hexagon:2611.00162v1

SCHMEISSER'S CONJECTURE IN LARGE DEGREE

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Abstract. We prove Schmeisser's conjecture for all sufficiently large degrees.

1. Introduction

Let P be a complex polynomial of degree $n \geq 2$, with zeros $z_1, \ldots, z_n$ counted with multiplicity. Its critical points are the zeros of P'. The Gauss–Lucas theorem places them in the convex hull of $z_1, \ldots, z_n$, the set of all sums $\sum_{j=1}^n t_j z_j$ with real $t_j \geq 0$ and $\sum_{j=1}^n t_j = 1$. Sendov's conjecture, proposed in 1958 [9] and recorded under Ilieff's name as Problem 4.5 in Hayman's book [2, 7], asks whether each zero is within distance one of a critical point whenever all zeros lie in the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. The constant one is attained by $P(z) = z^n - 1$, whose zeros have modulus one and whose only critical point is zero.

Early work established the estimate in small degree and for restricted zero configurations. Schmeisser treated hulls with every vertex on the unit circle [6], then studied polynomials with gaps in their powers [7]. His 1977 paper treated triangular hulls and asked whether, for zeros in the closed unit disk, the distance bound holds at every point of their hull. He recorded this convex-hull bound for degrees at most five, for hulls with all vertices on the circle, and for triangular hulls [8, §3]. The general question is Problem 4.30 in the anniversary edition of Hayman's book [3].

Pawłowski obtained a degree-dependent bound for the distance from the centroid $n^{-1}\sum_{j=1}^n z_j$ to the nearest critical point [5]. Zhang sharpened it to $\sqrt{(n-2)/(n-1)}$ by a second-moment estimate [12]. For Sendov's conjecture, Dégot proved the unit-distance bound in sufficiently large degree for a zero at a prescribed modulus in $(0, 1)$ [1]. Tao obtained a single degree threshold valid for all zeros [10]. Mazur later gave an all-degree proof, with a further exposition by Tao [4, 11]. In that exposition the convex-hull problem is Conjecture 18.

Theorem 1.1. *There exists an integer $N \geq 2$ such that the following holds for every complex polynomial P of degree $n \geq N$. If all zeros of P lie in the closed unit disk, then for every point a in their convex hull there is a zero ξ of P' with $|\xi - a| \leq 1$.*

Date: November 7, 2026.
2020 *Mathematics Subject Classification.* Primary 30C15; Secondary 30C10.
Key words and phrases. Schmeisser's conjecture, critical points, convex hulls.

A point violating the distance bound can be moved to an edge of the convex hull. Near the unit circle, the two endpoint zeros give equal-value relations for the polynomial. At intermediate moduli, a reciprocal-polynomial comparison yields a degree bound. Near the origin, points where the zeros meet their smallest enclosing circle give a lower bound on the scale at a rightmost zero. Estimates for the logarithmic derivative of P' then rule out both bounded and unbounded values at that scale.

2. Normalization and polynomial fibers

Write $\operatorname{conv} E$ for the convex hull of a set $E \subset \mathbb{C}$. List the critical points of P as $\xi_1, \dots, \xi_{n-1}$, with multiplicity. A strict counterexample consists of P with all zeros in $\overline{\mathbb{D}}$ and $a \in \operatorname{conv}\{z_1, \dots, z_n\}$ such that every $|\xi_j - a| > 1$.

In a strict counterexample the smallest disk containing the zeros has positive radius $R \le 1$, since a polynomial with just one distinct zero has a critical point there. If the disk has center c, replace $P(z)$ by a monic multiple of $P(c + Rz)$ and the witness by $(a - c)/R$. Critical distances are multiplied by $1/R \ge 1$, and affine maps preserve convex combinations. By Gauss–Lucas the new witness is nonzero. Rotate the configuration so that this witness, still denoted by a, is positive real. The inequalities $|\xi_j| \le 1$ and $|\xi_j - a| > 1$ imply $\operatorname{Re} \xi_j < a/2$. For real $s \ge a$, the derivative of $|s - \xi_j|^2$ in s is $2(s - \operatorname{Re} \xi_j) > 0$. Replacing a by the last point of the root hull on the ray $[a, \infty)$ preserves the strict inequalities. For sufficiently large n, Tao's theorem [10, Theorem 1.2] gives a critical point within distance one of every zero, so a cannot be a vertex. It lies in the relative interior of an edge with distinct zeros as endpoints. If the hull is a segment, use the segment itself. In either case $a < 1$, since a convex combination of points in the disk can have real part one only if all points used equal one.

If Theorem 1.1 fails, these reductions give a sequence of normalized strict counterexamples with $n \to \infty$. We use P, a, z_j, ξ_j for the data at degree n, suppressing the sequence index. Unless specified otherwise, asymptotic estimates refer to $n \to \infty$. Passing to a subsequence, we may assume that a converges in $[0, 1]$.

Lemma 2.1 ([1, Theorems 2 and 3])**.** *Let P be a complex polynomial of degree $n \ge 2$ and let $P'(b) \ne 0$. For each $w \in \mathbb{C}$, there is a point β with $P(\beta) = w$ and*

$$|\beta - b| \le n \frac{|w - P(b)|}{|P'(b)|}.$$

If $P(\beta) = P(\gamma)$ and $\beta \ne \gamma$, the perpendicular bisector of $[\beta, \gamma]$ intersects the convex hull of the critical points.

Proof. The case $w = P(b)$ follows by taking $\beta = b$. Otherwise let $\beta_1, \dots, \beta_n$ be the zeros of $P - w$ with multiplicity. Then $P'(b)/(P(b) - w) = \sum_j (b - \beta_j)^{-1}$. At least one summand has modulus at least $n^{-1}|P'(b)/(P(b) - w)|$, which gives the radius bound.

For the bisector assertion, take the two distinct points to -1 and 1 by a complex affine map. If their bisector misses the convex hull of the critical points, we can interchange the points so that the critical points all have positive real part. The equality $P(-1) = P(1)$ gives

$$\int_{-1}^{1} \prod_{j=1}^{n-1} (t - \xi_j)\, dt = 0.$$

As a polynomial in the ξ_j, this integral is symmetric and affine in each variable separately. Choose $\varepsilon > 0$ with $\operatorname{Re} \xi_j \geq \varepsilon$ for all j. For sufficiently large real R the disk with center R and radius $R - \varepsilon/2$ contains all ξ_j, since $|\xi_j - R|^2 \leq R^2 - 2R\varepsilon + |\xi_j|^2$. This disk lies in $\operatorname{Re} z \geq \varepsilon/2$. The Grace–Walsh–Szegő coincidence theorem [1, Theorem 1] gives a point ξ in the disk with $\int_{-1}^{1} (t - \xi)^{n-1}\, dt = 0$. Integration gives $(1 - \xi)^n = (-1 - \xi)^n$, hence $|1 - \xi| = |-1 - \xi|$ and $\operatorname{Re} \xi = 0$, a contradiction. □

We use the root and critical point probability measures

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j}, \qquad \nu_n = \frac{1}{n-1} \sum_{j=1}^{n-1} \delta_{\xi_j},$$

where δ_z is unit mass at z. For a compactly supported measure σ, its logarithmic potential off its support is $U_\sigma(w) = \int \log|w - z|\, d\sigma(z)$. Weak convergence means convergence of integrals against continuous functions on the common compact disk.

Lemma 2.2. *If μ_n and ν_n converge weakly to μ and ν, respectively, then $U_\mu = U_\nu$ on $|w| > 1$. The means of μ_n and ν_n agree for each n.*

Proof. The coefficient of z^{n-2} in P'/n is both $-(n-1)\sum_j z_j/n$ and $-\sum_j \xi_j$. Thus the means agree. For $|w| > 1$,

$$\frac{wP'(w)}{nP(w)} = \int \frac{1}{1 - z/w}\, d\mu_n(z).$$

The integrand has real part at least $1/2$ and modulus at most $1/(1 - 1/|w|)$. The product formulas give

$$(n-1)U_{\nu_n}(w) - nU_{\mu_n}(w) = \log\left|\frac{wP'(w)}{nP(w)}\right| - \log|w|.$$

The right side is bounded for fixed w. Divide by n and take the limit. □

3. Witnesses approaching the unit circle

Let σ be normalized arc length on the unit circle.

Lemma 3.1. *Let $P = P_n$ be monic of degree n along a sequence with $n \to \infty$, with all roots in the closed unit disk. Suppose $b_n \in [0, 1]$ is a root, $b_n \to b > 0$, and $|\xi_j - b_n| \geq r_n$ for all j, where $r_n \to 1$. Then $b = 1$, $\mu_n \to \sigma$, and $\nu_n \to \delta_0$ weakly. Also $|P(0)|^{1/n} \to 1$.*

Proof. For large n, $r_n > 0$ and b_n is simple. Fix $d \in (0, b)$. By Gauss–Lucas, $|\xi_j| \le 1$, so the exclusion implies $2\operatorname{Re}\xi_j \le (1 + b_n^2 - r_n^2)/b_n = b + o(1)$. In particular,

$$|\xi_j - b_n|^2 - |\xi_j - d|^2 = (b_n - d)(b_n + d - 2\operatorname{Re}\xi_j) \ge (b - d)d + o(1)$$

uniformly in j. There is a fixed $\eta > 0$, chosen also with $\eta < (b - d)/2$, such that this difference stays positive upon replacing b_n by any γ with $|\gamma - b_n| < \eta$. The fiber estimate in Lemma 2.1 gives a point γ with $P(\gamma) = P(d)$ and $|\gamma - b_n| \le n|P(d)|/|P'(b_n)|$. If the bound were smaller than η, the critical points would all lie strictly on the d-side of the perpendicular bisector, contrary to the second part of Lemma 2.1. Therefore

$$|P(d)| \ge \frac{\eta}{n}|P'(b_n)| \ge \eta r_n^{n-1}.$$

On the other hand,

$$|P(d)|^{2/n} \le \frac{1}{n}\sum_{j=1}^{n}|z_j - d|^2 \le 1 + d^2 - 2d\operatorname{Re}\int z\,d\mu_n(z).$$

First letting $n \to \infty$, then $d \downarrow 0$, bounds the limit superior of the real part of the mean by zero. By Lemma 2.2, the critical-point mean therefore has real part at most $b/8$ for large n. If the proportion of critical points with real part at most $b/4$ is p_n, that mean has real part at least $(1-p_n)b/4 - p_n$. Hence $p_n \ge b/(8 + 2b) > 0$.

Uniformly for $0 \le d \le b/2$, the squared-distance comparison gives $|d - \xi_j|/|b_n - \xi_j| \le 1 + o(1)$ for every j and $|d - \xi_j|/|b_n - \xi_j| \le 1 - \kappa$ on the positive proportion just identified, for some fixed $\kappa > 0$. The latter assertion uses $b_n - d \ge b/3$ and $b_n + d - 2\operatorname{Re}\xi_j \ge b/3$ for those j and all large n. Moving $d = b/4$ to a point z within distance εr_n increases each ratio by at most ε. Choose $\varepsilon < \kappa/2$ small enough that $(1 - \kappa/2)^{b/(8+2b)}(1 + 2\varepsilon) < 1$. Since $r_n \to 1$, this yields a fixed disk about $b/4$ on which the product of the ratios decays exponentially, as it does on the segment. Thus

$$|P'(z)| \le e^{-cn}|P'(b_n)|$$

on the segment and the disk, for some $c > 0$. Integrate from $d = b/4$ to zero and use the lower bound on $|P(d)|$ to obtain

$$1 \ge |P(0)| \ge \frac{\eta}{2n}|P'(b_n)| \ge \frac{\eta}{2}r_n^{n-1}.$$

In particular, $|P'(b_n)| = O(n)$, and integration also gives $P(z)/P(0) = 1 + O(ne^{-cn})$ uniformly on the small disk about $b/4$.

Any subsequential weak limit of μ_n is supported on the unit circle. In fact, for $0 < r < 1$, the product formula at zero gives $\mu_n(\{|z| \le r\})\log r \ge n^{-1}\log|P(0)|$. The logarithmic potential of the limit is zero on the small disk about $b/4$, by the approximation of P there. To pass to the limit in the logarithm at any point in this disk, use a smaller disk with no roots around that point and truncate the kernel there. Harmonic continuation gives zero potential throughout the open unit disk, and the power series for $\log|1 - z\overline{w}|$

for $|z| < 1$, $|w| = 1$, gives zero nonconstant Fourier coefficients. Thus the limit is σ.

Any subsequential weak limit of ν_n is supported on

$$A_b = \{z : |z| \le 1,\ |z - b| = 1\}.$$

Indeed, the support satisfies $|z-b| \ge 1$ in the limit, and $\sum_j \log(|b_n-\xi_j|/r_n) = \log(|P'(b_n)|/n) - (n-1)\log r_n = o(n)$ is a sum of nonnegative terms. For every fixed $\varepsilon > 0$, dividing this sum by $(n-1)\log((1+\varepsilon)/r_n)$ bounds the fraction of critical points with $|b_n - \xi_j| \ge 1 + \varepsilon$ by a quantity tending to zero. For any weak limit ν, its Cauchy transform $\int (z-w)^{-1}\, d\nu(w)$ agrees with $1/z$ outside the closed unit disk by differentiation of the potentials in Lemma 2.2. The complement of the proper circular arc A_b remains connected after removing the origin. Continuation would give a pole at the origin if $b < 1$, which is impossible because the origin would lie off the support. Thus $b = 1$. By continuation the logarithmic potential agrees with $\log|z|$ off A_1, since their real gradients agree there and both have the same behavior at infinity. The arc has planar measure zero, and the distributional Laplacian of $\log|z-w|$ is $2\pi\delta_w$. Consequently the limit is δ_0. □

Proposition 3.2. *There is no strict counterexample sequence of degrees $n \to \infty$ whose supporting-edge witnesses $a \in (0,1)$ tend to 1.*

We use the primitive estimate and Fourier comparison from Tao's boundary analysis [10, §5]. The fiber relations at the two endpoints of the edge give the needed bound at a by convexity.

Proof. Select a root b_n with $\operatorname{Re} b_n \ge a$, which is possible because a lies in the convex hull. Then $|b_n - a|^2 \le 1 - a^2$. Rotate by $-\arg b_n$ to apply Lemma 3.1 with $r_n = 1 - |b_n - a|$. The rotation tends to the identity, so in the original coordinates $\mu_n \to \sigma$, $\nu_n \to \delta_0$, and $|P(0)|^{1/n} \to 1$.

The convergence $\mu_n \to \sigma$ implies that every point of the unit circle is at distance $o(1)$ from some root, uniformly on the circle. This follows by a finite cover by disks of arbitrarily small radius, each with positive limiting mass. Hence the convex hull contains a centered disk of radius $1 - o(1)$. Indeed, in every unit direction e, a root at distance $o(1)$ from e gives $\max_j \operatorname{Re}(z_j \bar{e}) \ge 1 - o(1)$ uniformly in e. The intersection of the corresponding supporting half-planes contains that disk. For the outward unit normal e of the edge through a, this maximum is $\operatorname{Re}(a\bar{e})$. Hence $e \to 1$, and both endpoints u, v of the edge have real parts tending to one and moduli at most one. Thus $u, v \to 1$, and the unit edge direction $d = (v-u)/|v-u|$ satisfies $d^2 \to -1$.

Write $\mathbb{E}$ for averaging with respect to ν_n, and set

$$m = \mathbb{E}\xi, \qquad V = \mathbb{E}|\xi - m|^2, \qquad Q = \mathbb{E}|\xi|^2,$$

$$L(z) = \mathbb{E}\log|z - \xi|, \qquad c_n = \frac{\log|P(0)|}{n}.$$

Here ξ is the integration variable, $Q = V + |m|^2 \to 0$, and $c_n \le 0$. The potential L is finite off the critical points, and its logarithmic singularities

on the unit circle are integrable. Constants in the estimates can depend on the fixed contours or compact sets used in the proof.

We first derive a primitive estimate on suitable paths. Suppose z is joined to a point of modulus $1/4$ by a piecewise smooth path of uniformly bounded length whose distance from every critical point is bounded below by a positive constant. In the outward orientation and with arc-length parametrization $w(s)$, assume $d\log|w(s)|/ds \geq c > 0$ on each smooth piece. The paths used below satisfy $1/4 \leq |w| \leq 1$. The identity

$$\frac{1}{w-\xi} = \frac{1}{w-m} + \frac{\xi-m}{(w-m)^2} + \frac{(\xi-m)^2}{(w-m)^2(w-\xi)}$$

gives $\mathbb{E}(1/(w-\xi)) = 1/(w-m) + O(V)$. For $B(w) = P'(w)(w-m)/n$, the product formula for P' gives

$$B'(w) = P'(w)\left(\frac{1}{n} + \frac{n-1}{n}(w-m)\mathbb{E}\frac{1}{w-\xi}\right) = P'(w)(1+O(V)).$$

Also,

$$\frac{d}{ds}\log|P'(w(s))| = (n-1)\,\mathrm{Re}\left(w'(s)\mathbb{E}\frac{1}{w(s)-\xi}\right).$$

Here $\mathbb{E}(1/(w-\xi)) = 1/w + o(1)$ uniformly on the path and $|w'(s)| = 1$. The derivative is therefore at least $c(n-1)/2$ for large n. If z is the endpoint at arc length s_0, this gives $|P'(w(s))| \leq |P'(z)|e^{-c(n-1)(s_0-s)/2}$. Its integral is at most $2|P'(z)|/(c(n-1))$. On $|w| \leq 1/4$, the weak convergence $\nu_n \to \delta_0$ and the product formula give $\sup|P'(w)| = O(3^{-n})$: all but $o(n)$ factors have modulus at most $1/4 + 1/32$, and the remaining ones are at most $5/4$. In $P(z) - P(0) - B(z)$, integrate $P' - B'$ along $w(s)$ and add $P(w(0)) - P(0) - B(w(0))$. The integral is $O(V|P'(z)|/n)$, and the added term is $O(3^{-n})$ by integration along the segment from 0 to $w(0)$. Since $|z-m| \geq 1/8$ for large n, this gives

$$P(z) = P(0) + (1+O(V))\frac{P'(z)(z-m)}{n} + O(3^{-n}). \tag{1}$$

For a root λ to which this estimate applies, use $|P(0)| \geq (2/3)^n$ for large n to obtain

$$\frac{|P'(\lambda)|}{n}|\lambda - m| = |P(0)|(1 + O(V) + O(2^{-n})).$$

Taylor expansion of the logarithmic kernel about m gives $L(\lambda) = \log|\lambda - m| + O(V)$. The remainder bound is uniform even when some ξ are far from m. Expand for $|\xi - m|$ less than half the fixed lower bound for $|\lambda - m|$, and on the complementary region use boundedness of the kernel and the lower bound on $|\xi - m|$ to dominate the remainder by $C|\xi - m|^2$. Taking logarithms of the last displayed estimate and using $\log(|P'(\lambda)|/n) = (n-1)L(\lambda)$ gives

$$L(\lambda) = c_n + O(V/n) + O(2^{-n}). \tag{2}$$

For z near 1, with $|z| \leq 1$, a radial segment to $z/(4|z|)$ satisfies the path conditions. In fact these segments lie in a fixed compact subset of

$\{|w-1|<1\}$, separated from the critical points by $|\xi_j - a| > 1$ and $a \to 1$. We can thus apply (2) at u and v. For real x with $a + xd \in [u, v]$,

$$\frac{d^2}{dx^2} L(a+xd) = -\operatorname{Re} \mathbb{E} \frac{d^2}{(a+xd-\xi)^2} = 1 + o(1),$$

where the convergence is uniform by the separation near 1, $\nu_n \to \delta_0$, and $d^2 \to -1$. By convexity $L(a)$ is at most the convex combination of $L(u)$ and $L(v)$ with the weights that express a in terms of u, v. Since $L(a) > 0$ and $c_n \le 0$,

$$|c_n| + L(a) \le CV/n + O(2^{-n}). \tag{3}$$

In particular $|P(0)| \to 1$. Also the nonnegative sum $\sum_j \log|a - \xi_j| = (n-1)L(a)$ tends to zero, so the distance of every critical point from A_1 tends to zero uniformly.

We next apply (2) to roots near any compact subarc of the unit circle disjoint from A_1. From a nearby point with radius r_0 and argument θ_0, vary the argument at unit speed toward 0 if $\cos\theta_0 > 1/2$, or toward the nearer of $\pi, -\pi$ if $\cos\theta_0 < 1/2$. After an angular change of size s, use radius $r = r_0 - \kappa s$, with fixed small $\kappa > 0$. Then follow the corresponding real axis to radius $1/4$. At radius one the angular sweep stays uniformly away from A_1, whose intersections with the unit circle have arguments $\pm\pi/3$. The separation persists for small κ and r_0 close to one. The positive axis segment stays at least $1/4$ from the circle $|w-1| = 1$. The negative one stays at least $1/4$ from $A_1 \subset \{\operatorname{Re} w \ge 0\}$. The outward derivative of $\log r$ per unit arc length on the angular part is $\kappa/(r\sqrt{\kappa^2 + r^2})$, and on the axis it is $1/r$. These bounds are uniform on a neighborhood of the chosen subarc, and all critical points are uniformly close to A_1. From (3) and (2), all these roots satisfy

$$L(\lambda) = O(V/n) + O(2^{-n}). \tag{4}$$

For z uniformly separated from the critical points and bounded away from zero, expansion about $\xi = 0$ gives

$$L(z) = \log|z| - \operatorname{Re}(m/z) + O(Q).$$

As in the earlier kernel expansion, the remainder is bounded by $C|\xi|^2$ by using Taylor expansion for small ξ and bounded kernels for larger ξ. Choose roots approaching $1, i, -1, -i$. They satisfy (4). Since $\log|\lambda| \le 0$, the four bounds on $\operatorname{Re}(m/\lambda)$ bound both signs of $\operatorname{Re} m$ and $\operatorname{Im} m$ by $O(Q) + o(|m|) + O(2^{-n})$. Using $Q = V + |m|^2$ and $m \to 0$, we obtain

$$|m| = O(V) + O(2^{-n}), \qquad 1 - a = O(V) + O(2^{-n}). \tag{5}$$

The second bound follows by averaging $|a - \xi_j|^2 > 1$, which gives $1 - a^2 < Q - 2a \operatorname{Re} m$.

On a compact subarc I of the circle disjoint from A_1, choose a root λ at distance $o(1)$ from $e^{i\theta}$, uniformly for $\theta \in I$. Along the radial segment from λ to $\lambda/|\lambda|$ and the short arc to $e^{i\theta}$, the derivatives of $L - \log|z|$ are $O(|m| + Q)$, by expansion of $\mathbb{E}(1/(z-\xi))$ about zero. For the expansion, the algebraic identity used above for $1/(w-\xi)$ applies with m replaced by zero,

leaving the first moment as the linear term. Both comparisons are uniformly separated from A_1. The radial contribution from $\log|z|$ is nonnegative, so (4) and (5) give

$$L(e^{i\theta}) \geq -o(V) - O(2^{-n}) \quad (\theta \in I). \tag{6}$$

Let $d\sigma(\theta) = d\theta/(2\pi)$ denote the circle measure in angular coordinates. For a measurable set of angles J with $\sigma(J) = \delta$, one has

$$\int_J \left| \log|e^{i\theta} - \xi| + \operatorname{Re}(e^{-i\theta}\xi) \right| d\sigma(\theta) \leq C\sqrt{\delta}\,|\xi|^2 \qquad (|\xi| \leq 1).$$

For $|\xi| \leq 1/2$ this follows by Taylor expansion. For larger $|\xi|$, the logarithmic kernel has uniformly bounded L^2 norm. Indeed, after rotation, $|e^{i\theta} - |\xi||$ is at least $c|\theta|$ for $|\theta| \leq \pi$, by $|e^{i\theta} - r|^2 = (1-r)^2 + 4r\sin^2(\theta/2)$ for $r = |\xi| > 1/2$, and is at most 2. Cauchy–Schwarz now gives the estimate. The Fourier coefficients of $\log|e^{i\theta} - \xi|$ follow from the series of $\log(1 - \xi e^{-i\theta})$, first for $|\xi| < 1$, then for $|\xi| = 1$ by dominated convergence in the rotated coordinate. The constant term is zero, and only $-\operatorname{Re}(\xi^2 e^{-2i\theta})/2$ contributes to the cosine integral at frequency two. Averaging over the critical points gives

$$\int L(e^{i\theta})\, d\sigma(\theta) = 0, \qquad \operatorname{Re}\mathbb{E}\xi^2 = -4\int L(e^{i\theta})\cos(2\theta)\, d\sigma(\theta).$$

Omit fixed neighborhoods of the two angles $\pm\pi/3$ with total measure δ. On the omitted set the integral of $|L|$ is at most $C\sqrt{\delta}Q + \delta|m|$. On the remaining arcs use (6) for the negative part. Since the integral of $|L|$ is twice that of its negative part, we get

$$|\operatorname{Re}\mathbb{E}\xi^2| \leq (C\sqrt{\delta} + o(1))V + O(2^{-n}) \tag{7}$$

for every fixed sufficiently small $\delta > 0$. The $o(1)$ and exponential constant here may depend on δ, while C is absolute.

For each critical point set $\ell = \log|a - \xi| \geq 0$. The identity

$$\operatorname{Re}\xi = \frac{|\xi|^2 - (1-a^2) - (e^{2\ell} - 1)}{2a}$$

expresses its numerator as a difference of nonnegative quantities. As $(x-y)^2 \leq x^2 + y^2$ for $x, y \geq 0$, and $0 \leq \ell \leq \log 2$, we obtain, for large n,

$$\operatorname{Re}(\xi^2) = 2(\operatorname{Re}\xi)^2 - |\xi|^2 \leq -\tfrac{1}{4}|\xi|^2 + C((1-a)^2 + \ell^2).$$

Here the coefficient of $|\xi|^2$ before weakening is at most $1/(2a^2) - 1$, using $|\xi| \leq 1$. By (3), $\mathbb{E}\ell^2 \leq (\log 2)L(a) = o(V) + O(2^{-n})$, and $(1-a)^2 = o(V) + O(2^{-n})$ by (5). Averaging and applying (7) gives $Q/4 \leq (C\sqrt{\delta} + o(1))V + O(2^{-n})$. Choose δ so that $C\sqrt{\delta} < 1/8$, then use $V \leq Q$ to conclude $Q = O(2^{-n})$.

Set $\tau = \max_j |\xi_j| \leq \sqrt{(n-1)Q}$. The product formula for P' and a telescoping difference of products imply

$$\sup_{|z| \leq 1 + n^{-3}} |P'(z) - nz^{n-1}| \leq n(n-1)\tau(1 + n^{-3} + \tau)^{n-2} = O(n^2\tau).$$

Integration gives the same bound for $P(z) - (z^n + P(0))$ there. Put $R = |P(0)|^{1/n} \to 1$. At each root ω of $z^n + P(0)$, on the circle $|z - \omega| = n^{-3}$, expansion of $(1 + (z - \omega)/\omega)^n - 1$ gives a modulus at least $n^{-2}/(2R)$ for large n, since the terms of order at least two have total modulus at most $\sum_{k\geq 2}(n^{-2}/R)^k/k! = O(n^{-4})$. Multiplication by $|\omega|^n = |P(0)| \to 1$ shows that $z^n + P(0)$ dominates the error. By Rouché's theorem each disk contains one root of P, and any two distinct roots of P are separated by at least $2R\sin(\pi/n) - 2n^{-3} \geq c/n$ for a fixed $c > 0$.

By Tao's theorem [10, Theorem 1.2], for large n each endpoint u, v is within distance 1 of critical points c_u, c_v, respectively. Set $D = |c_u - c_v| \leq 2\tau = o(n^{-2})$. At the midpoint $w = (u + v)/2$,

$$|w - c_u|^2 = \frac{|u - c_u|^2 + |v - c_u|^2}{2} - \frac{|u - v|^2}{4} \leq 1 + D + D^2/2 - \frac{|u - v|^2}{4} \leq 1$$

for large n. Interchanging u, v gives the same for c_v. The two closed unit disks centered at c_u, c_v cover the two halves of the edge by convexity, contradicting the strict witness. □

4. Witnesses with an interior radial limit

Proposition 4.1. *A normalized strict counterexample sequence cannot have $a \to a_* \in (0, 1)$.*

Proof. Choose a zero b of maximal real part. Then $\operatorname{Re} b \geq a$ and $|b - a|^2 \leq 1 - a^2 < 1$, so $P'(b) \neq 0$. Pass to a subsequence with $b \to b_*$. If $b_* = a_*$, then $|\xi_j - b| \geq 1 - |b - a|$ for every j. Rotate each polynomial to place b on the positive real axis. Lemma 3.1 then forces $a_* = 1$.

Suppose $b_* \neq a_*$. Choose a fixed $t \in (0, 1)$ so close to one that $t - 1 + |b_* - ta_*|^2 > 0$. Put $\gamma = b - ta$ and $C = ta(2\operatorname{Re} b - (1 + t)a)$. Then $C \geq t(1 - t)a^2$, $|\gamma|^2 \leq 1 - t(2 - t)a^2 < 1$, and $t \geq 1 - |\gamma|^2$ eventually. Direct expansion gives

$$\begin{aligned} |b - \xi_j|^2 - |1 - \overline{\gamma}\xi_j|^2 &= t(|\xi_j - a|^2 - 1) + C \\ &\quad + (t - 1 + |\gamma|^2)(1 - |\xi_j|^2) \geq C. \end{aligned}$$

Since $|1 - \overline{\gamma}\xi_j| \leq 2$, this gives $|b - \xi_j| \geq \sqrt{1 + C/4}\,|1 - \overline{\gamma}\xi_j|$. The identity

$$|1 - \overline{\gamma}\xi_j|^2 - |\gamma - \xi_j|^2 = (1 - |\gamma|^2)(1 - |\xi_j|^2)$$

also gives $|b - \xi_j|^2 - |\gamma - \xi_j|^2 \geq C$. Set $\eta = \min(1, C/10)$. For $|\beta - b| \leq \eta$, replacing b by β reduces the difference by at most $2|\xi_j - b|\eta + \eta^2 \leq 4\eta + \eta^2 \leq C/2$. Hence $\beta \neq \gamma$ and all critical points are strictly closer to γ than to β. The bisector assertion in Lemma 2.1 then excludes $P(\beta) = P(\gamma)$ in this disk. The radius bound in the same lemma gives

$$|P(\gamma)| \geq \frac{\eta}{n}|P'(b)|.$$

For $F(z) = \sum_{k=0}^{d} f_k z^k$ of degree d, define the reciprocal polynomial $F^*(z) = \sum_{k=0}^{d} \overline{f_{d-k}} z^k$. For $q = P'/n$, the coefficient of z^k in q^* is $1 - k/n$ times the coefficient of z^k in P^*, for $0 \leq k \leq n - 1$. The same formula gives

zero for $k = n$, so $q^* = P^* - z(P^*)'/n$. Taking the logarithmic derivative of the product for P^* gives

$$P^*(\gamma) = \prod_{j=1}^{n}(1 - \gamma\overline{z_j}) \neq 0, \qquad \frac{q^*(\gamma)}{P^*(\gamma)} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{1-\gamma\overline{z_j}}.$$

Each summand has real part at least $1/2$. Also $|P(\gamma)| \leq |P^*(\gamma)|$, by the squared-distance identity above applied to each z_j. Multiplying the bounds on $|b - \xi_j|$ yields

$$|P'(b)| \geq n(1 + C/4)^{(n-1)/2}|q^*(\gamma)| \geq \frac{n}{2}(1 + C/4)^{(n-1)/2}|P^*(\gamma)|.$$

The inequalities for $|P(\gamma)|$ give $\eta(1 + C/4)^{(n-1)/2} \leq 2$. Since $C \geq t(1-t)a^2$ stays bounded away from zero, this contradicts $n \to \infty$. □

5. The logarithmic derivative at a rightmost zero

For a polynomial F of degree at least two and a simple zero b, the value $F''(b)/F'(b)$ is the sum of the reciprocals $1/(b-c)$ over its critical points c, counted with multiplicity. The next estimate bounds the corresponding sum of squared moduli when b has largest real part among the zeros.

Lemma 5.1. *There is an absolute constant C with the following property. Let b be a simple zero of a complex polynomial F of degree at least two. Suppose that every zero u of F satisfies $\operatorname{Re} u \leq \operatorname{Re} b$. With c running over the critical points of F, counted with multiplicity, set*

$$r = \min_c |b - c|, \qquad S = \sum_c \frac{1}{b-c}, \qquad Q = \sum_c \frac{1}{|b-c|^2}.$$

Then $Qr^2 \leq C(1 + r|S|)$.

Proof. Replacing $F(z)$ by $F(b + rz)$ puts the chosen zero at zero and the nearest critical point at distance one. The two sums become rS and r^2Q. If the estimate fails, there is a sequence in the normalized coordinates with $Q/(1 + |S|) \to \infty$. All limits in this proof refer to that sequence. Put $v_c = -1/c$, $h = |S|$, and $B = -\sum_c v_c^2$. By Gauss–Lucas, $\operatorname{Re} v_c \geq 0$, and by normalization $|v_c| \leq 1$. Thus $\sum_c(\operatorname{Re} v_c)^2 \leq \sum_c \operatorname{Re} v_c = \operatorname{Re} S \leq h$, and Cauchy–Schwarz gives

$$0 \leq Q - \operatorname{Re} B = 2\sum_c(\operatorname{Re} v_c)^2 \leq 2h, \qquad |\operatorname{Im} B| \leq 2\sqrt{hQ}.$$

In particular $B/Q \to 1$. Also, for $\operatorname{Re} z > 0$, the product over the zeros of F gives

$$\operatorname{Re}\left(\frac{F'(z)}{F(z)} - \frac{1}{z}\right) = \sum_{u\neq 0}\operatorname{Re}\frac{1}{z-u} \geq 0, \tag{8}$$

where the zeros u are counted with multiplicity. Both cases below use, for $|z| \leq 1/2$,

$$\frac{F'(z)}{F'(0)} = \prod_c(1 + v_c z) = \exp\bigl(Sz + Bz^2/2 + O(Q|z|^3)\bigr).$$

Indeed, the power series for $\log(1+w)$ at zero has remainder bounded by $2|w|^3$ after its quadratic term for $|w| \le 1/2$, and $\sum_c |v_c|^3 \le Q$.

Suppose first that $h/\sqrt{Q} \to \infty$ along a subsequence. Then $e = Q/h^2 \to 0$. For $z = \zeta/S$ with ζ in a fixed bounded set, the quadratic term is $O(e)$ and the remainder is $O(e/h)$. Expanding the exponential to first order in the quadratic term gives an error $O(e/h+e^2) = o(e)$. Since $F(0) = 0$, integration gives

$$\frac{SF(\zeta/S)}{F'(0)} = e^{\zeta} - 1 + \frac{B}{S^2} A(\zeta) + o(e),$$
$$A(\zeta) = \int_0^{\zeta} e^t \frac{t^2}{2}\, dt = \frac{e^{\zeta}(\zeta^2 - 2\zeta + 2) - 2}{2},$$

For $\zeta_0 = \pm 2\pi i$, Rouché's theorem on a small fixed circle about ζ_0 gives a zero $\zeta \to \zeta_0$. Since $e^{\zeta} - 1 = (\zeta - \zeta_0) + O(|\zeta - \zeta_0|^2)$, the equation at the zero first gives $\zeta - \zeta_0 = O(e)$, then $\zeta - \zeta_0 = -(B/S^2)A(\zeta_0) + o(e)$. Evaluating $A(\zeta_0) = \zeta_0^2/2 - \zeta_0$ and dividing by S gives zeros of F at

$$t_{\pm} = \frac{\pm 2\pi i}{S} + \frac{B}{S^3}(2\pi^2 \pm 2\pi i) + o(e/h).$$

The real parts of the leading terms are $\pm 2\pi \operatorname{Im} S/h^2$. Since $\operatorname{Re} t_{\pm} \le 0$ and the corrections are $O(e/h)$, we have $|\operatorname{Im} S|/h = O(e)$, and $\operatorname{Re} S \ge 0$ gives $S/h \to 1$. Adding the expansions and using $B/Q \to 1$ gives

$$\operatorname{Re}(t_+ + t_-) = (4\pi^2 + o(1))Q/h^3 > 0,$$

a contradiction.

Otherwise $h/\sqrt{Q}$ is bounded, and we pass to a subsequence with $S/\sqrt{Q} \to \eta \in \mathbb{C}$. At $z/\sqrt{Q}$ the product expansion has exponent

$$\frac{S}{\sqrt{Q}} z + \frac{B}{2Q} z^2 + O(|z|^3/\sqrt{Q}).$$

Integrating from zero shows that $\sqrt{Q}\, F(z/\sqrt{Q})/F'(0)$ and its derivative converge locally uniformly to $G(z)$ and $G'(z)$, where

$$G(z) = \int_0^z e^{\eta t + t^2/2}\, dt.$$

Applying (8) at $z/\sqrt{Q}$ and dividing by $\sqrt{Q}$ gives $\operatorname{Re}(G'(z)/G(z)) \ge \operatorname{Re}(1/z)$ in the limit whenever $\operatorname{Re} z > 0$ and $G(z) \ne 0$.

Along the rays of arguments $\pm\pi/3$, the function G has limits L_+ and L_-, respectively. We may rotate the integration rays to arguments $\pm\pi/2$: on the connecting arcs of radius R, the integrand is bounded by $e^{-R^2/4+|\eta|R}$, and the arc integrals tend to zero. It follows that

$$L_+ - L_- = i \int_{-\infty}^{\infty} e^{-u^2/2 + i\eta u}\, du = i\sqrt{2\pi}\, e^{-\eta^2/2}.$$

For the last equality, differentiating the real-line integral in η and integrating by parts shows that its derivative is $-\eta$ times the integral. Its value at $\eta = 0$ is the Gaussian integral. Hence at least one of the two limits is nonzero. On that ray, at $z = Re^{\pm \pi i/3}$, we have $G(z)$ bounded away from zero for large

R and $|G'(z)| \le e^{-R^2/4+|\eta|R}$. This contradicts $\operatorname{Re}(G'(z)/G(z)) \ge \operatorname{Re}(1/z) = 1/(2R)$. □

Lemma 5.2. *Let (P_j) be a sequence of polynomials of degree at least two with all zeros in the closed unit disk. Suppose $a_j > 0$ and every critical point c of P_j satisfies $|c - a_j| > 1$. Let $b_j = x_j + iy_j$ be a zero of P_j of largest real part, and suppose $x_j \ge a_j$ and $b_j \to i$. With $\rho_j = \max(x_j, 1 - y_j)$, the sequence*

$$\rho_j \left| \frac{P_j''(b_j)}{P_j'(b_j)} \right|$$

is bounded.

Proof. Since $|b_j - a_j|^2 \le 1 - 2a_jx_j + a_j^2 \le 1 - a_j^2$, each b_j is a simple zero. Suppose that the displayed sequence tends to infinity along a subsequence. Suppress the index j and set

$$F(u) = P(i + \rho u), \qquad \beta = (b - i)/\rho,$$
$$X = x/\rho, \qquad A = a/\rho.$$

Thus $0 < A \le X \le 1$ and $-1 \le \operatorname{Im}\beta \le 0$. Write v for the critical points of F, with multiplicity. Every critical point $c = i + \rho v$ satisfies $|c| \le 1$ by Gauss–Lucas. Comparing with $|c - a|^2 > 1$ gives $\operatorname{Re} v < A/2$. Define

$$K(z) = \frac{F''(z)}{F'(z)} = \sum_v \frac{1}{z - v},$$
$$S = K(\beta), \qquad h = |S|, \qquad Q = \sum_v \frac{1}{|\beta - v|^2}.$$

Our assumption is $h \to \infty$. If $X \ge 1/2$, then $|\beta - v| \ge X/2 \ge 1/4$. Otherwise $\rho = 1 - y$ and for all sufficiently small ρ,

$$|b - a|^2 \le \rho^2/4 + (1 - \rho)^2 \le (1 - \rho/2)^2.$$

The exclusion $|c - a| > 1$ implies $|b - c| \ge \rho/2$, so $|\beta - v| \ge 1/2$ in this case. Lemma 5.1 applied to F at β gives $Q \le C_1 h$ for all sufficiently large j, with a fixed $C_1 \ge 1$.

Let z be a zero with $|z - \beta| \le 1/16$ and $|K(z)| \ge h/2$. Since $|\beta - v| \ge 1/4$, we have $|z - v| \ge |\beta - v|/2 \ge 1/8$ and $\sum_v |z - v|^{-2} \le 4Q$. In particular z is simple. For $|\zeta| \le 8$, apply the series for $\log(1 + w)$ to each factor $1 + \zeta/(K(z)(z - v))$ in the quotient of derivatives. The linear terms sum to ζ, giving

$$\frac{F'(z + \zeta/K(z))}{F'(z)} = \exp(\zeta + O(h^{-1})).$$

For large h, each $|\zeta/(K(z)(z - v))| \le 1/2$. The error in the logarithm is then bounded by $|\zeta/K(z)|^2 \sum_v |z - v|^{-2}$. Integration from $\zeta = 0$ gives

$$\frac{K(z)F(z + \zeta/K(z))}{F'(z)} = e^{\zeta} - 1 + O(h^{-1}).$$

The errors are uniform in z. On either circle of radius M/h about $\pm 2\pi i$, $|e^{\zeta} - 1| \ge M/(2h)$ for large h by Taylor expansion. Choose the fixed M so

that this lower bound exceeds the error in the last display. Rouché's theorem gives zeros $z + t_+$ and $z + t_-$ with

$$t_\pm = \frac{\pm 2\pi i}{K(z)} + O(h^{-2}). \tag{9}$$

At $z = \beta$, the real parts in (9) are $\pm 2\pi \operatorname{Im} S/h^2 + O(h^{-2})$. Both are nonpositive, so $|\operatorname{Im} S| = O(1)$. Each $(\beta - v)^{-1}$ has positive real part, giving $\operatorname{Re} S > 0$ and $S/h \to 1$.

Fix $0 < \epsilon \le \min(1/16, 1/(16C_1))$. For $|z - \beta| \le \epsilon$, subtracting the reciprocals term by term gives $|K(z) - S| \le 2\epsilon Q \le h/8$, since $|z - v| \ge |\beta - v|/2$. Therefore $\operatorname{Re} K(z) \ge 3h/4$ and $|K(z)| \le 5h/4$ for large j. At any zero in the disk, the leading term of t_- in (9) has modulus at most $8\pi/(3h)$ and imaginary part $-2\pi \operatorname{Re} K(z)/|K(z)|^2 \le -24\pi/(25h)$. Including the error gives $|t_-| \le 10/h$ and $\operatorname{Im} t_- \le -2/h$. Take $\lfloor \epsilon h/20 \rfloor$ such steps from β. At each step the total distance traveled so far is at most $\epsilon/2$, so the next step is available until the last one. The imaginary parts strictly decrease. For large h there are at least $\epsilon h/80$ steps in the latter half, and each has index at least $\epsilon h/80$. Their endpoints satisfy $|u| \le |\beta| + \epsilon/2 < 2$ and $\operatorname{Im} u \le -\epsilon/40$.

We bound the number of zeros in this region by a circle identity. In the following sums, u runs over the zeros of F and v over the critical points, each with multiplicity. At the scaled point 2, put

$$L = F'(2)/F(2), \qquad D = K(2) - L,$$
$$R_0 = \sum_u \frac{1}{|2-u|^2}, \qquad R_1 = \sum_v \frac{1}{|2-v|^2},$$
$$E_0 = \sum_u \frac{1 - |i + \rho u|^2}{2\rho |2-u|^2}, \qquad E_1 = \sum_v \frac{1 - |i + \rho v|^2}{2\rho |2-v|^2}.$$

The scaled roots satisfy $\operatorname{Re} u \le 1$, so $\operatorname{Re} L \ge R_0$ and $|D| \le 1$ by $K(2) - L = (F'/F)'(2)/L$ and $|(F'/F)'(2)| \le R_0$. The bound $\operatorname{Re} v < A/2$ gives $R_1 \le \operatorname{Re} K(2)/(2 - A/2)$.

For a scaled point $w \ne 2$, expand $1 - |i + \rho w|^2 = -2\rho \operatorname{Im} w - \rho^2 |w|^2$ and use $|w|^2 = |(2 - w) - 2|^2$ to obtain

$$\operatorname{Im} \frac{1}{2-w} = -\frac{\rho}{2} + 2\rho \operatorname{Re} \frac{1}{2-w} - \frac{2\rho}{|2-w|^2} - \frac{1 - |i + \rho w|^2}{2\rho |2-w|^2}.$$

Subtracting the sum for the roots from the sum for the critical points, whose number is one less, yields

$$E_0 = \operatorname{Im} D - \frac{\rho}{2} - 2\rho \operatorname{Re} D + 2\rho(R_1 - R_0) + E_1.$$

For each $c = i + \rho v$, the exclusion inequality gives $1 - |c|^2 < a^2 - 2a \operatorname{Re} c$. Thus

$$E_1 \le \rho A \operatorname{Re} K(2) + \rho(A^2/2 - 2A) R_1.$$

Combining the two displays and substituting the bound for R_1 gives

$$0 \le E_0 \le \sqrt{1 + 4\rho^2} + \frac{4\rho}{4 - A} \operatorname{Re} K(2).$$

In this bound, $\operatorname{Im} D - 2\rho \operatorname{Re} D \le \sqrt{1+4\rho^2}|D|$, and the coefficient of R_1 before substitution is $\rho(2-A)^2/2$. Moreover $|K(2)-S| \le 6Q$. Indeed, $|2-\beta| \le \sqrt{5}$ and $|2-v| \ge 3/2$ give $|\beta - v|/|2-v| \le 1 + |2-\beta|/(3/2)$, so subtraction of the reciprocals gives

$$|K(2)-S| \le |2-\beta| \left(1 + \frac{|2-\beta|}{3/2}\right) Q < 6Q.$$

It follows that $K(2) = O(h)$ and $E_0/h \to 0$, since $\rho \to 0$ and $h \to \infty$.

For each of the at least $\epsilon h/80$ zeros constructed above with $|u| \le 2$ and $\operatorname{Im} u \le -\epsilon/40$, the contribution to E_0 is

$$\frac{-\operatorname{Im} u - \rho|u|^2/2}{|2-u|^2} \ge \frac{\epsilon}{1280}$$

for $\rho \le \epsilon/160$, since the numerator is at least $\epsilon/40 - 2\rho$ and the denominator is at most 16. All other terms are nonnegative by the unit disk condition on the original zeros. Thus $E_0/h \ge \epsilon^2/102400$, a contradiction. □

6. The location of the zeros near the origin

The semicircle weak limit follows from the exterior potential comparison, as in Tao's analysis of Sendov's conjecture [10, §2.1]. To locate every zero we use a differential equation for P/P'. For a nonempty compact set $E \subset \mathbb{C}$, write $\operatorname{dist}(w, E) = \min_{z \in E} |w - z|$.

Lemma 6.1. *In a normalized strict counterexample sequence with $a \to 0$, all zeros of P approach the left unit semicircle*

$$\Gamma = \{z \in \mathbb{C} : |z| = 1,\ \operatorname{Re} z \le 0\}$$

uniformly, in the sense that $\max_{1 \le j \le n} \operatorname{dist}(z_j, \Gamma) \to 0$.

Proof. All critical points approach Γ uniformly because $1 - a < |\xi_j| \le 1$ and $\operatorname{Re} \xi_j < a/2$. Pass to weak limits μ, ν of μ_n, ν_n, with ν supported on Γ. By Lemma 2.2, the exterior potentials agree. For $|w| > 1$ and $|z| \le 1$, expand $\log|w - z| = \log|w| - \operatorname{Re} \sum_{k \ge 1} z^k/(kw^k)$. The integral of this series against $\mu - \nu$ has zero real part and tends to zero at infinity, so all its coefficients vanish. Thus $\int z^k \, d\mu(z) = \int z^k \, d\nu(z)$ for integers $k \ge 1$, and the same holds for $k = 0$ because both measures have mass one.

The Poisson extension to the disk of the function $z \mapsto \operatorname{dist}(z, \Gamma)$ on the circle is continuous on the closed disk and harmonic inside. It is strictly positive off Γ by the Poisson formula. Approximation of the boundary function by trigonometric polynomials and the maximum principle approximate this extension uniformly by harmonic polynomials. These have equal integrals for μ and ν by the moment identities. Since the integral against ν is zero, μ is supported on Γ as well. Trigonometric polynomial approximation on the circle now gives $\mu = \nu$.

Put

$$r_n(w) = \frac{P(w)}{P'(w)}, \quad g_n(w) = \frac{P''(w)}{nP'(w)}, \quad g(w) = \int \frac{d\mu(z)}{w - z}.$$

On each compact subset of $\mathbb{C} \setminus \Gamma$, all critical points stay a positive distance away for large n. Thus r_n, g_n are holomorphic there and $g_n \to g$ locally uniformly by the product for P' and weak convergence of ν_n. For the same reason $n^{-1} \log |P'/n| \to U_\mu$ uniformly there. For P, take $\log 0 = -\infty$ and truncate $\log |w - z|$ below at a fixed constant smaller than its minimum for w in the compact set and $z \in \Gamma$. The resulting continuous kernels converge uniformly after integration against μ_n, by weak convergence and compactness of the family parametrized by w. Their limiting integral is $U_\mu(w)$. Thus $n^{-1} \log |P| \le U_\mu + o(1)$ and $|r_n| \le \exp(o(n))$ locally uniformly.

The equation $r_n' = 1 - n g_n r_n$ determines the limit where $g \neq 0$. Fix w_0 with $g(w_0) \neq 0$ and take primitives $\phi_n' = g_n$ with $\phi_n(w_0) = 0$. On a sufficiently small closed disk of radius r about w_0, $|g_n(w) - g(w_0)| \le |g(w_0)|/2$ for large n. Integrating along segments proves injectivity, since $\phi_n(w) - \phi_n(w')$ differs from $g(w_0)(w - w')$ by at most half its modulus. On the boundary the same estimate with $w' = w_0$ shows by Rouché's theorem that the image contains the disk of radius $|g(w_0)|r/2$ about zero. Choose $\delta = |g(w_0)|r/8$. For w in a fixed small neighborhood of w_0, $s = \phi_n(w)$ satisfies $|s| < \delta$. The chain rule gives

$$\frac{d}{ds}(r_n \circ \phi_n^{-1}) = \frac{1}{g_n \circ \phi_n^{-1}} - n(r_n \circ \phi_n^{-1}).$$

Multiply by e^{ns} and integrate along the segment from $s - \delta$ to s:

$$r_n(w) = e^{-n\delta} r_n(\phi_n^{-1}(s - \delta)) + \int_0^\delta \frac{e^{-nt}}{g_n(\phi_n^{-1}(s - t))}\, dt.$$

The first term is at most $\exp(-n\delta + o(n))$ in modulus. The functions $1/(g_n \circ \phi_n^{-1})$ have uniformly bounded derivatives on $|s| \le 2\delta$ by Cauchy's estimate. Replacing the reciprocal in the integrand by $1/g_n(w)$ gives $(1 - e^{-n\delta})/(n g_n(w))$, with error $O(\int_0^\delta t e^{-nt}\, dt) = O(n^{-2})$. Hence $n r_n \to 1/g$ locally uniformly wherever $g \neq 0$.

The complement of Γ is connected and $w g(w) \to 1$ at infinity, so g is not identically zero there. If g had a zero of order $k \ge 1$, take a small circle surrounding that zero alone and lying off Γ. Uniform convergence on the circle would give $n r_n$ the winding number $-k$ of $1/g$ there. This is impossible because r_n is holomorphic throughout the enclosed disk for large n. Therefore g never vanishes off Γ, and the convergence to $1/g$ excludes all zeros of P from every compact set disjoint from Γ. The argument applies to every weakly convergent subsequence, proving the uniform assertion. □

7. Witnesses approaching the origin

We consider monic polynomials P of degree n whose roots have minimum enclosing disk $\{|z| \le 1\}$. Let $a > 0$ be real, in the convex hull of the roots, and assume that $|\xi - a| > 1$ for every critical point ξ of P. Sums over roots and critical points include their algebraic multiplicities. By Gauss–Lucas,

$$|\xi| \le 1, \qquad |\xi| \ge 1 - a, \qquad \operatorname{Re} \xi < a/2. \tag{10}$$

Lemma 7.1. *If x is the largest real part of a root of P, then $nx > 1/100$.*

Proof. Suppose $nx \le 1/100$. We have $a \le x \le 1/200$. Call a root on the unit circle a contact root. The convex hull of these roots contains zero. Otherwise there is a unit direction e with $\operatorname{Re}(r\bar{e}) > 0$ for every contact root r. For sufficiently small $\varepsilon > 0$, each such root has $|r-\varepsilon e|^2 = 1-2\varepsilon \operatorname{Re}(r\bar{e})+\varepsilon^2 < 1$, and the finitely many other roots stay strictly inside the disk after the center moves to εe. This contradicts minimality. Thus the angular gap between successive contact roots is at most π. The gap containing 1 has endpoints $r_\pm$ with arguments $\theta_- < 0 < \theta_+$ satisfying

$$\arccos x \le \theta_+ \le \pi - \arccos x, \qquad -\pi + \arccos x \le \theta_- \le -\arccos x.$$

The inner bounds follow from $\operatorname{Re} r_\pm \le x$, and the outer bounds follow from the gap length.

Put $t = 2x$, $s = \sqrt{1-t^2}$, $\gamma = \arccos t$, $p = t - is$, and $q = t + is$. The angular distances from r_- to p and from r_+ to q are at most $\arcsin x + \arcsin(2x) \le 4x$, since $(\arcsin y)' \le 4/3$ for $0 \le y \le 1/2$. The Euclidean distances have the same bound. For $|z| < 1$ let $\omega(z)$ be the harmonic measure of the open right-hand arc from p to q, given by

$$\omega(z) = \frac{1}{2\pi} \int_{-\gamma}^{\gamma} \frac{1-|z|^2}{|e^{i\theta} - z|^2}\, d\theta.$$

Set $\omega = 0$ on the complementary arc away from p, q. The imaginary part of $(q-z)/(p-z)$ is $2s(t - \operatorname{Re} z)/|p-z|^2$. When $\operatorname{Re} z = t$ in the open disk, the ratio is negative real, since $-s < \operatorname{Im} z < s$. Hence the argument $\Theta(z) = \arg((q-z)/(p-z))$ in $(0, 2\pi)$ is harmonic in the disk and lies in $(0, \pi)$ when $\operatorname{Re} z < t$. The inscribed angle theorem makes its boundary values constant on each arc. At -1 the ratio is $e^{i\gamma}$, and at 1 it is $e^{i(\pi+\gamma)}$. The Poisson formula on concentric circles with radii tending to one, followed by bounded convergence, gives $\Theta(z) = \gamma + \pi\omega(z)$. The identity extends to the complementary arc, and every root and critical point has real part less than t.

On the unit circle away from the roots, write $H(z) = zP'(z)/P(z)$. For each root r, the real part of $z/(z-r)$ exceeds or equals $1/2$ because the difference is $(1-|r/z|^2)/(2|1-r/z|^2)$. Factorization gives

$$\frac{H(q)}{H(p)} = \frac{q}{p} \prod_{\xi} \frac{q-\xi}{p-\xi} \prod_{r} \frac{p-r}{q-r}.$$

Taking arguments, with $\arg H \in (-\pi/2, \pi/2)$, gives the following relation modulo 2π. The factor q/p contributes 2γ, while the difference of the numbers of critical points and roots contributes $-\gamma$.

$$\pi \sum_{\xi} \omega(\xi) = \pi \sum_{r} \omega(r) + \arg H(q) - \arg H(p) - \gamma. \tag{11}$$

To fix the multiple of 2π in this identity, multiply the roots by $u \in [0,1]$, which multiplies the critical points by the same u. All arguments in the

identity vary continuously, and the identity holds at $u = 0$, where $H \equiv n$ and $\omega(0) = \gamma/\pi$.

Let Ω_q be the sum of $\omega(r)$ over the roots with $|q-r| \le 1/2$. Every term in $H(q)$ has argument in $(-\arcsin t, \pi/2)$, since $\arg(q) = \gamma$, $\operatorname{Re}(q-r) > 0$, and $\operatorname{Re}(q/(q-r)) \ge 1/2$. If $\Omega_q \ge 1/2$, this gives $\arg H(q) + \pi\Omega_q \ge \pi/2 - 1/2$. For $|q-r| \le 1/2$, the angle $\arg(p-r)+\pi/2$ lies between zero and $\arcsin(1/(4s)) < 1/3$, because $p-r$ lies within $1/2$ of $-2is$ and has positive real part. Hence

$$\arg \frac{q}{q-r} = \frac{\pi}{2} - \pi\omega(r) - \big(\arg(p-r) + \pi/2\big).$$

If $\Omega_q < 1/2$, all these near terms lie in the sector with arguments between $\pi/2 - \pi\Omega_q - 1/3$ and $\pi/2$. Its width is less than $2\pi/3$, so projection onto its bisector bounds the modulus of their sum below by half the sum of their moduli. Since $|q - r_+| \le 4x < 1/2$, the near sum includes the term for r_+, of modulus at least $1/(4x)$. The remaining, far terms have total modulus at most $2n$. Their ratio to the near sum in modulus is at most $16nx \le 4/25$, so they change its argument by at most $\arcsin(4/25) < 1/6$. Again $\arg H(q) + \pi\Omega_q \ge \pi/2 - 1/2$. Applying the same bound to the polynomial with conjugate zeros gives $-\arg H(p) + \pi\Omega_p \ge \pi/2 - 1/2$, where Ω_p is defined with p in place of q. The two near sets are disjoint, and all harmonic measures are nonnegative. Thus (11) gives

$$\pi \sum_\xi \omega(\xi) \ge \pi - \gamma - 1 \ge \pi/2 - 1 > 1/2.$$

The vectors $p-\xi, q-\xi$ have first coordinate $t - \operatorname{Re}\xi$ and second coordinates $-s - \operatorname{Im}\xi, s - \operatorname{Im}\xi$. Their dot product is $|\xi|^2 - 2t\operatorname{Re}\xi + 2t^2 - 1$, and their determinant is $2s(t - \operatorname{Re}\xi) > 0$. Dividing and subtracting $(t-a)/s$ gives

$$\cot\Theta(\xi) - \frac{t-a}{s} = \frac{|\xi - a|^2 - 1 + 2at - a^2}{2s(t - \operatorname{Re}\xi)} > 0.$$

The cotangent decreases on $(0,\pi)$, so $\Theta(\xi) < \arctan(s/(t-a))$, where $t > a$. Subtracting $\gamma = \arctan(s/t)$ gives

$$\pi\omega(\xi) < \arctan\frac{as}{1-at} \le 2a.$$

This yields $\pi\sum_\xi \omega(\xi) < 2(n-1)x \le 1/50$, a contradiction. □

Lemma 7.2. *Let polynomials f_j converge locally uniformly on $\mathbb{C}$ to a nonconstant entire function f. Suppose that the roots of f_j have real part at most 1 and imaginary part at most 0, and that every zero of f' is real. Then*

$$f(z) = e^{\alpha z} g(z), \qquad f'(z) = e^{\alpha z} h(z),$$

where $\operatorname{Re}\alpha \ge 0$, $\operatorname{Im}\alpha \le 0$, and g, h are genus-zero products of minimal exponential type. Such a product has the form $Cz^m \prod_{\lambda\ne 0}(1 - z/\lambda)$, where $C \ne 0$, $m \ge 0$ is the multiplicity at zero, and the nonzero zeros are counted with multiplicity and satisfy $\sum_{\lambda\ne0} |\lambda|^{-1} < \infty$. Minimal exponential type means that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that the modulus is at most $C_\varepsilon e^{\varepsilon|z|}$ for all z. If $\operatorname{Im}\alpha < 0$, then f has no zeros.

Proof. Let $z_0 = 3 + i$. By Hurwitz $f(z_0) \neq 0$. The reciprocals $(z_0 - \lambda)^{-1}$ for the zeros λ of f_j lie in the closed fourth quadrant, so

$$\sum_{\lambda} \frac{1}{|z_0 - \lambda|} \leq \sqrt{2}\, |f_j'(z_0)/f_j(z_0)| = O(1).$$

This follows by projecting onto the direction $1 - i$. On disks whose boundaries avoid zeros of f, the zeros converge with multiplicities by Rouché's theorem, so $\sum_{\lambda: f(\lambda)=0} |z_0 - \lambda|^{-1} < \infty$. For zeros of f_j with $|z_0 - \lambda| > R > 2|z - z_0|$, the logarithm of the normalized factor $1 + (z - z_0)/(z_0 - \lambda)$ differs from $(z - z_0)/(z_0 - \lambda)$ by at most $|z - z_0|^2/|z_0 - \lambda|^2$. The sum of these errors is $O(|z - z_0|^2/R)$ by the reciprocal sum bound. For fixed R with the cutoff circle avoiding the zeros of f, the sum of the tail reciprocals tends to $f'(z_0)/f(z_0) - \sum_{|z_0-\lambda|<R,\ f(\lambda)=0} (z_0 - \lambda)^{-1}$. As $R \to \infty$, these limits converge to a number α in the closed fourth quadrant, since each tail sum lies there. Passing to the limits in the products gives

$$\frac{f(z)}{f(z_0)} = e^{\alpha(z - z_0)} \prod_{f(\lambda)=0} \left(1 + \frac{z - z_0}{z_0 - \lambda}\right).$$

For $\lambda \neq 0$, the factor is $(1 - z/\lambda)/(1 - z_0/\lambda)$. For $\lambda = 0$ the factor is z/z_0. Absolute convergence of the reciprocal sum permits the normalization factors to be absorbed in a nonzero constant. For minimal type, divide $\sum_\lambda \log(1 + (|z| + |z_0|)/|z_0 - \lambda|)$ by $|z|$. Each summand tends to zero and, for $|z| \geq |z_0|$, is bounded by $2/|z_0 - \lambda|$. Dominated convergence gives the required bound.

By Gauss–Lucas the same factorization applies to f_j', giving $f'(z) = e^{\alpha' z} h(z)$. For real $s > 1$, put $L_j(s) = f_j'(s)/f_j(s)$ and $K_j(s) = f_j''(s)/f_j'(s)$. The root factorization gives

$$|L_j'(s)| \leq \sum_{\lambda} |s - \lambda|^{-2} \leq \frac{\operatorname{Re} L_j(s)}{s - 1}, \qquad |K_j(s) - L_j(s)| = \left|\frac{L_j'(s)}{L_j(s)}\right| \leq \frac{1}{s - 1}.$$

Pass to the limit in j by Hurwitz and local uniform convergence of the functions and derivatives. Then let $s \to +\infty$. The logarithmic derivatives of g, h tend to zero there by their absolutely summable inverse distances to z_0. More explicitly, $|z_0 - \lambda|/|s - \lambda| \leq 1 + |z_0 - 3|/|s - \lambda|$ for $s \geq 3$, since $\operatorname{Re} \lambda \leq 1$. Thus $\alpha' = \alpha$.

Suppose $\tau = -\operatorname{Im} \alpha > 0$. Multiply f by a constant of modulus one so that h is real entire, meaning $h(\bar{z}) = \overline{h(z)}$. This is possible because all its zeros are real and its canonical factors are real. With $g^*(z) = \overline{g(\bar{z})}$, write $g = u + iv$ for the real entire functions $u = (g + g^*)/2$ and $v = (g - g^*)/(2i)$. The imaginary part of $h = g' + \alpha g$ on the real axis gives $v' + (\operatorname{Re} \alpha) v = \tau u$. In particular v is not identically zero, since otherwise $u \equiv 0$. On the real axis away from the zeros of g,

$$\operatorname{Im} \frac{g'(s)}{g(s)} = \sum_{\lambda: g(\lambda)=0} \operatorname{Im} \frac{1}{s - \lambda} \leq 0.$$

The left side equals $(uv' - u'v)/|g|^2$. Hence $u'v - uv' \geq 0$ on the real axis by continuity, and substituting $v' + (\operatorname{Re}\alpha)v = \tau u$ gives $vv'' - (v')^2 = \tau(u'v - uv') \geq 0$.

If v had a real zero s_0 of order $m \geq 1$ with leading coefficient $c \neq 0$, then $vv'' - (v')^2$ would have leading term $-mc^2(s - s_0)^{2m-2}$, contradicting nonnegativity near s_0. Thus $(\log|v|)'' = (vv'' - (v')^2)/v^2 \geq 0$ on $\mathbb{R}$. The minimal exponential type of g and g^* gives $\log|v(s)| \leq \log C_\varepsilon + \varepsilon|s|$ for each $\varepsilon > 0$. By convexity, every secant slope lies between $-\varepsilon$ and ε, as follows by comparing it with secants whose left or right endpoint tends to infinity in absolute value. Letting $\varepsilon \downarrow 0$ shows that $|v|$ is constant on the real axis. The function v is real and nonvanishing there, so it is constant by continuity and the identity theorem. The relation with u makes u and g constant. □

Proposition 7.3. *There is no sequence of the normalized counterexamples with $n \to \infty$ and $a \to 0$.*

Proof. Select a root $b = x + iy$ with maximal real part $x \geq a$. Lemma 6.1 implies that any limit of b is i or $-i$. Pass to a subsequence and conjugate if necessary so that $b \to i$. The inequality $|b - a|^2 \leq 1 - 2ax + a^2 < 1$ implies that b is simple. Use the scale and coordinates

$$\rho = \max(x, 1 - y), \qquad \beta = (b - i)/\rho,$$
$$X = x/\rho, \qquad A = a/\rho, \qquad v_\xi = (\xi - i)/\rho.$$

Here $0 < A \leq X \leq 1$, and $\rho \to 0$. Every scaled root $(r - i)/\rho$ has real part at most X and imaginary part at most zero, and $\operatorname{Re} v_\xi < A/2$. Set

$$S = \sum_\xi \frac{1}{\beta - v_\xi}, \qquad Q = \sum_\xi |\beta - v_\xi|^{-2}.$$

By Lemma 5.2, $S = \rho P''(b)/P'(b)$ is bounded. The scaled nearest-critical distance is at least $1/4$ for large n, by the two cases $X \geq 1/2$ and $X < 1/2$ in that proof. Lemma 5.1, applied to $u \mapsto P(i + \rho u)$, gives $Q = O(1)$. Expand

$$|\xi|^2 = |b|^2 - 2\operatorname{Re}(b\,\overline{b - \xi}) + |b - \xi|^2 \leq 1.$$

Multiply by $\rho/|b - \xi|^2$ and sum. The upper bound $1 - |b|^2 \leq 2\rho$ follows from $1 - |b|^2 = 2(1 - y) - (1 - y)^2 - x^2$.

$$(n - 1)\rho \leq \frac{1 - |b|^2}{\rho} Q + 2\operatorname{Re}(bS) \leq 2Q + 2|S|.$$

By Lemma 7.1, $n\rho \geq nx > 1/100$, so we can pass to a subsequence with $(n - 1)\rho \to N_0 > 0$.

The polynomials

$$f_n(u) = \frac{P(i + \rho u)}{\rho P'(b)}$$

satisfy $f_n(\beta) = 0$ and $f_n'(\beta) = 1$. By the product formula, $f_n'(u) = \prod_\xi (1 + (u - \beta)/(\beta - v_\xi))$. Applying $|1 + z|^2 \leq \exp(2\operatorname{Re} z + |z|^2)$ to each factor gives

$$|f_n'(u)| \leq \exp\left(\operatorname{Re}((u - \beta)S) + \frac{|u - \beta|^2 Q}{2}\right).$$

Since β is bounded, the derivatives are locally uniformly bounded, and integrating from β gives the same boundedness for f_n. By Montel's theorem pass to a subsequence with $f_n \to f$ locally uniformly and $\beta \to \beta_\infty$. Then $f(\beta_\infty) = 0$ and $f'(\beta_\infty) = 1$. Every scaled critical point satisfies

$$a \operatorname{Re} v_\xi - \frac{\rho}{2}|v_\xi|^2 - \frac{a^2}{2\rho} < \operatorname{Im} v_\xi \le -\frac{\rho}{2}|v_\xi|^2,$$

by the two disk inequalities for ξ. As $A \le 1$, every bounded critical-point limit is real and has real part at most $1/2$. By Hurwitz the zeros of f' have the same property. We may apply Lemma 7.2.

For real $s > 1$, define

$$K_n(s) = \sum_\xi (s - v_\xi)^{-1}, \quad R_n(s) = \sum_\xi |s - v_\xi|^{-2}, \quad E_n(s) = \sum_\xi \frac{1 - |\xi|^2}{2\rho|s - v_\xi|^2}.$$

All zeros of f' have real part at most $1/2$, so $f'(s) \neq 0$. Hence $K_n(s) = f_n''(s)/f_n'(s) \to f''(s)/f'(s)$, and $R_n(s) \le \operatorname{Re} K_n(s)/(s - 1/2)$ is bounded. Expanding $|\xi|^2 = |i + \rho v_\xi|^2$ and using

$$\frac{|v_\xi|^2}{|s - v_\xi|^2} = 1 - 2s \operatorname{Re} \frac{1}{s - v_\xi} + \frac{s^2}{|s - v_\xi|^2}$$

gives

$$\operatorname{Im} K_n(s) = -\frac{(n-1)\rho}{2} + \rho s \operatorname{Re} K_n(s) - \frac{\rho s^2 R_n(s)}{2} - E_n(s).$$

From $|\xi - a| > 1$ we have $0 \le 1 - |\xi|^2 < a^2 - 2a \operatorname{Re} \xi$, hence

$$0 \le E_n(s) < \rho A \operatorname{Re} K_n(s) + \rho(A^2/2 - As) R_n(s) \le \rho A \operatorname{Re} K_n(s).$$

Taking limits gives $\operatorname{Im}(f''(s)/f'(s)) = -N_0/2$. In the representation $f' = e^{\alpha u} h(u)$ of Lemma 7.2, the real zeros of h give $h'(s)/h(s) \in \mathbb{R}$. Thus $\operatorname{Im} \alpha = -N_0/2 < 0$, which forces f to be zero-free. This contradicts $f(\beta_\infty) = 0$. □

Proof of Theorem 1.1. If there were strict counterexamples in arbitrarily large degrees, the normalization in Section 2 would give a sequence with $n \to \infty$ and a converging to a point of $[0, 1]$. Propositions 3.2, 4.1, and 7.3 exclude the limit 1, any limit in $(0, 1)$, and the limit 0, respectively. Hence the degrees of strict counterexamples are bounded. Take N to be one more than the largest such degree if there is one, and $N = 2$ otherwise. □

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