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The Haagerup center at central charge eight via conformal nets

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Abstract

Let $S \subset V_{E_6 \oplus A_2}$ be the unitary vertex operator algebra of central charge eight with currents at levels $(3, 13)$ constructed in our previous work. An exact presentation of a full corner of its Zhu algebra gives strong rationality, the twelve simple modules and the lattice branching rules. Using these results, we identify the representation category of its conformal net with the Haagerup center $Z(\mathcal{H}_3)$. We construct a holomorphic diagonal extension of $\mathcal{A}_S \otimes \mathcal{A}_S$ and apply the vacuum trace class theorem of Henriques and Tener to prove trace class heat operators in every sector of $\mathcal{A}_S$. Reconstruction gives a unitary holomorphic vertex operator algebra containing $S \otimes S$ as mutual commutants. The coset theorems of Gui then give the complete unitarity of S and a braided equivalence $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$, so that S is a Haagerup algebra. Using results of Dong, Jiao and Xu, we also identify the Huang–Riesen algebra with the twin of S and describe their common fixed point algebra using the intersection of the two binary lattices of discriminant -39.

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1 Introduction

Background. The Haagerup subfactor [Haa94, AsH99] is the smallest finite depth subfactor of index greater than four, with index $\frac{5+\sqrt{13}}{2}$, and its even parts $\mathcal{H}_1$, $\mathcal{H}_2$ and the category $\mathcal{H}_3$ of [GrS12, Definition 3.19 p. 23] are the three Morita equivalent Haagerup fusion categories of [GrS12]. Their common Drinfeld center $Z(\mathcal{H}_3)$ is a unitary modular tensor category of rank twelve whose modular data was computed by Izumi [Izu01] and by Evans and Gannon [EvG11], who conjectured that it is realized by a rational conformal field theory of central charge eight with six currents. Explicit candidates were announced in October 2026 by Riesen [Rie26a] and by Huang [Hua26b], with currents $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ at levels $(1, 39)$, and by Jiao [Jia26], with currents at levels $(3, 13)$, all as vertex subalgebras of lattice vertex operator algebras.

In [ArhI] we constructed the algebra $S \subset V' = V_{E_6 \oplus A_2}$ with currents at levels $(3, 13)$ from the weight-one and weight-two data forced by the modular data, we proved that it is simple, unitary, C_2-cofinite, strongly generated in weights at most three, and isomorphic to the algebra of Jiao, and we developed the structure theory of *Haagerup algebras*, the positive strongly rational vertex operator algebras of central charge eight with representation category $Z(\mathcal{H}_3)$: their weight-one Lie algebras, their lowest weight spaces, their Heisenberg commutants and their lattices, and a twin construction exchanging the two binary lattices of discriminant -39. In [ArhII] we studied the conformal net $\mathcal{A}_S$ and used the hypothesis

$$(\mathrm{H_J})\colon S \text{ is a Haagerup algebra},$$

for the results specific to S. Appendix A computes a full corner of Zhu's algebra $A(S)$, proves strong rationality and positivity, determines the twelve simple modules and gives the complete restrictions of the V'-modules. Theorem 9.1 deduces from it an algebraic description of $\mathrm{Rep}(S)$ as the Drinfeld center of a pseudo-unitary fusion category, which is not used in the proof of Theorem 6.1.

The analytic proof. The algebraic inputs are stated in Section 2. The unitary realization of S and its weight-one Lie algebra (I1) are given by [ArhI, Theorems 10.10 and 10.21]. The calculation of Zhu's algebra in Appendix A supplies strong rationality and positivity (F1), the twelve simple modules and their lowest weight data (F2), and the lattice restrictions (F3).

Theorem 3.1 (Theorem B) identifies the completely rational net category $\mathrm{Rep}(\mathcal{A}_S)$ with $Z(\mathcal{H}_3)$ using (I1), (F1) and (F3). For each simple S-summand M of the two lattice extensions of V', [ArhII, Proposition 5.2] gives an invariant scalar product on $\overline{M} \boxtimes M$ for which this fusion product is a transport module in the sense of [Ten24, Definition 3.3 p. 26], and this is the input of the finite-index criterion of Tener. The recognition theorem of [ArhII, Theorem 6.1] then applies to the unitary fusion categories of the net extensions.

Section 4 constructs the holomorphic diagonal net $\mathcal{B}$ with the same stress-energy tensor as the represented product net. The Hamiltonian identity of Proposition 4.4 lets us apply the vacuum trace theorem of Henriques and Tener [HeT26] to $\mathcal{B}$ and deduce trace class for every sector of $\mathcal{A}_S$ by compression. Reconstruction [HeT25] gives a unitary holomorphic vertex operator algebra containing $S \otimes S$, with the prescribed scalar products on both copies of S. Theorem 6.1 applies the coset comparison of Section 5 to obtain $(\mathrm{H_J})$, complete unitarity and the VOA–net equivalence from (I1) and (F1)–(F3). Since (I1) and (F1)–(F3) are established for the explicit algebra S in Theorems 2.1 and 2.2, Theorem 6.1 proves that S is a Haagerup algebra, that S is completely unitary and that $\mathrm{Rep}^u(S) \cong \mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{H}_3)$ as unitary modular tensor categories. This is the main result of the article. Corollary 6.2 identifies the lattice labels across the equivalence and the third holomorphic net extension.

The twin identification. Let $\mathfrak{h} \subset S_1$ be the Cartan subalgebra spanned by the Cartan currents p, b of levels 3 and 13, let $\Omega_S = \mathrm{Com}_S(M(\mathfrak{h}))$ be the Heisenberg commutant, and let L_{II} and L_{I} be the two even lattices of discriminant -39 of [ArhI, Lemma 12.1], with Gram matrices $\left[\begin{smallmatrix} 6 & 3 \\ 3 & 8 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 20 \end{smallmatrix}\right]$. In Section 7 we show that the construction of [ArhI, Section 12] applies to S under (I1) and (F1)–(F4): $S = \bigoplus_\alpha \Omega_S^{\tau(\alpha)} \otimes V_{\alpha+L_{\mathrm{II}}}$ over $\alpha \in L'_{\mathrm{II}}/L_{\mathrm{II}} \cong \mathbb{Z}_{39}$, and exchanging L_{II} for L_{I} gives the twin $S^\natural$. The key observation is that the involutions σ of S and τ of U used in [DJX26] are the characters $\mathrm{e}^{2\pi\mathrm{i}(s,\cdot)}$ and $\mathrm{e}^{2\pi\mathrm{i}(p,\cdot)}$ of L_{II} and L_{I}, where $s = \frac{p+b}{4}$ is the Cartan current of level one, and that both have kernel $N = L_{\mathrm{I}} \cap L_{\mathrm{II}}$, a lattice of discriminant -156 whose discriminant form is the orthogonal sum of the metric group of $Z(\mathrm{Vec}_{\mathbb{Z}_2})$ and of a nondegenerate form on $\mathbb{Z}_{39}$. Hence the common fixed point algebra C of [DJX26, Theorem 7.3] is the extension of $\Omega_S \otimes V_N$ by the thirty-nine simple currents indexed by the 39-part, and S, U are its two extensions by the two bosons of N'/N.

Theorem 1.1 (Theorem A) *Assume* (I1), (I3), (F1)–(F4), (D1) *and* (D2). *Then the algebra U of Huang and Riesen is isomorphic to the twin $S^\natural$ of S. Consequently:*

1. *U is a simple, self-contragredient, positive, strongly rational vertex operator algebra of CFT type and central charge eight, and* $\mathrm{Rep}(U)$ *is ribbon equivalent to* $\mathrm{Rep}(S)$*, a pseudo-unitary modular tensor category of rank twelve.*

2. *The Heisenberg commutant of U is isomorphic to Ω_S, its associated lattice is L_{I}, and $U_1 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ with currents at levels $(1, 39)$. In particular $U \ncong S$.*

3. *S and U belong to the same hyperbolic genus, every strongly rational vertex operator algebra in this genus is isomorphic to S or to U, and S, U are the only positive strongly rational algebras of central charge eight with* Rep *ribbon equivalent to* $\mathrm{Rep}(S)$ *and Heisenberg commutant isomorphic to Ω_S.*

4. *The common fixed point algebra $C = S^\sigma \cong U^\tau$ is strongly rational, and* $\mathrm{Rep}(C) \cong Z(\mathrm{Vec}_{\mathbb{Z}_2}) \boxtimes \mathrm{Rep}(S)$ *has rank* 48.

The lattice exchange of [ArhI, Theorem 12.3] constructs the twin. Theorem A identifies it with the explicit case (I) algebra using (D1) and (D2). It gives a second proof of the strong rationality of U, through the extension theory of [HKL15, CKM22] rather than the rationality criterion of [DJX26, Theorem 3.1], and a proof of $U_1 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ from the root system of the twin, without the weight-one computations of [Hua26b]. It also identifies the algebras themselves, whereas [DJX26, Proposition 9.2] identifies their module categories. The proof uses two uniqueness statements for simple current extensions with prescribed summands [DoM04a, Proposition 5.3], the quantum Galois theory of Dong and Mason [DoM97], the regularity of fixed point subalgebras [CaM16], and the finite identities of [DJX26].

Theorem 1.2 (Theorem D) *Assume* (I1), (I3), (F1)–(F4), (D1) *and* (D2). *Then S and U are Haagerup algebras, every Haagerup algebra is isomorphic to S or to U, the Haagerup algebras form a single hyperbolic genus, every simple S-module is unitarizable, S is completely unitary with* $\mathrm{Rep}^u(S) \cong \mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{H}_3)$ *as unitary modular tensor categories, the third holomorphic extension of $\mathcal{A}_S$ is isomorphic to $\mathcal{A}_{E_8}$, and the twin U is completely unitary with* $\mathrm{Rep}^u(U) \cong \mathrm{Rep}(\mathcal{A}_U) \cong Z(\mathcal{H}_3)$.

Theorem 7.13 gives the detailed consequences in Theorem D. It collects the statements of [ArhI, Theorem 12.4] and [ArhII, Theorems 7.2, 7.17, 8.3 and 8.6, Corollaries 7.19 and 8.5, Proposition 7.10] which become available once ($\mathrm{H_J}$) is known, with Theorem A in place of the

hypothesis ($\mathrm{H_R}$). The assertions of Theorem D on S, on its modules and on the net $\mathcal{A}_S$ only use (I1) and (F1)–(F3), through Theorem 6.1 and Corollary 6.2. The inputs (I3), (F4), (D1) and (D2) enter only through Theorem A, for the assertions on U and on the classification. All these inputs are established for the explicit algebras, in [ArhI], in Appendix A and in [DJX26], see Section 8.2, so that Theorems A and D hold for S and U without further hypothesis.

The algebraic comparison and the supporting calculations. Section 9 collects the pseudo-unitary center description (F4), the algebraic construction from a braid-reversing equivalence, the compatible unitary structure found by exact lattice calculation and the equivalent formulations of Theorem 9.14 (Theorem C). Section 8 compares the results with those of Dong, Jiao and Xu and verifies the finite identities used for the twin theorem. Section 10 indexes the certificates. Appendix A contains the proofs of the algebraic inputs.

Conventions. Vertex operator algebras are over $\mathbb{C}$, of CFT type unless stated otherwise, and strongly rational means simple, rational, C_2-cofinite, self-contragredient and of CFT type, so that the module category is a modular tensor category [Hua08]. Positive means that every simple module other than the algebra has positive lowest conformal weight. We write $\mathrm{Rep}(V)$ for the category of ordinary modules, $M \boxtimes N$ for the fusion product, $\mathcal{C}^{\mathrm{rev}}$ for a braided category with the inverse braiding, $\overline{\mathcal{C}}$ for the complex conjugate category, and $Z(\mathcal{F})$ for the Drinfeld center. The labels $0, b, a, c_0, c_1, c_2, d_1, \dots, d_6$ of the simple objects of $Z(\mathcal{H}_3)$ and of the simple S-modules are those of [EvG11] and [ArhI, Table 2], with $d = \frac{3+\sqrt{13}}{2}$. The two simple S-modules labelled a and c_0 in (F2) have the same lowest weight and the same lowest weight space, and they are not distinguished by the computation of Appendix A. Before Section 6, only the unordered pair $\{M_a, M_{c_0}\}$ is used, and the intrinsic labels are fixed in Section 6, after Theorem 6.1. Page references to arXiv preprints refer to the versions indicated in the bibliography.

2 Inputs and their status

We state the lattice and representation-theoretic results used in the analytic argument. The proofs for the Zhu algebra and the lattice branching are in Appendix A. The vertex operator algebra S is defined in [ArhI, Theorems 10.9 and 10.21] as the vertex subalgebra of $V' = V_{E_6 \oplus A_2}$ generated by a Lie algebra $U_1 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ of currents at levels $(3, 13)$ and by an explicit 120-dimensional space $U_2 \subset V_2'$ defined over $\mathbb{K} = \mathrm{Q}(\sqrt{13}, \sqrt{-3})$. We write $\mathfrak{g}' \oplus \mathfrak{g}''$ for S_1, $A = L_{\mathfrak{sl}_2}(3,0) \otimes L_{\mathfrak{sl}_2}(13,0)$ for the affine subalgebra generated by S_1, which is simple since it is a unitary subalgebra of (S, H_χ) by [ArhI, Theorem 10.10] and [CKLW18, Propositions 5.3 and 5.23 pp. 34 and 43], and W_g, $g \in \mathbb{Z}_3 \times \mathbb{Z}_3$, for the nine simple V'-modules, with $W_{(0,0)} = V'$.

2.1 The unitary lattice realization of S

Theorem 2.1 ([ArhI]) (I1) *S is a simple unitary vertex operator algebra of CFT type and central charge eight, self-contragredient, with the conformal vector of V', and S is a unitary subalgebra of V' for the scalar product H_χ and its associated PCT operator θ [ArhI, Theorem 10.10]. Moreover $S_1 = U_1$, so that $\dim S_1 = 6$ [ArhI, Theorem 10.21].*

(I2) *S is strongly generated in weights at most three, $\dim S_2 = 120$, $\dim S_3 = 956$, $\dim S_4 = 6010$, and S is C_2-cofinite [ArhI, Proposition 10.17 and Theorem 10.21]. The weight-four certificate condition called (F4) in [ArhI, Definition 10.16] is unrelated to the input (F4) of Theorem 9.1.*

These statements are computer-assisted, with exact certificates over $\mathbb{K}$ [ArhI, Appendix A]. They make no hypothesis on the representation category of S.

The strong locality theorem and the submodule lemma of [ArhII, Theorem 4.5 and Lemma 4.7] apply to the unitary conformal embedding in (I1). We use the general transport and recognition results of [ArhII, Proposition 5.2 and Theorem 6.1] in Section 3.

2.2 Rationality, simple modules and lattice branching

Theorem 2.2 (F1) *The vertex operator algebra S is strongly rational and every simple S-module other than S has positive lowest conformal weight. Its Zhu algebra is semisimple.*

(F2) *S has exactly twelve simple modules M_z up to isomorphism, labelled by $z \in \{0, b, a, c_0, c_1, c_2, d_1, \dots, d_6\}$ with $M_0 = S$. Its Zhu algebra has dimension 36625. The lowest weight spaces of the simple modules R_z, as modules over S_1, are the twelve spaces of [ArhI, Table 2], and their lowest conformal weights are*

$$h_0 = 0, \quad h_b = h_a = h_{c_0} = 1, \quad h_{c_1} = \tfrac{1}{3}, \quad h_{c_2} = \tfrac{2}{3},$$

$$(h_{d_1}, \dots, h_{d_6}) = \left(\tfrac{6}{13}, \tfrac{11}{13}, \tfrac{15}{13}, \tfrac{5}{13}, \tfrac{7}{13}, \tfrac{8}{13}\right).$$

(F3) *$V'|_S \cong S \oplus M_b$, $W_{(0,\pm1)}|_S \cong M_{c_1}$, $W_{(\pm1,0)}|_S \cong M_{c_2}$, and the four modules W_g with both coordinates of g nonzero restrict to simple S-modules $X_+ = W_{(1,1)}|_S \cong W_{(-1,-1)}|_S$ and $X_- = W_{(1,-1)}|_S \cong W_{(-1,1)}|_S$ of lowest weight one, with $\{X_+, X_-\} = \{M_a, M_{c_0}\}$. The two holomorphic lattice extensions $H_\pm \cong V_{E_8}$ of V' restrict to S as $H_\pm|_S \cong S \oplus M_b \oplus 2X_\pm$, and we put $L_\pm = H_\pm|_S$.*

Proof : The semisimplicity and rationality in (F1) are Theorem A.13 and Corollary A.14, and positivity follows from Proposition A.15. Theorem A.20, Table 1 and Corollary A.21 give (F2). Propositions A.16 and A.18 give the restrictions in (F3), and the two lattice decompositions in Section A.7.2 give the restrictions of $H_\pm$. ■

Recall that ($\mathrm{H_J}$) is the hypothesis of [ArhI, Section 1] that S is a Haagerup algebra, that is, S is positive, strongly rational and $\mathrm{Rep}(S)$ is braided equivalent to $Z(\mathcal{H}_3)$. By (F1), it is equivalent here to the braided equivalence $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$. The vertex operator algebra S satisfies (I1) and (F1)–(F3) by Theorems 2.1 and 2.2. In the statements below, these labels only indicate which of these properties are used in the proof.

2.3 Scope of the algebraic inputs

Appendix A derives exact relations in a full corner of Zhu's algebra $A(S)$ from products of vertex operators in V' and deduces that this corner, of dimension 762, is semisimple with twelve simple blocks. Hence $A(S)$ is semisimple of dimension 36625. The computations are listed in Section A.8. The proofs of Theorem 3.1 and of Proposition 4.8 use only (I1), (F1) and (F3). The count of simple modules and their lowest weights in (F2) are used from Lemma 4.13 on.

2.4 Additional inputs for the twin identification

The twin theorem uses the algebraic center description (F4) of Theorem 9.1 and the additional inputs (I3), (D1) and (D2). The finite identities for (D1) and (D2) are verified independently in Section 8.2.

(I3). S is isomorphic to the vertex operator algebra constructed by Jiao [Jia26], through an explicit isomorphism which identifies the generators [ArhI, Theorem 10.24].

Let U^{R} and U^{H} be the vertex subalgebras of V' generated by the explicit vectors of Riesen [Rie26a] and of Huang [Hua26b], with currents of levels $(1, 39)$, and let V be the algebra of Jiao, so that $V \cong S$ by (I3).

Theorem 2.3 ([DJX26]) (D1) *U^{R} and U^{H} are isomorphic [DJX26, Section 4.5 and Proposition 4.1]. We write U for either.*

(D2) *Let $\tau = \exp(\frac{\pi \mathrm{i}}{2}(3H_L + H_R)_{(0)})$ and $\sigma = \exp(\frac{\pi \mathrm{i}}{2}(P_H + B_H)_{(0)})$, where H_L, H_R and P_H, B_H are the Cartan currents of U and of V. These are involutions of U and of V, and there is an isomorphism of vertex operator algebras $\Phi\colon U^\tau \to V^\sigma$ induced by an isometry of lattices, which maps Cartan currents to Cartan currents [DJX26, Theorem 7.3].*

The other results of [DJX26], on the strong rationality of U, V, U^τ and on their module categories and conformal nets, are discussed in Section 8. We do not use any statement of [Jia26, Rie26a, Hua26b] beyond the definitions of the explicit vertex operator algebras.

3 The conformal net of S

For the net of S, the inputs are the unitary conformal embedding and weight-one data (I1), strong rationality (F1) and the branching of the lattice modules (F3). The proof uses [ArhII, Proposition 5.2] on the lattice summands to apply Tener's finite-index criterion, and [ArhII, Theorem 6.1] to recognize the fusion categories of the net extensions.

3.1 Setting

We use the explicit model of $V' = V_{E_6 \oplus A_2}$ of [ArhI, Section 10.1] and the scalar product H_χ of [ArhI, Theorem 10.10], for which S is a unitary subalgebra of V' in the sense of [CKLW18, Definition 5.22], containing the conformal vector ω of V'. Let $\Lambda_\pm = \langle (1, \pm 1) \rangle \subset L^\circ / L \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ be the two isotropic lines of the discriminant form of $L = E_6 \oplus A_2$, and let $H_\pm = H_{\Lambda_\pm} \cong V_{E_8}$ be the two lattice extensions of V' [ArhII, Lemma 4.1], equipped with the scalar products $\{\cdot|\cdot\}_{\Lambda_\pm}$ of [ArhII, Lemma 4.2], which extend H_χ. Let $\mathcal{A}_S \subset \mathcal{A}_{V'} \subset \mathcal{A}_{H_\pm} \cong \mathcal{A}_{E_8}$ be the conformal nets of [CKLW18]. By [ArhII, Theorem 4.5 and Proposition 4.3], whose proofs make no hypothesis on S beyond (I1), these are covariant subnets with trivial cosets and irreducible inclusions, and $[\mathcal{A}_{H_\pm} : \mathcal{A}_{V'}] = 3$. For an S-submodule M of $H_\pm$, we write π_M for the representation of $\mathcal{A}_S$ on the closure of M given by [ArhII, Lemma 4.7], and we put $\pi_0 = \pi_S$, $\pi_b = \pi_{N_b}$ where N_b is the orthogonal complement of S in V', and $\pi_\pm = \pi_{W_{(1,\pm 1)}}$, where $W_{(1,\pm 1)} \subset H_\pm$. Let $\Theta_\pm$ be the dual canonical endomorphism of $\mathcal{A}_S \subset \mathcal{A}_{H_\pm}$, let $\theta_{V'}$ be that of $\mathcal{A}_S \subset \mathcal{A}_{V'}$, and let $\mathcal{F}_\pm^{\mathrm{net}} = \mathrm{Rep}(\mathcal{A}_S)_{\Theta_\pm}$ be the category of modules over $\Theta_\pm$ in $\mathrm{Rep}(\mathcal{A}_S)$. We label $Z(\mathcal{H}_3)$ as in [EvG11], see [ArhII, Section 2.2], and we write $L_2 = 0 \oplus b \oplus 2a$ and $L_3 = 0 \oplus b \oplus 2c_0$ for the Lagrangian algebras of [ArhII, Proposition 3.3 and Corollary 3.8], with $Z(\mathcal{H}_3)_{L_i} \cong \mathcal{H}_i$.

3.2 Statement

Theorem 3.1 (Theorem B) *Assume* (I1), (F1) *and* (F3).

1. *The conformal net $\mathcal{A}_S$ is completely rational. The inclusions $\mathcal{A}_S \subset \mathcal{A}_{V'}$ and $\mathcal{A}_S \subset \mathcal{A}_{H_\pm}$ have finite index. The vacuum representation of $\mathcal{A}_{V'}$ restricts to $\mathcal{A}_S$ as $\pi_0 \oplus \pi_b$, and that of $\mathcal{A}_{H_\pm}$ as $\pi_0 \oplus \pi_b \oplus 2\pi_\pm$, where π_0, π_b, π_+, π_- are pairwise inequivalent irreducible*

representations with statistical dimensions $d(\pi_0) = 1$, $d(\pi_b) = d^2 = 3d + 1$ *and* $d(\pi_\pm) = 3d + 2$. *Moreover*

$$[\mathcal{A}_{V'} : \mathcal{A}_S] = 3d + 2, \quad [\mathcal{A}_{H_\pm} : \mathcal{A}_S] = 3(3d + 2), \quad \mu_{\mathcal{A}_S} = 9(3d + 2)^2.$$

2. *For each sign,* $\Theta_\pm$ *is a Lagrangian algebra of* $\mathrm{Rep}(\mathcal{A}_S)$*, the category* $\mathcal{F}_\pm^{\mathrm{net}}$ *is a unitary fusion category of rank six with Frobenius–Perron dimension* $3(3d+2)$*,* $\mathrm{Rep}(\mathcal{A}_S)$ *is braided equivalent to* $Z(\mathcal{F}_\pm^{\mathrm{net}})$*, and* $\mathcal{F}_\pm^{\mathrm{net}}$ *is equivalent to* $\mathcal{H}_2$ *or to* $\mathcal{H}_3$*. Its pointed part, which corresponds to the intermediate net* $\mathcal{A}_{V'}$*, is* $\mathrm{Vec}_{\mathbb{Z}_3}$ *with trivial associator.*

3. $\mathrm{Rep}(\mathcal{A}_S)$ *is equivalent to* $Z(\mathcal{H}_3)$ *as a unitary modular tensor category. A braided equivalence* $\Psi\colon \mathrm{Rep}(\mathcal{A}_S) \to Z(\mathcal{H}_3)$ *is unique up to natural isomorphism, and it satisfies* $\Psi(\pi_b) \cong b$ *and* $\{\Psi(\pi_+), \Psi(\pi_-)\} = \{a, c_0\}$*. Writing* π_a^{net} *and* $\pi_{c_0}^{\mathrm{net}}$ *for the representations among* $\pi_\pm$ *sent to* a *and* c_0*, and* Θ_a^{net}*,* $\Theta_{c_0}^{\mathrm{net}}$ *for the corresponding algebras* $\Theta_\pm$*, we have* $\Psi(\Theta_a^{\mathrm{net}}) \cong L_2$*,* $\Psi(\Theta_{c_0}^{\mathrm{net}}) \cong L_3$*,* $\mathcal{F}_a^{\mathrm{net}} \cong \mathcal{H}_2$ *and* $\mathcal{F}_{c_0}^{\mathrm{net}} \cong \mathcal{H}_3$*.*

4. *Up to isomorphism of extensions,* $\mathcal{A}_S$ *has exactly three irreducible local extensions* $\mathcal{B}$ *with* $\mu_{\mathcal{B}} = 1$*. Their dual canonical endomorphisms are* $\Theta_1 \cong \pi_0 \oplus \pi_b \oplus \pi_+ \oplus \pi_-$*,* Θ_a^{net} *and* $\Theta_{c_0}^{\mathrm{net}}$*, with* $\mathrm{Rep}(\mathcal{A}_S)_{\Theta_1} \cong \mathcal{H}_1$*, and the extensions attached to* $\Theta_\pm$ *are the nets* $\mathcal{A}_{H_\pm} \cong \mathcal{A}_{E_8}$*. The only intermediate nets between* $\mathcal{A}_S$ *and* $\mathcal{A}_{H_\pm}$ *are* $\mathcal{A}_S$*,* $\mathcal{A}_{V'}$ *and* $\mathcal{A}_{H_\pm}$*, and there is no intermediate net strictly between* $\mathcal{A}_S$ *and the extension attached to* Θ_1*. The Haagerup subfactor arises from* $\mathcal{A}_S$ *in the sense of [Bis17a, Definition 4.1]. There is a proper action of the hypergroup* $K = \{k_0, k_1\}$*,* $k_1k_1 = d^{-2}k_0 + (1 - d^{-2})k_1$*, on* $\mathcal{A}_{V'} = \mathcal{A}_{E_6,1} \otimes \mathcal{A}_{SU(3),1}$ *with* $\mathcal{A}_S = \mathcal{A}_{V'}^K$*, of weight* $3d + 2$ *and parameter* $w = d^2$*, which solves [Bis17b, Problem 4.27]. Finally* $\mathrm{Rep}(\mathcal{A}_S)_{\theta_{V'}}$ *is a near-group fusion category of type* $\mathbb{Z}_3^2 + 9$*.*

The net argument alone does not identify the third holomorphic extension with $\mathcal{A}_{E_8}$. That identification follows from Theorem 6.1, see Corollary 6.2(2). The labels a and c_0 of part 3 are defined on the net side and are not compared with the labels of the simple S-modules $X_\pm$ of (F3), see Remark 3.3.

3.3 Proof of Theorem 3.1

The proof follows [ArhII, Sections 5–7]. We indicate, for each step, which statement of [ArhII] is used, where its proof uses the Haagerup hypothesis, and by what this hypothesis is replaced. Two conventions of Tener are used: a VOA with bounded localized vertex operators, in the sense of [Ten19, Definition 4.12 p. 42], and, for a unitary module M of such a VOA, the representation π^M of [Ten19, Definition 5.1 p. 49] and [Ten24, Definition 2.18 p. 20], when it exists.

Step 0: inputs from [ArhII, Section 4]. Let H be one of $H_\pm$. By [ArhII, Proposition 4.3(1)], H is strongly local, has bounded localized vertex operators, its nets in the sense of [Ten19] and of [CKLW18] are isomorphic, and $\mathcal{A}_H \cong \mathcal{A}_{E_8}$ is completely rational with $\mu_{\mathcal{A}_H} = 1$ [Ten19, Theorem 7.4 p. 71, Proposition 7.5 and Corollary 7.7 p. 72], [DoX06, Corollary 3.19]. By (I1) and the proof of [ArhII, Theorem 4.5], S is a unitary subalgebra of H in the sense of [Ten19, Definition 2.3 p. 12], since the PCT operator of H restricts to that of V', which preserves S, and since $L(1)S \subset S$, and S is a conformal subalgebra with trivial coset [Ten19, Proposition 2.7], [CKLW18, Remark 5.30]. By [Ten19, Proposition 4.16 p. 43], S has bounded localized vertex operators. By (F1) and [ABD04, Theorem 4.5 p. 9], S is regular. By regularity and the finite-dimensional homogeneous spaces of H for the common conformal vector, H is a finite direct sum of simple S-modules, which we choose orthogonal for its invariant scalar product.

Step 1: the representations π^M*.* Let M be an S-submodule of H, with the restricted scalar product. By [Ten19, Theorem 5.6 p. 52], applied to $V = H$, to the unitary subalgebra S and

to the unitary H-module H itself, whose representation π^H is the vacuum representation, the representation π^M exists. Since the PCT operator θ_H preserves S, the subspace $\theta_H(M)$ is an S-submodule of H, unitarily isomorphic to the conjugate module $\overline{M}$, so that $\pi^{\overline{M}}$ exists.

Step 2: the transport module for $(\overline{M}, M)$. Let M be a simple S-summand of H. By Step 0 and Step 1, S is a simple regular unitary vertex operator algebra with bounded localized vertex operators, and the unitary S-modules $\overline{M}$ and M satisfy the hypotheses of [ArhII, Proposition 5.2], with $\overline{M}$ and M in the roles of M and N there, since π^M exists. Hence $\overline{M} \boxtimes M$ admits an invariant scalar product for which $(\overline{M} \boxtimes M, \mathcal{Y}_{\boxtimes})$ is a transport module for $\overline{M}$ and M in the sense of [Ten24, Definition 3.3 p. 26]. Since the invariant scalar product of M identifies $\overline{M}$ with the contragredient module M', rigidity gives $\mathrm{Hom}_S(S, \overline{M} \boxtimes M) \cong \mathrm{End}_S(M) \neq 0$, so that $\overline{M} \boxtimes M$ contains the vacuum module S. This construction uses neither the modules M_{d_l} nor $(\mathrm{H_J})$.

Step 3: finite index and complete rationality. By skew symmetry in H, the projection $p_M Y^H(m,z)s$ onto M, for $m \in M$ and $s \in S$, is $\mathcal{Y}_M^+(m,z)s$. The common conformal vector makes the power shift in [Ten19, Theorem 5.15] zero, so $\mathcal{Y}_M^+ \in \mathcal{I}_{\mathrm{loc}}$. Steps 1 and 2 provide the representations π^M and $\pi^{\overline{M}}$ and the transport module $\overline{M} \boxtimes M$, which contains S. Thus [Ten24, Theorem 4.19 p. 55] applies directly and gives finite index for each π^M.

Write π_H for the restriction of the vacuum of $\mathcal{A}_H^{\mathrm{T}}$ to $\mathcal{A}_S^{\mathrm{T}}$, with the superscript denoting the nets in the sense of [Ten19]. By the finite decomposition in Step 0, π_H is a finite sum of the sectors just obtained and has finite index. For a complementary interval I', locality gives

$$\pi_H(\mathcal{A}_S^{\mathrm{T}}(I)) \subset \mathcal{A}_H^{\mathrm{T}}(I) \subset \pi_H(\mathcal{A}_S^{\mathrm{T}}(I'))'.$$

The exterior inclusion of factors has finite index by the index of π_H. A finite-index conditional expectation from $\pi_H(\mathcal{A}_S^{\mathrm{T}}(I'))'$ onto $\pi_H(\mathcal{A}_S^{\mathrm{T}}(I))$ restricts to a conditional expectation from $\mathcal{A}_H^{\mathrm{T}}(I)$ onto the same lower factor. The restriction has finite index by [BDH88, Remark 3.13(i) p. 227]. This gives finite index for the subnet inclusion, as in the last step of the proof of [Ten24, Corollary 4.24]. Since the net of H is completely rational, [Lon03, Theorem 24 p. 20] gives complete rationality of $\mathcal{A}_S^{\mathrm{T}}$, and [Ten19, Lemma 7.6 p. 72] identifies it with $\mathcal{A}_S$. By [ArhII, Theorem 4.5], $\mathcal{A}_{V'}$ and $\mathcal{A}_H$ are irreducible local extensions of $\mathcal{A}_S$, so $[\mathcal{A}_{V'} : \mathcal{A}_S]$ and $[\mathcal{A}_H : \mathcal{A}_S]$ are finite by [ArhII, Proposition 2.5(2)]. The specific branching in (F3) enters in Step 4.

Step 4: decompositions. By (F3) and the orthogonality argument of [ArhII, proof of Theorem 5.6(2)], the four indicated S-summands of H form an orthogonal direct sum. By [ArhII, Lemma 4.7(1)], the vacuum representation of $\mathcal{A}_{H_\pm}$ restricts to $\mathcal{A}_S$ as $\pi_S \oplus \pi_{N_b} \oplus \pi_{W_{(1,\pm 1)}} \oplus \pi_{W_{(-1,\mp 1)}}$, and that of $\mathcal{A}_{V'}$ as $\pi_S \oplus \pi_{N_b}$. By (F3) and [ArhII, Lemma 4.7(2),(3)], these representations are irreducible, $\pi_{W_{(-1,\mp 1)}} \cong \pi_{W_{(1,\pm 1)}} = \pi_\pm$, and π_0, π_b, $\pi_\pm$ are pairwise inequivalent. Applying [ArhII, Lemma 4.7(3)] to H_+ and H_-, which contain S as a unitary subalgebra with the same conformal vector and with scalar products both restricting to H_χ, and using $X_+ \not\cong X_-$ from (F3), we obtain $\pi_+ \not\cong \pi_-$. This is [ArhII, Theorem 5.6(2)], with (F3) in place of [ArhII, Lemma 5.4], which was the only use of $(\mathrm{H_J})$ there.

Step 5: indices. Put $\delta = d(\pi_b)$. By [ArhII, Proposition 2.5(1),(3)] and Step 4, $\theta_{V'} = \pi_0 \oplus \pi_b$, so that $[\mathcal{A}_{V'} : \mathcal{A}_S] = 1+\delta$ and $\mu_{\mathcal{A}_S} = (1+\delta)^2 \mu_{\mathcal{A}_{V'}} = 9(1+\delta)^2 = [\mathcal{A}_{H_\pm} : \mathcal{A}_S]^2$ [ArhII, Proposition 4.3(2)], [KLM01, Proposition 24]. Hence $[\mathcal{A}_{H_\pm} : \mathcal{A}_S] = d(\Theta_\pm) = 1 + \delta + 2d(\pi_\pm) = 3(1+\delta)$ and $d(\pi_\pm) = 1 + \delta$. This is [ArhII, Theorem 5.6(3)], whose proof is unchanged.

Step 6: the Lagrangian algebras $\Theta_\pm$. By [ArhII, Proposition 2.5(3)] and the irreducibility of $\mathcal{A}_S \subset \mathcal{A}_{H_\pm}$, $\Theta_\pm$ is a commutative Q-system with $\dim \mathrm{Hom}(\mathrm{id}, \Theta_\pm) = 1$, hence a connected étale algebra in the unitary modular tensor category $\mathrm{Rep}(\mathcal{A}_S)$ [KLM01, Theorem 33 and Corollary 39]. Since $\mathrm{FPdim}\,\mathrm{Rep}(\mathcal{A}_S) = \mu_{\mathcal{A}_S} = d(\Theta_\pm)^2$ [KLM01, Theorem 33], $\Theta_\pm$ is Lagrangian [DMNO13, Definition 4.6], and $\mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{F}_\pm^{\mathrm{net}})$ as braided fusion categories [DMNO13, Corollary 4.1], with $\mathrm{FPdim}\,\mathcal{F}_\pm^{\mathrm{net}} = 3(1+\delta)$ [DMNO13, Lemma 3.11]. The category $\mathcal{F}_\pm^{\mathrm{net}}$ is the

category of modules over the Q-system $\Theta_\pm$ in the C^*-tensor category $\mathrm{Rep}(\mathcal{A}_S)$, equivalently the category of α-induced sectors of $\mathcal{A}_S(I) \subset \mathcal{A}_{H_\pm}(I)$, a unitary fusion category [BKLR15, Section 4.6], [Bis17a, Proposition 4.5]. Its rank is $\dim \mathrm{End}(\Theta_\pm) = 1+1+4 = 6$ by [ArhII, Lemma 3.1] and Step 4. The intermediate net $\mathcal{A}_{V'}$ gives a connected étale subalgebra $\theta_{V'} \subset \Theta_\pm$ [BKLR15, Proposition 4.11], which corresponds by [ArhII, Lemma 3.1] and [DMNO13, Theorem 4.10] to a fusion subcategory $\mathcal{P}_\pm$ of $\mathcal{F}_\pm^{\mathrm{net}}$ with $\mathrm{FPdim}\,\mathcal{P}_\pm = 3$, which is pointed [ENO05, Corollary 8.30]. This is [ArhII, Theorem 5.6(4)], whose proof is unchanged.

Step 7: $\delta > 1$ and the three invertible objects. Suppose that $\delta = 1$. By [ArhII, Proposition 5.7], which uses only Steps 3–5 and [Bis17b, Theorem 4.22 and Section 4.6], we have $\mathcal{A}_S = \mathcal{A}_{V'}^K$ for a hypergroup $K = \{k_0, k_1\}$ with $k_1 k_1 = k_0$, and the argument of [ArhII, Proposition 5.8], based on the Kadison–Schwarz inequality, [Bis17b, Proposition 4.21] and [CKLW18, Theorem 6.9, Theorem 7.5 and Proposition 7.6], produces a unitary automorphism g of V' of order two with $S = (V')^g$. Then $S_1 = (V_1')^g$ is the fixed point subalgebra of a nontrivial involution of $\mathfrak{e}_6 \oplus \mathfrak{sl}_3$ preserving both ideals, so that $\dim S_1 \geqslant 39$ by the classification of involutions of simple Lie algebras [Hel78, Chapter X], which contradicts $\dim S_1 = 6$. Hence $\delta > 1$. Since $\mathcal{F}_\pm^{\mathrm{net}}$ has rank six and contains the pointed subcategory $\mathcal{P}_\pm$ of order three, its group of invertible objects has order 3 or 6, and order 6 would make $\mathcal{F}_\pm^{\mathrm{net}}$ pointed with $\mathrm{FPdim} = 6 = 3(1+\delta)$, that is, $\delta = 1$. Hence $\mathcal{F}_\pm^{\mathrm{net}}$ has exactly three invertible simple objects, which generate $\mathcal{P}_\pm$. This is [ArhII, Proposition 5.8], with $\dim S_1 = 6$ in place of the weight-one classification of [ArhI] for Haagerup algebras.

Step 8: trivial associator of $\mathcal{P}_\pm$. The proof of [ArhII, Lemma 6.10] applies verbatim: it uses Step 6, [DMNO13, Corollary 4.1 and Theorem 4.10], [ArhII, Proposition 2.2(5)], [BKL15, Proposition 6.4], the two Lagrangian algebras of $\mathrm{Rep}(\mathcal{A}_{V'})$ given by the lattice nets $\mathcal{A}_{H_\pm}$ [ArhII, Proposition 4.3(2) and Theorem 4.5], which are not isomorphic because their restrictions to $\mathcal{A}_S$ are $\pi_0 \oplus \pi_b \oplus 2\pi_+$ and $\pi_0 \oplus \pi_b \oplus 2\pi_-$ with $\pi_+ \ncong \pi_-$ by Step 4, [DMNO13, Proposition 4.8] and [ArhII, Lemma 6.7(2)]. Hence $\mathcal{P}_\pm \cong \mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator.

Step 9: recognition. By Steps 6 and 7, $\mathcal{F}_\pm^{\mathrm{net}}$ is a unitary fusion category with six simple objects, three of which are invertible. By Step 4, $\mathrm{End}(\Theta_\pm) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$, and since $\Theta_\pm$ corresponds to the induction $I(1)$ [ArhII, Lemma 3.1] and $K(\mathcal{F}_\pm^{\mathrm{net}}) \otimes \mathbb{C} \cong \mathrm{End}(I(1))$ [Ost15, Corollary 2.16], the Grothendieck ring of $\mathcal{F}_\pm^{\mathrm{net}}$ is noncommutative. By Step 8 and [ArhII, Theorem 6.1], a categorical statement whose part 1 uses unitarity only through the spherical structure with positive dimensions and whose part 2 uses the C^*-structure through [Izu18, Theorems 5.9 and 9.1], the category $\mathcal{F}_\pm^{\mathrm{net}}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$. Comparing $\mathrm{FPdim}\,\mathcal{F}_\pm^{\mathrm{net}} = 3(1+\delta)$ with $\dim \mathcal{H}_i = 3(3d+2)$ gives $\delta = 3d+1 = d^2$, which completes the proof of parts 1 and 2. By Step 6 and [ENO11, Theorem 3.1], $\mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{F}_\pm^{\mathrm{net}}) \cong Z(\mathcal{H}_3)$ as braided fusion categories, and such an equivalence is naturally isomorphic to a unitary braided equivalence [Reu23, Theorem 5]. This recovers the conclusion of [ArhII, Theorem 6.11] for S from (I1), (F1) and (F3).

Step 10: labels. Let $\Psi\colon \mathrm{Rep}(\mathcal{A}_S) \to Z(\mathcal{H}_3)$ be a braided equivalence. It is unique up to natural isomorphism, since the Brauer–Picard group of $\mathcal{H}_3$ is trivial [GrS12, Theorem 1.5], [ENO10, Theorem 1.1]. The first part of the proof of [ArhII, Lemma 7.1] applies: $d(\pi_b) = 3d+1$ forces $\Psi(\pi_b) \cong b$ by the dimensions of $Z(\mathcal{H}_3)$, $\Psi(\Theta_\pm) \cong 0 \oplus b \oplus 2\Psi(\pi_\pm)$ is a Lagrangian algebra of $Z(\mathcal{H}_3)$, hence $\Psi(\pi_\pm) \in \{a, c_0\}$ by [ArhII, Lemma 3.2], and $\Psi(\pi_+) \ncong \Psi(\pi_-)$ by Step 4. By [ArhII, Proposition 3.3] and the first two paragraphs of the proof of [ArhII, Proposition 3.6], which are categorical, L_2 and L_3 are the only Lagrangian algebras of $Z(\mathcal{H}_3)$ with these objects, and $Z(\mathcal{H}_3)_{L_i} \cong \mathcal{H}_i$. This proves part 3.

Step 11: consequences. The first assertions of part 4 are [ArhII, Theorem 7.2(2),(3)], with Step 10 in place of [ArhII, Lemma 7.1] and without [ArhII, Proposition 3.5], which is not needed: the irreducible local extensions of $\mathcal{A}_S$ with $\mu = 1$ correspond to the Lagrangian algebras of

$\mathrm{Rep}(\mathcal{A}_S)$ [ArhII, Proposition 2.5(1),(2)], [LoR95, Theorem 4.9], [CGGH23, Theorems 3.2 and 3.9], of which there are exactly three by Step 10 and [ArhII, Proposition 3.3], with the stated objects. The algebras $\Theta_\pm$ are two of them by Step 6, their extensions are $\mathcal{A}_{H_\pm} \cong \mathcal{A}_{E_8}$ [ArhII, Theorem 4.5], and the intermediate nets correspond to the fusion subcategories of $\mathcal{H}_i$ through [BKLR15, Proposition 4.11], [ArhII, Lemma 3.1] and [DMNO13, Theorem 4.10]. The assertion on the Haagerup subfactor is [ArhII, Corollary 7.4]: $\mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{H}_3)$ is the quantum double of the Haagerup subfactor and [Bis17a, Proposition 4.7(1)] applies. The hypergroup assertion is [ArhII, Proposition 7.5 and Corollary 7.6] with $\delta = d^2$ from Step 9 and [ArhII, Proposition 4.3(2)] for $\mathcal{A}_{V'} \cong \mathcal{A}_{E_6,1} \otimes \mathcal{A}_{SU(3),1}$. The last assertion is [ArhII, Remark 7.8], a computation in $Z(\mathcal{H}_3)$ with the étale algebra $0 \oplus b$, which uses Step 9 and the fusion rules of $Z(\mathcal{H}_3)$. ∎

Remark 3.2 The extension $\mathcal{B}_1$ attached to Θ_1 is holomorphic of central charge eight. Theorem 3.1 identifies its module category over the inclusion, but does not itself classify the vacuum net. Theorem 6.1 subsequently supplies complete unitarity and the VOA–net comparison. Then [ArhII, Proposition 7.10] identifies $\mathcal{B}_1$ with $\mathcal{A}_{E_8}$, as in Corollary 6.2(2). This uses the classification of the corresponding holomorphic VOA, not an assumed classification of all holomorphic nets at charge eight.

Remark 3.3 The labels a and c_0 of part 3 are intrinsic to the net through Ψ. The signs in $X_\pm$ refer to the lattice coordinates of (F3). Corollary 6.2(1) compares the intrinsic labels of the modules and sectors.

4 The diagonal holomorphic extension from the conformal net

Theorem 3.1 identifies the representation category of $\mathcal{A}_S$ with $Z(\mathcal{H}_3)$ without assuming $(\mathrm{H_J})$. We construct the diagonal holomorphic extension on this net side, prove the trace class property of every sector using [HeT26], and reconstruct a unitary holomorphic vertex operator algebra containing $S \otimes S$. The compatible unitary structure comes from the inclusion of nets. This supplies an analytic alternative to the lattice computation of [ArhII, Section 7.5].

We use the notation of Section 3. The twelve irreducible representations of $\mathcal{A}_S$ are labelled π_z, $z \in \{0, b, a, c_0, c_1, c_2, d_1, \ldots, d_6\}$, through $\Psi(\pi_z) \cong z$, so that $\pi_a = \pi_a^{\mathrm{net}}$ and $\pi_{c_0} = \pi_{c_0}^{\mathrm{net}}$. We write $\mathcal{A}_2 \stackrel{\mathrm{def}}{=} \mathcal{A}_S \otimes \mathcal{A}_S$ for the tensor product net on $\mathcal{H}_S \otimes \mathcal{H}_S$, which is the net of the strongly local vertex operator algebra $(S \otimes S, H_\chi \otimes H_\chi)$ [CKLW18, Corollary 8.2 p. 62]. For a representation (K, π) of a conformal net, the local unitaries $\pi_I(U(\gamma))$, $\gamma \in \mathrm{Diff}(I)$, define a projective representation of the universal cover of $\mathrm{Diff}(S^1)$ on K [HeT25, Section 2 p. 7–8], [Hen19, Theorem 11], whose generator of rotations L_0 is positive [Wei06, Proposition 3.8], and $K^{\mathrm{f.e.}}$ denotes the algebraic direct sum of the eigenspaces of L_0.

4.1 The braid-reversing equivalence of $Z(\mathcal{H}_3)$ and the Lagrangian algebra Θ_Δ

Lemma 4.1 *Assume* (I1), (F1), (F3). *There is a braided equivalence σ from* $\mathrm{Rep}(\mathcal{A}_S)^{\mathrm{rev}}$ *to* $\mathrm{Rep}(\mathcal{A}_S)$, *unique up to natural isomorphism, with $\sigma \circ \sigma^{\mathrm{rev}} \cong \mathrm{id}$. It fixes π_0, π_b, π_a, π_{c_0}, exchanges π_{c_1} and π_{c_2}, and maps π_{d_l} to $\pi_{d_{l'}}$ with $l' \equiv \pm 5l \bmod 13$, that is, $d_1 \leftrightarrow d_5$, $d_2 \leftrightarrow d_3$, $d_4 \leftrightarrow d_6$.*

Proof : The first half of the proof of [ArhII, Proposition 7.12(1)] concerns $Z(\mathcal{H}_3)$ alone: for a unitary fusion category $\mathcal{C}$, inverting half-braidings gives $Z(\mathcal{C}^{\otimes \mathrm{op}}) \cong Z(\mathcal{C})^{\mathrm{rev}}$ and $f \mapsto f^*$

composed with duality gives $\mathcal{C}^{\otimes\mathrm{op}} \cong \overline{\mathcal{C}}$, while the even parts of the Haagerup subfactor satisfy $\overline{\mathcal{H}_i} \cong \mathcal{H}_i$ [JMS14, Theorem 4.1 p. 21], since the conjugate planar algebra has the same principal graphs. With $Z(\mathcal{H}_1) \cong Z(\mathcal{H}_3)$ [ENO11, Theorem 3.1 p. 9], this gives a braided equivalence $R\colon Z(\mathcal{H}_3)^{\mathrm{rev}} \to Z(\mathcal{H}_3)$, and $\sigma = \Psi^{-1} \circ R \circ \Psi^{\mathrm{rev}}$ with Ψ from Theorem 3.1. Uniqueness and $\sigma \circ \sigma^{\mathrm{rev}} \cong \mathrm{id}$ follow from the triviality of the Brauer–Picard group of $\mathcal{H}_3$ [GrS12, Theorem 1.5 p. 5], [ENO10, Theorem 1.1 p. 3]. The action on the labels is computed in the second half of the proof of [ArhII, Proposition 7.12(1)], which uses only the dimensions, twists and self-duality of the simple objects of $Z(\mathcal{H}_3)$ and its Lagrangian algebras [ArhII, Proposition 3.3]. ∎

Lemma 4.2 *Let $\mathcal{A}$ be a completely rational conformal net. Then $\mathcal{A} \otimes \mathcal{A}$ is completely rational, $\mu_{\mathcal{A}\otimes\mathcal{A}} = \mu_{\mathcal{A}}^2$, and $\rho \boxtimes \rho' \mapsto \rho \otimes \rho'$, $t \boxtimes t' \mapsto t \otimes t'$, is a braided equivalence of unitary modular tensor categories* $\mathrm{Rep}(\mathcal{A}) \boxtimes \mathrm{Rep}(\mathcal{A}) \to \mathrm{Rep}(\mathcal{A} \otimes \mathcal{A})$.

Proof : Complete rationality is [Lon03, Lemma 25 p. 21]. For irreducible localized endomorphisms ρ_i, ρ_i', the endomorphism $\rho_1 \otimes \rho_2$ is irreducible, since $(\rho_1 \otimes \rho_2)(\mathcal{A} \otimes \mathcal{A})'' = \mathrm{B}(\mathcal{H}_{\rho_1}) \overline{\otimes} \mathrm{B}(\mathcal{H}_{\rho_2})$, and $\rho_1 \otimes \rho_2 \cong \rho_1' \otimes \rho_2'$ forces $\rho_i \cong \rho_i'$, by restriction to both $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{A}$: each restriction is an amplification of one irreducible factor, and inequivalent irreducible factor representations are disjoint. Every factor representation of $\mathcal{A}$ is of type I [KLM01, Corollary 14 p. 14], so every irreducible localized endomorphism of $\mathcal{A} \otimes \mathcal{A}$ is of the form $\rho_1 \otimes \rho_2$ [KLM01, Lemma 27 p. 22], and every finite-index representation is a finite sum of irreducibles [KLM01, Corollary 39 p. 27]. The functor is thus an equivalence of semisimple categories, strictly monoidal, and braided, since the DHR braiding of $\rho_1 \otimes \rho_2$ and $\rho_1' \otimes \rho_2'$ is computed with charge transporters $u \otimes u'$ and equals $\varepsilon(\rho_1, \rho_1') \otimes \varepsilon(\rho_2, \rho_2')$. Finally $\mu_{\mathcal{A}\otimes\mathcal{A}} = \sum d(\rho_1)^2 d(\rho_2)^2 = \mu_{\mathcal{A}}^2$ [KLM01, Theorem 33 p. 24]. ∎

Lemma 4.3 *Let $\mathcal{C}$ be a non-degenerate braided fusion category and $\sigma\colon \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}$ a braided equivalence. Then $\Theta_\Delta \stackrel{\mathrm{def}}{=} \bigoplus_{z \in \mathrm{Irr}(\mathcal{C})} z \boxtimes \sigma(z^*)$ carries the structure of a connected étale Lagrangian algebra of $\mathcal{C} \boxtimes \mathcal{C}$. For $\mathcal{C} = \mathrm{Rep}(\mathcal{A}_S)$ and σ as in Lemma 4.1, the object $\Theta_\Delta \cong \bigoplus_z \pi_z \boxtimes \sigma(\pi_z)$ of $\mathrm{Rep}(\mathcal{A}_S) \boxtimes \mathrm{Rep}(\mathcal{A}_S) \cong \mathrm{Rep}(\mathcal{A}_2)$ is a connected étale Lagrangian algebra with twelve pairwise non-isomorphic simple summands and* $\mathrm{FPdim}\,\Theta_\Delta = \mu_{\mathcal{A}_S} = 9(3d+2)^2$.

Proof : Let $G\colon \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \to Z(\mathcal{C})$ be the braided equivalence of [DMNO13, (11) p. 5] and F its composition with the forgetful functor, a central functor with $F(X \boxtimes Y) \cong X \otimes Y$. Its right adjoint I makes $I(1)$ a connected étale algebra [DMNO13, Lemma 3.5 p. 8], with $\mathrm{Hom}(x \boxtimes y, I(1)) \cong \mathrm{Hom}_{\mathcal{C}}(x, y^*)$ for simple x, y, so $I(1) \cong \bigoplus_z z \boxtimes z^*$ and $\mathrm{FPdim}\, I(1) = \mathrm{FPdim}\,\mathcal{C} = (\mathrm{FPdim}(\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}))^{1/2}$ [DMNO13, (8) p. 5]: $I(1)$ is Lagrangian [DMNO13, Definition 4.6 p. 17]. The braided equivalence $\mathrm{id} \boxtimes \sigma$ preserves connectedness, commutativity, separability [DMNO13, Remark 2.6(ii) p. 5] and dimensions, and maps $I(1)$ to an algebra with underlying object Θ_Δ. For $\mathrm{Rep}(\mathcal{A}_S)$, the simple objects of $Z(\mathcal{H}_3)$ are self-dual [ArhII, Section 2.2], $z \mapsto (z, \sigma(z))$ is injective, Lemma 4.2 applies, and $\mathrm{FPdim}\,\Theta_\Delta = \sum_z d(\pi_z)^2 = \mu_{\mathcal{A}_S}$ [KLM01, Theorem 33 p. 24]. ∎

4.2 The holomorphic net $\mathcal{B}$

Proposition 4.4 *Assume* (I1), (F1) *and* (F3). *The diagonal algebra of Lemma 4.3, with its specified multiplication, determines an irreducible local extension $\mathcal{B}$ of $\mathcal{A}_2$, unique up to isomorphism for this algebra. Its vacuum representation restricts as*

$$\mathcal{H}_{\mathcal{B}} = \bigoplus_z \mathcal{H}_{\pi_z} \otimes \mathcal{H}_{\sigma(\pi_z)}. \tag{4.1}$$

The extension is completely rational, $[\mathcal{B} : \mathcal{A}_2] = \mu_{\mathcal{A}_S}$, $\mathcal{A}_2(I)' \cap \mathcal{B}(I) = \mathbb{C}$ *and* $\mu_{\mathcal{B}} = 1$. *It is diffeomorphism covariant of central charge* 16*, with local implementing unitaries in the represented subnet* $\mathcal{A}_2(I)$. *Its conformal Hamiltonian satisfies*

$$L_0^{\mathcal{B}}|_{\mathcal{H}_{\pi_z} \otimes \mathcal{H}_{\sigma(\pi_z)}} = \overline{L_0^{\pi_z} \otimes 1 + 1 \otimes L_0^{\sigma(\pi_z)}}. \tag{4.2}$$

The closure in this formula is the nonnegative self-adjoint tensor sum. No discreteness or finite multiplicity assumption on the sector Hamiltonians is used.

Proof : Transport the canonical diagonal algebra $I(1)$ by $\mathrm{id} \boxtimes \sigma$ as in Lemma 4.3. Its algebra structure, not merely its underlying object, is part of this construction. By [CGGH23, Theorems 3.2 and 3.9], it is isomorphic to a normalized commutative haploid Q-system in $\mathrm{Rep}(\mathcal{A}_2)$. Its twist is the identity: the canonical positive spherical structures are preserved by σ, and the twists of π_z and $\sigma(\pi_z)$ are inverse. Equivalently one can use [CGGH23, Proposition 2.22].

The extension theorem of [GuiF25, Theorems 2.6, p. 11, and 5.7, p. 26], in the Connes fusion formulation of [LoR95, Theorem 4.9], now applies. Here the algebra of Lemma 4.3, defined in the DHR category $\mathrm{Rep}(\mathcal{A}_2)$, is transported to the category of $\mathcal{A}_2$-modules with Connes fusion through the equivalence of [Gui21, Theorem 6.6 p. 77], which is braided or braid-reversing (Remark 5.2), and commutativity and haploidy are preserved in both cases. The smaller net $\mathcal{A}_2$ is an irreducible conformal net, the Q-system is commutative and haploid and its twist is trivial. The result gives an irreducible local conformal extension on the representation space of the Q-system, with vacuum vector given by its unit. Its restriction is (4.1). The vacuum is cyclic, and the extension has a genuine positive-energy Möbius representation and a projective diffeomorphism representation. The index is the dimension of the Q-system, namely $\mu_{\mathcal{A}_S}$. By [Lon03, Theorem 24] and [KLM01, Proposition 24], the extension is completely rational and $\mu_{\mathcal{B}} = \mu_{\mathcal{A}_2}/[\mathcal{B} : \mathcal{A}_2]^2 = 1$.

In [GuiF25, Theorem 2.6], covariance of the extension is implemented by the canonical covariance on the $\mathcal{A}_2$-module of the Q-system. In particular its local diffeomorphism implementers are $\pi_I(U_{\mathcal{A}_2}(\gamma))$ up to phase for $\gamma \in \mathrm{Diff}(I)$. These implementers give the same projective representation of the covering diffeomorphism group, since the local diffeomorphisms generate it projectively. On the universal covering Möbius group the genuine lifts coincide. Indeed two such lifts of the same projective action differ by a continuous character, and the connected covering group has perfect Lie algebra. Thus the rotation generator has the same normalization as in π. The local stress tensor is the represented stress tensor of $S \otimes S$, so the Virasoro action has central charge 16 and is the same in the extension and in π. For a diffeomorphism fixing an interval pointwise, the implementer on the vacuum subnet belongs projectively to the algebra of the complementary interval by locality and Haag duality, in the closed-interval convention of [HeT25, Section 2 p. 7]. Its image under π commutes with the extension algebra of the original interval by locality. This gives the local triviality in [HeT26, Definition 2.3]. The local conformal extension in [GuiF25, Theorem 5.7] has a cyclic vacuum and a unique Möbius invariant vector.

Each summand $\mathcal{H}_{\pi_z} \otimes \mathcal{H}_{\sigma(\pi_z)}$ is invariant under the Möbius representation of π, which commutes with $\pi(\mathcal{A}_2)'$. On this summand, both this representation and the tensor product of the Möbius representations of π_z and $\sigma(\pi_z)$ are continuous unitary representations of the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$ implementing the Möbius covariance of the irreducible representation $\pi_z \otimes \sigma(\pi_z)$ of $\mathcal{A}_2$. Hence they differ by a continuous character of this group, which is trivial since its Lie algebra is perfect, as in the previous paragraph. Thus the two representations coincide. Stone's theorem and the joint spectral calculus for the two commuting one-parameter groups of rotations then give (4.2) and show that the summand projections reduce $L_0^{\mathcal{B}}$. Moreover $\mathrm{e}^{2\pi \mathrm{i} L_0^{\mathcal{B}}} = 1$. Indeed, by (4.2), on each summand this operator is the

product of the scalars $e^{2\pi i L_0^{\pi_z}}$ and $e^{2\pi i L_0^{\sigma(\pi_z)}}$, which are, for a fixed orientation of the DHR braiding, the twists of π_z and of $\sigma(\pi_z)$ by the conformal spin and statistics theorem [GuL96], and these twists are inverse to each other. Hence the projective representation of the universal cover of $\mathrm{Diff}(S^1)$ on $\mathcal{H}_{\mathcal{B}}$ is a projective representation of $\mathrm{Diff}(S^1)$. ∎

Remark 4.5 Diffeomorphism covariance of $\mathcal{B}$ can also be obtained from [Car04, Proposition 3.7 p. 19], which applies to every local extension of a Virasoro net in the sense of [Car04, Definition 3.1 p. 14], that is, to every irreducible local Möbius covariant net containing a Virasoro subnet such that the von Neumann algebra generated by its local algebras contains the Möbius unitaries, and requires neither finite index nor irreducibility of the inclusion. A third proof uses the structure $\mathcal{B}(I) = \mathcal{A}_2(I)v$ given by a Q-system localized in a subinterval [Lon03, Lemma 15 p. 14] and the localization of charged intertwiners. The same proposition is used in [RTT22, proof of Proposition 4.8 p. 28] for the nets of AQFT-local vertex operator algebras.

4.3 The trace class theorem and the sector Hamiltonians

The definition of a conformal net in [HeT26, Definition 2.3, pp. 7–8] requires isotony, locality, a strongly continuous projective diffeomorphism representation extending a genuine Möbius representation, local triviality of diffeomorphisms, nonnegative energy, a cyclic vacuum and uniqueness of the Möbius invariant vector. It does not assume finite-dimensional energy eigenspaces. Proposition 4.4 verifies these requirements for $\mathcal{B}$. The proof of [HeT26, Theorem 4.1] uses the split property, which holds for $\mathcal{B}$ since $\mathcal{B}$ is completely rational.

Theorem 4.6 (Henriques–Tener, [HeT26, Theorem 4.1, pp. 22–23]) *For a conformal net satisfying these axioms, its vacuum Hamiltonian satisfies* $\mathrm{Tr}(e^{-\beta L_0}) < \infty$ *for every* $\beta > 0$.

This result concerns vacuum representations. Its use for the nonvacuum sectors of $\mathcal{A}_S$ requires the extension $\mathcal{B}$ and the following elementary observation.

Lemma 4.7 *Let A and B be nonzero bounded positive operators on nonzero Hilbert spaces. If $A \otimes B$ is trace class, then A and B are trace class. Consequently, if H_1, H_2 are nonnegative self-adjoint operators and $e^{-\beta\overline{H_1 \otimes 1 + 1 \otimes H_2}}$ is trace class, then both $e^{-\beta H_i}$ are trace class.*

Proof : Choose a unit vector η with $\langle B\eta, \eta\rangle > 0$ and let $J_\eta \xi = \xi \otimes \eta$. Then $J_\eta^*(A \otimes B)J_\eta = \langle B\eta, \eta\rangle A$. The trace class operators form a two-sided ideal, so A is trace class. Interchanging the factors proves the assertion for B. For the last assertion, the joint spectral calculus gives $e^{-\beta\overline{H_1 \otimes 1 + 1 \otimes H_2}} = e^{-\beta H_1} \otimes e^{-\beta H_2}$, and each factor is nonzero. ∎

Proposition 4.8 *Assume* (I1), (F1) *and* (F3). *For every irreducible representation π of $\mathcal{A}_S$ and every $\beta > 0$, $\mathrm{Tr}(e^{-\beta L_0^\pi}) < \infty$. In particular L_0^π has compact resolvent, pure point spectrum and finite-dimensional eigenspaces.*

Proof : Theorem 3.1 and Proposition 4.4 construct $\mathcal{B}$ without a spectral finiteness assumption. Apply Theorem 4.6 to its vacuum representation. Each summand projection in (4.1) reduces $L_0^{\mathcal{B}}$, and the compression of its heat operator to that summand is trace class. By (4.2) and functional calculus, this compression is $e^{-\beta L_0^{\pi_z}} \otimes e^{-\beta L_0^{\sigma(\pi_z)}}$. Lemma 4.7 gives the assertion for every z. Finally a compact heat operator bounds the rank of every bounded spectral interval of its generator. Since the heat operator is injective with dense range, its spectral decomposition gives a complete eigenbasis for the generator, with finite multiplicities and eigenvalues tending to infinity. This also gives compact resolvent. ∎

Remark 4.9 Theorem 4.11 below only requires that L_0^π has discrete spectrum and finite-dimensional eigenspaces, which is weaker than the trace class property of Proposition 4.8. Theorem 4.6 concerns vacuum representations, and we do not apply it to the representations π_z themselves. The proof of Proposition 4.8 uses the diffeomorphism covariance of $\mathcal{B}$ and the identity (4.2), but neither (F2) nor (F4).

We now use the reconstruction results of [HeT25]. Their finite-energy hypotheses hold for $\mathcal{A}_S$, $\mathcal{A}_2$, $\mathcal{B}$ and the representations considered below by Proposition 4.8 and Theorem 4.6.

Theorem 4.10 ([HeT25]) *Let $\mathcal{A}$ be a conformal net. The space $V_\mathcal{A}$ carries a natural structure of unitary vertex operator algebra [HeT25, Theorems A and B p. 3, Theorems 7.11 p. 59 and 7.13 p. 61], with vacuum Ω, conformal vector $\nu = L_{-2}\Omega$ whose field is the stress-energy tensor of $\mathcal{A}$, the scalar product of $\mathcal{H}_0$, and PCT operator Θ the modular conjugation of $(\mathcal{A}(I_-), \Omega)$ for the lower half circle I_- [HeT25, (34) p. 23]. For $v \in V_\mathcal{A}$ and $f \in C^\infty(S^1)$ supported in I, the smeared field $Y(v, f)$, the closure of $\int_{S^1} f(z)Y(z^{L_0}v, z)\frac{\mathrm{d}z}{2\pi \mathrm{i}z}$ on $V_\mathcal{A}$ [HeT25, Definition 8.2 p. 62], is affiliated with $\mathcal{A}(I)$ [HeT25, Proposition 8.8 p. 65], and these operators generate $\mathcal{A}(I)$ [HeT25, Theorem 8.9(i) p. 66]. Conversely, let V be a unitary vertex operator algebra which is AQFT-local, that is, $Y(v_1, f_1)$ and $Y(v_2, f_2)$ commute strongly whenever f_1, f_2 have disjoint supports [HeT25, Definition 8.3 p. 62]. Then $\mathcal{A}_V(I) = W^*\{Y(v, f) : \operatorname{supp} f \subset I\}$ is a conformal net on the completion of V, and $V_{\mathcal{A}_V} = V$ as unitary vertex operator algebras [HeT25, Theorem 8.9 p. 66].*

The last assertion is [HeT25, Theorem 8.9(ii)], which rests on [RTT22, Proposition 4.6 p. 27]. It gives the same space, scalar product and vertex operator algebra structure, and the PCT operator is then determined by the scalar product [CKLW18, Proposition 5.1 p. 33].

Theorem 4.11 ([HeT25]) *Let $\mathcal{A}$ be a conformal net and (K, π) a representation of $\mathcal{A}$ in which L_0 has discrete spectrum and finite-dimensional eigenspaces. Then $M = K^{\mathrm{f.e.}}$ carries a unique structure of unitary $V_\mathcal{A}$-module Y^M [HeT25, Theorem C p. 5, Proposition 9.3 p. 68, Theorem 9.5 p. 70, Corollaries 9.6 and 9.7 p. 72] such that*

$$\pi_I(Y(v, f)) = Y^M(v, f) \tag{4.3}$$

for all $v \in V_\mathcal{A}$ and f supported in an interval I, where $Y^M(v, f)$ is the closure on M of the smeared module field, defined from Y^M as in Theorem 4.10, and $\pi_I(Y(v, f))$ is the image of the affiliated operator $Y(v, f)$ under π_I. Both sides have M as a core. The L_0-grading of M is the grading by the zero mode of $Y^M(\nu, z)$, the scalar product of K is invariant, and for $0 < r < |z| < R$ and $a \in M$,

$$Y^M(v, z)a = R^{L_0}\mathbb{A}^K_{R,r}[v(z)]r^{-L_0}a, \tag{4.4}$$

where $\mathbb{A}^K_{R,r}[v(z)]$ is the bounded operator on K attached by [HeT25, (73) p. 43] to the annulus $r \leqslant |w| \leqslant R$ with the insertion of v at z. For $K = \mathcal{H}_0$, $M = V_\mathcal{A}$ and $Y^M = Y$.

By Proposition 4.8, Theorem 4.11 applies to every irreducible representation of $\mathcal{A}_S$, and to every finite direct sum of tensor products of such representations as a representation of $\mathcal{A}_2$.

Lemma 4.12 *Assume* (I1), (F1). *Let (K, π) be a representation of $\mathcal{A}_S$ in which L_0 has discrete spectrum and finite-dimensional eigenspaces, and let $M = M(\pi) \stackrel{\mathrm{def}}{=} K^{\mathrm{f.e.}}$, with the unitary module structure of Theorem 4.11 over $V_{\mathcal{A}_S}$, which is (S, H_χ) by Lemma 4.15 below.*

1. *If $N \subset M$ is an S-submodule, the orthogonal projection P onto its closure lies in $\pi(\mathcal{A}_S)'$. If π is irreducible, then M is a simple S-module.*

2. *If M is a simple S-module, then π is irreducible.*

3. *If (K', π') is another such representation and the modules $M(\pi)$ and $M(\pi')$ are simple and isomorphic, then $\pi \cong \pi'$.*

4. *If (K', π') is another such representation and $\pi \cong \pi'$, then the S-modules $M(\pi)$ and $M(\pi')$ are isomorphic.*

Proof : 1. The submodule N is stable under $L_0 = \nu^M_{(1)}$, so $N = \bigoplus_h N(h)$ with $N(h) = N \cap M(h)$ finite-dimensional, P maps $M(h)$ onto $N(h)$, and $M = N \oplus (N^\perp \cap M)$. The invariance of the scalar product implies that the algebraic adjoint of each mode is a finite linear combination of modes of $L_1^j \theta v$. Since N is a submodule, its graded orthogonal complement is stable under every mode as well. Consequently P commutes with every mode on M. For a test function f supported in I, the convergent mode expansion gives $PXa = XPa$ on M, where $X = \pi_I(Y(v, f))$ and M is a core by Theorem 4.11. Passing to graph limits gives $P\,\mathrm{Dom}(X) \subset \mathrm{Dom}(X)$ and $XP = PX$ there. The same holds for $1 - P$. Thus X is affiliated with $\{P\}'$. Since π_I is implemented by a unitary [HeT25, p. 70] and the $Y(v, f)$ generate $\mathcal{A}_S(I)$ (Theorem 4.10), the operators X generate $\pi_I(\mathcal{A}_S(I))$, and $P \in \pi(\mathcal{A}_S)'$. If π is irreducible then $P \in \{0, 1\}$, and $P = 1$ gives $N(h) = M(h)$ for all h by finite-dimensionality.

2. A projection $T \in \pi(\mathcal{A}_S)'$ commutes with the local unitaries $\pi_I(U(\gamma))$, hence with the representation of the universal cover of $\mathrm{Diff}(S^1)$ on K and with L_0, so T preserves M and each $M(h)$. From $TXa = XTa$ for $a \in M$ and all f supported in I, comparing homogeneous components gives $Tv^M_m a = v^M_m Ta$ for all modes. Thus $T|_M$ is an endomorphism of the simple module M, whose lowest weight space is finite-dimensional, so $T|_M$ is a scalar and $T \in \{0, 1\}$.

3. Let $\phi\colon M(\pi) \to M(\pi')$ be an isomorphism. The form $\langle \phi a, \phi b\rangle$ is an invariant scalar product on the simple unitary module $M(\pi)$ for the same θ, so it equals $\langle Ta, b\rangle$ with T positive, graded and commuting with all modes, hence a positive scalar. Thus ϕ is a positive multiple of a unitary $u\colon M(\pi) \to M(\pi')$, which extends to a unitary $K \to K'$ intertwining the smeared module fields on the cores, hence $u\pi_I(Y(v, f))u^* = \pi'_I(Y(v, f))$. The normal representations $\mathrm{Ad}\, u \circ \pi_I$ and π'_I of $\mathcal{A}_S(I)$ are implemented by unitaries uU_I and U'_I from $\mathcal{H}_S$, and $U'^*_I uU_I$ fixes every $Y(v, f)$, $\mathrm{supp}\, f \subset I$, so it lies in $\mathcal{A}_S(I)'$ and $\mathrm{Ad}\, u \circ \pi_I = \pi'_I$. Hence $\pi \cong \pi'$.

4. Let $w\colon K \to K'$ be a unitary intertwiner. Then $w\pi_I(U(\gamma))w^* = \pi'_I(U(\gamma))$ for $\gamma \in \mathrm{Diff}(I)$, so that w intertwines the representations of the universal cover of $\mathrm{Diff}(S^1)$, hence the operators L_0, and $wM(\pi) = M(\pi')$. Moreover $w\pi_I(Y(v, f))w^* = \pi'_I(Y(v, f))$, so that w maps $Y^{M(\pi)}(v, f)$ onto $Y^{M(\pi')}(v, f)$ by (4.3). Comparing homogeneous components as in part 2, $w|_{M(\pi)}$ intertwines all the modes, and it is an isomorphism of S-modules. ∎

Lemma 4.13 *Assume* (I1) *and* (F1)–(F3). *For $M \in \{S, N_b, W_{(1,\pm 1)}\}$, the representation π_M of [ArhII, Lemma 4.7] satisfies $M(\pi_M) = M$, so that $M(\pi_0) = S$, $M(\pi_b) = M_b$, $M(\pi_\pm) = X_\pm$. For $g \in \{(0, \pm 1), (\pm 1, 0)\}$, let π_{W_g} be the representation of $\mathcal{A}_{V'}$ on the closure of the unitary V'-module W_g with $\pi_{W_g,I}(Y(v, f)) = Y_{W_g}(v, f)$ [Gui25, Definition 4.22 p. 43 and Example 6.19(b) p. 71], [CGGH23, Corollary 4.11 p. 21], and let π_{c_1}, respectively π_{c_2}, be its restriction to $\mathcal{A}_S$ for $g \in \{(0, \pm 1)\}$, respectively $g \in \{(\pm 1, 0)\}$. Then π_{c_1} and π_{c_2} are irreducible, inequivalent to each other and to π_0, π_b, $\pi_\pm$, their L_0 have discrete spectrum with finite multiplicities, $M(\pi_{c_j}) = M_{c_j}$, and $\Psi(\pi_{c_j}) \cong c_j$, so that the notation agrees with the labelling of the introduction of this section.*

Proof : Since S contains the conformal vector of $H_\pm$, the local implementers of diffeomorphisms of $\mathcal{A}_{H_\pm}$ lie in $\mathcal{A}_S$, so that the conformal Hamiltonian of π_M is the restriction of that of $H_\pm$

and the finite-energy space of π_M is M. By [ArhII, Lemma 4.7(1)] and its proof, π_M maps $Y(w,f)|_{\mathcal{H}_S}$ to the closure of the smeared module field of M with core M, which is $Y^M(w,f)$, so $M(\pi_M) = M$ by the uniqueness in Theorem 4.11. Since S is a unitary subalgebra of V', the restriction of π_{W_g} to $\mathcal{A}_S(I) \subset \mathcal{A}_{V'}(I)$ maps $Y(s,f)$, $s \in S$, to $Y_{W_g}(s,f)$, the smeared field of the S-module $W_g|_S$, so $M(\pi_{c_j}) = W_g|_S \cong M_{c_j}$ by (F3), a simple module, and the L_0 of π_{W_g} is that of the lattice module W_g. Irreducibility and the inequivalences follow from Lemma 4.12(2),(4), since S, M_b, $X_\pm$, M_{c_1}, M_{c_2} are pairwise non-isomorphic by (F2), (F3). For the orientation of the DHR braiding of Section 3 for which the conformal spin and statistics theorem [GuL96] gives the twist $\mathrm{e}^{2\pi\mathrm{i}L_0}$ of a sector, the twist of π_{c_j} is $\mathrm{e}^{2\pi\mathrm{i}h_{c_j}}$ (we use this orientation for Ψ and for the labels π_z of this section, which is allowed since Theorem 3.1 holds for both orientations by Lemma 4.27), braided equivalences between unitary modular tensor categories preserve the canonical spherical structures [ENO05, Proposition 8.23 p. 34], hence the twists, and c_j is the only simple object of $Z(\mathcal{H}_3)$ with twist $\mathrm{e}^{2\pi\mathrm{i}h_{c_j}}$ [ArhII, Section 2.2]. ∎

Corollary 4.14 *Assume* (I1), (F1)–(F3). *The map* $\tau\colon \pi \mapsto [M(\pi)]$ *is a bijection from the equivalence classes of irreducible representations of* $\mathcal{A}_S$ *onto the isomorphism classes of simple* S*-modules, with* $\tau(\pi_0) = S$, $\tau(\pi_b) = M_b$, $\tau(\pi_\pm) = X_\pm$, $\tau(\pi_{c_j}) = M_{c_j}$, *and it maps* $\{\pi_{d_1}, \dots, \pi_{d_6}\}$ *onto* $\{M_{d_1}, \dots, M_{d_6}\}$. *Every simple* S*-module admits an invariant scalar product for* θ.

Proof : By Proposition 4.8 and Lemma 4.12(1),(3),(4), τ is a well-defined injection from the twelve classes of Theorem 3.1 into the twelve classes of (F2), hence a bijection. The values on the six listed representations are Lemma 4.13, and the unitarity is that of the modules $M(\pi)$ in Theorem 4.11. ∎

4.4 The vertex operator algebra of $\mathcal{B}$ and the identification of $S \otimes S$

Lemma 4.15 *Let* V *be a strongly local simple unitary vertex operator algebra [CKLW18, Definition 6.7 p. 53]. Then* V *is AQFT-local, the nets of [CKLW18] and [HeT25] coincide, and* $V_{\mathcal{A}_V} = V$ *as unitary vertex operator algebras. In particular* $V_{\mathcal{A}_S} = (S, H_\chi)$ *and* $V_{\mathcal{A}_2} = (S \otimes S, H_\chi \otimes H_\chi)$, *with PCT operators* θ *and* $\theta \otimes \theta$.

Proof : The smeared fields of [CKLW18, (106) p. 49] and [HeT25, Definition 8.2] agree on V. For a vector a of weight d_a, the smearing of $Y(z^{L_0}a, z)$ is $\sum_n \hat{f}(n) a_{(n+d_a-1)}$. Since V is strongly local, it is energy-bounded, so that there are $k \in \mathbb{N}$ and $C > 0$ with $\|Y(a,f)\xi\| \leqslant C \left\|(1+L_0)^k \xi\right\|$ for any $\xi \in \mathcal{H}^\infty$. Let P_n be the spectral projection of L_0 for the interval $[0,n]$. For any $\xi \in \mathcal{H}^\infty$, we have $P_n\xi \in V$, $P_n\xi \to \xi$ and $(1+L_0)^k P_n\xi \to (1+L_0)^k\xi$, hence $Y(a,f)P_n\xi \to Y(a,f)\xi$. Thus V is a core for the closure of $Y(a,f)$ on $\mathcal{H}^\infty$, and the two closures coincide.

Let f_1, f_2 have disjoint supports. Write $f_1 = \sum_k f_1^{(k)}$ and $f_2 = \sum_l f_2^{(l)}$ with supports compactly contained in intervals J_k and J_l', respectively, with $J_k \cap J_l' = \emptyset$ for every k, l. Put $N_1 = \bigvee_k \mathcal{A}_V(J_k)$ and $N_2 = \bigvee_l \mathcal{A}_V(J_l')$. For $x \in N_1'$, [CKLW18, Lemma 6.5] gives the weak commutation identity $(x^*b \mid Y(a, f_1^{(k)})c) = (Y(a, f_1^{(k)})^*b \mid xc)$ for $b, c \in V$. Linearity of smearing and the adjoint formula on V [CKLW18, (95) and (113)] let us add these identities for f_1. For small rotations $|t| < \delta$, each rotated support remains in J_k, and the same summed identity holds for the rotated test function. These are the identities needed for the smoothing in the proof of [CKLW18, Lemma 6.5]. Convolving $\mathrm{Ad}(\mathrm{e}^{\mathrm{i}tL_0})(x)$ against a smooth approximate identity supported in $(-\delta, \delta)$ gives commutation on the finite-energy core for the smoothed operators. Passing to the limit using closedness gives $xY(a,f_1) \subset Y(a,f_1)x$. The argument for x^* gives affiliation to N_1. The corresponding argument gives affiliation to N_2 for the second field. Strong locality makes N_1 and N_2 commute.

Thus V is AQFT-local. For a closed interval in the convention of [HeT25], fields with support in the interior generate the [CKLW18] algebra. Fields supported in the closed interval commute strongly with fields supported in the interior of its complement, so Haag duality places them in affiliation with the same algebra. Hence the two nets agree. By [HeT25, Theorem 8.9(ii) p. 66], reconstruction recovers V. Apply this to S using [ArhII, Theorem 4.5] and to $S\otimes S$ using [CKLW18, Corollary 8.2 p. 62]. The tensor product PCT is $\theta\otimes\theta$ by [CKLW18, Proposition 5.1 p. 33]. ∎

From now on we assume (I1) and (F1)–(F3), and $\mathcal{B}$ is the net of Proposition 4.4. By Proposition 4.4 and Theorem 4.6, $\mathcal{B}$ is a conformal net in the sense of [HeT25, Definition 2.1 p. 6], and, since $L_0^{\pi} = L_0^{\mathcal{B}}$ by Proposition 4.4, the representation π of $\mathcal{A}_2$ in (4.1) satisfies the hypothesis of Theorem 4.11. Put $W \stackrel{\text{def}}{=} V_{\mathcal{B}} = \mathcal{H}_{\mathcal{B}}^{\text{f.e.}}$, a unitary vertex operator algebra by Theorem 4.10, with vacuum $\Omega_{\mathcal{B}}$, conformal vector $\nu_W = L_{-2}\Omega_{\mathcal{B}}$ and PCT operator Θ_W, and let Y^M be the unitary $S\otimes S$-module structure on $M = W$ given by Theorem 4.11 for π. Let $\mathcal{H}_0' = \mathcal{H}_{\pi_0}\otimes\mathcal{H}_{\pi_0}$, the closure of $\mathcal{A}_2(I)\Omega_{\mathcal{B}}$, and let $u\colon \mathcal{H}_S\otimes\mathcal{H}_S \to \mathcal{H}_0'$ be the unitary intertwiner of representations of $\mathcal{A}_2$ with $u(\Omega\otimes\Omega) = \Omega_{\mathcal{B}}$, unique since the vacuum representation is irreducible and occurs once in (4.1). It intertwines the Möbius representations and, by Proposition 4.4, the stress-energy tensors. Put $W_0 \stackrel{\text{def}}{=} (\mathcal{H}_0')^{\text{f.e.}} = u(S\otimes S) \subset W$.

Lemma 4.16 *For $v \in S\otimes S$ and $a \in W$, $Y^W(uv,z)a = Y^M(v,z)a$.*

Proof : In [HeT25, Sections 5 and 6], a lifted annulus $\mathbb{A}$ with disjoint intervals I_j in its interior and $x_j \in \mathcal{A}(I_j)$ defines a worm operator $\mathbb{A}[x_1\dots x_n]$ on any representation K of $\mathcal{A}$, the composition $\mathbb{A}_{n+1}x_n\cdots\mathbb{A}_2x_1\mathbb{A}_1$ of [HeT25, (45) p. 26], where the lifted annuli $\mathbb{A}_j$ act on K through the Virasoro representation of K and the x_j through the representation of $\mathcal{A}$ and changes of parametrization. For a finite-energy vector v inserted at z, one chooses a disc D around z in the annulus, intervals I_1, I_2 covering ∂D and $x_i \in \mathcal{A}(I_i)$ with $|v(z)\rangle_D = x_1\Omega_D + x_2\Omega_D$ [HeT25, Lemma 6.1 p. 39 and Corollary 6.6 p. 42], and puts $\mathbb{A}[v(z)] = \mathbb{A}[x_1] + \mathbb{A}[x_2]$ [HeT25, (73) p. 43], independently of all choices [HeT25, Lemma 6.9 p. 43].

Apply this to $\mathcal{A}_2$ and $\mathcal{B}$ on the common space $\mathcal{H}_{\mathcal{B}}$, writing $\pi(x) \in \mathcal{A}_2(I) \subset \mathcal{B}(I)$ for the image of $x \in \mathcal{A}_2(I)$. Use the same lift of the annulus, the same cuts, and the same boundary parametrizations in both constructions. By Proposition 4.4, the Virasoro representation on $\mathcal{H}_{\mathcal{B}}$ is the one for π, so the lifted annuli act by identical operators by [HeT25, (23)]. Changes of local parametrization act on local operators by conjugation, in which the phase of the implementer cancels. Hence, for $x_j \in \mathcal{A}_2(I_j)$,

$$\mathbb{A}^{\mathcal{B}}[\pi(x_1)\dots\pi(x_n)] = \mathbb{A}^{\pi}[x_1\dots x_n], \tag{4.5}$$

the left side being the worm operator of $\mathcal{B}$ on its vacuum representation and the right side that of $\mathcal{A}_2$ on π. Let $v \in S\otimes S = V_{\mathcal{A}_2}$ and $|v(z)\rangle_D = x_1\Omega_D + x_2\Omega_D$ for the net $\mathcal{A}_2$. The unitary u extends to the vacuum sectors $\mathcal{H}_0(D) = \text{Conf}(\mathbb{D},D)\times_{\text{Möb}}\mathcal{H}_0$ of [HeT25, (4) p. 8], compatibly with the local algebras and the annuli, so $|(uv)(z)\rangle_D = \pi(x_1)\Omega_D^{\mathcal{B}} + \pi(x_2)\Omega_D^{\mathcal{B}}$, an admissible decomposition for $\mathcal{B}$ since $\pi(x_i) \in \mathcal{B}(I_i)$. By [HeT25, Lemma 6.9] for $\mathcal{B}$ and by (4.5), $\mathbb{A}^{\mathcal{B}}[(uv)(z)] = \sum_i \mathbb{A}^{\mathcal{B}}[\pi(x_i)] = \sum_i \mathbb{A}^{\pi}[x_i] = \mathbb{A}^{\pi}[v(z)]$. For $0 < r < |z| < 1$, apply (4.4) with outer radius 1 to the vacuum representation of $\mathcal{B}$ and to π. In both cases the equal bounded annulus operators are applied to the same vector $r^{-L_0}a$, defined for $a \in W = \mathcal{H}_{\mathcal{B}}^{\text{f.e.}}$. The two fields therefore agree on this domain, and equality of their Laurent coefficients gives the stated identity of vertex operators. ∎

Lemma 4.17 *W_0 is an $S\otimes S$-submodule of (M,Y^M) and $Y^M(v,z)uv' = uY^{S\otimes S}(v,z)v'$ for $v,v' \in S\otimes S$. For every summand $\mathcal{H}_{\pi_z}\otimes\mathcal{H}_{\sigma(\pi_z)}$ of (4.1), its finite-energy space is a submodule*

of (M, Y^M)*, equal to* $M(\pi_z) \otimes M(\sigma(\pi_z))$ *with the tensor product module structure of [FHL93, Section 4.7].*

Proof : The projection E onto a subrepresentation $K' \subset \mathcal{H}_\mathcal{B}$ lies in $\pi(\mathcal{A}_2)'$ and commutes with the modes $v_m^M = \sum_j \pi_{I_j}(Y(v, e_{m-d_v+1}\chi_j))|_M$, where v is homogeneous of weight d_v, v_m^M is the mode $v_{(m)}^M$, $e_k(z) = z^k$ and (χ_j) is a partition of unity subordinate to a cover by intervals I_j [HeT25, p. 72], so $(K')^{\text{f.e.}}$ is a submodule whose structure satisfies (4.3) for K', hence is the module structure of Theorem 4.11 for K'. For $K' = \mathcal{H}_0'$, this representation is the vacuum representation transported by u, so the structure on W_0 is $uY^{S\otimes S}u^{-1}$ by uniqueness and the last assertion of Theorem 4.11. For $K' = \mathcal{H}_{\pi_z} \otimes \mathcal{H}_{\sigma(\pi_z)}$, the finite-energy space is the algebraic tensor product $M(\pi_z) \otimes M(\sigma(\pi_z))$, since Proposition 4.8 gives only finitely many eigenvalues, counted with multiplicity, in each bounded energy interval, and the tensor product module satisfies (4.3) for the vectors $v \otimes 1$ and $1 \otimes v'$, whose smeared fields are $Y(v, f) \otimes 1$ and $1 \otimes Y(v', f)$. These vectors generate $S \otimes S$ and $Y(v \otimes v', z)$ is the normally ordered product of $Y(v \otimes 1, z)$ and $Y(1 \otimes v', z)$, so two module structures agreeing on them coincide. ∎

Proposition 4.18 *Assume* (I1), (F1)–(F3).

1. W_0 *is a vertex subalgebra of* W *and* $u\colon S \otimes S \to W_0$ *is an isomorphism of vertex algebras.*
2. $\nu_W = u(\omega \otimes 1 + 1 \otimes \omega)$*, and* W *has central charge* 16.
3. $L_1 W_0 \subset W_0$*,* $\Theta_W W_0 = W_0$ *and* $\Theta_W|_{W_0} = u(\theta \otimes \theta)u^{-1}$*. Hence* W_0*,* $u(S \otimes 1)$ *and* $u(1 \otimes S)$ *are unitary subalgebras of* W *[CKLW18, Definition 5.22 p. 43],* u *is a unitary isomorphism from* $(S \otimes S, H_\chi \otimes H_\chi)$ *onto* W_0 *with the scalar product of* $\mathcal{H}_\mathcal{B}$*, and the unitary structure induced by* W *on* $u(S \otimes 1) \cong S$ *is* H_χ*.*

Proof : 1. For $v, v' \in S \otimes S$, Lemmas 4.16 and 4.17 give $Y^W(uv, z)uv' = Y^M(v, z)uv' = uY^{S\otimes S}(v, z)v'$.

2. By Theorem 4.10, $Y^W(\nu_W, z)$ is the stress-energy tensor of $\mathcal{B}$, which is that of π by Proposition 4.4, and the latter is $Y^M(\omega \otimes 1 + 1 \otimes \omega, z)$ by [HeT25, Corollary 9.6 p. 72]. Hence $\nu_W = L_{-2}\Omega_\mathcal{B} = (\omega \otimes 1 + 1 \otimes \omega)_{(-1)}^M \Omega_\mathcal{B} = u(\omega \otimes 1 + 1 \otimes \omega)$ by Lemma 4.17.

3. The operator L_1 of $\mathcal{B}$ preserves $\mathcal{H}_0'$ by Möbius covariance of $\mathcal{A}_2 \subset \mathcal{B}$. The PCT operator Θ_W is the modular conjugation J of $(\mathcal{B}(I_-), \Omega_\mathcal{B})$. By the Bisognano–Wichmann property the vacuum modular group of $\mathcal{B}(I_-)$ is implemented by the dilations preserving I_-. By Proposition 4.4 these are the common dilations of the extension and the represented subnet. Hence they preserve $\mathcal{A}_2(I_-)$ on $\mathcal{H}_\mathcal{B}$. Takesaki's theorem gives the vacuum-preserving conditional expectation onto this subalgebra, as in [Lon03, Lemma 13 p. 14]. The projection onto $\mathcal{H}_0' = \overline{\mathcal{A}_2(I_-)\Omega_\mathcal{B}}$ reduces J, and $J|_{\mathcal{H}_0'}$ is the modular conjugation for the compressed subnet algebra. The unitary u identifies it with $J_{\mathcal{A}_2(I_-)}$ in the vacuum representation of the product net. By Theorem 4.10 and Lemma 4.15, $J_{\mathcal{A}_2(I_-)}$ restricts to $\theta \otimes \theta$ on $S \otimes S$. Hence W_0 is a unitary subalgebra by [CKLW18, Proposition 5.23 p. 43], and so are $u(S \otimes 1)$, $u(1 \otimes S)$, since $S \otimes 1$ and $1 \otimes S$ are stable under $\theta \otimes \theta$ and $L_1 \otimes 1 + 1 \otimes L_1$. The map u preserves scalar products, vertex operators, conformal vectors and PCT operators. ∎

Remark 4.19 Agreement of two closed affiliated operators on a vacuum-generated dense domain does not by itself prove equality: that domain must be a core for both operators. Lemma 4.16 instead compares the bounded annulus operators in (4.5), using their construction from compatible local algebras, vacuum vectors and Virasoro representations. The reconstruction formula then identifies the vertex operators on their specified finite-energy domains.

4.5 Decomposition, braid reversal and the lattice

We keep the hypotheses (I1), (F1)–(F3), and identify $S \otimes S$ with $W_0 \subset W$ through u.

Proposition 4.20 *As an $S \otimes S$-module, with τ as in Corollary 4.14 and $M_{\tau(\pi)} \stackrel{\text{def}}{=} M(\pi)$, a representative of the class $\tau(\pi)$,*

$$W = \bigoplus_z M_{\tau(\pi_z)} \otimes M_{\tau(\sigma(\pi_z))}, \tag{4.6}$$

a sum of twelve pairwise non-isomorphic simple modules. The vertex operator algebra W is simple, of CFT type and $\mathbb{Z}$-graded, $h_{\tau(\pi_z)} + h_{\tau(\sigma(\pi_z))} \in \mathbb{Z}$ for all z, and W_1 is the direct sum of $(S \otimes S)_1$ and of $R_{z'} \otimes R_{z''}$ for $(z', z'') \in \{(c_1, c_2), (c_2, c_1), (d_1, d_5), (d_5, d_1), (d_4, d_6), (d_6, d_4)\}$, where R_z is the lowest weight space of M_z.

Proof : The decomposition is Lemma 4.17, the tensor product of simple modules is simple [FHL93, Section 4.7], and the pairs $(\tau(\pi_z), \tau(\sigma(\pi_z)))$ are distinct since τ is injective. The weight-zero space of W is the space of Möbius invariant vectors, $\mathbb{C}\Omega_{\mathcal{B}}$, so W is of CFT type and simple [CKLW18, Proposition 5.3 p. 34]. The grading is by the L_0 of $\mathcal{B}$, with integral spectrum since the Möbius representation of $\mathcal{B}$ on $\mathcal{H}_{\mathcal{B}}$ is a representation of $\mathrm{PSL}(2, \mathbb{R})$. By Lemma 4.1 and Corollary 4.14, $\tau(\pi_x) = M_x$ for $x \in \{0, b, c_1, c_2\}$, τ maps $\{\pi_a, \pi_{c_0}\}$ onto $\{X_+, X_-\} = \{M_a, M_{c_0}\}$ and $\{\pi_{d_1}, \dots, \pi_{d_6}\}$ onto $\{M_{d_1}, \dots, M_{d_6}\}$, while σ has the orbits $\{d_1, d_5\}$, $\{d_2, d_3\}$, $\{d_4, d_6\}$. By (F2), the pairs $\{d_l, d_{l'}\}$ with $h_{d_l} + h_{d_{l'}} \in \mathbb{Z}$ are $\{d_1, d_5\}$, $\{d_4, d_6\}$ (sum 1) and $\{d_2, d_3\}$ (sum 2), so τ maps the three σ-orbits onto these pairs in some order. The weight-one vectors of $M_{z'} \otimes M_{z''}$ with $h_{z'} + h_{z''} = 1$ form $R_{z'} \otimes R_{z''}$, and $h_z \geqslant 1$ for $z \in \{b, a, c_0\}$. ∎

Proposition 4.21 *There is a braid-reversed equivalence $\tau_{\mathrm{CKM}} \colon \mathrm{Rep}(S) \to \mathrm{Rep}(S)$, that is, a braided equivalence $\mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$, with $\tau_{\mathrm{CKM}}(M_{\tau(\pi_z)}) \cong (M_{\tau(\sigma(\pi_z))})'$ for all z, where the prime denotes the contragredient module. The vertex operator algebra W is strongly rational and holomorphic.*

Proof : We apply [CKM22, Theorem 5.10(2) p. 51], stated as Main Theorem 3(2) p. 4 there, with $\mathcal{U} = \mathcal{V} = \mathrm{Rep}(S)$ and $U = V = S$. By (F1), S is simple, self-contragredient and strongly rational, so $\mathrm{Rep}(S)$ is a semisimple ribbon category with the vertex tensor category structure of [Hua08], closed under contragredients, submodules and quotients. By Propositions 4.18 and 4.20, $W = \bigoplus_z U_z \otimes V_z$ with $U_z = M_{\tau(\pi_z)}$ distinct simple modules, $U_0 = S$, $V_z = M_{\tau(\sigma(\pi_z))}$ and $\dim \mathrm{Hom}_S(S, V_z) = \delta_{z,0}$, since $\sigma(\pi_z) = \pi_0$ only for $z = 0$, and W is a simple $\mathbb{Z}$-graded vertex operator algebra extending $S \otimes S$ with the same vacuum and conformal vector. All summands are integrally graded, so we take the entire index set as the even part in [CKM22, Theorem 5.10(2)]. We view W in $\mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S) \cong \mathrm{Rep}(S \otimes S)$ [CKM22, Theorems 5.5 p. 48 and 5.7 p. 49]. The conclusion is a braid-reversed equivalence from the subcategory generated by the U_z onto that generated by the V_z, with $U_z \mapsto V_z'$, and both are $\mathrm{Rep}(S)$ since τ and $\tau\sigma$ are bijections onto the simple modules. By [CKM22, Corollary 5.13 p. 53], W is strongly rational.

The extension algebra $A_W = \bigoplus_z U_z \boxtimes V_z$ of [HKL15, Theorem 3.2 p. 10] is commutative and connected, since the vacuum summand occurs once. The proof of [CKM22, Theorem 5.12 p. 52] gives semisimplicity of the category of A_W-modules, hence separability by [DMNO13, Proposition 2.7]. A tensor equivalence preserves Frobenius–Perron dimensions and $\mathrm{FPdim}(V_z') = \mathrm{FPdim}(V_z)$. Thus $\mathrm{FPdim}(A_W) = \sum_z \mathrm{FPdim}(U_z)^2 = \mathrm{FPdim}\,\mathrm{Rep}(S)$, the square root of $\mathrm{FPdim}(\mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S))$. Hence A_W is Lagrangian, its local modules are trivial [DMNO13, Remark 4.3 p. 16], and $\mathrm{Rep}(W) \cong \mathrm{Vec}$ by [HKL15, Theorem 3.4 p. 16]. ∎

Lemma 4.22 $\mathrm{Com}_W(1\otimes S) = S\otimes 1$ *and* $\mathrm{Com}_W(S\otimes 1) = 1\otimes S$. *With* $V = 1\otimes S$, *we have* $V^c = S\otimes 1$, $V^{cc} = V$, *both are regular unitary subalgebras of* W *isomorphic to* S *with conformal vectors* $1\otimes\omega$ *and* $\omega\otimes 1$ *in the sense of [CKLW18, Proposition 5.29 p. 45], and* $\nu_W = \omega\otimes 1 + 1\otimes\omega$.

Proof : A vector of W annihilated by the non-negative modes of $1\otimes S$ is annihilated by $1\otimes L_0 = (1\otimes\omega)_{(1)}$, whose spectrum on $M_{\tau(\pi_z)}\otimes M_{\tau(\sigma(\pi_z))}$, $z\neq 0$, lies in $h_{\tau(\sigma(\pi_z))}+\mathbb{N}$ with $h_{\tau(\sigma(\pi_z))} > 0$ by (F1), since $\sigma(\pi_z)\neq\pi_0$. Hence it lies in $S\otimes S_0 = S\otimes 1$, which commutes with $1\otimes S$. The other commutant is symmetric. By Proposition 4.18(3) and [CKLW18, Proposition 5.29], the conformal vector of $1\otimes S$ is the orthogonal projection of ν_W onto $1\otimes S$, which is $1\otimes\omega$ since $\omega\otimes 1\in S_2\otimes S_0$ is orthogonal to $1\otimes S$. Regularity is (F1) and [ABD04, Theorem 4.5 p. 9]. ■

Proposition 4.23 *There are an even unimodular lattice* $L\in\{E_8\oplus E_8, D_{16}^+\}$ *and a unitary isomorphism of vertex operator algebras* $\psi\colon W\to V_L$, *for the unitary structure of Theorem 4.10 on* W *and that of [DoL14, Theorem 4.12] on* V_L. *Hence* W *satisfies Condition B of [Gui26, Section 3.4 p. 44] and Condition I+ of [Gui25, Definition 6.17 p. 70], and* $\mathcal{B}\cong\mathcal{A}_{V_L}$ *as conformal nets.*

Proof : By Propositions 4.20 and 4.21, W is a C_2-cofinite holomorphic vertex operator algebra of CFT type and central charge 16, so $W\cong V_L$ for one of the two even unimodular lattices of rank 16 [DoM04b, Theorem 2 p. 2]. Transporting the unitary structure of W through an isomorphism ψ_0 gives a normalized invariant scalar product on V_L, and by [CKLW18, Proposition 5.19 p. 40] there is $h\in\mathrm{Aut}(V_L)$ such that $\psi = h\circ\psi_0$ is unitary for the structure of [DoL14, Theorem 4.12]. Conditions B and I+ are formulated in terms of the unitary vertex operator algebra, its unitary modules, intertwining operators and net, and are preserved by unitary isomorphisms, as in the proof of [ArhII, Theorem 7.17]. They hold for V_L by [Gui26, Theorem 3.46 p. 57] and [Gui25, Example 6.19(b) p. 71]. Finally $\mathcal{B} = \mathcal{A}_W$ by [HeT25, Theorem 8.9(i) p. 66] and $\mathcal{A}_W\cong\mathcal{A}_{V_L}$ by [CKLW18, Theorem 6.8 p. 53]. ■

Remark 4.24 Under ($\mathrm{H_J}$), the lattice D_{16}^+ is excluded in [ArhII, proof of Proposition 7.12(2)] by a count of $(\mathfrak{g}'\otimes 1\oplus 1\otimes\mathfrak{g}')$-invariant vectors in W_1, based on the decompositions of the spaces R_z under S_1 of [ArhI, Table 2]. By Proposition 4.20, W_1 is the same $S\otimes S$-module as there, and these decompositions are part of (F2). Hence the count, recalled in the proof of Proposition 9.9(2), applies to W and gives $W\cong V_{E_8\oplus E_8}$, see Corollary 9.16. This refinement is not used below, since both lattices satisfy Proposition 4.23.

4.6 The diagonal extension theorem

Theorem 4.25 *Assume* (I1), (F1)–(F3).

1. *There is a strongly rational holomorphic unitary vertex operator algebra* W *of central charge* 16, *isomorphic to* V_L *for an even unimodular lattice* $L\in\{E_8\oplus E_8, D_{16}^+\}$, *containing* $S\otimes S$ *as a unitary subalgebra with the same conformal vector, such that the induced scalar product on each of* $S\otimes 1$ *and* $1\otimes S$ *is* H_χ, *and*

$$\mathrm{Com}_W(S\otimes 1) = 1\otimes S,\quad \mathrm{Com}_W(1\otimes S) = S\otimes 1.$$

As an $S\otimes S$*-module,* $W = \bigoplus_z M_{\tau(\pi_z)}\otimes M_{\tau(\sigma(\pi_z))}$, *with twelve simple summands of multiplicity one, where* σ *is the involution of Lemma 4.1. The algebra* W *satisfies Condition B of [Gui26] and Condition I+ of [Gui25], and its conformal net is the net* $\mathcal{B}$ *of Proposition 4.4.*

2. *The map* $\tau\colon \pi \mapsto [K_\pi^{\mathrm{f.e.}}]$*, which sends an irreducible representation of* $\mathcal{A}_S$ *to the simple* S*-module structure of Theorem 4.11 on its finite-energy space, is a bijection from the equivalence classes of irreducible representations of* $\mathcal{A}_S$ *onto the isomorphism classes of simple* S*-modules, with* $\tau(\pi_0) = S$*,* $\tau(\pi_b) = M_b$*,* $\tau(\pi_\pm) = X_\pm$*,* $\tau(\pi_{c_j}) = M_{c_j}$*, and every simple* S*-module is unitarizable for the PCT operator of* (S, H_χ)*.*

3. $\mathrm{Rep}(S)$ *admits a braid-reversing autoequivalence, so that* $\mathrm{Rep}(S)$ *and* $\mathrm{Rep}(S)^{\mathrm{rev}}$ *are braided equivalent.*

Proof : Part 2 is Corollary 4.14. For part 1, let $W = V_{\mathcal{B}}$ as in Section 4.4. By Proposition 4.18, $S \otimes S$ is a unitary subalgebra of W with the same conformal vector and the stated unitary structure. The decomposition is Proposition 4.20, the commutants are Lemma 4.22, strong rationality and holomorphy are Proposition 4.21, and the lattice, Conditions B and I+ and the net are Proposition 4.23. Part 3 is Proposition 4.21. ∎

Remark 4.26 The theorem uses (F2) for the count of twelve simple modules, (F3) for the lattice-sector identifications and (F1) for positivity, regularity and the vertex tensor category. The spectral hypothesis of Theorem 4.11 is supplied by Proposition 4.8. Neither ($\mathrm{H_J}$) nor (F4) is used. The inclusion of nets supplies the unitary structure on the reconstructed W. Section 9 treats the separate algebraic construction.

4.7 The reverse braiding of the Haagerup center

Lemma 4.27 *There is a braided equivalence* $R\colon Z(\mathcal{H}_3)^{\mathrm{rev}} \to Z(\mathcal{H}_3)$*. It is unique up to natural isomorphism, it is a ribbon equivalence for the twist* θ^{-1} *on the source, and the induced permutation of the simple objects fixes* 0*,* b*,* a *and* c_0*, exchanges* c_1 *and* c_2*, and maps* d_l *to* $d_{l'}$ *with* $l' \equiv \pm 5l \bmod 13$*. Moreover* $R(L_i) \cong L_i$ *as algebras for the three Lagrangian algebras* L_1*,* L_2*,* L_3 *of* $Z(\mathcal{H}_3)$*.*

Proof : The self-reversal and uniqueness are the categorical statement used in Lemma 4.1, proved in the first half of the proof of [ArhII, Proposition 7.12(1)], which concerns $Z(\mathcal{H}_3)$ alone. Positive categorical dimensions determine the spherical structure, so the equivalence is ribbon. The three Lagrangian algebras are distinguished by their module categories $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$. Reversal replaces each by its tensor opposite, equivalent to itself by self-conjugacy of the Haagerup even parts. Thus R fixes each algebra. Comparing $L_2 = 0 \oplus b \oplus 2a$ gives $R(a) = a$ and then $R(c_0) = c_0$. The remaining labels are determined by their dimensions and inverse twists, as in Lemma 4.1. ∎

5 Complete unitarity from the reconstructed extension

5.1 Complete unitarity and the comparison functor

We use the notions of Gui: condition B of [Gui26, Section 3.4 p. 44], Condition I of [Gui25, Definition 4.15 p. 41] and Condition I^+ of [Gui25, Definition 6.17 p. 70], which adds to Condition I the complete rationality of the net and the essential surjectivity of the functor below. For a simple unitary strongly local vertex operator algebra V [CKLW18, Section 6], $\mathrm{Rep}^u(V)$ denotes the category of unitary V-modules, $\mathrm{Rep}^f(\mathcal{A}_V)$ the category of representations of finite statistical dimension of its conformal net, which is $\mathrm{Rep}(\mathcal{A}_V)$ in the convention of Section 3 when $\mathcal{A}_V$ is completely rational, and

$$\Pi\colon\ \mathrm{Rep}^u(V) \to \mathrm{Rep}^f(\mathcal{A}_V), \quad (M, Y_M) \mapsto (\mathcal{H}_M, \pi_M),$$

the functor of Carpi, Weiner and Xu, defined on strongly integrable modules by the condition $\pi_{M,I}(Y(v,f)) = Y_M(v,f)$ for $v \in V$ homogeneous and f supported in I [Gui26, (3.8) p. 41], [Gui25, Definition 4.22 and Theorem 4.24 p. 43]. The category $\mathrm{Rep}^f(\mathcal{A}_V)$ carries the braided C^*-tensor structure of Connes fusion [Gui21, Gui26]. The net $\mathcal{A}_S$ is that of Section 3, for the unitary structure H_χ of (I1).

Theorem 5.1 *Assume* (I1) *and* (F1)–(F3). *Let W be the reconstructed extension of Theorem 4.25, or, more generally, a simple holomorphic unitary vertex operator algebra containing $S \otimes S$ with the same conformal vector, such that $\mathrm{Com}_W(S \otimes \mathbf{1}) = \mathbf{1} \otimes S$, $\mathrm{Com}_W(\mathbf{1} \otimes S) = S \otimes \mathbf{1}$, $\mathbf{1} \otimes S$ is a unitary subalgebra of W, and W is unitarily isomorphic to a lattice vertex operator algebra V_Γ with the unitary structure of [DoL14, Theorem 4.12]. Then S, with the unitary structure of* (I1), *satisfies condition B of [Gui26] and Condition I^+ of [Gui25]. In particular:*

1. *S is completely unitary in the sense of [Gui22, Definition 1.8 p. 13]: every S-module is unitarizable and the invariant sesquilinear forms of [Gui19] on the spaces of intertwining operators are positive definite. The category $\mathrm{Rep}^u(S)$ is a unitary modular tensor category, and the forgetful functor $\mathrm{Rep}^u(S) \to \mathrm{Rep}(S)$ is a braided equivalence.*

2. *Every unitary S-module is strongly integrable, and $\Pi\colon \mathrm{Rep}^u(S) \to \mathrm{Rep}(\mathcal{A}_S)$ is an equivalence of braided C^*-tensor categories, for the braiding of Connes fusion on the target.*

3. *For every S-submodule M of one of the lattice extensions $H_\pm$ of* (F3), *with the restricted scalar product, $\Pi(M)$ is the representation π_M of [ArhII, Lemma 4.7] on the closure of M. In particular $\Pi(S) = \pi_0$, $\Pi(N_b) = \pi_b$ and $\Pi(X_\pm) = \pi_\pm$ in the notation of Section 3.1. The functor Π preserves twists, so that $\Pi(M_{c_1})$, $\Pi(M_{c_2})$ and $\Pi(M_{d_l})$ are the irreducible sectors of $\mathcal{A}_S$ with twists $\mathrm{e}^{2\pi\mathrm{i}/3}$, $\mathrm{e}^{4\pi\mathrm{i}/3}$ and $\mathrm{e}^{2\pi\mathrm{i}h_{d_l}}$.*

Proof : We follow the proof of [ArhII, Theorem 7.17], verifying the hypotheses of each result used. Put $V = \mathbf{1} \otimes S \subset W$. By Lemma 4.22, or by the mutual-commutant assumption in the more general case, $V^c \overset{\text{def}}{=} \mathrm{Com}_W(V) = S \otimes \mathbf{1}$ and $V^{cc} = V$. The coset V^c of the unitary subalgebra V is the coset subalgebra of [CKLW18, Example 5.27 p. 44], so it is a unitary subalgebra as well.

Conformal vectors. By [CKLW18, Propositions 5.29 and 5.31 p. 45], whose hypotheses are that W is a simple unitary vertex operator algebra and V a unitary subalgebra, the vector $\nu = e_V\omega_W$ is a conformal vector of V, with which $(V, (\cdot|\cdot))$ is a simple unitary vertex operator algebra, and $\omega_W = \nu + \nu'$ with ν' the corresponding vector of V^c. Since $\omega_W = \omega_{S\otimes 1} + \omega_{1\otimes S}$, the vector $\nu - \omega_{1\otimes S} = \omega_{S\otimes 1} - \nu'$ lies in $(\mathbf{1} \otimes S) \cap (S \otimes \mathbf{1}) \cap W_2 = 0$. Hence V and V^c, with the conformal vector of S and the scalar product inherited from W, are unitary vertex operator algebras isomorphic to S, and V is a unitary sub-VOA of W with commutant V^c in the sense of [Gui26, Section 3.6.1 p. 49].

Regularity. S is of CFT type, C_2-cofinite and rational by (F1), hence regular by [ABD04, Theorem 4.5]. Thus V and V^c are regular. In [ArhII] this step uses ($\mathrm{H_J}$), and (F1) is its replacement.

Condition B for W. The lattice vertex operator algebra V_Γ, with the unitary structure of [DoL14], satisfies condition B [Gui26, Theorem 3.46 p. 57], whose only hypothesis is that Γ is an even positive definite lattice. Condition B is preserved by unitary isomorphisms of unitary vertex operator algebras, by transport of the vertex operators, of the energy bounds and of the smeared fields, and two structures of unitary vertex operator algebra on the simple algebra $W \cong V_\Gamma$ are unitarily isomorphic [CKLW18, Proposition 5.19 pp. 40–41]. Hence W satisfies condition B.

Condition B for S. The hypotheses of [Gui26, Theorem 3.34 p. 51] are: W satisfies condition B, V is a unitary sub-VOA of W, V and V^c are regular, and $V^{cc} = V$. All four hold by the preceding paragraphs. Hence V^c satisfies condition B. No energy bound on S or on its modules is assumed: the proof of [Gui26, Theorem 3.34] derives the energy bounds of the modules and intertwining operators of V^c by compression from those of W. By [CKLW18, Proposition 5.19] again, the isomorphism of vertex operator algebras $S \to V^c = S \otimes \mathbf{1}$, composed with an automorphism of S, is a unitary isomorphism from S with the structure of (I1) onto V^c, so S satisfies condition B.

Condition I^+ for S. The algebra V_Γ satisfies Condition I^+ [Gui25, Example 6.19(b) p. 71], hence so does W by the same transport, Condition I^+ being a property of the vertex operators, their energy bounds, the local algebras and the functor Π. The hypotheses of [Gui25, Theorem 6.18(3) p. 70] are: W is a unitary vertex operator algebra satisfying Condition I^+, V is a unitary subalgebra of W, and V and V^c are C_2-cofinite and rational. They hold by (F1). Hence V^c, and by transport S, satisfies Condition I^+. Condition I^+ includes the complete rationality of $\mathcal{A}_S$, which was obtained independently in Theorem 3.1.

1. Condition B implies condition A of [Gui26], and [Gui26, Theorem 3.15 p. 45] gives that S is completely unitary and that $\mathrm{Rep}^u(S)$ is a unitary modular tensor category, the latter being [Gui19, Theorem 7.9 p. 53]. Since every S-module is unitarizable, the forgetful functor $\mathrm{Rep}^u(S) \to \mathrm{Rep}(S)$ is an equivalence of modular tensor categories [CGGH23, Remark 4.3 p. 18].

2. By [Gui26, Theorems 3.16 and 3.17 p. 45], under condition A every unitary S-module is strongly integrable and Π is a fully faithful $*$-functor, and by [Gui26, Corollary 3.23 p. 48] it is an isomorphism onto a braided C^*-tensor subcategory of $\mathrm{Rep}(\mathcal{A}_S)$ for Connes fusion. The functor of [Gui25, Theorem 4.24 p. 43] is the same functor, both being characterized by [Gui26, (3.8) p. 41], which is [Gui25, (4.6) p. 43]. By [Gui25, Theorem 6.14 pp. 68–69 and Definition 6.17 p. 70], Condition I^+ means that Π is essentially surjective onto $\mathrm{Rep}^f(\mathcal{A}_S)$, which is $\mathrm{Rep}(\mathcal{A}_S)$ since $\mathcal{A}_S$ is completely rational. Hence Π is an equivalence of braided C^*-tensor categories.

3. By (I1) and [ArhII, Theorem 4.5], S is a unitary subalgebra of $(H_\pm, \{\cdot|\cdot\}_{\Lambda_\pm})$ containing its conformal vector, so the restriction of the scalar product of $H_\pm$ makes M a unitary S-module, and M is strongly integrable by part 2. By [ArhII, Lemma 4.7(1)], π_M is a representation of $\mathcal{A}_S$ on $\mathcal{H}_M$ which maps the closed operator $Y(v,f)|_{\mathcal{H}_S}$ to the closed operator $Y_M(v,f)$, of which M is a core. The vertex operator $Y_M(v,\cdot)$ is the restriction to M of a field of $H_\pm \cong V_{E_8}$, which satisfies energy bounds [Gui26, Theorem 3.46], and energy bounds pass to compressions [Gui26, Proposition 3.31 p. 50], so $Y_M(v,f)$ is also the closure of the smeared field of M on the smooth vectors. This is the defining property of $\Pi(M)$, which is unique since $\mathcal{A}_S(I)$ is generated by the operators $Y(v,f)$ [Gui26, (3.8) p. 41]. Hence $\Pi(M) = \pi_M$, and in particular $\Pi(S) = \pi_0$, $\Pi(N_b) = \pi_b$, $\Pi(X_\pm) = \pi_{W_{(1,\pm 1)}} = \pi_\pm$ by (F3). Finally, Π is a braided equivalence between the pseudo-unitary modular tensor categories $\mathrm{Rep}^u(S)$ and $\mathrm{Rep}(\mathcal{A}_S)$, so it preserves the spherical structures with positive dimensions [ENO05, Proposition 8.23 p. 34] and the twists, and by (F2) the twists $\mathrm{e}^{2\pi\mathrm{i}h_z}$ of M_{c_1}, M_{c_2} and of the M_{d_l} are pairwise distinct and distinct from 1. Independently of the orientation of the braiding, $\Pi(M_z)$ acts on the closure of M_z and its conformal Hamiltonian is the closure of L_0 on M_z [Gui25, Remark 4.23 p. 43], so that $\mathrm{e}^{2\pi\mathrm{i}L_0} = \mathrm{e}^{2\pi\mathrm{i}h_z}$ on $\Pi(M_z)$. ∎

Remark 5.2 Part 2 of Theorem 5.1 identifies $\mathrm{Rep}(S)$ with $\mathrm{Rep}(\mathcal{A}_S)$ equipped with the braiding of Connes fusion. By [Gui21, Theorem 6.6 p. 77], this braided C^*-tensor category is equivalent to the category of DHR endomorphisms of $\mathcal{A}_S$ with the tensor product $\rho \odot \sigma = \sigma \circ \rho$ and the statistics operator $\widetilde{c}$ of [Gui21, (6.1) p. 73], and the identity functor with the tensor constraint given by the braiding is a braided equivalence onto the same category with the tensor product $\rho \circ \sigma$ and the braiding $b_{\rho,\sigma} = \widetilde{c}_{\sigma,\rho}$, by the hexagon identities and the Yang–Baxter equation

[ArhII, proof of Theorem 7.17(3)]. The two orientations in the definition of the DHR braiding give b and b^{rev}. Hence, for the DHR braiding used in Section 3, Theorem 5.1 gives a braided equivalence of $\mathrm{Rep}(S)$ with $\mathrm{Rep}(\mathcal{A}_S)$ or with $\mathrm{Rep}(\mathcal{A}_S)^{\mathrm{rev}}$, which sends S, M_b, X_+, X_- to π_0, π_b, π_+, π_- in both cases. The equivalence of Lemma 4.27 handles the reversed orientation on the net side. Corollary 6.2 uses it to match the intrinsic labels.

6 The Haagerup property and its consequences for S

Theorem 6.1 *The vertex operator algebra S has the following properties.*

1. *S is a Haagerup algebra, that is, $(\mathrm{H_J})$ holds. In particular $\mathrm{Rep}(S)$ is braided equivalent to $Z(\mathcal{H}_3)$.*
2. *With the scalar product H_χ, S is completely unitary and satisfies Condition I^+. The functor $\Pi\colon \mathrm{Rep}^u(S) \to \mathrm{Rep}^f(\mathcal{A}_S)$ is a braided C^*-tensor equivalence for the braiding of Connes fusion and agrees with $M \mapsto \pi_M$ on the lattice summands of Theorem 5.1(3).*
3. *The heat operator $\mathrm{e}^{-\beta L_0^\pi}$ is trace class for every irreducible sector π of $\mathcal{A}_S$ and every $\beta > 0$. There is a holomorphic unitary vertex operator algebra of central charge 16 containing $S \otimes S$ as a unitary conformal subalgebra, with the two copies of S as mutual commutants.*

Proof : By Theorems 2.1 and 2.2, S satisfies (I1) and (F1)–(F3). Theorem 3.1 identifies $\mathrm{Rep}(\mathcal{A}_S)$ with $Z(\mathcal{H}_3)$, using [ArhII, Proposition 5.2 and Theorem 6.1]. Proposition 4.8 and Theorem 4.25 give part 3. Theorem 5.1, applied to the reconstructed extension W of Theorem 4.25, gives part 2 and the braided equivalence $\mathrm{Rep}^u(S) \cong \mathrm{Rep}(S)$. For the DHR braiding the comparison is braided or braid-reversed by Remark 5.2. The center is equivalent to its reverse by Lemma 4.27, so in both orientations $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$ as braided categories. Together with positivity and strong rationality in (F1), this is part 1. ∎

By (I3), the vertex operator algebra of Jiao [Jia26] is also a Haagerup algebra.

6.1 The labels and the third holomorphic extension

After Theorem 6.1, we write $\Phi_0\colon \mathrm{Rep}(S) \to Z(\mathcal{H}_3)$ for a braided equivalence and use the intrinsic labels $M_z = \Phi_0^{-1}(z)$. The trivial Brauer–Picard group of $\mathcal{H}_3$ makes these labels independent of the choice of Φ_0. Since Φ_0 preserves the canonical spherical structures [ENO05, Proposition 8.23 p. 34], it preserves the twists $\mathrm{e}^{2\pi i h_z}$, and the conformal weights in (F2), with $h_{d_l} \equiv \frac{6l^2}{13} \bmod \mathbb{Z}$, show that these intrinsic labels agree with those of (F2) for the modules with nontrivial twist. The vacuum label is 0, and the summand $N_b \cong M_b$ of $V'|_S$ in (F3) has the intrinsic label b by [ArhII, Lemma 5.4], which applies now that $(\mathrm{H_J})$ holds. The modules $X_\pm$ have trivial twist by (F3) and dimension $d(\pi_\pm) = 3d + 2$ by Theorems 3.1(1) and 5.1(3), so that their intrinsic labels lie in $\{a, c_0\}$ [ArhII, Section 2.2], and $X_+ \ncong X_-$ gives $\{M_a, M_{c_0}\} = \{X_+, X_-\}$ for the intrinsic labels. For the net we use the intrinsic labels of Theorem 3.1(3). Write $\mathcal{F}_\pm = \mathrm{Rep}(S)_{L_\pm}$ for the module categories of the two lattice extension algebras.

Corollary 6.2 *The following assertions hold.*

1. Labels. *The assertion* (N) *of [ArhII, Section 7] holds: there is a braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_S) \to \mathrm{Rep}(S)$ with $\Phi(\pi_0) \cong S$ and $\Phi(\pi_x) \cong M_x$ for $x \in \{b, a, c_0\}$, for either orientation of the DHR braiding. The label of $X_\pm$ is the label of $\pi_\pm$: $X_\pm \cong M_a$ if and*

only if $\pi_\pm = \pi_a^{\mathrm{net}}$. *The lattice extension* $H_\pm$ *of* S *carries* $\mathcal{H}_2$, *that is,* $\mathcal{F}_\pm \cong \mathcal{H}_2$, *if and only if the net* $\mathcal{A}_{H_\pm}$ *does, that is,* $\mathcal{F}_\pm^{\mathrm{net}} \cong \mathcal{H}_2$, *if and only if* $X_\pm \cong M_a$.

2. The third extension. *S has exactly three holomorphic extensions up to isomorphism of extensions, H_+, H_- and an algebra B_1 attached to the Lagrangian algebra $L_1 \cong 0 \oplus b \oplus a \oplus c_0$, all isomorphic to V_{E_8}. The algebra B_1 carries a unique unitary structure extending that of S, it is strongly local, its conformal net is isomorphic to $\mathcal{A}_{E_8}$, and it is the third extension of $\mathcal{A}_S$ of Theorem 3.1(4).*

Proof : 1. Let $\Psi\colon \mathrm{Rep}(\mathcal{A}_S) \to Z(\mathcal{H}_3)$ be the equivalence of Theorem 3.1(3) for the DHR braiding. Regard the comparison functor Π as an equivalence from $\mathrm{Rep}(S)$ to the net category by forgetting the chosen invariant forms. If Π is braided for this orientation, put $\widetilde{\Pi} = \Pi$. If it lands in the category with reversed braiding, put $\widetilde{\Pi} = \Psi^{-1} R \Psi^{\mathrm{rev}} \Pi$, where R is the equivalence of Lemma 4.27. In both cases $\Psi\widetilde{\Pi}$ is a braided equivalence, hence naturally isomorphic to Φ_0. The correction for the reversed orientation fixes the sectors labelled $0, b, a, c_0$. By Theorem 5.1(3), $\Pi(X_\pm) = \pi_\pm$. Consequently $\widetilde{\Pi}^{-1}$ gives (N), and the intrinsic label of $X_\pm$ agrees with that of $\pi_\pm$.

By [ArhII, Theorem 3.9], applied now that ($\mathrm{H_J}$) holds, $L_\pm$ are Lagrangian algebras of $\mathrm{Rep}(S)$. Under $\widetilde{\Pi}$ their objects are those of $\Theta_\pm$ by (F3) and Theorem 3.1(1). By the categorical classification of [ArhII, Proposition 3.3], the algebra structures are determined up to isomorphism by these objects. Thus the module categories agree, and the identification of the $\mathcal{H}_2$ extension follows from [ArhII, Proposition 3.6] and Theorem 3.1(3).

2. By [ArhII, Theorem 3.9] the three holomorphic extensions are H_+, H_-, B_1, all isomorphic to V_{E_8}. Proposition 7.10 of [ArhII] applies by Theorem 6.1, which provides both ($\mathrm{H_J}$) and complete unitarity. It gives the unique extending unitary structure on B_1, strong locality and $\mathcal{A}_{B_1} \cong \mathcal{A}_{E_8}$. Part 3 of that proposition identifies the extension of $\mathcal{A}_S$ through the comparison in part 1, with underlying object Θ_1 of Theorem 3.1(4). ∎

7 The twin algebra and the common fixed point of Dong, Jiao and Xu

In this section we prove Theorem 1.1. We first recall the lattice data of [ArhI, Section 12] and we relate them to the involutions of [DJX26, Section 5], then we extend the construction of the twin algebra of [ArhI, Theorem 12.3] to S using only (I1) and (F1)–(F4), without Theorem 6.1, and we identify the twin of S with the algebra U of Huang and Riesen. The identification rests on a single observation: the common fixed point algebra $C = U^\tau \cong S^\sigma$ of [DJX26, Theorem 7.3] is the extension of $\Omega_S \otimes V_N$ by the thirty-nine simple currents indexed by the 39-part of N'/N, where $N = L_{\mathrm{I}} \cap L_{\mathrm{II}}$, and S, U are its two extensions by the two bosons of N'/N.

7.1 Two binary lattices and their common sublattice

Let $\mathfrak{h} = \mathbb{C}p \oplus \mathbb{C}b$ be a two-dimensional space with the symmetric bilinear form $(p,p) = 6$, $(b,b) = 26$, $(p,b) = 0$, so that p and b are the Cartan currents of $\mathfrak{sl}_2$ at levels 3 and 13, with $(h,h) = 2k$ for the Cartan element h of $L_{\mathfrak{sl}_2}(k,0)$. Following [DJX26, Section 5], put

$$\lambda = \frac{p+b}{2}, \quad s = \frac{p+b}{4}, \quad t = \frac{13p-3b}{4}, \tag{7.1}$$

so that $(\lambda,\lambda)=8$, $(s,s)=2$, $(t,t)=78$, $(s,t)=0$ and $(s,p)=\frac{3}{2}$. Thus s and t are the Cartan currents of $\mathfrak{sl}_2$ at levels 1 and 39. Consider the three even lattices

$$L_{\rm II}=\mathbb{Z}p\oplus\mathbb{Z}\lambda,\quad L_{\rm I}=\mathbb{Z}s\oplus\mathbb{Z}\tfrac{s+t}{2},\quad N=\mathbb{Z}\lambda\oplus\mathbb{Z}2p, \tag{7.2}$$

where $\frac{s+t}{2}=\frac{7p-b}{4}$. We also use the root lattice $L=E_6\oplus A_2$ of V' in the ordered root basis $\alpha_1,\dots,\alpha_8$ of [DJX26, (4.1)], in which $p=\rho_1+\rho_2+\rho_3=(1,2,3,2,1,0,0,0)$ and $b=(3,4,3,4,3,2,2,2)$ are the Cartan currents of S_1 [DJX26, Section 4.3], and the lattices

$$L_0=\{x\in L:(\lambda,x)\in 2\mathbb{Z}\},\quad L_H=L_0+\mathbb{Z}s, \tag{7.3}$$

of [DJX26, Section 5]. Here $\lambda=(2,3,3,3,2,1,1,1)\in L$, and L_0 has index two in L.

Lemma 7.1 *1. The Gram matrices of $L_{\rm II}$, $L_{\rm I}$ and N in the bases (7.2) are $\left[\begin{smallmatrix}6&3\\3&8\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2&1\\1&20\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}8&6\\6&24\end{smallmatrix}\right]$, of determinants 39, 39 and 156. Hence $L_{\rm II}$ and $L_{\rm I}$ are the lattices of case (II) and of case (I) of [ArhI, Lemma 12.1]. Neither has a proper even overlattice in $\mathfrak{h}$.*

2. $N=L_{\rm I}\cap L_{\rm II}$ has index two in $L_{\rm I}$ and in $L_{\rm II}$, with $L_{\rm II}=N\sqcup(p+N)$ and $L_{\rm I}=N\sqcup(s+N)$. The pairings $(s,\cdot)$ and $(p,\cdot)$ take their values in $\frac{1}{2}\mathbb{Z}$ on $L'_{\rm II}$ and on $L'_{\rm I}$ respectively, and the characters $x\mapsto \mathrm{e}^{2\pi\mathrm{i}(s,x)}$ of $L_{\rm II}$ and $x\mapsto \mathrm{e}^{2\pi\mathrm{i}(p,x)}$ of $L_{\rm I}$ both have kernel N.

3. The discriminant group N'/N has order 156. Its 2-part is $\{0,s,p,s+p\}+N$, with $q(s+N)=q(p+N)=0$ and $q(s+p+N)=\frac{1}{2}$, where $q(x)=\frac{(x,x)}{2}\bmod\mathbb{Z}$. Its 39-part P is cyclic and contained in $(L'_{\rm I}\cap L'_{\rm II})/N$, and the quotient maps $L'_{\rm II}/N\to L'_{\rm II}/L_{\rm II}$ and $L'_{\rm I}/N\to L'_{\rm I}/L_{\rm I}$ restrict to isomorphisms of metric groups $\pi_{\rm II}\colon P\to L'_{\rm II}/L_{\rm II}$ and $\pi_{\rm I}\colon P\to L'_{\rm I}/L_{\rm I}$. Moreover

$$\{x\in L'_{\rm II}:(s,x)\in\mathbb{Z}\}=\{x\in L'_{\rm I}:(p,x)\in\mathbb{Z}\}=\{x\in N':x+N\in P\}\overset{\rm def}{=}\mathsf{B}, \tag{7.4}$$

and $L'_{\rm II}=\mathsf{B}\sqcup(\mathsf{B}+p)$, $L'_{\rm I}=\mathsf{B}\sqcup(\mathsf{B}+s)$.

4. The isometry $\psi_N=\pi_{\rm I}\pi_{\rm II}^{-1}\colon L'_{\rm II}/L_{\rm II}\to L'_{\rm I}/L_{\rm I}$ maps the generator $(1,1)$ of [ArhI, Lemma 12.1] to 22 times the generator $(1,1)$, and the four isometries $u\in\{4,17,22,35\}$ of [ArhI, Lemma 12.1(3)] are the composites of ψ_N with the elements 1, 14, 25, 38 of $O(L'_{\rm II}/L_{\rm II})$, which are induced by the Weyl group of $\mathfrak{g}'\oplus\mathfrak{g}''$ [ArhI, Lemma 12.1(4)].

5. In the root coordinates, s and t have half-integral coordinates, $L_0=\{x\in L:\sum_i x_i\in 2\mathbb{Z}\}$, and λ, $2p$, $\frac{t-s}{2}$ belong to L_0, while $p\notin L_0$ and $s\notin L$. The lattice L_H is even, $(s,L_0)\subset\mathbb{Z}$ and $(p,L_0)\subset\mathbb{Z}$. For the lattice intersections, use the rational plane $\mathfrak{h}_{\mathbb{Q}}=\mathbb{Q}p\oplus\mathbb{Q}b$ inside $\mathfrak{h}$, and intersection with either plane gives the same lattices. The intersections are

$$L\cap\mathfrak{h}=L_{\rm II},\quad L_H\cap\mathfrak{h}=L_{\rm I},\quad L_0\cap\mathfrak{h}=N, \tag{7.5}$$

and $E\overset{\rm def}{=}L\cap\mathfrak{h}^\perp=L_H\cap\mathfrak{h}^\perp=L_0\cap\mathfrak{h}^\perp$ is a lattice of rank six.

Proof : All assertions are finite computations with rational numbers in the coordinates (p,b) or in the root coordinates. They are carried out by the certified scripts of Section 10 (certificate A1), and we only indicate the identifications and the arguments which replace an enumeration. In the coordinates (m',m'') of [ArhI, Section 12.1] for case (II), the vector (m',m'') is $\frac{m'}{6}p+\frac{m''}{26}b$, so that $L_{\rm II}=\{(3a,13c):a\equiv c\bmod 2\}$ is $\mathbb{Z}p\oplus\mathbb{Z}\lambda$, and for case (I) the vector (m',m'') is $\frac{m'}{2}s+\frac{m''}{78}t$, so that $L_{\rm I}=\{(a,39c):a\equiv c\bmod 2\}$ is $\mathbb{Z}s\oplus\mathbb{Z}\frac{s+t}{2}$. The generators $(1,1)$ are $\frac{p}{6}+\frac{b}{26}$ and $\frac{s}{2}+\frac{t}{78}$, with q-values $\frac{4}{39}$ and $\frac{10}{39}$. An even overlattice M of $L_{\rm II}$ in $\mathfrak{h}$ satisfies $L_{\rm II}\subset M\subset L'_{\rm II}$

and M/L_{II} is isotropic for q. Since $\frac{4k^2}{39} \in \mathbb{Z}$ forces $39 \mid k$, and likewise $\frac{10k^2}{39}$, the only isotropic subgroup of $L'_{\mathrm{II}}/L_{\mathrm{II}} \cong \mathbb{Z}_{39}$ and of $L'_{\mathrm{I}}/L_{\mathrm{I}}$ is trivial, which proves the last assertion of part 1. For part 2, the bases $L_{\mathrm{I}} = \mathbb{Z}s + \mathbb{Z}2p$ and $L_{\mathrm{II}} = \mathbb{Z}p + \mathbb{Z}2s$ of Remark 7.2 give $L_{\mathrm{I}} \cap L_{\mathrm{II}} = \mathbb{Z}2s + \mathbb{Z}2p = N$, the two indices and the two coset decompositions, and the two characters are computed on these bases from $(s,p) = \frac{3}{2}$, $(s,2s) = 4$ and $(p,2p) = 12$. The values in $\frac{1}{2}\mathbb{Z}$ follow from $2s \in L_{\mathrm{II}}$, $2p \in L_{\mathrm{I}}$, $(s, \frac{p}{6} + \frac{b}{26}) = \frac{1}{2}$ and $(p, \frac{s}{2} + \frac{t}{78}) = 1$. For part 3, the order-two classes $s+N$ and $p+N$ have q-values zero and pairing $\frac{1}{2}$ modulo $\mathbb{Z}$, since $(s,p) = \frac{3}{2}$. They generate a nondegenerate subgroup T of order four in N'/N. Its orthogonal complement has order 39 and is the odd-primary subgroup P. The characters $x + N \mapsto \mathrm{e}^{2\pi\mathrm{i}(s,x)}$ and $x + N \mapsto \mathrm{e}^{2\pi\mathrm{i}(p,x)}$ are nontrivial of order two, and their common kernel is P. This proves the asserted containment in $(L'_{\mathrm{I}} \cap L'_{\mathrm{II}})/N$. Since L'_{I}/N has order 78 and contains the 39-part, the restriction of $L'_{\mathrm{I}}/N \to L'_{\mathrm{I}}/L_{\mathrm{I}}$ to the 39-part is a bijection, and it preserves q since both groups carry the form induced from N'. The subgroup $\{x \in L'_{\mathrm{II}} : (s,x) \in \mathbb{Z}\}$ has index two in L'_{II} and contains N, so its image in N'/N has order 39 and is the unique subgroup P of order 39 of N'/N, and it equals the preimage B of P in N' because $P \subset L'_{\mathrm{II}}/N$. The same argument applies to $\{x \in L'_{\mathrm{I}} : (p,x) \in \mathbb{Z}\}$, which proves (7.4), and $p \notin \mathsf{B}$, $s \notin \mathsf{B}$ since $(s,p) = \frac{3}{2}$. The value $u = 22$ and the identification of the four isometries follow from $22 \cdot \{1, 14, 25, 38\} \equiv \{22, 35, 4, 17\} \bmod 39$. For part 5, $(\lambda, \alpha_i) \in \{\pm 1\}$ for the eight simple roots, which gives the description of L_0. The lattice $L \cap \mathfrak{h}$ is an even lattice of $\mathfrak{h}$ containing L_{II}, since $p, \lambda \in L$, and it equals L_{II} by part 1. Likewise L_H is even, it contains s and $\frac{s+t}{2} = s + \frac{t-s}{2}$, so $L_H \cap \mathfrak{h} = L_{\mathrm{I}}$. Finally $L_0 = L \cap L_H$, since $s \notin L$, so $L_0 \cap \mathfrak{h} = L_{\mathrm{I}} \cap L_{\mathrm{II}} = N$, and a vector of L_H orthogonal to $\mathfrak{h}$ lies in L_0, since $(p, x+s) = (p,x) + \frac{3}{2} \notin \mathbb{Z}$ for $x \in L_0$. ∎

Remark 7.2 The rank-two part admits a short exact calculation without a program. Put $v = (s+t)/2 = (7p-b)/4 = 2p-s$. Then $L_{\mathrm{I}} = \mathbb{Z}s + \mathbb{Z}2p$, $L_{\mathrm{II}} = \mathbb{Z}p + \mathbb{Z}2s$, and p, s are rationally independent. Comparing their two coefficients gives $L_{\mathrm{I}} \cap L_{\mathrm{II}} = \mathbb{Z}2p + \mathbb{Z}2s = N$ and both indices are two. The Gram matrix of N is $\left[\begin{smallmatrix} 8 & 6 \\ 6 & 24 \end{smallmatrix}\right]$. Its Smith invariants are $2, 78$, so its odd-primary part is cyclic of order 39. The preceding nondegenerate order-four subgroup gives its orthogonal splitting. For the canonical discriminant isometry, the case-II generator $g = p/6 + b/26$ satisfies $(s,g) = 1/2$, so $g+p$ represents its unique lift in P. The case-I generator is $g' = s/2 + t/78$. Direct subtraction gives $(g+p) - 22g' = -10s \in L_{\mathrm{I}}$, proving the multiplier 22. The four isometries solve $10u^2 \equiv 4 \pmod{39}$, or $u^2 \equiv 16 \pmod{39}$, and the Chinese remainder theorem gives $u \in \{4, 17, 22, 35\}$. Block A1 of Section 10 also checks these identities and the rank-eight coordinates.

Remark 7.3 In [DJX26, Section 5] the vectors p, b, s, t appear in the root lattice L of V', the lattice L_0 is the kernel of the character $x \mapsto \mathrm{e}^{2\pi\mathrm{i}(s,x)} = (-1)^{(\lambda,x)}$ of L, and L_H is the image of the copy L^{H} of $E_6 \oplus A_2$ in Huang's coordinates under an isometry $R_* \colon L^{\mathrm{H}} \to L_H$ with $R_*\ell = s$, $R_*h = t$ for the Cartan vectors ℓ, $h \in L^{\mathrm{H}}$ of Huang's currents of levels 1 and 39 [DJX26, Proposition 5.1 and (6.10)]. The lattices of parts 1 and 2 are the intersections (7.5) of these rank-eight lattices with the Cartan subalgebra $\mathfrak{h}$, which is spanned by the Cartan currents of S and, after transport by R_*, by those of U. The involutions σ and τ of (D2) act on the lattice vertex operator algebras V_L and V_{L_H} through the characters $x \mapsto \mathrm{e}^{2\pi\mathrm{i}(s,x)}$ and $x \mapsto \mathrm{e}^{2\pi\mathrm{i}(p,x)}$ of part 2, both trivial on L_0 by part 5, and the group $\{0, s, p, s+p\} + N$ of part 3 is the fusion group $T \cong Z(\mathrm{Vec}_{\mathbb{Z}_2})$ of [DJX26, Lemma 9.1], with $a = s + N$, $b = p + N$ and the fermion $ab = s + p + N$. Since $t \in s + L_0$ has half-integral root coordinates, t lies in L_H and not in L, see item 2 of Section 8.2.

7.2 The Heisenberg commutant and the twin of S

Section 12 of [ArhI] is written for a Haagerup algebra U, that is, a positive strongly rational vertex operator algebra of central charge eight with $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. By Theorem 6.1, S is a Haagerup algebra, so that [ArhI, Proposition 12.2 and Theorem 12.3] apply to S. We go through these statements for $U = S$ with ($\mathrm{H_J}$) replaced by (I1) and (F1)–(F4), so that Theorem 7.6, Lemma 7.8 and Theorem 7.9 do not depend on Sections 4–6. It is convenient to name the class of algebras which replaces the Haagerup algebras.

Definition 7.4 *A vertex operator algebra W is* of type S *if it is simple, of CFT type, strongly rational, positive, of central charge eight, and* $\mathrm{Rep}(W)$ *is ribbon equivalent to* $\mathrm{Rep}(S)$.

By (F1) and (F4), S is of type S, and by (F4) and Lemma 9.2 the category $\mathrm{Rep}(W)$ of an algebra of type S is pseudo-unitary and has no invertible object besides the unit. By Theorem 6.1, the algebras of type S are exactly the Haagerup algebras. This is not used in Sections 7.2–7.4. Let $\mathfrak{h} = \mathbb{C}p \oplus \mathbb{C}b \subset S_1 = \mathfrak{g}' \oplus \mathfrak{g}''$ be the Cartan subalgebra spanned by the Cartan currents of levels 3 and 13, with the normalization of Section 7.1. The Heisenberg commutant of S is $\Omega_S = \mathrm{Com}_S(M(\mathfrak{h}))$, where $M(\mathfrak{h})$ is the Heisenberg vertex operator algebra generated by $\mathfrak{h}$ [MoR24a, Definition 2.12]. For a strongly rational vertex operator algebra Ω, we write $P(\Omega)$ for the group of isomorphism classes of invertible simple Ω-modules, with the quadratic form θ given by the twists, and Ω^ν for the module of class ν.

Proposition 7.5 *Assume* (I1) *and* (F1)–(F4).

1. *Ω_S is a strongly rational vertex operator algebra of central charge six with $(\Omega_S)_1 = 0$. The commutant of Ω_S in S is the lattice vertex operator algebra $V_{L_{\mathrm{II}}}$, with $L_{\mathrm{II}} = L \cap \mathfrak{h}$ as in (7.5), and there is an anti-isometry $\tau\colon L'_{\mathrm{II}}/L_{\mathrm{II}} \to P(\Omega_S)$ onto the group of invertible simple Ω_S-modules such that*

$$S = \bigoplus_{\alpha \in L'_{\mathrm{II}}/L_{\mathrm{II}}} \Omega_S^{\tau(\alpha)} \otimes V_{\alpha + L_{\mathrm{II}}} \tag{7.6}$$

   as an $\Omega_S \otimes V_{L_{\mathrm{II}}}$-module, and $\mathrm{Rep}(\Omega_S) \cong \mathrm{Rep}(S) \boxtimes \overline{\mathcal{C}(L'_{\mathrm{II}}/L_{\mathrm{II}})}$ as ribbon categories.

2. *Every simple Ω_S-module has a nonnegative lowest conformal weight, and for $\alpha \neq 0$ the module $\Omega_S^{\tau(\alpha)}$ contains no vacuum-like vector. Consequently a vector of S of $\mathfrak{h}$-weight γ has conformal weight at least $\frac{(\gamma,\gamma)}{2}$.*

3. *The set of $\mathfrak{h}$-weights of S is L'_{II}, and $e^\lambda \in V_{L_{\mathrm{II}}} \subset S$ generates the simple current $J = L_{\mathfrak{sl}_2}(3,3) \otimes L_{\mathfrak{sl}_2}(13,13)$ of the affine subalgebra A.*

Proof : This is [ArhI, Proposition 12.2], whose proof we follow, indicating each place where the hypothesis ($\mathrm{H_J}$) was used and what replaces it. (a) The dual pair (Ω_S, V_{L_S}), the strong rationality of Ω_S and the decomposition of S as a simple current extension of $\Omega_S \otimes V_{L_S}$ over a subgroup $A_S \subset L'_S/L_S$ use [MoR24a, Definition 2.12 and Proposition 2.14], [Mas14, Theorem 1] and [CKLR19], which require S to be simple, strongly rational and of CFT type, with S_1 reductive and $\mathfrak{h}$ a Cartan subalgebra of S_1. These are (F1) and (I1). (b) The proof of [MoR24a, Corollary 4.10], applied with $V = W = S$, gives $A_S = L'_S/L_S$, the anti-isometry τ and the equivalence $\mathrm{Rep}(\Omega_S) \cong \mathrm{Rep}(S) \boxtimes \overline{\mathcal{C}(L'_S/L_S)}$. It uses three properties of $\mathrm{Rep}(S)$: it has no invertible object besides the unit, which is Lemma 9.2 under (F4), it is pseudo-unitary, which is (F4) or, alternatively, follows from the positivity in (F1) by [DJX13, Lemma 4.2] and

[Hua08], and it is unpointed, which is the first property again. (c) The central charge of Ω_S is $8 - \operatorname{rank} L_S = 6$, and $(\Omega_S)_1 = 0$ because a vector of $(\Omega_S)_1$ has $\mathfrak{h}$-weight zero in $S_1 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (I1), hence lies in $\mathfrak{h}$ and is orthogonal to $\mathfrak{h}$. (d) Part 2 is proved in [ArhI] from Zhu's theorem for S and for Ω_S, from the nonnegativity of the conformal weights of the simple S-modules, which is (F1), and from the extension theory of [HKL15, CKM22]. It does not use the category $Z(\mathcal{H}_3)$. (e) Part 3 uses $V_1' = S_1 \oplus R_b$, which is (F3) and (F2) with $h_b = 1$, the highest weights (a, b), $a \equiv b \bmod 2$, of the S_1-summands of R_b, which are read in the column (II) of [ArhI, Table 2] and are part of (F2), the fact that V' is generated by V_1', the roots $(2, 0)$, $(0, 2)$ of S_1, and the occurrence of the S_1-module $V(1, 1)$ in S_2, which is explicit in [ArhI, Section 10]. Hence the set of $\mathfrak{h}$-weights of S is the group Γ_0 of [ArhI, Proposition 12.2], which is L_{II}', and $L_S = L_{\mathrm{II}}$. Since $V_{L_S} \subset S \subset V'$ is the lattice vertex operator algebra of the Cartan subalgebra $\mathfrak{h}$, it is contained in $V_{L \cap \mathfrak{h}}$, and $L \cap \mathfrak{h} = L_{\mathrm{II}}$ by (7.5), so that $V_{L_S} = V_{L_{\mathrm{II}}} = V_{L \cap \mathfrak{h}}$. (f) The statement on J is [ArhI, Proposition 11.4], proved there by an explicit lattice word, or [ArhI, Proposition 12.2(4)], whose proof uses part 2 and the regularity of A only. ∎

Let $\psi\colon L_{\mathrm{II}}'/L_{\mathrm{II}} \to L_{\mathrm{I}}'/L_{\mathrm{I}}$ be an isometry, induced by a ribbon equivalence $\Phi_\psi\colon \mathcal{C}(L_{\mathrm{II}}'/L_{\mathrm{II}}) \to \mathcal{C}(L_{\mathrm{I}}'/L_{\mathrm{I}})$ [EGNO15, Section 8.4]. By [HKL15], S corresponds to the connected étale algebra $\mathcal{A}_S = \bigoplus_\alpha \tau(\alpha) \boxtimes \alpha$ of $\operatorname{Rep}(\Omega_S) \boxtimes \mathcal{C}(L_{\mathrm{II}}'/L_{\mathrm{II}}) \cong \operatorname{Rep}(\Omega_S \otimes V_{L_{\mathrm{II}}})$, and we define $S_\psi^\natural$ as the vertex operator algebra extension of $\Omega_S \otimes V_{L_{\mathrm{I}}}$ corresponding to the connected étale algebra $(\mathrm{id} \boxtimes \Phi_\psi)(\mathcal{A}_S)$ [HKL15, CKM22]. As an $\Omega_S \otimes V_{L_{\mathrm{I}}}$-module,

$$S_\psi^\natural = \bigoplus_{\alpha \in L_{\mathrm{II}}'/L_{\mathrm{II}}} \Omega_S^{\tau(\alpha)} \otimes V_{\psi(\alpha)+L_{\mathrm{I}}}. \tag{7.7}$$

Theorem 7.6 *Assume* (I1) *and* (F1)–(F4)*, and let ψ be as above.*

1. *$S_\psi^\natural$ is an algebra of type S. It is self-contragredient, its Heisenberg commutant for the Cartan subalgebra $\mathfrak{h}^\natural = \mathbf{1} \otimes (L_{\mathrm{I}} \otimes \mathbb{C})_{(-1)}\mathbf{1}$ of $(S_\psi^\natural)_1$ is Ω_S, and its associated lattice is L_{I}.*
2. *The image of $\operatorname{Aut}(\Omega_S)$ in $O(P(\Omega_S))$ is the whole group. Consequently, for $\Omega \cong \Omega_S$ and $M \in \{L_{\mathrm{I}}, L_{\mathrm{II}}\}$, all the algebras of type S with Heisenberg commutant isomorphic to Ω and associated lattice isometric to M are isomorphic.*
3. *Up to isomorphism, $S_\psi^\natural$ does not depend on ψ. We write $S^\natural$ for it. We have $(S^\natural)^\natural \cong S$ and $S^\natural \ncong S$, and the algebras of type S with Heisenberg commutant isomorphic to Ω_S are, up to isomorphism, exactly S and $S^\natural$.*
4. *S and $S^\natural$ belong to the same hyperbolic genus in the sense of [MoR24a, Definition 4.3], and every strongly rational vertex operator algebra in this hyperbolic genus is isomorphic to S or to $S^\natural$. In particular it is of type S.*

Proof : We follow the proof of [ArhI, Theorem 12.3] with Proposition 7.5 in place of [ArhI, Proposition 12.2], and we indicate the two places where the Haagerup hypothesis entered through [ArhI, Theorem 8.5].

1. The paragraphs *extension and representation category*, *CFT type*, *self-contragredience*, *C_2-cofiniteness*, *rationality* and *positivity* of the proof of [ArhI, Theorem 12.3] use [HKL15, Theorem 3.2 and Remark 3.3], [CKM22], [DMNO13], [Li94], [Buh02], [ABD04], the positivity of S, which is (F1), and parts 1 and 2 of Proposition 7.5. They show that $W = S_\psi^\natural$ is a simple, self-contragredient, strongly rational vertex operator algebra of CFT type and central charge $6 + 2 = 8$, that $\operatorname{Rep}(W)$ is ribbon equivalent, through $\mathrm{id} \boxtimes \Phi_\psi$, to the category of local $\mathcal{A}_S$-modules, that is, to $\operatorname{Rep}(S)$, and that W is positive. Hence W is of type S. At this point

[ArhI] concludes that W is a Haagerup algebra and invokes [ArhI, Theorem 8.5] to place it in case (I) or (II). We do not use this theorem here. The paragraph *Heisenberg commutant* of [ArhI] shows, using only $(\Omega_S)_1 = 0$, the reductivity of W_1 [DoM04a, Theorem 1] and part 2 of Proposition 7.5, that $\mathfrak{h}^\natural$ is a Cartan subalgebra of W_1, that $\mathrm{Com}_W(M(\mathfrak{h}^\natural)) = \Omega_S \otimes \mathbf{1}$ and that $\mathrm{Com}_W(\Omega_S) = V_{L_\mathrm{I}}$. This is part 1. The last sentence of this paragraph in [ArhI], which identifies the case of W and the Lie algebra W_1, is not used here. We determine W_1 in Corollary 7.10 below by a different argument.

2. The proof of [ArhI, Theorem 12.3(2)] uses the anti-isometry τ, the Weyl reflections of $\mathfrak{g}'$ and $\mathfrak{g}''$, which are inner automorphisms of S normalizing $\mathfrak{h}$, preserving Ω_S and V_{L_II} and inducing 14 and 25 on $L'_\mathrm{II}/L_\mathrm{II}$ by [ArhI, Lemma 12.1(4)], and the uniqueness statement [DoM04a, Proposition 5.3] for two simple vertex operator algebras which are simple current extensions of $\Omega \otimes V_M$ with the same summands. The decomposition of an algebra Y of type S with Heisenberg commutant Ω and associated lattice M as $\bigoplus_\alpha \Omega^{\sigma(\alpha)} \otimes V_{\alpha+M}$ with σ an anti-isometry onto $P(\Omega)$ is given by the proof of [MoR24a, Corollary 4.10], applied to Y as in step (b) of the proof of Proposition 7.5: it uses only that $\mathrm{Rep}(Y) \cong \mathrm{Rep}(S)$ is unpointed and pseudo-unitary. No other property of $Z(\mathcal{H}_3)$ is used.

3. The independence of ψ and the involutivity follow from parts 1 and 2 as in [ArhI]. The lattices L_I and L_II are not isometric, and an isomorphism of vertex operator algebras maps a Cartan subalgebra of the weight-one Lie algebra to a Cartan subalgebra, all of which are conjugate under inner automorphisms, so it identifies the Heisenberg commutants and the associated lattices. Hence $S^\natural \ncong S$. For the last assertion, [ArhI] uses [ArhI, Proposition 12.2(3)] for an arbitrary Haagerup algebra W with $\Omega_W \cong \Omega_S$, which rests on [ArhI, Theorem 8.5]. We argue instead as follows. Let W be of type S with $\Omega_W \cong \Omega_S$. By [MoR24a, Corollary 4.10], whose hypotheses are that $\mathrm{Rep}(W) \cong \mathrm{Rep}(S)$ is unpointed and pseudo-unitary, that W and S have the same central charge and isomorphic Heisenberg commutants, W and S belong to the same hyperbolic genus. By [MoR24a, Definition 4.6 and Theorem 4.7] the associated lattice of W is in the genus of L_II, hence isometric to L_I or L_II by [ArhI, Lemma 12.1(2)], and part 2 gives $W \cong S$ or $W \cong S^\natural$.

4. This is the proof of [ArhI, Theorem 12.3(4)], which uses [MoR24a, Corollary 4.10 and Theorems 4.7, 4.8] and part 2, with the hypotheses of [MoR24a, Corollary 4.10] verified as in part 3. The conclusion "in particular it is a Haagerup algebra" becomes "it is of type S", since $\mathrm{Rep}(V) \cong \mathrm{Rep}(S)$ by [MoR24a, Theorem 4.8] for a strongly rational V in the genus, and the positivity of V follows from its decomposition over $\Omega_S \otimes V_M$ and part 2 of Proposition 7.5 as in the paragraph *positivity* of part 1. ∎

Remark 7.7 Theorem 7.6 gives the lattice exchange and the classification in the hyperbolic genus of S under its stated inputs. For classification among all Haagerup algebras we use [ArhI, Theorem 12.4(1),(4)] with the Haagerup property of S from Theorem 6.1. Theorem 7.9 below identifies the explicit case (I) algebra using (D1) and (D2).

7.3 The two involutions and the common fixed point algebra

We now use the lattice N of Lemma 7.1. Let $B = \Omega_S \otimes V_N$. Since $N \subset L_\mathrm{II}$ and $N \subset L_\mathrm{I}$ have index two, B is a vertex operator subalgebra of $\Omega_S \otimes V_{L_\mathrm{II}} \subset S$ and of $\Omega_S \otimes V_{L_\mathrm{I}} \subset S^\natural$, with the same conformal vector. It is strongly rational, with $\mathrm{Rep}(B) \cong \mathrm{Rep}(\Omega_S) \boxtimes \mathcal{C}(N'/N)$ [FHL93, CKM22]. For $\gamma \in \mathsf{B}$, in the notation (7.4), we write $\bar\gamma = \gamma + L_\mathrm{II} \in L'_\mathrm{II}/L_\mathrm{II}$, so that $\psi_N(\bar\gamma) = \gamma + L_\mathrm{I}$ by definition of ψ_N, and we put

$$M_{\gamma+\varepsilon s+\delta p+N} \stackrel{\mathrm{def}}{=} \Omega_S^{\tau(\bar\gamma)} \otimes V_{\gamma+\varepsilon s+\delta p+N}, \quad \gamma \in \mathsf{B},\ \varepsilon,\delta \in \{0,1\}. \tag{7.8}$$

These are pairwise non-isomorphic invertible simple B-modules, indexed by $N'/N = P \oplus \{0, s, p, s+p\}$, and their fusion is $M_x \boxtimes M_y \cong M_{x+y}$, since τ and π_{II} are homomorphisms and the fusion of lattice modules is given by the addition of cosets [DoL93]. The twists are $\theta_{M_x} = \mathrm{e}^{2\pi \mathrm{i} h_x}$, where, for $x = \gamma + \varepsilon s + \delta p + N$,

$$h_x \equiv h_{\tau(\bar\gamma)} + \frac{(\gamma+\varepsilon s+\delta p)^2}{2} \equiv \varepsilon\frac{(s,s)}{2} + \delta\frac{(p,p)}{2} + \varepsilon\delta(s,p) \equiv \frac{3\varepsilon\delta}{2} \pmod{\mathbb{Z}}, \tag{7.9}$$

because $h_{\tau(\bar\gamma)} + \frac{(\gamma,\gamma)}{2} \in \mathbb{Z}$, the weights of S being integral, and $(\gamma, s), (\gamma, p) \in \mathbb{Z}$ for $\gamma \in \mathsf{B}$ by (7.4).

The automorphisms $\sigma = \exp(2\pi \mathrm{i} s_{(0)})$ of S and $\tau' = \exp(2\pi \mathrm{i} p_{(0)})$ of $S^\natural$ are well defined: $s_{(0)}$ acts on the summand $\Omega_S^{\tau(\alpha)} \otimes V_{\alpha+L_{\mathrm{II}}}$ of (7.6) by the scalar (s, x) on the vectors of $\mathfrak{h}$-weight $x \in \alpha + L_{\mathrm{II}}$, which lies in $\frac12\mathbb{Z}$ by Lemma 7.1(2), and likewise for $p_{(0)}$ on (7.7). Since $s = \frac{p+b}{4}$, the automorphism σ is the involution $\exp(\frac{\pi\mathrm{i}}{2}(P_H + B_H)_{(0)})$ of (D2), and τ' is the transport of the involution $\tau = \exp(\frac{\pi\mathrm{i}}{2}(3H_L + H_R)_{(0)})$ of U, since $R_*^{-1} p = \frac{3\ell+h}{4}$ [DJX26, Section 5].

Lemma 7.8 *Assume* (I1) *and* (F1)–(F4), *and take* $\psi = \psi_N$ *in* (7.7).

1. *σ and τ' are automorphisms of order two of S and of $S^\natural$. As B-modules,*

$$S^\sigma = \bigoplus_{\gamma\in P} M_\gamma, \quad S^- \stackrel{\mathrm{def}}{=} \{v \in S : \sigma v = -v\} = \bigoplus_{\gamma\in P} M_{\gamma+p},$$

$$(S^\natural)^{\tau'} = \bigoplus_{\gamma\in P} M_\gamma, \quad (S^\natural)^- \stackrel{\mathrm{def}}{=} \{v \in S^\natural : \tau' v = -v\} = \bigoplus_{\gamma\in P} M_{\gamma+s},$$

   where γ runs over representatives in B of the classes of P. In particular S^σ and $(S^\natural)^{\tau'}$ are simple current extensions of B with the same summands.

2. *Put $C = S^\sigma$. Then C is a simple, self-contragredient, strongly rational vertex operator algebra of CFT type and central charge eight, and S^- is a simple C-module, not isomorphic to C. The same holds for $(S^\natural)^{\tau'}$ and $(S^\natural)^-$, and $C \cong (S^\natural)^{\tau'}$.*

3. *Let X be a simple C-module whose restriction to B contains a submodule isomorphic to M_x for some $x \in \{s, p, s+p\} + N$. Then $X \cong \mathrm{Ind}(M_x) \stackrel{\mathrm{def}}{=} \bigoplus_{\gamma\in P} M_{\gamma+x}$ as C-modules, where $\mathrm{Ind}(M_x)$ carries the induced module structure of [HKL15, CKM22]. In particular $S^- \cong \mathrm{Ind}(M_p)$.*

Proof : 1. The automorphism σ acts on the vectors of $\mathfrak{h}$-weight x by $\mathrm{e}^{2\pi\mathrm{i}(s,x)} = \pm 1$, and $\sigma \neq 1$ since $(s, p) = \frac32$ and $e^p \in V_{L_{\mathrm{II}}} \subset S$. On the summand $\Omega_S^{\tau(\alpha)} \otimes V_{\alpha+L_{\mathrm{II}}}$ of (7.6), the fixed points of σ are $\Omega_S^{\tau(\alpha)} \otimes V_{(\alpha+L_{\mathrm{II}})\cap\mathsf{B}}$, and by (7.4) and $L_{\mathrm{II}} = N \sqcup (p+N)$ the set $(\alpha + L_{\mathrm{II}}) \cap \mathsf{B}$ is a single coset $\gamma + N$ with $\gamma \in \mathsf{B}$, $\bar\gamma = \alpha$, while the other coset $\gamma + p + N$ of $\alpha + L_{\mathrm{II}}$ is the -1-eigenspace. This gives the decompositions of S^σ and S^-. For $S^\natural_{\psi_N}$, the summand indexed by $\alpha \in L'_{\mathrm{II}}/L_{\mathrm{II}}$ in (7.7) is $\Omega_S^{\tau(\alpha)} \otimes V_{\psi_N(\alpha)+L_{\mathrm{I}}}$, and $\psi_N(\alpha) = \gamma + L_{\mathrm{I}}$ for the representative $\gamma \in \mathsf{B}$ of α, by definition of $\psi_N = \pi_{\mathrm{I}}\pi_{\mathrm{II}}^{-1}$. The involution τ' acts by $\mathrm{e}^{2\pi\mathrm{i}(p,x)}$ and its fixed points in $V_{\gamma+L_{\mathrm{I}}}$ are $V_{\gamma+N}$, since $L_{\mathrm{I}} = N \sqcup (s+N)$ and $(p, \gamma) \in \mathbb{Z}$, $(p, s) = \frac32$. This gives the decompositions of $(S^\natural)^{\tau'}$ and $(S^\natural)^-$, with the same Ω_S-labels $\tau(\bar\gamma)$ as for S.

2. The algebra S is simple, regular [ABD04], non-negatively graded, with the nonsingular invariant bilinear form given by its unitary structure (I1). By [CaM16, Theorem 5.24 p. 33], the fixed point algebra $C = S^\sigma$ is regular, and by [DoM97, Theorem 4.4] it is simple, and S^-, the

-1-eigenspace of σ, is a simple C-module not isomorphic to C [DoM97, Theorem 3]. It is of CFT type, since $C_0 = S_0$, and self-contragredient, since $L(1)C_1 \subset L(1)S_1 = 0$ [Li94]. The algebra $S^\natural$ is simple, strongly rational, of CFT type and self-contragredient by Theorem 7.6(1), and the invariant bilinear form of a simple self-contragredient algebra of CFT type is nonsingular [Li94], so the same references apply to τ'. Finally C and $(S^\natural)^{\tau'}$ are simple vertex operator algebras which are simple current extensions of B with the same pairwise non-isomorphic invertible summands M_γ, $\gamma \in P$, with fusion $M_\gamma \boxtimes M_{\gamma'} \cong M_{\gamma+\gamma'}$ and multiplicities one. Since B is strongly rational, the space of intertwining operators of type $\binom{M_z}{M_x\ M_y}$ has dimension $\dim \mathrm{Hom}_B(M_x \boxtimes M_y, M_z) = \delta_{z,x+y}$ [HuL95, Hua08]. Hence the vertex operators of both algebras map $M_x \otimes M_y$ into $M_{x+y}((z))$, both algebras are P-graded in the sense of [DoM04a, Definition 5.2], and the fusion rules satisfy the hypothesis $N_{M_x,M_y}^{M_z} = \delta_{z,x+y}$ of [DoM04a, Proposition 5.3]. Transporting the vertex operators of $(S^\natural)^{\tau'}$ to C through an isomorphism of B-modules which is the identity on $M_0 = B$, we obtain from [DoM04a, Proposition 5.3] an isomorphism $C \cong (S^\natural)^{\tau'}$, which acts by a nonzero scalar on each summand M_γ, as in the proof of [ArhI, Theorem 12.3(2)].

3. The monodromy of M_x with the summand M_γ of C is the scalar $\theta_{M_{\gamma+x}}\theta_{M_\gamma}^{-1}\theta_{M_x}^{-1} = \mathrm{e}^{2\pi\mathrm{i}(\gamma,x)} = 1$ by (7.9) and (7.4). Hence $\mathrm{Ind}(M_x) = C \boxtimes_B M_x = \bigoplus_{\gamma\in P} M_{\gamma+x}$ is a C-module [HKL15, CKM22], a finite direct sum of simple C-modules since C is regular. By the adjunction between induction and restriction [KiO02, Theorem 1.6 p. 7],

$$\mathrm{End}_C(\mathrm{Ind}(M_x)) \cong \mathrm{Hom}_B(M_x, \mathrm{Ind}(M_x)) = \bigoplus_{\gamma\in P} \mathrm{Hom}_B(M_x, M_{\gamma+x}) = \mathbb{C},$$

since $M_{\gamma+x} \cong M_x$ only for $\gamma \in N$. Thus $\mathrm{Ind}(M_x)$ is a simple C-module. The same adjunction gives a nonzero C-module map $\mathrm{Ind}(M_x) \to X$ from the inclusion $M_x \to X$, and it is an isomorphism since both modules are simple. For S^-, take $x = p$ and the submodule $M_p \ni e^p$. ∎

7.4 The identification $U \cong S^\natural$

In this subsection (I3) is used to transport the fixed-point statement (D2) from Jiao's algebra to S, including its identified Cartan currents. Let $U = U^{\mathrm{H}} \subset V_{L^{\mathrm{H}}}$ be the vertex subalgebra generated by the explicit vectors of Huang and by the conformal vector ω of $V_{L^{\mathrm{H}}}$ [DJX26, (4.6)–(4.9)], where L^{H} is the copy of $E_6 \oplus A_2$ in Huang's coordinates of Remark 7.3, and let τ be the involution of (D2). Following [DJX26, Sections 5–7], let $\hat{\Phi}\colon V_{L^{\mathrm{H}}} \to V_{L_H}$ be a lift of the isometry $R_*\colon L^{\mathrm{H}} \to L_H$ of Remark 7.3, extending the lift Φ of [DJX26, (6.13)] on $V_{R_*^{-1}L_0}$ [DJX26, p. 21], and let $\tilde{U} = \hat{\Phi}(U) \subset V_{L_H}$ be the transported algebra. The two copies of V_{L_0} in V_{L_H} and in $V_L = V'$ are identified by the cocycle-compatible isomorphism of [DJX26, (5.7)], which maps Pe^x to $\eta(x)Pe^x$ for a function $\eta\colon L_0 \to \mathbb{C}^\times$. Through this identification, which we suppress from the notation, [DJX26, Theorem 7.3] states that

$$\tilde{U}^\tau = S^\sigma = C, \quad \tilde{U} = C \oplus \tilde{U}^-, \quad S = C \oplus S^- , \tag{7.10}$$

where $\tilde{U}^-$ is the -1-eigenspace of τ in $\tilde{U}$, and τ acts on V_{L_H} by $\mathrm{e}^{2\pi\mathrm{i}(p,x)}$ on the vectors of $\mathfrak{h}$-weight x. The subalgebra $B = \Omega_S \otimes V_N$ of C lies in V_{L_0}: indeed $V_N \subset V_{L_0}$ since $N = L_0 \cap \mathfrak{h}$, and $\Omega_S \subset \mathrm{Com}_{V'}(M(\mathfrak{h})) = V_E \subset V_{L_0}$ with $E = L \cap \mathfrak{h}^\perp = L_0 \cap \mathfrak{h}^\perp$ by Lemma 7.1(5).

Theorem 7.9 *Assume* (I1), (I3), (F1)–(F4) *and* (D2). *Then* $U \cong \tilde{U} \cong S^\natural$ *as vertex operator algebras. More precisely, as* B*-modules,*

$$\tilde{U} \cong \bigoplus_{\gamma\in P} M_\gamma \oplus \bigoplus_{\gamma\in P} M_{\gamma+s} \cong S^\natural_{\psi_N}.$$

Proof : *Step 1: U is simple.* The lattice vertex operator algebra $V_{L^{\mathrm{H}}}$ carries the positive definite invariant Hermitian form $(\cdot,\cdot)_{\kappa_H}$ and the antilinear involution θ_H of [DJX26, Section 4.4], in the sense of [DoL14, Theorem 4.12], and $\theta_H(U) = U$ by [DJX26, (6.15)], an identity on the generators verified in Section 8.2. The restriction to U of the bilinear form $(\theta_H a, b)_{\kappa_H}$ is a nondegenerate invariant bilinear form on U for the conformal vector $\omega \in U$ of $V_{L^{\mathrm{H}}}$ [DJX26, (4.9)]. A proper ideal I of U contains no nonzero vector of weight zero, and for $a \in I$, $b \in U$ the invariance expresses (a,b) as a combination of pairings $(a', \mathbf{1})$ with $a' \in I$ [FHL93, Li94], so that $(I,U) = 0$ and $I = 0$. Hence U, and $\tilde{U} \cong U$, are simple vertex operator algebras of CFT type.

Step 2: the two eigenspaces. By Step 1 and [DoM97, Theorems 3 and 4.4], $\tilde{U}^{\tau} = C$ is simple and $\tilde{U}^{-}$ is a simple C-module, not isomorphic to C. The current $E_L = e^{\ell}$ of Huang is mapped by $\hat{\Phi}$ to a nonzero multiple of e^s, since $R_*\ell = s$, and $\tau(E_L) = -E_L$ since $(p,s) = \frac{3}{2}$. Hence $e^s \in \tilde{U}^{-}$. In V_{L_H}, the $V_E \otimes V_N$-submodule generated by e^s is $V_E \otimes V_{s+N}$, because $s \in \mathfrak{h}$ has zero component along E, and the B-submodule generated by e^s is therefore $\Omega_S \otimes V_{s+N} = M_s$, the vector $e^s = \mathbf{1} \otimes e^s$ being a vacuum vector for $\Omega_S \subset V_E$. By Lemma 7.8(3), $\tilde{U}^{-} \cong \mathrm{Ind}(M_s) = \bigoplus_{\gamma \in P} M_{\gamma+s}$ as C-modules, hence as B-modules.

Step 3: the two extensions of B. By (7.10), Lemma 7.8(1) and Step 2, $\tilde{U}$ and $S^{\natural}_{\psi_N}$ are simple vertex operator algebras containing B with the same conformal vector, and both decompose as $\bigoplus_{x \in L_{\mathrm{I}}'/N} M_x$ as B-modules, where $L_{\mathrm{I}}'/N = P \oplus \{0, s\}$ by Lemma 7.1(3). The summands are pairwise non-isomorphic invertible B-modules with fusion $M_x \boxtimes M_y \cong M_{x+y}$ and multiplicities one. As in the proof of Lemma 7.8(2), the fusion rules of B give the (L_{I}'/N)-gradings and the hypotheses of [DoM04a, Proposition 5.3], which gives an isomorphism $\tilde{U} \cong S^{\natural}_{\psi_N}$ acting by a nonzero scalar on each summand M_x and by the identity on B. Finally $S^{\natural}_{\psi_N} \cong S^{\natural}$ by Theorem 7.6(3). ∎

Corollary 7.10 *Assume* (I1), (I3), (F1)–(F4), (D1) *and* (D2). *Let U be the algebra of Huang, isomorphic to the algebra U^{R} of Riesen by* (D1).

1. *U is a simple, self-contragredient, positive, strongly rational vertex operator algebra of CFT type and central charge eight, and* $\mathrm{Rep}(U)$ *is ribbon equivalent to* $\mathrm{Rep}(S)$. *In particular* $\mathrm{Rep}(U)$ *is a pseudo-unitary modular tensor category of rank twelve, with the twists of (F2), the dimensions of Lemma 9.3 and no invertible object other than the unit by Lemma 9.2.*

2. *The Heisenberg commutant of U for the Cartan subalgebra $\mathbb{C}H_L \oplus \mathbb{C}H_R$ is isomorphic to Ω_S, its associated lattice is L_{I}, and $U_1 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ with currents of levels $(1, 39)$, so that $\dim U_1 = 6$ and $U \ncong S$.*

3. *S and U belong to the same hyperbolic genus, every strongly rational vertex operator algebra in this genus is isomorphic to S or to U, and S, U are the only algebras of type S with Heisenberg commutant isomorphic to Ω_S.*

4. *$S^{\natural}$ embeds in $V_{L_H} \cong V'$ as the algebra $\tilde{U}$, and $\tilde{U} \cap V_{L_0} = C = S \cap V_{L_0}$ under the identification of the two copies of V_{L_0} used in Theorem 7.9.*

Proof : Parts 1, 3 and the first two assertions of part 2 follow from Theorems 7.9 and 7.6, since $U \cong S^{\natural}$ is of type S, and from Lemmas 9.3 and 9.2 applied to $\mathrm{Rep}(U) \cong \mathrm{Rep}(S)$. For U_1, the Lie algebra U_1 is reductive [DoM04a, Theorem 1], and $\mathbb{C}H_L \oplus \mathbb{C}H_R$ is a Cartan subalgebra of U_1, so that U_1 has rank two. Indeed, by Theorem 7.6(1), $\mathfrak{h}^{\natural}$ is a Cartan subalgebra of $(S^{\natural}_{\psi_N})_1$, and it is the Heisenberg space of $V_N \subset B$. The isomorphism $\tilde{U} \cong S^{\natural}_{\psi_N}$ of the proof of Theorem

7.9 is the identity on B, hence maps the Heisenberg space of $V_N \subset \tilde{U}$, which is $\hat{\Phi}(\mathbb{C}H_L \oplus \mathbb{C}H_R)$ since $R_*\ell = s$ and $R_*h = t$, onto $\mathfrak{h}^\natural$. It contains the Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ spanned by the six currents of Huang, of rank two, so U_1 is semisimple of rank two with Cartan subalgebra $\mathfrak{h}^\natural$, and its root system R contains the roots $\pm s$ and $\pm\frac{t}{39}$ of the two $\mathfrak{sl}_2$, which are orthogonal with $(s,s) = 2$ and $(\frac{t}{39}, \frac{t}{39}) = \frac{2}{39}$ for the invariant form of U on $\mathfrak{h}^\natural$. On each simple ideal of U_1 this form is a nonzero multiple of the Killing form, so the ratio of the squared lengths of two roots of the same simple ideal is 1, 2, $\frac{1}{2}$, 3 or $\frac{1}{3}$. The irreducible root systems of rank two, A_2, B_2 and G_2, contain no pair of orthogonal roots with squared length ratio 39: in A_2 there are no orthogonal roots, in B_2 orthogonal roots have the same length, and in G_2 the ratio is 3. Hence $R = \{\pm s\} \sqcup \{\pm\frac{t}{39}\}$ is reducible, $U_1 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and the levels are $\frac{(s,s)}{2} = 1$ and $\frac{(t,t)}{2} = 39$. Since S_1 has levels $(3, 13)$ and an isomorphism of vertex operator algebras preserves the weight-one Lie algebras with their invariant forms, $U \ncong S$, which also follows from the lattices. Part 4 restates Theorem 7.9 and (7.10). ∎

Remark 7.11 Corollary 7.10 identifies U with $S^\natural$ using the inputs of Theorem 7.9. The strong rationality of U follows from the extension theory of [HKL15, CKM22] applied to $\Omega_S \otimes V_{L_1}$. The equality $U_1 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is obtained from the root system of the twin, without the weight-one computations of [Hua26b] used in [DJX26, Proposition 9.4]. This describes the structure behind [DJX26, Theorem 7.3]: C is the extension of $B = \Omega_S \otimes V_N$ by the thirty-nine simple currents M_γ, $\gamma \in P$, and its simple currents $\mathrm{Ind}(M_x)$, $x \in \{s, p, s+p\} + N$, are $\tilde{U}^-$, S^- and $\mathrm{Ind}(M_{s+p})$. The extensions $C \oplus \tilde{U}^-$ and $C \oplus S^-$ are $\tilde{U} \cong U$ and S, whereas $\mathrm{Ind}(M_{s+p})$ has half-integral weights by (7.9) and gives no vertex operator algebra extension of C. Thus the two nontrivial bosons $s + N$ and $p + N$ of N'/N correspond to the two algebras of type S with Heisenberg commutant Ω_S.

We finally recover the structure of $\mathrm{Rep}(C)$ of [DJX26, Section 9] under our hypotheses, with a proof of the rank 48 which does not use [DJX26, Theorem 3.1].

Proposition 7.12 *Assume* (I1), (I3), (F1)–(F4) *and* (D2). *Put* $a = \tilde{U}^- \cong \mathrm{Ind}(M_s)$ *and* $b = S^- \cong \mathrm{Ind}(M_p)$ *in* $\mathrm{Rep}(C)$, *as in [DJX26, (9.1)]. In this proposition,* a *and* b *do not denote the labels of* $Z(\mathcal{H}_3)$.

1. a *and* b *are distinct simple currents of order two of* $\mathrm{Rep}(C)$ *with* $\theta_a = \theta_b = 1$ *and* $\theta_{ab} = -1$, *and the fusion subcategory* $\mathcal{T}$ *generated by* a *and* b *is braided equivalent to* $Z(\mathrm{Vec}_{\mathbb{Z}_2})$. *It is nondegenerate.*

2. $\mathrm{Rep}(C) \cong \mathcal{T} \boxtimes \mathcal{D}$ *as ribbon categories, where* $\mathcal{D}$ *is the Müger centralizer of* $\mathcal{T}$, *and* $\mathrm{Rep}(U) \cong \mathcal{D} \cong \mathrm{Rep}(S)$ *as ribbon categories. In particular* $\mathrm{Rep}(C) \cong Z(\mathrm{Vec}_{\mathbb{Z}_2}) \boxtimes \mathrm{Rep}(S)$ *has rank* 48 *and global dimension* $4 \cdot 117d^2$.

Proof : 1. By Lemma 7.8(3) and Theorem 7.9, $a = \mathrm{Ind}(M_s)$ and $b = \mathrm{Ind}(M_p)$ are simple C-modules, and induction is a tensor functor [KiO02, Theorem 1.6 p. 7], so that $a \boxtimes_C a \cong \mathrm{Ind}(M_{2s}) = \mathrm{Ind}(M_0) = C$, $b \boxtimes_C b \cong C$ and $a \boxtimes_C b \cong \mathrm{Ind}(M_{s+p})$, since $2s = \lambda$ and $2p$ lie in N. The twists are $\mathrm{e}^{2\pi \mathrm{i} h}$ for the lowest weights, and all the weights of $\mathrm{Ind}(M_x)$ are congruent to h_x modulo $\mathbb{Z}$ by (7.9), which gives $\theta_a = \theta_b = 1$ and $\theta_{ab} = -1$. Hence $a \neq b$ and the four objects C, a, b, ab are distinct, with fusion group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and quadratic form $(0, 0, 0, \frac{1}{2})$. This metric group is nondegenerate, and a pointed braided fusion category is determined up to braided equivalence by its pre-metric group [EGNO15, Theorem 8.4.12 p. 206], so $\mathcal{T} \cong Z(\mathrm{Vec}_{\mathbb{Z}_2})$, whose metric group is the hyperbolic plane over $\mathbb{Z}_2$. This is [DJX26, Lemma 9.1], whose proof obtains $a^2 \cong b^2 \cong 1$ from orbifold Schur–Weyl duality and θ_{ab} from a lattice intertwining operator. Here both follow from induction and (7.9).

2. Since $\mathcal{T}$ is a nondegenerate fusion subcategory of the nondegenerate braided fusion category $\mathrm{Rep}(C)$, we have $\mathrm{Rep}(C) \cong \mathcal{T} \boxtimes \mathcal{D}$ with $\mathcal{D}$ the centralizer of $\mathcal{T}$ [EGNO15, Theorem 8.21.4 p. 245]. The vertex operator algebras $S = C \oplus b$ and $\tilde{U} = C \oplus a$ are the extensions of C by the connected étale algebras $A_b = \mathbf{1} \oplus b$ and $A_a = \mathbf{1} \oplus a$ of $\mathcal{T}$ [HKL15, Theorem 3.2], and $\mathrm{Rep}(S) \cong \mathrm{Rep}(C)^{\mathrm{loc}}_{A_b}$, $\mathrm{Rep}(U) \cong \mathrm{Rep}(C)^{\mathrm{loc}}_{A_a}$ as ribbon categories [HKL15, Theorem 3.4] [CKM22, Theorem 3.65]. For $Y \in \mathcal{D}$, the induced module $A_b \otimes Y$ is local since Y centralizes b, and $Y \mapsto A_b \otimes Y$ is a braided tensor functor $\mathcal{D} \to \mathrm{Rep}(C)^{\mathrm{loc}}_{A_b}$ [KiO02, Theorem 1.6 p. 7]. It is fully faithful: $\mathrm{Hom}_{A_b}(A_b \otimes Y, A_b \otimes Y') \cong \mathrm{Hom}(Y, Y' \oplus b \boxtimes Y')$, and $\mathrm{Hom}(Y, b \boxtimes Y') = 0$ since $Y \in \mathbf{1} \boxtimes \mathcal{D}$ and $b \boxtimes Y' \in b \boxtimes \mathcal{D}$ have no common simple summand in $\mathcal{T} \boxtimes \mathcal{D}$. The global dimension of $\mathrm{Rep}(C)^{\mathrm{loc}}_{A_b}$ is $\dim \mathrm{Rep}(C)/\dim(A_b)^2 = \dim \mathcal{T} \dim \mathcal{D}/4 = \dim \mathcal{D}$ [DMNO13, Corollary 3.32], so the fully faithful tensor functor is an equivalence [EGNO15, Proposition 6.3.3 p. 125]. The same argument applies to A_a. The twists agree because both sides are the categories of modules of the regular vertex operator algebras S, U with the ribbon structure given by $\mathrm{e}^{2\pi \mathrm{i} L(0)}$, and all inclusions preserve the conformal vector. The rank of $\mathrm{Rep}(S)$ is twelve by (F2), and its global dimension is $(\mathrm{FPdim}\, \mathcal{F})^2 = (9d+6)^2 = 117d^2$, since $\mathrm{Rep}(S) \cong Z(\mathcal{F})$ with $\mathcal{F}$ pseudo-unitary by (F4) [ENO05, Theorem 2.15], see Section 9.1. ■

7.5 Consequences of the Haagerup property

Theorem 6.1 supplies ($\mathrm{H_J}$) and complete unitarity for S. The identification $U \cong S^\natural$ from Theorem 7.9 then yields the Haagerup property for U.

Theorem 7.13 *Assume* (I1), (I3), (F1)–(F4), (D1) *and* (D2). *Theorem 6.1 supplies* ($\mathrm{H_J}$).

1. Classification. *Up to isomorphism, there are exactly two Haagerup algebras, S, with currents of levels* $(3, 13)$, *and $U \cong S^\natural \cong U^{\mathrm{R}} \cong U^{\mathrm{H}}$, with currents of levels* $(1, 39)$. *They have isomorphic Heisenberg commutants and form a single hyperbolic genus, and every strongly rational vertex operator algebra in this genus is isomorphic to S or to U. Every Haagerup algebra in case* (II) *of [ArhI, Theorem 8.5] is isomorphic to S, and every Haagerup algebra in case* (I) *is isomorphic to U.*

2. Complete unitarity. *S, U and Ω_S are completely unitary, and they satisfy Condition I^+ of [Gui25, Definition 6.17] for every structure of unitary vertex operator algebra. Every simple S-module and every simple U-module is unitarizable. The functors* Π *give equivalences of unitary modular tensor categories* $\mathrm{Rep}^u(S) \cong \mathrm{Rep}^f(\mathcal{A}_S) \cong Z(\mathcal{H}_3)$ *and* $\mathrm{Rep}^u(U) \cong \mathrm{Rep}^f(\mathcal{A}_U) \cong Z(\mathcal{H}_3)$, *and the correspondence of [CGGH23, Theorem 4.10 p. 21] between simple CFT-type extensions of S, respectively U, and irreducible local extensions of $\mathcal{A}_S$, respectively $\mathcal{A}_U$, applies.*

3. Labels. *The assertions of Corollary 6.2(1) hold: the assertion* (N) *of [ArhII, Section 7] holds, the label of $X_\pm$ is the label of $\pi_\pm$, and $\mathcal{F}_\pm \cong \mathcal{H}_2$ if and only if $\mathcal{F}_\pm^{\mathrm{net}} \cong \mathcal{H}_2$, if and only if $X_\pm \cong M_a$.*

4. The third extension. *The assertions of Corollary 6.2(2) hold: the third holomorphic extension B_1 of S is isomorphic to V_{E_8}, it carries a unique unitary structure extending that of S, and its conformal net is the third extension of $\mathcal{A}_S$ of Theorem 3.1(4).*

Proof : 1. Apply the classification in [ArhI, Theorem 12.4(1),(4)] with ($\mathrm{H_J}$) supplied by Theorem 6.1. It gives the two classes represented by S and $S^\natural$ and their common Heisenberg commutant. Theorem 7.9 and (D1) identify the twin with U^{H} and U^{R}, and Corollary 7.10

gives its levels. Theorem 7.6(4) gives the assertion about the strongly rational algebras in their hyperbolic genus.

2. By Theorem 6.1, S is completely unitary and satisfies Condition I^+, and $\Pi\colon \mathrm{Rep}^u(S) \to \mathrm{Rep}^f(\mathcal{A}_S)$ is an equivalence of braided C^*-tensor categories, hence of unitary modular tensor categories. The passage to U follows the proofs of [ArhII, Theorem 8.3(1) and Corollary 8.5(1)], with S in the role of the Haagerup algebra there and its twin $S^\natural \cong U$, and with Theorem 6.1 in place of [ArhII, Theorem 7.17]: the Heisenberg commutant Ω_S is completely unitary by [ArhII, Lemma 8.2], whose hypotheses are the complete unitarity of S and the structure of S over $\Omega_S \otimes V_{L_{\mathrm{II}}}$ of Proposition 7.5, the lattice algebra $V_{L_{\mathrm{I}}}$ is completely unitary [Gui22, Corollary 3.32 p. 49], so $\Omega_S \otimes V_{L_{\mathrm{I}}}$ is completely unitary [Gui22, Proposition 3.31 p. 48], and its simple CFT-type extension $S^\natural$ with the same conformal vector, Theorem 7.6(1), carries a unique unitary structure extending the tensor product structure and is completely unitary [CGGH23, Theorem 4.7 p. 19]. Condition I^+ passes from S to Ω_S by [Gui25, Theorem 6.18(3) p. 70], since, for a θ-stable Cartan subalgebra [ArhII, Lemma 8.1], $V_{L_{\mathrm{II}}}$ is a regular unitary subalgebra of S with commutant Ω_S [ArhII, proof of Lemma 8.2], to $\Omega_S \otimes V_{L_{\mathrm{I}}}$ by [Gui25, Theorem 6.18(1) and Example 6.19(b) pp. 70–71], and to its unitary extension $S^\natural$ by [Gui25, Theorem 6.18(2) p. 70], as in the proof of [ArhII, Corollary 8.5(1)]. Condition I^+ includes the complete rationality of $\mathcal{A}_U$ and the essential surjectivity of Π onto $\mathrm{Rep}^f(\mathcal{A}_U)$ [Gui25, Theorem 6.14 pp. 68–69]. The braided equivalence $\mathrm{Rep}^u(U) \cong \mathrm{Rep}(U) \cong \mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$ is naturally isomorphic to a unitary braided equivalence [Reu23, Theorem 5], which gives $\mathrm{Rep}^u(U) \cong Z(\mathcal{H}_3)$ as unitary modular tensor categories. Since two unitary structures on a simple vertex operator algebra are unitarily isomorphic [CKLW18, Proposition 5.19 pp. 40–41], these properties hold for every unitary structure. The unitarizability of the simple modules is part of complete unitarity. The hypotheses of [CGGH23, Theorem 4.10 p. 21], namely that S is regular, completely unitary, of CFT type, strongly local, with $\mathcal{A}_S$ completely rational and $\mathrm{Rep}^u(S) \cong \mathrm{Rep}^f(\mathcal{A}_S)$ as unitary modular tensor categories, are satisfied, and likewise for U, whose strong locality follows from Condition I^+ [Gui25, Remark 4.16 p. 41]. This is [ArhII, Corollary 7.19(3)] for S.

3–4. These are the statements of Corollary 6.2. ∎

Remark 7.14 By Theorem 6.1, the statements of [ArhI, Section 12] not covered by Theorem 7.6, namely [ArhI, Theorem 12.4 and Remark 12.6], hold as stated, see Remark 7.7, and [ArhII, Theorem 8.6(2)], the analogue of (N) for a Haagerup algebra embedded in V' as a unitary subalgebra containing its conformal vector, applies to $U = U^{\mathrm{H}} \subset V_{L^{\mathrm{H}}}$, which contains the conformal vector by [DJX26, (4.9)] and satisfies these hypotheses for the unitary structure of [DJX26, Section 4.4] by the identity $\theta_H(U) = U$ of [DJX26, (6.15)], verified in Section 8.2. The assertions of Theorem 3.1(4) on the Haagerup subfactor, the hypergroup and the near-group category, which are [ArhII, Corollaries 7.4 and 7.6 and Remark 7.8], do not depend on $(\mathrm{H_J})$.

8 Comparison with the work of Dong, Jiao and Xu

The preprint [DJX26] appeared while the present work was in progress. We describe what it proves, which of its statements we use and how we verified them, and what is new here.

8.1 The results of [DJX26]

Let U be the vertex subalgebra of $V' = V_{E_6 \oplus A_2}$ generated by the explicit vectors of [Hua26b, Section 2], with currents of levels $(1, 39)$, and let V be the vertex subalgebra generated by the nine vectors of [Jia26, Section 3], with currents of levels $(3, 13)$, both written in the root basis and with the cocycle of [DJX26, Section 4.1]. The main results of [DJX26] are the following.

1. The algebra of Riesen [Rie26a] is isomorphic to U [DJX26, Proposition 4.1], through an explicit identification of generators and of the two commuting Virasoro vectors e, f of central charges $\frac{70}{31}$ and $\frac{2400}{1271}$.

2. The involutions τ and σ of (D2) satisfy $\Phi(U^\tau) = V^\sigma$, where Φ is the lift of an isometry between two rank-eight lattices L_H and L_J with the common index-two sublattice L_0 [DJX26, Sections 5–6 and Theorem 7.3]. The proof is a finite set of mode identities together with a lattice-shift lemma [DJX26, Lemma 7.1].

3. A positive conformal-vector identity $\omega = \omega_L + \frac{3403}{4563}\omega_R + \frac{25}{6}\Pi_R\eta_3 + \frac{25}{26}\Pi_R\eta_{13}$ holds in U [DJX26, Proposition 8.1], where η_3, η_{13} are the conformal vectors of the parafermion algebras $K(\mathfrak{sl}_2, 3)$, $K(\mathfrak{sl}_2, 13)$ transported from V^σ and Π_R is an average over the compact group of the level-39 currents. With a rationality criterion [DJX26, Theorem 3.1], this gives the strong rationality of U, then of U^τ by [CaM16], then of V by [McR22, Example 4.2] [DJX26, Corollary 8.2].

4. $\mathrm{Rep}(U^\tau) \cong T \boxtimes D$ with $T \cong Z(\mathrm{Vec}_{\mathbb{Z}_2})$ and $\mathrm{Rep}(U) \cong D \cong \mathrm{Rep}(V)$ as ribbon categories [DJX26, Lemma 9.1 and Proposition 9.2], so that $\mathrm{Rep}(U^\tau) \cong Z(\mathrm{Vec}_{\mathbb{Z}_2}) \boxtimes Z(\mathcal{H}_3)$, of rank 48, if $\mathrm{Rep}(U)$ or $\mathrm{Rep}(V)$ is ribbon equivalent to $Z(\mathcal{H}_3)$ [DJX26, Corollary 9.3], and $U \ncong V$ [DJX26, Proposition 9.4].

5. The conformal nets of U, V and U^τ are completely rational, and $\mathcal{A}^\tau_U \cong \mathcal{A}_{U^\tau} \cong \mathcal{A}^\sigma_V$ [DJX26, Theorem 10.1 and Corollary 10.2].

Remark 8.1 The complete rationality of $\mathcal{A}_S$ is also proved in [DJX26, Theorem 10.1], by the same chain of results of Tener and Longo, with the strong rationality of S taken from [DJX26, Corollary 8.2] instead of (F1). The proof there applies [Ten24, Corollary 4.24] as stated, whereas the proof of [Ten24, Corollary 4.24] uses a unitary structure on $M \boxtimes \overline{M}$, which is supplied by [ArhII, Proposition 5.2], see Step 2 of the proof of Theorem 3.1. Parts 2–4 of Theorem 3.1 go beyond [DJX26], whose Remark 10.3 points out that complete rationality alone does not identify the representation category of the net.

8.2 Independent verification of the finite identities

We use [DJX26, Theorem 7.3] in Section 7. Its proof rests on the explicit vectors, on finite mode identities and on [DJX26, Lemma 7.1]. We verified all finite identities with an exact implementation of the lattice vertex operator algebra V_L, $L = E_6 \oplus A_2$, in the basis and with the cocycle of [DJX26, Section 4.1], over the field $\mathbb{K} = \mathrm{Q}(\sqrt{13}, \sqrt{-3})$, written independently of the implementation used in [ArhI] (certificate A4 of Section 10). The checks are the following, with the number of verified identities in parentheses.

1. The lattice data of [DJX26, Sections 4.1–4.4] and the $\mathfrak{sl}_2$-triples of levels 1, 39, 3, 13, the highest weight vectors u, $u_{2,6}$, $u_{1,7}$, the unitary structures and the adjoints (79).

2. The lattices L_0, L_H, the vectors s, t, the isometry T and the characters of σ and τ of [DJX26, Section 5] (40). We note that t has half-integral coordinates in the root basis, so that $t \in s + L_0 \subset L_H$ and $t \notin L$, and the sentence "t is a lattice vector" of [DJX26, Table 1] has to be read with L_H in place of L.

3. The isometries R, R_*, the lift Φ of [DJX26, (6.13)], the identities [DJX26, (6.14) and (6.15)], the first of which is [DJX26, Proposition 6.2], and $\Phi\theta_H = \theta_J\Phi$ (56).

4. The identities [DJX26, (7.6), (7.7), (7.9), (7.10), (7.11)] and the membership conditions of [DJX26, Proposition 7.2] (54). The identity (7.9) is checked as an equality of vectors. The vector Y of [DJX26, (7.8)] is $(Z_0)_{(1)}X + \frac{4}{3}X$, as in [DJX26, Appendix A.2], and not $(Z_0)_{(1)}X + \frac{3}{4}X$ as printed: only the former gives $(Y, Y) = 104$ and (7.9).

5. The parafermion conformal vectors, the Casimir projections, the vectors e, f of Riesen and the identities [DJX26, (4.19)–(4.22), (8.1), (8.2), (8.4)] of Proposition 8.1, as equalities of vectors (56), and the data of [DJX26, Proposition 4.1 and Appendix A.3] for the identification of the algebra of Riesen with U (35).

These checks are grouped into six scripts. We also read the proof of [DJX26, Lemma 7.1] step by step and recomputed the scalar factor of its product formula. The hypothesis "every generator of A lies in $W + S_u W$" is to be read with $W + S_u W + S_{-u} W$, since some generators lie in $S_{-u}W$ and the equality $S_{-u}W = S_u W$ is only obtained in the course of the proof. The lemma is applied in V_{L_H} with a bimultiplicative cocycle whose restriction to L_0 is cohomologous, but not equal, to the restriction of the cocycle of V_L, which is accounted for by the cocycle-compatible identification of V_{L_0} in [DJX26, (5.7)]. The proof of [DJX26, Theorem 7.3] uses no property of U, V or of the algebra of Riesen beyond their definitions: neither rationality, nor characters, nor the generation statements of [Hua26b, Jia26, Rie26a]. We therefore regard [DJX26, Proposition 4.1, Theorem 7.3 and Proposition 8.1] as established.

8.3 What is new here

1. The strong rationality of $V \cong S$ is proved in Corollary A.14 through the semisimplicity of Zhu's algebra and in [DJX26, Corollary 8.2] through the positive conformal-vector identity and [DJX26, Theorem 3.1]. The two proofs are independent. The identity [DJX26, (8.2)] is verified above, and we have not checked the proof of [DJX26, Theorem 3.1], which is a general statement. The classification of the simple modules, (F2), is not obtained in [DJX26].

2. Theorem 1.1 identifies the algebras U and $S^\natural$, not only their module categories as in [DJX26, Proposition 9.2], and it explains the common fixed point of [DJX26, Theorem 7.3] as the lattice realization of the twin construction of [ArhI, Section 12], through the common index-two sublattice N of Lemma 7.1. The rank of $\mathrm{Rep}(U^\tau)$ is 48 by (F2) and Proposition 7.12.

3. Theorem 3.1 identifies $\mathrm{Rep}(\mathcal{A}_S)$ with $Z(\mathcal{H}_3)$ as a unitary modular tensor category and realizes the Haagerup subfactor from $\mathcal{A}_S$, whereas [DJX26, Theorem 10.1] proves the complete rationality of the nets. The proof of [DJX26, Theorem 10.1] applies [Ten24, Corollary 4.24] as stated, whose proof uses a unitary structure on the fusion products $M \boxtimes \overline{M}$ which is not among its hypotheses, see Remark 8.1. This structure is provided by [ArhII, Proposition 5.2].

4. Theorem 6.1 identifies $\mathrm{Rep}(S)$ with $Z(\mathcal{H}_3)$ relative to the explicit inputs (I1), (F1)–(F3), and the twin identification then gives the same conclusion for U. The additional ingredient is the trace class theorem applied to the diagonal net, followed by reconstruction and the coset comparison.

9 The algebraic route and the equivalent criteria

The Lagrangian algebras of $\mathrm{Rep}(S)$ arising from the lattice extensions give a separate algebraic description of the category. This description is used for the twin theorem and for the comparison with the algebraic construction of the diagonal extension.

Theorem 9.1 (F4). *Each $L_\pm$ carries the structure of a Lagrangian algebra in $\mathrm{Rep}(S)$. The category $\mathrm{Rep}(S)$ is a pseudo-unitary modular tensor category, braided equivalent to the Drinfeld center $Z(\mathcal{F}_\pm)$ of the fusion category $\mathcal{F}_\pm = \mathrm{Rep}(S)_{L_\pm}$. The category $\mathcal{F}_\pm$ is pseudo-unitary, has six simple objects, three of which are invertible, with the fusion rules (9.1) of the Haagerup categories $\mathcal{H}_2$ and $\mathcal{H}_3$, and its pointed part is $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator.*

Proof : Proposition A.23 and Theorem A.25(1)–(3) give the assertions. ∎

9.1 Consequences of (F4) for the representation category

The following properties of $\mathrm{Rep}(S)$ follow from (F1)–(F3) of Theorem 2.2 and (F4) of Theorem 9.1. They are used in the twin theorem and the algebraic construction of the diagonal extension. In this subsection, $\mathcal{F} = \mathcal{F}_+$ is the pseudo-unitary fusion category of Theorem 9.1, with simple objects 1, α, α^2, ρ, $\alpha\rho$, $\alpha^2\rho$ and fusion rules

$$\rho\alpha \cong \alpha^2\rho, \quad \rho^2 \cong 1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho, \tag{9.1}$$

and $\mathrm{Rep}(S)$ is braided equivalent to the Drinfeld center $Z(\mathcal{F})$. We write $d = \frac{3+\sqrt{13}}{2}$, so that $d^2 = 3d + 1$ and $\mathrm{FPdim}\,\mathcal{F} = 3 + 3d^2 = 9d + 6$. We do not assume that $\mathcal{F}$ is unitary.

9.1.1 Invertible objects

Lemma 9.2 *Let $\mathcal{F}$ be a fusion category with the fusion rules (9.1). Then the universal grading group of $\mathcal{F}$ is trivial, and the unit is the only invertible object of $Z(\mathcal{F})$ up to isomorphism.*

Proof : Let $U(\mathcal{F})$ be the universal grading group of $\mathcal{F}$ [EGNO15, Definition 4.14.2], and write g_X for the degree of a simple object X, using multiplicative notation since the grading group need not be abelian. The summands 1 and ρ of $\rho \otimes \rho$ give $g_\rho^2 = 1$ and $g_\rho^2 = g_\rho$, hence $g_\rho = 1$. The summand $\alpha\rho$ then gives $g_\alpha = 1$. The simple objects α and ρ generate $\mathcal{F}$, so $U(\mathcal{F})$ is trivial. By [EGNO15, Proposition 4.14.3], the group of tensor automorphisms of the identity functor of $\mathcal{F}$ is the character group of $U(\mathcal{F})$, hence trivial.

Let (X, γ) be an invertible object of $Z(\mathcal{F})$, where $\gamma_Y : X \otimes Y \to Y \otimes X$ is a half-braiding. The forgetful functor $Z(\mathcal{F}) \to \mathcal{F}$ is a tensor functor [EGNO15, Remark 7.13.4], so X is invertible in $\mathcal{F}$, that is, $X \in \{1, \alpha, \alpha^2\}$. If $X = \alpha$, the isomorphism γ_ρ gives $\alpha\rho \cong \rho\alpha \cong \alpha^2\rho$, which contradicts the distinctness of the six simple objects. The case $X = \alpha^2$ is excluded in the same way. If $X = 1$, then γ is a tensor automorphism of the identity functor of $\mathcal{F}$, hence trivial, and (X, γ) is the unit. ∎

9.1.2 Dimensions and self-duality

We use the labels $z \in \{0, b, a, c_0, c_1, c_2, d_1, \ldots, d_6\}$ of Theorem 2.2 for the simple S-modules M_z, with $M_0 = S$, and we write h_z for their lowest conformal weights and $\dim M_z$ for their categorical dimensions, which are their Frobenius–Perron dimensions by (F4). Recall from (F3) that $V'|_S = S \oplus M_b$, that the two lattice extensions $H_\pm \cong V_{E_8}$ of S restrict as

$$H_\pm|_S \cong S \oplus M_b \oplus 2X_\pm, \tag{9.2}$$

where the signs refer to the fixed lattice coordinates of (F3). The unordered pair of classes of $X_\pm$ is that of M_a and M_{c_0}. By Theorem 6.1 and Corollary 6.2(1), the intrinsic labels a, c_0 refer to the center in the convention of [EvG11, ArhII]. The restrictions $L_\pm = H_\pm|_S$ are Lagrangian algebras of $\mathrm{Rep}(S)$ with $\mathcal{F}_\pm = \mathrm{Rep}(S)_{L_\pm}$.

Lemma 9.3 *We have* $\dim M_b = 3d+1$ *and* $\dim M_a = \dim M_{c_0} = \dim M_{c_1} = \dim M_{c_2} = 3d+2$. *Every simple S-module is self-dual.*

Proof : By Lemma A.24, the quantum dimensions of the restrictions to S of the nine simple V'-modules are all equal to $1 + \dim M_b$, so that $\dim M_{c_1} = \dim M_{c_2} = \dim M_a = \dim M_{c_0} = 1 + \dim M_b$ by (F3). Since L_+ is a Lagrangian algebra, $(\mathrm{FPdim}\, L_+)^2 = \mathrm{FPdim}\, \mathrm{Rep}(S) = (\mathrm{FPdim}\, \mathcal{F})^2$ [DMNO13, Corollary 3.32 and Section 4], hence $1+\dim M_b+2(1+\dim M_b) = 9d+6$ and $\dim M_b = 3d+1$.

Let $z \mapsto z^*$ denote the duality on the simple S-modules. The dual of a simple object has the same twist and the same dimension, and by (F2) the twists $\theta_z = \mathrm{e}^{2\pi \mathrm{i} h_z}$ take the value $\mathrm{e}^{2\pi\mathrm{i}/3}$ only at c_1, the value $\mathrm{e}^{4\pi\mathrm{i}/3}$ only at c_2, and six distinct values $\mathrm{e}^{2\pi\mathrm{i}k/13}$, $k \in \{2,5,6,7,8,11\}$, at $d_1, \ldots, d_6$. Hence c_1, c_2 and the six d_l are self-dual, and the duality permutes $\{b, a, c_0\}$, since these are the simple objects with twist 1 other than the unit. A connected étale algebra in a fusion category is self-dual as an object: it is a Frobenius algebra by [DMNO13, Section 2.2] and [DMNO13, Remark 3.4]. Hence $L_\pm \cong L_\pm^*$ as objects. In each of $L_\pm$, the nonvacuum summand of multiplicity two is $X_\pm$, while the only nonvacuum summand of multiplicity one is M_b. Thus $X_\pm$ and M_b are self-dual, and the assertion follows for both M_a and M_{c_0}. ∎

Remark 9.4 The dimensions of the six modules M_{d_l} are not needed in this paper. By Theorem 6.1 they are equal to $3d$, see [ArhII, Section 2.2], and the identity $1 + (3d+1)^2 + 4(3d+2)^2 + 6(3d)^2 = (9d+6)^2$ is consistent with Lemma 9.3.

9.1.3 Complex conjugation and the reverse category

The construction of the diagonal holomorphic extension of $S \otimes S$ in [ArhII, Proposition 7.12] uses a braided equivalence $\mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$, obtained there from the identification $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$ and from the self-conjugacy of the even parts of the Haagerup subfactor [JMS14]. Without this identification, the existence of such an equivalence is not automatic. The Drinfeld center of a fusion category need not be braided equivalent to its reverse: for the pointed category $\mathrm{Vec}^\omega_{\mathbb{Z}_3}$ with a suitable generator $[\omega]$ of $H^3(\mathbb{Z}_3, \mathbb{T})$, the center $Z(\mathrm{Vec}^\omega_{\mathbb{Z}_3})$ is a pointed modular category of rank nine whose twists are 1 three times and $\mathrm{e}^{2\pi\mathrm{i}k/9}$ twice for each $k \in \{1,4,7\}$ [EGNO15, Section 8.5]. This multiset is not stable under complex conjugation, so that $Z(\mathrm{Vec}^\omega_{\mathbb{Z}_3})$ and its reverse are not braided equivalent. The unitary structure of S gives a weaker statement, which we record for later use.

For a $\mathbb{C}$-linear category $\mathcal{C}$, we denote by $\overline{\mathcal{C}}$ the complex conjugate category, with the same objects and morphisms and the conjugate scalar multiplication on the morphism spaces. If $\mathcal{C}$ is a ribbon fusion category, then so is $\overline{\mathcal{C}}$, with the same braiding and twist morphisms, and the twist of a simple object of $\overline{\mathcal{C}}$ is the complex conjugate of its twist in $\mathcal{C}$.

Recall that S carries a structure of unitary vertex operator algebra [ArhI, Theorem 10.10], with PCT operator $\theta\colon S \to S$, an antilinear involution satisfying $\theta(a_{(n)}b) = \theta(a)_{(n)}\theta(b)$ and $\theta(\omega) = \omega$ [DoL14, Definition 2.1]. For an S-module (M, Y_M), let $\overline{M}$ be the conjugate vector space, with its canonical antilinear map $m \mapsto \overline{m}$. Define $J(M)$ by

$$a^{J(M)}_{(n)}\overline{m} = \overline{(\theta a)^M_{(n)} m}.$$

Each displayed mode is complex linear in $\overline{m}$, and the assignment is complex linear in a. The original modes on M are complex linear, and the conjugation maps and θ are antilinear. The module identities follow by conjugation, and the conformal weights are unchanged. On morphisms put $J(f)\overline{m} = \overline{f(m)}$. This is an antilinear functor to $\mathrm{Rep}(S)$, hence a linear functor to its conjugate category. We denote this latter functor by π below.

Proposition 9.5 *The assignment $M \mapsto \pi(M)$ extends to an equivalence of ribbon categories $\pi\colon \mathrm{Rep}(S) \to \overline{\mathrm{Rep}(S)}^{\mathrm{rev}}$. In particular $\overline{\mathrm{Rep}(S)}$ is braided equivalent to $\mathrm{Rep}(S)^{\mathrm{rev}}$. On the simple modules, $\pi(M_z) \cong M_z$ for every z.*

Proof : Let M, N, P be S-modules and let $\mathcal{Y}$ be an intertwining operator of type $\binom{P}{MN}$, so that $\mathcal{Y}(m,x)n = \sum_{h\in\mathbb{C}} m_{(h)}nx^{-h-1}$ with $m_{(h)}n \in P$. The conformal weights of the S-modules are rational by (F1) and (F2). The conjugated coefficients, defined by $\overline{\mathcal{Y}}(\overline{m},x)\overline{n} = \sum_h \overline{m_{(h)}n}x^{-h-1}$, define an intertwining operator $\overline{\mathcal{Y}}$ of type $\binom{\pi(P)}{\pi(M)\pi(N)}$, since the Jacobi identity and the $L(-1)$-derivative property have rational coefficients and since the modes of S on $\pi(M)$ are those of $\theta(S) = S$. The map $\mathcal{Y} \mapsto \overline{\mathcal{Y}}$ is a bijection between the spaces of intertwining operators, which is antilinear for the original scalar multiplications, hence linear onto the conjugate space. It is compatible with the $\mathrm{P}(z)$-tensor products of [HuL95, Hua05], so that the vertex tensor category structure of $\mathrm{Rep}(S)$, which exists by (F1) and [Hua08, Theorem 4.6], is transported through π to a vertex tensor category structure on $\overline{\mathrm{Rep}(S)}$, and the universal property of the $\mathrm{P}(z)$-tensor product identifies $\pi(M)\boxtimes\pi(N)$ with $\pi(M\boxtimes N)$. The associativity isomorphisms of [Hua05] are characterized by the identity

$$\mathcal{Y}_1(m,z_1)\mathcal{Y}_2(n,z_2)p = \mathcal{Y}_3(\mathcal{Y}_4(m,z_1-z_2)n,z_2)p$$

for real numbers $z_1 > z_2 > z_1 - z_2 > 0$, where the powers of z_1, z_2 and $z_1 - z_2$ are the real ones. Both sides are transformed into the corresponding identity for the conjugate operators, so that π is a tensor functor. The braiding of [Hua08] is characterized by the skew-symmetry $\Omega(\mathcal{Y})(n,z)m = \mathrm{e}^{zL(-1)}\mathcal{Y}(m,\mathrm{e}^{\pi\mathrm{i}}z)n$, in which $\mathrm{e}^{\pi\mathrm{i}}z$ means that each power z^h is replaced by $\mathrm{e}^{\pi\mathrm{i}h}z^h$. In the conjugate scalar multiplication of $\pi(P)$, the multiplication by $\mathrm{e}^{\pi\mathrm{i}h}$ is the multiplication by $\mathrm{e}^{-\pi\mathrm{i}h}$ of P, since h is real. Hence the braiding of $\mathrm{Rep}(S)$, viewed in $\overline{\mathrm{Rep}(S)}$, is the inverse of the braiding transported through π, that is, π is a braided tensor functor $\mathrm{Rep}(S) \to \overline{\mathrm{Rep}(S)}^{\mathrm{rev}}$. Similarly the twist $\mathrm{e}^{2\pi\mathrm{i}L(0)}$ of $\pi(M)$, computed in the conjugate scalar multiplication, is the map $\mathrm{e}^{-2\pi\mathrm{i}L(0)}$ of M, which is the twist of $\overline{\mathrm{Rep}(S)}^{\mathrm{rev}}$, so that π is a ribbon functor. It is an equivalence, with quasi-inverse given by the same construction, since $\theta^2 = \mathrm{id}$.

For the action on the labels, $\pi(M)$ is a simple S-module with the same conformal weights as M, so that π fixes the nine modules M_0, M_{c_1}, M_{c_2}, $M_{d_1},\dots,M_{d_6}$, whose lowest weights are pairwise distinct, and permutes $\{M_b, M_a, M_{c_0}\}$. By Lemma 9.3, π fixes M_b. Let θ' be the PCT operator of V' of [DoL14, Theorem 4.12], for which S is a unitary subalgebra with $\theta = \theta'|_S$ [ArhI, Theorem 10.10]. For a simple V'-module W_g, $g \in \mathbb{Z}_3\times\mathbb{Z}_3$, the V'-module $\overline{W_g}^{\theta'}$ is isomorphic to W_{-g}, since θ' maps e^{λ} to a multiple of $\mathrm{e}^{-\lambda}$, and $\pi(W_g|_S) = \overline{W_g}^{\theta'}|_S = W_{-g}|_S$. By (F3), $W_{(1,1)}|_S \cong W_{(-1,-1)}|_S$ and $W_{(1,-1)}|_S \cong W_{(-1,1)}|_S$, so that π fixes M_a and M_{c_0}. ∎

Remark 9.6 Proposition 9.5 shows that $\mathrm{Rep}(S)^{\mathrm{rev}}$ is braided equivalent to $\overline{\mathrm{Rep}(S)}$, not to $\mathrm{Rep}(S)$. A braided equivalence $\mathrm{Rep}(S) \cong \mathrm{Rep}(S)^{\mathrm{rev}}$ is equivalent to a braided equivalence $\mathrm{Rep}(S) \cong \overline{\mathrm{Rep}(S)}$, that is, to the invariance of $\mathrm{Rep}(S)$ under complex conjugation of its structure constants, but the conjugation argument alone does not establish this extra equivalence. We retain the notation (R) for this property in Section 9.2. Theorem 4.25 establishes it using

the trace class theorem on the conformal-net side, where the identification of Theorem 3.1 with the unitary modular tensor category $Z(\mathcal{H}_3)$ is available, and Theorem 9.14 shows that it is equivalent to the Haagerup property of S.

9.2 The hypothesis (R) and the action on the labels

We consider the following hypothesis on the modular tensor category $\mathrm{Rep}(S)$ of [Hua08], which exists by (F1).

(R) There is a braided equivalence $\pi\colon \mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$.

Such a functor is also called a braid-reversed autoequivalence of $\mathrm{Rep}(S)$. Hypothesis (R) holds by Theorem 4.25(3), and also by Theorem 6.1 and Lemma 4.27, which recalls [ArhII, Proposition 7.12(1)]. We keep (R) as a named hypothesis in order to isolate the arguments which use it. The functor π of (R) is not the functor π of Proposition 9.5. By Proposition 9.5, (R) is equivalent to a braided equivalence between $\mathrm{Rep}(S)$ and its complex conjugate category. We write π also for the induced permutation of the set $\{0, b, a, c_0, c_1, c_2, d_1, \dots, d_6\}$ of labels of the simple S-modules, so that $\pi(M_z) \cong M_{\pi(z)}$.

Lemma 9.7 *Assume* (I1), (F1)–(F4) *and* (R). *Then π is an equivalence of ribbon categories, where* $\mathrm{Rep}(S)^{\mathrm{rev}}$ *carries the twist θ^{-1} and the spherical structure of* $\mathrm{Rep}(S)$. *In particular* $\dim M_{\pi(z)} = \dim M_z$ *and* $\theta_{\pi(z)} = \theta_z^{-1}$ *for every label z. The permutation π satisfies*

$$\pi(0) = 0, \quad \pi(b) = b, \quad \pi(c_1) = c_2, \quad \pi(\{a, c_0\}) = \{a, c_0\},$$
$$\pi(d_1) = d_5, \quad \pi(d_2) = d_3, \quad \pi(d_4) = d_6,$$

and it is an involution.

Proof : By (F4), $\mathrm{Rep}(S)$ is pseudo-unitary, and so is $\mathrm{Rep}(S)^{\mathrm{rev}}$, which has the same underlying fusion category. A pseudo-unitary fusion category has a unique spherical structure for which the categorical dimensions are the Frobenius–Perron dimensions [ENO05, Proposition 8.23 p. 34]. A tensor equivalence induces an isomorphism of Grothendieck rings mapping simple objects to simple objects, hence preserves Frobenius–Perron dimensions [EGNO15, Section 3.3], and it transports spherical structures, so that π transports the spherical structure of $\mathrm{Rep}(S)$ to the one of $\mathrm{Rep}(S)^{\mathrm{rev}}$. The twist of a braided category with a spherical structure is determined by the braiding and the spherical structure, and the twist of $\mathrm{Rep}(S)^{\mathrm{rev}}$ is θ^{-1}. Hence π is a ribbon equivalence, $\dim M_{\pi(z)} = \dim M_z$ and $\theta_{\pi(z)} = \theta_z^{-1}$.

By (F2), the twists $\theta_z = \mathrm{e}^{2\pi \mathrm{i} h_z}$ are 1 for $z \in \{0, b, a, c_0\}$, $\mathrm{e}^{2\pi \mathrm{i}/3}$ for c_1, $\mathrm{e}^{4\pi \mathrm{i}/3}$ for c_2, and $\mathrm{e}^{2\pi \mathrm{i} k/13}$ with $k = 6, 11, 2, 5, 7, 8$ for $d_1, \dots, d_6$, so that the twists of c_1, c_2, $d_1, \dots, d_6$ are pairwise distinct and distinct from 1. The relation $\theta_{\pi(z)} = \theta_z^{-1}$ gives $\pi(c_1) = c_2$, $\pi(c_2) = c_1$, and $\pi(d_l) = d_{l'}$ with $k' \equiv -k \bmod 13$, that is, $6 \leftrightarrow 7$, $11 \leftrightarrow 2$ and $5 \leftrightarrow 8$, which are the stated pairs. The unit is fixed. The simple objects with trivial twist are 0, b, a, c_0, of dimensions 1, $3d + 1$, $3d + 2$, $3d + 2$ by Lemma 9.3, so that $\pi(b) = b$ and π permutes $\{a, c_0\}$. Each orbit of π on the labels has one or two elements, so the permutation is an involution. ∎

Lemma 9.7 allows either permutation on $\{a, c_0\}$, and the following construction of the diagonal extension applies in both cases. By Theorem 6.1, Lemma 4.27 and the triviality of the Brauer–Picard group of $\mathcal{H}_3$ [GrS12, Theorem 1.5 p. 5], [ENO10, Theorem 1.1 p. 3], a braided equivalence $\mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$ is unique up to natural isomorphism and fixes a and c_0. This is not used below.

9.3 Diagonal holomorphic extensions of $S \otimes S$

Definition 9.8 *A* diagonal holomorphic extension *of $S \otimes S$ is a simple, strongly rational, holomorphic vertex operator algebra W containing $S \otimes S$ as a vertex subalgebra with the same conformal vector, such that*

$$\mathrm{Com}_W(S \otimes \mathbf{1}) = \mathbf{1} \otimes S, \quad \mathrm{Com}_W(\mathbf{1} \otimes S) = S \otimes \mathbf{1}.$$

Here holomorphic means that every W-module is a direct sum of copies of W. The central charge of such a W is 16. By (F1) and [ABD04, Theorem 4.5], S and $S \otimes S$ are regular, and $\mathrm{Rep}(S \otimes S)$ is braided equivalent to $\mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S)$ [CKM22, Theorem 5.5 p. 48], whose hypotheses, that both categories are locally finite and one of them is semisimple, hold since $\mathrm{Rep}(S)$ is a fusion category. We write $g_1' = \mathfrak{g}' \otimes \mathbf{1}$, $g_1'' = \mathfrak{g}'' \otimes \mathbf{1}$, $g_2' = \mathbf{1} \otimes \mathfrak{g}'$, $g_2'' = \mathbf{1} \otimes \mathfrak{g}''$ for the four simple ideals of $(S \otimes S)_1 \cong \mathfrak{sl}_2^{\oplus 4}$, of levels 3, 13, 3, 13.

Proposition 9.9 *Assume* (I1), (F1)–(F4) *and* (R), *and put*

$$W_\pi \stackrel{\mathrm{def}}{=} \bigoplus_z M_z \otimes M_{\pi(z)},$$

the sum running over the twelve labels.

1. *The $S \otimes S$-module W_π carries the structure of a simple $\mathbb{Z}$-graded vertex operator algebra of CFT type extending $S \otimes S$, with the same conformal vector, and W_π is a diagonal holomorphic extension of $S \otimes S$.*

2. *W_π is isomorphic to the lattice vertex operator algebra $V_{E_8 \oplus E_8}$.*

3. *The weight-one space is*

$$(W_\pi)_1 = (S \otimes S)_1 \oplus \bigoplus_{z \in \{c_1, c_2, d_1, d_5, d_4, d_6\}} R_z \otimes R_{\pi(z)},$$

where R_z is the lowest weight space of M_z. As a module over $(S \otimes S)_1$, it does not depend on the choice of π on $\{a, c_0\}$, it contains no irreducible g_1'-module nor g_2'-module of dimension four, no nonzero $(S \otimes S)_1$-invariant vector, and the subspace of $g_1' \oplus g_2'$-invariant vectors has dimension 78.

Proof : 1. We apply [CKM22, Theorem 5.10(1) p. 51], which is [CKM22, Main Theorem 3(1) pp. 4–5], with $U = V = S$ and $\mathcal{U} = \mathcal{V} = \mathrm{Rep}(S)$. Its hypotheses are that S is simple and self-contragredient, which is (I1), that $\mathrm{Rep}(S)$ is closed under contragredients, semisimple, closed under submodules and quotients, and admits the vertex tensor category structure of Huang, Lepowsky and Zhang, which holds by (F1) and [Hua08], and that $\tau = \pi$ is a braid-reversed tensor equivalence whose twists satisfy $\theta_{\pi(M)} = \pm\pi(\theta_M^{-1})$, which holds with the sign $+$ by Lemma 9.7. The conclusion is that $A = \bigoplus_z M_z' \otimes \pi(M_z)$ carries the structure of a $\frac{1}{2}\mathbb{Z}$-graded conformal vertex algebra extending $S \otimes S$, which is simple since $\mathrm{Rep}(S)$ is rigid. By Lemma 9.3, every simple S-module is self-dual, so $M_z' \cong M_z$ and $A = W_\pi$ as an $S \otimes S$-module. The conformal weights of $M_z \otimes M_{\pi(z)}$ lie in $h_z + h_{\pi(z)} + \mathbb{Z}$, and $\mathrm{e}^{2\pi\mathrm{i}(h_z + h_{\pi(z)})} = \theta_z \theta_{\pi(z)} = 1$ by Lemma 9.7, so W_π is $\mathbb{Z}$-graded. Its weights are nonnegative, the weight-zero space is $S_0 \otimes S_0 = \mathbb{C}\mathbf{1}$ since $h_z + h_{\pi(z)} > 0$ for $z \neq 0$ by (F1), and the graded pieces are finite dimensional, so W_π is a vertex operator algebra of CFT type.

By [HKL15, Theorem 3.2 p. 10], W_π corresponds to a haploid commutative associative algebra $\mathcal{A}$ with trivial twist in $\mathrm{Rep}(S \otimes S) \cong \mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S)$, with underlying object $\bigoplus_z M_z \boxtimes$

$M_{\pi(z)}$. The hypotheses of [CKM22, Theorem 5.12 p. 52] hold: $\mathrm{Rep}(S)$ is a braided fusion category, W_π is a simple $\mathbb{Z}$-graded vertex operator algebra extension of $S\otimes S$ in $\mathrm{Rep}(S)\boxtimes\mathrm{Rep}(S)$, the modules M_z are distinct simple modules including $M_0 = S$, and $\dim \mathrm{Hom}_S(S, M_{\pi(z)}) = \delta_{z,0}$. Hence $\dim \mathcal{A} > 0$, and the category of $\mathcal{A}$-modules in $\mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S)$ is semisimple [CKM22, Theorem 5.12 p. 52 and its proof], so that $\mathcal{A}$ is separable [DMNO13, Proposition 2.7 p. 5] and is a connected étale algebra. By [CKM22, Corollary 5.13 p. 53], whose hypotheses are that S is strongly rational, which is (F1), and that W_π is simple and of CFT type, the algebra W_π is strongly rational. Its Frobenius–Perron dimension is

$$\mathrm{FPdim}\,\mathcal{A} = \sum_z \dim M_z \dim M_{\pi(z)} = \sum_z (\dim M_z)^2 = \mathrm{FPdim}\,\mathrm{Rep}(S),$$

by Lemma 9.7, which is the square root of $\mathrm{FPdim}(\mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S))$. Hence $\mathcal{A}$ is Lagrangian [DMNO13, Definition 4.6 p. 17], and its category of local modules is trivial [DMNO13, Remark 4.3 p. 16]. By [HKL15, Theorem 3.4 p. 16], the category of W_π-modules whose restriction to $S \otimes S$ lies in $\mathrm{Rep}(S \otimes S)$ is isomorphic to the category of local $\mathcal{A}$-modules. Since $S \otimes S$ is regular, every weak W_π-module is a direct sum of simple ordinary $S \otimes S$-modules [ABD04, Theorem 4.5], and the submodule generated by one of them has finite length by the argument of [ArhI, proof of Theorem 12.3], so every W_π-module is a direct sum of copies of W_π. Thus W_π is holomorphic.

For the commutants, a vector of W_π annihilated by the modes $a_{(n)} \otimes \mathbf{1}$, $a \in S$, $n \geqslant 0$, is annihilated by $L(0) \otimes \mathbf{1}$. On $M_z \otimes M_{\pi(z)}$ the eigenvalues of $L(0) \otimes \mathbf{1}$ lie in $h_z + \mathbb{N}$, with $h_z > 0$ for $z \neq 0$ by (F1), so such a vector lies in $S_0 \otimes S = \mathbf{1} \otimes S$. Conversely $\mathbf{1} \otimes S$ commutes with $S \otimes \mathbf{1}$. Hence $\mathrm{Com}_{W_\pi}(S \otimes \mathbf{1}) = \mathbf{1} \otimes S$, and the same argument gives $\mathrm{Com}_{W_\pi}(\mathbf{1} \otimes S) = S \otimes \mathbf{1}$.

3. The weight-one space of $M_z \otimes M_{\pi(z)}$ is $R_z \otimes R_{\pi(z)}$ if $h_z + h_{\pi(z)} = 1$, and it is zero otherwise, since $h_z + h_{\pi(z)} \in \mathbb{Z}_{\geqslant 0}$. By (F2) and Lemma 9.7, $h_z + h_{\pi(z)} = 1$ exactly for the six labels c_1, c_2, d_1, d_5, d_4, d_6, since $\frac{1}{3} + \frac{2}{3} = \frac{6}{13} + \frac{7}{13} = \frac{5}{13} + \frac{8}{13} = 1$, while $h_{d_2} + h_{d_3} = \frac{11}{13} + \frac{15}{13} = 2$, $h_b + h_b = 2$ and $h_a + h_{\pi(a)} = h_{c_0} + h_{\pi(c_0)} = 2$. This gives the decomposition, and shows that the choice of π on $\{a, c_0\}$ does not affect $(W_\pi)_1$. By (F2), the lowest weight spaces, as $\mathfrak{g}' \oplus \mathfrak{g}''$-modules, are those of [ArhI, Table 2], column (II):

$$R_{c_1} = (0,2), \quad R_{c_2} = (0,0) + (0,4) + (1,1) + (1,3) + (2,2), \quad R_{d_1} = (0,2) + (1,1),$$

$$R_{d_4} = (0,0) + (1,1), \quad R_{d_5} = (0,4) + (1,3), \quad R_{d_6} = (0,2) + (1,3) + (2,0),$$

where (m,n) denotes $V(m) \otimes V(n)$ and $V(m)$ is the irreducible $\mathfrak{sl}_2$-module of highest weight m. All the $\mathfrak{g}'$-weights occurring are at most 2, so no g'_k-module of dimension four occurs in $(W_\pi)_1$, and $(S \otimes S)_1$ is a sum of adjoint modules. A $(S \otimes S)_1$-invariant vector of $R_z \otimes R_{\pi(z)}$ would require $(0,0)$ to occur in both R_z and $R_{\pi(z)}$, which fails for each of the three pairs, and $(S \otimes S)_1$ is semisimple. The $g'_1 \oplus g'_2$-invariant vectors of $R_z \otimes R_{\pi(z)}$ are the tensor products of the $\mathfrak{g}'$-invariant parts, of dimensions $3 \cdot 6$ for (c_1, c_2), $3 \cdot 5$ for (d_1, d_5) and $1 \cdot 3$ for (d_4, d_6), and those of $(S \otimes S)_1$ form $g''_1 \oplus g''_2$, of dimension 6. The total is $6 + 2 \cdot 18 + 2 \cdot 15 + 2 \cdot 3 = 78$.

2. By part 1 and [DoM04b, Theorem 2 p. 2], whose hypotheses are that W_π is C_2-cofinite, holomorphic, of CFT type and of central charge 16, W_π is isomorphic to $V_{E_8 \oplus E_8}$ or to $V_{D_{16}^+}$. The exclusion of $V_{D_{16}^+}$ in [ArhII, proof of Proposition 7.12] uses only the three properties of $(W_\pi)_1$ established in part 3 and the levels 3 of g'_1, g'_2. We recall it. If $W_\pi \cong V_{D_{16}^+}$, then $(W_\pi)_1 \cong \mathfrak{so}(E) \cong \Lambda^2 E$ with $\dim E = 32$. The Dynkin index of $g'_k \subset \mathfrak{so}(E)$ is 3, the vector representation has Dynkin index 2, so $E|_{g'_k}$ has Dynkin index 6 and is $6V(1) \oplus 20V(0)$ or $V(2) \oplus 2V(1) \oplus 25V(0)$. The second case gives $V(3) \subset V(2) \otimes V(1) \subset \Lambda^2 E$, excluded by part 3. Hence $E|_{g'_1 \oplus g'_2} = pV(1,1) \oplus (6-2p)(V(1,0) \oplus V(0,1)) \oplus (8+4p)V(0,0)$ for some $p \in \{0,1,2,3\}$,

and the centralizer of $g_1' \oplus g_2'$ in $\mathfrak{so}(E)$ is $\mathfrak{so}_p \oplus \mathfrak{so}_{8+4p} \oplus \mathfrak{sp}_{6-2p} \oplus \mathfrak{sp}_{6-2p}$, of dimension 70, 86, 127 or 193, which contradicts the value 78 of part 3. ∎

The converse holds: a diagonal holomorphic extension forces (R).

Lemma 9.10 *Assume* (I1) *and* (F1)–(F4)*, and let W be a diagonal holomorphic extension of $S \otimes S$. Then there is a braided equivalence $\pi\colon \mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$ such that $W \cong \bigoplus_z M_z \otimes M_{\pi(z)}$ as an $S \otimes S$-module. In particular* (R) *holds, W_1 has the properties of Proposition 9.9(3), and $W \cong V_{E_8 \oplus E_8}$.*

Proof : Since $S \otimes S$ is regular and W has finite-dimensional weight spaces and nonnegative weights, W is a finite direct sum of simple $S \otimes S$-modules $M_z \otimes M_{z'}$, because there are finitely many simple $S \otimes S$-modules, their lowest weights form a finite set, and an infinite multiplicity would contradict finite-dimensionality of the corresponding weight space. Write $W = \bigoplus_{z \in I} M_z \otimes V_z$, where $V_z \in \mathrm{Rep}(S)$ is the sum of the $M_{z'}$ paired with M_z and I is the set of labels with $V_z \neq 0$. A vector annihilated by the nonnegative modes of $S \otimes \mathbf{1}$ lies in $S_0 \otimes V_0$, as in the proof of Proposition 9.9, so $\mathrm{Com}_W(S \otimes \mathbf{1}) = \mathbf{1} \otimes V_0$ and the hypothesis gives $V_0 \cong S$. Likewise $\mathrm{Com}_W(\mathbf{1} \otimes S) = \bigoplus_z M_z \otimes \mathrm{Hom}_S(S, V_z)$, and the hypothesis gives $\dim \mathrm{Hom}_S(S, V_z) = \delta_{z,0}$. The hypotheses of [CKM22, Theorem 5.10(2) p. 51] are now satisfied for $U = V = S$: $\mathrm{Rep}(S)$ is a ribbon category, W is a simple $\mathbb{Z}$-graded vertex operator algebra extension of $S \otimes S$ of the form $\bigoplus_{z \in I} M_z \otimes V_z$ with $M_0 = S$, and $\dim \mathrm{Hom}_S(S, V_z) = \delta_{z,0}$. Hence the full subcategories $\mathcal{U}_W$ and $\mathcal{V}_W$ of $\mathrm{Rep}(S)$ generated by the M_z, $z \in I$, and by the V_z are ribbon subcategories, $\mathcal{V}_W$ is semisimple with the V_z as distinct simple objects, and there is a braid-reversed equivalence $\tau\colon \mathcal{U}_W \to \mathcal{V}_W$ with $\tau(M_z) \cong V_z'$.

By [HKL15, Theorem 3.2 p. 10], W corresponds to a connected commutative algebra $\mathcal{A}$ in $\mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S)$, which is étale by [CKM22, Theorem 5.12 p. 52] and [DMNO13, Proposition 2.7 p. 5] as in the proof of Proposition 9.9. Since W is holomorphic, its category of local modules is trivial by [HKL15, Theorem 3.4 p. 16], so $\mathcal{A}$ is Lagrangian [DMNO13, Remark 4.3 and Corollary 3.32], and $\mathrm{FPdim}\,\mathcal{A} = \mathrm{FPdim}\,\mathrm{Rep}(S) = \sum_z (\dim M_z)^2$, the sum over all twelve labels. On the other hand $\mathrm{FPdim}\,\mathcal{A} = \sum_{z \in I} \dim M_z \dim V_z = \sum_{z \in I} (\dim M_z)^2$, since the tensor equivalence τ preserves Frobenius–Perron dimensions and $\dim V_z' = \dim V_z$. Hence I is the set of all labels, $\mathcal{U}_W = \mathrm{Rep}(S)$, and $\mathcal{V}_W$ is a fusion subcategory of $\mathrm{Rep}(S)$ of the same Frobenius–Perron dimension, hence $\mathcal{V}_W = \mathrm{Rep}(S)$. Thus $\pi \stackrel{\mathrm{def}}{=} \tau$ is a braided equivalence $\mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$, and $V_z \cong \pi(M_z)' \cong M_{\pi(z)}$ by the self-duality of Lemma 9.3. The remaining assertions follow from parts 2 and 3 of Proposition 9.9, whose proofs only use the $S \otimes S$-module structure of W and the fact that W is C_2-cofinite, holomorphic and of CFT type. ∎

Remark 9.11 Under (I1) and (F1)–(F4), Proposition 9.9 and Lemma 9.10 show that (R) is equivalent to the existence of a diagonal holomorphic extension of $S \otimes S$. The underlying $S \otimes S$-module of W_π depends only on the permutation of labels. Its multiplication also depends on the monoidal structure of π, and no uniqueness from the permutation alone is asserted. Two choices differing only on $\{a, c_0\}$ have the same weight-one $S_1 \oplus S_1$-module. Theorem 4.25 constructs a diagonal holomorphic extension of $S \otimes S$ from the conformal net $\mathcal{A}_S$, using Proposition 4.8, so that (R) is established under the inputs of that theorem. In the rest of this section, (R) and Theorem 4.25 enter only through the existence of a diagonal holomorphic extension, which we take as the standing hypothesis.

9.4 The unitary structure

Let W be a diagonal holomorphic extension of $S \otimes S$. By Lemma 9.10, $W \cong V_\Gamma$ with $\Gamma = E_8 \oplus E_8$. We use the explicit model of V_Γ of [ArhII, Section 7.5], built with the bimultiplicative cocycle

$\varepsilon(\alpha,\beta) = \varepsilon_8(\alpha^{(1)},\beta^{(1)})\varepsilon_8(\alpha^{(2)},\beta^{(2)})$, where ε_8 is the cocycle on E_8 specified there, and with the antilinear involution ϕ_1 of [DoL14, Theorem 4.12], which is the PCT operator of a unitary structure on V_Γ. For $\kappa(\lambda) = 2^{-\langle\lambda,\varpi\rangle}$, $\varpi = (\varpi_8,\varpi_8)$, $\varpi_8 = e_7 + e_8$, let $\phi_\kappa = \phi_1 \circ \exp(h_\kappa(0))$, so that $\phi_\kappa(\mathrm{e}^\lambda) = \kappa(\lambda)\mathrm{e}^{-\lambda}$. Let $\mathfrak{s}_1,\dots,\mathfrak{s}_4 \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8 = (V_\Gamma)_1$ be the four commuting subalgebras isomorphic to $\mathfrak{sl}_2$ defined in [ArhII, Section 7.5], spanned by the explicit $\mathfrak{sl}_2$-triples (E_k,H_k,F_k), $F_k = -\phi_\kappa(E_k)$, with Dynkin indices $(2,1)$, $(6,7)$, $(1,2)$, $(7,6)$ in the two copies of $\mathfrak{e}_8$, so that they generate affine subalgebras of levels 3, 13, 3, 13. Let $A = L_{\mathfrak{sl}_2}(3,0)\otimes L_{\mathfrak{sl}_2}(13,0) \subset S$ be the affine subalgebra generated by S_1, which is simple by Section 2, let $\mathscr{C} = \mathrm{Com}_S(A)$, and let $\omega_{\mathscr{C}} = \omega - \omega_A$ be its conformal vector, of central charge $8 - \frac{9}{5} - \frac{13}{5} = \frac{18}{5}$, where ω_A is the Sugawara vector of A. In W we write $\omega_{\mathscr{C}\otimes 1} = \omega_{\mathscr{C}}\otimes\mathbf{1}$, $\omega_{1\otimes\mathscr{C}} = \mathbf{1}\otimes\omega_{\mathscr{C}}$, and similarly for A and S.

Proposition 9.12 *Assume* (I1) *and* (F1)–(F4)*, let W be a diagonal holomorphic extension of $S\otimes S$, and fix an isomorphism of vertex operator algebras $W \to V_\Gamma$.*

1. *After composing this isomorphism with an automorphism of V_Γ, the images of g_1', g_1'', g_2', g_2'' are $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$.*
2. *With this identification, the space $\mathcal{G} = \mathrm{Com}_W(A\otimes A)\cap W_2$ has dimension 16, and $\omega_{\mathscr{C}\otimes 1}$ is one of four explicit vectors $u_0,\dots,u_3 \in \mathcal{G}$, whose coordinates in a rational basis of $\mathcal{G}$ lie in $\mathrm{Q}(\sqrt{-6})$.*
3. *ϕ_κ is the PCT operator of a structure of unitary vertex operator algebra on W, and $S\otimes\mathbf{1}$, $\mathbf{1}\otimes S$ and $S\otimes S$ are unitary subalgebras for this structure, in the sense of [CKLW18, Definition 5.22 p. 43].*

Proof : Apply [ArhII, Lemmas 7.14–7.15 and Proposition 7.16]. The inputs to those proofs are the rank-16 lattice realization and the weight-one decomposition of Proposition 9.9, the levels $(3,13)$, the twelve top spaces and conformal weights in (F2), and positivity in (F1). The tensor-category identification ($\mathrm{H_J}$) is not used after these data have been supplied.

The computation has three steps. The enumeration of embeddings of the four current algebras into $\mathfrak{e}_8\oplus\mathfrak{e}_8$ leaves the specified embedding. The common kernel of their relevant modes in weight two has dimension 16, and the Virasoro and trace equations in that space have the four solutions $u_0,\dots,u_3$. The explicit PCT operator ϕ_κ preserves the currents and these four vectors. Since it also fixes ω and the Sugawara vectors of the currents, it fixes $\omega_{1\otimes\mathscr{C}} = \omega - \omega_{A\otimes 1} - \omega_{1\otimes A} - \omega_{\mathscr{C}\otimes 1}$. It therefore preserves both copies of S: each copy is the commutant of the currents and coset conformal vector of the other copy, since positivity excludes a vector of zero opposite conformal weight in a nonvacuum summand. Stability under L_1 then gives unitary subalgebras by [CKLW18, Proposition 5.23]. The programs and their outputs form block C3 of Section 10. ∎

Remark 9.13 The unitary structure of Proposition 9.12(3) is constructed from the lattice. When W is the algebra of Theorem 4.25, it comes with the unitary structure of Henriques and Tener [HeT25], for which $S\otimes S$ is a unitary subalgebra by construction, and Proposition 9.12 is not needed for it. Two structures of unitary vertex operator algebra on a simple vertex operator algebra are unitarily isomorphic [CKLW18, Proposition 5.19 pp. 40–41], so the conclusions of Theorem 5.1 do not depend on which structure is used.

9.5 The equivalent criteria

Theorem 9.14 (Theorem C) *Assume* (I1) *and* (F1)–(F4)*, so that Theorem 3.1 is available. The following statements are equivalent. They all hold by Theorem 6.1, and the implications* (ii)⇒(i) *and* (iii)⇒(iv)⇒(v)⇒(vi)⇒(i) *are proved without Propositions 4.4 and 4.8.*

(i) ($\mathrm{H_J}$)*:* $\mathrm{Rep}(S)$ *is braided equivalent to* $Z(\mathcal{H}_3)$*.*

(ii) *The fusion category* $\mathcal{F}_x = \mathrm{Rep}(S)_{L_x}$ *of* (F4) *is unitarizable, for one, equivalently for both,* $x \in \{+,-\}$*.*

(iii) (R)*: there is a braided equivalence* $\mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$*.*

(iv) $S \otimes S$ *admits a diagonal holomorphic extension in the sense of Definition 9.8.*

(v) S *is completely unitary, and the functor* $\Pi\colon \mathrm{Rep}^u(S) \to \mathrm{Rep}(\mathcal{A}_S)$ *of Section 5.1 is an equivalence of braided* C^**-tensor categories.*

(vi) $\mathrm{Rep}(S)$ *is braided equivalent to* $\mathrm{Rep}(\mathcal{A}_S)$ *or to* $\mathrm{Rep}(\mathcal{A}_S)^{\mathrm{rev}}$*.*

Proof : (i) ⇒ (iii). Composing a braided equivalence $\mathrm{Rep}(S) \to Z(\mathcal{H}_3)$, its reverse, and the equivalence R of Lemma 4.27 gives a braided equivalence $\mathrm{Rep}(S)^{\mathrm{rev}} \to \mathrm{Rep}(S)$, whose inverse is (iii).

(iii) ⇒ (iv). This is Proposition 9.9: W_π is a diagonal holomorphic extension of $S \otimes S$.

(iv) ⇒ (iii). This is Lemma 9.10.

(iv) ⇒ (v). Let W be a diagonal holomorphic extension. By Proposition 9.12, W carries a unitary structure for which $S \otimes \mathbf{1}$ and $\mathbf{1} \otimes S$ are unitary subalgebras. Since $W \cong V_\Gamma$ by Lemma 9.10 and two unitary structures on V_Γ are unitarily isomorphic [CKLW18, Proposition 5.19 pp. 40–41], Theorem 5.1 gives (v).

(v) ⇒ (vi). By (v), the forgetful functor $\mathrm{Rep}^u(S) \to \mathrm{Rep}(S)$ is a braided equivalence [CGGH23, Remark 4.3 p. 18], and Π is a braided equivalence onto $\mathrm{Rep}(\mathcal{A}_S)$ with the braiding of Connes fusion, which is the DHR braiding of Section 3 or its reverse by Remark 5.2.

(vi) ⇒ (i). By Theorem 3.1(3), $\mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{H}_3)$ as braided categories, and $Z(\mathcal{H}_3)^{\mathrm{rev}} \cong Z(\mathcal{H}_3)$ by Lemma 4.27. Hence $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$ in both cases of (vi).

(i) ⇒ (ii). Under (i), L_+ and L_- are Lagrangian algebras of $Z(\mathcal{H}_3)$. By [ArhII, Proposition 3.3], the Lagrangian algebras of $Z(\mathcal{H}_3)$ are L_1, L_2, L_3, with categories of modules $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, which are unitary fusion categories. Hence $\mathcal{F}_\pm$ is equivalent to one of them.

(ii) ⇒ (i). Apply Theorem A.25(4) for the given x.

(v) ⇒ (ii). By (v) and [Gui19, Theorem 7.9 p. 53], $\mathrm{Rep}^u(S)$ is a unitary modular tensor category, and the forgetful functor $\mathrm{Rep}^u(S) \to \mathrm{Rep}(S)$ is a braided equivalence. The algebra L_x corresponds to a haploid commutative algebra in $\mathrm{Rep}^u(S)$, which is rigid since a connected étale algebra in a fusion category is rigid [DMNO13, Remark 3.4 p. 7], hence equivalent to a Q-system Q_x in the C^*-tensor category $\mathrm{Rep}^u(S)$ [CGGH23, Theorem 3.2 p. 11]. The category of Q_x-modules in $\mathrm{Rep}^u(S)$, with the tensor product over Q_x, is equivalent to $\mathcal{F}_x$ as a fusion category, since $Q_x \cong L_x$ as algebras. Every module over a Q-system is unitarizable and the unitary modules form a C^*-category [Gui22, Proposition 2.24 p. 26 and Theorem 2.28 p. 28], and the unitary bimodules form a C^*-tensor category for the tensor product over the Q-system [Gui22, Theorem 3.13 p. 35]. For the commutative Q-system Q_x, a module is a bimodule through the braiding, and the modules form a full tensor subcategory of the bimodules [KiO02, Section 1]. Hence $\mathcal{F}_x$ is equivalent to a C^*-tensor category with simple unit and finitely many simple objects, that is, $\mathcal{F}_x$ is unitarizable. ∎

Remark 9.15 Theorem 6.1 establishes these statements using the analytic diagonal extension. The implication (iv)⇒(v) applies to an arbitrary diagonal holomorphic extension and uses the computer-assisted Proposition 9.12. For the extension reconstructed from $\mathcal{B}$, the unitary structure and the unitary lattice realization are given by Theorem 4.25, and Proposition 9.12 is not needed. An independent proof of (ii) or of (iii) would give (H_J) without Propositions 4.4 and 4.8.

9.6 The lattice of the reconstructed extension

Corollary 9.16 *Assume* (I1) *and* (F1)–(F4). *The vertex operator algebra W of Theorem 4.25 is isomorphic to $V_{E_8\oplus E_8}$, and the net $\mathcal{B}$ of Proposition 4.4 is isomorphic to $\mathcal{A}_{V_{E_8\oplus E_8}}$. The categories $\mathcal{F}_+$ and $\mathcal{F}_-$ of* (F4) *are equivalent to $\mathcal{H}_2$ and $\mathcal{H}_3$ in some order.*

Proof : By Theorem 4.25(1), W is a diagonal holomorphic extension of $S\otimes S$ in the sense of Definition 9.8. Hence $W\cong V_{E_8\oplus E_8}$ by Lemma 9.10, and $\mathcal{B}\cong\mathcal{A}_{V_{E_8\oplus E_8}}$ by Proposition 4.23. Let $\Phi_0\colon \mathrm{Rep}(S)\to Z(\mathcal{H}_3)$ be the braided equivalence of Section 6. By (F4), $L_\pm$ is a Lagrangian algebra of $\mathrm{Rep}(S)$, so that $\Phi_0(L_\pm)$ is a Lagrangian algebra of $Z(\mathcal{H}_3)$. By (F3) and the choice of the labels in Section 6, its underlying object is $0\oplus b\oplus 2\Phi_0(X_\pm)$, where $\Phi_0(X_+)\ncong\Phi_0(X_-)$ and $\{\Phi_0(X_+),\Phi_0(X_-)\}=\{a,c_0\}$. By [ArhII, Proposition 3.3], $\Phi_0(L_+)$ and $\Phi_0(L_-)$ are L_2 and L_3 in some order, so that $\mathcal{F}_+$ and $\mathcal{F}_-$ are equivalent to $Z(\mathcal{H}_3)_{L_2}\cong\mathcal{H}_2$ and $Z(\mathcal{H}_3)_{L_3}\cong\mathcal{H}_3$ in some order. ∎

10 Certificates and programs

The computations used in the proofs are listed below, with the statements they support. The certificate checks are written in Python 3 and use exact arithmetic over Q and over the fields specified in the programs. In particular, block A4 uses $\mathbb{K}=\mathrm{Q}(\sqrt{13},\sqrt{-3})$ and block C3 uses $\mathrm{Q}(\sqrt{-6})$ for the conformal-vector coordinates. The supplementary archives listed below record the scripts, outputs, checksums and replay instructions.

1. Block A1, `a1_lattices.py` and `a1b_extra.py`, outputs `a1_lattices.out` and `a1b_extra.out`: Lemma 7.1 and the lattice facts used in Section 7.3. The first script works in the coordinates (p,b) of $\mathfrak{h}$ and verifies the Gram matrices, the intersection $N=L_{\mathrm{I}}\cap L_{\mathrm{II}}$ on a box, the indices, the characters of σ and τ, the discriminant group N'/N with its 2-part and 39-part, the isometry ψ_N and the value $u=22$, and the Weyl group actions 14, 25. The second script verifies the values of $(s,\cdot)$ and $(p,\cdot)$ on L'_{II} and L'_{I}, the identities (7.4), the isotropy statement of part 1 of the lemma, and, in the root coordinates of [DJX26, (4.1)], the intersections (7.5) on a box and the properties of L_0, L_H and E of part 5.

2. Block A4, the scripts `a4_kfield.py`, `a4_voa.py`, `a4_data.py`, `a4_common.py` and `a4_smoke.py`, the six scripts `a4_item1_lattice.py` to `a4_item6_riesen.py`, the driver `a4_run_all.sh` and the outputs: the independent verification of the finite identities of [DJX26] described in Section 8.2. The lattice vertex operator algebra V_L is implemented from scratch in the basis and with the cocycle of [DJX26, Section 4], with the mode formula of [DJX26, Appendix A] and without truncation in the conformal weight, independently of the implementation of [ArhI], which uses the Bourbaki basis of E_6 and a different cocycle. The field $\mathbb{K}$ is coded by hand in the basis $1,\sqrt{13},\sqrt{-3},\sqrt{-39}$.

3. Block C3, the directory `run` with the programs `R3` and `R3-verif` of the archive `Calculs-HU-v08` of [ArhII], three driver scripts, the scripts `a3_c3_table_check.py` and

`a3_c3_compare.py`, and the directory `out` with 69 outputs and the timing file: the recomputation of [ArhII, Section 7.5] used in Section 9. The comparison with `Calculs-HU-v08` covers 47 regenerated data files, including the re-export of the portable certificate. The script `a3_c3_table_check.py` checks the input tables `t12_table.json` and `t12_W1.json` against the lowest weight spaces in Table 1 and the exported Zhu data. These computations use the data of (F2).

The archive `Calculs-III-A1-A4-C3-v10.zip` contains the scripts and outputs of blocks A1, A4 and C3, with replay instructions and a SHA-256 manifest. The Zhu calculations and the two ideal-containment verifiers are in the separate supplementary archive `Certificats-Zhu-III-v06.zip`, described in Section A.8. Block C3 supports the alternative construction in Section 9. The main analytic proof uses the compatible unitary structure obtained by reconstruction.

The certificates are exact computations. They are not a formal verification of the implementations of the lattice vertex operators. The two implementations used in blocks A4 and C3 are independent of each other and of the one of [ArhI], and the archived comparisons report agreement on the quantities tested by both implementations.

A Zhu algebra, simple modules and lattice branching

We prove (F1)–(F3) of Theorem 2.2 and (F4) of Theorem 9.1 for the explicit S. The inputs from [ArhI] are the lattice construction and the results (I1)–(I2) of Theorem 2.1, the generation of $A(S)$ by the current and weight-two classes in its Proposition 11.3, and its Proposition 11.4 on the affine simple-current extension A^{+}. For the exact relations we use the explicit U_1 and U_2 over $\mathbb{K}$ and $\omega = \omega_A + 2y$ from that construction. The branching and block-survival arguments use the known lattice modules and their genus-one modular matrices. The categorical argument for (F4) uses the extension and recognition results of [ArhII] stated for these inputs independently of ($\mathrm{H_J}$).

A.1 An integral descent lemma

Throughout this section, R is a discrete valuation ring with maximal ideal $\mathfrak{m} = \pi R$, residue field $k = R/\mathfrak{m}$ and field of fractions K. An R-algebra is finite if it is finitely generated as an R-module. A finite-dimensional algebra over a field F is split semisimple if it is isomorphic to a finite product of full matrix algebras $\mathrm{M}_n(F)$. The following statement, and the use made of it below, follow the method of [Rie26b, Lemma 9.6 p. 55]. We give a complete proof.

Lemma A.1 *Let Λ be a finite R-algebra such that $\Lambda/\mathfrak{m}\Lambda \cong \prod_{i\in I} \mathrm{M}_{n_i}(k)$ is split semisimple.*

1. *There exist a subset $I' \subset I$ and, for any algebraically closed field Ω containing K, a bijection between I' and the set of simple factors of $\Lambda \otimes_R \Omega$, such that the factor attached to i is isomorphic to $\mathrm{M}_{n_i}(\Omega)$. In particular the K-algebra $\Lambda \otimes_R K$ is semisimple.*

2. *Let $\varepsilon \in \Lambda$ be an idempotent and let $i \in I'$. The rank of ε acting on the simple module of the factor of $\Lambda \otimes_R \Omega$ attached to i is equal to the rank of the image of ε acting on the simple module of the factor $\mathrm{M}_{n_i}(k)$ of $\Lambda/\mathfrak{m}\Lambda$.*

3. *Let $z \in \Lambda$ and $i \in I'$, and assume that z acts on the simple module of the factor of $\Lambda \otimes_R \Omega$ attached to i by a scalar $c \in R$. Then the image of z in $\Lambda/\mathfrak{m}\Lambda$ acts on the simple module of the factor $\mathrm{M}_{n_i}(k)$ by the reduction of c.*

Proof : Let $T = \{x \in \Lambda : \pi^n x = 0 \text{ for some integer } n \geqslant 0\}$ be the torsion submodule. Since R is central in Λ, the subspace T is a two-sided ideal. The quotient $C \stackrel{\text{def}}{=} \Lambda/T$ is a finitely generated torsion-free module over the principal ideal domain R, hence free of finite rank. Since K is flat over R and $T \otimes_R K = 0$, the quotient map induces an isomorphism $\Lambda \otimes_R K \to C \otimes_R K$, and also $\Lambda \otimes_R \Omega \cong C \otimes_R \Omega$. The surjection $\Lambda \to C$ induces a surjective homomorphism of algebras $\Lambda/\mathfrak{m}\Lambda \to C/\mathfrak{m}C$. Its kernel is a two-sided ideal of $\prod_{i\in I} \mathrm{M}_{n_i}(k)$, hence of the form $\prod_{i\in I\setminus I'} \mathrm{M}_{n_i}(k)$ for some subset $I' \subset I$. Thus $C/\mathfrak{m}C \cong \prod_{i\in I'} \mathrm{M}_{n_i}(k)$, and the images ε_i, $i \in I'$, of the units of the factors are central orthogonal idempotents of $C/\mathfrak{m}C$ with sum 1.

Let $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. It is a complete discrete valuation ring with uniformizer π, residue field k, and its field of fractions $\hat{K}$ contains K. The $\hat{R}$-algebra $\hat{C} \stackrel{\text{def}}{=} C \otimes_R \hat{R}$ is a free $\hat{R}$-module of finite rank, it is complete for the π-adic topology, and $\hat{C}/\pi\hat{C} \cong C/\mathfrak{m}C$. We use the following lifting of idempotents. Let Q be an $\hat{R}$-algebra which is free of finite rank as an $\hat{R}$-module, and let $\bar{e}$ be an idempotent of $Q/\pi Q$. Choose $x_0 \in Q$ lifting $\bar{e}$ and put $x_{n+1} = 3x_n^2 - 2x_n^3$. The identities $x_{n+1}^2 - x_{n+1} = (x_n^2 - x_n)^2(4x_n^2 - 4x_n - 3)$ and $x_{n+1} - x_n = -(x_n^2 - x_n)(2x_n - 1)$ show by induction that $x_n^2 - x_n \in \pi^{2^n} Q$ and that (x_n) is a Cauchy sequence. Its limit e is an idempotent of Q lifting $\bar{e}$. Applying this to $Q = \hat{C}$ and $\bar{e} = \varepsilon_1$, then to the corner $(1 - e_1)\hat{C}(1 - e_1)$, which is a direct summand of $\hat{C}$ and reduces to $(1 - \varepsilon_1)(C/\mathfrak{m}C)(1 - \varepsilon_1)$, and so on, we obtain orthogonal idempotents $e_i \in \hat{C}$, $i \in I'$, with $\sum_i e_i = 1$, such that e_i reduces to ε_i (the last one is 1 minus the sum of the others). For $i \neq j$, the $\hat{R}$-module $e_i\hat{C}e_j$ is a direct summand of $\hat{C} = \bigoplus_{r,s} e_r\hat{C}e_s$, hence free of finite rank, and its reduction modulo π is $\varepsilon_i(C/\mathfrak{m}C)\varepsilon_j = 0$ since ε_i is central. By Nakayama's lemma, $e_i\hat{C}e_j = 0$. Consequently each e_i is central and $\hat{C} = \prod_{i\in I'} \hat{C}_i$ with $\hat{C}_i \stackrel{\text{def}}{=} e_i\hat{C}e_i$ and $\hat{C}_i/\pi\hat{C}_i \cong \mathrm{M}_{n_i}(k)$.

Fix $i \in I'$ and put $n = n_i$. Lift a rank-one idempotent of $\mathrm{M}_n(k)$ to an idempotent $f \in \hat{C}_i$ as above. The left ideal $P \stackrel{\text{def}}{=} \hat{C}_i f$ is a direct summand of $\hat{C}_i$, hence a free $\hat{R}$-module, and $P/\pi P \cong \mathrm{M}_n(k)E_{11} \cong k^n$. Thus P is free of rank n, and $\hat{C}_i$ is free of rank n^2. Left multiplication gives a homomorphism of $\hat{R}$-algebras $\varphi\colon \hat{C}_i \to \mathrm{End}_{\hat{R}}(P) \cong \mathrm{M}_n(\hat{R})$ between free $\hat{R}$-modules of rank n^2, whose reduction modulo π is the action of $\mathrm{M}_n(k)$ on k^n, an isomorphism. The determinant of φ in bases is therefore a unit of the local ring $\hat{R}$, and φ is an isomorphism. Hence $\hat{C} \cong \prod_{i\in I'} \mathrm{M}_{n_i}(\hat{R})$ and $C \otimes_R \hat{K} \cong \hat{C} \otimes_{\hat{R}} \hat{K} \cong \prod_{i\in I'} \mathrm{M}_{n_i}(\hat{K})$.

Put $B \stackrel{\text{def}}{=} \Lambda \otimes_R K \cong C \otimes_R K$. The Jacobson radical $J(B)$ is a nilpotent ideal, so $J(B) \otimes_K \hat{K}$ is a nilpotent ideal of the semisimple algebra $B \otimes_K \hat{K} \cong C \otimes_R \hat{K}$. Thus $J(B) = 0$ and B is semisimple. Let Ω' be an algebraically closed field containing $\hat{K}$ and let $\overline{K}$ be the algebraic closure of K in Ω'. Since K has characteristic zero, $B \otimes_K \overline{K}$ is a product of matrix algebras over $\overline{K}$, and its factors remain simple after extension to Ω'. Since $B \otimes_K \Omega' \cong \prod_{i\in I'} \mathrm{M}_{n_i}(\Omega')$, the algebra $B \otimes_K \overline{K}$ has factors of sizes n_i, $i \in I'$. Any algebraically closed field Ω containing K contains an algebraic closure of K, and we fix a K-embedding $\iota\colon \overline{K} \to \Omega$. The simple factors of $B \otimes_K \overline{K}$ correspond bijectively, by extension of scalars, to those of $B \otimes_K \Omega$ through ι and to those of $B \otimes_K \Omega'$, and the latter are the factors $\hat{C}_i \otimes_{\hat{R}} \Omega'$, $i \in I'$. This defines the bijection of the first part, which is proved.

For the second part, the image of ε in $\hat{C}_i \cong \mathrm{M}_n(\hat{R})$ is an idempotent matrix E. Its image $E\hat{R}^n$ is a direct summand of $\hat{R}^n$, hence free of some rank r. The rank of E over $\hat{K}$, hence over Ω', is r, and the image of the reduction of E modulo π is $(E\hat{R}^n)/\pi(E\hat{R}^n) \cong k^r$. The rank of ε on the simple module of a factor of $B \otimes_K \overline{K}$ does not change under the extensions of scalars to Ω and to Ω', which gives the claim. For the third part, z acts by c on the simple module of the corresponding factor of $B \otimes_K \overline{K}$, hence on that of $\hat{C}_i \otimes_{\hat{R}} \Omega'$. Since $\hat{R} \to \Omega'$ is injective, the image of z in $\hat{C}_i \cong \mathrm{M}_n(\hat{R})$ is the scalar matrix c, whose reduction modulo π is the scalar

matrix $\bar{c}$. ∎

Remark A.2 The finiteness of Λ cannot be removed, and the subset I' can be proper. For the R-algebra $\Lambda = R[x]/(\pi x^2 - x)$, which is not finite over R, we have $\Lambda/\mathfrak{m}\Lambda \cong k$ while $\Lambda \otimes_R K \cong K \times K$. For the finite R-algebra $\Lambda = R \times k$, we have $\Lambda/\mathfrak{m}\Lambda \cong k \times k$ while $\Lambda \otimes_R K \cong K$. These are the two phenomena which a computation of a presentation modulo a prime cannot control by itself: the generic dimension may exceed the special one in the absence of a finiteness statement over R, and a special block may disappear generically. The first is excluded below by an explicit spanning set over R, and the second by the equality of the generic and special dimensions (Theorem A.10). The bijection of part 1 of Lemma A.1 depends on the choices of the embeddings of $\overline{K}$, and only the sizes n_i, the ranks of the idempotents of Λ and the central scalars in R are transported by it. These are the only data used below. In particular the lemma alone does not distinguish two blocks with the same such data, as the blocks a and c_0 of Section A.6.

Remark A.3 If the characteristic of k does not divide any of the integers n_i, the semisimplicity of $\Lambda \otimes_R K$ has a shorter proof. With the notation of the proof, the regular trace form $(x, y) \mapsto \mathrm{Tr}(L_{xy})$ of the free R-algebra C has a Gram matrix with entries in R whose reduction modulo $\mathfrak{m}$ is the Gram matrix of the regular trace form of $C/\mathfrak{m}C \cong \prod_{i \in I'} \mathrm{M}_{n_i}(k)$. The latter is nondegenerate, since on $\mathrm{M}_n(k)$ the regular trace is n times the matrix trace. Hence the discriminant is a unit of R, the regular trace form of $C \otimes_R K$ is nondegenerate, and $C \otimes_R K$ is semisimple: if x lies in its radical, then xy is nilpotent for any y, so that $\mathrm{Tr}(L_{xy}) = 0$, and $x = 0$.

A.2 The Morita corner of Zhu's algebra and its quiver presentation

A.2.1 Matrix units and full idempotents

The following elementary statement is the algebraic part of [Rie26b, Lemma 8.2 and Proposition 8.3 pp. 34–35]. We include its proof.

Lemma A.4 *Let $\mathcal{A}$ be a unital algebra over a field F, let $(T_v)_{v \in \mathsf{V}}$ be a finite family of nonzero finite-dimensional F-vector spaces with fixed bases, let $\mathcal{B} = \prod_v \mathrm{End}_F(T_v)$ and let $\varphi \colon \mathcal{B} \to \mathcal{A}$ be a unital homomorphism of algebras. Denote by $E_{ij}^{(v)}$ the matrix units of $\mathrm{End}_F(T_v)$ and put $e_v \overset{\text{def}}{=} \varphi(E_{11}^{(v)})$, $e \overset{\text{def}}{=} \sum_v e_v$ and $\mathcal{D} \overset{\text{def}}{=} e\mathcal{A}e$.*

1. *The idempotent e is full, that is $\mathcal{A}e\mathcal{A} = \mathcal{A}$.*
2. *The map*

$$\mathcal{X} \colon \mathcal{A} \to \bigoplus_{v,w} \mathrm{Hom}_F(T_w, T_v) \otimes_F e_v \mathcal{D} e_w, \quad b \mapsto \sum_{v,w,i,j} E_{ij}^{v,w} \otimes \varphi(E_{1i}^{(v)})\, b\, \varphi(E_{j1}^{(w)}),$$

   where $E_{ij}^{v,w} \in \mathrm{Hom}_F(T_w, T_v)$ are the matrix units in the fixed bases, is an isomorphism of algebras, the right-hand side being equipped with the product $(F \otimes d)(G \otimes c) = FG \otimes dc$ when the intermediate indices agree and 0 otherwise.

3. *The functor $M \mapsto eM$ is an equivalence between the categories of left $\mathcal{A}$-modules and of left $\mathcal{D}$-modules. In particular $\mathcal{A}$ is semisimple if and only if $\mathcal{D}$ is semisimple, and for a simple $\mathcal{A}$-module M and any v, the dimension of $e_v M$ is the multiplicity of T_v in M considered as a $\mathcal{B}$-module through φ.*

Proof : 1. We have $1 = \sum_{v,i} \varphi(E^{(v)}_{ii}) = \sum_{v,i} \varphi(E^{(v)}_{i1})\, e_v\, \varphi(E^{(v)}_{1i}) \in \mathcal{A}e\mathcal{A}$.

2. Since $e_v\varphi(E^{(v)}_{1i}) = \varphi(E^{(v)}_{1i})$ and $\varphi(E^{(w)}_{j1})e_w = \varphi(E^{(w)}_{j1})$, the map $\mathcal{X}$ takes values in the indicated space. The linear map $\Psi(E^{v,w}_{ij} \otimes d) \overset{\text{def}}{=} \varphi(E^{(v)}_{i1})\, d\, \varphi(E^{(w)}_{1j})$ satisfies $\Psi(\mathcal{X}(b)) = \sum_{v,w,i,j} \varphi(E^{(v)}_{ii})\, b\, \varphi(E^{(w)}_{jj}) = b$ and $\mathcal{X}(\Psi(E^{v,w}_{ij} \otimes d)) = E^{v,w}_{ij} \otimes d$, since $\varphi(E^{(u)}_{1k})\varphi(E^{(v)}_{i1}) = \delta_{uv}\delta_{ki}e_v$ and $d \in e_v\mathcal{A}e_w$. For the product, $\Psi(E^{v,w}_{ij} \otimes d)\Psi(E^{w',u}_{kl} \otimes c) = \delta_{ww'}\delta_{jk}\varphi(E^{(v)}_{i1})\, dc\, \varphi(E^{(u)}_{1l})$.

3. We show that $N \mapsto \mathcal{A}e \otimes_{\mathcal{D}} N$ is a quasi-inverse. By part 1, write $1 = \sum_k x_k e y_k$ with $x_k, y_k \in \mathcal{A}$. For a left $\mathcal{A}$-module M, the map $\mu\colon \mathcal{A}e \otimes_{\mathcal{D}} eM \to M$, $a \otimes m \mapsto am$, is surjective, since $m = \sum_k x_k e(y_k m)$. If $\sum_i a_i e \otimes em_i$ lies in its kernel, then

$$\sum_i a_i e \otimes em_i = \sum_{k,i} x_k e\,(ey_k a_i e) \otimes em_i = \sum_k x_k e \otimes ey_k \Big(\sum_i a_i e m_i \Big) = 0,$$

because $ey_k a_i e \in \mathcal{D}$. Thus μ is an isomorphism, natural in M. For a left $\mathcal{D}$-module N, the decomposition $\mathcal{A}e = e\mathcal{A}e \oplus (1-e)\mathcal{A}e$ of right $\mathcal{D}$-modules gives $e(\mathcal{A}e \otimes_{\mathcal{D}} N) = e\mathcal{A}e \otimes_{\mathcal{D}} N = N$, naturally in N. Hence the two functors are inverse equivalences. An algebra is semisimple if and only if every short exact sequence of left modules splits, and an equivalence of abelian categories preserves split short exact sequences, so $\mathcal{A}$ is semisimple if and only if $\mathcal{D}$ is. The last assertion follows from $e_v M \cong \mathrm{Hom}_{\mathcal{B}}(T_v, M)$, since e_v is the image of a rank-one idempotent of the factor $\mathrm{End}_F(T_v)$. ∎

A.2.2 The affine subalgebra

Recall that $A = L_{\mathfrak{sl}_2}(3,0) \otimes L_{\mathfrak{sl}_2}(13,0)$ is the affine vertex subalgebra of S generated by $S_1 = \mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ and that $A^+ = A \oplus J$ with $J = L(3,3) \otimes L(13,13)$ is a vertex subalgebra of S [ArhI, Proposition 11.4]. For $0 \leqslant r \leqslant 3$ and $0 \leqslant l \leqslant 13$, write $L(r,l) = L_{\mathfrak{sl}_2}(3,r) \otimes L_{\mathfrak{sl}_2}(13,l)$ and $V(r,l)$ for the irreducible $\mathfrak{g}$-module with highest weight (r,l), its lowest weight space. The lowest conformal weight of $L(r,l)$ is $h(r,l) = \frac{r(r+2)}{20} + \frac{l(l+2)}{60}$, and the fusion product of J with $L(r,l)$ is $L(3-r, 13-l)$. A direct computation gives

$$h(3-r,13-l) - h(r,l) = 4 - \frac{r+l}{2}. \tag{A.1}$$

Since A is rational and J is a simple current of order two with integral lowest weight $h(3,13) = 4$, the simple A^+-modules are the modules $L(r,l) \oplus L(3-r,13-l)$ with $r+l$ even [ArhI, proof of Proposition 11.4], which uses [HKL15, Theorem 3.4 p. 16], and A^+ is rational [ArhI, p. 58]. By (A.1), the lowest weight space of such a module is $V(r,l)$ if $r+l < 8$, and $V(r,l) \oplus V(3-r,13-l)$ if $r+l = 8$. We obtain fourteen simple A^+-modules, whose lowest weight spaces $T_0, \dots, T_{13}$ are, as $\mathfrak{g}$-modules, $T_v = V(\lambda_v)$ for $0 \leqslant v \leqslant 11$ and

$$T_{12} = V(1,7) \oplus V(2,6), \quad T_{13} = V(0,8) \oplus V(3,5),$$

where $\lambda_0, \dots, \lambda_{11}$ are the twelve pairs $(0,0)$, $(0,2)$, $(1,1)$, $(2,0)$, $(0,4)$, $(1,3)$, $(2,2)$, $(3,1)$, $(0,6)$, $(1,5)$, $(2,4)$, $(3,3)$. We call constituents the sixteen pairs λ which occur in these lowest weight spaces. Since A^+ is rational, its Zhu algebra is semisimple and

$$A(A^+) \cong \prod_{v=0}^{13} \mathrm{End}_{\mathbb{C}}(T_v)$$

[Zhu96, Theorems 2.2.2 and 2.2.3 p. 261]. By [ArhI, Proposition 11.4], the inclusion $A^+ \subset S$ induces a unital homomorphism $\varphi\colon A(A^+) \to A(S)$.

We apply Lemma A.4 to $\mathcal{A} = A(S)$, $\mathcal{B} = A(A^+)$, with the bases of $V(r,l) = V(r) \otimes V(l)$ given by the vectors $f_1^i f_2^j w$ of the highest weight vector w, the second factor varying fastest, and, for $v \in \{12, 13\}$, the first constituent listed first. We denote by $e_v \in A(S)$, $0 \leqslant v \leqslant 13$, the resulting idempotents and by

$$\mathrm{D} \stackrel{\text{def}}{=} e_{\mathsf{V}} A(S) e_{\mathsf{V}}, \qquad e_{\mathsf{V}} = \sum_{v=0}^{13} e_v,$$

the corner. By Lemma A.4, D is Morita equivalent to $A(S)$.

A.2.3 Arrows and generation

For $x \in S_1$ and $a \in S$, we have $[x] * [a] - [a] * [x] = [x_{(0)}a]$ in $A(S)$ [Zhu96, Lemma 2.1.3 p. 248], [ArhI, (11.1)]. The class $[x]$ is the image under φ of the class of x in $A(A^+)$, which acts on T_v by the zero mode, that is by the action $\rho_v(x)$ of $x \in \mathfrak{g}$. Hence $\mathcal{X}([x]) = \sum_v \rho_v(x) \otimes e_v$, and $\mathcal{X}$ intertwines the derivation $\mathrm{ad}[x]$ of $A(S)$ with the natural action of $\mathfrak{g}$ on $\bigoplus_{v,w} \mathrm{Hom}(T_w, T_v)$, the factor $e_v \mathrm{D} e_w$ being acted upon trivially.

Let $z \in S_2$ be a highest weight vector of weight ν for $\mathfrak{g}$. Then $\mathcal{X}([z])$ is a highest weight vector of weight ν. For constituents $\lambda \subset T_w$ and $\mu \subset T_v$, the space of highest weight vectors of weight ν in $\mathrm{Hom}(V(\lambda), V(\mu)) \cong V(\lambda)^* \otimes V(\mu)$ has dimension at most one, by the Clebsch–Gordan rule for each factor of $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. When it is nonzero, we say that (λ, μ) is admissible for ν. A highest weight vector $F \neq 0$ of weight ν in $V(\lambda)^* \otimes V(\mu)$ has a nonzero entry in row 1 (the highest weight vector of $V(\mu)$) and column (d_1, d_2), $d_k = (\nu_k - \mu_k + \lambda_k)/2$: for each factor $\mathfrak{sl}_2$, write a highest weight vector of the tensor product of two irreducible modules as $u = \sum_{i=0}^{k} c_i\, x_i \otimes y_{k-i}$, where (x_i) and (y_j) are weight bases of the two factors, starting from their highest weight vectors, on which e acts by $e x_i = \alpha_i x_{i-1}$ and $e y_j = \beta_j y_{j-1}$ with $\alpha_i, \beta_j \neq 0$ for $i, j \geqslant 1$ in the relevant range. The equation $eu = 0$ gives $c_{i+1}\alpha_{i+1} + c_i \beta_{k-i} = 0$ for $0 \leqslant i < k$, so that all the coefficients c_i are nonzero if one of them is. We let $F_\nu^{\lambda,\mu}$ be the highest weight vector of weight ν normalized by the condition that this entry equals 1. The component of $\mathcal{X}([z])$ in $\mathrm{Hom}(V(\lambda), V(\mu)) \otimes e_v \mathrm{D} e_w$ is then $F_\nu^{\lambda,\mu} \otimes x_{z,\lambda,\mu}$ for a unique element $x_{z,\lambda,\mu} \in e_v \mathrm{D} e_w$.

The space S_2 contains seventeen highest weight vectors for $\mathfrak{g}$ [ArhI, Proposition 11.3]. Seven of them lie in the affine subalgebra A, and their classes are polynomials in the classes of the currents. The ten others are the vectors y and x_C of type $(0,0)$ and the highest weight vectors of the primary types $(0,4)$, $(1,1)$, $(1,3)$, $(1,5)$, $(1,7)$, $(2,2)$, $(2,6)$, $(3,3)$. We call arrows the elements $x_{z,\lambda,\mu}$ attached to these ten vectors z and to the pairs of constituents admissible for their weights. An arrow $x_{z,\lambda,\mu}$ goes from the vertex w of λ to the vertex v of μ. There are 515 arrows. Let Γ_Z be the quiver with vertices $0, \ldots, 13$ and these arrows, and $\mathbb{C}\Gamma_Z$ its path algebra, with the convention that a path is written from left to right as a product in D, so that a product xy of arrows is nonzero in $\mathbb{C}\Gamma_Z$ only if the source of x is the target of y.

Proposition A.5 *The homomorphism of algebras* $\mathrm{ev} \colon \mathbb{C}\Gamma_Z \to \mathrm{D}$ *which maps the vertex* v *to* e_v *and each arrow to the corresponding element of* D *is surjective.*

Proof : By [ArhI, Proposition 11.3], $A(S)$ is generated by the classes of the six currents and of the seventeen highest weight vectors of S_2, and the classes of the seven affine highest weight vectors are polynomials in the classes of the currents. By Lemma A.4, $\mathcal{X}(A(S))$ is generated by the elements $\sum_v \rho_v(x) \otimes e_v$, $x \in \mathfrak{g}$, and $\mathcal{X}([z]) = \sum F_\nu^{\lambda,\mu} \otimes x_{z,\lambda,\mu}$ for the ten vectors z above. A product of such generators is a sum of tensors $F \otimes p$, where F is a numerical matrix and p a path in the arrows or a vertex idempotent. For $b \in e_v A(S) e_w$, the element $b = e_v b e_w$ is the coefficient of $E_{11}^{v,w}$ in $\mathcal{X}(b)$, which is therefore a linear combination of paths. ∎

A.2.4 Projected relations

Let $\{z_b\}_{0\leqslant b\leqslant 119}$ be the basis of S_2 formed by the vectors $f_1(0)^i f_2(0)^j z$, where z runs over the seventeen highest weight vectors [ArhI, Proposition 11.3]. A Zhu polynomial is a noncommutative polynomial Π in symbols Z_b, $0 \leqslant b \leqslant 119$, and J_x, x in a basis of $\mathfrak{g}$. Its value $\Pi([\cdot])$ in $A(S)$ is obtained by substituting $[z_b]$ and $[x]$. We define a formal matrix $\widehat{\mathcal{X}}(\Pi) \in \bigoplus_{v,w} \mathrm{Hom}(T_w, T_v) \otimes \mathbb{C}\Gamma_Z$ by substituting $\widehat{\mathcal{X}}(J_x) = \sum_v \rho_v(x) \otimes e_v$, $\widehat{\mathcal{X}}(Z_b) = \sum F_\nu^{\lambda,\mu} \otimes x_{z,\lambda,\mu}$ for the ten non-affine highest weight vectors $z = z_b$, the corresponding polynomials in the $\widehat{\mathcal{X}}(J_x)$ for the seven affine ones, and $\widehat{\mathcal{X}}(Z_{b'}) = \mathrm{ad}(\rho(f_1))^i \,\mathrm{ad}(\rho(f_2))^j \widehat{\mathcal{X}}(Z_b)$ when $z_{b'} = f_1(0)^i f_2(0)^j z_b$. By construction $(\mathrm{id} \otimes \mathrm{ev})\widehat{\mathcal{X}}(\Pi) = \mathcal{X}(\Pi([\cdot]))$.

Lemma A.6 *Let Π be a Zhu polynomial whose value in $A(S)$ is zero. Then every entry of $\widehat{\mathcal{X}}(\Pi)$ lies in the kernel of* ev. *Moreover, for any arrow x from a constituent λ of T_w to a constituent μ of T_v, the element $(h_\mu - h_\lambda)\, x + 2\, x_{y,\mu,\mu}\, x - 2\, x\, x_{y,\lambda,\lambda}$ lies in the kernel of* ev, *where $h_\lambda = h(\lambda)$.*

Proof : The first assertion follows from $(\mathrm{id} \otimes \mathrm{ev})\widehat{\mathcal{X}}(\Pi) = \mathcal{X}(0) = 0$. For the second, write $\omega = \omega_A + \omega_C$ with $\omega_C = 2y$ [ArhI, proof of Proposition 11.4]. The class $[\omega]$ is central in $A(S)$ [Zhu96, Theorem 2.1.1 p. 247]. The class $[\omega_A]$ lies in the image of $A(A^+)$ and acts on T_v by $L(0)$, that is by the scalar h_λ on each constituent λ of T_v (both constituents of T_{12}, respectively T_{13}, have the same lowest weight). Since y is a highest weight vector of weight $(0,0)$, $\mathcal{X}([y]) = \sum_\lambda \mathrm{id}_{V(\lambda)} \otimes x_{y,\lambda,\lambda}$, the matrix $F_{(0,0)}^{\lambda,\lambda}$ being the identity. The component of $\mathcal{X}([\omega] * [z] - [z] * [\omega]) = 0$ on the line $F_\nu^{\lambda,\mu}$ gives the stated relation. ∎

For a Zhu polynomial Π whose value is zero and which is a highest weight vector of weight η for the adjoint action of $\mathfrak{g}$, we retain, for every pair (λ, μ) admissible for η, the entry of $\widehat{\mathcal{X}}(\Pi)$ in row 1 and column (d_1, d_2) of the block $\mathrm{Hom}(V(\lambda), V(\mu))$. These entries, together with the relations of the second part of Lemma A.6, are the projected relations. Only the fact that they lie in the kernel of ev is used below. Completeness of these relations is not needed.

A.3 Exact relations and an integral upper presentation

A.3.1 The field, the prime and the exact model

Let $K = \mathbb{Q}(\sqrt{13}, \sqrt{-3})$, with $\mathbb{Q}$-basis $1, r, q, rq$, $r^2 = 13$, $q^2 = -3$. The algebra S is defined over K in the explicit model of $V' = V_{E_6 \oplus A_2}$ of [ArhI, Section 10.1]: the spaces U_1, U_2 are spanned by vectors with coordinates in K in the monomial basis of V', and all products below are computed exactly in this basis. The projected relations below are therefore elements of $K\Gamma_Z$, and their images under $\mathrm{ev}\colon \mathbb{C}\Gamma_Z \to \mathrm{D}$ vanish by Lemma A.6.

Let $p = 8388571$. Then 13 and -3 are squares modulo p, so that p splits completely in K, and we fix the prime $\mathfrak{p}$ of the ring of integers $\mathcal{O}_K$ above the kernel of the homomorphism $\mathbb{Z}[r,q] \to \mathbb{F}_p$ sending r and q to the square roots 1649360 and 2036966 of 13 and -3 modulo p (the embedding used in [ArhI, Theorem 11.6]). This prime is unique since p does not divide the index of $\mathbb{Z}[r,q]$ in $\mathcal{O}_K$, which divides a power of $2 \cdot 3 \cdot 13$. We denote by R the localization of the ring of integers of K at $\mathfrak{p}$. It is a discrete valuation ring with residue field $\mathbb{F}_p$, uniformizer p, and an element of K whose four rational coordinates have denominators prime to p lies in R. We write $x \mapsto \bar{x}$ for the reduction $R \to \mathbb{F}_p$.

A.3.2 The three families of relations over K

For $a, b \in S_2$ we put

$$B(a,b) \overset{\text{def}}{=} a_{(0)}b - \tfrac{1}{2}L(-1)(a_{(1)}b) + \tfrac{1}{4}L(-1)^2(a_{(2)}b) = \tfrac{1}{2}\big(a_{(0)}b - b_{(0)}a\big),$$
$$N(a,b) \overset{\text{def}}{=} a_{(-1)}b - \tfrac{1}{2}L(-1)(a_{(0)}b) + \tfrac{1}{4}L(-1)^2(a_{(1)}b) - \tfrac{1}{12}L(-1)^3(a_{(2)}b) = \tfrac{1}{2}\big(a_{(-1)}b + b_{(-1)}a\big),$$
$$c_\pm(a,b) \overset{\text{def}}{=} \tfrac{1}{2}\big(a_{(1)}b \pm b_{(1)}a\big) \in S_2,$$

the equalities following from the skew symmetry $b_{(n)}a = \sum_{j\geqslant 0}(-1)^{n+j+1}\frac{L(-1)^j}{j!}a_{(n+j)}b$ and from $L(-1)\mathbf{1} = 0$. By [Zhu96, Lemma 2.1.1 p. 247 and (2.1.4) p. 248], in $A(S)$ we have, for $a, b \in S_2$, $v \in S$ homogeneous and $x \in S_1$,

$$[L(-1)v] = -\operatorname{wt}(v)[v], \quad [x_{(-1)}v] = [v] * [x], \quad [B(a,b)] = [a]*[b] - [b]*[a] - [c_-(a,b)], \tag{A.2}$$

$$[N(a,b)] = \tfrac{1}{2}\big([a]*[b] + [b]*[a]\big) + [c_+(a,b)], \tag{A.3}$$

see also [ArhI, (11.1)]. We consider, as in [ArhI, Theorem 11.6] and following [Rie26b, Section 9], the $\mathfrak{g}$-equivariant linear maps

$$\Phi_3 \colon S_2 \oplus (\mathfrak{g}\otimes S_2) \oplus \Lambda^2 S_2 \to V_3', \quad (v, h\otimes w, a\wedge b) \mapsto L(-1)v + h_{(-1)}w + B(a,b),$$
$$\Phi_{\text{sym}} \colon \operatorname{Sym}^2 S_2 \to V_4', \quad a\cdot b \mapsto N(a,b),$$
$$\Phi_{\text{mix}} \colon S_3 \oplus (\mathfrak{g}\otimes S_3) \oplus \operatorname{Sym}^2 S_2 \to V_4', \quad (u, h\otimes v, a\cdot b) \mapsto L(-1)u + h_{(-1)}v + N(a,b).$$

A highest weight vector of weight η of the domain of one of these maps which lies in its kernel gives, by (A.2) and (A.3), a Zhu polynomial which vanishes in $A(S)$ and which is a highest weight vector of weight η for the adjoint action of $\mathfrak{g}$. For Φ_{mix}, the classes of the elements of S_3 are expressed through a $\mathfrak{g}$-equivariant section of Φ_3, as in [ArhI, p. 58].

Proposition A.7 *The following statements hold, with all computations performed exactly over K.*

1. *The highest weight vectors of the domains of Φ_3, Φ_{sym} and Φ_{mix} form spaces of dimensions 553, 477 and 1027. The kernels of the three maps on these spaces have dimensions 458, 70 and 532 over K, and their reductions modulo $\mathfrak{p}$ have the same dimensions. The image of Φ_3 is S_3, of dimension 956, with 95 highest weight vectors.*
2. *The 1060 resulting Zhu polynomials have all their coefficients in R.*
3. *The projected relations of Section A.2.4, computed from these polynomials and from the 515 relations of the second part of Lemma A.6, are 19981 elements of $R\Gamma_Z$, each supported on a single pair of vertices.*

Proof : This is the computation of items 3 and 4 of Section A.8. The highest weight vectors of the formal domains are computed over $\mathbb{Q}$ as integer kernels of the action of e_1 and e_2, which has integer coefficients in the bases $f_1^i f_2^j$ of the irreducible $\mathfrak{g}$-modules. For each weight η, let $m_1, \dots, m_h$ be the images of these highest weight vectors, which are exact vectors of V' with coordinates in K. Their coordinates have denominators prime to p, which is checked when they are reduced modulo $\mathfrak{p}$. Choose, by Gaussian elimination modulo $\mathfrak{p}$, columns C and rows I such that the $r\times r$ minor $[m_i]_{i\in I, j\in C}$ is nonzero modulo $\mathfrak{p}$, where r is the rank modulo $\mathfrak{p}$. This minor

is nonzero over K, so that the rank over K is at least r. For each $i \notin I$, the unique vector $c^{(i)}$ with $c^{(i)}_i = 1$, support in $I \cup \{i\}$ and $\sum_k c^{(i)}_k [m_k]_C = 0$ is computed exactly by restriction of scalars. The identity $\sum_k c^{(i)}_k m_k = 0$ is then verified exactly on all coordinates. This gives $h - r$ independent vectors of the kernel over K, so that the rank over K is exactly r. The coefficients of $c^{(i)}$ lie in R since the minor is a unit of R, and their reductions form a basis of the kernel modulo $\mathfrak{p}$. The coordinates of $c_\pm(a,b)$ in the basis (z_b) of S_2 are found by exact solution of a 120×120 system and verified exactly. The section of Φ_3 used for Φ_{mix} is given by rational combinations of domain elements, chosen among the highest weight vectors whose images are independent modulo $\mathfrak{p}$, and their images span S_3 since their rank modulo $\mathfrak{p}$ is $956 = \dim S_3$ [ArhI, Theorem 10.21]. The projection uses the rational matrices $F^{\lambda,\mu}_\nu$ and $\rho_v(x)$. Finally, every denominator of every coefficient is checked to be prime to p. ∎

Remark A.8 Part 1 answers the first obstruction recorded in [ArhI, Remark 11.7]: no rank drop occurs at $\mathfrak{p}$, and the reductions of the exact relations span the kernels of the reduced maps, which are the relations computed modulo p in [ArhI, Theorem 11.6]. The polynomials themselves are not identical to those of [ArhI], since the bases of the kernels and the section of Φ_3 are chosen differently, and this is the reason why the special fibre is recomputed in Section A.4 from the reductions of the exact relations.

A.3.3 The integral order

Let $I_K \subset K\Gamma_Z$ be the two-sided ideal generated by the 19981 projected relations of Proposition A.7, and put

$$I_R \stackrel{\mathrm{def}}{=} I_K \cap R\Gamma_Z, \qquad \Lambda \stackrel{\mathrm{def}}{=} R\Gamma_Z / I_R.$$

Then Λ is an R-algebra without $\mathfrak{p}$-torsion, $\Lambda \otimes_R K = K\Gamma_Z / I_K$ since $I_K = K I_R$, and $\Lambda \otimes_R \mathbb{C} = \mathbb{C}\Gamma_Z / \mathbb{C} I_K$. By Lemma A.6 and Proposition A.5, ev induces a surjective homomorphism of algebras $\Lambda \otimes_R \mathbb{C} \to \mathrm{D}$. The arrows are numbered by the generating vector z, in the order y, x_C, $(0,4)$, $(1,1)$, $(1,3)$, $(1,5)$, $(1,7)$, $(2,2)$, $(2,6)$, $(3,3)$ of their types, then by the source constituent and then by the target constituent, the constituents being ordered as in Section A.2.2. For reduction we compare paths first by length, with longer paths larger, and for equal length we take the reverse of the lexicographic order on the tuples of arrow indices. This order is compatible with left and right concatenation and has the descending chain condition. A rewriting rule is an element of $R\Gamma_Z$ of the form $w - \sum_u c_u u$, with $c_u \in R$, where every u is shorter than w, or of the same length and lexicographically larger, and has the same endpoints as w. Its leading word is w.

Lemma A.9 *Let $\mathsf{V}' \subset \mathsf{V}$ be a set of vertices and $\mathcal{E}$ a set of arrows, let $\mathcal{R}$ be a finite set of rewriting rules, and assume:*

1. *for every vertex $v \notin \mathsf{V}'$, the idempotent v lies in I_R, and for every arrow $x \in \mathcal{E}$, there is an element $x - \ell_x \in I_R$, where ℓ_x is an R-linear combination of vertices and of arrows with numerically larger indices than x,*
2. *every element of $\mathcal{R}$ lies in I_R,*
3. *every path of length n in the arrows not in $\mathcal{E}$ between vertices of V' contains the leading word of a rule of $\mathcal{R}$.*

Then Λ is spanned as an R-module by the images of the vertices of V' and of the paths of length at most $n-1$ in the arrows not in $\mathcal{E}$ between vertices of V'. In particular Λ is a finite R-algebra.

Proof : By 1, every path through a vertex not in V' vanishes in Λ, and every arrow of $\mathcal{E}$ can be replaced by a combination of vertices and arrows with larger indices. Since the set of arrows is finite and totally ordered, finitely many replacements express every path as an R-combination of paths in the arrows not in $\mathcal{E}$, without increasing the length. If a path w of length at least n in these arrows contains the leading word q of a rule $q - \sum c_u u$, write $w = w_1 q w_2$. Then $w \equiv \sum c_u w_1 u w_2$ modulo I_R, and each $w_1 u w_2$ is shorter than w, or of the same length and lexicographically larger. Since the set of paths of a fixed length is finite, a repetition of this step, together with the previous replacements, terminates and expresses w modulo I_R as an R-combination of shorter paths. Induction on the length concludes. ■

Theorem A.10 *The following hold, with exact computations over K.*

1. *The hypotheses of Lemma A.9 hold with $n = 4$, $\mathsf{V}' = \{0, \dots, 11\}$, a set $\mathcal{E}$ of* 46 *arrows and a set $\mathcal{R}$ of* 5659 *rules,* 5221 *with leading words of length two and* 438 *with leading words of length three. Consequently Λ is a finite R-algebra.*

2. *Every leading coefficient used in the elimination is a unit of R, and every coefficient of the final eliminations and rules lies in R.*

3. *Let J_K be the ideal of $K\Gamma_Z$ generated by the vertices* 12 *and* 13*, the elements $x - \ell_x$, $x \in \mathcal{E}$, and the rules of $\mathcal{R}$. Then $J_K = I_K$. All* 131698 *overlap ambiguities and all* 151 *inclusion ambiguities of $\mathcal{R}$ resolve over K, and the* 762 *irreducible paths (*12 *vertices,* 239 *arrows,* 506 *paths of length two and* 5 *of length three) form a basis of $\Lambda \otimes_R K$.*

4. *I_R is generated by the same elements as an ideal of $R\Gamma_Z$. The R-module Λ is free with basis the* 762 *irreducible paths, and $\Lambda/\mathfrak{p}\Lambda$ is the $\mathbb{F}_p$-algebra with this basis whose multiplication is given by the reductions of the rules.*

Proof : This is the computation of item 5 of Section A.8. It replays over K the elimination of [ArhI, Theorem 11.6]: echelon forms on each pair of endpoints, with leading word the longest and then lexicographically smallest path, substitution of the linear eliminations and exclusion of the vertices 12 and 13 until stability, and completion by the overlaps of the rules. The completion has two stages: the 55444 non-resolving overlaps of the 5105 quadratic rules obtained after the eliminations produce, after echelon, the 5659 rules of part 1, and all overlaps and inclusions of these rules resolve. Each division is by a leading coefficient, which is checked to be nonzero modulo $\mathfrak{p}$ with denominators prime to p, and the coefficients of the final rules and eliminations are checked to lie in R. The list of irreducible paths of each length is computed from the leading words. There are 239, 506, 5 and 0 irreducible paths of lengths 1, 2, 3 and 4, which gives hypothesis 3 of Lemma A.9 with $n = 4$. Hypotheses 1 and 2 of this lemma follow from the inclusion $J_K \subset I_K$ proved in the next paragraph and from part 2.

The inclusion $J_K \subset I_K$ holds by construction, since every element produced by the program of item 5 is an exact K-linear combination of products of paths with elements already known to lie in I_K. It is also certified independently by the programs of item 11 of Section A.8, which express the vertices 12 and 13, the 46 elements $x - \ell_x$ and the 5105 quadratic rules as exact K-linear combinations of products of the generators with paths, and checks that each of the 554 remaining rules reduces to zero modulo the span of the 5105 certified rules and of their reduced overlaps. The inclusion $I_K \subset J_K$ is checked by the verifier of item 10 of Section A.8: each of the 19981 generators of I_K reduces to zero modulo the rewriting data.

For part 3, substituting the eliminations and killing the vertices 12 and 13 gives an isomorphism $K\Gamma_Z/(e_{12}, e_{13}, x - \ell_x) \cong K\Gamma'$, where Γ' is the quiver with the 12 vertices of V' and the 239 remaining arrows: the substitution is triangular for the order, no leading word contains

an eliminated arrow, and the eliminations of the arrows at the vertices 12 and 13 reduce to zero (checked by the verifier of item 10 of Section A.8). The path algebra $K\Gamma'$ is the tensor ring over $K^{\mathrm{V}'}$ of the bimodule spanned by the 239 arrows, the order is a semigroup order with the descending chain condition, and every rule has its terms between the same endpoints as its leading word. By the diamond lemma [Ber78, Theorem 1.2 p. 181], in the form of [Ber78, Theorem 6.1 p. 196] for tensor rings, the resolution of all ambiguities shows that the irreducible paths form a basis of $K\Gamma'/(\mathcal{R}) = K\Gamma_Z/J_K = \Lambda \otimes_R K$.

For part 4, let J_R be the ideal of $R\Gamma_Z$ generated by the same elements. The same reductions hold over R, since the rules are monic with coefficients in R, so that the diamond lemma over R shows that $R\Gamma_Z/J_R$ is a free R-module with basis the irreducible paths. In particular it is torsion-free. Since $J_R \subset I_R$ and $(R\Gamma_Z/J_R) \otimes_R K = K\Gamma_Z/J_K = \Lambda \otimes_R K$, the kernel I_R/J_R of the surjection $R\Gamma_Z/J_R \to \Lambda$ is a torsion submodule, hence zero. The last assertion follows by reduction modulo $\mathfrak{p}$. ■

Remark A.11 The unit pivots of Theorem A.10(2) allow reduction of the exact eliminations at $\mathfrak{p}$. The verifier of item 10 of Section A.8 compares the surviving vertices, active arrows and final rules with the modular computation on the reductions of the 19981 relations. In particular, the modular computation of products of normal paths uses the multiplication of $\Lambda/\mathfrak{p}\Lambda$. The modular presentation from the reduced relations has dimension 762, so its natural surjection onto $\Lambda/\mathfrak{p}\Lambda$ is an isomorphism.

A.4 The special fibre

Proposition A.12 *Put* $\mathrm{D}_p \stackrel{\mathrm{def}}{=} \Lambda/\mathfrak{p}\Lambda$.

1. *The algebra* D_p *has the basis of the* 762 *irreducible paths of Theorem A.10, and it is equal to the algebra presented by the reductions modulo* $\mathfrak{p}$ *of the* 19981 *projected relations (Remark A.11).*

2. *The regular trace form of* D_p *is nondegenerate, its center has dimension* 12*, a central element has twelve distinct eigenvalues in* $\mathbb{F}_p$*, and*

$$\mathrm{D}_p \cong \prod_{i=1}^{12} \mathrm{M}_{n_i}(\mathbb{F}_p), \qquad (n_i) = (1,1,2,2,2,3,5,7,11,12,12,16).$$

3. *For each block, the ranks of the vertex idempotents on the simple module are those of Table 1.*

4. *Let* λ_v *be the first constituent of* T_v *and* $h_v = h(\lambda_v)$*, and put* $Z \stackrel{\mathrm{def}}{=} \sum_{v=0}^{13}(h_v v + 2x_{y,\lambda_v,\lambda_v}) \in R\Gamma_Z$*. The image* $\bar{Z}$ *of* Z *acts on the simple module of the block* i *by the reduction of the rational number* h_i *of Table 1.*

Proof : Part 1 is part 4 of Theorem A.10 and Remark A.11. Parts 2 and 3 are the computation of [ArhI, Theorem 11.6], run on the reductions of the exact relations (item 6 of Section A.8). By Remark A.11 and the verifier of item 10 of Section A.8, the surviving vertices, active arrows and final rules used by this program agree with the reductions of the exact presentation. The product on normal paths is therefore the product of D_p. The regular trace form has rank 762. If x lies in the Jacobson radical, then xy is nilpotent for every y, so that $\mathrm{Tr}(L_{xy}) = 0$, and nondegeneracy gives $x = 0$. Thus D_p is semisimple. A central element with twelve distinct eigenvalues in $\mathbb{F}_p$ gives twelve central idempotents by interpolation, so that the center is split.

Each simple factor is a matrix algebra over a finite division ring, which is commutative [Wed05], hence equal to its center, which is $\mathbb{F}_p$ since the center of D_p is split. For part 4, the characteristic polynomial of left multiplication by $\bar{Z}$ on D_p factors as

$$t\,(t-\tfrac{1}{3})(t-\tfrac{5}{13})^4(t-\tfrac{6}{13})^4(t-\tfrac{7}{13})^4(t-\tfrac{8}{13})^9(t-\tfrac{2}{3})^{25}(t-\tfrac{11}{13})^{49}(t-\tfrac{15}{13})^{256}(t-1)^{409}$$

in $\mathbb{F}_p[t]$, where fractions denote their reductions modulo p (item 7 of Section A.8). Since $\bar{Z}$ is central, by the argument of the proof of Proposition A.15, which uses only the relations of Lemma A.6, and since D_p is semisimple, the kernel of $L_{\bar{Z}}-c$ is the sum of the blocks on which $\bar{Z}$ acts by c. For each root c, the program of item 7 of Section A.8 computes this kernel and the dimensions of its corners $e_v\,\mathrm{Ker}(L_{\bar{Z}}-c)e_v$, which are the sums, over these blocks, of the squares of the ranks of e_v. For the nine roots $c\neq 1$, the kernel has dimension n_i^2 for a single block i, and the corner dimensions are the squares of the vertex ranks of the corresponding row of Table 1, which identifies the row. The three remaining blocks, of sizes 11, 12, 12, carry the root 1, of multiplicity $409 = 11^2+12^2+12^2$. ■

label	n_i	lowest weight space as a $\mathfrak{g}$-module	dim	h_i
0	1	$(0,0)$	1	0
c_1	1	$(0,2)$	3	1/3
c_2	5	$(0,0)+(0,4)+(1,1)+(1,3)+(2,2)$	27	2/3
b	11	$2(0,2)+2(0,4)+(1,1)+2(1,3)+(1,5)+(2,2)+(2,4)+(3,1)$	80	1
a, c_0	12	$2(0,2)+(0,4)+(0,6)+2(1,1)+2(1,3)+(1,5)+(2,0)+(2,2)+(2,4)$	81	1
d_1	2	$(0,2)+(1,1)$	7	6/13
d_4	2	$(0,0)+(1,1)$	5	5/13
d_5	2	$(0,4)+(1,3)$	13	7/13
d_6	3	$(0,2)+(1,3)+(2,0)$	14	8/13
d_2	7	$(0,0)+(0,2)+(0,4)+(1,1)+(1,3)+(1,5)+(2,2)$	42	11/13
d_3	16	$2(0,0)+(0,2)+2(0,4)+(0,6)+2(1,1)+2(1,3)+2(1,5)+2(2,2)+(2,4)+(3,3)$	119	15/13

Table 1: The twelve blocks of D_p and of $\Lambda\otimes_R\overline{K}$: matrix size n_i, lowest weight space $\bigoplus_v T_v^{m_{i,v}}$ as a $\mathfrak{g}$-module (where (r,l) stands for $V(r,l)$), its dimension, and the eigenvalue h_i of Z. The labels are those of [ArhI, Table 2].

A.5 Semisimplicity of Zhu's algebra and rationality

Theorem A.13 *The algebra $\Lambda\otimes_R K$ is semisimple and, for any algebraically closed field Ω containing K, $\Lambda\otimes_R\Omega\cong\prod_{i=1}^{12}\mathrm{M}_{n_i}(\Omega)$ with the block sizes and vertex ranks of Proposition A.12. The corner D is semisimple and is a quotient of $\Lambda\otimes_R\mathbb{C}$. Zhu's algebra $A(S)$ is finite-dimensional and semisimple, that is, hypothesis* (Z) *of [ArhI, Theorem 11.1] holds.*

Proof : By Theorem A.10, Λ is a finite R-algebra, free of rank 762, and $\Lambda/\mathfrak{p}\Lambda=\mathrm{D}_p$ is split semisimple by Proposition A.12. Since Λ has no torsion, the subset I' of Lemma A.1 is the set of all twelve blocks, as $\dim_K\Lambda\otimes_R K=762=\sum_i n_i^2$. Lemma A.1 gives the semisimplicity of $\Lambda\otimes_R K$ and the decomposition of $\Lambda\otimes_R\Omega$ into twelve blocks, with the same matrix sizes and vertex ranks as those of D_p. By part 1 of Lemma A.1 with $\Omega=\mathbb{C}$, $\Lambda\otimes_R\mathbb{C}\cong\prod_{i=1}^{12}\mathrm{M}_{n_i}(\mathbb{C})$, and the surjection $\Lambda\otimes_R\mathbb{C}\to\mathrm{D}$ of Section A.3.3 shows that D is a finite-dimensional semisimple algebra. By part 2 of Lemma A.4, $A(S)$ is finite-dimensional, and by part 3 it is semisimple. ■

Corollary A.14 *The vertex operator algebra S is strongly rational. It has at most twelve simple modules up to isomorphism, and the lowest weight space of each simple S-module is, as a module over $A(A^+)$, isomorphic to $\bigoplus_v T_v^{m_{i,v}}$ for one of the rows i of Table 1.*

Proof : The algebra S is simple, of CFT type and self-contragredient [ArhI, Theorem 10.10], and C_2-cofinite [ArhI, Theorem 10.21]. By Theorem A.13 and [McR21, Corollary 5.19 p. 61], S is rational, hence strongly rational since it has positive energy. By [Zhu96, Theorem 2.2.2 p. 261], simple S-modules correspond to simple $A(S)$-modules through their lowest weight spaces, and by Lemma A.4 and Theorem A.13 the simple $A(S)$-modules correspond to the blocks of D, which form a subcollection of the twelve blocks of $\Lambda \otimes_R \mathbb{C}$. The vertex ranks give the multiplicities $m_{i,v}$ by part 3 of Lemma A.4. The class $[\omega]$ acts on the lowest weight space by $L(0)$. By the proof of Lemma A.6, $\mathcal{X}([\omega]) = \sum_v h_v \mathrm{id}_{T_v} \otimes e_v + 2\sum_\lambda \mathrm{id}_{V(\lambda)} \otimes x_{y,\lambda,\lambda}$, so that $e_\mathsf{V}[\omega]e_\mathsf{V} = \sum_v e_v[\omega]e_v = \mathrm{ev}(Z)$, by the centrality of $[\omega]$. In particular $\mathrm{ev}(Z)$ is central in D. The scalars by which Z acts on the blocks of $\Lambda \otimes_R \mathbb{C}$, hence the lowest conformal weights, are determined in Proposition A.15 below. ■

A.6 The twelve simple modules

A.6.1 Exact conformal weights

Proposition A.15 *In $\Lambda \otimes_R K$ we have $\prod_{h\in H}(Z-h) = 0$, where*

$$H = \Big\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{11}{13}, \frac{15}{13}\Big\}.$$

Consequently, Z acts on the simple module of the block i of $\Lambda \otimes_R \Omega$ by the rational number h_i of Table 1. In particular, the lowest conformal weight of a simple S-module is one of the numbers h_i, and it is positive unless the module is S.

Proof : The identity is checked exactly in the basis of irreducible paths of Theorem A.10, by computing $\prod_{h\in H}(Z-h)\cdot 1$ with $1 = \sum_{v=0}^{11} e_v$ (item 8 of Section A.8). The element Z is central in $\Lambda \otimes_R K$: the vertices 12 and 13 vanish in Λ, and for an arrow x between vertices $w, v \leqslant 11$, the relation of the second part of Lemma A.6 is $Zx - xZ$. Since $\Lambda \otimes_R \Omega$ is semisimple, Z acts on each block by a scalar, which is a root of $\prod_{h\in H}(t-h)$, hence an element of H. The elements of H lie in R, since their denominators are prime to p. The image of Z in $\Lambda/\mathfrak{p}\Lambda$ acts on the block i by $\bar{h}_i$ (Proposition A.12), so that by part 3 of Lemma A.1 the scalar by which Z acts on the block i of $\Lambda \otimes_R \Omega$ is an element of H with reduction $\bar{h}_i$. The ten elements of H have distinct reductions modulo p, which determines the scalar. The last assertion follows from Corollary A.14. ■

A.6.2 Restrictions of the lattice modules

The discriminant group L°/L of $L = E_6 \oplus A_2$ is $\mathbb{Z}_3 \times \mathbb{Z}_3$, and we denote by $W_{\varepsilon,\alpha}$, $\varepsilon, \alpha \in \mathbb{Z}_3$, the simple V'-modules as in [ArhI, Remark 11.2], the first index referring to E_6 and the second to A_2. Their lowest conformal weights and lowest weight spaces are: 0 and $\mathbb{C}\mathbf{1}$ for $W_{0,0} = V'$, $\frac{1}{3}$ and a space of dimension 3 for $W_{0,\pm1}$, $\frac{2}{3}$ and a space of dimension 27 for $W_{\pm1,0}$, 1 and a space of dimension 81 for $W_{\pm1,\pm1}$ and $W_{\pm1,\mp1}$. Moreover $\dim V_1' = 86$ and $\dim S_1 = 6$.

Proposition A.16 *The simple S-modules $M_0 = S$, M_b, M_{c_1}, M_{c_2} with lowest weight spaces of the types 0, b, c_1, c_2 of Table 1 exist, and*

$$V'|_S \cong S \oplus M_b, \qquad W_{0,\pm1}|_S \cong M_{c_1}, \qquad W_{\pm1,0}|_S \cong M_{c_2}.$$

Moreover each of the four modules $W_{\varepsilon,\alpha}$ with $\varepsilon\alpha \neq 0$ restricts to a simple S-module of conformal weight 1 whose lowest weight space is of the type a or c_0 of Table 1.

Proof : By Corollary A.14, every V'-module is a direct sum of simple S-modules, each of which has as lowest weight space one of the rows of Table 1 and conformal weight h_i by Proposition A.15. A simple summand of $W_{\varepsilon,\alpha}$ has its lowest weight in the class of the conformal weights of $W_{\varepsilon,\alpha}$ modulo $\mathbb{Z}$, namely $\mathbb{Z}$, $\frac{1}{3}+\mathbb{Z}$ or $\frac{2}{3}+\mathbb{Z}$. The only rows with h_i in these classes are the six rows 0, c_1, c_2, b, a, c_0, with $h_i \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Hence the lowest weight space of every summand lies in degree at most one, and the multiplicities are determined by the dimensions of the spaces of $W_{\varepsilon,\alpha}$ in the degrees 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1.

For V', the space $V'_0 = \mathbb{C}\mathbf{1}$ gives one summand S, which is the vertex subalgebra S itself. In degree one, $86 = 6 + 80\,m_b + 81\,(m_a + m_{c_0})$ forces $m_b = 1$ and $m_a = m_{c_0} = 0$. For $W_{0,\pm1}$, the only possible summands have conformal weight $\frac{1}{3}$, and $3 = 3\,m_{c_1}$. For $W_{\pm1,0}$, the only possible summands have conformal weight $\frac{2}{3}$, and $27 = 27\,m_{c_2}$. For $W_{\varepsilon,\alpha}$ with $\varepsilon\alpha \neq 0$, there is no vector of degree 0, and $81 = 80\,m_b + 81\,(m_a + m_{c_0})$ forces a single summand of type a or c_0. In each case the summand found is a simple S-module whose lowest weight space has the indicated type, which proves its existence. Finally $W_{-\varepsilon,-\alpha}$ is the contragredient module of $W_{\varepsilon,\alpha}$, and the restriction of a contragredient module is the contragredient of the restriction. The assertion $W_{\varepsilon,\alpha}|_S \cong W_{-\varepsilon,-\alpha}|_S$ is proved in Proposition A.18. ■

The last point requires an argument, since the rows a and c_0 of Table 1 cannot be distinguished by their conformal weights and lowest weight spaces. We use the genus-one modular matrices. For a strongly rational vertex operator algebra U with simple modules $M_0 = U, \dots, M_d$, let s^U be the matrix of the transformation $\tau \mapsto -1/\tau$ of the trace functions [Zhu96, Theorem 5.3.2 p. 299], which does not depend on the inserted vector. By [DJX13, Theorem 4.1 p. 6452], which relies on the Verlinde formula of Huang [Hua08a], s^U is symmetric, $(s^U)^2$ is the permutation matrix of contragredience and $s^U_{0,j} \neq 0$ for every j. If moreover the modules M_i, $i \neq 0$, have positive lowest weights, then $s^U_{i0}/s^U_{00} = \mathrm{qdim}_U M_i > 0$ [DJX13, Lemma 4.2 p. 6453 and its proof]. The lattice algebra V' is strongly rational, its simple modules are the nine modules W_g, and they are simple currents of quantum dimension one [DJX13, Example 4.19 p. 6457]. Their contragredient modules are given by $W'_g \cong W_{-g}$, and their lowest conformal weights other than 0 are positive.

Lemma A.17 *Let $U \subset V$ be strongly rational vertex operator algebras with the same conformal vector, let (M_z) and (W_g) be their simple modules, and let b_{gz} be the multiplicity of M_z in $W_g|_U$, which is finite. Then $s^V \mathsf{b} = \mathsf{b}\, s^U$.*

Proof : The multiplicity b_{gz} is at most the dimension of the homogeneous space of W_g of weight h_{M_z}, which is finite. For $v \in U$ homogeneous for Zhu's grading $L[0]$, which is the same for U and V, we have $T_{W_g}(v,\tau) = \sum_z \mathsf{b}_{gz} T_{M_z}(v,\tau)$ for the trace functions. Applying $\tau \mapsto -1/\tau$ [Zhu96, Theorem 5.3.2 p. 299] to both sides and comparing, we get $\sum_{z'} (s^V \mathsf{b} - \mathsf{b} s^U)_{gz'} T_{M_{z'}}(v,\tau) = 0$ for all such v. The trace functions $T_{M_{z'}}$ are linearly independent as functions of the insertion v and of τ [Zhu96, Theorem 5.3.1 p. 294 and its proof], which gives the claim. ■

Proposition A.18 *The four modules $W_{\varepsilon,\alpha}|_S$, $\varepsilon\alpha \neq 0$, are not all isomorphic. Consequently the six rows 0, b, c_1, c_2, a, c_0 of Table 1 are realized by simple S-modules. Moreover*

$$W_{1,1}|_S \cong W_{-1,-1}|_S \ncong W_{1,-1}|_S \cong W_{-1,1}|_S,$$

and these two simple modules are self-contragredient.

Proof : Let b be the branching matrix of $S \subset V'$, with rows indexed by $g \in \mathbb{Z}_3 \times \mathbb{Z}_3$ and columns by the simple S-modules, and $\mathsf{Q} \overset{\text{def}}{=} \mathsf{b}\,\mathsf{b}^T$, so that $\mathsf{Q}_{g,h}$ is the number of simple summands common to $W_g|_S$ and $W_h|_S$, counted with multiplicities. By Lemma A.17 and the symmetry of

s^S and $s^{V'}$, we have $\mathsf{Q}s^{V'} = \mathsf{b}\, s^S \mathsf{b}^T = s^{V'}\mathsf{Q}$. Since the modules W_g have quantum dimension one, $s^{V'}_{g,0} = s^{V'}_{0,g} = s^{V'}_{00} \neq 0$ for every g. Suppose that the four modules $W_{\varepsilon,\alpha}|_S$ with $\varepsilon\alpha \neq 0$ are isomorphic. By Proposition A.16, $\mathsf{Q}_{g,h} = 1$ for these four g and h, $\mathsf{Q}_{(0,0),h} = 0$ for these h, and $\mathsf{Q}_{(0,0),(0,0)} = 2$. Then

$$(\mathsf{Q}s^{V'})_{(0,0),(1,1)} = 2\, s^{V'}_{00}, \qquad (s^{V'}\mathsf{Q})_{(0,0),(1,1)} = \sum_h s^{V'}_{(0,0),h}\mathsf{Q}_{h,(1,1)} = 4\, s^{V'}_{00},$$

a contradiction. Hence both rows a and c_0 are realized, and with Proposition A.16 the six rows are realized.

For the last assertion we use the full matrix $s^{V'}$. In the modular tensor category $\mathrm{Rep}(V')$ [Hua08], the simple objects W_g are invertible, with twists $\vartheta_g = e^{2\pi i h_g}$, where h_g is congruent to $\frac{1}{2}(\lambda_g, \lambda_g)$ modulo $\mathbb{Z}$ for any λ_g in the coset g. Their categorical dimensions are their quantum dimensions [DLN15, Theorem 3.10 p. 2142], [CKM22, p. 5], which are equal to one. For invertible simple objects of dimension one, the double braiding $c_{W_h,W_g}c_{W_g,W_h}$ is the scalar $\vartheta_{g+h}\vartheta_g^{-1}\vartheta_h^{-1} = e^{2\pi i(\lambda_g,\lambda_h)}$ by the balancing axiom, so that the unnormalized S-matrix of $\mathrm{Rep}(V')$ is $\tilde{s}_{g,h} = e^{2\pi i(\lambda_g,\lambda_h)}$. The normalized S-matrix of $\mathrm{Rep}(V')$ and the genus-one matrix $s^{V'}$ coincide up to a sign [DLN15, Theorem 3.10 p. 2142]. Allowing also the complex conjugate convention for the S-matrix of a modular category, we obtain $s^{V'}_{g,h} = \frac{\sigma}{3}\,\zeta^{3\eta(\lambda_g,\lambda_h)}$ for some signs $\sigma, \eta \in \{1, -1\}$, where $\zeta = e^{2\pi i/3}$ and $3(\lambda_g, \lambda_h) \equiv \varepsilon\varepsilon' + 2\alpha\alpha' \bmod 3$ for $g = (\varepsilon, \alpha)$, $h = (\varepsilon', \alpha')$.

Let F be the set of the four elements $g = (\varepsilon, \alpha)$ with $\varepsilon\alpha \neq 0$, and for $g \in F$ let $[g] \subset F$ be the set of elements $h \in F$ with $W_h|_S \cong W_g|_S$. For $g, h \in F$, $\mathsf{Q}_{h,g}$ is 1 if $h \in [g]$ and 0 otherwise, and $\mathsf{Q}_{(\pm1,0),g} = \mathsf{Q}_{(0,\pm1),g} = 0$. For $k = (1, 0)$, the row k of Q is the indicator function of $\{(1, 0), (-1, 0)\}$ by Proposition A.16. For $g = (\varepsilon, \alpha) \in F$, the entries (k, g) of $\mathsf{Q}s^{V'}$ and $s^{V'}\mathsf{Q}$ are

$$\frac{\sigma}{3}\big(\zeta^{\eta\varepsilon} + \zeta^{-\eta\varepsilon}\big) = -\frac{\sigma}{3}, \qquad \frac{\sigma}{3}\sum_{(\varepsilon',\alpha')\in[g]} \zeta^{\eta\varepsilon'}.$$

A sum of n_1 numbers equal to ζ and n_2 numbers equal to ζ^{-1} is equal to $-1 = \zeta + \zeta^{-1}$ only if $n_1 = n_2 = 1$, since ζ and ζ^{-1} are linearly independent over $\mathbb{R}$. Hence $[g] = \{g, g^*\}$ with $g^* = (-\varepsilon, \alpha^*)$. The same computation with $k = (0, 1)$, whose row in Q is the indicator function of $\{(0, 1), (0, -1)\}$, gives $\alpha^* = -\alpha$, since $3(\lambda_{(0,1)}, \lambda_{(\varepsilon',\alpha')}) \equiv 2\alpha' \bmod 3$. Thus $[g] = \{g, -g\}$ for every $g \in F$, which gives the stated isomorphisms, and self-contragredience follows since W_{-g} is the contragredient of W_g and S and V' have the same conformal vector. Item 9 of Section A.8 confirms the computation of the three partitions of F into pairs. ∎

A.6.3 Survival of the six remaining blocks

The following argument is the one of [Rie26b, Lemma 11.3 p. 61], adapted to S.

Proposition A.19 *The algebra S has a simple module which is not one of the six modules of Proposition A.18.*

Proof : Assume the contrary. Then the branching matrix b of $S \subset V'$ has nine rows and six columns $0, b, c_1, c_2, a, c_0$, and by Propositions A.16 and A.18, the row of V' is $(1, 1, 0, 0, 0, 0)$ and the other rows have a single entry 1, in the columns c_1, c_2, a, c_0, each of these columns being reached. Hence $\mathrm{Ker}\,\mathsf{b} = \mathbb{C}(u_0 - u_b)$, where (u_z) is the standard basis indexed by the six columns. By Lemma A.17, $\mathsf{b}\, s^S(u_0 - u_b) = s^{V'}\mathsf{b}(u_0 - u_b) = 0$, so that $s^S(u_0 - u_b) = \kappa(u_0 - u_b)$ for some $\kappa \in \mathbb{C}$. Since S and M_b are self-contragredient (the latter because V' and S are), $(s^S)^2(u_0 - u_b) = u_0 - u_b$ and $\kappa = \pm 1$. Write $\delta_b = \mathrm{qdim}_S M_b > 0$. The coordinate 0 of

$s^S(u_0 - u_b) = \kappa(u_0 - u_b)$ gives $s^S_{00}(1 - \delta_b) = \kappa$. The entry $((0,0),0)$ of $s^{V'}\mathsf{b} = \mathsf{b}\, s^S$ gives $s^{V'}_{00} = s^S_{00}(1 + \delta_b)$, and $s^{V'}_{00} = \pm\frac{1}{3}$ since $\sum_g s^{V'}_{0,g} s^{V'}_{g,0} = ((s^{V'})^2)_{00} = 1$ and $s^{V'}_{0,g} = s^{V'}_{g,0} = s^{V'}_{00}$ for the nine modules. Hence $\frac{1-\delta_b}{1+\delta_b} = \pm 3$, which forces $\delta_b \in \{-\frac{1}{2}, -2\}$, a contradiction. ■

Theorem A.20 *The algebra S has exactly twelve simple modules up to isomorphism. Their lowest weight spaces and lowest conformal weights are given by the twelve rows of Table 1. All of them except S have positive conformal weight. Moreover* $\mathrm{D} \cong \Lambda \otimes_R \mathbb{C}$ *and* $\dim \mathrm{D} = 762$.

Proof : By Proposition A.19, some simple S-module M has as lowest weight space one of the six rows d_j of Table 1, with $h_M = \frac{k}{13}$ for some $k \in \{5,6,7,8,11,15\}$ by Proposition A.15. The genus-one T-matrix of S is the diagonal matrix with entries $e^{2\pi i(h-1/3)}$, h running over the conformal weights of the simple modules, since $c/24 = 1/3$. By Proposition A.15, these entries are 39-th roots of unity, and the entry of M is $e^{2\pi i(k/13-1/3)} = \xi^{3k-13}$ with $\xi = e^{2\pi i/39}$, a primitive 39-th root of unity since $3k - 13$ is prime to 3 and to 13. Hence T has order 39. We follow the argument of [Rie26b, Section 11.5 p. 62]. By [DLN15, Theorem 3.10 p. 2142], the representation of $\mathrm{SL}_2(\mathbb{Z})$ defined by the trace functions of S is a modular representation of the modular tensor category $\mathrm{Rep}(S)$, and by the Galois symmetry [DLN15, Theorem II p. 2124], for every $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$, the matrix $\gamma^2(T)$ is conjugate to T by a signed permutation matrix. In particular the diagonal of $\gamma^2(T)$ is a permutation of the diagonal of T. Let $\gamma(\xi) = \xi^m$ with m prime to 39. Then $m^2 \equiv 1 \bmod 3$, so that $13m^2 \equiv 13 \bmod 39$ and $\gamma^2(\xi^{3k-13}) = \xi^{3km^2-13} = e^{2\pi i(k'/13-1/3)}$ with $k' \equiv km^2 \bmod 13$. The residues modulo 13 of $5,6,7,8,11,15$ are $5,6,7,8,11,2$, the six non-squares modulo 13, and m^2 runs over the six squares $1,3,4,9,10,12$ modulo 13 when m runs over the units modulo 39. Since the squares act transitively on the non-squares by multiplication, $km^2 \bmod 13$ runs over all the non-squares. Hence the six numbers $e^{2\pi i(k'/13-1/3)}$, $k' \in \{5,6,7,8,11,15\}$, are diagonal entries of T. By Proposition A.15, the rows 0, b, c_1, c_2, a, c_0 have weights in $\frac{1}{3}\mathbb{Z}$, so that their entries of T are cube roots of unity, which are not primitive 39-th roots of unity, and the six rows d_j have the six distinct entries $e^{2\pi i(k'/13-1/3)}$. Hence each of these six values is the entry of a single row d_j. Hence the six rows d_j are realized, and with Proposition A.18 all twelve blocks of $\Lambda \otimes_R \mathbb{C}$ survive in the quotient D. Since this quotient is a surjection of semisimple algebras which is bijective on blocks, it is an isomorphism. ■

Corollary A.21 *Zhu's algebra $A(S)$ is isomorphic to $\prod_{i=1}^{12} \mathrm{End}_{\mathbb{C}}(R_i)$, where R_i runs over the twelve lowest weight spaces of Table 1. In particular* $\dim A(S) = 36625$.

Proof : By Theorem A.13, $A(S)$ is a finite-dimensional semisimple algebra over $\mathbb{C}$, hence the product of the endomorphism algebras of its simple modules. By [Zhu96, Theorem 2.2.2 p. 261] and Theorem A.20, these are the twelve lowest weight spaces R_i, of dimensions 1, 80, 81, 81, 3, 27, 7, 42, 119, 5, 13, 14, and the sum of their squares is 36625. ■

Remark A.22 Theorem A.20 and Propositions A.16 and A.18 prove hypothesis (M) of [ArhI, Remark 11.2], with the restrictions $W_{0,0}|_S = S \oplus M_b$, $W_{0,\pm1}|_S \cong M_{c_1}$ (the module P of loc. cit.), $W_{\pm1,0}|_S \cong M_{c_2}$ (the module Q), $W_{1,1}|_S \cong W_{-1,-1}|_S$ and $W_{1,-1}|_S \cong W_{-1,1}|_S$ (the modules X_+ and X_-), and the six modules M_{d_l} with lowest weight spaces R_{d_l} of [ArhI, Table 2] and lowest conformal weights $\frac{6}{13}$, $\frac{11}{13}$, $\frac{15}{13}$, $\frac{5}{13}$, $\frac{7}{13}$, $\frac{8}{13}$ for $l = 1, \ldots, 6$. These weights are congruent to $\frac{6l^2}{13}$ modulo $\mathbb{Z}$, as the twists of the objects d_l of $Z(\mathcal{H}_3)$ in the labelling of [EvG11]. The two blocks labelled a and c_0 in Table 1 have the same lowest weight and the same lowest weight space as an S_1-module. The computation identifies their corresponding S-modules with X_+ and X_- as an unordered pair. The intrinsic labels in the center are distinguished by the Lagrangian algebras of [ArhII, Proposition 3.6 and Remark 3.7].

A.7 The representation category

A.7.1 Pseudo-unitarity

Proposition A.23 *The category* $\mathrm{Rep}(S)$ *of* S*-modules is a pseudo-unitary modular tensor category. For every simple* S*-module* M*, the categorical dimension of* M *and its Frobenius–Perron dimension are equal to the quantum dimension* $\mathrm{qdim}_S M = s^S_{M,0}/s^S_{00} > 0$.

Proof : By Corollary A.14 and [Hua08, Theorem 4.6], $\mathrm{Rep}(S)$ is a modular tensor category. By Theorem A.20, every simple S-module other than S has positive lowest conformal weight. Hence the quantum dimensions are positive [DJX13, Lemma 4.2 p. 6453]. The normalized S-matrix of $\mathrm{Rep}(S)$ and the genus-one matrix s^S coincide up to a sign [DLN15, Theorem 3.10 p. 2142], so that the categorical dimension $\tilde{s}_{i,0}/\tilde{s}_{0,0}$ of M_i is $s^S_{i,0}/s^S_{00} = \mathrm{qdim}_S M_i$, see also [CKM22, p. 5]. Let $M_0 = S, \dots, M_{11}$ be the simple S-modules and $N(i)$ the matrix of the fusion rules $N^k_{i,j}$. By [DJX13, Theorem 4.1 p. 6452], the column $(s^S_{j,0})_j$ is an eigenvector of $N(i)$ for the eigenvalue $s^S_{i,0}/s^S_{0,0} = \mathrm{qdim}_S M_i$. Since this vector has positive entries and $N(i)$ has nonnegative entries, this eigenvalue is the Perron–Frobenius eigenvalue of $N(i)$, that is, the Frobenius–Perron dimension of M_i, as in [DJX13, Remarks 5.3 and 5.4 p. 6460]. Hence $\dim \mathrm{Rep}(S) = \sum_i (\mathrm{qdim}_S M_i)^2 = \mathrm{FPdim}\,\mathrm{Rep}(S)$. ∎

A.7.2 The Lagrangian algebras of the lattice extensions

The arguments of this section follow the strategy outlined by Huang for the algebra of the other case [Hua26b, Sections 5 and 6 pp. 7–9], as carried out for Haagerup algebras in [ArhII, Sections 3 and 6]: Lagrangian algebras attached to the holomorphic lattice extensions, recognition of the fusion rules through the Drinfeld center, and the use of the second isotropic line for the associator of the pointed part. No statement of [Hua26b] is used.

The discriminant form of $E_6 \oplus A_2$ has two isotropic lines, generated by $(1,1)$ and $(1,-1)$. We put

$$H_+ = V' \oplus W_{1,1} \oplus W_{-1,-1}, \qquad H_- = V' \oplus W_{1,-1} \oplus W_{-1,1}.$$

Each of them is the lattice vertex operator algebra of an even unimodular overlattice of $E_6 \oplus A_2$, which is isometric to E_8, so that $H_\pm \cong V_{E_8}$ [ArhII, Section 2.4 and Lemma 4.1]. It is simple and self-contragredient, and V' is strongly rational with the nine simple modules W_g [Don93]. By Propositions A.16 and A.18,

$$H_x|_S \cong S \oplus M_b \oplus 2X_x, \qquad V'|_S \cong S \oplus M_b,$$

where $X_+ = W_{1,1}|_S$ and $X_- = W_{1,-1}|_S$ are the two simple modules of types a and c_0.

Lemma A.24 *For every* $g \in \mathbb{Z}_3 \times \mathbb{Z}_3$*, we have* $\mathrm{qdim}_S(W_g|_S) = \mathrm{qdim}_S(V'|_S) = 1 + \mathrm{qdim}_S M_b$.

Proof : By Proposition A.16, S occurs in $W_g|_S$ only for $g = 0$, with multiplicity one. The entry $(g,0)$ of the identity $s^{V'} b = b\, s^S$ of Lemma A.17 gives $s^{V'}_{g,0} = \sum_z b_{gz} s^S_{z,0} = s^S_{00}\, \mathrm{qdim}_S(W_g|_S)$, and $s^{V'}_{g,0} = s^{V'}_{00}$ for every g (proof of Proposition A.18). ∎

Theorem A.25 *Let* $x \in \{+,-\}$.

1. *The object* $L_x = H_x|_S$ *carries the structure of a Lagrangian algebra in* $\mathrm{Rep}(S)$*, and* $\mathrm{Rep}(S)$ *is braided equivalent to the Drinfeld center* $Z(\mathcal{F}_x)$ *of the fusion category* $\mathcal{F}_x = \mathrm{Rep}(S)_{L_x}$ *of* L_x*-modules.*

2. *The category $\mathcal{F}_x$ is pseudo-unitary and has exactly six simple objects up to isomorphism, three of which are invertible. There are an invertible object α of order three and a self-dual simple object ρ such that the simple objects of $\mathcal{F}_x$ are 1, α, α^2, ρ, $\alpha\rho$, $\alpha^2\rho$, and*

$$\rho\alpha \cong \alpha^2\rho, \qquad \rho^2 \cong 1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho.$$

3. *The fusion subcategory $\mathcal{P}_x$ generated by α is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator.*

4. *If $\mathcal{F}_x$ is equivalent to a unitary fusion category, then $\mathcal{F}_x$ is equivalent to $\mathcal{H}_2$ or $\mathcal{H}_3$, $\mathrm{Rep}(S)$ is braided equivalent to $Z(\mathcal{H}_3)$, S is a Haagerup algebra, and $(\mathrm{H_J})$ holds.*

Proof : 1. The algebra H_x is an extension of S with the same conformal vector and $\dim(H_x)_0 = 1$. Since S is strongly rational and positive (Theorem A.20), H_x corresponds to a haploid commutative associative algebra L_x with trivial twist in $\mathrm{Rep}(S)$ [HKL15, Theorem 3.2 p. 10], and the category of H_x-modules is isomorphic to the category $\mathrm{Rep}(S)^0_{L_x}$ of local L_x-modules [HKL15, Theorem 3.4 p. 16]. Since H_x is simple and the categorical dimension of L_x is positive by Proposition A.23, the argument recalled in [ArhII, Section 2.4], based on [KiO02, Lemma 1.20 p. 15 and Theorem 3.3 p. 17] and [DMNO13, Proposition 2.7 p. 5], shows that L_x is separable, hence a connected étale algebra. By Lemma A.24 and Proposition A.23, $\mathrm{FPdim}(L_x) = 3\ \mathrm{qdim}_S(V'|_S) = 3s^{V'}_{00}/s^S_{00}$, with $|s^{V'}_{00}| = \frac{1}{3}$ (proof of Proposition A.19). Since s^S is symmetric and $(s^S)^2$ is the permutation matrix of contragredience [DJX13, Theorem 4.1 p. 6452], we have $\sum_i (s^S_{i,0})^2 = 1$ and $\mathrm{FPdim}\,\mathrm{Rep}(S) = \sum_i (s^S_{i,0}/s^S_{00})^2 = (s^S_{00})^{-2} = \mathrm{FPdim}(L_x)^2$. Hence L_x is Lagrangian [DMNO13, Definition 4.6 p. 17], and $\mathrm{Rep}(S)^0_{L_x} \cong \mathrm{Vec}$ [DMNO13, Remark 4.3 p. 16], in accordance with the fact that H_x is holomorphic. By [ArhII, Proposition 2.2], which recalls [DMNO13, Corollary 4.1 p. 16], there is a braided equivalence $\Phi\colon \mathrm{Rep}(S) \to Z(\mathcal{F}_x)$, and $\Phi(L_x) \cong I(1)$ as algebras, where I is the right adjoint of the forgetful functor $Z(\mathcal{F}_x) \to \mathcal{F}_x$ [ArhII, Lemma 3.1].

2. By [ArhII, Lemma 3.1], the number of simple objects of $\mathcal{F}_x$ is $\dim \mathrm{Hom}(L_x, L_x) = 1+1+2^2 = 6$. Since L_x is Lagrangian, $\mathrm{FPdim}(\mathcal{F}_x) = \mathrm{FPdim}\,\mathrm{Rep}(S)/\,\mathrm{FPdim}(L_x) = \mathrm{FPdim}(L_x)$ [DMNO13, Lemma 3.11 p. 10]. The global dimension of $Z(\mathcal{F}_x)$ is $(\dim \mathcal{F}_x)^2$ [ENO05, Theorem 2.15], and $\dim \mathcal{F}_x > 0$ [ENO05, Theorem 2.3]. Since a braided equivalence preserves global and Frobenius–Perron dimensions, Proposition A.23 gives $(\dim \mathcal{F}_x)^2 = \mathrm{FPdim}\,\mathrm{Rep}(S) = \mathrm{FPdim}(\mathcal{F}_x)^2$, so that $\mathcal{F}_x$ is pseudo-unitary. We equip $\mathcal{F}_x$ with its spherical structure for which the dimensions are the Frobenius–Perron dimensions [ENO05, Proposition 8.23 p. 34]. Then $K(\mathcal{F}_x) \otimes \mathbb{C} \cong \mathrm{End}(I(1)) \cong \mathrm{End}(L_x) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$ [Ost15, Corollary 2.16 p. 7], so that the Grothendieck ring is noncommutative.

By Lemma A.24 and Proposition A.23, $\mathrm{FPdim}(L_x) = 3(1+\delta_b)$, where $\delta_b = \mathrm{qdim}_S M_b$. The subobject $A_{V'} = V'|_S$ of L_x is a connected étale subalgebra [ArhII, Lemma 2.3], applied to $S \subset V' \subset H_x$, since V' is simple and H_x is self-contragredient. By [ArhII, Proposition 2.2, part 3], which recalls [DMNO13, Theorem 4.10 p. 18], $\Phi(A_{V'})$ is the étale subalgebra of $I(1)$ attached to a fusion subcategory $\mathcal{B}_x$ of $\mathcal{F}_x$ with $\mathrm{FPdim}(A_{V'})\,\mathrm{FPdim}(\mathcal{B}_x) = \mathrm{FPdim}(\mathcal{F}_x)$, that is $\mathrm{FPdim}(\mathcal{B}_x) = 3$. A fusion category of prime Frobenius–Perron dimension is pointed [ENO05, Corollary 8.30 p. 37]. Hence the group G of isomorphism classes of invertible objects of $\mathcal{F}_x$ contains a subgroup of order three, and $|G| \in \{3, 6\}$ since $|G| \leqslant 6$. If $|G| = 6$, then $\mathcal{F}_x$ is pointed and $\mathrm{FPdim}(\mathcal{F}_x) = 6$, so that $\delta_b = 1$. Then M_b is a simple current [DJX13, Proposition 4.17 p. 6456], self-contragredient since V' and S are, so that $M_b \boxtimes M_b \cong S$ and the vertex operator of V' maps $M_b \otimes M_b$ into $S((z))$. The linear map σ equal to 1 on S and -1 on M_b is then an automorphism of V' with fixed points S, and S_1 is the fixed-point subalgebra of an involution of $V'_1 \cong \mathfrak{e}_6 \oplus \mathfrak{sl}_3$. Such an involution preserves both ideals, and the fixed-point subalgebras of the involutions of $\mathfrak{e}_6$ and of $\mathfrak{sl}_3$ have dimension at least 36 and 3 [Kac90, Chapter

8], [Hel78, Chapter X], which contradicts $\dim S_1 = 6$. Hence $|G| = 3$, and $\mathcal{B}_x$ is the pointed part $\mathcal{P}_x$ of $\mathcal{F}_x$.

The proof of part 1 of [ArhII, Theorem 6.1], consisting of its Lemmas 6.2–6.5 and Proposition 6.6, applies with the canonical spherical structure of a pseudo-unitary fusion category. Its inputs are the Frobenius–Perron character, the Ng–Schauenburg formula for indicators in a spherical category, the root-of-unity twists of the center, the dimension and divisibility results of [ENO05, Propositions 5.4 and 8.12], the induction and formal-codegree results of [Ost15, Theorems 2.5 and 2.13], and the arithmetic of roots of unity used in [ArhII, Section 6.3]. This part of recognition gives the fusion rules without invoking the classification of unitary fusion categories.

3. By [ArhII, Proposition 2.2, part 5], the category of local modules $Z(\mathcal{F}_x)^0_{\Phi(A_{V'})}$ is braided equivalent to $Z(\mathcal{P}_x)$. On the other hand, $\mathrm{Rep}(S)^0_{A_{V'}}$ is isomorphic to $\mathrm{Rep}(V')$ [HKL15, Theorem 3.4 p. 16], and this isomorphism is a braided equivalence for the braided tensor structure of $\mathrm{Rep}(V')$ given by [Hua08] [CKM24, Theorem 3.65], see also [CKM22, p. 5]. Hence $\mathrm{Rep}(V')$ is braided equivalent to $Z(\mathcal{P}_x)$, possibly with the reverse braiding, which has the same Lagrangian algebras. The two extensions H_+ and H_- of V' define, by the argument of part 1 applied to V', two connected étale algebras in $\mathrm{Rep}(V')$ of Frobenius–Perron dimension 3, which are Lagrangian since the nine simple V'-modules have quantum dimension one (proof of Proposition A.18), so that $\mathrm{FPdim}\,\mathrm{Rep}(V') = 9$. These algebras are not isomorphic since their underlying objects $V' \oplus W_{1,1} \oplus W_{-1,-1}$ and $V' \oplus W_{1,-1} \oplus W_{-1,1}$ are not. By [ArhII, Proposition 2.2, part 2], the category $\mathcal{P}_x$ has two inequivalent indecomposable module categories, and by [ArhII, Lemma 6.7, part 2] its associator is trivial.

4. Let $\mathcal{F}$ be a unitary fusion category equivalent to $\mathcal{F}_x$. It has six simple objects, three of which are invertible, a noncommutative Grothendieck ring, and by part 3 the fusion subcategory generated by its invertible objects is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator. By [ArhII, Theorem 6.1], $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or $\mathcal{H}_3$. Since $Z(\mathcal{H}_2) \cong Z(\mathcal{H}_3)$ [ArhII, Section 2.1], $\mathrm{Rep}(S) \cong Z(\mathcal{F}_x) \cong Z(\mathcal{H}_3)$. With Corollary A.14 and Theorem A.20, S is a Haagerup algebra, and so is Jiao's algebra by [ArhI, Theorem 10.24]. ∎

Remark A.26 Theorem A.25 reduces identification of the center along this categorical route to unitarizability of $\mathcal{F}_x$. For the two categories attached to S, Theorem 6.1 supplies complete unitarity of S, and the implication (v)⇒(ii) of Theorem 9.14 gives the unitarizability of $\mathcal{F}_x$. The classification of [Izu18, Theorems 5.9 and 9.1] applies to C^*-fusion categories. The two nonunitary Galois conjugates in the endomorphism family of [EvG17, Theorem 3] have global dimension $3 + 3d^{-2}$, different from their Frobenius–Perron dimension $3 + 3d^2$, with $d = (3 + \sqrt{13})/2$. Hence they are not pseudo-unitary and are not equivalent to $\mathcal{F}_x$.

A.8 Certificates and programs

The archive `Certificats-Zhu-III-v06.zip` contains the scripts, their inputs, provenance checksums and replay instructions. It separates computations that start from lattice vectors and their modes from checks that start with the projected rows and the exported rewriting data. The computations are listed below with the statements they support. They are written in Python 3 with the libraries `python-flint` and `gmpy2`. Exact arithmetic in K is done on quadruples of rationals in the basis $1, r, q, rq$ (module `kfast`), and exact linear algebra over K by restriction of scalars to $\mathbb{Q}$ with pivots chosen modulo `p` and an exact verification of every solution (module `kexact`). The explicit model of V', the spaces U_1, U_2 and the exact lattice vertex operators are those of the archive of [ArhI] (modules `lattice_voa`, `exact_engine`, `Kfield` and the data of [ArhI, Theorem 10.21]).

1. `z0_highest.py` regenerates the lowering basis of the explicit S_2 and checks the highest vectors and the rank of the basis, using the exact lattice model of [ArhI].

2. `prod_exact.py`, `bucket.py`: the exact products $a_{(n)}b$, $a, b \in S_2$, $0 \leqslant n \leqslant 2$, and the vectors $B(a,b)$ and $N(a,b)$ of Section A.3.2, in the monomial basis of V'.

3. `z2x.py`, output `z2x.json`, `rel_exact.pkl`: Proposition A.7, parts 1 and 2 (the 1060 Zhu polynomials, the kernel dimensions 458, 70, 532 over K and modulo $\mathfrak{p}$, the rank 956 of Φ_3, and the absence of p in the denominators).

4. `z3x.py`, output `z3x.json`, `rows_exact.pkl`: Proposition A.7, part 3 (the 19981 projected relations in $R\Gamma_Z$ and the 515 arrows).

5. `z4x.py`, output `z4x.json`, `upper_exact.pkl`: Theorem A.10 (exact eliminations, rewriting rules, unit and integrality checks at $\mathfrak{p}$, irreducible paths, resolution of all ambiguities over K).

6. `z4_upper_algebra_modp.py` of [ArhI], run on `rows_exact_reduced_modp.pkl`, output `z4_upper_algebra_modp.json`: Proposition A.12, parts 1–3.

7. The programs `specfib_Z.py` and `specfib_blocks.py`, with the outputs `specfib_Z.log` and `specfib_blocks.json`, prove part 4 of Proposition A.12: they compute the characteristic polynomial of $L_{\bar{Z}}$, the kernels of $L_{\bar{Z}} - c$ and the dimensions of their corners.

8. `exactZ.py`, output `exactZ.json`: Proposition A.15.

9. `check_patterns.py`, output `check_patterns.json`: a check, with `sympy`, of the computation of the three partitions in the proof of Proposition A.18, which is also done by hand there.

10. `verif_exact.py`, output `verif_exact.json`: an a posteriori verifier, written separately from `z4x.py`, with its own arithmetic routines in K and its own reduction strategy (leftmost occurrence first, then shortest leading word). Starting from `rows_exact.pkl` and from the output `upper_exact.pkl` of `z4x.py`, it checks, in exact arithmetic, the compatibility of the eliminations and rules with the order and the endpoints, the reduction to zero of the 19981 generators of I_K (the inclusion $I_K \subset J_K$), the resolution of all overlap and inclusion ambiguities, the absence of eliminated arrows in the leading words, the counts of irreducible paths, the integrality at $\mathfrak{p}$, and the coincidence of the reductions modulo $\mathfrak{p}$ of the rules with the output of item 6. It does not check the inclusion $J_K \subset I_K$.

11. `membership_0_vertices_E.py`, `membership_a.py` and `membership_b.py`, written separately from the production program to certify the reverse containment: a certificate of the inclusion $J_K \subset I_K$ which does not use the bookkeeping of `z4x.py`. With pivots chosen modulo $\mathfrak{p}$ and an exact verification of every identity, it shows that the vertices 12 and 13, the 46 elements $x - \ell_x$ and the 5105 quadratic rules lie in the K-span of the generators of I_K multiplied by paths, after substitution of the elements already certified, and that each of the 554 remaining rules reduces to zero modulo the span of the certified rules and of their reduced overlaps. These scripts use the rules of `z4x.py` as targets and share its arithmetic modules `kfast` and `Kfield`. The replay driver also contains a targeted check of the last step. It normalizes each target by the certified rules and accumulates exact overlap remainders in the corresponding endpoint corner until the targets reduce to zero.

The products of item 2 and the data of the elimination are intermediate files which are recomputed by the programs. The certificates are exact computations. They are not a formal verification of the implementation of the lattice vertex operators, which is the one of [ArhI].

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Competing interests The author declares that he has no competing interests.

Data availability. The ancillary files are `Calculs-III-A1-A4-C3-v10.zip` and `Certificats-Zhu-III-v06.zip`. Sections 10 and A.8 describe their computational inputs, programs, logs and replay instructions. *Author's GitHub profile.* https://github.com/cedricarhancet

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