 2$ derivatives in L^2 at a scale ρ, read at a wavelength λ, cost the fixed power λ^{-Z} where the maximum in (14.8) is bounded.

Lemma 14.3 (Sobolev bounds read at a wavelength)**.** *There exists $C(N_{\mathrm{rep}}) < \infty$ such that, for every $0 < \lambda \leq 1$, $0 < \rho \leq 1$ and every field $F \in H^{N_{\mathrm{rep}}+2}(\mathbb{T}^2)^V$ of a family,*

$$\lambda^{-s}\|F\|_{C^{N_{\mathrm{rep}}}_{\lambda}(\mathbb{T}^2)} \leq C\max\Big\{1, \frac{\lambda^{1+\gamma/2+\delta}}{\rho}\Big\}^{N_{\mathrm{rep}}+1}\lambda^{-Z}\,\|F\|_{H^{N_{\mathrm{rep}}+2}_{\rho,\omega}(\mathbb{T}^2)}\,. \tag{14.8}$$

The pair $(F, \emptyset)$, with base F and no profile, is a representation of F at wavelength λ with $\mathrm{size}_\lambda((F,\emptyset)) \leq 4\lambda^{-s}\|F\|_{C^{N_{\mathrm{rep}}}_{\lambda}(\mathbb{T}^2)}$.

Proof. For a scalar function g, a multi-index j and $\rho \leq 1$, the Cauchy–Schwarz inequality in Fourier gives $\|\partial^j g\|_{L^\infty(\mathbb{T}^2)} \leq C\rho^{-|j|-1}\|g\|_{H^{|j|+2}_\rho(\mathbb{T}^2)}$. Applied to each component, it bounds the left side of (14.8) by $C\sum_{i=0}^{N_{\mathrm{rep}}}\lambda^{i-s}\rho^{-i-1}\|F\|_{H^{N_{\mathrm{rep}}+2}_{\rho,\omega}(\mathbb{T}^2)}$, and for $0 \leq i \leq N_{\mathrm{rep}}$, since $\lambda \leq 1$,

$$\lambda^{i-s}\rho^{-i-1} = \Big(\frac{\lambda^{1+\gamma/2+\delta}}{\rho}\Big)^{i+1}\lambda^{-(1+s+(\gamma/2+\delta)(i+1))} \leq \max\Big\{1, \frac{\lambda^{1+\gamma/2+\delta}}{\rho}\Big\}^{N_{\mathrm{rep}}+1}\lambda^{-Z}\,.$$

Finally, $F \in C^{N_{\mathrm{rep}}}(\mathbb{T}^2)^V$ by the Sobolev embedding, and the size of $(F,\emptyset)$ is bounded as in the proof of Lemma 14.2(a). □

At the time t, the fresh smoothing of the new sources of a successor created at level m is the product of two averages (6.2) along ellipses of width r_m,

$$S(t) \coloneqq S_{\ell,r_m,\Omega_1(t)}\,S_{\ell,r_m,\Omega_2(t)}\,, \tag{14.9}$$

where ℓ is a fixed integer with $2\ell > N_{\mathrm{mom}}$, depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$, and $\Omega_1(t)$, $\Omega_2(t)$ are the maps of the ellipses (11.18). By Step 7 of the proof of Lemma 11.7, they are invertible with $|\Omega_i(t)|_\infty \leq C_\Omega$, and, at the levels finer than the top level of Subsection 11.1, their long axes $\Omega_1(t)e_1$, $\Omega_2(t)e_2$ are uniformly transverse in the sense of Proposition 6.1.

A passage splits $Y(t)$ with this smoothing. By Lemma 14.4(a), $S(t)$ moves derivatives onto its kernel at the cost r_m^{-1} each, uniformly in the eccentricity, as every derivative combines derivatives along the two long axes; by (b), the vanishing moments of the kernel make $(I - S)Y(t)$ small.

Lemma 14.4 (The smoothing)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T) < \infty$ such that the following holds for every $M \geq 1$, $\kappa \in \mathcal{K}_M$, $m_0 < m < M$, $t \in \mathbb{R}$ and every successor G' of a family of level m.*

(a) *For every pair of integers* $1 \le N_1 \le N_2 \le N_{\mathrm{rep}} + N_{\mathrm{cor}} + 2p$ *and every field* F *of* G' *of class* C^{N_1} *on* $\mathbb{T}^2$,

$$\sum_{|i|=N_2} \|\partial^i S(t) F\|_{L^\infty(\mathbb{T}^2)} \le C\, r_m^{N_1-N_2} \sum_{|j|=N_1} \|\partial^j F\|_{L^\infty(\mathbb{T}^2)}\,.$$

(b) *For every integer* $0 \le N \le N_{\mathrm{ret}} - N_{\mathrm{mom}} - 2$, *every* $\rho \ge r_m$ *and every field* $F \in H^{N+N_{\mathrm{mom}}+2}(\mathbb{T}^2)^{V'}$ *of* G', *where* V' *is the set of components of* G',

$$\|F - S(t)F\|_{H^N_{\rho,G'}(\mathbb{T}^2)} \le C\Big(\frac{r_m}{\rho}\Big)^{N_{\mathrm{mom}}} \|F\|_{H^{N+N_{\mathrm{mom}}+2}_{\rho,G'}(\mathbb{T}^2)}\,.$$

Proof. (a) This is Step 1 of the proof of Proposition 6.1, with r_m in place of r, $\Omega_i(t)$ in place of Ω_i, and $L^\infty(\mathbb{T}^2)$ in place of $L^2(\mathbb{T}^2)$, applied to the derivatives $\partial^j F$, $|j| = N_1$: every ∂^i with $|i| = N_2$ is a combination of products of $N_2 - N_1$ derivatives $\partial_{(1)}$ or $\partial_{(2)}$ with one ∂^j, $|j| = N_1$, with coefficients that are products of entries of the inverse of the matrix of the long axes $\Omega_1(t)e_1$, $\Omega_2(t)e_2$, which are at most 2 in absolute value. Only this inverse enters, not the inverse of the short axes. Each $\partial_{(1)}$ or $\partial_{(2)}$ costs r_m^{-1}, and C depends only on N_2 and on norms in $L^1(\mathbb{R}^2)$ of derivatives of W_ℓ of orders at most $N_2 - N_1$. (b) is the third bound of Proposition 6.1 at the order $N_{\mathrm{mom}} < 2\ell$, applied to the derivatives $\partial^j F_v$, $|j| \le N$, which commute with $S(t)$, and summed with the weights of $H^N_{\rho,G'}(\mathbb{T}^2)$; the factorial weights cost a constant depending on $N + N_{\mathrm{mom}} \le N_{\mathrm{ret}}$, and the constant of (b) contains $C_\Omega^{N_{\mathrm{mom}}}$. Both parts use the bounds on $\Omega_i(t)$ recalled after (14.9), which hold for every $t \in \mathbb{R}$, as $m > m_0$. □

14.3 Proof of Proposition 5.4

Proof of Proposition 5.4. The top level $m_0 \ge 2$ (so that $m \ge 2$, as Lemma 7.1(a) and (14.7) require) is so large that, for $m > m_0$, $\rho(m) \le 1$ and $C\varepsilon_{m-1}^{5\eta/4} \le 1$, with the constant C of Proposition 14.1(i); then $|\Upsilon_m| \le 2$. Let t_{pr} and Π be those of the paragraph after Proposition 14.1. Fix a datum g, a time $t \in [t_{\mathrm{pr}}, t_{\mathrm{f}}]$ and a representation R' of $Y(t)$ at wavelength ε_{m-1}, and write S for $S(t)$. The proof writes $U(t)$ as $\Upsilon_m Y(t)$ plus five profiles (Step 2), bounds their profile norms (Steps 3–6), and adds them (Step 7).

Step 1: Regularity. By Proposition 13.10(i), Proposition 14.1(ii) and the Sobolev embedding, $Y(t)$ is of class $C^{N_{\mathrm{ret}}-2}$ on $\mathbb{T}^2$, with $N_{\mathrm{ret}} - 2 \ge N_{\mathrm{rep}} + N_{\mathrm{cor}} + 2p + 2$ by (14.3), and $U(t)$ is of class $C^{2N_{\mathrm{rep}}+2}$ on $\mathbb{T}^2$; this is the first claim of Proposition 5.4. Since S keeps regularity, $\exp_p(\Gamma_m(t) : \nabla^2)$ costs $2p$ derivatives and Corr_m maps fields of class $C^{N_{\mathrm{rep}}+N_{\mathrm{cor}}}$ to fields of class $C^{N_{\mathrm{rep}}}$, the five profiles defined in Step 2 are of class $C^{N_{\mathrm{rep}}}$ on $\mathbb{T}^2$.

Step 2: The exact split. The operator $\exp_p(\Gamma_m(t) : \nabla^2)$ acts on every component by the same scalar differential operator with constant coefficients, so it commutes with the constant matrix Υ_m. For a field F of G' of class $C^{N_{\mathrm{rep}}+N_{\mathrm{cor}}+2p+2}$ on $\mathbb{T}^2$, define, as in (13.2), $F^\flat := \exp_p(\Gamma_m(t) : \nabla^2)\,\Upsilon_m F$, and its exponential and corrector parts

$$F^{\exp} := \Upsilon_m\big(\exp_p(\Gamma_m(t) : \nabla^2) - I\big)F\,, \qquad F^{\mathrm{cor}} := (\mathrm{Corr}_m - I)\,F^\flat\,. \tag{14.10}$$

Then the lift (5.10) of F is $F^\sharp = \mathrm{Corr}_m F^\flat = \Upsilon_m F + F^{\exp} + F^{\mathrm{cor}}$, since $F^\flat = \Upsilon_m F + F^{\exp}$. Both parts are linear in F, so the splitting $Y(t) = SY(t) + (I - S)Y(t)$ gives the exact identity

$$\begin{aligned} U(t) = \Upsilon_m Y(t) &+ (SY(t))^{\exp} + (SY(t))^{\mathrm{cor}} \\ &+ ((I-S)Y(t))^{\exp} + ((I-S)Y(t))^{\mathrm{cor}} + \big(U(t) - Y^\sharp(t)\big). \end{aligned} \tag{14.11}$$

We give the two exponential parts the wavelength ε_{m-1}, and the two corrector parts and $U(t) - Y^\sharp(t)$ the wavelength ε_m. Lemma 14.2(b), with $\lambda' = \varepsilon_{m-1}$ and $\lambda = \varepsilon_m$, turns R' and these five profiles

into a representation R of $U(t)$ at wavelength ε_m, whose size is at most $|\Upsilon_m|\,\mathrm{size}_{\varepsilon_{m-1}}(R')$ plus the five profile norms bounded in Steps 3–6.

Step 3: The exponential part of $SY(t)$. We show

$$\varepsilon_{m-1}^{-s}\big\|(SY(t))^{\exp}\big\|_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^{\delta-2\Theta}\,\varepsilon_{m-1}^{-s}[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}\,. \tag{14.12}$$

Every term carries $\Gamma_m(t)$ at least once, which is small compared with r_m^2 by (14.6), and r_m^2 absorbs the two derivatives that come with it: on S each costs r_m^{-1}, and on $Y(t)$ each costs $\varepsilon_{m-1}^{-1}\le Cr_m^{-1}$. In detail, every term of $\exp_p(\Gamma_m(t):\nabla^2)-I$ is $(\Gamma_m(t):\nabla^2)^l/l!$ with $1\le l\le p$, and $(\Gamma_m(t):\nabla^2)^l$ is a sum of 4^l derivatives of order $2l$ with coefficients of absolute value at most $|\Gamma_m(t)|^l$. Let j be a multi-index with $|j|\le N_{\mathrm{rep}}$, and put $N_2:=|j|+2l$ and $N_1:=\min\{N_{\mathrm{rep}},N_2\}\ge 1$. Lemma 14.4(a) keeps N_1 derivatives on $Y(t)$ and puts the other N_2-N_1 on S:

$$\begin{aligned}\big\|\partial^j(\Gamma_m(t):\nabla^2)^l SY(t)\big\|_{L^\infty(\mathbb{T}^2)} &\le 4^l|\Gamma_m(t)|^l\sum_{|i|=N_2}\|\partial^i SY(t)\|_{L^\infty(\mathbb{T}^2)}\\ &\le C|\Gamma_m(t)|^l\,r_m^{N_1-N_2}\sum_{|i|=N_1}\|\partial^i Y(t)\|_{L^\infty(\mathbb{T}^2)}\,.\end{aligned}$$

We multiply by $\varepsilon_{m-1}^{|j|-s}$ and write $|\Gamma_m(t)|^l r_m^{N_1-N_2}\varepsilon_{m-1}^{|j|}=(|\Gamma_m(t)|/r_m^2)^l(r_m/\varepsilon_{m-1})^{N_1-|j|}\varepsilon_{m-1}^{N_1}$. Here $(|\Gamma_m(t)|/r_m^2)^l\le C\varepsilon_{m-1}^{\delta-2\Theta}$ by (14.6) and $2\Theta<\delta$, and $(r_m/\varepsilon_{m-1})^{N_1-|j|}\le C$ by (14.7), since $N_1\ge|j|$. As $1\le N_1\le N_{\mathrm{rep}}$, for $|i|=N_1$ we have $\varepsilon_{m-1}^{N_1}\|\partial^i Y(t)\|_{L^\infty(\mathbb{T}^2)}\le i![Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}$, so

$$\varepsilon_{m-1}^{|j|-s}\big\|\partial^j(\Gamma_m(t):\nabla^2)^l SY(t)\big\|_{L^\infty(\mathbb{T}^2)}\le C\varepsilon_{m-1}^{\delta-2\Theta}\,\varepsilon_{m-1}^{-s}[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}\,.$$

The readout commutes with derivatives and $|\Upsilon_m|\le 2$; dividing by $j!$ and summing over $|j|\le N_{\mathrm{rep}}$ and $1\le l\le p$ gives (14.12).

Step 4: The corrector part of $SY(t)$. We show

$$\varepsilon_m^{-s}\big\|(SY(t))^{\mathrm{cor}}\big\|_{C^{N_{\mathrm{rep}}}_{\varepsilon_m}(\mathbb{T}^2)}\le C\varepsilon_{m-1}^{\mu}\,\varepsilon_{m-1}^{-s}[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}\,. \tag{14.13}$$

The correction costs the corrector amplitude times one derivative of $Y(t)$ at the wavelength ε_{m-1}; the length $\rho(m)$ cancels, and (5.2) says that this is small when measured at the wavelength ε_m. This is where $s<\frac12(1-\alpha)$ is used. We apply Lemma 13.3 to the field $(SY(t))^\flat$ of G. Its derivatives at the scale $\rho(m)$ are controlled by the seminorm of $Y(t)$ at the wavelength ε_{m-1}:

$$\sum_{1\le|i|\le N_{\mathrm{rep}}+N_{\mathrm{cor}}}\rho(m)^{|i|}\big\|\partial^i(SY(t))^\flat\big\|_{L^\infty(\mathbb{T}^2)}\le C\,\frac{\rho(m)}{\varepsilon_{m-1}}\,[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}\,. \tag{14.14}$$

Indeed, we expand $\exp_p(\Gamma_m(t):\nabla^2)$ as in Step 3, now with $0\le l\le p$; the readout commutes with derivatives, $|\Upsilon_m|\le 2$, and $|\Gamma_m(t)|\le Cr_m^2$ by (14.6), as $2\Theta<\delta$. For a multi-index i with $1\le|i|\le N_{\mathrm{rep}}+N_{\mathrm{cor}}$, Lemma 14.4(a) with $N_1:=\min\{N_{\mathrm{rep}},|i|+2l\}\ge1$ bounds the term of degree l by $C\rho(m)^{|i|}r_m^{N_1-|i|}\sum_{|j|=N_1}\|\partial^j Y(t)\|_{L^\infty(\mathbb{T}^2)}$. If $|i|\le N_{\mathrm{rep}}$, then $N_1\ge|i|$ and $\rho(m)^{|i|}r_m^{N_1-|i|}\le\rho(m)^{N_1}\le C\rho(m)\,\varepsilon_{m-1}^{N_1-1}$ by (14.7). If $|i|>N_{\mathrm{rep}}$, then $N_1=N_{\mathrm{rep}}$, and the $|i|-N_{\mathrm{rep}}\le N_{\mathrm{cor}}$ derivatives beyond N_{rep} fall on S, each at the cost $\rho(m)/r_m=\varepsilon_{m-1}^{-\Theta}$. The power $\rho(m)^{N_{\mathrm{rep}}-1}\le C\varepsilon_{m-1}^{N_{\mathrm{rep}}-1}\varepsilon_{m-1}^{(\gamma/2+\delta)(N_{\mathrm{rep}}-1)}$ (by (14.7)) absorbs them:

$$\rho(m)^{|i|}r_m^{N_{\mathrm{rep}}-|i|}=\rho(m)^{N_{\mathrm{rep}}}\Big(\frac{\rho(m)}{r_m}\Big)^{|i|-N_{\mathrm{rep}}}\le C\rho(m)\,\varepsilon_{m-1}^{N_{\mathrm{rep}}-1}\,\varepsilon_{m-1}^{(\gamma/2+\delta)(N_{\mathrm{rep}}-1)-\Theta N_{\mathrm{cor}}}\,,$$

and the last exponent is nonnegative by (14.4). In both cases the term is at most

$$C\,\frac{\rho(m)}{\varepsilon_{m-1}}\,\varepsilon_{m-1}^{N_1}\sum_{|j|=N_1}\|\partial^j Y(t)\|_{L^\infty(\mathbb{T}^2)} \le C\,\frac{\rho(m)}{\varepsilon_{m-1}}\,[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}\,,$$

which proves (14.14). Now Lemma 13.3 and (14.14) give, for $|j| \le N_{\mathrm{rep}}$,

$$\begin{aligned}\varepsilon_m^{|j|-s}\big\|\partial^j (SY(t))^{\mathrm{cor}}\big\|_{L^\infty(\mathbb{T}^2)} &\le C\varepsilon_m^{-s}\,\frac{\bar\varepsilon_m}{\rho(m)}\,\frac{\rho(m)}{\varepsilon_{m-1}}\,[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}\\ &= C\left(\varepsilon_m^{-s}\,\bar\varepsilon_m\,\varepsilon_{m-1}^{s-1}\right)\varepsilon_{m-1}^{-s}[Y(t)]_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)}\,,\end{aligned}$$

and the first factor is at most $C\varepsilon_{m-1}^{\mu}$ by (5.2). Summing over j proves (14.13).

Step 5: The parts of $(I-S)Y(t)$. We show

$$\varepsilon_{m-1}^{-s}\big\|((I-S)Y(t))^{\exp}\big\|_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)} + \varepsilon_m^{-s}\big\|((I-S)Y(t))^{\mathrm{cor}}\big\|_{C^{N_{\mathrm{rep}}}_{\varepsilon_m}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^2\|g\|_{L^2_G(\mathbb{T}^2)}\,. \tag{14.15}$$

By Lemma 14.4(b) with $\rho=\rho(m)$, the bound $r_m/\rho(m)\le\varepsilon_{m-1}^{\Theta}$, (14.3) and Proposition 13.10(i),

$$\|(I-S)Y(t)\|_{H^{N_{\mathrm{rep}}+N_{\mathrm{cor}}+2p+2}_{\rho(m),G'}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^{\Theta N_{\mathrm{mom}}}\|Y(t)\|_{H^{N_{\mathrm{ret}}}_{\rho(m),G'}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^{\Theta N_{\mathrm{mom}}}\|g\|_{L^2_G(\mathbb{T}^2)}\,. \tag{14.16}$$

For the exponential part, in the norms $H^N_{\rho(m),\omega}(\mathbb{T}^2)$ each operator $\Gamma_m(t):\nabla^2$ costs two orders (Lemma 8.1(b)) and the factor $C|\Gamma_m(t)|/\rho(m)^2\le C\varepsilon_{m-1}^{\delta}$ (Lemma 10.3(b)), and the readout costs $|\Upsilon_m|\le 2$. Lemma 14.3 with $\lambda=\varepsilon_{m-1}$ and $\rho=\rho(m)$, where $\varepsilon_{m-1}^{1+\gamma/2+\delta}/\rho(m)\le C$ by (14.7), the bound (14.5) and (14.16) give

$$\begin{aligned}\varepsilon_{m-1}^{-s}\big\|((I-S)Y(t))^{\exp}\big\|_{C^{N_{\mathrm{rep}}}_{\varepsilon_{m-1}}(\mathbb{T}^2)} &\le C\varepsilon_{m-1}^{-Z}\big\|((I-S)Y(t))^{\exp}\big\|_{H^{N_{\mathrm{rep}}+2}_{\rho(m),\omega}(\mathbb{T}^2)}\\ &\le C\varepsilon_{m-1}^{\delta-Z}\|(I-S)Y(t)\|_{H^{N_{\mathrm{rep}}+2p+2}_{\rho(m),G'}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^{\Theta N_{\mathrm{mom}}-Z}\|g\|_{L^2_G(\mathbb{T}^2)}\,.\end{aligned}$$

For the corrector part, the inequality of the proof of Lemma 14.3, with the costs of $\exp_p(\Gamma_m(t):\nabla^2)$ and Υ_m just used and (14.5), bounds $\rho(m)^{|i|}\|\partial^i((I-S)Y(t))^\flat\|_{L^\infty(\mathbb{T}^2)}$, for $|i|\le N_{\mathrm{rep}}+N_{\mathrm{cor}}$, by $C\rho(m)^{-1}$ times the left side of (14.16). Moreover, $\bar\varepsilon_m\le C\rho(m)$ and $\rho(m)\ge c\varepsilon_m$ by (13.7), and $\varepsilon_m\ge c\varepsilon_{m-1}^q$ by Lemma 7.1(a). Lemma 13.3 and (14.16) then give

$$\varepsilon_m^{-s}\big\|((I-S)Y(t))^{\mathrm{cor}}\big\|_{C^{N_{\mathrm{rep}}}_{\varepsilon_m}(\mathbb{T}^2)} \le C\varepsilon_m^{-s}\rho(m)^{-1}\varepsilon_{m-1}^{\Theta N_{\mathrm{mom}}}\|g\|_{L^2_G(\mathbb{T}^2)} \le C\varepsilon_{m-1}^{\Theta N_{\mathrm{mom}}-q(1+s)}\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

The moments of S beat both losses: by (13.5), and as $Q\ge 4+2qZ$ by (14.3), $Z\ge 1+s$ and $q>1$,

$$\Theta N_{\mathrm{mom}} > Q+q\gamma+4 \ge 2+\max\{Z,q(1+s)\}\,,$$

which proves (14.15).

Step 6: The comparison error. We show

$$\varepsilon_m^{-s}\big\|U(t)-Y^\sharp(t)\big\|_{C^{N_{\mathrm{rep}}}_{\varepsilon_m}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^2\|g\|_{L^2_G(\mathbb{T}^2)}\,. \tag{14.17}$$

Proposition 14.1(ii) bounds $U(t)-Y^\sharp(t)$ in $L^2_\omega(\mathbb{T}^2)$ by $C\varepsilon_{m-1}^Q\|g\|_{L^2_G(\mathbb{T}^2)}$, and, with Proposition 13.10(ii), in $H^{2N_{\mathrm{rep}}+4}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)$ by $C\|g\|_{L^2_G(\mathbb{T}^2)}$. Interpolation between these two bounds (in Fourier, by the Cauchy–Schwarz inequality) gives $N_{\mathrm{rep}}+2$ derivatives with half the smallness:

$$\big\|U(t)-Y^\sharp(t)\big\|_{H^{N_{\mathrm{rep}}+2}_{\rho_{\mathrm{mol}}(m),\omega}(\mathbb{T}^2)} \le C\varepsilon_{m-1}^{Q/2}\|g\|_{L^2_G(\mathbb{T}^2)}\,. \tag{14.18}$$

Since $m < M$, Proposition 2.1(b) gives $\kappa_m \geq c\varepsilon_m^{\beta+\gamma}$, hence, by (11.1), $c\varepsilon_m^{1+\gamma/2+\delta} \leq \rho_{\mathrm{mol}}(m) \leq 1$. Lemma 14.3 with $\lambda = \varepsilon_m$ and $\rho = \rho_{\mathrm{mol}}(m)$, (14.18) and $\varepsilon_m \geq c\varepsilon_{m-1}^q$ (Lemma 7.1(a)) give

$$\varepsilon_m^{-s} \big\| U(t) - Y^\sharp(t) \big\|_{C_{\varepsilon_m}^{N_{\mathrm{rep}}}(\mathbb{T}^2)} \leq C\varepsilon_m^{-Z}\varepsilon_{m-1}^{Q/2}\|g\|_{L^2_G(\mathbb{T}^2)} \leq C\varepsilon_{m-1}^{Q/2-qZ}\|g\|_{L^2_G(\mathbb{T}^2)}\,,$$

and $Q/2 - qZ \geq 2$ by (14.3).

Step 7: Conclusion. By Lemma 14.2(a) applied to R', $\varepsilon_{m-1}^{-s}[Y(t)]_{C_{\varepsilon_{m-1}}^{N_{\mathrm{rep}}}(\mathbb{T}^2)} \leq \mathrm{size}_{\varepsilon_{m-1}}(R')$. Inserting this, (14.12)–(14.17) and Proposition 14.1(i) into the bound of Step 2 gives

$$\mathrm{size}_{\varepsilon_m}(R) \leq \big(1 + C(\varepsilon_{m-1}^{5\eta/4} + \varepsilon_{m-1}^{\delta-2\Theta} + \varepsilon_{m-1}^{\mu})\big)\,\mathrm{size}_{\varepsilon_{m-1}}(R') + C\varepsilon_{m-1}^2\|g\|_{L^2_G(\mathbb{T}^2)}\,.$$

By (2.3), $\eta = \min\{\delta/8, \mu/4\}$, and by (14.4), $\delta - 2\Theta > \frac{5}{6}\delta$, so each of $5\eta/4$, $\delta - 2\Theta$ and μ is at least η. Since $\varepsilon_{m-1} \leq 1$, this proves Proposition 5.4. □

14.4 Proof of Proposition 5.2

Proof of Proposition 5.2. Each cycle starts once the previous preparation is complete and the admissibility window has moved forward by $3\tau_j$.

Step 1: Levels finer than m_0. By the choice of m_0 (item (4) of Subsection 11.1), $\rho(j) \leq 1$ for every level $j > m_0$. An admissible seed of level $j > m_0$ on a window $[t_{\mathrm{S}}, t_{\mathrm{f}}]$ with $t_{\mathrm{S}} + \frac{1}{2}t_0 < t_{\mathrm{f}}$ is an admissible family of level j on $[t_{\mathrm{S}}, t_{\mathrm{f}}]$ (Definition 11.4, with no successor). For every $j \geq m_0$, the row and column sums of the coupling matrices of an admissible family of level j, for the masses ω_v and ω_v/Ψ_v^2, are at most $\frac{1}{8}$ (Proposition 11.5). By the bounds (2.5) on τ_j and h_j (Lemma 7.1(b)), and since $\varepsilon_j \leq \Lambda^{-j}$ (Lemma 7.1(a)), the series of the $3\tau_j + 4h_j$ converges, and item (4) takes m_0 so large that, for every $m > m_0$,

$$\sum_{j=m_0+1}^{m} (3\tau_j + 4h_j) < \tfrac{1}{2}t_0\,. \tag{14.19}$$

Step 2: The homogenization cascade. Let m, κ, G, t_{S}, t_{f}, g and U be as in Proposition 5.2. Set $G_m := G$, $U_m := U$, $t_{(m)} := t_{\mathrm{S}}$ and $t_{\mathrm{A}}^{(m)} := t_{\mathrm{S}}$; by Step 1, G_m is an admissible family of level m on $[t_{\mathrm{A}}^{(m)}, t_{\mathrm{f}}]$. For $j = m, m-1, \dots, m_0+1$ in turn, given an admissible family G_j of level j on $[t_{\mathrm{A}}^{(j)}, t_{\mathrm{f}}]$ and a solution U_j of G_j on $[t_{(j)}, t_{\mathrm{f}}]$, put

$$t_{\mathrm{c}}^{(j)} := \max\big\{t_{(j)},\, t_{\mathrm{A}}^{(j)} + 3\tau_j\big\}\,. \tag{14.20}$$

Let G_j' be the successor of G_j in the descent of the seed G given by Lemma 11.8(iv), and let $t_{\mathrm{pr}}^{(j)} \in [t_{\mathrm{c}}^{(j)}, t_{\mathrm{c}}^{(j)} + 4h_j]$ and Π_j be the time and the map constructed in Section 12 for G_j and G_j', with $t_{\mathrm{A}} = t_{\mathrm{A}}^{(j)}$ and $t_{\mathrm{c}} = t_{\mathrm{c}}^{(j)}$. By the paragraph after Proposition 14.1, Propositions 5.3, 14.1, 13.10 and 5.4 hold for this pair. The hypothesis $t_{\mathrm{c}}^{(j)} + 4h_j \leq t_{\mathrm{f}}$ of Proposition 5.3 follows from (14.21) at level j, whose proof below uses only the times $t_{\mathrm{pr}}^{(i)}$ of the levels $i > j$, constructed before. By Proposition 5.3(b), $G_{j-1} := G_j'$ is an admissible family of level $j-1$ on $[t_{\mathrm{A}}^{(j-1)}, t_{\mathrm{f}}]$, where $t_{\mathrm{A}}^{(j-1)} := t_{\mathrm{A}}^{(j)} + 3\tau_j$. Let U_{j-1} be the solution of G_{j-1} from the retained trace of $\Pi_j U_j(t_{\mathrm{c}}^{(j)})$ at time $t_{(j-1)} := t_{\mathrm{pr}}^{(j)}$. By the uniqueness in Proposition 6.2 and (8.3), subtracted at the times $t_{\mathrm{c}}^{(j)}$ and $t \geq t_{\mathrm{c}}^{(j)}$, the restriction of U_j to $[t_{\mathrm{c}}^{(j)}, t_{\mathrm{f}}]$ is the solution of G_j from $U_j(t_{\mathrm{c}}^{(j)})$ at time $t_{\mathrm{c}}^{(j)}$. Hence these four propositions apply at level j to the datum $U_j(t_{\mathrm{c}}^{(j)})$, with $U = U_j$ and $Y = U_{j-1}$.

The times satisfy, for $m_0 < j \le m$,

$$t_{\mathrm{c}}^{(j)} + 4h_j \le t_{\mathrm{S}} + \sum_{i=j}^{m} (3\tau_i + 4h_i) < t_{\mathrm{S}} + \tfrac{1}{2}t_0 < t_{\mathrm{f}}\,. \tag{14.21}$$

For $j = m$, $t_{\mathrm{c}}^{(m)} = t_{\mathrm{S}} + 3\tau_m$. For $j < m$, $t_{(j)} = t_{\mathrm{pr}}^{(j+1)} \le t_{\mathrm{c}}^{(j+1)} + 4h_{j+1}$ and $t_{\mathrm{A}}^{(j)} + 3\tau_j = t_{\mathrm{S}} + 3\sum_{i=j}^{m}\tau_i$, which gives the first inequality by induction on j, for $j = m, m-1, \dots$ in turn; the second is (14.19). In particular, for $m_0 < j \le m$, $t_{\mathrm{pr}}^{(j)} \le t_{\mathrm{c}}^{(j)} + 4h_j < t_{\mathrm{S}} + \frac{1}{2}t_0$. Moreover, $t_{(m)} = t_{\mathrm{A}}^{(m)}$ and, for $m_0 \le j < m$,

$$t_{(j)} = t_{\mathrm{pr}}^{(j+1)} \ge t_{\mathrm{c}}^{(j+1)} \ge t_{\mathrm{A}}^{(j+1)} + 3\tau_{j+1} = t_{\mathrm{A}}^{(j)}\,,$$

so U_j lives in the window $[t_{\mathrm{A}}^{(j)}, t_{\mathrm{f}}]$ of G_j, on which Step 1 bounds its coupling matrices.

Step 3: The marked amplitudes. We show that, for $m_0 \le j \le m$,

$$\|U_j\|_{L^\infty([t_{(j)},t_{\mathrm{f}}];L^2_{G_j}(\mathbb{T}^2))} \le \prod_{i=j+1}^{m} \big(1 + C(\varepsilon_{i-1}^{Q} + \varepsilon_{i-1}^{\delta})\big)\|g\|_{L^2_\omega(\mathbb{T}^2)} \le C\|g\|_{L^2_\omega(\mathbb{T}^2)}\,, \tag{14.22}$$

which is (5.3) iterated; the product is bounded as in Subsection 5.2. For $j = m$, this is Proposition 6.2 for G_m, since the markers of a seed are 1; the proposition applies by Step 1 and Definition 11.2(iv). Let $m_0 < j \le m$. Since U_j is continuous in time (Proposition 6.2), the norm of the datum $U_j(t_{\mathrm{c}}^{(j)})$ in $L^2_{G_j}(\mathbb{T}^2)$ is at most the left side of (14.22) at level j. Proposition 5.3(b) at level j, with $Y = U_{j-1}$, bounds $\|U_{j-1}\|_{L^\infty([t_{(j-1)},t_{\mathrm{f}}];L^2_{G_{j-1}}(\mathbb{T}^2))}$ by $1 + C(\varepsilon_{j-1}^{Q} + \varepsilon_{j-1}^{\delta})$ times this norm, which gives (14.22) at level $j-1$.

Step 4: The top level. Let $t \in [t_{\mathrm{S}} + \frac{1}{2}t_0, t_{\mathrm{f}}]$; by (14.21), $t \ge t_{\mathrm{pr}}^{(m_0+1)} = t_{(m_0)}$. At level $m_0 + 1$ the solution Y is U_{m_0}, so Proposition 13.10(i) and (14.22) give

$$\|U_{m_0}(t)\|_{H^{N_{\mathrm{ret}}}_{\rho(m_0+1),G_{m_0}}(\mathbb{T}^2)} \le C\|U_{m_0+1}(t_{\mathrm{c}}^{(m_0+1)})\|_{L^2_{G_{m_0+1}}(\mathbb{T}^2)} \le C\|g\|_{L^2_\omega(\mathbb{T}^2)}\,.$$

Since $\rho(m_0+1) \le 1$ and $\varepsilon_{m_0}^{1+\gamma/2+\delta}/\rho(m_0+1) \le C$ by (14.7), Lemma 14.3 with $\lambda = \varepsilon_{m_0}$, together with (14.5) and $N_{\mathrm{ret}} \ge N_{\mathrm{rep}} + 2$, shows that $R_{m_0}(t) \coloneqq (U_{m_0}(t), \emptyset)$ is a representation of $U_{m_0}(t)$ at wavelength ε_{m_0} with

$$\mathrm{size}_{\varepsilon_{m_0}}(R_{m_0}(t)) \le C\varepsilon_{m_0}^{-Z}\|U_{m_0}(t)\|_{H^{N_{\mathrm{rep}}+2}_{\rho(m_0+1),\omega}(\mathbb{T}^2)} \le C\varepsilon_{m_0}^{-Z}\|g\|_{L^2_\omega(\mathbb{T}^2)}\,. \tag{14.23}$$

Step 5: The regularity cascade. For $j = m_0+1, \dots, m$ in turn, $t \in [t_{\mathrm{pr}}^{(j)}, t_{\mathrm{f}}]$ by (14.21), and Proposition 5.4 at level j, applied to the representation $R_{j-1}(t)$ of $U_{j-1}(t)$ at wavelength ε_{j-1}, gives a representation $R_j(t)$ of $U_j(t)$ at wavelength ε_j. The datum of this passage is $U_j(t_{\mathrm{c}}^{(j)})$, whose norm is at most $C\|g\|_{L^2_\omega(\mathbb{T}^2)}$ by (14.22), so

$$\mathrm{size}_{\varepsilon_j}(R_j(t)) \le \big(1 + C\varepsilon_{j-1}^{\eta}\big)\,\mathrm{size}_{\varepsilon_{j-1}}(R_{j-1}(t)) + C\varepsilon_{j-1}\|g\|_{L^2_\omega(\mathbb{T}^2)}\,,$$

which is (5.4) together with (14.22). Iterating, with $\prod_j(1 + C\varepsilon_{j-1}^{\eta}) \le \exp(C\sum_j \varepsilon_{j-1}^{\eta})$, the bounded sums of Subsection 5.2 and (14.23),

$$\begin{aligned}\mathrm{size}_{\varepsilon_m}(R_m(t)) &\le \exp\Big(C\sum_{j=m_0+1}^{m}\varepsilon_{j-1}^{\eta}\Big)\Big(\mathrm{size}_{\varepsilon_{m_0}}(R_{m_0}(t)) + C\sum_{j=m_0+1}^{m}\varepsilon_{j-1}\|g\|_{L^2_\omega(\mathbb{T}^2)}\Big)\\ &\le C\big(\varepsilon_{m_0}^{-Z} + 1\big)\|g\|_{L^2_\omega(\mathbb{T}^2)}\,.\end{aligned}$$

Since m_0 depends only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$, the representation $R(t) \coloneqq R_m(t)$ proves Proposition 5.2, for every $t \in [t_{\mathrm{S}} + \frac{1}{2}t_0, t_{\mathrm{f}}]$. □

15. Initialization at the finest levels

In this section we prove Proposition 5.1, the entry, which initializes the induction at the finest levels: it produces the admissible seed of level $M-1$, depending only on M, κ and t_{in}, from which the homogenization cascade (Proposition 5.2) starts. A weak solution θ of (4.1), whose velocity field $\mathbf{b}_{M+n}$ has n levels finer than M, only provides the datum of the seed. Throughout, $\kappa \in \mathcal{K}_M$, so that $\kappa_M = \kappa$, $t_{\mathsf{in}} \in \mathbb{R}$ and $t_{\mathrm{f}} := t_{\mathsf{in}} + T$; C and c denote positive constants depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$, which may change from line to line; and $M \geq M_*(\alpha, s, q, \zeta, \Lambda, n, t_0, T)$, the threshold of Proposition 5.1, fixed last, above the finitely many levels beyond which the inequalities below hold.

The route. The entry walks up from level $M+n$ to level $M-1$ in three stages, and then back down.

1. *Weak passages*, from level i to level $i-1$ for $i = M+n, \dots, M+1$. At the levels finer than M, the Péclet number $\varepsilon_i^{\beta}/\kappa$ is small (Lemma 15.1), so a passage keeps the diffusivity κ: the enhancement of level i has the relative size $(\varepsilon_i^{\beta}/\kappa)^2$, and it is kept as a source of the coarser family instead of being added to the diffusivity. On the quiet margins of level i the velocity field is $\mathbf{b}_{i-1}$, and molecular diffusion over a margin reaches the wavelength ε_{i-1}; the solution regains its derivatives there, and the next passage starts at the block boundary between two margins.

2. *One smoothing*, at level M: the sources are averaged over discs of radius r_{w}, slightly below ε_M, so that the terminal cycle can use Sections 12 and 13.

3. *The terminal cycle* at level M, by the method of Proposition 5.3, at the bounded contrast $\kappa_{M-1}/\kappa \leq 2$. It is prepared over a time of order h_M and creates one generation of components. Its successor, restarted h_M later with its components declared of generation 0, is the seed.

Back down, the terminal lift and the n weak lifts carry a representation at ε_{M-1} to one at ε_{M+n} at a constant cost, and θ differs from the lifted solution by little in $C^{0,s}(\mathbb{T}^2)$.

One ladder of families. We write G_i and U_i, $M \leq i \leq M+n$, for the families of the weak levels and their solutions, and $t_{(i)}$ for the time at which U_i starts; $U_{M+n} = \theta$. This continues upward the notation G_j, U_j, $t_{(j)}$ of the proof of Proposition 5.2: the seed is $G_{M-1} = G$, and its solution is $U_{M-1} = U$, from $t_{(M-1)} = t_{\mathrm{e}}$. Between them stand $\widetilde{G}_M$, the family G_M with smoothed sources, its solution $\widetilde{U}_M$, and the terminal successor G' of $\widetilde{G}_M$, of level $M-1$, with its solution Y (Subsection 15.3). Figure 15.1 shows the times.

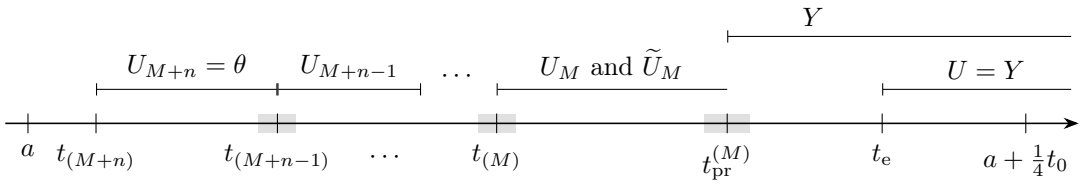


Figure 15.1: The times of the entry, not to scale. Shaded: the quiet margins, the two of level i around $t_{(i-1)}$, forming $[t_{(i-1)} - \tau_i', t_{(i-1)} + \tau_i']$, $M < i \leq M+n$, and the two of level M around $t_{\mathrm{pr}}^{(M)}$. Bars: the solutions of the ladder from their starts, with the preparation time $t_{\mathrm{pr}}^{(M)}$ and $t_{\mathrm{e}} = t_{\mathrm{pr}}^{(M)} + h_M$; a bar ends where the next solution takes over, or at t_{f}. Dots: the levels between $M+n-1$ and M.

15.1 Parameters and scales

Parameters. The entry uses the readout weight $\varepsilon_{i-1}^{\delta/16}$ of a letter at level i, the mass factor $(4nN_{\mathrm{cor,w}})^{-1}\varepsilon_{i-1}^{\delta/32}$ of a letter of a weak passage at level i, with $N_{\mathrm{cor,w}}$ of item (1), and the width

$$r_{\mathrm{w}} \coloneqq \varepsilon_M^{1+\gamma/4+\delta/2}$$

of the smoothing at level M. Since $\delta \le \frac{1}{4}(q-1)$ and $\gamma < \frac{2}{3}(q-1)$ by (2.3), $\frac{\gamma}{4}+\frac{\delta}{2} < \frac{7}{24}(q-1)$. With $L \coloneqq \Lambda^{q^M/(q-1)}$, (2.4) gives $\varepsilon_{M+1} \le L^{-q}$ and $\varepsilon_M \ge (2L)^{-1}$, and $L^{q-1-\gamma/4-\delta/2} \ge \Lambda^{17/24} \ge 4$ as $\Lambda \ge 128$, so that $\varepsilon_{M+1} \le r_{\mathrm{w}} \le \varepsilon_M$. The entry then fixes the following orders and exponents, in this order, depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$ (Table 11.1).

(1) *The weak orders*: the corrector order $N_{\mathrm{cor,w}}$ and word length $N_{\mathrm{len,w}}$ of a weak passage, with

$$(q-1)N_{\mathrm{cor,w}} > 5 + 2q^n\,, \qquad 2(q+2)\gamma\,(N_{\mathrm{len,w}}+1) > 4 + 3q^n\,. \tag{15.1}$$

They absorb the cell tail and the frontier of a weak passage (Lemma 15.7). With them, θ is recovered from a field of G_M at the order $nN_{\mathrm{cor,w}}+5$ (Lemma 15.13(b)).

(2) *The terminal orders*: the accuracy $Q_{\mathrm{e}} \coloneqq 2\big(5+q^{n+1}(1+s)\big)$ of the terminal cycle; then its corrector order $N_{\mathrm{cor,e}}$, truncation order p_{e}, word length $N_{\mathrm{len,e}}$ and number $N_{\mathrm{cl,e}}$ of antidivergence steps, with

$$\begin{aligned} (q\gamma+3\delta)N_{\mathrm{cor,e}} &> Q_{\mathrm{e}}+2\,, \qquad & \delta p_{\mathrm{e}} &> Q_{\mathrm{e}}+2\,,\\ \tfrac{1}{2}\delta(N_{\mathrm{len,e}}+1) &> Q_{\mathrm{e}}+2\,, \qquad & (q\gamma+3\delta)N_{\mathrm{cl,e}} &> 3\,; \end{aligned} \tag{15.2}$$

then the largest birth order of the sources the terminal cycle creates and the width exponent of their smoothing,

$$N_{\mathrm{src,e}} \coloneqq N_{\mathrm{cor,e}} + nN_{\mathrm{cor,w}} + 4p_{\mathrm{e}} + 1\,, \qquad \Theta_{\mathrm{e}} \coloneqq \frac{\delta}{16(N_{\mathrm{src,e}}-1)}\,. \tag{15.3}$$

The accuracy absorbs the loss $\varepsilon_{M+n}^{-1-s} \le C\varepsilon_{M-1}^{-q^{n+1}(1+s)}$ of the recovery of θ (Lemma 15.14), and (15.2) the residual of the terminal cycle (Table 15.1).

(3) *The representation order*

$$N_{\mathrm{rep}} \coloneqq nN_{\mathrm{cor,w}} + N_{\mathrm{cor,e}} + 2p_{\mathrm{e}} + 4\,. \tag{15.4}$$

This is Part I's N_{rep}: the terminal lift spends $N_{\mathrm{cor,e}}+2p_{\mathrm{e}}+1$ derivatives and the weak lifts $nN_{\mathrm{cor,w}}$, which leaves three. The orders of items (1) and (2) fix the seed's bounds: birth orders at most $N_{\mathrm{src,e}}$, at most $N_{\mathrm{path,e}} \coloneqq (N_{\mathrm{len,e}}+2)(nN_{\mathrm{len,w}}+1)$ components along a path of sources, that is, at most $B \coloneqq N_{\mathrm{path,e}}-1$ sources on a path, the bound B of item (1) of Subsection 11.1. Section 11 then fixes the remaining parameters of its item (1).

(4) After these: the moment orders $N_{\mathrm{mom,w}}$ of the smoothing at level M and $N_{\mathrm{mom,e}}$ of the smoothing of the sources the terminal cycle creates, with

$$\begin{aligned} &\delta - \tfrac{\delta}{16} - \tfrac{\delta}{64} - \Theta_{\mathrm{e}}(N_{\mathrm{src,e}}-2) + \Theta_{\mathrm{e}}N_{\mathrm{mom,e}} > Q_{\mathrm{e}}+2\,,\\ &q\Big(2(q+2)\gamma - \tfrac{5\delta}{64} + \big(\tfrac{\gamma}{4}+\tfrac{\delta}{2}\big)N_{\mathrm{mom,w}}\Big) > 2Q_{\mathrm{e}}+4\,; \end{aligned} \tag{15.5}$$

they absorb the smoothings in Table 15.1. Since $2(q+2)\gamma < 1$ and $q < \frac{17}{16}$ by (2.3) and (2.1), the second condition gives $(\frac{\gamma}{4}+\frac{\delta}{2})N_{\mathrm{mom,w}} > (2Q_{\mathrm{e}}+4)/q - 1 > 2+2q^n$, which Lemma 15.14 uses. Then the recovery order $N_{\mathrm{rd,e}} \ge nN_{\mathrm{cor,w}}+5$ of the terminal comparison (Proposition 15.10(c)),

which, with $N_{\mathrm{rec,e}} \coloneqq 4(N_{\mathrm{cor,e}} + N_{\mathrm{src,e}}) + 2N_{\mathrm{cl,e}} + 16$ and $\bar{N}_{\mathrm{mom}} \coloneqq \max\{N_{\mathrm{mom,w}}, N_{\mathrm{mom,e}}, N_{\mathrm{mom}}\}$, is also at least the floor (13.6) of the comparison at $m = M$, with the terminal orders,

$$\begin{aligned} N_{\mathrm{rd,e}} &\geq \max\{N_{\mathrm{rec,e}} + N_{\mathrm{src,e}} + \bar{N}_{\mathrm{mom}} + 6p_{\mathrm{e}},\ N_{\mathrm{src,e}} - 2 + \bar{N}_{\mathrm{mom}} + N_{\mathrm{cor,e}} + 2p_{\mathrm{e}} + 1\}\,, \\ N_{\mathrm{rd,e}} &\geq N_{\mathrm{rec,e}} + 2N_{\mathrm{src,e}}N_{\mathrm{path,e}} \\ &\quad + \max\{N_{\mathrm{src,e}} + \bar{N}_{\mathrm{mom}} + N_{\mathrm{cor,e}} + 6p_{\mathrm{e}} + 4,\ (N_{\mathrm{path,e}} + 3)(N_{\mathrm{cor,e}} + N_{\mathrm{src,e}} + 12) + 2p_{\mathrm{e}}\}\,; \end{aligned} \tag{15.6}$$

then the output order $N_{\mathrm{out,e}}$ of the terminal preparation and the trace order $N_{\mathrm{tr,w}}$ of a weak passage,

$$\begin{aligned} N_{\mathrm{out,e}} &\coloneqq \max\{N_{\mathrm{rep}},\ 2N_{\mathrm{rd,e}} + 2,\ N_{\mathrm{cor,e}} + 2p_{\mathrm{e}} + N_{\mathrm{rd,e}} + 2\}\,, \\ N_{\mathrm{tr,w}} &\geq \max\{N_{\mathrm{mom,w}} + nN_{\mathrm{cor,w}} + 1 + N_{\mathrm{rd,e}},\ 2N_{\mathrm{rd,e}} + 2\}\,. \end{aligned} \tag{15.7}$$

As $N_{\mathrm{out,e}} \geq 2N_{\mathrm{rd,e}} + 2$, the proof of Proposition 15.10 runs at the order $N_{\mathrm{out,e}}$, above this floor. The condition on $N_{\mathrm{tr,w}}$ in (15.7) leaves room for the smoothing and the sources of G_M in Lemma 15.8, above $N_{\mathrm{rd,e}}$. The order $N_{\mathrm{tr,w}}$ is also at least the order N_{buf} of (12.2) that the preparation reads at the terminal level to produce Π at $N_{\mathrm{out,e}}$, above the order $2N_{\mathrm{rd,e}}$ at which the proof of Proposition 15.10(a) interpolates. Then the seed's constants (Section 11, item (2)).

(5) The threshold M_*, last.

Lemma 15.1 separates the weak levels from the terminal one. At the levels finer than M, by (a), the Péclet number is small, the strain of $\mathbf{b}_{i-1}$ is small over the time diffusion needs to reach ε_{i-1}, the quiet margins are long enough for item 1 of the route (as $\gamma > 3\delta$), and a pulse lasts many cell diffusion times. At level M, by (b), the margins are too short, and the contrast is at most 2.

Lemma 15.1 (Scales of the entry)**.** *There exist $\Lambda_0(\alpha, s, q, \zeta) \geq 128$ and $C(\alpha, s, q, \zeta)$ with the following property. For every $\Lambda \geq \Lambda_0$ there exists an integer $M_2(\alpha, s, q, \zeta, \Lambda) \geq 2$ such that, for every $M \geq 2$ and $\kappa \in \mathcal{K}_M$, the following hold.*

(a) *For every $M < i \leq M + n$,*

$$\frac{\varepsilon_i^{\beta}}{\kappa} \leq C\varepsilon_{i-1}^{(q+2)\gamma}\,, \quad \frac{\varepsilon_{i-1}^{\beta}}{\kappa} \leq C\varepsilon_{i-1}^{\gamma}\,, \quad \frac{\varepsilon_{i-1}^2}{\kappa\tau_i'} \leq C\varepsilon_{i-1}^{\gamma-3\delta}\,, \quad \frac{\kappa\tau_i}{\varepsilon_i^2} \geq \frac{1}{C}\varepsilon_{i-1}^{-(2(q-1)+\gamma-4\delta)}\,.$$

(b) *At level M, $\kappa\tau_M' \asymp \varepsilon_{M-1}^{2+\gamma+3\delta}$ and*

$$\frac{\varepsilon_M^{\beta}}{\kappa} \leq C\varepsilon_M^{\gamma}\,, \quad \frac{\bar{\varepsilon}_M}{\rho(M)} \asymp \varepsilon_{M-1}^{q\gamma+3\delta}\,, \quad \frac{\kappa h_M}{\varepsilon_M^2} \asymp \varepsilon_{M-1}^{-2(q\gamma+3\delta)}\,, \quad \frac{\kappa\tau_M}{\varepsilon_M^2} \asymp \varepsilon_{M-1}^{-(2q\gamma+4\delta)}\,.$$

Moreover $\kappa_{M-1}/\kappa \geq 1$, and, if $M \geq M_2$, then $\kappa_{M-1}/\kappa \leq 2$.

Proof. All the bounds are arithmetic. By (2.18) and Lemma 7.1(a), for $i - 1 \geq M$,

$$\kappa \geq \tfrac{1}{2}\varepsilon_{i-1}^{\beta-\gamma}\,, \qquad \varepsilon_i \leq 2\varepsilon_{i-1}^{q}\,, \qquad \kappa \asymp \varepsilon_M^{\beta-\gamma} \asymp \varepsilon_{M-1}^{\beta+\gamma}\,.$$

With (2.5), they give the exponents of (a) and (b) by two identities: $q(\beta - \gamma) = \beta + \gamma$, which turns $q\beta - \beta + \gamma$ into $(q+2)\gamma$, and $q - 1 - \frac{\gamma}{2} - \delta = q\gamma + 3\delta$ (Lemma 7.1(c), as $\beta = 1 + \alpha$), which gives $\bar{\varepsilon}_M/\rho(M) = \varepsilon_M(\kappa h_M)^{-1/2} \asymp \varepsilon_{M-1}^{q-1-\gamma/2-\delta}$ and $\kappa\tau_M/\varepsilon_M^2 \asymp \varepsilon_{M-1}^{2\delta-2(q-1-\gamma/2-\delta)}$. By Proposition 2.1(c), and by Proposition 2.1(a) and Lemma 7.1(a),

$$1 \leq \frac{\kappa_{M-1}}{\kappa} \leq 1 + C\frac{\varepsilon_M^{2\beta}}{\kappa^2} \leq 1 + C\varepsilon_M^{2\gamma} \leq 1 + C\Lambda^{-2\gamma M}\,,$$

which is at most 2 for $M \geq M_2$, the least integer $M_2 \geq 2$ with $C\Lambda^{-2\gamma M_2} \leq 1$. □

15.2 The levels finer than M, at small Péclet number

The two-dimensional cell inverse. Let $M < i \le M+n$ and let k be a pulse of level i. In the chart of k (Section 7), with $z = y/\varepsilon_i$, the shear added at level i is $\nabla^\perp\psi_i(t,y)$ of (7.11), of size $2\pi\varepsilon_i^{\beta-1}$; in the cell time $\kappa t/\varepsilon_i^2$ it is the Péclet number $\varepsilon_i^\beta/\kappa$ times a bounded field. A weak passage solves its cell problems with the two-dimensional cell inverse below, as Definition 11.2(iii) allows: once, from zero, on the window of each pulse. As the divergence-free shear drops out of the energy, solutions without forcing decay at the cell rate of level i, which absorbs the switching in Lemma 15.7.

Lemma 15.2 (The two-dimensional cell inverse)**.** *Let $N \ge 1$ be one of the orders used below; they depend only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$, and M_* is fixed after them. There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T, N)$ such that the following hold for every $M \ge M_*$, $M < i \le M+n$, every pulse k of level i, every conductivity $\mathbf{K}(t,y)$ on the window of k as in Section 10, and every pair F^0, F of two-scale forms on the window, of zero average in z, measurable in t, of class C^{N+3} in z with bounded derivatives. A solution on an interval I of the equation below is a two-scale form $\mathcal{A}$ with $\mathcal{A}(\cdot,y,\cdot) \in C(I; L^2([0,1]^2))$ for every $y \in \mathbb{R}^2$ that solves it in the integrated form in t, tested against smooth periodic functions of z, as in Lemma 10.4.*

(a) *There exists exactly one solution $\mathcal{A}$ on the window, of zero average in z and with $\nabla_z\mathcal{A}(\cdot,y,\cdot)$ square integrable over the window times $[0,1]^2$ for every $y \in \mathbb{R}^2$, of*

$$\begin{cases} \partial_t\mathcal{A} + \varepsilon_i^{-1}(\nabla^\perp\psi_i)(t,\varepsilon_i z)\cdot\nabla_z\mathcal{A} - \kappa\varepsilon_i^{-2}\nabla_z\cdot(\mathbf{K}\nabla_z\mathcal{A}) & \\ \qquad = \kappa\varepsilon_i^{-2}\big(F^0 + \nabla_z\cdot F\big) & \text{in the window of } k\,, \\ \mathcal{A} = 0 & \text{at its start, } t = (k - \tfrac{61}{48})\tau_i\,; \end{cases}$$

it is of class C^N in z.

(b) *Let $F^0 = F = 0$, and let $\mathcal{A}$ be a solution on an interval $[t_1,t_2]$ within the window, of class C^N in z, with $\mathcal{A}(t_1)$ of zero average and of class C^{N+2} in z, and with $\nabla_z\mathcal{A}(\cdot,y,\cdot)$ square integrable over $[t_1,t_2]\times[0,1]^2$ for every $y \in \mathbb{R}^2$. Then, for every $t \in [t_1,t_2]$ and $y \in \mathbb{R}^2$,*

$$\|\mathcal{A}(t,y,\cdot)\|_{L^2([0,1]^2)} \le C\exp\big(-\pi^2\kappa(t-t_1)/\varepsilon_i^2\big)\|\mathcal{A}(t_1,y,\cdot)\|_{L^2([0,1]^2)}\,.$$

Proof. The solution of (a) is constructed by Galerkin approximation. The shear is divergence free, so it drops out of the energy identity; with $\mathbf{K} \ge \frac{3}{4}I$ and $\|\nabla_z\mathcal{A}\|^2_{L^2([0,1]^2)} \ge 4\pi^2\|\mathcal{A}\|^2_{L^2([0,1]^2)}$ for forms of zero average, the identity gives (b), with $C = 1$, and, for the difference of two solutions, the uniqueness in (a). The derivatives of the shear in z are at most $C\varepsilon_i^\beta/\kappa \le C$, by Lemma 15.1(a); so, differentiating in z, the energy identity and the Sobolev embedding give the regularity in (a). □

The weak corrector operator. The *weak finite lift* of level i is the finite lift of Definition 10.5 for the pulses of level i, with κ for both diffusivities, the corrector order $N_{\mathrm{cor,w}}$, the slow length ε_{i-1} and the two-dimensional cell inverse: the inverse of a right side $H^0 + \nabla_z\cdot H$ is the solution of Lemma 15.2(a) with $F^0 = \varepsilon_i^2\kappa^{-1}H^0$ and $F = \varepsilon_i^2\kappa^{-1}H$. The definition, existence and uniqueness of the finite lift use only the cell inverse (Section 10): the correctors exist at every order, each solved once on the window of its pulse from zero at its start, and the lift is determined by its values on the window. The *weak corrector operator* $\mathrm{Corr}_i^{\mathrm{w}}(t)$ is the corrector operator of Definition 10.8 built from the weak finite lift. For a field U of (t,x), $\mathrm{Corr}_i^{\mathrm{w}}U$ is the field $t \mapsto \mathrm{Corr}_i^{\mathrm{w}}(t)U(t)$.

In Definition 15.3, the means of the weak finite lift along the paths of sources become the inherited sources (ii), and those along the empty path, less the molecular flux, become the letters (iii).

Definition 15.3 (The weak families)**.** The family G_{M+n} has one component, of mass $\frac{1}{2}$, and no source. For $M < i \le M+n$, the *weak family* G_{i-1}, of level $i-1$, with the diffusivity κ, the velocity field $\mathbf{b}_{i-1}$ and no smoothing, is defined from G_i as follows.

(i) *Components:* the components v of G_i, with their masses, and their *children* v', labelled by the nonempty words of length $\#v' \le N_{\mathrm{len,w}}$ in the letters $2, \dots, N_{\mathrm{cor,w}}+1$, with the masses $\omega_{v'} \coloneqq \big((4nN_{\mathrm{cor,w}})^{-1}\varepsilon_{i-1}^{\delta/32}\big)^{\#v'}\omega_v$. All markers are 1. The components *over* v are v, with the empty word, and its children.

(ii) *Inherited sources:* for every word of length at most $N_{\mathrm{len,w}}$, every nonempty path of sources of G_i from v_1 to v_2 and every $2 \le d \le N_{\mathrm{cor,w}}+1+(M+n-i)N_{\mathrm{cor,w}}$, a source of birth order d from the component over v_1 with this word to the component over v_2 with this word. Its coefficients are the means of degree d of the weak finite lift along the path (Definition 10.5), transported to x (11.8); its factor is the product of the factors along the path.

(iii) *Letters:* for every component with a word of length less than $N_{\mathrm{len,w}}$ and every letter $2 \le d \le N_{\mathrm{cor,w}}+1$, a source of birth order d to the component over the same v whose word is extended by d. Its coefficients are $\varepsilon_{i-1}^{-\delta/16}$ times the mean of degree d of the weak finite lift along the empty path, less $\mathbf{1}_{\{d=2\}}\kappa\mathbf{K}$ (the eddy diffusivity at degree d), transported to x; its factor is $\varepsilon_{i-1}^{-\delta/16}(\varepsilon_i^{\beta}/\kappa)^2$.

Every source of G_{i-1} has the majorant 1 and the birth radius ε_i. For $M \le i \le M+n$, every component of G_i has the reserve $\mathrm{res}_i \coloneqq (N_{\mathrm{gen}}+3+i-M)/(N_{\mathrm{gen}}+2+N_{\mathrm{off}})$, at most 1 for $i < M+n$ as $N_{\mathrm{off}} \ge n$ (item (2) of Subsection 11.1), which replaces $\mathrm{res}(v)$ in (11.9) for this family; the smoothed family of Lemma 15.4(b) has the components of G_M, at the reserve res_M.

The power $\varepsilon_{i-1}^{-\delta/16}$ in the coefficients of a letter is the inverse of the readout weight it adds, as for a successor (5.10). The readout of the weak passage is accordingly $(\Upsilon_i^{\mathrm{w}}F)_v \coloneqq F_v + \sum_{v'}\varepsilon_{i-1}^{\delta\#v'/16}F_{v'}$, over the children v' of v, and the *weak lift* of level i is the map $F \mapsto \mathrm{Corr}_i^{\mathrm{w}}(t)\Upsilon_i^{\mathrm{w}}F$. The birth orders of G_i are at most $1+(M+n-i)N_{\mathrm{cor,w}}$, and its paths have at most $(M+n-i)N_{\mathrm{len,w}}$ sources.

By Lemma 15.4(a), a source of birth order d_σ has coefficients of that order at most $C\kappa$ times its factor and $\varepsilon_i^{d_\sigma-2}$, each derivative and each lower order costing ε_{i-1}^{-1}. The factor of a letter is small because the shear is weak: the first corrector, forced by the shear alone, has the size of the Péclet number, so the eddy diffusivity at degree 2, the average of the shear against it, is κ times the square of the Péclet number times a bounded form. The masses and readout weights turn this gain into the small coupling of (b), with which the smoothed family of level M satisfies the invariant.

Lemma 15.4 (The weak coefficients)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T)$ such that, for every $M \ge M_*$, the following hold.*

(a) *For every $M < i \le M+n$, every source σ of G_{i-1}, of birth order d_σ, every $2 \le d \le d_\sigma$ and every order N used below,*

$$\sum_{j_1 \le N} \frac{\varepsilon_{i-1}^{j_1}}{j_1!} \sup_{\mathbb{R}\times\mathbb{T}^2} |\nabla^{j_1} A_{\sigma,d}| \le C\,\mathrm{fac}(\sigma)\,\kappa\,\varepsilon_i^{d_\sigma-2}\varepsilon_{i-1}^{d-d_\sigma}\,.$$

(b) *The family G_M has total mass at most $\frac{2}{3}$. The family $\widetilde{G}_M$, which is G_M with every source smoothed by the paired smoothing S_{w} of width r_{w} over round discs (Subsection 15.3), satisfies the clauses of Definition 11.1 on $[t_{(M)}, t_{\mathrm{f}}]$ with constants depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0$, T (in clause (b1) the majorant of a source is multiplied by such a constant), the time $t_{(M)}$ being that of Proposition 15.5; the row and column sums of its priced majorant matrix and*

of its coupling matrices (8.4) *of every order used below, at every scale* $\rho \le \varepsilon_M$*, are at most* $C\varepsilon_M^{2(q+2)\gamma-5\delta/64} \le C\varepsilon_M^{q-1}$.

Proof. (a) By induction on $M+n-i$, from G_{M+n}, which has no source. In the chart of a pulse k of level i, with $z = y/\varepsilon_i$, the system of G_i is a system on the window of pulse k in the sense of Section 10, with κ for κ_m and κ_{m-1}, ε_i for ε_m and ε_{i-1} for ε_{m-1}: by (a) at the level $i+1$ and the clauses of Definition 11.1 for G_i, proved below at every level together with (a), its forms are transverse and satisfy (10.17). On transverse forms the shear term of the cell operator vanishes, so the inverse of Lemma 15.2 is that of Lemma 10.4, by the uniqueness in Lemma 15.2(a). The proofs of Propositions 10.6 and 10.7 then apply with ε_i in place of $\bar\varepsilon_m$: the four products (10.22) become $(\varepsilon_i^\beta/\kappa)\varepsilon_i$, ε_i, $\varepsilon_i^{j'}$ and ε_i^2, with $\varepsilon_i^\beta/\kappa \le C$ by Lemma 15.1(a). Along the empty path the bracket of degree 1 is the shear term alone and $\mathcal{L}_2 Y$ has no part of zero average, as $\mathbf{K}$ does not depend on z; so every corrector of positive degree along $\emptyset$ carries one power of the Péclet number $\varepsilon_i^\beta/\kappa$, and the means along $\emptyset$ differ from $\mathbf{1}_{\{j=2\}}\kappa\mathbf{K}$ by the average of the shear, of size $(\kappa/\varepsilon_i)(\varepsilon_i^\beta/\kappa)$, against such a corrector. Hence, for every path $\sigma_1\cdots\sigma_J$ of G_i, $2 \le d \le j$ and $j_1 \le N$,

$$\varepsilon_{i-1}^{j_1}\big\|\nabla_y^{j_1}\mathcal{M}_{j,d}[\sigma_1\cdots\sigma_J]\big\|_{L^\infty([(k-\frac{61}{48})\tau_i,(k+\frac{61}{48})\tau_i]\times\mathbb{R}^2)} \le C\,\mathrm{fac}(\sigma_1)\cdots\mathrm{fac}(\sigma_J)\,\kappa\,\varepsilon_i^{j-2}\varepsilon_{i-1}^{d-j}\,,$$
$$\varepsilon_{i-1}^{j_1}\big\|\nabla_y^{j_1}\big(\mathcal{M}_{j,d}[\emptyset]-\mathbf{1}_{\{j=2\}}\kappa\mathbf{K}\big)\big\|_{L^\infty([(k-\frac{61}{48})\tau_i,(k+\frac{61}{48})\tau_i]\times\mathbb{R}^2)} \le C\kappa\,(\varepsilon_i^\beta/\kappa)^2\,\varepsilon_i^{j-2}\varepsilon_{i-1}^{d-j}\,.$$

These means of degree d_σ, with the power $\varepsilon_{i-1}^{-\delta/16}$ for the letters, are the coefficients of Definition 15.3, transported to x with the derivatives of the charts at the scale ε_{i-1} (Lemma 7.7); this gives (a).

(b) *Row and column sums.* Every source of G_M was created at the weak passage from level $M+1$, with birth radius $\varepsilon_{M+1} \le r_{\mathrm{w}}$, so its price in (11.3) is 1. By (a) with $i = M+1$, its coefficients of order d are at most $C\,\mathrm{fac}(\sigma)\kappa r_{\mathrm{w}}^{d-2}$, with derivatives at the scale ε_M; so the entries of the coupling matrices (8.4) of $\widetilde G_M$ of every order used below, at the scale ε_M, are at most C times those of the majorant matrix (11.3). These entries increase with ρ, so the same holds at every scale $\rho \le \varepsilon_M$, among them $\rho_{\mathrm{mol}}(M) = c_{\mathrm{mol}}\varepsilon_M$ (by (11.1), as $\kappa\varepsilon_M^{2-\beta} \ge \frac12\varepsilon_M^{2-\gamma} \ge \varepsilon_M^2$ by the proof of Lemma 15.1 and $\varepsilon_M^\gamma \le \frac12$ for $M \ge M_*$). Along a letter from a component to its child, created at the weak passage from a level $i > M$, the square root of the mass ratio and the factor give

$$(4nN_{\mathrm{cor,w}})^{1/2}\varepsilon_{i-1}^{-\delta/64}\cdot\varepsilon_{i-1}^{-\delta/16}\Big(\frac{\varepsilon_i^\beta}{\kappa}\Big)^2 \le C\varepsilon_{i-1}^{2(q+2)\gamma-5\delta/64} \le C\varepsilon_M^{2(q+2)\gamma-5\delta/64}\,,$$

by Lemma 15.1(a). For an inherited source, the factor is the product of the factors along its path, the square roots of the mass ratios telescope, and the word multiplies the masses at both ends by the same number; so its entry is the product of the entries of the letters in its history. Since G_{M+n} has no source, every source of G_M has at least one letter, and at most $nN_{\mathrm{len,w}}$, in its history, and every entry is at most $C\varepsilon_M^{2(q+2)\gamma-5\delta/64}$. A row or a column has a number of entries bounded by the numbers of words, paths and degrees; this gives the sums, and $2(q+2)\gamma - \frac{5\delta}{64} \ge q-1$ since $(q+2)\gamma \ge q-1$ (as $\beta > 1$) and $\frac{5}{64}\delta < q-1$. The masses: a weak passage multiplies the total mass by at most $1 + \sum_{L\ge1}(4n)^{-L} = 1 + 1/(4n-1)$, and $(1+1/(4n-1))^n \le \frac43$.

The other clauses of Definition 11.1 are proved at every level, for G_{i-1}, $M < i \le M+n$, and $\widetilde G_M$, together with (a), which uses them at the level i. They are checked as for the successors in Subsection 11.6, the weak finite lift replacing the finite lift of Section 10, with the inverse of Lemma 10.4 on transverse forms, as in (a): clauses (a1)–(a7) by Steps 1 and 2 there, the weak cell solutions being solved once on their windows and unique there (Lemma 15.2(a)). Clause (b1) holds by Steps 4 and 5, with $\mathrm{res}(v) = \mathrm{res}_i$ and $\mathrm{res}(\tilde v) = \mathrm{res}_{i-1} = \mathrm{res}_i - 1/(N_{\mathrm{gen}}+2+N_{\mathrm{off}})$ for every component

of G_{i-1} (within G_i all reserves are equal, which replaces the monotone generations of Step 4), a weak passage taking at most $nN_{\mathrm{cor,w}}-1\le N_{\mathrm{src}}-2$ slow derivatives ($N_{\mathrm{src}}\ge N_{\mathrm{src,e}}\ge nN_{\mathrm{cor,w}}+6$, as N_{src} counts from the seed's birth orders, item (1) of Subsection 11.1). The entry across the pulse of the conductivity has real part at least $\frac{9}{10}$ on the outer domain by (2.13) at the level i, so the bounds of (a) hold in the analytic norms; with the birth radius ε_i, the left side of (b1) is at most $C(\varepsilon_i/\varepsilon_{i-1})^{d_\sigma-d}$. Clauses (b2) and (b2′) hold by Cauchy's estimate, as in Step 6, the birth radii being $\varepsilon_i\le\varepsilon_{i-1}$, and, for $\widetilde{G}_M$, $\varepsilon_{M+1}\le\rho_{\mathrm{mol}}(M)=c_{\mathrm{mol}}\varepsilon_M$; (b3) and (d) by the sums above; and, for $\widetilde{G}_M$, (c1) as the long axes of S_{w} have the inverse $2^{1/2}I$, (c3) by Lemma 8.2, and (c2) as $r_{\mathrm{w}}/\rho_{\mathrm{mol}}(M)=\varepsilon_M^{\gamma/4+\delta/2}/c_{\mathrm{mol}}\le C\varepsilon_M^{\delta}$ and $r_{\mathrm{w}}/\rho(M)\le C\varepsilon_{M-1}^{3\delta+\frac{5}{4}q\gamma+\frac{1}{2}q\delta}\le C\varepsilon_{M-1}^{4\delta}$, by $3\delta<\gamma$, Proposition 2.1(c) and $q-1-\frac{\gamma}{2}-\delta=q\gamma+3\delta$. □

Proposition 15.5 carries out item 1 of the route: items (i)–(iii) restart each solution in the next family, with the new components starting from zero, and (iv) follows from Lemma 15.6.

Proposition 15.5 (The weak reduction). *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)$ such that, for every $M\ge M_*$, the families G_i do not depend on t_{in}, θ_{in} or θ, and there exists $t_{(M)}\in[t_{\mathsf{in}},t_{\mathsf{in}}+6nh_{M+1}]$, depending only on M, κ and t_{in}, with the following property. For every $\theta_{\mathsf{in}}\in L^2(\mathbb{T}^2)$ and every weak solution θ of (4.1) there exist times $t_{\mathsf{in}}<t_{(M+n)}<t_{(M+n-1)}<\cdots<t_{(M)}$ and solutions U_i of the systems (5.6) of G_i, with κ in place of their diffusivity κ_m and without smoothing, on $[t_{(i)},t_{\mathrm{f}}]$, in the class of Proposition 8.6, such that:*

(i) *$U_{M+n}=\theta$ on $[t_{(M+n)},t_{\mathrm{f}}]$;*

(ii) *for every $M<i\le M+n$, $t_{(i-1)}$ is the block boundary at the centre of the first pair of quiet margins $[(l+\frac{1}{2})h_i-\tau_i',(l+\frac{1}{2})h_i+\tau_i']$, $l\in\mathbb{Z}$, whose left end is at least $t_{(i)}$; so $t_{(i)}+\tau_i'\le t_{(i-1)}\le t_{(i)}+2h_i$, and $\mathbf{b}_i=\mathbf{b}_{i-1}$ on these margins;*

(iii) *for every $M<i\le M+n$, $U_{i-1}(t_{(i-1)})$ is $U_i(t_{(i-1)})$ on the components of G_i and 0 on their children;*

(iv) *for every $M\le i\le M+n$, $\|U_i\|_{L^\infty([t_{(i)},t_{\mathrm{f}}];L^2_\omega(\mathbb{T}^2))}\le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$, and*

$$\|U_M\|_{L^\infty([t_{(M)},t_{\mathrm{f}}];H^{N_{\mathrm{tr,w}}}_{\varepsilon_M,\omega}(\mathbb{T}^2))}+\kappa^{1/2}\|\nabla U_M\|_{L^2((t_{(M)},t_{\mathrm{f}});H^{N_{\mathrm{tr,w}}}_{\varepsilon_M,\omega}(\mathbb{T}^2))}\le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,.$$

Lemma 15.6 is the regain of derivatives on the quiet margins of item 1 of the route.

Lemma 15.6 (A complete trace on the quiet margins). *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)$ such that the following holds for every $M\ge M_*$, $M<i\le M+n$, θ_{in} and θ. For a component v of G_{i-1}, let N_v be $N_{\mathrm{tr,w}}$ plus $nN_{\mathrm{cor,w}}-1$ times the largest number of sources on a path of G_{i-1} that starts at v (the components of G_i are among those of G_{i-1}). Then the solutions of Proposition 15.5 satisfy*

$$\|U_i(t_{(i-1)})\|_{H^{(N_v)}_{\varepsilon_{i-1},\omega}(\mathbb{T}^2)}\le C\|U_i\|_{L^\infty([t_{(i)},t_{(i-1)}];L^2_\omega(\mathbb{T}^2))}\,.$$

Proof. The *heat interval* $[t_{(i-1)}-\varepsilon_{i-1}^2/\kappa,t_{(i-1)}]$ lies in the second half of the quiet margin that ends at $t_{(i-1)}$, since $\varepsilon_{i-1}^2/\kappa\le C\varepsilon_{i-1}^{\gamma-3\delta}\tau_i'\le\frac{1}{2}\tau_i'$ by Lemma 15.1(a); and every time within $\frac{6}{5}h_{i+1}$ of it lies in the two quiet margins around $t_{(i-1)}$, since $\frac{6}{5}h_{i+1}\le\frac{1}{2}\tau_i'$ by (2.5) and Lemma 7.1(a), (c).

Step 1: the coefficients on the heat interval. Let $N_*:=\max_v N_v+nN_{\mathrm{len,w}}(1+nN_{\mathrm{cor,w}})$, an auxiliary order depending only on $\alpha,s,q,\zeta,\Lambda,n,t_0,T$ (M_* is taken after it). On the heat interval the coefficients of G_i have their slow derivatives at the wavelength ε_{i-1}, up to the order N_*: for a

source $\sigma\colon v' \to v$ of G_i, of birth order d_σ, let

$$\widetilde{B}_\sigma \coloneqq \Big(\frac{\omega_{v'}}{\omega_v}\Big)^{1/2} \sum_{d=2}^{d_\sigma} \frac{1}{\kappa \varepsilon_{i-1}^{d-2}} \sum_{j=0}^{N_*} \frac{\varepsilon_{i-1}^j}{j!} \sup |\nabla^j A_{\sigma,d}| \,,$$

the supremum over the heat interval and $\mathbb{T}^2$; then the matrix $\widetilde{B}_{vv'} \coloneqq \sum_{\sigma\colon v' \to v} \widetilde{B}_\sigma$ has row and column sums at most C. Indeed, a coefficient of G_i at a time t reads the flow of $\mathbf{b}_i$ only over the histories of the pulses of level $i+1$ whose cutoff is positive at t, which lie within $\frac{6}{5}h_{i+1}$ of t (the nesting after (11.19)), hence, for t in the heat interval, in the margins, where $\mathbf{b}_i = \mathbf{b}_{i-1}$. So there the coefficients do not depend on the fast variable of level i, by (a4), and their slow derivatives cost ε_{i-1}^{-1} at every order, by Cauchy's estimate in the chart parameters from (b1) and by the transport (11.8); (a4) and (b1) hold for G_i by the proof of Lemma 15.4. With the sizes of Lemma 15.4(a) at the level $i+1$ and $\varepsilon_{i+1}^{d_\sigma-2}\varepsilon_i^{d-d_\sigma} \le \varepsilon_{i-1}^{d-2}$, this gives $\widetilde{B}_\sigma \le C(\omega_{v'}/\omega_v)^{1/2}\operatorname{fac}(\sigma)$, whose row and column sums are at most C by the telescoping in the proof of Lemma 15.4(b).

Step 2: one path. We rescale time so that the heat interval is $[0,1]$, $t = t_{(i-1)} - (1-\underline{t})\varepsilon_{i-1}^2/\kappa$, write $\underline{U}_v(\underline{t}) \coloneqq \omega_v^{-1/2} U_{i,v}(t)$, and measure lengths at the scale ε_{i-1}, in the norms $H^N_{\varepsilon_{i-1}}(\mathbb{T}^2)$, negative orders N being the dual norms. The drift $\mathbf{b}_{i-1}$ then has derivatives at most $C\varepsilon_{i-1}^\beta/\kappa \le C\varepsilon_{i-1}^\gamma$ at the unit scale, by Lemma 15.1(a) and the bound $\|\nabla^j \mathbf{b}_{i-1}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \le C^j (j-1)!\, \varepsilon_{i-1}^{\beta-1-j}$ of Section 12. Let $\mathcal{E}(\underline{t},\underline{t}')$ be the evolution of the scalar part of the system of G_i, drift and diffusion, and $\mathsf{F}_\sigma(\underline{t})g \coloneqq (\varepsilon_{i-1}^2/\kappa)(\omega_{v'}/\omega_v)^{1/2}\nabla\cdot\sum_d A_{\sigma,d}\nabla^{d-1}g$ the source term of σ. The rate $\kappa/\varepsilon_{i-1}^2$ of a source cancels against the length of the heat interval, and the product rule and its dual give, for $|N|, |N-d_\sigma| \le N_*$,

$$\|\mathsf{F}_\sigma(\underline{t})g\|_{H^{N-d_\sigma}_{\varepsilon_{i-1}}(\mathbb{T}^2)} \le C\widetilde{B}_\sigma \|g\|_{H^N_{\varepsilon_{i-1}}(\mathbb{T}^2)} \,:$$

at a fixed time a source loses its whole birth order d_σ, not the $d_\sigma - 2$ of the energy estimates. As the sources of G_i form no cycle and its paths have at most $(M+n-i)N_{\mathrm{len,w}}$ sources, Duhamel's formula, iterated in the spaces $H^N_{\varepsilon_{i-1}}(\mathbb{T}^2)$, $|N| \le N_*$, writes $\underline{U}_v(1)$ as the sum, over the paths $\sigma_1\cdots\sigma_J$ of G_i from a component v' to v, of

$$\int_{0<t_1<\cdots<t_J<1} \mathcal{E}(1,t_J)\mathsf{F}_{\sigma_J}(t_J)\mathcal{E}(t_J,t_{J-1})\cdots\mathsf{F}_{\sigma_1}(t_1)\mathcal{E}(t_1,0)\,\underline{U}_{v'}(0)\,\mathrm{d}t_1\cdots\mathrm{d}t_J \,,$$

the empty path, $J=0$, giving $\mathcal{E}(1,0)\underline{U}_v(0)$. Fix the path and the times. One of the $J+1$ intervals between $0, t_1, \dots, t_J, 1$ has length at least $1/(J+1)$; let d_- and d_+ be the sums of the birth orders of the sources before and after it, so that $N_v + d_- + d_+ \le N_*$. Before this long interval we follow the integrand in $H^{-N}_{\varepsilon_{i-1}}(\mathbb{T}^2)$, down to $N = d_-$, and after it in $H^{N_v+N}_{\varepsilon_{i-1}}(\mathbb{T}^2)$, from $N = d_+$ down to 0, spaces on which the scalar evolutions are bounded, uniformly in the lengths of their intervals, by the energy estimate and its dual. Across the long interval, split at its midpoint, the adjoint evolution maps $L^2(\mathbb{T}^2)$ to $H^{d_-}_{\varepsilon_{i-1}}(\mathbb{T}^2)$ over the first half and $\mathcal{E}$ maps $L^2(\mathbb{T}^2)$ to $H^{N_v+d_+}_{\varepsilon_{i-1}}(\mathbb{T}^2)$ over the second, by energy estimates with time weights, with constants $C(J)$, as both halves have length at least $1/(2(J+1))$ and the orders are at most N_*. So the integrand is at most $C(J)\widetilde{B}_{\sigma_1}\cdots\widetilde{B}_{\sigma_J}\|\underline{U}_{v'}(0)\|_{L^2(\mathbb{T}^2)}$ in $H^{N_v}_{\varepsilon_{i-1}}(\mathbb{T}^2)$, and the ordered simplex has volume $1/J! \le 1$.

Step 3: the sum over paths. Summing over the paths,

$$\|\underline{U}_v(1)\|_{H^{N_v}_{\varepsilon_{i-1}}(\mathbb{T}^2)} \le \sum_{v'} \Bigg(\sum_{J=0}^{(M+n-i)N_{\mathrm{len,w}}} C(J)(\widetilde{B})^J \Bigg)_{vv'} \|\underline{U}_{v'}(0)\|_{L^2(\mathbb{T}^2)} \,.$$

The row and column sums of this matrix are at most C, as its degree, the $C(J)$ and the row and column sums of $\widetilde{B}$ (Step 1) depend only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$. So, by Schur's test, $\|U_i(t_{(i-1)})\|_{H^{(N_v)}_{\varepsilon_{i-1},\omega}(\mathbb{T}^2)} \le C\|U_i(t_{(i-1)} - \varepsilon_{i-1}^2/\kappa)\|_{L^2_\omega(\mathbb{T}^2)}$, and $t_{(i-1)} - \varepsilon_{i-1}^2/\kappa \ge t_{(i)}$. □

Proof of Proposition 15.5. Let $t_{(M+n)}$ be the left end of the first pair of quiet margins of level $M+n$ whose left end is greater than t_{in}, so that $t_{\mathsf{in}} < t_{(M+n)} \le t_{\mathsf{in}} + h_{M+n}$, and let $U_{M+n} := \theta$ on $[t_{(M+n)}, t_{\mathrm{f}}]$, the solution of the single equation of G_{M+n}, with $\|\theta\|_{L^\infty([t_{\mathsf{in}},t_{\mathrm{f}}];L^2(\mathbb{T}^2))} \le \|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. Since $\mathbf{b}_{M+n}$ is smooth, θ is smooth for $t > t_{\mathsf{in}}$, by parabolic regularity, so it lies in the class of Proposition 8.6 at every order on $[t_{(M+n)}, t_{\mathrm{f}}]$. Let $M < i \le M+n$, let U_i be constructed, with $\|U_i\|_{L^\infty([t_{(i)},t_{\mathrm{f}}];L^2_\omega(\mathbb{T}^2))} \le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$, and define $t_{(i-1)}$ by (ii); on the quiet margins of level i the pulses of level i vanish, so $\mathbf{b}_i = \mathbf{b}_{i-1}$ there. By Lemma 15.6, the datum (iii) has the orders N_v of that lemma at the scale ε_{i-1}, with norm at most $C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. Proposition 8.6 applies to G_{i-1} at these orders, which decrease along every source σ by at least $nN_{\mathrm{cor,w}} - 1 \ge d_\sigma - 2$, as it asks, and at the scale ε_{i-1}: its drift condition holds by Step 2 of the proof of Lemma 15.6, and its matrix is at most C times the majorant matrix, by Lemma 15.4(a), with row and column sums at most $C\varepsilon_M^{q-1} \le \frac{1}{8}$ as in the proof of Lemma 15.4(b). It gives U_{i-1} on $[t_{(i-1)}, t_{\mathrm{f}}]$, with the energy of that proposition at most C times the norm of the datum. After n restarts this gives (iv), the constant growing by a fixed factor per restart, as the orders of the components of G_M are at least $N_{\mathrm{tr,w}}$. By (ii), the times $t_{(i)}$ are determined by t_{in} and the margins; as $h_i \le h_{M+1}$, $t_{(M)} \le t_{(M+n)} + 2nh_{M+1} \le t_{\mathsf{in}} + 6nh_{M+1} < t_{\mathrm{f}}$ for $M \ge M_*$. □

For $M \le i \le M+n$ we write $N_i := (M+n-i)N_{\mathrm{cor,w}} + 5$: the composed weak lifts from level i to level $M+n$ spend $(M+n-i)N_{\mathrm{cor,w}}$ derivatives.

A weak passage is the comparison of Section 13 without the truncated exponential. Of its residual, the cell tail is small by the separation of scales, the flux into the omitted words by the square of the Péclet number for each letter, and the switching as a pulse lasts many cell diffusion times; by (15.1) and Lemma 15.1(a), their size absorbs the loss of Lemma 15.13(b) in Lemma 15.14.

Lemma 15.7 (One weak passage)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T)$ such that, for every $M \ge M_*$, $M < i \le M+n$, θ_{in} and θ, the solutions of Proposition 15.5 satisfy*

$$\left\|U_i - \mathrm{Corr}_i^{\mathrm{w}}\, \Upsilon_i^{\mathrm{w}} U_{i-1}\right\|_{L^\infty([t_{(i-1)},t_{\mathrm{f}}];H^{N_i}_{\varepsilon_i,\omega}(\mathbb{T}^2))} \le C\varepsilon_M^{3+2q^n} \|U_i\|_{L^\infty([t_{(i)},t_{\mathrm{f}}];L^2_\omega(\mathbb{T}^2))}\,.$$

Proof. Let $k := M+n-i$. For a component v of G_i, let $P_i(v)$ be the largest number of sources on a path of G_i that starts at v, and $\widetilde{N}_v := N_i + (nN_{\mathrm{cor,w}} - 1)P_i(v)$, the orders of Lemma 15.6 for G_i, with N_i in place of $N_{\mathrm{tr,w}}$. Let $R := 4(N_{\mathrm{cor,w}} + 1 + kN_{\mathrm{cor,w}}) + 18 + N_{\mathrm{cor,w}} - 1$, the order $N_{\mathrm{rec}} = 4(N_{\mathrm{cor}} + N_{\mathrm{src}}) + 2N_{\mathrm{cl}} + 16$ of the reconstruction at $N_{\mathrm{cor}} = N_{\mathrm{cor,w}}$, $N_{\mathrm{src}} = 1 + kN_{\mathrm{cor,w}}$ and $N_{\mathrm{cl}} = 1$, plus the $d - 2 \le N_{\mathrm{cor,w}} - 1$ of a frontier letter. As $k \le n-1$, $N_{\mathrm{src,e}} \ge nN_{\mathrm{cor,w}} + 6$ by (15.3), and $N_{\mathrm{rd,e}} \ge N_{\mathrm{rec,e}} + 2N_{\mathrm{src,e}}N_{\mathrm{path,e}} \ge 8N_{\mathrm{src,e}}$ by (15.6), and $N_{\mathrm{tr,w}} \ge 2N_{\mathrm{rd,e}} + 2$ by (15.7),

$$N_i + R = (5k+5)N_{\mathrm{cor,w}} + 26 \le 5nN_{\mathrm{cor,w}} + 26 < 16N_{\mathrm{src,e}} + 2 \le 2N_{\mathrm{rd,e}} + 2 \le N_{\mathrm{tr,w}}\,.$$

A component of U_{i-1} of order $K \ge N_{\mathrm{tr,w}}$ in Lemma 15.6 is bounded in $H^K(\mathbb{T}^2)$ and, by its equation and the order condition of Proposition 8.6, Lipschitz in t with values in $H^{K-2}(\mathbb{T}^2)$. So, by the Sobolev embedding, $U_{i-1}(t)$ is of class $C^{\max\{nN_{\mathrm{cor,w}}, R+1\}}$, these derivatives being Lipschitz in t, uniformly in x: the classes that the weak lift and Lemmas 13.2 and 13.4–13.6 ask for; the corrector coefficients are absolutely continuous in t, with a bounded derivative at each mode.

Step 1: zero initial difference. At the block boundary $t_{(i-1)}$, between two quiet margins, the weak corrector operator is the identity: the correctors of the pulses whose cutoff is positive at $t_{(i-1)}$ are

solved from zero on windows within $\frac{121}{48}\tau_i$ of $t_{(i-1)}$, which lie in these margins as $\frac{121}{48}\tau_i+\frac{1}{24}\tau_i \le 5\tau_i \le \tau_i'$; there the shear of level i vanishes and, by (a4), the forms of G_i do not depend on the fast variable, so the correctors of positive degree vanish (Proposition 10.6(e), as in Lemma 13.1). Since $\Upsilon_i^{\mathrm{w}} U_{i-1}(t_{(i-1)}) = U_i(t_{(i-1)})$ by Proposition 15.5(iii), the difference vanishes at $t_{(i-1)}$.

Step 2: the residual. The computations of Lemmas 13.2 and 13.5 and of the readout identity of Lemma 13.8 use only the recursion of Definition 10.5, the equations of the cell problems, and the masses and factors of the new components. As these lemmas are stated in the setting of the lift, we rerun their proofs, and those of Lemmas 13.4–13.6 for the sizes, with $\varepsilon_m = \bar\varepsilon_m = \varepsilon_i$, $\rho(m) = \varepsilon_{i-1}$, $\kappa_m = \kappa_{m-1} = \kappa$, $\kappa_m h_m = \varepsilon_{i-1}^2$, $N_{\mathrm{cor}} = N_{\mathrm{cor,w}}$, $N_{\mathrm{src}} = 1 + kN_{\mathrm{cor,w}}$, $N_{\mathrm{cl}} = 1$, with τ_i in (13.14) and the bounds of the proof of Lemma 15.4(a) for (13.8). With the weak finite lift and Definition 15.3 they show that $U_i - \mathrm{Corr}_i^{\mathrm{w}}\, \Upsilon_i^{\mathrm{w}} U_{i-1}$ solves the system of G_i on $[t_{(i-1)}, t_{\mathrm{f}}]$ with a forcing in three parts:

- the *cell tail*, the degrees above $N_{\mathrm{cor,w}}$ of the weak finite lift, written in divergence form as in Lemma 13.4, with one antidivergence step at the wavelength ε_{i-1}, of size $(\varepsilon_i/\varepsilon_{i-1})^{N_{\mathrm{cor,w}}}$ times the cleanup factor $1 + (\kappa T)^{1/2}/\varepsilon_{i-1}$;
- the *frontier*, the letters out of the words of length $N_{\mathrm{len,w}}$, whose targets are not components of G_{i-1}, of size $(1 + (\varepsilon_i/\varepsilon_{i-1})^2(1 + (\kappa T)^{1/2}/\varepsilon_{i-1}))\, \varepsilon_M^{2(q+2)\gamma(N_{\mathrm{len,w}}+1)}$, the first factor being the cost of a coarse flux under the weak lift (Lemma 13.6 with κ for both diffusivities); read out, each of its $N_{\mathrm{len,w}} + 1$ letters contributes the relative size $(\varepsilon_i^\beta/\kappa)^2$ of its eddy diffusivity;
- the *switching*, the time derivatives of the cutoffs between consecutive pulses, where the correctors of the earlier pulse have decayed for a time at least $\frac{1}{24}\tau_i$, of size $\varepsilon_{i-1}^{-C} \exp(-c\kappa\tau_i/\varepsilon_i^2)$ (Step 3 of the proof of Lemma 13.5 at the level i, with (a4) for G_i, and with Lemma 15.2(b) on each equation of the triangular system, Duhamel's formula and induction on the path length and the degree in place of the decay of the cell inverse on transverse modes).

The forcing is measured in the norm of the right side of Proposition 8.6 at the orders $\widetilde N_v$ and the scale ε_i; in it each part is at most its size times the energy of U_{i-1} at the orders of Lemma 15.6 and the scale ε_{i-1} (for the frontier, layer by layer), which Proposition 8.6 and Lemma 15.6 bound by $C\|U_i\|_{L^\infty([t_{(i)},t_{(i-1)}];L^2_\omega(\mathbb{T}^2))}$. Indeed, each term of the three parts is a two-scale form read at $(y, y/\varepsilon_i)$ and carried to x by the chart, times a derivative of the field or of a flux, and a derivative at the scale ε_i costs a constant on the two-scale form (Proposition 10.7, with (10.17) from (b1) by Cauchy's estimate), at most $C\varepsilon_i/\varepsilon_{i-1}$ on a slow coefficient or a chart (Lemmas 15.4(a) and 7.7), and, on the field, $\varepsilon_i/\varepsilon_{i-1}$ times one derivative at the scale ε_{i-1}. So only the order at which U_{i-1} is read rises: the forcing in v at the order $\widetilde N_v$ reads it at the order $\widetilde N_v + R$ at the components (v', w), for every word w and every v' with a path of $J \ge 0$ sources of G_i into v; as this path, followed by a longest path from v, lifts with the word w to inherited sources (Definition 15.3(ii)), $N_{(v',w)} \ge N_{\mathrm{tr,w}} + (nN_{\mathrm{cor,w}} - 1)P_i(v) \ge \widetilde N_v + R$.

For the frontier we argue as for the omitted length in the proof of Lemma 13.9, layer by layer in the length l of the words. The layer of length l receives sources only from itself and, by letters, from the layer of length $l-1$; its matrix in Proposition 8.6 at the orders $N_{(v,w)}$ and the scale ε_{i-1} has sums at most $\frac{1}{8}$ (proof of Proposition 15.5), and its datum vanishes for $l \ge 1$. Let D_l be $\kappa^{1/2}$ times the norm of ∇U_{i-1} on it in $L^2((t_{(i-1)}, t_{\mathrm{f}}); H^{(N_{(v,w)})}_{\varepsilon_{i-1},\omega}(\mathbb{T}^2))$. A letter $(v, w) \to (v, wd)$ has coefficients at most $C\varepsilon_{i-1}^{-\delta/16}(\varepsilon_i^\beta/\kappa)^2\kappa$ times the powers of Lemma 15.4(a), and $N_{(v,w)} \ge N_{(v,wd)} + nN_{\mathrm{cor,w}} - 1 \ge N_{(v,wd)} + d - 2$, as a path from (v, wd) extends by this letter. With Lemma 8.1, the square root $(4nN_{\mathrm{cor,w}})^{1/2}\varepsilon_{i-1}^{-\delta/64}$ of the mass ratio and Schur's test (a child has one parent, a parent $N_{\mathrm{cor,w}}$ children), Proposition 8.6 gives $D_l \le C\varepsilon_{i-1}^{-5\delta/64}(\varepsilon_i^\beta/\kappa)^2 D_{l-1}$ for $l \ge 1$, and bounds D_0 by the datum, which Lemma 15.6 bounds. The frontier flux F^{fr} carries the readout weight $\varepsilon_{i-1}^{\delta N_{\mathrm{len,w}}/16}$, the square

root $(4nN_{\mathrm{cor,w}})^{-N_{\mathrm{len,w}}/2}\varepsilon_{i-1}^{\delta N_{\mathrm{len,w}}/64}$ of the mass ratio and the eddy coefficient $\kappa(\varepsilon_i^\beta/\kappa)^2$; so, in the norm of the forcing and before the coarse-flux cost, the powers of ε_{i-1} cancel and, by Lemma 15.1(a),

$$\kappa^{-1/2}\|F^{\mathrm{fr}}\| \le C\varepsilon_{i-1}^{5\delta N_{\mathrm{len,w}}/64}\Big(\frac{\varepsilon_i^\beta}{\kappa}\Big)^2 D_{N_{\mathrm{len,w}}} \le C\Big(\frac{\varepsilon_i^\beta}{\kappa}\Big)^{2(N_{\mathrm{len,w}}+1)} D_0 \le C\varepsilon_M^{2(q+2)\gamma(N_{\mathrm{len,w}}+1)} D_0\,.$$

Step 3: the sizes. We use $\varepsilon_i \le 2\varepsilon_{i-1}^q$ and $c\varepsilon_M^{q^{n-1}} \le \varepsilon_{i-1} \le \varepsilon_M$ (Lemma 7.1(a)), and $(\kappa T)^{1/2}/\varepsilon_{i-1} \le C\varepsilon_{i-1}^{-1}$. The cell tail is at most $C\varepsilon_{i-1}^{(q-1)N_{\mathrm{cor,w}}-1} \le C\varepsilon_M^{4+2q^n}$ by the first condition of (15.1). The frontier is at most $C\varepsilon_{i-1}^{-1}\varepsilon_M^{2(q+2)\gamma(N_{\mathrm{len,w}}+1)} \le C\varepsilon_M^{4+3q^n-q^{n-1}} \le C\varepsilon_M^{4+2q^n}$ by the second. The switching is at most $C\varepsilon_{i-1}^{-C}\exp(-c\varepsilon_{i-1}^{-(2(q-1)+\gamma-4\delta)}) \le C\varepsilon_M^{4+2q^n}$ by Lemma 15.1(a), since $2(q-1)-4\delta>0$.

Step 4: energy. Proposition 8.6 for G_i at the scale ε_i and the orders $\widetilde{N}_v$, which decrease along the sources of G_i as it asks ($d_\sigma - 2 \le kN_{\mathrm{cor,w}} - 1 \le nN_{\mathrm{cor,w}} - 1$), bounds the energy of the difference by C times its forcing; its hypotheses hold by Lemmas 15.1(a) and 15.4, and its uniqueness identifies the difference with the solution from zero, as the difference lies in its class: the lift by the count of Step 2, and U_i by the proof of Proposition 15.5, as $N_{\mathrm{tr,w}} \ge N_i$ (for $i = M+n$, θ is smooth). With Steps 1 and 3 this gives the lemma, as $\widetilde{N}_v \ge N_i$, with one power of ε_M to spare. □

15.3 Level M: smoothing and a homogenization cycle

The smoothing. The weak families have unsmoothed sources: Proposition 8.6 solves their systems at orders that decrease along the paths of sources, and the derivatives are regained on each quiet margin. The terminal cycle uses Sections 12 and 13, which need the clauses of the invariant, and clause (c) asks for paired smoothings with vanishing moments and widths below $\rho_{\mathrm{mol}}(M)$ and $\rho(M)$; so the sources are smoothed once, at level M.

Let $S_{\mathrm{w}} := S_{\ell_{\mathrm{w}},r_{\mathrm{w}},2^{-1/2}I}\,S_{\ell_{\mathrm{w}},r_{\mathrm{w}},2^{-1/2}I}$, the product of two averages (6.2) over round discs of radius $2^{-1/2}r_{\mathrm{w}}$, with $2\ell_{\mathrm{w}} - 1 \ge \max\{N_{\mathrm{mom,w}}, N_{\mathrm{mom}}\}$. Its matrix of long axes, $2^{-1/2}I$, has the inverse $2^{1/2}I$, of $|\cdot|_\infty$-norm $2^{1/2} \le 2$, so Proposition 6.1 applies. Let $\widetilde{G}_M$ be the family G_M with every source smoothed by S_{w}, and let $\widetilde{U}_M$ be the solution of $\widetilde{G}_M$ from $U_M(t_{(M)})$ at time $t_{(M)}$. By Lemma 15.8, $\widetilde{U}_M$ keeps the derivatives of U_M at the scale ε_M, and, at a lower order, differs from it by at most a constant times the gain of the moments of S_{w} times the small coupling of Lemma 15.4(b).

Lemma 15.8 (The smoothing at level M)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T)$ such that, for every $M \ge M_*$, θ_{in} and θ, the component energy (13.3) of $\widetilde{U}_M$ of order $N_{\mathrm{tr,w}}$ at the scale ε_M, with the diffusivity κ, on $[t_{(M)}, t_{\mathrm{f}}]$ is at most $C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$, and, with $N := N_{\mathrm{tr,w}} - N_{\mathrm{mom,w}} - nN_{\mathrm{cor,w}} - 1$,*

$$\|U_M - \widetilde{U}_M\|_{L^\infty([t_{(M)},t_{\mathrm{f}}];H^N_{\varepsilon_M,\omega}(\mathbb{T}^2))} \le C\varepsilon_M^{q-1+(\gamma/4+\delta/2)N_{\mathrm{mom,w}}}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,.$$

Proof. Proposition 8.3 applies to $\widetilde{G}_M$ at every order up to $N_{\mathrm{tr,w}}$ and the scale ε_M: its condition (i) holds by the bound on $\nabla^j \mathbf{b}_M$ stated after (12.1) and $\varepsilon_M^\beta/\kappa \le C\varepsilon_M^\gamma$ (Lemma 15.1(b)), and (ii) and (iii) by Lemma 15.4(a) at $i = M+1$ and the order $N_{\mathrm{tr,w}}$, with the sums of the proof of Lemma 15.4(b). The solution $\widetilde{U}_M$ has no forcing, and its datum $U_M(t_{(M)})$ has norm at most $C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ in $H^{N_{\mathrm{tr,w}}}_{\varepsilon_M,\omega}(\mathbb{T}^2)$ by Proposition 15.5(iv); so part (c) gives the first bound.

The difference $U_M - \widetilde{U}_M$ vanishes at $t_{(M)}$ and solves the system of $\widetilde{G}_M$ with the forcing $\nabla\cdot\sum_\sigma\sum_d A_{\sigma,d}\nabla^{d-2}(I - S_{\mathrm{w}})\nabla U_{M,v'}$, the sums over the sources $\sigma\colon v' \to v$. By the third bound of Proposition 6.1 at the scale ε_M and the order $N_{\mathrm{mom,w}} < 2\ell_{\mathrm{w}}$, $I - S_{\mathrm{w}}$ costs $C(r_{\mathrm{w}}/\varepsilon_M)^{N_{\mathrm{mom,w}}} = C\varepsilon_M^{(\gamma/4+\delta/2)N_{\mathrm{mom,w}}}$ and $N_{\mathrm{mom,w}}$ more derivatives at the scale ε_M, and $d-2 \le nN_{\mathrm{cor,w}} - 1$, as the birth orders of G_M are at most $1 + nN_{\mathrm{cor,w}}$. By Lemma 15.4, $\kappa^{-1}\varepsilon_M^{2-d}$ times the coefficients, with the mass

ratios, has row and column sums at most $C\varepsilon_M^{q-1}$. So the forcing, measured as in (13.4) at the order N, is at most $C\varepsilon_M^{q-1+(\gamma/4+\delta/2)N_{\mathrm{mom,w}}}$ times $\kappa^{1/2}\|\nabla U_M\|_{L^2((t_{(M)},t_{\mathrm{f}});H^{N_{\mathrm{tr,w}}}_{\varepsilon_M,\omega}(\mathbb{T}^2))}$, which Proposition 15.5(iv) bounds by $C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. Proposition 8.3(a) at the order N then gives the second bound. □

The terminal successor of Definition 15.9 is the successor of Section 13 at the level M, with four changes: the orders; one generation of children, with the marker 1, so that it can be declared a seed without change (Proposition 15.11); the coefficients, majorant and smoothing of the new sources; and the covariance, anchored at the preparation time so that the retained trace there is the identity on the components of $\widetilde{G}_M$. The preparation time depends on $t_{(M)}$ alone: with $l \in \mathbb{Z}$ such that $(l-\frac{1}{2})h_M \le t_{(M)} < (l+\frac{1}{2})h_M$, it is $t_{\mathrm{pr}}^{(M)} \coloneqq (l+\frac{3}{2})h_M$, the time (12.17) at $m=M$ with $t_1 = t_{(M)}$.

Definition 15.9 (The terminal successor)**.** The *terminal successor* G' of $\widetilde{G}_M$ is the family of level $M-1$ that the comparison of Section 13 constructs at $m=M$ (Definition 11.3), with these changes.

(i) The orders are $N_{\mathrm{cor,e}}$ and p_{e}; the mean and covariance letters run over the degrees up to $N_{\mathrm{cor,e}}+1+nN_{\mathrm{cor,w}}$ and the paths of sources of $\widetilde{G}_M$, so that the birth orders are at most $N_{\mathrm{src,e}}$.

(ii) One generation: the components of $\widetilde{G}_M$, with their masses, and their children v', with nonempty words of length $\#v' \le N_{\mathrm{len,e}}$, each with the mass $c_{\mathrm{e}}\varepsilon_{M-1}^{\delta/32}\omega_v$ divided by the number of these words, for a constant $c_{\mathrm{e}}(\alpha,s,q,\zeta,\Lambda,n,t_0,T)>0$, the readout weight $\varepsilon_{M-1}^{\delta\#v'/16}$ in (5.10), and the marker 1.

(iii) The coefficients of the new sources are multiplied by $\varepsilon_{M-1}^{-\delta/16}$. The new sources have the birth radius $\rho(M)$, the majorant $C_{\mathrm{maj}}\varepsilon_{M-1}^{\delta-\delta/16}$ times the product of the majorants along the path, and the round smoothing of S_{w}, with the width $\rho(M)\varepsilon_{M-1}^{\Theta_{\mathrm{e}}}$ in place of r_{w} and with the kernel order ℓ_{e} in place of ℓ_{w}, where $2\ell_{\mathrm{e}}-1 \ge \max\{N_{\mathrm{mom,e}}, N_{\mathrm{mom}}\}$.

(iv) The covariance is anchored at the preparation time $t_{\mathrm{pr}}^{(M)}$: $\Gamma_M(t)-\Gamma_M(t_{\mathrm{pr}}^{(M)})$ replaces $\Gamma_M(t)$ in Definition 11.3, in (13.20) (the defect and covariance letters) and in the lift.

The *terminal lift* of a field Y of G' is

$$Y^{\sharp}(t) \coloneqq \mathrm{Corr}^{\mathrm{e}}(t)\Big[\exp_{p_{\mathrm{e}}}\big((\Gamma_M(t)-\Gamma_M(t_{\mathrm{pr}}^{(M)})):\nabla^2\big)\,\Upsilon^{\mathrm{e}}Y(t)\Big], \tag{15.8}$$

where Υ^{e} is the readout (5.10) of G', $\exp_{p_{\mathrm{e}}}$ is the truncated exponential (5.11) with p_{e} in place of p, and $\mathrm{Corr}^{\mathrm{e}}$ is the corrector operator Corr_M of Definition 10.8 for $\widetilde{G}_M$ with the corrector order $N_{\mathrm{cor,e}}$.

The anchored covariance is small: by Lemmas 10.3(b) and 15.1(b),

$$|\Gamma_M(t)-\Gamma_M(t_{\mathrm{pr}}^{(M)})| \le C\kappa_{M-1}\tau'_M \le C\varepsilon_{M-1}^{\delta}\rho(M)^2 \le C\varepsilon_{M-1}^{\gamma+3\delta}\varepsilon_{M-1}^2\,.$$

Proposition 15.10 is Proposition 5.3 at the terminal level, with a preparation for $\widetilde{G}_M$, which is not admissible but satisfies the clauses of the invariant (Lemma 15.4(b)); the contrast κ_{M-1}/κ is at most 2, and the datum already has derivatives at the scale $\rho_{\mathrm{mol}}(M) = c_{\mathrm{mol}}\varepsilon_M$. The last bound of (a), by interpolation, and the comparison (c) carry $N_{\mathrm{rd,e}}$ derivatives at the scale ε_M at half the accuracy of the L^2 bound; the weak lifts read (c) in Lemma 15.14. Parts (b) and (c) use sums of G' bounded in the proof of Proposition 15.11, which does not use (b) or (c).

Proposition 15.10 (The terminal cycle)**.** *There exist $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)$ and $c(\alpha,s,q,\zeta,\Lambda,n,t_0,T)>0$ such that, for every $M \ge M_*$, there exist $t_{\mathrm{pr}}^{(M)}$ and a linear map $\Pi\colon H^{N_{\mathrm{tr,w}}}(\mathbb{T}^2)^V \to H^{N_{\mathrm{out,e}}}(\mathbb{T}^2)^V$, V the set of components of $\widetilde{G}_M$, both depending only on M, κ and t_{in}, with the*

following properties. For every θ_{in} *and* θ*, let* Y *be the solution of* G' *on* $[t_{\mathrm{pr}}^{(M)}, t_{\mathrm{f}}]$ *from the datum equal to* $\Pi\widetilde{U}_M(t_{(M)})$ *on the components of* $\widetilde{G}_M$ *and to* 0 *on their children.*

(a) $t_{(M)} + ch_M \le t_{\mathrm{pr}}^{(M)} \le t_{(M)} + Ch_M$, $\mathbf{b}_M = \mathbf{b}_{M-1}$ *on* $[t_{\mathrm{pr}}^{(M)} - \tau'_M, t_{\mathrm{pr}}^{(M)} + \tau'_M]$, *and*

$$\begin{aligned}
\|\Pi\widetilde{U}_M(t_{(M)})\|_{H^{N_{\mathrm{out,e}}}_{\rho(M),\omega}(\mathbb{T}^2)} &\le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,,\\
\|\widetilde{U}_M(t_{\mathrm{pr}}^{(M)}) - \Pi\widetilde{U}_M(t_{(M)})\|_{L^2_\omega(\mathbb{T}^2)} &\le C\varepsilon_{M-1}^{Q_{\mathrm{e}}}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,,\\
\|\widetilde{U}_M(t_{\mathrm{pr}}^{(M)}) - \Pi\widetilde{U}_M(t_{(M)})\|_{H^{N_{\mathrm{rd,e}}}_{\varepsilon_M,\omega}(\mathbb{T}^2)} &\le C\varepsilon_{M-1}^{Q_{\mathrm{e}}/2}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,.
\end{aligned}$$

(b) *The component energy* (13.3) *of* Y *of order* $N_{\mathrm{out,e}}$ *at the scale* $\rho(M)$*, with the diffusivity* κ_{M-1}*, on* $[t_{\mathrm{pr}}^{(M)}, t_{\mathrm{f}}]$ *is at most* $C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$.

(c) *For every* $t \in [t_{\mathrm{pr}}^{(M)}, t_{\mathrm{f}}]$, $Y^\sharp(t) \in H^{N_{\mathrm{rd,e}}}(\mathbb{T}^2)^V$, *and*

$$\|\widetilde{U}_M - Y^\sharp\|_{L^\infty([t_{\mathrm{pr}}^{(M)},t_{\mathrm{f}}];H^{N_{\mathrm{rd,e}}}_{\varepsilon_M,\omega}(\mathbb{T}^2))} \le C\varepsilon_{M-1}^{Q_{\mathrm{e}}/2}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}\,.$$

part of the residual	size, over $\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$	bounded by	coarse flux
cell tail	$\big(\bar\varepsilon_M/\rho(M)\big)^{N_{\mathrm{cor,e}}}$ × cleanup factor	Lemma 13.4	
frontier	$\varepsilon_{M-1}^{\delta(N_{\mathrm{len,e}}+1)/2}$	as Lemma 13.9	yes
truncation of $\exp_{p_{\mathrm{e}}}$	$\varepsilon_{M-1}^{\delta p_{\mathrm{e}}}$	(13.19), Lemma 10.3(b)	yes
new smoothings	$\varepsilon_{M-1}^{\delta-\delta/16-\delta/64-\Theta_{\mathrm{e}}(N_{\mathrm{src,e}}-2)+\Theta_{\mathrm{e}}N_{\mathrm{mom,e}}}$	Proposition 6.1	yes
old smoothings at $\rho(M)$	$\varepsilon_M^{2(q+2)\gamma-5\delta/64}\big(r_{\mathrm{w}}/\rho(M)\big)^{N_{\mathrm{mom,w}}}$	Proposition 6.1	yes
old smoothings at $c_{\mathrm{mol}}\varepsilon_M$	$\varepsilon_M^{2(q+2)\gamma-5\delta/64}\big(r_{\mathrm{w}}/\varepsilon_M\big)^{N_{\mathrm{mom,w}}}$	Proposition 6.1	
switching	$\varepsilon_{M-1}^{-C}\exp(-c\kappa\tau_M/\varepsilon_M^2)$	Lemma 13.5	

Table 15.1: The residual of the terminal lift $Y^\sharp$ in the system of $\widetilde{G}_M$ on $[t_{\mathrm{pr}}^{(M)}, t_{\mathrm{f}}]$, measured as in (13.4) at the order $N_{\mathrm{rd,e}}$ and the scale ε_M. The cell tail has the degrees above $N_{\mathrm{cor,e}}$; the frontier the words of length $N_{\mathrm{len,e}}+1$, each letter gaining $\varepsilon_{M-1}^{\delta/2}$ (proof of Proposition 15.11); the new smoothings are bounded by the priced majorant entry of that proof times the gain $\varepsilon_{M-1}^{\Theta_{\mathrm{e}}N_{\mathrm{mom,e}}}$ of the moments. The cleanup factor $1+(\kappa T)^{1/2}\varepsilon_M^{-1}(\varepsilon_M/\rho(M))^{N_{\mathrm{cl,e}}}$ is at most C, as $(q\gamma+3\delta)N_{\mathrm{cl,e}} > 3$, and so is the cost of a coarse flux under the lift (Lemma 13.6; fourth column), as $\kappa_{M-1}/\kappa \le 2$. The truncation row includes the Taylor part, as in the proof of Proposition 13.11.

Proof. (a) The preparation of Section 12 runs at the terminal level for $\widetilde{G}_M$ from $t_1 = t_{(M)}$, the start of the interval $[t_{(M)}, t_{\mathrm{f}}]$ on which $\widetilde{G}_M$ satisfies the invariant (Lemma 15.4(b)), so that $t_{\mathrm{A}} = t_{(M)}$ and, for $M \ge M_*$, $t_1 + 2h_M \le t_{\mathsf{in}} + 6nh_{M+1} + 2h_M \le t_{\mathrm{f}}$, as Proposition 12.7 asks. It uses that proposition at $m = M$ for $\widetilde{G}_M$, with $N_{\mathrm{out,e}}$ in place of N_{out} and the constants of Lemma 15.4(b), and Propositions 12.4–12.6 and Lemmas 12.8–12.15, which hold at $m = M$ (Remark 12.16), where the contrast κ_{M-1}/κ is at most 2 (Lemma 15.1(b)), so that their powers of the contrast factor are constants. Assembled as in Subsection 12.10, with the bound $\|\widetilde{U}_M(t_{(M)})\|_{H^{N_{\mathrm{buf}}}_{\rho_{\mathrm{mol}}(M),\omega}(\mathbb{T}^2)} \le C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ of Proposition 15.5(iv) ($\widetilde{U}_M(t_{(M)}) = U_M(t_{(M)})$, $\rho_{\mathrm{mol}}(M) = c_{\mathrm{mol}}\varepsilon_M$, $N_{\mathrm{buf}} \le N_{\mathrm{tr,w}}$) in place of the molecular smoothing, they give Π, depending only on $\widetilde{G}_M$ and $t_{(M)}$, the first two bounds, the second as $Q \ge 4 + 2qZ > Q_{\mathrm{e}}$ by (14.2), (14.3), (15.4) and (15.2), and the time $t_{\mathrm{pr}}^{(M)}$, a block boundary

between two quiet margins, with $t_{(M)}+h_M<t_{\rm pr}^{(M)}\le t_{(M)}+2h_M$. Here $\widetilde G_M$ has paths of at most $nN_{\rm len,w}\le N_{\rm path}$ sources (the remark after Definition 15.3, and item (1) of Subsection 11.1), as Proposition 12.7 asks; and that proposition holds with $N_{\rm out,e}$ in place of $N_{\rm out}$, as its proof uses $N_{\rm out}$ only as the order J of Lemma 12.15, which holds for every J, in Step 4, and as the order of the norms of Step 5, which the chart at $t_{\rm pr}$ changes by at most a constant. The third bound follows by interpolation in Fourier between the second bound and the bounds of $\widetilde U_M(t_{\rm pr}^{(M)})$ in $H^{N_{\rm tr,w}}_{\varepsilon_M,\omega}(\mathbb{T}^2)$ (Lemma 15.8) and of $\Pi\widetilde U_M(t_{(M)})$ in $H^{N_{\rm out,e}}_{\rho(M),\omega}(\mathbb{T}^2)$ (the first bound), as $2N_{\rm rd,e}\le\min\{N_{\rm tr,w},N_{\rm out,e}\}$ and $\varepsilon_M\le\rho(M)$.

(b) Proposition 8.3(c) applies to G' at the order $N_{\rm out,e}$, the scale $\rho(M)$ and the diffusivity κ_{M-1}. Its condition (i) holds since $\rho(M)\le\varepsilon_{M-1}$ and $\rho(M)^2\varepsilon_{M-1}^{\beta-2}=\kappa_{M-1}h_M\varepsilon_{M-1}^{\beta-2}\le C\varepsilon_{M-1}^{2\delta}\kappa_{M-1}$ by (2.5); (ii) as in the proof of Proposition 13.10(i); and the coupling sums of G' are at most $C\varepsilon_{M-1}^{\delta/2}$ by clauses (b3) and (d) in the proof of Proposition 15.11, which bound the matrix (8.4) at the order $N_{\rm out,e}$ and the scale $\rho(M)$, as it increases with both, $\rho(M)\le\rho_{\rm mol}(M-1)$ and the seed's derivative order is at least $N_{\rm out,e}$ (item (2) of Subsection 11.1). That proof uses Lemma 15.4(b), on $[t_{(M)},t_{\rm f}]$, and bounds that hold at every time, so these clauses hold on $[t_{(M)}+3\tau_M,t_{\rm f}]\supset[t_{\rm pr}^{(M)},t_{\rm f}]$, as $t_{\rm pr}^{(M)}>t_{(M)}+h_M\ge t_{(M)}+3\tau_M$ (proof of (a), (2.4)). The datum is bounded by (a).

(c) *The identities.* Lemmas 13.1, 13.2 and 13.4–13.6 hold at $m=M$ for $\widetilde G_M$, with $N_{\rm cor,e}$, $N_{\rm src,e}$, $N_{\rm cl,e}$, $N_{\rm rec,e}$, $nN_{\rm len,w}$ in place of $N_{\rm cor}$, $N_{\rm src}$, $N_{\rm cl}$, $N_{\rm rec}$, $N_{\rm path}$: $\widetilde G_M$ is in the setting of the lift (Section 13) with these orders, as it satisfies the clauses of the invariant (Lemma 15.4(b)), its birth orders are at most $1+nN_{\rm cor,w}\le N_{\rm src,e}$, its paths have at most $nN_{\rm len,w}$ sources (proof of (a)), and its birth radii, those of G_M, are $\varepsilon_{M+1}\le2\varepsilon_M^{q-1}\varepsilon_M\le\varepsilon_M$ for $M\ge M_*$ (Definition 15.3, Lemma 7.1(a)). The identity of Lemma 13.7 is exact, and the anchor does not change it, as only the derivative $\mathbf{J}_M^\kappa-\kappa_{M-1}I$ of the covariance appears in it; the readout identity of G' is that of Lemma 13.8, with the same proof. By (b), $\Upsilon^{\rm e}Y$ and its anchored exponential have the classes of Lemmas 13.7(ii) and 13.5, as in Step 2 of the proof of Proposition 13.11 with $N_{\rm out,e}$, $N_{\rm src,e}$, $p_{\rm e}$ in place of $N_{\rm ret}+3$, $N_{\rm src}$, p: by (15.7) and the first line of (15.6), $N_{\rm out,e}-N_{\rm src,e}-2p_{\rm e}-2\ge N_{\rm rec,e}+1$, which gives the orders $2p_{\rm e}+1$ and $N_{\rm cor,e}+1$ of these lemmas. As in Step 2 of the proof of Lemma 15.7, with the cost $C\varepsilon_M/\rho(M)$ of a derivative at the scale ε_M on slow factors, the proofs of Lemmas 13.4–13.6 give, for $N\le N_{\rm rd,e}$, the same bounds with $H^N_{\varepsilon_M,\omega}(\mathbb{T}^2)$ in place of $L^2_\omega(\mathbb{T}^2)$ in the norm (13.4), reading N more orders. At $N=N_{\rm rd,e}$ the residual reads Y at most at the order $2N_{\rm rd,e}\le N_{\rm out,e}-2$, and the row of the old smoothings at $c_{\rm mol}\varepsilon_M$ in Table 15.1 reads $Y^\sharp$ at the order $N_{\rm rd,e}+N_{\rm mom,w}+nN_{\rm cor,w}-1\le N_{\rm out,e}-N_{\rm cor,e}-2p_{\rm e}$ (first line of (15.6)), up to which the component energy of $Y^\sharp$ at the scale $\rho_{\rm mol}(M)$, with the diffusivity κ, is at most $C\|\theta_{\sf in}\|_{L^2(\mathbb{T}^2)}$, by the proof of Proposition 13.10(ii) with the sizes of Lemma 15.1(b). So $Y^\sharp$ solves the system of $\widetilde G_M$ on $[t_{\rm pr}^{(M)},t_{\rm f}]$ with a forcing, the residual, listed in Table 15.1; by (15.2), (15.5) and Lemma 15.1(b), each is at most $C\varepsilon_{M-1}^{Q_{\rm e}+2}\|\theta_{\sf in}\|_{L^2(\mathbb{T}^2)}$, the energy of Y being bounded by (b).

The initial difference. At $t_{\rm pr}^{(M)}$ the anchored exponential is the identity, Y vanishes on the children, and the correctors of the pulses whose cutoff is positive vanish, by Lemma 13.1, as the velocity added at level M vanishes on $[t_{\rm pr}^{(M)}-\tau'_M,t_{\rm pr}^{(M)}+\tau'_M]$ and $t_{\rm pr}^{(M)}>t_{(M)}+3\tau_M$. So $Y^\sharp(t_{\rm pr}^{(M)})=\Pi\widetilde U_M(t_{(M)})$, and by (a) the difference $\widetilde U_M-Y^\sharp$ at $t_{\rm pr}^{(M)}$ is at most $C\varepsilon_{M-1}^{Q_{\rm e}/2}\|\theta_{\sf in}\|_{L^2(\mathbb{T}^2)}$ in $H^{N_{\rm rd,e}}_{\varepsilon_M,\omega}(\mathbb{T}^2)$.

Energy. Proposition 8.3(a) for $\widetilde G_M$ at the order $N_{\rm rd,e}$ and the scale ε_M, whose hypotheses hold as in the proof of Lemma 15.8, bounds the left side of (c) by C times the sum of the initial difference and the residual, and $\varepsilon_{M-1}^{Q_{\rm e}+2}\le\varepsilon_{M-1}^{Q_{\rm e}/2}$. By (b), $Y^\sharp(t)\in H^{N_{\rm rd,e}}(\mathbb{T}^2)^V$, as $N_{\rm out,e}\ge N_{\rm cor,e}+2p_{\rm e}+N_{\rm rd,e}+2$ and the lift spends $N_{\rm cor,e}+2p_{\rm e}$ derivatives, and two more in the Sobolev embedding. □

15.4 The first family of the induction

Proposition 15.11 makes the terminal successor, restarted at t_{e}, an admissible seed $G = G_{M-1}$. The reserve of generation 0 is the one the terminal cycle reaches from res_M, and the markers of G' are 1, so declaring its components of generation 0 changes no coefficient, smoothing or mass. As G' does not descend from a seed, Lemma 11.7 does not apply; the main estimate is the recursion for (b3): a new source gains $\varepsilon_{M-1}^{\delta}$ from its majorant and loses less than $\varepsilon_{M-1}^{-\delta/2}$.

Proposition 15.11 (The seed)**.** *Let $M \geq M_*$ and $t_{\mathrm{e}} := t_{\mathrm{pr}}^{(M)} + h_M$. Then $t_{\mathrm{e}} \leq t_{\mathsf{in}} + \frac{1}{4}t_0$, and the family G of level $M-1$ obtained from G' by declaring all its components of generation 0 is an admissible seed of level $M-1$ on $[t_{\mathrm{e}}, t_{\mathrm{f}}]$ (Definition 11.2), with the bounds of item (3) of Subsection 15.1 and constants depending only on $\alpha, s, q, \zeta, \Lambda, n, t_0, T$.*

Proof. By Propositions 15.5 and 15.10(a), and as $M \geq M_*$, $t_{\mathrm{e}} \leq t_{\mathsf{in}} + 6nh_{M+1} + (C+1)h_M \leq t_{\mathsf{in}} + \frac{1}{4}t_0$, and $t_{\mathrm{e}} \geq t_{\mathrm{pr}}^{(M)} \geq t_{(M)} \geq t_{\mathsf{in}}$, so $t_{\mathrm{f}} - t_{\mathrm{e}} \leq T$; we check Definition 11.2 with $t_{\mathrm{S}} = t_{\mathrm{e}}$.

(i). The components of $\widetilde{G}_M$ have total mass at most $\frac{2}{3}$ (Lemma 15.4(b)), their children at most $c_{\mathrm{e}}\varepsilon_{M-1}^{\delta/32}$ times that. The sources of $\widetilde{G}_M$ have birth orders at most $1 + nN_{\mathrm{cor,w}}$, and the new ones at most $N_{\mathrm{src,e}}$. Along a path, at most $N_{\mathrm{len,e}} + 1$ new sources separate at most $N_{\mathrm{len,e}} + 2$ runs of sources of $\widetilde{G}_M$, each of at most $nN_{\mathrm{len,w}}$ sources, so a path has at most $(N_{\mathrm{len,e}} + 2)(nN_{\mathrm{len,w}} + 1)$ components. The widths are $r_{\mathrm{w}} \leq \varepsilon_M$ and $\rho(M)\varepsilon_{M-1}^{\Theta_{\mathrm{e}}} \leq \varepsilon_{M-1}$, and the birth radii of the sources from $\widetilde{G}_M$ are at most ε_{M+1}, those of the new ones $\rho(M) \leq \varepsilon_{M-1}$.

(ii). The coefficients of G are those of G'. By (b1) of (iv) and the transport, as in Steps 6 and 8 of Subsection 11.6, each derivative of a coefficient costs at most ε_{M-1}^{-1} (the fast variable of level $M-1$; the slow ones cost ε_{M-2}^{-1}) and the transport costs $\varepsilon_{M-2}^{d-d'} \leq \rho_\sigma^{d-d'}$; so the sum in (ii) at $\rho = \varepsilon_{M-1}$, which increases with ρ, is at most $C\,\mathrm{fac}(\sigma)\,\mathrm{maj}(\sigma)\kappa_{M-1}\rho_\sigma^{d-2} \leq C\,\mathrm{maj}(\sigma)\kappa_{M-1}\rho_\sigma^{d-2}$, as $\mathrm{fac}(\sigma) \leq 1$: it is 1 or a product of factors of letters of the G_{i-1} (Definitions 11.3, 15.9 and the proof of Lemma 15.4(b)), each at most $C\varepsilon_{i-1}^{2(q+2)\gamma-\delta/16} \leq 1$ for $M \geq M_*$ (Lemma 15.1(a)).

(iii). The cell solutions are those of the weak finite lifts, finitely many two-dimensional cell problems per pulse (Lemma 15.2), each solved once, from zero, on the window of its pulse, in the cell time $\kappa t/\varepsilon_i^2$ of its level $i > M$, depending, among the pulses of level $i+1$, only on those whose charts are used there; the terminal cycle solves none. Their invariance when the phases move along the pulses and their determination on the window are proved as in Steps 1 and 2 of Subsection 11.6.

(iv): clauses (a), (b1), (b2), (b2′). As in Steps 1, 2 and 4–6 of Subsection 11.6, for $\widetilde{G}_M$ and G' at $m = M$, with Lemma 15.4(b) in place of the invariant, with the reserve res_M on all components of $\widetilde{G}_M$, one step $1/(N_{\mathrm{gen}} + 2 + N_{\mathrm{off}})$ above generation 0, and with the supremum in time, as every bound used holds at every time. Three scale facts change: the correctors and the covariance gain $\bar{\varepsilon}_M/\rho(M) \leq C\varepsilon_{M-1}^{q\gamma+3\delta}$ (Lemma 15.1(b)) and $|\Gamma_M(t) - \Gamma_M(t_{\mathrm{pr}}^{(M)})|/\rho(M)^2 \leq C\varepsilon_{M-1}^{\delta}$ (after (15.8)); in Step 5, the cell average of a source σ of $\widetilde{G}_M$ subtracted by a mean letter along σ gains $(\rho_\sigma/\rho(M))^{d'-2}$ or $(\rho_\sigma/\varepsilon_{M-1})^{d-d'}$, at most $C\varepsilon_{M-1}^{q\gamma+3\delta}$ as $\rho_\sigma \leq \varepsilon_{M+1}$, in place of κ_M/κ_{M-1}, which is not small here; and in Step 6 the birth radii ε_{M+1} and $\rho(M) \leq C\varepsilon_{M-1}^{1+\gamma/2+\delta}$ are below $\rho_{\mathrm{mol}}(M-1) \geq c\varepsilon_{M-1}^{1+\gamma/2}$, and the widths r_{w}, $\rho(M)\varepsilon_{M-1}^{\Theta_{\mathrm{e}}}$ below ε_{M-1}.

(iv): the recursion for (b3). The markers are 1, so the two choices ω_v and ω_v/Ψ_v^2 of the masses agree. Step 3 of Subsection 11.6 uses Definition 11.3 only through two facts, which hold for $\widetilde{G}_M$ and G' by Definition 15.9: the copies keep the masses of the family, and the entry of a new source is at most a fixed number times the product along its path. It uses the invariant only through the old priced sums, which must be at most $\frac{1}{2}$ and are at most $C\varepsilon_M^{2(q+2)\gamma-5\delta/64}$ here, with prices 1 (Lemma 15.4(b)).

So the computation of (11.20) applies, with C depending only on the numbers of words and letters and on C_{maj}. A new source gains $\varepsilon_{M-1}^{\delta}$ from its majorant $C_{\mathrm{maj}}\varepsilon_{M-1}^{\delta-\delta/16}$, which clause (b1) justifies as in Step 5 of Subsection 11.6: the mean letters gain powers of $\bar{\varepsilon}_M/\rho(M)$, and the covariance and defect letters $\varepsilon_{M-1}^{\delta}$, by the scale facts above (for the defect letter, the bound after (15.8) replaces Lemma 10.3(b)). It costs the power $\varepsilon_{M-1}^{-\delta/16}$ of its coefficients, the square root $C\varepsilon_{M-1}^{-\delta/64}$ of the mass ratio, and the price $\varepsilon_{M-1}^{-\Theta_{\mathrm{e}}(N_{\mathrm{src,e}}-2)}$ of its width $\rho(M)\varepsilon_{M-1}^{\Theta_{\mathrm{e}}}$. Since $\Theta_{\mathrm{e}}(N_{\mathrm{src,e}}-2)<\frac{1}{16}\delta$ by (15.3), the exponent $\delta-\frac{\delta}{16}-\frac{\delta}{64}-\Theta_{\mathrm{e}}(N_{\mathrm{src,e}}-2)$ exceeds $\frac{55}{64}\delta>\frac{1}{2}\delta$, and, as $\varepsilon_M^{q-1}\le C\varepsilon_{M-1}^{q(q-1)}$ and $q(q-1)\ge 4\delta$, every row and column sum of the priced majorant matrix of G' is at most

$$C\varepsilon_M^{2(q+2)\gamma-5\delta/64}+C\varepsilon_{M-1}^{\delta-\delta/16-\delta/64-\Theta_{\mathrm{e}}(N_{\mathrm{src,e}}-2)}\le C\varepsilon_{M-1}^{\delta/2}\le C_{\mathrm{seed}}\varepsilon_{M-1}^{\delta/4}\,.$$

(iv): clause (c). (c1): the sources from $\widetilde{G}_M$ have the smoothing S_{w}, and the new ones the round smoothings of Definition 15.9(iii); all have the inverse $2^{1/2}I$ of the matrix of long axes, of $|\cdot|_\infty$-norm $2^{1/2}\le 2$, and the order ℓ of the kernel W_ℓ of their smoothings satisfies $2\ell-1\ge N_{\mathrm{mom}}$. (c2): by Lemma 7.1(a), (c) and Proposition 2.1(c), $\rho_{\mathrm{mol}}(M-1)\ge c\varepsilon_{M-1}^{1+\gamma/2}$ and $\rho(M-1)\ge c\varepsilon_{M-2}^{1+\gamma/2+\delta}$, and, with $q-1-\frac{\gamma}{2}=4\delta+q\gamma$,

$$\frac{r_{\mathrm{w}}}{\rho_{\mathrm{mol}}(M-1)}\le C\varepsilon_{M-1}^{q\gamma+4\delta+q(\gamma/4+\delta/2)}\,,\qquad \frac{\rho(M)\varepsilon_{M-1}^{\Theta_{\mathrm{e}}}}{\rho_{\mathrm{mol}}(M-1)}\le C\varepsilon_{M-1}^{\delta+\Theta_{\mathrm{e}}}\,,$$
$$\frac{r_{\mathrm{w}}}{\rho(M-1)}\le C\varepsilon_{M-2}^{q^2(1+\gamma/4+\delta/2)-1-\gamma/2-\delta}\,,\qquad \frac{\rho(M)\varepsilon_{M-1}^{\Theta_{\mathrm{e}}}}{\rho(M-1)}\le C\varepsilon_{M-2}^{(q-1)(1+\gamma/2+\delta)+q\Theta_{\mathrm{e}}}\,.$$

The exponents of the first line are at least δ, those of the second at least $q-1\ge 4\delta$, as $\frac{\gamma}{2}+\delta<\frac{7}{12}(q-1)$. (c3): Lemma 8.2, with the constant matrices $2^{-1/2}I$.

(iv): clause (d). As in Step 8 of Subsection 11.6, the coupling matrices of G at the scale $\rho_{\mathrm{mol}}(M-1)$ are at most C times the priced majorant matrix, entry by entry, so their row and column sums are at most $C\varepsilon_{M-1}^{\delta/2}\le C\varepsilon_{M-2}^{q\delta/2}\le C_{\mathrm{seed}}\varepsilon_{M-2}^{q\delta/4}$, by Lemma 7.1(a). □

15.5 From the first family to the solution θ

Representations of any order. Each lift spends derivatives, so the representations on the way back down have the orders $nN_{\mathrm{cor,w}}+3$ (at ε_M) down to 3 (at ε_{M+n}): for $N\ge 1$, a *representation of order* N at wavelength λ and its size $\mathrm{size}_{\lambda,N}(R)$ are defined by Subsection 5.6 and (5.13) with N in place of N_{rep} ($\mathrm{size}_\lambda=\mathrm{size}_{\lambda,N_{\mathrm{rep}}}$), and Lemma 14.2 holds for them, with the same proof.

The composed weak lift. For $t\in\mathbb{R}$, let $\mathcal{W}_{M+n}(t)U:=U$ for a field U of G_{M+n}, a function, and, for $M<i\le M+n$, $\mathcal{W}_{i-1}(t)U:=\mathcal{W}_i(t)\,\mathrm{Corr}_i^{\mathrm{w}}(t)\Upsilon_i^{\mathrm{w}}U$ for a field U of G_{i-1} of class $C^{(M+n-i+1)N_{\mathrm{cor,w}}+1}$; so $\mathcal{W}_M(t)U$ is a function for every field U of G_M of class $C^{nN_{\mathrm{cor,w}}+1}$.

Lemma 15.12 carries a representation through the terminal lift at a constant cost, harmless as only $n+1$ lifts remain: the anchored covariance is small against ε_{M-1}^2 and the contrast bounded.

Lemma 15.12 (The terminal lift of representations)**.** *There exists $C(\alpha,s,q,\zeta,\Lambda,n,t_0,T)$ such that, for every $M\ge M_*$, every field Y of G' on $[t_{\mathrm{pr}}^{(M)},t_{\mathrm{f}}]$, every $t\in[t_{\mathrm{pr}}^{(M)},t_{\mathrm{f}}]$ and every representation R of $Y(t)$ at wavelength ε_{M-1} (of order N_{rep}), there exists a representation $R^\sharp$ of its terminal lift $Y^\sharp(t)$ (15.8) of order $nN_{\mathrm{cor,w}}+3$ at wavelength ε_M with $\mathrm{size}_{\varepsilon_M,nN_{\mathrm{cor,w}}+3}(R^\sharp)\le C\,\mathrm{size}_{\varepsilon_{M-1}}(R)$.*

Proof. Let E be the base of R and $N:=nN_{\mathrm{cor,w}}+3$; we follow the three steps of (15.8).

Step 1: the readout. Scaled by $\omega_v^{-1/2}$, Υ^{e} is the identity plus, for each child v' of v, one entry $\varepsilon_{M-1}^{\delta\#v'/16}(\omega_{v'}/\omega_v)^{1/2} \le C\varepsilon_{M-1}^{5\delta/64}$, so $|\Upsilon^{\mathrm{e}}| \le 2$ for the norms $|\cdot|_\omega$, by Schur's test, and it maps R to a representation of size at most $2\,\mathrm{size}_{\varepsilon_{M-1}}(R)$ (Subsection 5.6).

Step 2: the exponential. For $1 \le j \le p_{\mathrm{e}}$, on a profile of wavelength $\lambda_P \ge \varepsilon_{M-1}$, $((\Gamma_M(t) - \Gamma_M(t_{\mathrm{pr}}^{(M)})) : \nabla^2)^j$ costs $(|\Gamma_M(t) - \Gamma_M(t_{\mathrm{pr}}^{(M)})|/\lambda_P^2)^j \le C$ (after (15.8)) and $2j$ derivatives; so it maps the profiles, and the base read as a profile of wavelength ε_{M-1}, to profiles of the same wavelengths, of order $N + N_{\mathrm{cor,e}} + 1$ and size at most $C\,\mathrm{size}_{\varepsilon_{M-1}}(R)$.

Step 3: the correctors. $\mathrm{Corr}^{\mathrm{e}}(t)P - P$ is the sum, over the pulses k with $\mathsf{H}_{M,k}(t) \neq 0$, of the correctors of degrees $1, \dots, N_{\mathrm{cor,e}}$ of P, read in the chart of k at $z = y/\varepsilon_M$. By the computation of Proposition 10.7(a), with the bounds of Lemma 15.4(b) in place of its hypotheses and $\bar\varepsilon_M \le 2^{1/2}\varepsilon_M$ (as $\kappa_{M-1}/\kappa \le 2$), the coefficient of $\nabla^d P$ in a corrector of degree $j \ge d \ge 1$ is at most $C(2^{1/2}\varepsilon_M)^j \varepsilon_{M-1}^{d-j} \le C\varepsilon_M^d$, and each of its derivatives costs $C\varepsilon_M^{-1}$; the chart has bounded derivatives at the scale ε_{M-1}. So, for a profile P of wavelength $\lambda_P \ge \varepsilon_{M-1}$, $\mathrm{Corr}^{\mathrm{e}}(t)P - P$ is a profile of wavelength ε_M with $\varepsilon_M^{-s}\|\mathrm{Corr}^{\mathrm{e}}(t)P - P\|_{C^N_{\varepsilon_M}(\mathbb{T}^2)} \le C\sum_{d\ge 1}\varepsilon_M^{-s}(\varepsilon_M/\lambda_P)^d [P]_{C^{N+N_{\mathrm{cor,e}}}_{\lambda_P}(\mathbb{T}^2)} \le C\lambda_P^{-s}[P]_{C^{N+N_{\mathrm{cor,e}}}_{\lambda_P}(\mathbb{T}^2)}$, since $d \ge 1 > s$. The profiles of Step 2 keep their wavelengths, and the base, read at the wavelength ε_M, has $\varepsilon_M^{-s}[E]_{C^N_{\varepsilon_M}(\mathbb{T}^2)} \le \varepsilon_{M-1}^{-s}[E]_{C^N_{\varepsilon_{M-1}}(\mathbb{T}^2)}$, as $(\varepsilon_M/\varepsilon_{M-1})^{1-s} \le 1$. The three steps use at most $N + N_{\mathrm{cor,e}} + 2p_{\mathrm{e}} + 1 = N_{\mathrm{rep}}$ derivatives, which gives the lemma. □

By Lemma 15.13(a), the weak lifts do the same, each spending $N_{\mathrm{cor,w}}$ derivatives, as their correctors take derivatives of the field with a factor ε_i each. By (b), a Sobolev bound at the wavelength ε_i becomes, after the lifts, a Hölder bound with the loss ε_{M+n}^{-1-s}, offset by the accuracies of the weak passages, of the smoothing and of the terminal cycle.

Lemma 15.13 (The weak lifts of representations)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T)$ such that, for every $M \ge M_*$, $M \le i \le M + n$ and $t \in \mathbb{R}$, the following hold.*

(a) *For every representation R of a field U of G_i of order $(M + n - i)N_{\mathrm{cor,w}} + 3$ at wavelength ε_i, there exists a representation $R^\sharp$ of $\mathcal{W}_i(t)U$ of order 3 at wavelength ε_{M+n} with $\mathrm{size}_{\varepsilon_{M+n},3}(R^\sharp) \le C\,\mathrm{size}_{\varepsilon_i,(M+n-i)N_{\mathrm{cor,w}}+3}(R)$.*

(b) *For every field U of G_i with components in $H^{N_i}(\mathbb{T}^2)$,*

$$\|\mathcal{W}_i(t)U\|_{C^{0,s}(\mathbb{T}^2)} \le C\varepsilon_{M+n}^{-1-s}\|U\|_{H^{N_i}_{\varepsilon_i,\omega}(\mathbb{T}^2)}.$$

Proof. (a) By induction on $M + n - i$: $|\Upsilon_i^{\mathrm{w}}| \le 2$ as in Step 1 of the proof of Lemma 15.12, and the coefficient of $\nabla^d P$ in a corrector of degree $j \ge d \ge 1$ of $\mathrm{Corr}_i^{\mathrm{w}}$ is at most $C\varepsilon_i^j\varepsilon_{i-1}^{d-j} \le C\varepsilon_i^d$, each derivative costing $C\varepsilon_i^{-1}$ (proof of Lemma 15.4(a)); so Step 3 there, with ε_i for ε_M and no exponential, gives the passage, which spends $N_{\mathrm{cor,w}}$ derivatives. (b) As in the proof of Lemma 14.3, U, with no profile, is a representation of order $N_i - 2 = (M + n - i)N_{\mathrm{cor,w}} + 3$ at wavelength ε_i of size at most $C\varepsilon_i^{-1-s}\|U\|_{H^{N_i}_{\varepsilon_i,\omega}(\mathbb{T}^2)}$; (a) and Lemma 14.2(a) with the order 3 give (b), as $\varepsilon_i \ge \varepsilon_{M+n}$. □

In Lemma 15.14, the difference between θ and the solution of the terminal successor, lifted through the terminal cycle and the n weak passages, telescopes into the errors of Lemmas 15.7 and 15.8 and Proposition 15.10(c), each read in $C^{0,s}(\mathbb{T}^2)$ by Lemma 15.13(b).

Lemma 15.14 (Recovering θ)**.** *There exists $C(\alpha, s, q, \zeta, \Lambda, n, t_0, T)$ such that, for every $M \ge M_*$, θ_{in} and θ, with $(\mathcal{W}_M Y^\sharp)(t) := \mathcal{W}_M(t)Y^\sharp(t)$,*

$$\|\theta - \mathcal{W}_M Y^\sharp\|_{L^\infty([t_{\mathsf{in}} + \frac{1}{4}t_0, t_{\mathrm{f}}]; C^{0,s}(\mathbb{T}^2))} \le C\varepsilon_{M-1}^2\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}.$$

Proof. Let $t \in [t_{\mathsf{in}} + \frac{1}{4}t_0, t_{\mathrm{f}}]$, so $t \geq t_{\mathrm{e}} \geq t_{(M)}$. As $N_{\mathrm{tr,w}} \geq N_{\mathrm{rd,e}} \geq nN_{\mathrm{cor,w}} + 5$ (item (4) of Subsection 15.1), by the Sobolev embedding $\mathcal{W}_{i-1}(t)$ applies to $U_{i-1}(t)$ and $\mathcal{W}_M(t)$ to $Y^\sharp(t)$, whose components lie in $H^{N_{\mathrm{tr,w}}}(\mathbb{T}^2)$ and $H^{N_{\mathrm{rd,e}}}(\mathbb{T}^2)$ (proof of Lemma 15.7, Proposition 15.10(c)). By Proposition 15.5(i) and the definition of the $\mathcal{W}_i$,

$$\theta(t) - \mathcal{W}_M(t)Y^\sharp(t) = \sum_{i=M+1}^{M+n} \mathcal{W}_i(t)\big[U_i(t) - \mathrm{Corr}_i^{\mathrm{w}}(t)\Upsilon_i^{\mathrm{w}}U_{i-1}(t)\big] + \mathcal{W}_M(t)\big[U_M(t) - Y^\sharp(t)\big],$$

and $U_M - Y^\sharp = (U_M - \widetilde{U}_M) + (\widetilde{U}_M - Y^\sharp)$. We bound each term by Lemma 15.13(b), with $\varepsilon_{M+n}^{-1-s} \leq C\varepsilon_M^{-2q^n}$ and $\varepsilon_{M+n}^{-1-s} \leq C\varepsilon_{M-1}^{-q^{n+1}(1+s)}$ (Lemma 7.1(a), $s < 1$). By Lemma 15.7 and Proposition 15.5(iv), each term of the sum is at most $C\varepsilon_M^{-2q^n}\varepsilon_M^{3+2q^n}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$; by Lemma 15.8, whose order $N_{\mathrm{tr,w}} - N_{\mathrm{mom,w}} - nN_{\mathrm{cor,w}} - 1$ is at least $N_{\mathrm{rd,e}} \geq N_M$, and as $(\frac{\gamma}{4} + \frac{\delta}{2})N_{\mathrm{mom,w}} > 2 + 2q^n$ (item (4)), the term with $U_M - \widetilde{U}_M$ is at most $C\varepsilon_M^{-2q^n}\varepsilon_M^{q+1+2q^n}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$; by Proposition 15.10(c), the term with $\widetilde{U}_M - Y^\sharp$ is at most $C\varepsilon_{M-1}^{Q_{\mathrm{e}}/2-q^{n+1}(1+s)}\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)} = C\varepsilon_{M-1}^5\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. As $\varepsilon_M \leq \varepsilon_{M-1}$, the sum is at most $C\varepsilon_{M-1}^2\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$. □

15.6 Proof of Proposition 5.1

Proof of Proposition 5.1. Let $M \geq M_*$, $\kappa \in \mathcal{K}_M$ and $t_{\mathsf{in}} \in \mathbb{R}$. The times $t_{(M)}$, $t_{\mathrm{pr}}^{(M)}$, $t_{\mathrm{e}} \in [t_{\mathsf{in}}, t_{\mathsf{in}} + \frac{1}{4}t_0]$, the map Π and the admissible seed G of level $M-1$ on $[t_{\mathrm{e}}, t_{\mathrm{f}}]$ depend only on M, κ and t_{in} (Propositions 15.5, 15.10 and 15.11). Given θ_{in} and θ, let Y be the solution of G' of Proposition 15.10 and $g := Y(t_{\mathrm{e}})$, a datum of G with $\|g\|_{L^2_\omega(\mathbb{T}^2)} \leq C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ by (b) there. As the systems of G and G' coincide, the solution U of G from g at time t_{e} is Y on $[t_{\mathrm{e}}, t_{\mathrm{f}}]$ (Proposition 6.2). For $[t_1, t_2]$ and Ξ as in the statement and $t \in [t_1, t_2]$ at which $R(t)$ exists (so $t \geq t_{\mathrm{e}} \geq t_{\mathrm{pr}}^{(M)}$), Lemma 15.12, applied to the representation $R(t)$ of $U(t) = Y(t)$, and then Lemma 15.13(a) at $i = M$ give a representation of $\mathcal{W}_M(t)Y^\sharp(t)$ of order 3 at wavelength ε_{M+n} of size at most $C\Xi$, so $\|\mathcal{W}_M(t)Y^\sharp(t)\|_{C^{0,s}(\mathbb{T}^2)} \leq C\Xi$ by Lemma 14.2(a). With Lemma 15.14, $\|\theta(t)\|_{C^{0,s}(\mathbb{T}^2)} \leq C\Xi + C\|\theta_{\mathsf{in}}\|_{L^2(\mathbb{T}^2)}$ for almost every $t \in [t_1, t_2]$. □

Index of notation and terms

The main symbols, by topic, then a glossary of terms, each with the place of its definition; local symbols are left out. Throughout, k indexes pulses, l renewal blocks and m levels (a second level is i: in a chain, in Definition 11.4 and in the entry; Subsection 14.4 writes j and i); M is the terminal level ($\kappa \in \mathcal{K}_M$).

General notation

$x \asymp y$	$C^{-1}y \leq x \leq Cy$, with C depending only on α, q, ζ	§2
$x \ll y$, $y \gg x$	for $m \geq 2$: $x \leq C\varepsilon_{m-1}^c y$, with $c > 0$ and C depending only on α, q, ζ	§2
$\lvert\cdot\rvert$, $\lvert\cdot\rvert_\infty$	Euclidean norm; largest modulus of a coordinate; their operator norms	§2
$(g)_E$	the average $⨍_E g$ of g over E	§2
$\lVert\cdot\rVert_{C^{0,s}}$, $[\cdot]_{C^{0,s}}$	Hölder norm and seminorm	(3.1)
$\nabla^\perp$; $\nabla^j g$	$(-\partial_2, \partial_1)$; the array of the 2^j ordered derivatives of order j of g	§2, §8

Parameters and exponents

α, β	Hölder exponent of the velocity field $\mathbf{b}$, $0 < \alpha < \frac{1}{3}$; $\beta := 1 + \alpha$	§2
s	Hölder exponent of Theorem A, $0 < s < \frac{1}{2}(1-\alpha)$	§2
q	supergeometric scale separation parameter, $\varepsilon_m \asymp \varepsilon_{m-1}^q$ for $m \geq 2$	(2.1)
Λ	minimal scale separation, $\varepsilon_{m-1}/\varepsilon_m \geq \Lambda \geq 128$	§2
ζ	the cutoff function, which switches the pulses on and off	(2.2)

γ	$\kappa_m \asymp \varepsilon_m^{\beta+\gamma}$ for $1 \le m < M$, and $\kappa_M \asymp \varepsilon_M^{\beta-\gamma}$	(2.3)
δ, μ, η	time-scale separation; cost exponent of a level (5.2); readout exponent	(2.3)
Λ_0	threshold for Λ, declared with its dependence in each statement	Thm. A
The reduction and the main results		
t_{in}, θ_{in}, θ, κ; t_0, T	initial time, datum, solution, diffusivity; waiting time, horizon	Thm. A
$\mathcal{S}$	the solution operator $\theta_{\mathsf{in}} \mapsto \theta(t_{\mathsf{in}}+1,\cdot)$ of Theorem B	§3
n	number of levels kept finer than M	Prop. 4.1
ϖ; ϖ_0, ϖ_1	relative size of a skew perturbation; its thresholds	§4
Φ, $\widetilde{\Phi}$; $\widetilde{\theta}$	stream functions, $\widetilde{\Phi}$ smooth and close to Φ; the solution for $\widetilde{\Phi}$	Prop. 4.2
$\mathbf{a}$, $\mathbf{b}_*$	the coefficient, with $\mathbf{a}-I$ skew, and the divergence-free drift	Lem. 4.3
$\mathcal{T}$, r_{tube}, r_{obs}	tube of Theorem C and its radius; radius of the observed disc	(9.11)
θ^{per}, U^{per}	the periodic comparison solution; the same in the coordinates y	Lem. 9.3
Scales and windows		
ε_m	wavelength of level m; $\varepsilon_0 \coloneqq 1$	(2.4)
h_m, τ'_m, τ_m	renewal time (the length of a block), margin time, pulse time	(2.4)
$l_{m,k}$	the renewal block of pulse k	(2.8)
$\zeta_{m,k}$, $\widehat{\zeta}_{m,l}$, $\widetilde{\zeta}_{m,k}$	fast cutoff of pulse k; slow cutoff of block l; envelope of pulse k	(2.7)–(2.10)
H; $\mathsf{H}_{m,k}$	smooth step; time cutoff of pulse k, on its window $\lvert t-k\tau_m\rvert \le \frac{61}{48}\tau_m$	(2.9), (10.2)
$\rho(m)$	$(\kappa_{m-1}h_m)^{1/2}$, the effective diffusion length over a renewal time	(5.5)
$\rho_{\mathrm{mol}}(m)$, c_{mol}	the molecular radius and its constant	(11.1)
r_m, Θ	width $\rho(m)\varepsilon_{m-1}^{\Theta}$ of the fresh smoothings of level m	(14.1)
r_{w}, S_{w}	width $\varepsilon_M^{1+\gamma/4+\delta/2}$ and smoothing (over round discs) at level M	§15.1, §15.3
$S_{\ell,r,\Omega}$	the average over the ellipse $\{y - r\Omega z : \lvert z\rvert < 1\}$ with the kernel W_ℓ	(6.2)
Times (Figure 5.1)		
t_{e}	entry time, $t_{\mathsf{in}} \le t_{\mathrm{e}} \le t_{\mathsf{in}} + \frac{1}{4}t_0$: the solution of the seed starts there	Prop. 5.1
t_{S}; m_{S}	start of the window $[t_{\mathrm{S}}, t_{\mathrm{f}}]$ of an admissible seed; its level	Def. 11.2
t_{A}, $t_{\mathrm{A}}^{(i)}$	start of the admissibility window $[t_{\mathrm{A}}, t_{\mathrm{f}}]$; at level i	Def. 11.4
t_{c}	start of a cycle, $t_{\mathrm{c}} \ge t_{\mathrm{A}} + 3\tau_m$	Prop. 5.3
t_{pr}	preparation time, a block boundary in $[t_{\mathrm{c}}, t_{\mathrm{c}}+4h_m]$; Y starts there	Prop. 5.3
$t_{\mathrm{pr}}^{(M)}$	preparation time of the terminal cycle; start of Y; $t_{\mathrm{e}} = t_{\mathrm{pr}}^{(M)} + h_M$	Prop. 15.10
$t_{(m)}$	start of the solution U_m, also for $M \le m \le M+n$ (Section 15)	§5.2
t_{f}	final time of the windows and the tube; $t_{\mathsf{in}} + T$ in Prop. 5.1 and §15	§5.2
The velocity field and its flows		
$\psi_{m,k}$; $e_{(k)}$	shear stream of pulse k; its direction of flow	(2.6), (2.16)
$e_{(k)}^{\perp}$	the coordinate vector other than $e_{(k)}$	§10
ϕ_m, $\mathbf{b}_m$; ϕ, $\mathbf{b}$	stream function and velocity of the levels up to m; their limits	(2.12), (2.14)
$X_m(t,x,t')$	position at time t of the trajectory of $\mathbf{b}_m$ through x at time t'	(2.11)
$\mathbf{K}_{m,l}$; $\psi_m(t,y)$	conductivity of the chart of block l; the level-m stream in a chart	(7.8), (7.11)
chain $\mathcal{C}$; m_{c}	chart parameters across the coarser levels; the carrier level	(11.7)
Diffusivities		
ν, $\mathbf{J}_m^{\nu}(t)$	a generic diffusivity (as in Prop. 6.2); the averaged flux of level m	(2.16)
$\overline{K}_m(\nu)$	the average of $\mathbf{J}_m^{\nu}$ over a period	(2.17)
$\mathcal{K}_M$, $\mathcal{K}$	the bands; their union, the permissible diffusivities	(2.18)
κ_m	effective diffusivity of level m: $\kappa_M \coloneqq \kappa$, $\kappa_{m-1} \coloneqq \overline{K}_m(\kappa_m)$	(2.19)
$\bar{\varepsilon}_m$	corrector amplitude $\varepsilon_m(\kappa_{m-1}/\kappa_m)^{1/2}$, with the contrast κ_{m-1}/κ_m	§5.1
$\Gamma_m(t)$	the covariance of the level-m pulses	(5.11)
Families and their norms		
G; V, v; ω_v, Ψ_v	a family; its components; the mass and the marker of v	§5.4
$\sigma\colon v' \to v$; S_σ, r_σ	a source; its smoothing and the width of the smoothing	§5.4
W_ℓ, ℓ	kernel of the smoothings, with vanishing moments of orders $1,\dots,2\ell-1$	(6.1)

$A_{\sigma,d}$; g, U	coefficient of order d of σ, $2 \le d \le N_{\mathrm{src}}$; a datum and the solution	(5.6)
L^2_ω, L^2_G, $H^N_{\rho,G}$	weighted norm, marked norm and Sobolev norm of a family	(5.7)
H^N_ρ; $H^N_{\rho,\omega}$, $H^{(N_v)}_{\rho,\omega}$	N derivatives at scale ρ; weighted, at one order or per component	(5.8), (8.1)
$B_{vv'}$, $B^{N,\rho}$	the coupling matrix; the coupling matrix of order N at the scale ρ	(5.9), (8.4)
F, F^0	flux forcing and scalar forcing of a system	(8.2)
G'; $\#v'$	the successor; the number of letters of the word of v'	§5.4, (5.10)
G_m, U_m	the families of the homogenization cascade and of the entry, and their solutions	§5.2, §15
$\widetilde{G}_M$, $\widetilde{U}_M$	G_M with its sources smoothed by S_{w}; its solution from $U_M(t_{(M)})$	§15.3
d_σ, ρ_σ, fac, maj	birth order, birth radius, factor $\mathrm{fac}(\sigma)$ and majorant $\mathrm{maj}(\sigma)$	§11.1
$\mathcal{A}(t,y,z)$	two-scale form, read at $z = y/\varepsilon_m$; transverse if a function of $z \cdot e^\perp_{(k)}$	§10
$\lVert\mathcal{A}\rVert_{N,j_1}$, $\lVert\mathcal{A}\rVert_{\mathrm{an},v}$	Wiener norm on the window of a pulse (10.18); analytic norm	(11.9)

The lift, the correctors and the representations

Υ_m, $Y^\sharp$, Corr_m	readout; lift of a field Y of the successor; corrector operator	(5.10)
$\exp_p$	the exponential series truncated after the degree p	(5.11)
$Y^\flat$	$\exp_p(\Gamma_m : \nabla^2)\Upsilon_m Y$, so that $Y^\sharp = \mathrm{Corr}_m Y^\flat$	(13.2)
Π	the preparation map	Prop. 5.3
$\chi^\nu_{m,k}$, χ^ν_m	the corrector of pulse k; the glued corrector (Lemma 10.2)	(10.3)
$\mathbf{K}(t,y)$; $\mathcal{L}_0$	a conductivity on the window of a pulse; the cell operator	§10.3, (10.10)
$\mathcal{V}_j$, $\mathcal{M}_{j,d}$	correctors of degree j and means of the finite lift, per path or summed	Def. 10.5
$\mathrm{Corr}_{m,k}$, $\mathcal{M}_m$	the corrector operator on pulse k; the mean flux	(10.23)
$\lvert U(x)\rvert_\omega$, C^N_λ	pointwise norm of a field of a family; norm, seminorm at scale λ	(5.12)
R; E, $P \in \mathcal{P}$, λ_P	representation at wavelength λ_*; base, profiles, their wavelengths	§5.6
$R^\sharp$	the representation of a lifted field, built from R (Lemmas 15.12, 15.13)	§15.5
Ξ	a bound on the sizes of the representations of $U(t)$	Prop. 5.1

The preparation (Section 12)

y_0, $\mathbf{K}(t)$	the point at which the conductivity is frozen; the frozen conductivity	§12.2
$L^2_{\omega,N'}$, $H^N_{\rho,\omega,N'}$	the moment norms about y_0	(12.8)
$\mathcal{R}_\xi$, $\mathcal{Z}_\xi$	the principal branch and its symbol	Prop. 12.4
$\mathsf{H}^{\mathrm{lo}}_m(\xi)$, c_{lo}	low-frequency cutoff $\mathsf{H}(2 - \bar\varepsilon_m\lvert\xi\rvert/c_{\mathrm{lo}})$, c_{lo} of Prop. 12.5	§12.8
$\mathcal{I}$	an insertion: a source term with polynomial coefficients	(12.18)
$U^{\{j\}}$	the terms of the Duhamel expansion of a family	§12.6
$U^{[0]}, \dots, U^{[N]}$	a Duhamel term of length N	§12.6
$U^\dagger$, $U^\dagger_{\mathrm{br}}$	a test, a solution of the adjoint fibre equation; the branch test	(12.19)
$\lvert\widehat{F}\rvert_{\dots;\xi}$, $\lVert g\rVert_{\dots;\xi}$	the normalized ξ-jet and the moment data at ξ, of orders N_1, N'	§12.8
Π_{blk}, Π_+, Π_{ren}	models of a Duhamel term over a block, the future block, the renewal	§§12.8–12.9
$H^{-N}_{\rho,\omega,N'+2}$, N_{neg}	negative norms, dual to the moment norms; the negative order	§12.9

Orders (chosen in the order of Table 11.1, which gives their conditions)

N_{cor}, p	corrector order (largest corrector degree); truncation order of $\exp_p$	§5.3
N_{ret}, N_{out}	order controlled by a cycle; order of the preparations	§5.3
N_{rep}, N_{src}, Q	representation order; source order; accuracy exponent	§5.3
Z; N_{mom}	power-loss exponent; moment order of the smoothings	(14.2), §11.1
N_{gen}, N_{len}	generations kept; length of the longest word	§11.1
N_{word}, N_{path}	number of words; path bound	(11.5), §11.1
N_{coef}, N_{an}, N_{off}	derivative-bound order; weight order and offset of the analytic norm	§11.1
N_{dep}, N_{buf}	depth of the Duhamel expansion; order of the molecular smoothing	(12.2)
N_{cl}, N_{rec}	cleanup steps for the cell tail; order of the reconstruction	(13.5)
$N_{\mathrm{cor,w}}$, $N_{\mathrm{len,w}}$	corrector order and word length of the weak passages	(15.1)
Q_{e}, $N_{\mathrm{src,e}}$, Θ_{e}	accuracy, largest new birth order, width exponent of the terminal cycle	(15.3)
$N_{\mathrm{cor,e}}$, p_{e}	corrector order and truncation order of the terminal cycle	(15.2)

$N_{\mathrm{len,e}}$, $N_{\mathrm{cl,e}}$	word length and antidivergence steps of the terminal cycle	(15.2)
$N_{\mathrm{mom,w}}$, $N_{\mathrm{mom,e}}$	moment orders of the smoothings at level M and of the terminal cycle	(15.5)
$N_{\mathrm{rd,e}}$, $N_{\mathrm{tr,w}}$, $N_{\mathrm{out,e}}$	terminal recovery order; weak trace order; terminal output order	(15.7)
N_i	$(M+n-i)N_{\mathrm{cor,w}}+5$, the order at which a field of G_i is read back down	§15.2

Constants and thresholds

C, c	generic constants; dependence declared in each section and statement	each section
m_0	the top level, the largest threshold on the level in Sections 11–14	§5.3, §11.1
M_*, M_1	thresholds for the terminal level (M_1: Proposition 4.1)	Prop. 5.1
C_{seed}, C_{adm}, C_{maj}	seed constant; bound on majorant and coupling sums; cycle factor	§11.1
C_Ω, C_{wid}, C_{der}, C_{jet}	constants of the invariant (with the c_j): smoothings, widths, derivatives	§11.1
c_{an}	the strip constant $1/20000$ of the chart parameters	§11.3

Terms

activity weight	the weight $w_{N_{\mathrm{act}}}(t)$ of the norm $\|\cdot\|_{H^J(\mathbb{T}^2);N_{\mathrm{act}},t}$	(12.23)
admissible family	a family descended by cycles from an admissible seed	Def. 11.4
admissible seed	a seed that satisfies Definition 11.2	Def. 11.2
allowed chain	a chain of levels at least m_{c} whose centres satisfy $\|y^{(i)}\|_1 < 3c_{\mathrm{an}}\varepsilon_i/4$	(11.7)
averaged flux	$\mathbf{J}_m^\nu$, the spatial average of the flux of the correctors of the pulses	(2.16)
base, profile	the smooth part and the oscillating parts of a representation	§5.6
Bloch sum	$\mathcal{B}g(\xi,z)$, the sum (12.9) of a field g about y_0	(12.9)
branch test	the test from 1 at a quiet time (off-branch: orthogonal to the principal branch)	§12.6
cell average	the average in z of a two-scale form of a source, carried by its transport	Def. 11.3
cell inverse	the solution of a cell problem across a pulse; $\chi_{\mathbf{K}}$, the corrector in $\mathbf{K}$	Lem. 10.4
cell solution	a function built from two-dimensional cell problems of a family	Def. 11.1
cell tail	$\mathcal{T}_k$, the operator left by the correctors of pulse k, of degrees above N_{cor}	Lem. 13.2
cell time	ε_m^2/κ_m, the time diffusion takes to cross a period cell	§2.5
child, parent	a component created by a cycle; the component it comes from	§5.4
comparison	the bound on $U - Y^\sharp$ in a cycle	Prop. 5.3
component energy	the quantity (13.3) of a field, of an order, at a scale, with a diffusivity	(13.3)
contrast factor	$1+\bar{\varepsilon}_m/\varepsilon_m = 1+(\kappa_{m-1}/\kappa_m)^{1/2}$	§12.6
corrector amplitude	$\bar{\varepsilon}_m$, the size of the correctors of level m per derivative	§5.1
covariance	$\Gamma_m(t)$, the accumulated surplus of the averaged flux over $\kappa_{m-1}I$	(5.11)
covariance letter	a letter for the means along a path conjugated by powers of $\mp\Gamma_m : \nabla^2$	Def. 11.3
cycle	one homogenization step, from level m to level $m-1$	Prop. 5.3
defect, defect letter	the mean flux along the empty paths minus $\mathbf{J}_m^{\kappa_m}(t)\nabla$; its letter	§10.5
entry	from θ at level $M+n$ up to the seed, and the lift back down	Prop. 5.1
family	a coupled system of equations of one level	§5.4
fibre equation	the equation of the periodic part of a Bloch wave at the frequency ξ	(12.5)
finite lift	the correctors and means of a system on the window of one pulse	Def. 10.5
frontier flux	F^{fr}, the readout of the flux into the omitted components	Lem. 13.8
generation	$\mathrm{gen}(v)$, the number of cycles that created the component v	§11.1
halving interval	a few cell times in which solutions and tests halve; $\#[t',t'']$ counts them	§12.7
homogenization cascade	the entry and the cycles, up the levels, forward in time	§5.2
invariant	the bounds satisfied by the admissible families	Def. 11.1
lift	$Y^\sharp$, from a field of the successor to the components of G	(5.10)
marked amplitude, A_m	the norm $\|U_m\|_{L^\infty([t_{(m)},t_{\mathrm{f}}];L^2_{G_m}(\mathbb{T}^2))}$, carried up the levels	§5.2
marker	the number Ψ_v, which grows with the generation; marked norm: $\|\cdot\|_{L^2_G}$	§5.4
mass	the number ω_v of a component	§5.4
mean letter	the letter for the means of the lift along a path	Def. 11.3
moment term	$U^{[j,i]}$, a term $U^{[j]}$ of a Duhamel term times $((y-y_0)/\rho(m))^i$	§12.6

outer domain	$\mathcal{D}_v$ with 2 in place of $\mathrm{res}(v)$; its real points are the allowed chains	(11.9)
passage	a cycle read back down on representations, at a fixed time	Prop. 5.4
path	sources $\sigma_1 \cdots \sigma_i$, each starting where the previous one ends	§10.3
preparation	Πg, close to the solution at t_{pr} and smooth at the scale $\rho(m)$	Prop. 5.3
price	$\max\{1, (\rho_\sigma / r_\sigma)^{d_\sigma - 2}\}$, which enters the priced majorant matrix	(11.3)
pulse	an odd slot k of level m, active for a time of about τ_m around $k\tau_m$	(2.6)
quiet margin	the pulse-free end of a renewal block, of length τ'_m	(2.10)
readout	Υ_m, which adds each new component to its parent	(5.10)
regularity cascade	the passages and the lifts, back down the levels, at a fixed time	Prop. 5.2
renewal block	$[(l - \frac{1}{2})h_m, (l + \frac{1}{2})h_m]$, with the Lagrangian coordinates of $\mathbf{b}_{m-1}$	(2.8)
renewal chart	$y \mapsto X_{m-1}(t, y, lh_m)$, inverse to the Lagrangian label y	§7.3
representation	a base plus profiles, each measured at its wavelength	§5.6
reserve	$\mathrm{res}(v)$, which fixes the radii of the domain $\mathcal{D}_v$ of the analytic norm	(11.9)
retained trace	the datum of G' obtained by inverting the truncated exponential	§5.5
seed	a family whose components have generation 0 and marker 1	§5.4
shear displacement	the map $y \mapsto y + (\int_t^{t'} \widetilde{\zeta}_{m,k}) \nabla^\perp \psi_{m,k}(y)$ of (7.13)	§7.3
size	$\mathrm{size}_{\lambda_*}(R)$, which bounds the $C^{0,s}$ norm	(5.13)
source	a coupling term between two components, with a smoothing	§5.4
successor	the family of level $m - 1$ that a cycle produces	Def. 11.3
terminal cycle	the cycle of the entry at level M, which passes to κ_{M-1}	§5.7
terminal level	the level M of the band $\mathcal{K}_M$ that contains κ	(2.18)
terminal lift	the lift of a field of G' with the anchored covariance $\Gamma_M(t) - \Gamma_M(t_{\mathrm{pr}}^{(M)})$	(15.8)
terminal successor	the successor G' of $\widetilde{G}_M$, of level $M - 1$	Def. 15.9
test	a solution of the adjoint fibre equation, paired with Duhamel terms	(12.19)
transport	coefficients carried from y to x (Piola identity, chain rule)	(11.8)
weak family	the family G_{i-1}, $M < i \leq M + n$, made from G_i at the diffusivity κ	Def. 15.3
weak lift	$\mathrm{Corr}_i^{\mathrm{w}}(t) \Upsilon_i^{\mathrm{w}}$, weak readout, then correctors; $\mathcal{W}_i(t)$, the composed lifts	§15.2
weak passage	a step of the entry at a level finer than M, at the diffusivity κ	§5.7
word, letter	the label of a created component; a letter names a source term	§5.4
zone	the frequencies with $\varepsilon_m \lvert\xi\rvert_\infty \leq \pi$, the cell frequencies	§12.2
zone part, F^{lo}	F^{zone}, the part of a field at the frequencies $\varepsilon_m \lvert\xi\rvert < \pi/2$; $\widehat{F^{\mathrm{lo}}} := \mathsf{H}_m^{\mathrm{lo}} \widehat{F}$	§12.9

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