Irrational even beta values in shifted intervals

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Abstract

For every positive odd integer m, we prove that at least one of

$$\beta(m+1), \beta(m+3), \ldots, \beta(19m-1)$$

is irrational, where β is the Dirichlet beta function. More generally, for each positive integer B we give an explicit constant C_B such that at least B even beta values with indices between $m+1$ and $C_B m - 1$ are irrational, for every positive odd m. Specific parameter choices give at least two irrational values among $\beta(2), \beta(4), \ldots, \beta(70)$, at least three through $\beta(186)$, and at least four through $\beta(360)$.

1 Introduction

For $v > 1$, let

$$\beta(v) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^v} = 1 - \frac{1}{3^v} + \frac{1}{5^v} - \cdots$$

be the Dirichlet beta function. Its values at positive odd integers are nonzero rational multiples of powers of π. The arithmetic of its even values is much less understood. We study how many irrational even values must occur in an interval whose starting index is allowed to vary.

Theorem 1.1. *For every positive odd integer m, at least one of*

$$\beta(m+1), \beta(m+3), \ldots, \beta(19m-1)$$

is irrational.

An earlier version of this paper [10] proved Theorem 1.1 with 67 in place of 19. More generally, that paper gave a bound for any prescribed number B of irrational values. The two statements below compare that general bound with our improved one. In both statements, φ is Euler's totient function and log is the natural logarithm.

[10, Theorem 1.1]. *For every positive odd integer m and every integer $B \geq 1$, at least B of*

$$\beta(m+1), \beta(m+3), \ldots, \beta(C_B^{(1)} m - 1)$$

are irrational, where

$$D_B^{(1)} = \sum_{d=1}^{B} \varphi(d), \qquad C_B^{(1)} = 1 + 2\left\lceil D_B^{(1)} \left(33 + \frac{45}{2} \log B\right)\right\rceil.$$

Theorem 1.2. *For every positive odd integer m and every integer $B \geq 1$, at least B of*

$$\beta(m+1), \beta(m+3), \ldots, \beta(C_B m - 1)$$

are irrational, where

$$D_B = \sum_{\substack{1 \leq d \leq 2B-1 \\ d \text{ odd}}} \varphi(d), \qquad C_B = 1 + 2\left\lceil D_B \left(9 + \frac{40}{9} \log(2B-1)\right)\right\rceil. \tag{1.1}$$

Both bounds give an upper endpoint linear in the starting index for each count B. For $B = 1, 2, 3$, the multipliers decrease from $67, 197, 463$ to $19, 85, 229$, respectively. The new rational functions need zeros only on grids with odd numerators and denominators, and their poles are spaced by 2 instead of 4. These choices reduce the numerator degree and the cost of clearing denominators.

Choosing parameters for particular starting indices and counts gives the endpoints below. The uniform column uses Theorem 1.2, which gives $19m - 1$ when $B = 1$.

First even index	Last even index		# irrational values
	Specific parameters	Uniform bound	(at least)
4	54	56	1
6	92	94	1
8	128	132	1
20	352	360	1
20	1344	1614	2
20	3538	4350	3
100	1838	1880	1

The rational parameter choices are given in Section 8. Appendix B refines the uniform one-value bound and explains the size of its multiplier.

1.1 Comparison with earlier results

Let $I(s)$ count the irrational values among $\beta(2), \beta(4), \ldots, \beta(s)$. Our earlier paper [9] proves, for every even $s \geq 160$,

$$I(s) \geq \left\lfloor \pi \sqrt{\frac{2}{135}} \sqrt{\frac{s}{\log s}} \right\rfloor. \tag{1.2}$$

To obtain B irrational values starting at $\beta(m+1)$ from such a count, we must allow for all $(m-1)/2$ preceding values to be irrational. We therefore need a count of at least $(m-1)/2 + B$.

For fixed B, reaching this target with (1.2) requires an endpoint of order $m^2 \log m$: quadratic growth in the starting index, with an additional logarithmic factor. Theorem 1.2 gives linear growth.

For example, the earlier paper also proves the sharper count $I(3828) \geq 10$. At most nine of these irrational values can have indices below 20, so at least one of $\beta(20), \beta(22), \ldots, \beta(3828)$ is irrational. The present construction reduces this upper endpoint to 352. The table compares endpoints guaranteeing one irrational value from each listed starting index:

First even index	Endpoint from [9]	Endpoint from [10]	Endpoint here
4	160	198	54
6	364	330	92
20	3828	1272	352
100	187 340	6632	1838

The column from [9] follows from the bounds $I(160) \geq 2$, $I(364) \geq 3$, $I(3828) \geq 10$ and $I(187\,340) \geq 50$. Discarding the first 1, 2, 9 and 49 even beta values, respectively, leaves at least one irrational value in each interval.

The column from [10] uses the specific parameter choices in Section 8.2 of that paper at starting indices 4 and 6, and the bound $67m - 1$ at 20 and 100. The last column uses the individual parameter choices in Section 8.2.

Sorokin [8, Theorem S] treats shifted intervals directly. For every sufficiently large odd m and every even $q > 100m \log m$, he proves that at least one of

$$\beta(m+1), \beta(m+3), \ldots, \beta(m+q-1)$$

is irrational. Huttner [1, Remark 5] gives the same conclusion. Taking the smallest allowable q gives an upper endpoint of order $m \log m$. Theorem 1.1 removes this logarithmic factor and applies to every positive odd m.

At the fixed starting index 4, Krisman [2] gives a shorter interval: at least one of $\beta(4), \beta(6), \ldots, \beta(36)$ is irrational. Earlier finite results starting at $\beta(2)$ reached the endpoint 10 in Lai and Zhou [6, Theorem 6.1]; those one-value statements are now subsumed by the irrationality of $\beta(2)$ [7].

Taking $m = 1$ in the present construction also improves the finite counts in [9]:

# irrational values	Last even index, starting at 2		
(at least)	Endpoint from [9]	Endpoint from [10]	Endpoint here
2	160	196	70
3	364	462	186
4	588	772	360
5	1030	1386	554
10	3828	5428	2794

The column from [10] uses that paper's Theorem 1.1 with $m = 1$. The specific estimates in [9] give smaller endpoints. The last column uses the parameter choices in Section 8.2.

1.2 How the proof locates the irrational values

We construct B sequences of linear forms supported on the desired interval, whose even upper index we denote by s. The integer m fixes the starting point, and B fixes the count; the sequence index n tends to infinity. The forms have the shape

$$F_{n,b} = q_b(n) + \sum_{\substack{m+1 \leq \nu \leq s \\ \nu \text{ even}}} b^\nu \rho_\nu(n) \beta(\nu), \qquad b = 1, 3, \ldots, 2B-1.$$

Here $q_b(n)$ and $\rho_\nu(n)$ are rational. If at most $B-1$ of the beta values in the interval were irrational, a fixed integer combination of the B forms would cancel their coefficients at every n. The resulting sequence would be rational. We arrange that clearing its denominators gives integers tending to zero, with infinitely many nonzero terms. A nonzero integer has absolute value at least 1, so this is impossible.

To remove the values below $\beta(m+1)$, we differentiate $m-1$ times. For example, normalized second differentiation changes

$$\frac{c_i}{(t+2h)^i} \quad \text{into} \quad \binom{i+1}{2}\frac{c_i}{(t+2h)^{i+2}}.$$

Reflection removes odd indices, leaving 4 as the first even exponent. The zeros that control the rational denominators must survive this operation, so we start with triple zeros. In general, we pair $(m-1)$st differentiation with zeros of multiplicity m. Krisman [2, Sections 2–4] uses the second-derivative and triple-zero device in the beta setting and attributes the differentiation method to Rivoal.

Our poles are spaced by 2. On a grid j/b with j and b odd, shifting past the hth pole changes the alternating sign by $(-1)^{bh} = (-1)^h$. This sign is the same on every grid, so the beta coefficients retain their common row. Zeros are needed only on these odd grids. This reduces the numerator degree and arithmetic cost that determine the endpoints. Integer zeros and half-integer poles, equivalent after rescaling to this parity choice, also occur in [2].

The remaining task is to keep a nonzero rational constant after elimination. At the seed index $n = 1$, each choice of a polynomial multiplier produces a vector of B rational constants, one for each grid. We construct B such multipliers and put their vectors in the columns of a matrix. A Gram representation using independent polynomials and a positive weight proves that the matrix is nonsingular. Combining the multipliers therefore lets us prescribe a seed vector whose last coordinate alone is nonzero. Prime dilation then propagates that constant to infinitely many indices. Section 3.1 shows the choice explicitly for two grids.

The polynomial multipliers, grouped denominator argument and prime-dilation calculation adapt those of [9]. The rational-grid and coefficient-elimination framework appears in Lai and Yu [5, Sections 2 and 5], who credit the earlier elimination method of Fischler, Sprang and Zudilin, and is adapted to real Dirichlet characters by Kumar and Tosi [3, Section 2]. The auxiliary-prime nonvanishing principle is the Archimedean criterion of Lai and Sprang [4, Lemma 2.3]. We supply the arguments needed here.

Section 2 defines the functions and forms. Section 3 constructs the seed, and Sections 4–6 establish denominator clearing, decay and prime dilation. Section 7 gives the contradiction; Section 8 chooses the endpoints. We use elementary prime valuations, standard complex analysis, Stirling's formula and the prime number theorem.

2 Rational functions on odd grids

2.1 Zeros, poles and differentiation

Fix a positive odd integer m, a count $B \geq 1$ and an integer $r \geq 1$. These parameters remain fixed as the positive odd integer n tends to infinity. Put

$$k = m-1, \qquad M = 2B-1, \qquad L = 2r+1.$$

The bth form will be an alternating sum over the grid j/b with j odd, for $b = 1, 3, \ldots, M$. We need zeros on these grids both to remove initial summands and to control the denominators of the rational constants. We place the poles at $0, -2, \ldots, -2n$ and the grid zeros in

$(-(2r+2)n, 2rn)$. This interval is centered at $-n$ and has length $2Ln$; r controls how far it extends beyond the poles.

The reduced fraction classes modulo 2 are

$$\Theta = \left\{ \frac{v}{d} : 1 \le d \le M \text{ odd},\ 1 \le v < 2d \text{ odd},\ (v,d) = 1 \right\}.$$

There are $\varphi(d)$ classes for each d, including the single class 1 for $d = 1$. Thus $|\Theta| = D_B$; write $D = D_B$. The integer D is odd, since $\varphi(d)$ is even for odd $d > 1$. Each class contributes Ln points to the zero interval, spaced by 2. The polynomial P_n below therefore has DLn simple zeros. Define

$$A_d = (2d)^L \prod_{p \mid d} p^{\lceil L/(p-1) \rceil}, \qquad K = \prod_{\theta \in \Theta} A_{\mathrm{den}(\theta)}. \tag{2.1}$$

The product over primes supplies the factorial divisibility proved in Section 4.

Choose an even pole order $a > mDL$. Set

$$P_n(t) = \prod_{\theta \in \Theta} \prod_{j=0}^{Ln-1} (t - 2rn + 2j + \theta), \tag{2.2}$$

$$\kappa_n = 2^{an} K^{mn} (n!)^{a-mDL}, \qquad R_n(t) = \frac{\kappa_n P_n(t)^m}{\prod_{h=0}^{n} (t+2h)^a}. \tag{2.3}$$

The factorial in κ_n balances the growth of the products; the powers of 2 and K give the coefficient divisibility proved in Section 4. None of these grid zeros is a pole. The map $\theta \mapsto 2 - \theta$ preserves Θ, so the root set is invariant under $t \mapsto -2n - t$. Since DLn and m are odd and a is even,

$$P_n(-2n - t) = -P_n(t), \qquad R_n(-2n - t) = -R_n(t).$$

Multiplication by an integer polynomial will let us adjust the rational constants of the forms, particularly at the seed index $n = 1$, while retaining all grid zeros. We choose the polynomial even about $-n$ to preserve reflection. Section 3 constructs such a polynomial $W_n(t)$, homogeneous in n, t of total degree

$$E = m(D - 1). \tag{2.4}$$

Homogeneity will control its scaling at prime indices. Define

$$T_n(t) = W_n(t) R_n(t), \qquad V_n(t) = \frac{1}{k!} T_n^{(k)}(t). \tag{2.5}$$

Both functions are antisymmetric about $-n$, because k is even. Every grid zero has multiplicity at least $m = k + 1$ in T_n, so V_n still vanishes there. The degree of T_n at infinity is at most

$$-(a - mDL)n - a + E.$$

Since $a > mDL$ and $L \ge 3$, we have $a > E$; the functions are proper and the sums below converge absolutely.

2.2 The linear forms

Write the partial fractions before and after differentiation as

$$T_n(t) = \sum_{h=0}^{n} \sum_{i=1}^{a} \frac{c_{i,h}(n)}{(t+2h)^i},$$

$$V_n(t) = \sum_{h=0}^{n} \sum_{\nu=m}^{a+k} \frac{A_{\nu,h}(n)}{(t+2h)^\nu}. \tag{2.6}$$

Differentiation relates their coefficients by

$$A_{i+k,h}(n) = \binom{i+k-1}{k} c_{i,h}(n), \qquad 1 \le i \le a.$$

The coefficients $A_{\nu,h}$ are zero outside $m \le \nu \le a+k$. Put

$$\rho_\nu(n) = \sum_{h=0}^{n} (-1)^h A_{\nu,h}(n).$$

Antisymmetry gives $A_{\nu,n-h} = (-1)^{\nu+1} A_{\nu,h}$. Since n is odd, $\rho_\nu = (-1)^\nu \rho_\nu$, and every odd coefficient vanishes.

For odd integers j write $\chi_4(j) = (-1)^{(j-1)/2}$. For each odd $b \le M$, put

$$F_{n,b} = \sum_{\substack{j \ge 1 \\ j \text{ odd}}} \chi_4(j) V_n(j/b).$$

In the term from the pole $-2h$, shift the character index by $2bh$. Since b is odd, the sign changes by $(-1)^h$. The omitted initial terms give

$$F_{n,b} = q_b(n) + \sum_{\substack{m+1 \le \nu \le a+m-1 \\ \nu \text{ even}}} b^\nu \rho_\nu(n) \beta(\nu), \tag{2.7}$$

$$q_b(n) = -\sum_{h=0}^{n} (-1)^h \sum_{\nu=m}^{a+k} A_{\nu,h}(n) b^\nu \sum_{\substack{1 \le j < 2bh \\ j \text{ odd}}} \frac{\chi_4(j)}{j^\nu}. \tag{2.8}$$

At $m = 1$, the exponent-one partial-fraction series converge conditionally. Only finitely many series are combined, and their total $\beta(1)$ coefficient is zero; no infinite rearrangement is involved.

2.3 A rational criterion for the endpoints

The positive rational u below bounds the location of the largest summands, measured beyond $2rn$ in units of n. The quantity U bounds the square of the exponential rate after clearing denominators.

Proposition 2.1. *With the parameters above, let $e_* > \mathrm{e}$ be rational. Suppose a positive rational u satisfies*

$$f(u) = \left(\frac{u+2L}{u}\right)^{mD} \left(\frac{u+2r}{u+2r+2}\right)^{a} < 1, \tag{2.9}$$

$$U = e_*^{2(a+k)} K^{2m} (u+2L)^{2mDL} 2^{2a} \left(\frac{(2r)^{2r}}{(2r+2)^{2r+2}}\right)^{a} < 1. \tag{2.10}$$

Then at least B of $\beta(m+1), \beta(m+3), \ldots, \beta(a+m-1)$ are irrational.

The exponent $a+k$ pays for denominators after differentiation. To finish the criterion, we also need a seed constant that survives the elimination of the possible irrational values. We will choose one multiplier W_n shared by all the forms so that

$$(q_1(1), q_3(1), \ldots, q_M(1)) = \eta e_B, \qquad \eta \in \mathbb{Q} \setminus \{0\}. \tag{2.11}$$

Here $e_B = (0, \ldots, 0, 1)$ has B coordinates. The elimination can be performed with nonzero weight on the last form, so this seed vector retains a nonzero constant.

3 A Gram matrix for the seed constants

We construct B candidate multipliers. Each produces a vector of B seed constants, one for each odd grid. These vectors form the columns of a rational matrix $\mathcal{M}$. If $\mathcal{M}$ is nonsingular, a rational combination of its columns gives e_B; clearing denominators gives the seed vector (2.11) with integer coefficients on the multipliers. We adapt the Gram representation of [9, Section 3], first showing the choice with two grids.

3.1 A worked example with two grids

Take $B = 2$, $m = 1$, $r = 2$ and $a = 70$, the parameters that will give two irrational values through $\beta(70)$. There is no differentiation, since $k = 0$. The grids have $b = 1, 3$. We want $q_1(1) = 0$ and $q_3(1) = \eta \neq 0$. Indeed, for any possible irrational value $\beta(\nu)$, the fixed combination $3^\nu F_{n,1} - F_{n,3}$ cancels its coefficient at every n. Its seed constant is then $-\eta \neq 0$.

At $n = 1$, put $z = t + 1$. The two kernels are

$$H_1(t) = \frac{1}{z}, \qquad H_3(t) = \frac{1}{z - 2/3} - \frac{1}{z} + \frac{1}{z + 2/3}.$$

Expansion at $t = 0$ gives $q_b(1) = \mathrm{Res}_{t=0} T_1(t) H_b(t)$. We will choose independent polynomials whose pairwise products, integrated against one positive weight, give the seed matrix up to a common sign. The line $z = iy$ lies midway between the poles $t = 0, -2$ and turns every symmetric factor $z^2 - \alpha^2$ into $-(y^2 + \alpha^2)$, with fixed sign. Write $x = y^2$. On this line, $H_b(-1 + iy) = iy\, g_b(x)$, with

$$g_1(x) = -\frac{1}{x}, \qquad g_3(x) = \frac{1}{x} - \frac{2}{x + 4/9}.$$

Multiplying by their common denominator $x(9x + 4)$ gives two independent polynomials,

$$h_1(x) = -9x - 4, \qquad h_3(x) = 4 - 9x.$$

To obtain products $h_b h_j$ in the integral, choose multipliers whose values on this line reproduce the polynomials: $W_{1,j}(-1 + iy) = h_j(y^2)$. Substituting $-(t + 1)^2$ for x and homogenizing in n gives

$$W_{n,1}(t) = 9(t + n)^2 - 4n^2, \qquad W_{n,3}(t) = 9(t + n)^2 + 4n^2.$$

These are integer polynomials, even about $-n$ and homogeneous of degree 2.

Let $\mathcal{M}$ have rows indexed by the seed constants and columns indexed by these multipliers. Closing the midpoint contour clockwise gives $\mathcal{M} = -\mathcal{N}$, where

$$\mathcal{N}_{b,j} = \int_{-\infty}^{\infty} w(y) h_b(y^2) h_j(y^2)\, dy, \qquad b, j \in \{1, 3\},$$

and the positive weight is

$$w(y) = \frac{2^{85} 3^{16}}{18\pi} \frac{\displaystyle\prod_{\alpha \in \{4/3,\, 2,\, 8/3,\, 10/3,\, 4,\, 14/3\}} (y^2 + \alpha^2)}{(1 + y^2)^{70}}.$$

The numerator's central zero cancels the kernel pole at $z = 0$; its factor $z^2 - 4/9$ cancels the paired poles at $z = \pm 2/3$. On the midpoint line, P_1 is $-iy$ times a positive product, since it has seven symmetric root pairs. Multiplying by the kernel's iy factor removes that sign, and the clockwise contour supplies the minus sign in $\mathcal{M} = -\mathcal{N}$. Section 3.4 gives the general calculation.

Put $\mu_\ell = \int_{-\infty}^{\infty} w(y) y^{2\ell}\, dy$ for $\ell = 0, 1, 2$. The matrix is explicitly

$$\mathcal{N} = \begin{pmatrix} 16\mu_0 + 72\mu_1 + 81\mu_2 & -16\mu_0 + 81\mu_2 \\ -16\mu_0 + 81\mu_2 & 16\mu_0 - 72\mu_1 + 81\mu_2 \end{pmatrix}.$$

Its determinant is

$$\det \mathcal{N} = 5184(\mu_0\mu_2 - \mu_1^2) > 0.$$

The inequality is strict Cauchy–Schwarz: y^2 is not constant under the positive weight w. The rational combination

$$W_n = \mathcal{N}_{1,3} W_{n,1} - \mathcal{N}_{1,1} W_{n,3}$$

has seed vector $(0, \det \mathcal{N})^{\mathsf{T}}$. Its coefficients are rational because $\mathcal{M}$ is a matrix of rational residues. Clearing their fixed denominators gives an integer multiplier with seed vector $(0, \eta)$, $\eta \neq 0$. We now make the same choice with B independent polynomials.

3.2 Moving the derivatives onto the kernels

At $n = 1$, the poles are 0 and -2. For a proper rational function V with these poles and antisymmetric about -1, its partial-fraction coefficients satisfy $A_{\nu,1} = (-1)^{\nu+1} A_{\nu,0}$. For odd $b \le M$, define

$$H_b(t) = \sum_{\substack{1 \le j < 2b \\ j \text{ odd}}} \frac{\chi_4(j)}{t + j/b}.$$

Expansion at 0 and (2.8) give $q_b = \operatorname{Res}_{t=0} V(t) H_b(t)$. If $V = T^{(k)}/k!$, repeated integration by parts for residues gives

$$q_b = \operatorname{Res}_{t=0} T(t) \frac{H_b^{(k)}(t)}{k!}. \tag{3.1}$$

The sign is positive because k is even. Transferring the derivatives to the kernel leaves the numerator of T intact.

Put $c = (m+1)/2$ and $z = t + 1$. The central kernel term is $\chi_4(b)/z$. The terms at distances $\pm\alpha$, $\alpha = 2j/b$, have equal weights $\chi_4(b-2j) = \chi_4(b+2j)$. On $z = iy$ we therefore have

$$\frac{H_b^{(k)}(-1+iy)}{k!} = iy\, g_b(y^2),$$

where

$$g_b(x) = \frac{(-1)^c \chi_4(b)}{x^c} - 2 \sum_{j=1}^{(b-1)/2} \chi_4(b-2j) \frac{\displaystyle\sum_{\ell=0}^{c-1} (-1)^\ell \binom{m}{2\ell+1} \alpha^{m-2\ell-1} x^\ell}{(x+\alpha^2)^m}, \qquad \alpha = \frac{2j}{b}. \tag{3.2}$$

This follows by combining $(iy - \alpha)^{-m} + (iy + \alpha)^{-m}$. For $b = 1$ the sum is empty: $g_1 = -1/x$ when $m = 1$, and $g_1 = 1/x^2$ when $m = 3$.

3.3 Independent integer polynomials

Let $\mathcal{A}$ be the distinct positive distances in (3.2). In reduced form they are v/d with $d \le M$ odd, v even and $0 < v < d$. Thus

$$N = |\mathcal{A}| = \frac{D-1}{2}.$$

Use reduced numerators and denominators to define

$$Q(x) = \prod_{v/d \in \mathcal{A}} (d^2 x + v^2), \qquad h_b(x) = x^c Q(x)^m g_b(x), \qquad J = mN. \tag{3.3}$$

Each h_b is an integer polynomial of degree at most J. The central term contributes $(-1)^c \chi_4(b) Q^m$. For a paired term with $\alpha = v/d$, clearing $(x + \alpha^2)^m$ leaves the integer numerator coefficients

$$-2\chi_4(b-2j)(-1)^\ell \binom{m}{2\ell+1} v^{m-2\ell-1} d^{m+2\ell+1}.$$

After multiplication by x^c and the other factors of Q^m, its degree is at most $c + (c-1) + m(N-1) = mN$.

Lemma 3.1. *The polynomials $h_1, h_3, \ldots, h_M$ are linearly independent over $\mathbb{R}$.*

Proof. In a nonzero relation among the g_b, choose the largest b with nonzero coefficient. If $b > 1$, its pole at $x = -((b-1)/b)^2$ has exact order m and is absent from every earlier kernel: it is farther from 0 than any of their negative poles. The order is exact because the corresponding term of $H_b^{(k)}/k!$ has nonzero principal coefficient, and $x = -z^2$ has nonzero derivative there. If $b = 1$, then g_1 is nonzero. Thus no relation is possible. Multiplication by the common denominator preserves independence. □

Homogenize the polynomials to obtain

$$\begin{aligned} W_{n,b}(t) &= n^{2J} h_b\left(-\frac{(t+n)^2}{n^2}\right) \\ &= \sum_{j=0}^{J} h_{b,j} (-1)^j n^{2J-2j} (t+n)^{2j}. \end{aligned} \tag{3.4}$$

They have integer coefficients, are even about $-n$, and have total degree $E = 2J = m(D-1)$, as required in (2.4). In particular,

$$W_{p,b}(pt) = p^E W_{1,b}(t). \tag{3.5}$$

3.4 The common positive weight

At $n = 1$, the symmetric simple roots give

$$P_1(t) = z \prod_{\alpha \in \Lambda} (z^2 - \alpha^2), \qquad z = t + 1, \qquad |\Lambda| = \frac{DL-1}{2}, \qquad \mathcal{A} \subseteq \Lambda.$$

All α in Λ are positive. Let $\mathcal{M}_{b,j}$ be the bth seed constant associated with $W_{1,j}$. Every pole of $H_b^{(k)}/k!$ in $(-2, 0)$, including the central pole, has order at most m and is canceled by P_1^m. Thus the product in (3.1) has poles only at $0, -2$.

Close the upward line $t = -1 + iy$ by a large right semicircle, which encloses 0 clockwise. Since $H_b^{(k)}(t) = O(t^{-m})$, the integrand has degree at most $mDL + E - 2a - m < -1$ at infinity. The arc tends to zero. On the midpoint line, P_1^m contributes $i^m(-1)^{|\Lambda|} y^m$ times a positive product. The kernel contributes $iy\, h_b(y^2)/(y^{2c} Q(y^2)^m)$. Since $2c = m + 1$, the powers of y cancel. The clockwise orientation supplies the final minus sign, giving

$$\mathcal{M}_{b,j} = \frac{\sigma \kappa_1}{2\pi} \int_{-\infty}^{\infty} \omega(y) h_b(y^2) h_j(y^2)\, dy, \qquad \sigma = (-1)^{c+(DL-1)/2+1}, \tag{3.6}$$

where

$$\omega(y) = \frac{\prod_{\alpha\in\Lambda}(y^2+\alpha^2)^m}{(y^2+1)^a Q(y^2)^m}.$$

This weight is strictly positive for all real y. The central singularity has canceled, and the degree estimate proves absolute convergence. Lemma 3.1 therefore makes $\sigma\mathcal{M}$ positive definite: its quadratic form is the integral of the square of a nonzero polynomial against this weight.

The entries of $\mathcal{M}$ are rational by (3.1). Solve $\mathcal{M}\mathbf{c} = e_B$ over $\mathbb{Q}$ and clear denominators. This gives integer coefficients c_j and a nonzero rational η such that $\mathcal{M}\mathbf{c} = \eta e_B$. Set

$$W_n = \sum_{\substack{1\le j\le M\\ j \text{ odd}}} c_j W_{n,j}.$$

This proves (2.11). The coefficients remain fixed as n varies, so their sizes do not affect an exponential rate.

4 Clearing denominators

4.1 The progression estimate

We first clear the coefficients of a single numerator block. This supplies the divisibility needed when we expand at each pole. Put $d_n = \operatorname{lcm}(1,\dots,n)$ and $\delta_n = 2d_n$. For $\theta = v/d \in \Theta$ in lowest terms, consider

$$B_\theta(t) = \frac{A_d^n}{(n!)^L}\prod_{j=0}^{Ln-1}(t - 2rn + 2j + v/d).$$

Lemma 4.1. *For every integer v_0 and every $\ell \ge 0$,*

$$d_n^\ell [t^\ell] B_\theta(t + v_0) \in \mathbb{Z}.$$

Proof. Clear d from each factor using d^{Ln} in A_d^n. At an odd prime $q \nmid d$, the constants form a progression with step $2d$, a unit modulo q. Split the Ln factors into L blocks of n. At each level $q^e \le n$, at least $L\lfloor n/q^e\rfloor$ constants are divisible. Extracting degree ℓ replaces ℓ constants by variable terms, losing at most ℓ contributions at that level. The valuation is at least

$$Lv_q(n!) - \ell v_q(d_n).$$

Levels above n can only improve the bound. A term retaining a zero constant is zero and needs no estimate.

If $q \mid d$, the explicit factor $q^{n\lceil L/(q-1)\rceil}$ pays for $Lv_q(n!)$. At $q = 2$, the polynomial after clearing d has integer coefficients, and the factor 2^{Ln} pays for $Lv_2(n!)$. These bounds prove the claim. □

Products and integer powers preserve this coefficient rule by convolution.

4.2 The coefficient bounds

Proposition 4.2. *For every positive odd n,*

$$\delta_n^{a-i} c_{i,h}(n) \in \mathbb{Z}, \tag{4.1}$$

$$\delta_n^{a+k-\nu} A_{\nu,h}(n),\ \delta_n^{a+k-\nu}\rho_\nu(n) \in \mathbb{Z}, \tag{4.2}$$

$$\delta_n^{a+k} q_b(n) \in \mathbb{Z} \qquad (b = 1,3,\dots,M). \tag{4.3}$$

Proof. Write $B_n = K^{mn}P_n^m/(n!)^{mDL}$ and suppress the index n on the coefficients. Expanding at $-2h$ gives

$$c_{i,h} = \binom{n}{h}^a [t^{a-i}] \left\{ W_n(t-2h)B_n(t-2h) \prod_{\ell \neq h} \left(1 + \frac{t}{2(\ell-h)}\right)^{-a} \right\}. \tag{4.4}$$

The constant factors from the other poles have product $2^{an}(h!(n-h)!)^a$, since a is even. Their cancellation with κ_n gives the displayed prefactor. Each $2(\ell-h)$ divides δ_n. The progression lemma, the integer coefficients of W_n, and the binomial series prove (4.1). Formula (2.6) then gives (4.2), because $a+k-(i+k) = a-i$.

The finite sums defining q_b may contain prime powers not covered by δ_n, so individual summands need not be integral after scaling. We use the surviving zeros to clear them in groups, as in [9, Section 4]. Fix an odd $j < 2bh$ and put

$$t_0 = j/b - 2h, \qquad x_\ell = j/b + 2(\ell - h).$$

The point t_0 is a non-pole grid zero in $(-2n, 0)$. Thus $V_n(t_0) = 0$ gives

$$\sum_\nu \frac{A_{\nu,h}}{x_h^\nu} = -\sum_{\ell \neq h} \sum_\nu \frac{A_{\nu,\ell}}{x_\ell^\nu}. \tag{4.5}$$

At a prime q put $v = v_q(\delta_n)$. At most one x_ℓ has valuation greater than v: the difference of two such numbers would be the nonzero integer $2(\ell - \ell')$, which divides δ_n. This argument also applies when q divides the grid denominator b. From (4.2),

$$v_q\left(\frac{\delta_n^{a+k} A_{\nu,\ell}}{x_\ell^\nu}\right) \geq \nu(v - v_q(x_\ell)).$$

If $v_q(x_h) \leq v$, the left group in (4.5) is integral after scaling. Otherwise every group on the right is integral after scaling, so the left group is too. Apply this at every prime, and sum the groups with their signs in (2.8). This proves (4.3). □

The common multiplier is δ_n^{a+k}. Differentiation shifts an order-i coefficient to order $i+k$, while its denominator bound remains δ_n^{a-i}. This explains the additional exponent k.

5 Decay after clearing denominators

Since $\delta_n^{1/n} \to \mathrm{e}$, clearing denominators costs an exponential factor e^{a+k}. We bound the nth-root limsup of the forms below its reciprocal.

5.1 The exponential profile

Every grid summand through $2rn$ vanishes. Write a remaining point as $2rn + t$, where $t \geq 1/M$. The products give

$$P_n(2rn+t) = \prod_{\theta \in \Theta} 2^{Ln} \frac{\Gamma((t+\theta)/2 + Ln)}{\Gamma((t+\theta)/2)},$$

$$\prod_{h=0}^{n} (2rn + t + 2h) = 2^{n+1} \frac{\Gamma(((2r+2)n + t)/2 + 1)}{\Gamma((2rn+t)/2)}.$$

For $x = t/n$, Stirling's formula gives the profile

$$\begin{aligned}\Phi(x) = m\log K + \frac{mD}{2}\big((x+2L)\log(x+2L) - x\log x\big) \\ - \frac{a}{2}\big((x+2r+2)\log(x+2r+2) - (x+2r)\log(x+2r)\big) + a\log 2.\end{aligned} \tag{5.1}$$

Here $0\log 0 = 0$. Since $(a - mDL) + mDL - a = 0$, both the $n\log n$ and linear Stirling terms cancel.

For each fixed $A > 0$, uniformly on $1/M \le t \le An$,

$$\log R_n(2rn + t) = n\Phi(t/n) + O_A(\log n). \tag{5.2}$$

Only the gamma factors in the denominator of P_n can have bounded arguments. Uniformly for $t \ge 1/M$ and θ in the fixed finite set Θ,

$$\log\Gamma((t+\theta)/2) = \frac{t}{2}\log(t/2) - \frac{t}{2} + O(\log(2+t)).$$

For large t this is Stirling's formula; on the remaining compact range it follows from continuity. The other gamma arguments grow proportionally to n. This proves (5.2) even when $t/n \to 0$.

5.2 Differentiation and the infinite tail

Use Cauchy's estimate on the circle of radius $\varepsilon = 1/(2M)$ around $2rn + t$. The numerator factors are positive at the real center. If a factor has value c there, its absolute-value ratio on the circle is at most $1 + \varepsilon/c$. Using $1 + x \le \exp(x)$ and multiplying these bounds gives a cost $\exp(O(\log n))$, since the sum of the reciprocals is $O(\log n)$. The inverse denominator ratios cost $\exp(O(1))$, since the factors are at least $2rn + t$ and their reciprocal sum is $O(1)$. Hence, on the bounded scaled range,

$$\sup_{|z|=\varepsilon} |R_n(2rn + t + z)| \le n^C R_n(2rn + t)$$

for a fixed C and sufficiently large n. Homogeneity gives $|W_n(2rn + t + z)| = O_A(n^E)$. Cauchy's estimate

$$|V_n(2rn + t)| \le \varepsilon^{-k} \sup_{|z|=\varepsilon} |T_n(2rn + t + z)|$$

therefore shows that multiplication and differentiation cost only a power of n. There are $O_A(n)$ terms in the bounded range, so their absolute sum has nth-root limsup at most $\exp(\max\Phi)$. Here m, B, r, a remain fixed.

For $t \ge An$ with A sufficiently large, every numerator factor on the circle is at most $3t$, each denominator factor is at least t, and $|W_n| \le C' t^E$. Thus

$$|V_n(2rn + t)| \le C''\kappa_n 3^{mDLn} t^{-(a-mDL)n-a+E}.$$

Put $q_n = (a - mDL)n + a - E > 1$. The odd summation grid has spacing $2/b$. Including the intervening grid points gives an upper bound on a grid of spacing $1/b$. A first term and an integral then give

$$\sum_{\substack{t \text{ on the odd grid} \\ t \ge An}} t^{-q_n} \le (An)^{-q_n} + b\frac{(An)^{1-q_n}}{q_n - 1}.$$

Stirling's formula gives the tail nth-root bound

$$2^a K^m 3^{mDL} \mathrm{e}^{-(a-mDL)} A^{-(a-mDL)}.$$

The factor n^{a-mDL} from the factorials cancels the factor $n^{-(a-mDL)}$ from the lower tail endpoint An. The bound tends to zero as A grows because $a > mDL$. Together with the bounded range this proves

$$\limsup_{\substack{n\to\infty \\ n \text{ odd}}} |F_{n,b}|^{1/n} \le G := \exp\left(\max_{x\ge 0} \Phi(x)\right). \tag{5.3}$$

5.3 The criterion implies cleared decay

The function f in (2.9) satisfies $2\Phi'(x) = \log f(x)$. After multiplication by a positive denominator, $(\log f)'$ has numerator

$$2\big((a - mDL)x^2 + 2L(a - mDL)x - mDL(2r)(2r + 2)\big).$$

This quadratic has one positive zero. Thus $\log f$ first decreases and then increases. It starts at $+\infty$ and tends to 0 from below, with leading term $-2(a-mDL)/x$. It has one zero $x_0 > 0$, where Φ has its global maximum. Substituting $\Phi'(x_0) = 0$ into (5.1) yields

$$G = K^m(x_0 + 2L)^{mDL}2^a \left(\frac{(x_0 + 2r)^{2r}}{(x_0 + 2r + 2)^{2r+2}}\right)^{a/2}. \tag{5.4}$$

If $f(u) < 1$, then $x_0 < u$. The ratio in parentheses decreases for $x \ge 0$, since its logarithmic derivative is $-2x/((x + 2r)(x + 2r + 2))$. Using $e_* > \mathrm{e}$ in (5.4) therefore gives

$$(\mathrm{e}^{a+k}G)^2 < U < 1.$$

The prime number theorem and (5.3) imply

$$\limsup_{\substack{n\to\infty \\ n \text{ odd}}} |\delta_n^{a+k} F_{n,b}|^{1/n} < 1. \tag{5.5}$$

Every fixed integer combination of the cleared forms also tends exponentially to zero.

6 Prime dilation and nonvanishing

The forms now tend to zero after denominator clearing. We must ensure that the final rational combination is nonzero infinitely often. At prime indices, its constant coefficient will survive a scaling modulo the prime while all beta coefficients vanish. This is the principle of Lai and Sprang's auxiliary-prime criterion [4, Lemma 2.3]; we adapt the dilation calculation of [9, Section 6] to the present odd grids.

Put

$$\tau_0 = a - E > 0, \qquad \tau = \tau_0 + k > k.$$

These exponents describe scaling at prime indices; the common denominator exponent remains $a + k$.

Proposition 6.1. *For every sufficiently large odd prime p,*

$$p^\tau \rho_\nu(p) \in p^\nu \mathbb{Z}_{(p)}, \tag{6.1}$$

$$p^\tau q_b(p) \equiv \chi_4(p) q_b(1) \pmod{p} \qquad (b = 1, 3, \dots, M). \tag{6.2}$$

Here $\mathbb{Z}_{(p)}$ consists of rationals with denominator prime to p.

6.1 The rational-function congruence

The factors divisible by p will reproduce the seed at $n = 1$ after rescaling; the remaining factors contribute units modulo p. For a single grid, $B = 1$ and $\Theta = \{1\}$, so

$$P_p(pt) = \prod_{j=0}^{Lp-1} \big(p(t-2r) + 2j + 1\big).$$

For an odd prime p, the selected indices are exactly $j = (p-1)/2 + \ell p$, $0 \le \ell < L$. At these indices,

$$p(t-2r) + 2j + 1 = p(t - 2r + 2\ell + 1),$$

so their product is $p^L P_1(t)$. With several grids, division by p can change the fraction class. The set Θ contains all the classes needed to keep the same mechanism valid.

Now take $p > M$ and larger than the fixed denominator primes. In a block of $P_p(pt)$ with class v/d, a factor is selected by p when $2dj + v \equiv 0 \pmod p$. After division by p, its new class v'/d satisfies

$$pv' \equiv v \pmod{2d}.$$

Multiplication by p permutes the odd reduced classes modulo $2d$. Each block has L selected factors, and together they give $p^{DL}P_1(t)$. The unselected factors in each block cover every nonzero residue L times. Wilson's theorem therefore gives their total unit as $(-1)^{DL} = -1$.

The normalization κ_p has valuation $a - mDL$. Using Wilson's theorem for $(p-1)!$ and Fermat's theorem for the fixed factors, its unit is

$$\kappa_p / p^{a-mDL} \equiv (-1)^{a-mDL}\kappa_1 = -\kappa_1 \pmod p.$$

Raising the numerator to odd m cancels the two minus signs. The selected denominator indices are $h = 0, p$, contributing $p^{2a}[t(t+2)]^a$; the remaining denominator unit is $(-1)^a = 1$. The net valuation is $-a$, so

$$p^a R_p(pt) \equiv R_1(t), \qquad p^{a-E}T_p(pt) \equiv T_1(t) \pmod p. \tag{6.3}$$

The second congruence uses $W_p(pt) = p^E W_1(t)$.

6.2 Local coefficient extraction

To extract coefficients, reduction must commute with the local series expansion. At a selected pole put $t = -2y + z$, where $y = 0, 1$. Removing the pole gives

$$z^a p^{a-E} T_p(p(-2y+z)) \in \mathbb{Z}_{(p)}[[z]].$$

After the displayed powers are canceled, every remaining denominator factor has a unit constant term. Its inverse has integral series coefficients in $\mathbb{Z}_{(p)}$, so reduction commutes with Taylor extraction.

The order-i term from the selected pole is

$$\frac{c_{i,py}(p)}{(p(-2y+z)+2py)^i} = p^{-i}c_{i,py}(p)z^{-i}.$$

After multiplication by $z^a p^{a-E}$, its coefficient of z^{a-i} is $p^{a-E-i}c_{i,py}(p)$. Terms from other poles are analytic at $z = 0$; after multiplication by z^a they start at degree a and cannot contribute, since $i \ge 1$. Comparing coefficients in (6.3) gives

$$p^{a-E-i}c_{i,py}(p) \equiv c_{i,y}(1) \pmod p.$$

Now multiply both sides by the integer $\binom{i+k-1}{k}$. With $\nu = i + k$ and $\tau = a - E + k$, we have $a - E - i = \tau - \nu$. Thus

$$p^{\tau-\nu} A_{\nu,py}(p) \equiv A_{\nu,y}(1) \pmod{p}. \tag{6.4}$$

At a nonselected pole $-2h$, $1 \le h \le p-1$, all other denominator factors are units. The numerator contains mDL constant factors divisible by p, so its degree-j coefficient has valuation at least $mDL - j$. Multiplication by κ_p adds $a - mDL$. The remaining denominator series has integral coefficients, so the order-i coefficient of R_p has valuation at least i. Multiplication by the integer polynomial W_p preserves this bound: its degree-j Taylor coefficient multiplies an order-$(i+j)$ coefficient. Differentiation gives

$$v_p(A_{\nu,h}(p)) \ge \nu - k \qquad (1 \le h \le p-1). \tag{6.5}$$

The estimate for a numerator coefficient remains valid when $j > mDL$, when its lower bound is negative. Equations (6.4)–(6.5) imply (6.1).

6.3 The surviving rational constant

Multiply (2.8) at $n = p$ by p^τ and take $p > 2M$. Every finite-sum denominator $j < 2bp$ is less than p^2. At nonselected poles, even the terms with $p \mid j$ have valuation at least $\tau - k > 0$. At selected poles, (6.4) gives valuation at least $\nu > 0$ whenever $p \nmid j$.

Only $h = p$ and $j = pj'$ survive, with $1 \le j' < 2b$ odd. The pole $h = 0$ has an empty finite sum. Since p is odd, $(-1)^p = -1$ and $\chi_4(pj') = \chi_4(p)\chi_4(j')$. Using (6.4), the surviving sum is $\chi_4(p)$ times the seed constant formula. This proves (6.2) and Proposition 6.1.

7 The irrationality contradiction

We prove Proposition 2.1. Suppose that at most $B-1$ of the beta values in (2.7) are irrational. Choose $B-1$ distinct even indices $\nu_1 < \cdots < \nu_{B-1}$ containing all possible irrational indices, adding other indices if necessary. There are enough choices because $a > mDL \ge 3B$.

The coefficients c_j have already fixed the common multiplier W_n and its seed vector. We now choose weights on the B forms to cancel the possible irrational values. Following the coefficient-elimination framework of [5, Section 5], choose fixed integer weights w_b by the signed maximal minors of the matrix whose columns are

$$(1^{\nu_j}, 3^{\nu_j}, \dots, M^{\nu_j})^{\mathsf{T}}, \qquad 1 \le j \le B-1.$$

They satisfy

$$\sum_{\substack{1 \le b \le M \\ b \text{ odd}}} w_b b^{\nu_j} = 0, \qquad w_M \ne 0.$$

To see that the last minor is nonzero, regard it as a generalized Vandermonde determinant at $1^2, 3^2, \dots, (M-2)^2$ with exponents $\nu_j/2$. A vanishing determinant would give a nonzero polynomial with at most $B-1$ monomials and $B-1$ distinct positive roots. A polynomial with q monomials has at most $q-1$ such roots: factor out its smallest power and apply Rolle's theorem inductively to its derivative. This proves nonsingularity. For $B = 1$, take $w_1 = 1$.

Set

$$H_n = \sum_{\substack{1 \le b \le M \\ b \text{ odd}}} w_b F_{n,b}, \qquad q_w(n) = \sum_{\substack{1 \le b \le M \\ b \text{ odd}}} w_b q_b(n).$$

The possible irrational coefficients cancel at every n, so H_n is rational under the assumption. The seed choice (2.11) gives $q_w(1) = w_M\eta \neq 0$. Let D_0 be a positive common denominator of the remaining rational beta values. Proposition 4.2 and (5.5) give

$$D_0\delta_n^{a+k}H_n \in \mathbb{Z}, \qquad D_0\delta_n^{a+k}H_n \longrightarrow 0 \quad (n \to \infty,\ n \text{ odd}). \tag{7.1}$$

Exclude the finitely many primes dividing D_0 or the numerator or denominator of $q_w(1)$. By Proposition 6.1, for every remaining sufficiently large prime,

$$p^\tau q_w(p) \equiv \chi_4(p)q_w(1) \not\equiv 0 \pmod{p}.$$

Every remaining beta term of $p^\tau H_p$ is divisible by p in $\mathbb{Z}_{(p)}$, by (6.1) and the assumed rationality. Hence $H_p \neq 0$ for infinitely many primes. Equation (7.1) gives infinitely many nonzero integers tending to zero, a contradiction.

8 Choosing the endpoints

8.1 A uniform choice for every count

For fixed B, r, u and e_*, the numerator contribution to $\frac{1}{2}\log U$ in (2.10) is proportional to m. Differentiation adds the denominator cost $(m-1)\log e_*$. Put $a_B = C_B - 1$. We take the pole order $a = ma_B$ to offset both costs. Its endpoint is $a + m - 1 = m(a_B + 1) - 1 = C_B m - 1$, linear in m. The following estimates show that this scale suffices.

Take $r = 2$, $L = 5$, $u = 1/100$, $e_0 = 68/25$ and $a = ma_B$. Every denominator d is odd, so its prime divisors are at least 3 and $\lceil 5/(p-1)\rceil \leq 3$. Consequently

$$A_d = (2d)^5 \prod_{p\mid d} p^{\lceil 5/(p-1)\rceil} \leq 2^5 d^8, \qquad K \leq 2^{5D}M^{8D}. \tag{8.1}$$

We have $a_B \geq 18D > 5D$. The exact rational inequality

$$1001\left(\frac{401}{601}\right)^{18} < 1 \tag{8.2}$$

therefore gives

$$f(u) \leq \left[1001\left(\frac{401}{601}\right)^{18}\right]^{mD} < 1.$$

Put $A = \log(27/(4e_0))$. The exponential series gives $163/60 < \mathrm{e} < 68/25$. The following exact comparisons control the numerator growth and the decay gained from a:

$$\frac{1001}{50} < \left(\frac{163}{60}\right)^3, \qquad \log(20.02) < 3, \tag{8.3}$$

$$4^{10}e_0^{19} < 27^{10}, \qquad A > \frac{9}{10}, \tag{8.4}$$

$$e_0 < \sum_{j=0}^{5} \frac{(11/10)^j}{j!}, \qquad \log e_0 < \frac{11}{10}. \tag{8.5}$$

For (8.4), the rational comparison implies $10A > 9\log e_0 > 9$.

Using (2.10) and (8.1),

$$\begin{aligned}\frac{1}{2}\log U &= m(\log K + 5D\log(10.01) - Aa_B) + (m-1)\log e_0 \\ &< m\left(15D + 8D\log M - \frac{9}{10}a_B\right) + \frac{11}{10}(m-1).\end{aligned}$$

Definition (1.1) gives $a_B \geq 18D + (80/9)D \log M$. The coefficient $80/9$ cancels the logarithmic cost because $(9/10)(80/9) = 8$. The constant 18 leaves a margin of $(6/5)mD$, since $(9/10)18 - 15 = 6/5$. Because $D \geq 1$, this absorbs the derivative cost $(11/10)(m-1)$. Indeed,

$$\frac{1}{2}\log U < -\frac{6}{5}mD + \frac{11}{10}(m-1) \leq -\frac{m}{10} - \frac{11}{10} < 0.$$

These choices give the coefficients 9 and $40/9$ in (1.1); the factor 2 ensures that $a_B = C_B - 1$ is even. Proposition 2.1 proves Theorem 1.2. At $B = 1$, we have $D_1 = 1$, $a_1 = 18$ and $C_1 = 19$, giving Theorem 1.1.

8.2 Sharper individual intervals

For the individual intervals below, use the tighter rational upper bound

$$e_* = \sum_{j=0}^{20} \frac{1}{j!} + \frac{1}{20 \cdot 20!} > \mathrm{e}. \tag{8.6}$$

The tail bound follows by replacing all subsequent factorial ratios by a geometric series.

The following choices satisfy $a > mDL$ and the two rational inequalities in Proposition 2.1. All rows use $r = 2$, the bound (8.6), and $u = t/10^{12}$ with the integer t shown. The endpoint is $s = a + m - 1$.

m	B	a	s	t	U (approx.)
1	2	70	70	779643543	0.09965
1	3	186	186	209514585	0.03463
1	4	360	360	132973689	0.96654
1	5	554	554	73388877	0.16375
1	10	2794	2794	11811809	0.26283
3	1	52	54	8990232707	0.49983
5	1	88	92	8058760219	0.10513
7	1	122	128	8645692609	0.84652
19	1	334	352	8128616097	0.10743
19	2	1326	1344	802179083	0.58430
19	3	3520	3538	218654559	0.08821
99	1	1740	1838	8139253515	0.07226

The decimals illustrate the margins; all comparisons are exact over $\mathbb{Q}$. These rows prove the individual intervals in the introduction. For example, $m = 19$, $B = 2$ gives two irrational values among $\beta(20), \beta(22), \ldots, \beta(1344)$. The accompanying endpoint file contains 21 certificates in total, including additional choices for counts 4, 5 and 10.

A Exact checks and audit scope

A.1 The identities and endpoint certificates

The finite checks compare the proof's formulas with separate exact calculations. They supplement the arguments for arbitrary parameters; the proof does not extrapolate from the checked cases.

For example, Gram entries can be calculated either from partial fractions or from moments of the positive weight. With each $\alpha = v/d$ reduced, put

$$\Delta = \prod_{v/d \in \Lambda} d^2, \qquad P_*(x) = \prod_{v/d \in \Lambda \setminus \mathcal{A}} (d^2 x + v^2).$$

Then

$$\frac{\prod_{\alpha\in\Lambda}(x+\alpha^2)}{Q(x)} = \frac{P_*(x)}{\Delta}.$$

The elementary beta integral gives, for $0 \le j < a$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{y^{2j}}{(1+y^2)^a}\,dy = \frac{(2j)!(2a-2j-2)!}{2^{2a-1}j!(a-j-1)!(a-1)!}. \tag{A.1}$$

Thus $\mathcal{M}_{b,j}/\kappa_1$ is σ/Δ^m times the moment functional in (A.1) applied to $P_*^m h_b h_j$. Its degree is at most $m(D(L+1)/2-1) < a$, so every required moment converges.

The structural checker uses nine seed cases with $m \in \{1,3,5\}$, $B \in \{1,2,3,4\}$ and $r \in \{1,2\}$, and nine prime cases with $3 \le p \le 17$. The complete parameter lists are in the accompanying results file. These checks use $a = mDL+1$ to keep the arithmetic small; decay is checked separately at the claimed endpoints.

Exact check	Number passed
Differentiated-kernel evaluations	57
Gram entries, residues versus moments	49
Positive leading principal minors	19
Integer seed-vector identities	9
Rational-function reconstructions	54
Reflection identities	1308
Differentiated grid zeros	332
Grouped integrality	4516
Selected-pole congruences	404
Other-pole valuation bounds	1940
Constant congruences	19
Beta-coefficient valuation bounds	202
Total	8909

Run `checks/checks.py` for these identities. Run `checks/endpoints.py` for the 21 individual interval certificates, the six uniform parameter choices recorded in its output, and the rational comparisons in Section 8 and Appendix B. Floating point locates rational trial values of u; the inequalities are then checked exactly over $\mathbb{Q}$. The separate program `checks/worked_example.py` checks the two-grid example in Section 3.1. All three programs use only the Python standard library and should be run without `-O`, since their assertions perform the checks.

A.2 Independent audit

An independent mathematical audit reported on 6 November 2026 found no substantive gap in the underlying plain-text proof, identified by its hash in the accompanying audit record. The audit covered the general Gram construction, denominator clearing, decay, prime dilation, coefficient elimination and uniform endpoints. It independently reproduced all 8909 structural checks, all 21 endpoint certificates, the six uniform parameter records and the seven integer-r comparisons in Appendix B, without running or importing the supplied programs.

The audit concerns the underlying proof and its certificates. The subsequent typeset exposition, including the worked example, was checked separately. The audit was not formal verification.

B The one-value endpoint multiplier

The endpoint $19m-1$ can be sharpened slightly. We first give an explicit refinement and then compare its multiplier with the transition of the decay estimate.

B.1 A sharper uniform bound

Proposition B.1. *For every positive odd integer m, put*

$$s(m)=m-1+2\left\lceil\frac{879m-50}{100}\right\rceil<18.58m. \tag{B.1}$$

At least one of $\beta(m+1),\beta(m+3),\dots,\beta(s(m))$ is irrational.

Proof. For $B=1$, put

$$a=2\left\lceil\frac{879m-50}{100}\right\rceil.$$

Then $a\ge(879/50)m-1$ and $a/m\ge52/3$. Indeed, $m=1$ gives $a=18$, $m=3$ gives $a=52$, and $m\ge5$ gives $a/m\ge879/50-1/5>52/3$. With $r=2$ and $u=1/100$, the comparison

$$1001^3\left(\frac{401}{601}\right)^{52}<1 \tag{B.2}$$

implies $f(u)<1$ for every m.

With e_* as in (8.6), put $A_*=\log(27/(4e_*))$ and $C=5\log(20.02)$. Then

$$\begin{aligned}\frac12\log U&=mC-A_*a+(m-1)\log e_*\\&\le m\left(C+\log e_*-\frac{879}{50}A_*\right)+A_*-\log e_*.\end{aligned}$$

Both terms on the last line are negative. For the first, take logarithms of the exact rational comparison

$$e_*^{929}2^{250}\left(\frac{1001}{100}\right)^{250}\left(\frac{4}{27}\right)^{879}<1. \tag{B.3}$$

For the second, use $27/4<(163/60)^2<e_*^2$. The criterion gives the stated interval. Finally, $\lceil x\rceil<x+1$ gives $s(m)<18.58m$. □

B.2 The limiting multiplier

The multiplier 18.58 comes from a rational estimate close to the transition of the decay bound. The limiting profile also explains the choice $r=2$.

For $B=1$, write $\mu=a/m$ and $y=x/2$. Since $D=1$, $K=2^L$ and $k=m-1$, the profile in (5.1) satisfies the exact identity

$$a+k+\Phi(2y)=m\Psi_r(\mu,y)-1, \tag{B.4}$$

where

$$\begin{aligned}\Psi_r(\mu,y)&=\mu+1+2L\log2+(y+L)\log(y+L)-y\log y\\&\quad-\mu\big((y+r+1)\log(y+r+1)-(y+r)\log(y+r)\big),\qquad L=2r+1.\end{aligned} \tag{B.5}$$

The constant $+1$ in Ψ_r comes from the additional denominator cost of $m-1$ derivatives; the final -1 in (B.4) accounts for the difference between $m-1$ and m. Taking the maximum gives $\log(\mathrm{e}^{a+k}G)=m\max_{y\ge0}\Psi_r(\mu,y)-1$. After division by m, the correction is $-1/m$ and

vanishes in the limit. Decay in the limiting profile corresponds to $\max_{y\geq 0}\Psi_r(\mu,y)<0$. For $r=2$, numerical solution of $\Psi_r=0$ and $\partial\Psi_r/\partial y=0$ gives

$$\mu\approx 17.5722528999,\qquad y\approx 0.004075478609.$$

Since the endpoint is $a+m-1$, the corresponding limiting multiplier is approximately 18.5722528999. These decimals locate the transition; Proposition B.1 proves the uniform bound 18.58.

The choice $r=2$ can also be justified against every other integer r within these estimates. For $r=1,3,4,5,6,7,8$, the exact comparisons

$$\left(\frac{163}{60}\right)^{18}2^{2L}L^{L}\left(\frac{r^{r}}{(r+1)^{r+1}}\right)^{18}>1 \tag{B.6}$$

give $\Psi_r(18,0)>1$, since $163/60<\mathrm{e}$. Thus, at $a=18m$, the logarithmic bound already satisfies

$$m\Psi_r(18,0)-1>m-1\geq 0.$$

It cannot give decay, even with the finite-m saving of 1. It also fails for smaller a, because the coefficient of a in that profile is $1+\log(r^r/(r+1)^{r+1})<0$. For $r\geq 9$, the degree condition $a>m(2r+1)\geq 19m$ itself excludes $a\leq 18m$.

Thus $r=2$ is the only integer choice that can reach this scale with the present factorial normalization and absolute-value decay estimate. The program `checks/optimization.py` reproduces the numerical transition; `checks/endpoints.py` proves (B.6) over $\mathbb{Q}$.

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