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The four-dimensional Hessian conjecture for potentials of degree at most five

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Abstract

The Hessian conjecture HC_n asserts that if $f \in \mathbb{C}[x_1, \dots, x_n]$ has $\det \mathrm{Hess}(f) \in \mathbb{C}^\times$, then ∇f is a polynomial automorphism. It is known for $n \le 3$, false for $n \ge 5$, and open for $n = 4$, where it implies the plane Jacobian conjecture. Ni proved HC_4 for potentials of degree at most four, and the author treated degree five when the top form depends on exactly three linear forms. We prove HC_4 for every potential of degree at most five. The remaining top forms are a binary form or a fifth power of a linear form. Both cases are reduced, by weighted layers of the Hessian determinant and exact computer algebra over $\mathbb{Q}$, to four elementary mechanisms: a Schur complement in one variable (which reduces to HC_3), a partial Legendre transform in two variables (which reduces to HC_2), a "dead direction" lemma (which reduces to HC_2 and the plane Jacobian conjecture in degrees at most $(3, 4)$, known by Moh), and explicit triangular inverses. Five self-contained verifiers are supplied as ancillary files. The paper has not been refereed.

1 Introduction

For $f \in \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ write $\mathrm{Hess}(f) = (\partial_i \partial_j f)$. The Hessian conjecture HC_n [9] states that if $\det \mathrm{Hess}(f) \in \mathbb{C}^\times$, then $\nabla f \colon \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial automorphism. It holds for $n = 2$ [5] and $n = 3$ [2], and fails for every $n \ge 5$ [10]. For $n = 4$ it is open and implies the plane Jacobian conjecture through $\Phi(x, y) = y^{\mathsf{T}} F(x)$, $\det \mathrm{Hess}\, \Phi = (\det JF)^2$ [10, 14]. Ni [14] proved HC_4 for $\deg f \le 4$; the author [12] proved it for $\deg f = 5$ when the top form depends on exactly three linear forms. Liu [8] proved HC_4 for real potentials of degree at most five whose Hessian has Lorentzian signature. Partial reductions for a degree-five branch are recorded in [6].

Theorem 1. *Let $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ with $\deg f \le 5$ and $\det \mathrm{Hess}(f) \in \mathbb{C}^\times$. Then ∇f is a polynomial automorphism of $\mathbb{C}^4$.*

Status. Statements marked *[C, . . .]* are certified by exact computation over $\mathbb{Q}$ (polynomial identities with symbolic coefficients, Gröbner bases, minimal associated primes, and the Rabinowitsch criterion $1 \in I + (1 - Tg) \Leftrightarrow g \in \sqrt{I}$). Each label is the label printed by one of the ancillary verifiers (Section 6); all 247 checks pass. Where a hypothesis "φ non-degenerate" is used, it is imposed by running the test once for each coefficient of $\det \mathrm{Hess}\, \varphi$ declared nonzero. The paper has not been refereed.

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2 Tools

Affine changes of coordinates preserve the hypothesis and the conclusion, and terms of degree ≤ 1 may be discarded. We use: (GN) a form in $n \leq 4$ variables with vanishing Hessian determinant depends on at most $n-1$ linear forms [7, 3]; (HC_2, HC_3) [5, 2], with transfer to characteristic-zero fields and descent of inverses [14, Lemmas 2.6, 2.7]; (dBvdE) over a field of characteristic zero, a bivariate polynomial without terms of degree ≤ 1 and with vanishing Hessian determinant is a polynomial in one linear form [4, Thm. 3.1]; (Moh) the plane Jacobian conjecture holds for maps of degree at most 100 [11]; (BBR) an injective polynomial self-map of $\mathbb{C}^n$ is bijective [1]; a bijective polynomial self-map of $\mathbb{C}^n$ is birational and, by Zariski's Main Theorem, an automorphism [13, Ch. 4]. We refer to this combination as (BBR).

Lemma 2 (Schur complement)**.** *If, in some linear coordinates $(y, w) \in \mathbb{C}^3 \times \mathbb{C}$, $f = aw^2 + w\,g(y) + h(y)$ with $a \in \mathbb{C}^\times$ and $\det \mathrm{Hess}\, f \in \mathbb{C}^\times$, then ∇f is an automorphism.*

Proof. This is [14, Prop. 6.1, branch $a \neq 0$], whose proof does not use the degree of g; see also [12, Lemma 3]. Put $k = h - g^2/(4a)$ and $w' = w + g/(2a)$; then $f_w = 2aw'$ and $f_y = \nabla k + w'\nabla g$, and substituting $w = \tau - g/(2a)$ shows $\det \mathrm{Hess}(k + \tau g) = \det \mathrm{Hess}\, f/(2a)$ for every τ. Over $\mathbb{C}(\tau)$, HC_3 and descent give a polynomial inverse of $\nabla(k + \tau g)$, whence the inverse of ∇f. □

Lemma 3 (partial Legendre transform)**.** *Let $f = \frac{1}{2} z^{\mathsf{T}} A(X) z + b(X)^{\mathsf{T}} z + c(X)$ with $X, z \in \mathbb{C}^2$ and $\det A \in \mathbb{C}^\times$. If $\det \mathrm{Hess}\, f \in \mathbb{C}^\times$, then ∇f is an automorphism.*

Proof. For a target (Y_X, Y_z) the equation $f_z = Y_z$ gives $z(X) = A^{-1}(Y_z - b)$, polynomial in X. With $\Phi(X) = f(X, z(X)) - Y_z^{\mathsf{T}} z(X)$ one has $\nabla\Phi = f_X(X, z(X))$ Differentiating $f_z(X, z(X)) = Y_z$ gives $\partial z/\partial X = -A^{-1} f_{zX}$, hence $\mathrm{Hess}\,\Phi = f_{XX} - f_{Xz} A^{-1} f_{zX}$ is the Schur complement of A in $\mathrm{Hess}\, f$ along $z = z(X)$, and $\det \mathrm{Hess}\, f = \det A \cdot \det \mathrm{Hess}\,\Phi$ there (an instance is checked in *[C, A4]*). Hence $\det \mathrm{Hess}\,\Phi \in \mathbb{C}^\times$, and by HC_2 the equation $\nabla\Phi(X) = Y_X$ has exactly one solution. So ∇f is injective, and (BBR) concludes. □

Lemma 4 (dead direction)**.** *Let $f = P(u_1, u_2, w) + sR(u_1, u_2)$ in linear coordinates (u_1, u_2, w, s), with $\deg f \leq 5$ and $\det \mathrm{Hess}\, f = c \in \mathbb{C}^\times$. Then ∇f is an automorphism.*

Proof. One has the identity *[C, A3]*

$$\det \mathrm{Hess}\, f = -P_{ww}(Q_0 + sQ_1) + (R_1 P_{2w} - R_2 P_{1w})^2, \qquad Q_1 = R_2^2 R_{11} - 2R_1 R_2 R_{12} + R_1^2 R_{22}.$$

R is not constant, since otherwise $\det \mathrm{Hess}\, f = 0$. If $P_{ww} \neq 0$, then $Q_1 = 0$. Where $R_2 \neq 0$, the derivative of R_1/R_2 along the field $(R_2, -R_1)$, which is tangent to the level curves, equals $Q_1/R_2^2 = 0$ (similarly where $R_1 \neq 0$). Hence on the open set where $\nabla R \neq 0$ each level curve has constant normal direction, so it is an open piece of a line; taking Zariski closures, every irreducible component of a general fibre $\{R = t\}$ is a line. Lines in distinct fibres are disjoint, hence parallel, so R is constant along a fixed direction and $R = g(l)$ for a linear form l; take $l = u_1$. Then $\det \mathrm{Hess}\, f = -g'^2(P_{22}P_{ww} - P_{2w}^2)$, so $g' \in \mathbb{C}^\times$, $R = \alpha u_1 + \beta$, and $P(u_1, \cdot, \cdot)$ has constant nonzero Hessian determinant in (u_2, w) for each u_1. Now f_s gives u_1; HC_2 applied fibrewise (with descent) gives (u_2, w) from (f_{u_2}, f_w); finally $f_{u_1} = P_1 + \alpha s$ gives s. This is a polynomial inverse.

If $P_{ww} = 0$, then $P = P_0 + wP_1$, and $\det \mathrm{Hess}\, f = J(R, P_1)^2$, so (R, P_1) is a plane Keller map of degree at most 4, hence an automorphism by (Moh). Then (u_1, u_2) is determined by (f_s, f_w). The remaining equations are linear in (w, s) with matrix $[\nabla P_1\ \nabla R]$, whose determinant is $\pm J(R, P_1)$. □

Lemma 5. *Let $f = w\,g(y) + h(y)$ with $y \in \mathbb{C}^3$, $\deg g \leq 2$, $\deg f \leq 5$ and $\det \mathrm{Hess}\, f \in \mathbb{C}^\times$. Then ∇f is an automorphism.*

Proof. Write g with constant Hessian B and $\nabla g = By + \beta$. The coefficient of w^2 in $\det \operatorname{Hess} f$ is $-\nabla g^{\mathsf{T}} \operatorname{adj}(B) \nabla g$, so it vanishes. If $\operatorname{rk} B = 3$, its top part is $-\det B \cdot y^{\mathsf{T}} B y \neq 0$, which is impossible. If $\operatorname{rk} B = 2$, then $\operatorname{adj} B = \lambda v v^{\mathsf{T}}$ with $Bv = 0$, so $v \cdot \beta = 0$. If $\operatorname{rk} B \leq 1$, choose $v \in \ker B$ with $v \cdot \beta = 0$. In every case $\partial_v g = 0$, and Lemma 4 applies with $s = w$ and $R = g$. (This is also the branch $a = 0$ of [14, Prop. 6.1].) □

3 The top form

Let $\det \operatorname{Hess} f = c \neq 0$ and $\deg f = 5$ (degree ≤ 4 is [14]). The top degree layer gives $\det \operatorname{Hess} f_5 = 0$, so by (GN) f_5 depends on at most three linear forms. The ternary case is [12]. Two cases remain: the *binary* case $f_5 = \varphi(x_1, x_2)$ with $\det \operatorname{Hess} \varphi \not\equiv 0$, and the *unary* case $f_5 = x_1^5$ (a binary form with vanishing Hessian is a fifth power).

4 The binary case

Write $X = (x_1, x_2)$, $z = (x_3, s)$ with $s = x_4$, and $f = \varphi + f_4 + f_3 + f_2$.

Proposition 6. *After a linear change of coordinates, $f = P(x') + sR(x') + s^2 L(x') + e\, s^3$ with $x' = (x_1, x_2, x_3)$, L affine, the cubic part of R a binary form $B_3(X)$, the quintic part of P equal to φ, and $A_4(X, x_3)$ the quartic part of P. Moreover:*

1. *if $e \neq 0$, then $\partial_3^2 A_4 = 0$;*
2. *if $e = 0$, the coefficient of $s^2 x_3$ vanishes; if moreover L is not constant, then after normalising to $L = x_1$, $\partial_3^2 A_4 = 0$.*

Proof. The degree-10 layer of $\det \operatorname{Hess} f$ equals $\det \operatorname{Hess} \varphi \cdot \det \operatorname{Hess}_z(f_4)$, so $\det \operatorname{Hess}_z f_4 = 0$. By (dBvdE) over $K = \mathbb{C}(X)$, $f_4 = G(X, \mu(X) \cdot z) + (\text{linear in } z)$. Take μ primitive in $\mathbb{C}[X]^2$ (Gauss's lemma), homogeneous of degree d. A term $g_k(X)(\mu \cdot z)^k$ with $k \geq 2$ has degree $\deg g_k + k(d+1) = 4$, so either no such term exists, or $d = 0$, or $d = 1$, $k = 2$ and g_2 is constant. This gives two possibilities. Either μ is constant (B1: $f_4 = A_4(X, x_3) + sB_3(X)$), or μ is linear with independent entries (B2: $f_4 = c\,(x_1 x_3 + x_2 s)^2 + \ldots$, $c \neq 0$). In case B2, with weights $(1, 1, \frac{3}{2}, \frac{3}{2})$ the top part is $\varphi + cq^2$ with $q = x_1 x_3 + x_2 s$, and $\det \operatorname{Hess}(\varphi + cq^2) = -16c^3 q^2 (10\varphi - 3cq^2) \neq 0$ *[C, P1]*, which is impossible. In case B1 the s-degree of f is at most 3, which gives the stated form. With weights $(1, 1, 1, 2)$ the layer of weight 10 equals $\det \operatorname{Hess}_X(\varphi + sB_3) \cdot (6es\, \partial_3^2 A_4 + 12es^2 l_{11} - 4s^2 l_{10}^2)$ *[C, P2]*, where l_{10}, l_{11} are the coefficients of $s^2 x_3$ and $s x_3^2$. Its first factor equals $\det \operatorname{Hess} \varphi \neq 0$ at $s = 0$, which gives (1) and the first part of (2). The second part of (2) is the weight-9 layer *[C, P3]*. □

Notation. In R we write $m_0, m_1, m_2, m_3, m_4, m_5$ for the coefficients of $x_3^2, x_2 x_3, x_2^2, x_1 x_3, x_1 x_2, x_1^2$ and n_0, n_1, n_2 for those of x_3, x_2, x_1; Q is the quadratic part of R. In P we write c_3 for the coefficient of $x_1^3 x_3$, q_0, q_4 for those of $x_3^3, x_1 x_3^2$, and r_0, r_3, r_4 for those of $x_3^2, x_1 x_3, x_1 x_2$. The verifiers use the same names for all coefficients.

4.1 The branch $e \neq 0$

Normalise $e = 1$. The substitution $s \mapsto s - L/3$ gives $f = P + sR + s^3$ *[C, A1]*, with A_4 still linear in x_3. Expanding in s,

$$\det \operatorname{Hess} f = 6s \det(H_P + sH_R) - \nabla R^{\mathsf{T}} \operatorname{adj}(H_P + sH_R) \nabla R,$$

so the coefficients of $s^4, s^3, s^2, s, 1$ give five systems. We split by the type of B_3 under GL_2, then by normal forms of the quadratic part Q of R. The normal forms use only translations, $x_3 \mapsto \alpha x_3 + \lambda(X)$, $GL_2(X)$ preserving the current B_3, and scalings. The outcomes are listed in Table 1. All incoherence, implication and identity statements are certified (part B of `verify_hc4_binary.py`).

B_3	case	outcome
3 distinct roots	($m_2 = m_5 = 0$ by translation)	s^4, s^3: P linear in x_3; s^2: f independent of x_3; impossible
$x_1^2 x_2$	$x_1 x_3 \in Q$	incoherent
	$Q = x_2^2$	one family, $n_0^2 = -12$, explicit triangular inverse (below)
	$Q = 0$	incoherent
x_1^3	$\det H_Q = 0$, $m_1^2 = 4 m_0 m_2$ forced	
	$Q = x_3^2,\ x_2^2$	φ degenerate (unary)
	$Q = x_1 x_3,\ x_1 x_2,\ x_3^2 + x_1 x_2,\ x_2^2 + x_1 x_3$	incoherent
	$R = x_1^3 + x_2$	triangular inverse
	$R = x_1^3 + x_3$	$\varphi = x_1^5$ (unary; argument below)
	$R = x_1^3 + n_2 x_1$	incoherent
0	$\operatorname{rk} Q = 3$	excluded by s^4
	R without linear part	$\nabla R(0) = 0$: row s of Hess f vanishes at 0
	$R = x_1$	fibrewise HC_2 (below)
	$R = x_1^2 + x_2$	triangular inverse
	other normal forms	incoherent or φ degenerate

Table 1: Branch $e \neq 0$, $f = P + sR + s^3$.

The family for $B_3 = x_1^2 x_2$. Here $\omega^2 = -3$ and $n_0 = \pm 2\omega$. The coefficients obey thirteen linear relations, leaving seven free parameters *[C, B2b]*, and $\det \mathrm{Hess}\, f = 4\kappa^2$ with $\kappa = n_2 - \omega r_4 \neq 0$. In the coordinates $(x_1,\ z = x_2 + \omega s,\ x_3,\ s)$ one has

$$f = 2 m x_3 + G(x_1, z) + s\,(\kappa x_1 - m^2 - n_1 m), \qquad m = z + \tfrac{1}{2} x_1^2.$$

So $f_{x_3} = 2m$ and f_s determine m and x_1, hence z. The remaining two equations are affine in (x_3, s) with determinant 2κ *[C, B2b]*. The sign $n_0 = -2\omega$ is the Galois conjugate.

$R = x_1^3 + x_3$. The s^2 coefficient gives $\mathrm{adj}_{11}(H_P) = P_{22} P_{33} - P_{23}^2 = 0$, and the s coefficient gives $\det H_P = x_1 P_{22}$ *[C, B3]*. If $P_{22} \neq 0$, the identity $P_{22} \det H_P = -\mathrm{adj}_{13}^2$ (valid when $\mathrm{adj}_{11} = 0$, *[C, A2]*) gives $x_1 P_{22}^2 = -\mathrm{adj}_{13}^2$, which is impossible by comparing x_1-adic valuations. Hence $P_{22} = P_{23} = 0$, $P = a(x_1, x_3) + x_2 b(x_1)$, and $\det \mathrm{Hess}\, f = -(6s\, a_{33} - 1) b'^2$ *[C, P5]*. So b' is constant, the coefficient of $x_1^4 x_2$ vanishes, and φ is a fifth power.

$R = x_1$. Here $\det \mathrm{Hess}\, f = 6s \det H_P - \mathrm{adj}_{11}(H_P)$, so $\det H_P = 0$ and $P_{22} P_{33} - P_{23}^2 = -c$. The map $\Psi = (x_1, P_2, P_3)$ is, for each fixed x_1, the gradient of a plane potential with constant nonzero Hessian determinant. By HC_2 it is bijective on each fibre, so Ψ is bijective and, by (BBR), an automorphism. Since dx_1, dP_2, dP_3 are independent and $\det H_P = 0$, dP_1 lies in the span of dP_2, dP_3, so $P_1 \circ \Psi^{-1} = \Theta(v, w)$ does not depend on the first coordinate. Then $\nabla f = (P_1 + s, P_2, P_3, x_1 + 3s^2)$ is inverted by Θ, $s = Y_1 - \Theta(Y_2, Y_3)$, $x_1 = Y_4 - 3s^2$, and Ψ^{-1}.

4.2 The branch $e = 0$

If $L = a \neq 0$ is constant, then $f_{ss} = 2a$ and Lemma 2 applies.

L **non-constant.** Normalise $L = x_1$ by $GL_2(X)$ and a translation; then A_4 is linear in x_3 (Proposition 6). The steps below are taken in order; each assumes the vanishings established before it (part C of the verifier).

C1 If $m_0 \neq 0$: normalise $m_0 = 1$ and $m_1 = m_3 = 0$ (shift of x_3); then $m_2 = b_0 = b_1 = b_2 = 0$ and $p_0 = \cdots = p_4 = 0$, so φ is degenerate.

C2 So $m_0 = 0$, which forces $m_1 = 0$; the substitution $s \mapsto s + \lambda x_3$ removes sx_1x_3, so $m_3 = 0$. If $q_0 \neq 0$: normalise $q_0 = 1$, $q_1 = q_4 = r_0 = 0$ (shifts of x_3); then $\varphi = 0$.

C3 So $q_0 = 0$, which forces $q_1 = 0$. If $q_4 \neq 0$ (normalised to 1): incoherent.

C4 So $q_4 = 0$, which forces $c_0 = c_1 = c_2 = 0$. If $r_0 \neq 0$ (normalised to 1): φ is degenerate.

C5 So $r_0 = 0$; now P is linear in x_3 and f is quadratic in (x_3, s) with $f_{zz} = \begin{pmatrix} 0 & n_0 \\ n_0 & 2x_1 \end{pmatrix}$, and $c_3b_0 = c_3b_1 = 0$ is forced. If $n_0 \neq 0$, Lemma 3 applies. If $n_0 = 0$ and $c_3 \neq 0$: incoherent. If $n_0 = c_3 = 0$: on every component with φ non-degenerate, $f_{x_3} = r_3x_1$ with $r_3 \neq 0$, and R is affine in x_2 *[C, C5]*. Then $x_1 = Y_3/r_3$; the pair (f_{x_2}, f_s) is affine in (x_2, s), and its matrix has determinant equal to $-\det \mathrm{Hess}\, f/r_3^2$, a nonzero constant, so (x_2, s) follows; finally $f_{x_1} = r_3x_3 + (\dots)$ gives x_3.

$L = 0$**.** Here $f = P + sR$, with A_4 arbitrary. The s^2 coefficient $-\nabla R^{\mathsf{T}} \operatorname{adj}(H_R) \nabla R$ depends only on R.

- B_3 with distinct roots: it forces R to be independent of x_3 *[C, D1]*. $B_3 = x_1^2x_2$: it forces Q to be independent of x_3 *[C, D2]*, and after normalising, the remaining term n_0x_3 is incoherent *[C, P4]*. In both cases Lemma 4 applies.
- $B_3 = x_1^3$: it forces $m_1^2 = 4m_0m_2$, $m_0 = 0 \Rightarrow m_1 = 0$, and $m_0 = 1 \Rightarrow m_2 = 0$. All normal forms with x_3 in Q are incoherent, and $R = x_1^3 + Q(X) + x_3$ forces φ degenerate *[C, P4]*. Otherwise R is independent of x_3 (Lemma 4).
- $B_3 = 0$: it forces $\det Q = 0$ and $n^{\mathsf{T}} \operatorname{adj}(Q) n = 0$ *[C, D4]*, so $\ker Q$ meets $n^{\perp}$ and Lemma 4 applies ($R \equiv 0$ is impossible).

5 The unary case

Let $f = x_1^5 + f_4 + f_3 + f_2$, with the term x_1^4 removed by translating x_1. Write $u = (x_2, x_3, x_4)$ and $f_4 = A_4(u) + x_1A_3(u) + x_1^2A_2(u) + x_1^3A_1(u)$. The admissible coordinate changes are $u \mapsto Mu + \nu x_1 + \mu$ and translations of x_1.

The degree-9 layer gives $\det \mathrm{Hess}\, A_4 = 0$, so A_4 is a cone. When $A_4 = 0$, the same layers force $\det \mathrm{Hess}\, A_3 = 0$ *[C, W0]*; when moreover $A_3 = 0$ they force $\det A_2 = 0$ *[C, W3]*; when $A_2 = 0$ as well, they force $\det \mathrm{Hess}\, B_u = 0$ for the cubic u-part B_u of f *[C, W4]*, and, if $B_u = 0$, $\det Q = 0$ for the coefficient $Q(u)$ of x_1. By (GN) each of these forms is a cone. Each normal form $GL_2(x_3, x_4)$ used below is applied when all previously normalised (x_3, x_4)-dependent forms vanish, so it does not disturb them. The analysis then proceeds by normal forms, as summarised in Table 2. At each node, the perfect-square or perfect-cube conditions that justify the next normal form are certified to lie in the radical of the relevant layers. The verifiers `verify_hc4_unary_*.py` run a deterministic procedure: they solve linear equations and force variables to zero by Rabinowitsch tests on the equations with few unknowns. Every leaf is of one of three kinds.

(I) Incoherent.

(D) x_4 is a dead direction: all monomials with x_4-degree ≥ 2 other than x_4^2 vanish, and so do all monomials containing x_4 once whose total degree is at least 4. Then $f_{44} = 2c_0$ is constant, and the coefficient of x_4 has degree ≤ 2. If $c_0 \neq 0$, Lemma 2 applies; if $c_0 = 0$, Lemma 5 applies.

(Z) f is quadratic in $z = (x_3, x_4)$ with $f_{zz} = A_0 + x_1 A_1$ for constant symmetric matrices A_0, A_1, and the rank-2 normal form of A_1 is incoherent. In the remaining cases:

- $A_1 = \mathrm{diag}(0,1)$ and $(A_0)_{33} \neq 0$: f_{33} is constant and Lemma 2 applies.
- $A_1 = \mathrm{diag}(0,1)$ and $(A_0)_{33} = 0$: $\det f_{zz} = -(A_0)_{34}^2$, so Lemma 3 applies if $(A_0)_{34} \neq 0$; otherwise f is linear in x_3 with a coefficient depending on (x_1, x_2) only, and Lemma 4 applies.
- $A_1 = 0$: Lemma 3 applies if $\det A_0 \neq 0$, Lemma 2 if $\mathrm{rk}\, A_0 = 1$, and Lemma 4 if $A_0 = 0$.

node	leaves
A_4 binary, $\det \mathrm{Hess}_2 \neq 0$ (4 root types)	$A_1 \ni x_1^3 x_4$: (I); otherwise (D)
$A_4 = x_2^4$	normal forms of $x_1 C(x_3,x_4)$, $x_1 x_2 Q$, $x_1 x_2^2 l$, $x_1^2 Q$, $x_1^2 x_2 l$, the cubic in (x_3,x_4) and $x_2 Q(x_3,x_4)$: leaves (D), (I), one (Z)
$A_4 = 0$, A_3 binary	(I)/(D)
$A_4 = 0$, $A_3 = x_2^3$	as for x_2^4: (D), one (Z)
$A_4 = A_3 = 0$, $\mathrm{rk}\, A_2 = 2$	(I)/(D)
$A_4 = A_3 = 0$, $A_2 = x_2^2$	(D), (I), one (Z)
$A_4 = A_3 = A_2 = 0$, $A_1 = 0$	cone types of the cubic u-part B_u: (D); $B_u = 0$: f quadratic in u, rank of the x_1-coefficient Q: 2 (I); 1: f linear in (x_3,x_4), Lemma 4; 0 (D)
$A_4 = A_3 = A_2 = 0$, $A_1 = x_1^3 x_2$	B_u depends on ≤ 2 forms; if their span and x_2 span all u-forms: (I); otherwise coordinates with $B_u \in \mathbb{C}[x_2, x_3]$: x_4 dead, or (Z)

Table 2: The unary tree.

6 Certificates

The ancillary files run with `python3` and no arguments. They use the free Python packages `passagemath-singular`, `python-flint` and `sympy`. Each prints a line `[OK]` or `[ECHEC]` for every check.

file	checks	time
`verify_hc4_prelim.py`	19	12 s
`verify_hc4_binary.py`	148	91 s
`verify_hc4_unary_1.py`	12	8 s
`verify_hc4_unary_2.py`	21	8 s
`verify_hc4_unary_3.py`	47	12 s

The three unary verifiers load the common file `unary_core.py`. The ternary case is certified by the verifier of [12].

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