Irrational even beta values in shifted intervals

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Abstract

For every positive odd integer m, we prove that at least one of

$$\beta(m+1), \beta(m+3), \ldots, \beta(67m-1)$$

is irrational, where β is the Dirichlet beta function. More generally, for each positive integer B we give an explicit constant C_B such that at least B even beta values with indices between $m+1$ and $C_B m - 1$ are irrational, for every positive odd m. We also give sharper endpoints for particular intervals.

1 Introduction

For $v > 1$, let

$$\beta(v) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^v} = 1 - \frac{1}{3^v} + \frac{1}{5^v} - \cdots$$

be the Dirichlet beta function. Its values at positive odd integers are nonzero rational multiples of powers of π. The arithmetic of its even values is much less understood. We study how many irrational even values must occur in an interval whose starting index is allowed to vary.

For an integer $B \geq 1$, put

$$D_B = \sum_{d=1}^{B} \varphi(d), \qquad s_B = 2\left\lceil D_B\left(33 + \frac{45}{2}\log B\right)\right\rceil, \tag{1.1}$$

where φ is Euler's totient function and log is the natural logarithm.

Theorem 1.1. *For every positive odd integer m and every integer $B \geq 1$, at least B of the numbers*

$$\beta(m+1), \beta(m+3), \ldots, \beta\big(m(s_B+1)-1\big) \tag{1.2}$$

are irrational. In particular, at least one of

$$\beta(m+1), \beta(m+3), \ldots, \beta(67m-1)$$

is irrational for every positive odd m.

The upper endpoint grows linearly with the starting index, with a multiplier depending only on B. Choosing the construction's parameters for specific starting indices and counts gives the following smaller upper endpoints (Section 8), compared here with those from Theorem 1.1:

First even index	Last even index		# irrational values
	Specific parameters	Theorem 1.1	(at least)
4	198	200	1
6	330	334	1
8	462	468	1
20	3080	3742	2
20	6950	8796	3

1.1 Comparison with earlier results

For any fixed number B of irrational values, Theorem 1.1 gives an upper endpoint that grows linearly with the starting index $m+1$.

Let $I(s)$ count the irrational values among $\beta(2), \beta(4), \ldots, \beta(s)$. Our earlier paper [9] proves that, for every even $s \geq 160$,

$$I(s) \geq \left\lfloor c\sqrt{\frac{s}{\log s}} \right\rfloor, \qquad c = \pi\sqrt{\frac{2}{135}}. \tag{1.3}$$

To use this count for an interval starting at $\beta(m+1)$, we must allow for all $(m-1)/2$ preceding values to be irrational. We therefore need a count of at least

$$\frac{m-1}{2} + B$$

to ensure that B irrational values remain in the interval. For fixed B, this target grows proportionally to m. Reaching it with the square-root bound (1.3) requires s of order $m^2 \log m$: quadratic growth in the starting index, with an additional logarithmic factor.

For example, the earlier paper also proves the sharper count $I(3828) \geq 10$. At most nine of these irrational values can have indices below 20, so at least one of $\beta(20), \beta(22), \ldots, \beta(3828)$ is irrational. Theorem 1.1 reduces this upper endpoint to 1272. The table compares endpoints guaranteeing one irrational value from each listed starting index:

First even index	Endpoint from [9]	Endpoint here
4	160	198
6	364	330
20	3828	1272
100	187 340	6632

The middle column follows from the bounds $I(160) \geq 2$, $I(364) \geq 3$, $I(3828) \geq 10$ and $I(187\,340) \geq 50$ in [9]. Discarding the first 1, 2, 9 and 49 even beta values, respectively, leaves at least one irrational value in each interval.

The last column uses the individual parameter choices in Section 8 at the starting indices 4 and 6, and the bound $67m - 1$ from Theorem 1.1 at 20 and 100.

Sorokin [8, Theorem S] treats shifted intervals directly. For every sufficiently large odd m and every even $q > 100m \log m$, he proves that at least one of

$$\beta(m+1), \beta(m+3), \ldots, \beta(m+q-1)$$

is irrational. Huttner [1, Remark 5] gives the same conclusion. Taking the smallest allowable q gives an upper endpoint of order $m \log m$. The case $B = 1$ of Theorem 1.1 removes this logarithmic factor: $q = 66m$ suffices for every positive odd m.

Earlier finite results guaranteed an irrational value in a block beginning at $\beta(2)$, with the endpoint reduced to $\beta(10)$ by Lai and Zhou [6, Theorem 6.1]. These statements are now subsumed by the irrationality of $\beta(2)$ [7]. At the starting index 4, Krisman [2] gives a sharper result than either bound in the table: at least one of $\beta(4), \beta(6), \ldots, \beta(36)$ is irrational.

1.2 How the proof locates the irrational values

We construct B sequences of linear forms, all supported on the desired interval. The integer m fixes its starting point, and B fixes the count; the sequence index n tends to infinity. Their beta coefficients have the common shape

$$F_{n,b} = q_b(n) + \sum_{\substack{m+1 \le \nu \le s \\ \nu \text{ even}}} b^\nu \rho_\nu(n) \beta(\nu), \qquad 1 \le b \le B.$$

The dependence on the grid index b is through the powers b^ν. If at most $B-1$ of these beta values were irrational, a fixed integer combination of the B forms would cancel their coefficients at every n. This would produce a sequence of rational numbers. We arrange that clearing their denominators gives integers tending to zero, and that infinitely many of them are nonzero. Since a nonzero integer has absolute value at least 1, that is impossible.

To remove the values below $\beta(m+1)$, we differentiate $m-1$ times. For example, normalized second differentiation changes

$$\frac{a_i}{(t+4h)^i} \quad \text{into} \quad \binom{i+1}{2} \frac{a_i}{(t+4h)^{i+2}}.$$

Reflection removes odd indices, so the first surviving even exponent is 4. The zeros used to control rational denominators must survive this operation; we therefore start with triple zeros. The general construction pairs $(m-1)$st differentiation with zeros of multiplicity m. The second-derivative and triple-zero device is also used in the beta setting by Krisman [2].

The remaining question is whether the rational constants still provide enough freedom after differentiation. We express the constants at the seed index $n = 1$ as a Gram matrix of independent polynomials against a positive weight. Differentiating the kernels in the Gram construction of [9, Section 3] preserves this positive representation. It lets us prescribe a nonzero seed constant that survives the final cancellation. Prime dilation then propagates that constant to infinitely many indices.

Our starting point is the construction in [9]. We extend its polynomial multipliers, grouped denominator argument and prime-dilation calculation to the differentiated functions. The rational-grid construction and coefficient elimination follow the methods used for odd zeta values by Lai and Yu [5, Sections 2 and 5] and adapted to real Dirichlet characters by Kumar and Tosi [3, Section 2].

Section 2 defines the functions and linear forms. Section 3 constructs the seed, and Sections 4–6 prove the denominator, decay and nonvanishing estimates. Section 7 completes the contradiction; Section 8 chooses the parameters.

2 Rational functions for a shifted interval

2.1 Repeated zeros and differentiation

Fix a positive odd integer m and an integer $B \ge 1$, and put $k = m - 1$. The bth form will be an alternating sum over the points j/b with j odd. We need zeros on these grids both to remove initial summands and to control the denominators of the rational constants. The signs on odd summation indices repeat after a shift by 4, so poles at $0, -4, \ldots, -4n$ allow the

partial-fraction sums to be shifted back to the beta series without changing their signs. A polynomial multiplier will provide freedom to choose the seed constants.

The fraction classes needed for all $1 \le b \le B$ are

$$\Theta = \left\{ \frac{v}{d} : 1 \le d \le B,\ 1 \le v \le d,\ (v,d) = 1 \right\}, \qquad D = |\Theta| = D_B.$$

The integer class is represented by 1.

Choose an integer $r \ge 3$ and put $L = 2r + 4$. At index n, the rightmost zero will be rn; reflection about $-2n$ puts the leftmost zero at $-(r+4)n$. The length of this interval is Ln. Let S be an odd integer satisfying $S > mDL$, and put $s = S - 1$. Define

$$A_d = d^L \prod_{p \mid d} p^{\lceil L/(p-1) \rceil}, \qquad K = \prod_{\theta \in \Theta} A_{\mathrm{den}(\theta)}. \tag{2.1}$$

The first product is over primes; the progression estimate in Section 4 explains the normalization.

For every positive odd integer n, define

$$P_n(t) = (t - rn) \prod_{\theta \in \Theta} \prod_{j=0}^{Ln-1} (t - rn + j + \theta), \tag{2.2}$$

$$C_n = 4^{Sn} K^{mn} \frac{((n+1)!)^S}{(n!)^{mDL}}, \qquad R_n(t) = \frac{C_n P_n(t)^m}{\prod_{h=0}^{n} (t+4h)^S}. \tag{2.3}$$

At $m = 1$, these are the rational functions of [9, Section 2]. All roots of P_n are simple. They include the required grid points in $[-(r+4)n, rn]$ and every denominator point $0, -4, \ldots, -4n$. Thus the poles of R_n have order at most $S - m$. The root set is reflection invariant and P_n has odd degree. Since m, n, S are odd,

$$P_n(-4n - t) = -P_n(t), \qquad R_n(-4n - t) = -R_n(t).$$

We will choose an integer polynomial $W_n(t)$, odd about $-2n$ and homogeneous of a fixed total degree E in n, t. Set

$$T_n(t) = W_n(t) R_n(t), \qquad V_n(t) = \frac{1}{k!} T_n^{(k)}(t). \tag{2.4}$$

The degree will satisfy $S - m - E > 0$. Then T_n is proper, with degree at infinity at most $-(S - mDL)n - (S - m - E)$. Both T_n and V_n are reflection invariant, because k is even. At every non-pole grid point in the zero interval, T_n has a zero of order at least $m = k + 1$, so V_n vanishes there. Its poles have order at most $S - m + k = S - 1$.

2.2 The linear forms

Write the partial fractions before and after differentiation as

$$T_n(t) = \sum_{h=0}^{n} \sum_{i=1}^{S-m} \frac{a_{i,h}(n)}{(t+4h)^i},$$

$$V_n(t) = \sum_{h=0}^{n} \sum_{\nu=1}^{S-1} \frac{A_{\nu,h}(n)}{(t+4h)^\nu}. \tag{2.5}$$

The only possibly nonzero differentiated coefficients are

$$A_{i+k,h}(n) = \binom{i+k-1}{k} a_{i,h}(n), \qquad 1 \le i \le S - m. \tag{2.6}$$

Reflection gives $A_{\nu,n-h}(n) = (-1)^\nu A_{\nu,h}(n)$. Consequently

$$\rho_\nu(n) = \sum_{h=0}^{n} A_{\nu,h}(n) = 0 \quad \text{for odd } \nu.$$

To write the alternating signs compactly, let $\chi_4(j)$ be 0 for even j, 1 for $j \equiv 1 \pmod 4$, and -1 for $j \equiv 3 \pmod 4$. For $1 \le b \le B$, put

$$F_{n,b} = \sum_{j \ge 1} \chi_4(j) V_n(j/b). \tag{2.7}$$

In the term belonging to the pole $-4h$, shift the summation index by $4bh$. The character is unchanged, and the omitted initial terms give a rational constant. Thus

$$F_{n,b} = q_b(n) + \sum_{\substack{m+1 \le \nu \le S-1 \\ \nu \text{ even}}} b^\nu \rho_\nu(n) \beta(\nu), \tag{2.8}$$

$$q_b(n) = -\sum_{h=0}^{n} \sum_{\nu=1}^{S-1} A_{\nu,h}(n) b^\nu \sum_{j=1}^{4bh} \frac{\chi_4(j)}{j^\nu}. \tag{2.9}$$

For $k \ge 2$ the individual series are absolutely convergent. When $k = 0$, the individual exponent-one series converge conditionally and their total beta coefficient is zero. There are only finitely many partial-fraction series, so the decomposition is valid in this case too.

2.3 A rational criterion for the endpoints

The criterion below states the two estimates we will need. The positive rational u bounds the location of the largest terms, measured beyond rn in units of n. The quantity U bounds the fourth power of the exponential rate after clearing denominators. Put $e_0 = 68/25$, a rational upper bound for e.

Proposition 2.1. *With the parameters above, suppose that a positive rational u satisfies*

$$f(u) = \left(\frac{u+L}{u}\right)^{4mD} \left(\frac{u+r}{u+r+4}\right)^{S} < 1, \tag{2.10}$$

$$U = e_0^{4(S+k)} K^{4m} (L+u)^{4mDL} 4^{4S} \left(\frac{r^r}{(r+4)^{r+4}}\right)^{S} < 1. \tag{2.11}$$

Then at least B of $\beta(m+1), \beta(m+3), \ldots, \beta(S-1)$ are irrational.

The extra factor e_0^{4k} in (2.11) pays for the denominators introduced by differentiation. We still need a seed constant that survives the elimination of the possible irrational values.

For example, when $B = 2$, the combination

$$2^\nu F_{n,1} - F_{n,2}$$

eliminates $\beta(\nu)$ at every n, by (2.8). If $q_1(1) = 0$ and $q_2(1) = \eta \ne 0$, its rational constant at $n = 1$ is $-\eta$, whichever index ν must be eliminated.

For general B, the elimination in Section 7 can be performed with a nonzero coefficient on the last form. We therefore choose W_n so that

$$(q_1(1), \ldots, q_B(1)) = \eta e_B, \qquad \eta \in \mathbb{Q} \setminus \{0\}, \tag{2.12}$$

where $e_B = (0, \ldots, 0, 1)$. The rational constant of the combined form at $n = 1$ is then η times its nonzero last coefficient. This is the seed constant used by the prime-dilation argument in Section 6.

3 A Gram matrix for the differentiated seed

To obtain (2.12), we first construct B candidate polynomial multipliers $W_{n,1}, \dots, W_{n,B}$. Each produces a vector of B seed constants, one for each summation grid. These vectors form the columns of a rational matrix M. If M is nonsingular, a rational combination of its columns gives e_B; clearing denominators then gives the desired seed vector ηe_B with integer coefficients on the multipliers. We choose the candidates so that M has a Gram representation.

3.1 Moving the derivatives onto the kernels

We adapt the Gram representation of [9, Section 3] by transferring the derivatives from the rational function to its kernel. At $n = 1$ the only poles are 0 and -4. For a proper rational function V invariant under $t \mapsto -4 - t$, write

$$V(t) = \sum_{\nu} \frac{A_\nu}{t^\nu} + \sum_{\nu} \frac{(-1)^\nu A_\nu}{(t+4)^\nu}.$$

Its rational constant in the character sum over j/b is

$$q_b = -\sum_{\nu} (-1)^\nu A_\nu b^\nu \sum_{j=1}^{4b} \frac{\chi_4(j)}{j^\nu}.$$

Introduce

$$H_b(t) = \sum_{\substack{1 \le j < 4b \\ j \text{ odd}}} \frac{\chi_4(j)}{t + j/b}.$$

Expanding each summand at 0 shows that $q_b = \operatorname{Res}_{t=0} V(t) H_b(t)$. If $V = T^{(k)}/k!$, the residue of a derivative is zero, so repeated integration by parts gives

$$q_b = \operatorname{Res}_{t=0} T(t) \frac{H_b^{(k)}(t)}{k!}. \tag{3.1}$$

The sign is positive because k is even. This identity lets us leave the numerator of T intact and differentiate the simpler kernel instead.

Put $a = (m+1)/2$. On the midpoint line $t = -2 + iy$, pair the kernel terms whose distances from -2 are $\pm\alpha$, where $\alpha = (2j-1)/b$. Their character weights have opposite signs. In $H_b^{(k)}/k!$, with $k = m - 1$, the paired terms are

$$(-1)^{b-j}\big((t + 2 - \alpha)^{-m} - (t + 2 + \alpha)^{-m}\big).$$

It follows that $H_b^{(k)}(-2 + iy)/k! = g_{b,k}(y^2)$, where

$$g_{b,k}(x) = -2 \sum_{j=1}^{b} (-1)^{b-j} \frac{\displaystyle\sum_{\ell=0}^{a-1} (-1)^\ell \binom{m}{2\ell} \alpha^{m-2\ell} x^\ell}{(x + \alpha^2)^m}, \qquad \alpha = \frac{2j-1}{b}. \tag{3.2}$$

For example, at $b = 1$ the original and twice-differentiated kernels are

$$g_{1,0}(x) = -\frac{2}{x+1}, \qquad g_{1,2}(x) = \frac{6x - 2}{(x+1)^3}.$$

The numerator changes sign, but the denominator stays positive on $x \ge 0$. We will use these numerators as multipliers, so that the constant matrix becomes an integral of products against one positive weight.

3.2 Independent integer polynomials

Let

$$\mathcal{A} = \left\{ \frac{2j-1}{b} : 1 \le j \le b \le B \right\}, \qquad N = |\mathcal{A}|.$$

For $\alpha = v/d \in \mathcal{A}$ use its reduced numerator and denominator, and define

$$Q(x) = \prod_{v/d \in \mathcal{A}} (d^2 x + v^2), \qquad h_b(x) = Q(x)^m g_{b,k}(x), \qquad J = mN - a. \tag{3.3}$$

Each h_b is an integer polynomial of degree at most J. Indeed, with denominator $(d^2x + v^2)^m$, the corresponding numerator in (3.2) is

$$-2(-1)^{b-j} \sum_{\ell=0}^{a-1} (-1)^\ell \binom{m}{2\ell} v^{m-2\ell} d^{m+2\ell} x^\ell.$$

Its coefficients are integers. After multiplication by Q^m, the degree is at most $m(N-1)+a-1 = mN - a$.

Lemma 3.1. *The polynomials $h_1, \dots, h_B$ are linearly independent over $\mathbb{R}$. Moreover,*

$$N = D + \sum_{\substack{d \le B \\ d \text{ even}}} \varphi(d) \le 2D. \tag{3.4}$$

Proof. In a nonzero relation among the $g_{b,k}$, choose the largest index b with nonzero coefficient. Its kernel has a pole of exact order m at $x = -1/b^2$, and every earlier kernel is regular there. The order is exact because the corresponding term of $H_b^{(k)}/k!$ has a nonzero principal coefficient, and the change $x = -(t+2)^2$ has nonzero derivative at that pole. The relation is therefore impossible. Multiplication by Q^m preserves independence.

The distinct distances are precisely the reduced fractions v/d with $d \le B$, v odd and $0 < v < 2d$. There are $\varphi(d)$ such numerators for odd d, and $2\varphi(d)$ for even d. This proves (3.4). □

At the seed, $-(t+2)^2 = y^2$ on the midpoint line, so $(t+2)h_b(-(t+2)^2)$ gives the desired numerator there and is odd about -2. To lift it to all odd indices, make it homogeneous:

$$\begin{aligned} W_{n,b}(t) &= n^{2J}(t+2n) h_b\left(-\frac{(t+2n)^2}{n^2} \right) \\ &= \sum_{j=0}^{J} h_{b,j} (-1)^j n^{2J-2j} (t+2n)^{2j+1}. \end{aligned} \tag{3.5}$$

The expanded formula has integer coefficients. Its total degree is

$$E = 2J + 1 = m(2N-1) \le m(4D-1), \tag{3.6}$$

and it satisfies

$$W_{\ell p,b}(pt) = p^E W_{\ell,b}(t). \tag{3.7}$$

Because $S > mDL$ and $L \ge 10$, the two exponents

$$\tau_0 = S - m - E > 0, \qquad \tau = \tau_0 + k = S - 1 - E > k \tag{3.8}$$

are positive. The first ensures properness and tail convergence; the second will control prime dilation after differentiation.

3.3 The common positive weight

Let $M_{b,j}$ be the bth rational constant associated with $(W_{1,j}R_1)^{(k)}/k!$. We prove that M is nonsingular. Writing $z = t + 2$, the simple symmetric roots of P_1 give

$$P_1(t) = z \prod_{\alpha \in \Lambda} (z^2 - \alpha^2), \qquad |\Lambda| = DL/2,$$

where all α are positive and $\mathcal{A} \subseteq \Lambda$. Substitution on the midpoint line gives

$$W_{1,j}(-2+iy)R_1(-2+iy) = (-1)^{mDL/2+a+1} C_1 \frac{y^{m+1} \prod_{\alpha \in \Lambda} (y^2+\alpha^2)^m}{(y^2+4)^S} h_j(y^2).$$

Every pole of $H_b^{(k)}/k!$ in $(-4, 0)$ has order m and is canceled by P_1^m. Thus the product in (3.1) has poles only at $0, -4$. Close the upward midpoint line by a large right semicircle. It encloses 0 clockwise, so the residue theorem gives

$$M_{b,j} = \frac{\sigma C_1}{2\pi} \int_{-\infty}^{\infty} \omega(y) h_b(y^2) h_j(y^2)\, dy, \qquad \sigma = (-1)^{DL/2+a}, \tag{3.9}$$

where

$$\omega(y) = \frac{y^{m+1} \prod_{\alpha \in \Lambda} (y^2+\alpha^2)^m}{(y^2+4)^S Q(y^2)^m}.$$

The orientation supplies a minus sign, and m is odd, which explains the sign σ. The character weights in H_b sum to zero, so $H_b^{(k)}(t) = O(t^{-k-2})$. The contour integrand therefore has degree at most $mDL + E - 2S - 1 < -2$ at infinity. The arc vanishes and the integral converges absolutely.

The weight is positive for $y \neq 0$, since $m + 1$ is even. For a nonzero vector $v \in \mathbb{R}^B$, Lemma 3.1 implies that $\sum_b v_b h_b$ is a nonzero polynomial. Therefore

$$\sigma v^{\mathsf{T}} M v = \frac{C_1}{2\pi} \int_{-\infty}^{\infty} \omega(y) \left(\sum_b v_b h_b(y^2) \right)^2 dy > 0.$$

Thus σM is positive definite. Its entries are rational by (3.1), so fixed integers $c_1, \dots, c_B$ can be chosen with

$$Mc = \eta e_B, \qquad \eta \in \mathbb{Q} \setminus \{0\}.$$

For instance, clear the denominators of M and use the last column of the resulting integer matrix's adjugate. From now on set

$$W_n = \sum_{j=1}^{B} c_j W_{n,j}. \tag{3.10}$$

This proves (2.12). The coefficients c_j fix one multiplier shared by all B sequences. They remain fixed as n varies, so their sizes do not affect an exponential decay rate.

4 Denominators after differentiation

4.1 A progression estimate

Put $d_n = \operatorname{lcm}(1, \dots, n)$ and $\delta_n = 4d_n$. We first control the Taylor coefficients of the numerator blocks. Taking a coefficient of degree j removes at most j divisible constants at each prime-power level; d_n^j pays for that loss.

Lemma 4.1. *For $\theta = v/d \in \Theta$ in lowest terms, define*

$$B_\theta(t) = \frac{A_d^n}{(n!)^L} \prod_{j=0}^{Ln-1} (t - rn + j + v/d).$$

For every integer v_0 and every $j \geq 0$,

$$d_n^j [t^j] B_\theta(t + v_0) \in \mathbb{Z}. \tag{4.1}$$

Proof. Clear d from each factor using the factor d^{Ln} in A_d^n. Fix a prime $q \nmid d$. At each level $q^e \leq n$, split the Ln progression terms into L blocks of length n. Each block has at least $\lfloor n/q^e \rfloor$ divisible constants. A term in the coefficient of t^j replaces j constants by variable terms and loses at most j contributions at that level. Its valuation is therefore at least

$$Lv_q(n!) - jv_q(d_n).$$

A term retaining a zero constant is zero and needs no estimate. If $q \mid d$, the cleared constants are units and the extra factor in A_d^n has valuation

$$n \left\lceil \frac{L}{q-1} \right\rceil \geq \frac{Ln}{q-1} \geq Lv_q(n!).$$

These bounds prove (4.1). Products and integer powers preserve the same coefficient rule by convolution. □

4.2 Partial-fraction coefficients

Proposition 4.2. *For every positive odd n,*

$$\delta_n^{S-i} a_{i,h}(n) \in \mathbb{Z}, \tag{4.2}$$

$$\delta_n^{S+k-\nu} A_{\nu,h}(n) \in \mathbb{Z}, \qquad \delta_n^{S+k-\nu} \rho_\nu(n) \in \mathbb{Z}, \qquad \delta_n^{S+k} q_b(n) \in \mathbb{Z}. \tag{4.3}$$

Proof. Write $B_n(t) = K^{mn} P_n(t)^m / (n!)^{mDL}$ and suppress the index n on the coefficients. Expanding at $-4h$ gives

$$a_{i,h} = (n+1)^S (-1)^{Sh} \binom{n}{h}^S \times [t^{S-i}] \left\{ W_n(t-4h) B_n(t-4h) \prod_{\ell \neq h} \left(1 + \frac{t}{4(\ell - h)}\right)^{-S} \right\}. \tag{4.4}$$

The constants from the other poles have product $4^{Sn}(-1)^{Sh}(h!(n-h)!)^S$. Their cancellation with C_n leaves the displayed integer prefactor. Each $4(\ell - h)$ divides δ_n, and W_n has integer coefficients. Lemma 4.1 and the binomial series prove (4.2). Formula (2.6) then proves the first two bounds in (4.3).

The finite sums in (2.9) can introduce prime powers not covered by δ_n, so the coefficient bounds alone do not prove integrality term by term. We use the zeros that survive differentiation to cancel these denominators after grouping, as in [9, Section 4]. Fix an odd $j \leq 4bh$ and put $t_0 = j/b - 4h$. This is a grid point in $(-4n, 0)$ and is not a pole, so $V_n(t_0) = 0$. With $x_\ell = j/b + 4(\ell - h)$, this says

$$\sum_\nu \frac{A_{\nu,h}}{x_h^\nu} = -\sum_{\ell \neq h} \sum_\nu \frac{A_{\nu,\ell}}{x_\ell^\nu}. \tag{4.5}$$

At a prime q put $v = v_q(\delta_n)$. At most one x_ℓ has valuation greater than v: otherwise the difference of two such terms, a nonzero $4(\ell - \ell')$ dividing δ_n, would too. The coefficient bound gives

$$v_q\left(\frac{\delta_n^{S+k} A_{\nu,\ell}}{x_\ell^\nu}\right) \geq \nu(v - v_q(x_\ell)). \tag{4.6}$$

If $v_q(x_h) \leq v$, the left group in (4.5) is integral after scaling. Otherwise every group on the right is integral after scaling, so the same conclusion follows from (4.5). This holds at every prime. Summing the integral groups with the signs in (2.9) proves the last bound in (4.3). □

The exponent in the common multiplier is $S + k$. Multiplicity m in the numerator preserves the zeros needed for the grouped argument, while differentiation shifts a coefficient of order i to order $i + k$. Its denominator bound remains δ_n^{S-i}, which is $\delta_n^{S+k-(i+k)}$. This explains the additional arithmetic cost of differentiation.

5 Decay of the differentiated sums

Clearing denominators multiplies a form by δ_n^{S+k}. Since $\delta_n^{1/n} \to \mathrm{e}$, it suffices to bound its nth-root limsup strictly below $\mathrm{e}^{-(S+k)}$. The initial grid terms through rn vanish. We estimate the remaining terms by their absolute values, first on a bounded scaled interval and then on the infinite tail.

5.1 The exponential profile

For $t > 0$ the products in (2.2)–(2.3) give

$$P_n(rn + t) = t \prod_{\theta \in \Theta} \frac{\Gamma(t + Ln + \theta)}{\Gamma(t + \theta)},$$
$$\prod_{h=0}^{n} (rn + t + 4h) = 4^{n+1} \frac{\Gamma(((r+4)n + t)/4 + 1)}{\Gamma((rn + t)/4)}.$$

Set $x = t/n$. Stirling's formula, including the normalization C_n, gives the exponential profile

$$\begin{aligned}\Phi(x) = {} & m \log K + mD((x + L)\log(x + L) - x \log x) \\ & - \frac{S}{4}((x + r + 4)\log(x + r + 4) - (x + r)\log(x + r)) + S \log 4.\end{aligned} \tag{5.1}$$

The coefficient of $n \log n$ in the factorial normalization is $S - mDL$; the products contribute its negative. The linear Stirling terms cancel as well.

For fixed $A > 0$, the estimate

$$\log R_n(rn + t) = n\Phi(t/n) + O_A(\log n), \qquad 1/B \leq t \leq An, \tag{5.2}$$

is uniform. Only the factors $\Gamma(t + \theta)$ can have bounded arguments. Stirling's formula for $t \geq 1$ and continuity on $[1/B, 1]$ give, uniformly for $\theta \in \Theta$,

$$\log \Gamma(t + \theta) = t \log t - t + O_B(\log(2 + t)), \qquad t \geq 1/B.$$

The other gamma arguments are proportional to n, and the extra $m \log t$ term is also $O_A(\log n)$. This proves (5.2) down to the first possible nonzero grid point beyond rn.

5.2 Why differentiation preserves the exponential rate

Use Cauchy's estimate on the circle $|z| = \varepsilon = 1/(2B)$ about $rn + t$, where $t \geq 1/B$. Every numerator factor is positive at the real center. If its value is c, its absolute-value ratio on the circle is at most $1 + \varepsilon/c$. The sum of these reciprocals is $O(\log n)$ on $1/B \leq t \leq An$, including the extra factor t. The denominator factors are at least $rn + t$, and their reciprocal sum is $O(1)$. Thus, for a fixed constant C,

$$\sup_{|z|=\varepsilon} |R_n(rn + t + z)| \leq n^C R_n(rn + t).$$

Homogeneity gives $|W_n(rn + t + z)| = O_A(n^E)$ on this range. Consequently

$$|V_n(rn + t)| \leq \varepsilon^{-k} \sup_{|z|=\varepsilon} |T_n(rn + t + z)|. \tag{5.3}$$

There are $O_A(n)$ grid points in the bounded scaled interval. By (5.2), their absolute sum has nth-root limsup at most $\exp(\max \Phi)$. Here m, and hence the derivative order k, is fixed while $n \to \infty$.

We also need a bound over the infinite tail. Take $A \geq \max(L, r + 2, 1)$ and $t \geq An$. On the same circle each numerator factor is at most $3t$ in absolute value, each denominator factor is at least t, and $|W_n(rn + t + z)| \leq C' t^E$. Equation (5.3) gives

$$|V_n(rn + t)| \leq C'' C_n 3^{mDLn+m} t^{-(S-mDL)n-\tau_0}.$$

For $q_n = (S - mDL)n + \tau_0 > 1$, a first term and an integral bound the sum on any grid of spacing $1/b$:

$$\sum_{\substack{t \text{ on the grid} \\ t \geq An}} t^{-q_n} \leq (An)^{-q_n} + b\frac{(An)^{1-q_n}}{q_n - 1}.$$

Stirling's formula for C_n gives the tail nth-root bound

$$4^S K^m 3^{mDL} \mathrm{e}^{-(S-mDL)} A^{-(S-mDL)}. \tag{5.4}$$

The powers of n^{S-mDL} from the factorials cancel those from the lower tail endpoint. Since $S > mDL$, (5.4) tends to zero as A grows. Together with the bounded range, this proves

$$\limsup_{\substack{n\to\infty \\ n \text{ odd}}} |F_{n,b}|^{1/n} \leq G := \exp\left(\max_{x \geq 0} \Phi(x)\right). \tag{5.5}$$

5.3 The two inequalities imply cleared decay

The function f in (2.10) satisfies $4\Phi'(x) = \log f(x)$. The derivative of $\log f$ has a positive denominator and numerator

$$4\big((S - mDL)x^2 + L(S - mDL)x - mDLr(r + 4)\big).$$

This quadratic has exactly one positive zero. Thus $\log f$ first decreases and then increases. It starts at $+\infty$ and tends to 0 from below, with leading term $-4(S - mDL)/x$. It has exactly one zero $x_0 > 0$, where Φ reaches its global maximum. Substituting $\Phi'(x_0) = 0$ into (5.1) gives

$$G = K^m (x_0 + L)^{mDL} 4^S \left(\frac{(x_0 + r)^r}{(x_0 + r + 4)^{r+4}}\right)^{S/4}. \tag{5.6}$$

If $f(u) < 1$, then $x_0 < u$. The last ratio decreases for $x \geq 0$, since its logarithmic derivative is $-4x/((x+r)(x+r+4))$. Finally, the exponential series gives

$$\mathrm{e} < \frac{163}{60} + \frac{7}{4320} < \frac{68}{25} = e_0.$$

Using these bounds in (5.6) yields $(\mathrm{e}^{S+k}G)^4 < U < 1$. The prime number theorem and (5.5) therefore prove

$$\limsup_{\substack{n\to\infty \\ n \text{ odd}}} |\delta_n^{S+k} F_{n,b}|^{1/n} < 1. \tag{5.7}$$

The conclusion also holds for every fixed integer combination of the B forms.

6 Prime dilation and nonvanishing

The previous section makes the forms small after their denominators are cleared. To obtain a contradiction, we must also ensure that the eventual rational combination is nonzero infinitely often. At indices divisible by a large prime, the constant coefficient and the beta coefficients have different valuations. The seed constant survives a common scaling while the beta coefficients vanish modulo the prime. This separation is the principle behind the auxiliary-prime criterion of Lai and Sprang [4, Lemma 2.3]. We obtain it by extending the prime-dilation calculation of [9, Section 6] to the differentiated functions.

Write $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}$ and let v_p be the usual prime valuation. Recall $\tau_0 = S - m - E$ and $\tau = \tau_0 + k = S - 1 - E > k$ from (3.8).

Proposition 6.1. *Fix a positive odd integer ℓ. For all sufficiently large primes p,*

$$p^{\tau}\rho_\nu(\ell p) \in p^{\nu}\mathbb{Z}_{(p)}, \tag{6.1}$$

$$p^{\tau}q_b(\ell p) \equiv \chi_4(p)(\ell+1)^{-S}q_b(\ell) \pmod{p}. \tag{6.2}$$

6.1 The undifferentiated function

Take p large enough in terms of the fixed parameters and ℓ; in particular, it divides none of their denominators and $p > \ell+1$. Both $(\ell p)!$ and $(\ell p+1)!$ have p-valuation ℓ. After removing p^ℓ, their units reduce to $(-1)^\ell \ell!$, by Wilson's theorem. Fermat's theorem treats the fixed powers of 4 and K. Thus $C_{\ell p}$ has valuation $(S - mDL)\ell$, and its unit is congruent to

$$(-1)^{\ell(S-mDL)}\frac{C_\ell}{(\ell+1)^S}. \tag{6.3}$$

In a numerator block of fraction class v/d, a factor of $P_{\ell p}(pz)$ is selected by p when $dj + v \equiv 0 \pmod{p}$. After division by p, its fraction class becomes v'/d, where $pv' \equiv v \pmod{d}$. This permutes the reduced classes for each denominator, and hence permutes Θ. Each block has $L\ell$ selected factors. Together with the extra factor $p(z - r\ell)$, they give $p^{DL\ell+1}P_\ell(z)$. The unselected numerator units contribute $(-1)^{DL\ell}$, since each complete block of nonzero residues has product -1. For $P_{\ell p}^m$, raise both contributions to the mth power.

The selected denominator indices are $h = py$, $0 \leq y \leq \ell$. They give

$$p^{S(\ell+1)}\prod_{y=0}^{\ell}(z+4y)^S,$$

and the unselected denominator unit is $(-1)^{S\ell}$. The signs cancel, while the net valuation is

$$(S - mDL)\ell + m(DL\ell+1) - S(\ell+1) = m - S.$$

Using homogeneity (3.7), we obtain

$$p^{\tau_0}T_{\ell p}(pz) \equiv (\ell+1)^{-S}T_\ell(z) \pmod{p}. \tag{6.4}$$

6.2 Local coefficient extraction

To extract coefficients from (6.4), we need reduction modulo p to commute with the local series expansion. At a selected pole $h = py$, put $z = -4y + u$. The factorization above gives

$$u^S p^{\tau_0} T_{\ell p}(p(u - 4y)) \in \mathbb{Z}_{(p)}[[u]]. \tag{6.5}$$

After the displayed powers are removed, every other denominator factor has a unit constant term. Its inverse has p-integral series coefficients, and reduction commutes with Taylor extraction. The coefficient of u^{S-i} in (6.5) is $p^{\tau_0 - i} a_{i,py}(\ell p)$, so

$$p^{\tau_0 - i} a_{i,py}(\ell p) \equiv (\ell + 1)^{-S} a_{i,y}(\ell) \pmod{p}. \tag{6.6}$$

Now use the integer binomial coefficient in (2.6). With $\nu = i + k$ and $\tau = \tau_0 + k$, this gives

$$p^{\tau - \nu} A_{\nu,py}(\ell p) \equiv (\ell + 1)^{-S} A_{\nu,y}(\ell) \pmod{p}. \tag{6.7}$$

Thus the passage to differentiated coefficients is made after the justified local extraction.

Consider next a pole $h = x + py$ with $1 \le x < p$, and substitute $t = pz - 4x$. The extra numerator factor is a unit, the numerator blocks have $DL\ell$ selected factors, and the denominator has ℓ selected factors, each of order S. Including $C_{\ell p}$, the net valuation is zero. Local expansion of $R_{\ell p}$ at $-4h$ shows that its partial-fraction coefficient of order i has valuation at least i. Multiplication by $W_{\ell p}$ preserves this inequality: its jth Taylor coefficient is an integer and multiplies a coefficient of $R_{\ell p}$ of order $i + j$. Differentiation then gives

$$v_p(A_{\nu,h}(\ell p)) \ge \nu - k \qquad (p \nmid h). \tag{6.8}$$

This weaker bound suffices because $\tau > k$. Combining (6.7) and (6.8) over all poles proves (6.1).

6.3 The surviving rational constant

Use (2.9), taking $p > 4B\ell$ as well. Then every $j \le 4b\ell p$ is less than p^2, so $v_p(j) \le 1$. At a nonselected pole, multiplication by p^τ leaves valuation at least $\tau - k > 0$, even if $p \mid j$. At a selected pole, a term with $p \nmid j$ has valuation at least $\nu > 0$, by (6.7).

Only the terms with $h = py$ and $j = pj'$ survive. Their upper limit becomes $j' \le 4by$, and $\chi_4(pj') = \chi_4(p)\chi_4(j')$. Equation (6.7) therefore turns the surviving sum into $\chi_4(p)(\ell + 1)^{-S}$ times the constant formula at index ℓ. This is (6.2), completing the proof of Proposition 6.1.

7 The irrationality contradiction

We now prove Proposition 2.1. Assume its two inequalities hold and suppose that at most $B - 1$ of $\beta(m + 1), \beta(m + 3), \dots, \beta(S - 1)$ are irrational. Choose $B - 1$ distinct even indices $\nu_1 < \cdots < \nu_{B-1}$ containing all possible irrational indices, adding further indices if necessary. There are enough choices because $S > mDL$ and $D \ge B$.

The coefficients c_j have already fixed the common multiplier W_n and the seed vector. As in the elimination argument of [5, Section 5], we choose fixed integer weights $w_1, \dots, w_B$ on the B forms to annihilate the columns

$$(1^{\nu_j}, 2^{\nu_j}, \dots, B^{\nu_j})^{\mathsf{T}}, \qquad 1 \le j \le B - 1.$$

Use their signed maximal minors. The last weight w_B is nonzero: its minor is a generalized Vandermonde determinant at $1^2, \dots, (B - 1)^2$ with exponents $\nu_j/2$. A vanishing minor would

give a nonzero polynomial with at most $B-1$ monomials and $B-1$ distinct positive roots. A polynomial with q monomials has at most $q-1$ such roots: factor out its smallest power and apply Rolle's theorem inductively to its derivative. This proves nonsingularity. For $B=1$, take $w_1=1$.

Set

$$H_n=\sum_{b=1}^{B} w_b F_{n,b}, \qquad q_w(n)=\sum_{b=1}^{B} w_b q_b(n).$$

The chosen beta coefficients cancel at every index. All remaining beta values are rational under the assumption, so H_n is a rational sequence. Let D_0 clear the denominators of those rational beta values. Proposition 4.2 and (5.7) give

$$D_0\delta_n^{S+k}H_n\in\mathbb{Z}, \qquad D_0\delta_n^{S+k}H_n\longrightarrow 0 \quad (n\to\infty,\ n \text{ odd}). \tag{7.1}$$

The seed choice (2.12) gives $q_w(1)=w_B\eta\neq 0$. Exclude the finitely many primes dividing D_0 or the numerator or denominator of $q_w(1)$, and take p large enough for Proposition 6.1. Its constant congruence at $\ell=1$ gives

$$p^{\tau}q_w(p)\equiv\chi_4(p)2^{-S}q_w(1)\not\equiv 0 \pmod p.$$

Every remaining beta term of $p^{\tau}H_p$ is divisible by p in $\mathbb{Z}_{(p)}$, by (6.1) and the assumed rationality. Thus $H_p\neq 0$ for infinitely many primes. Equation (7.1) gives infinitely many nonzero integers tending to zero, which is impossible. This proves Proposition 2.1.

8 Choosing the endpoints

8.1 A choice valid for every starting index and count

To balance the mth power in the numerator, we also make the pole exponent S proportional to m. Take

$$r=7, \qquad L=18, \qquad u=\frac{1}{50}, \qquad S_0=s_B+1, \qquad S=mS_0.$$

The integer S is odd, and the resulting endpoint is $S-1=m(s_B+1)-1$. Let f_0,U_0 denote the expressions (2.10)–(2.11) with multiplicity 1, derivative order 0 and denominator exponent S_0. Then

$$f=f_0^m, \qquad U=U_0^m e_0^{4(m-1)}. \tag{8.1}$$

Thus $f_0<1$ will give the first condition for every m. For the second, we need enough decay in U_0 to pay the extra derivative cost $e_0^{4(m-1)}$.

To see how S_0 is chosen, put

$$A=\frac{11\log 11-7\log 7}{4}-\log 4-\log e_0.$$

From (2.1), the inequality $\lceil 18/(p-1)\rceil\le 18$ gives

$$A_d\le d^{36}, \qquad K\le B^{36D}.$$

Consequently

$$\begin{aligned}\frac{1}{4}\log U_0&=\log K+18D\log(18+u)-AS_0\\&\le 36D\log B+18D\log(18+u)-AS_0.\end{aligned}$$

The first two terms account for the denominator normalization and the numerator growth. The negative term is the decay gained from the pole exponent. We will verify the elementary bounds

$$\log(18+u) < \frac{29}{10}, \qquad A > \frac{4}{5} \tag{8.2}$$

below. They give

$$\frac{1}{4}\log U_0 < 36D\log B + \frac{261}{5}D - \frac{4}{5}S_0.$$

The coefficient 45 in S_0 cancels the logarithmic cost because $(4/5)45 = 36$. The coefficient 66 leaves a further margin because $(4/5)66 = 264/5 > 261/5$. These choices account for 33 and $45/2$ in (1.1). Indeed, that definition gives

$$S_0 \geq 66D + 45D\log B + 1 > 18D, \qquad \frac{1}{4}\log U_0 < -\frac{3}{5}D - \frac{4}{5}.$$

This margin will absorb the cost of differentiation.

To verify the bounds just used and the first condition $f_0 < 1$, we use three rational inequalities:

$$901^4\left(\frac{351}{551}\right)^{66} < 1, \tag{8.3}$$

$$(901/50)^{10} < (163/60)^{29}, \tag{8.4}$$

$$\left((4e_0)^4\frac{7^7}{11^{11}}\right)^5 e_0^{16} < 1. \tag{8.5}$$

Here $f_0 = 901^{4D}(351/551)^{S_0}$, so $S_0 \geq 66D$ and (8.3) give $f_0 < 1$. Since $18 + u = 901/50$ and $163/60 < \mathrm{e}$, (8.4) proves the first bound in (8.2). The logarithm of the left side of (8.5) is $-20A + 16\log e_0$. Since $\log e_0 > 1$, it gives $A > 4/5$, the second bound. Finally,

$$e_0 < \sum_{j=0}^{5}\frac{(11/10)^j}{j!} < \mathrm{e}^{11/10}, \qquad \text{so } \log e_0 < \frac{11}{10}. \tag{8.6}$$

All the rational comparisons are checked in the accompanying code.

Using (8.1), (8.6) and $D \geq 1$, we obtain

$$\frac{1}{4}\log U < -m\left(\frac{3}{5}D + \frac{4}{5}\right) + (m-1)\frac{11}{10} \leq -\frac{3}{10}m - \frac{11}{10} < 0.$$

Thus $f < 1$ and $U < 1$ for every positive odd m and every $B \geq 1$. Proposition 2.1 proves Theorem 1.1. At $B = 1$ we have $D_1 = 1$ and $s_1 = 66$, giving $67m - 1$.

8.2 Sharper individual intervals

The criterion can be checked directly without the upper bound $K \leq B^{36D}$. The following rational choices satisfy $S > mDL$, $f(u) < 1$ and $U < 1$:

m	B	$s = S - 1$	r	u	U(approx)
3	1	198	7	21/2000	0.00275
5	1	330	7	103/10 000	0.00659
7	1	462	7	13/1250	0.01585
19	2	3080	7	19/10 000	0.00183
19	3	6950	7	59/100 000	0.02403

The displayed decimals illustrate the margin; the inequalities are verified exactly over $\mathbb{Q}$. In the row with $m = 19$ and $B = 2$, the forms are supported on $\beta(20), \beta(22), \dots, \beta(3080)$, and the criterion guarantees at least two irrational members. These choices prove the intervals listed in the introduction.

A Exact checks and audit scope

A.1 The kernel, Gram and arithmetic identities

The proof of rank in Section 3 applies to every m and B. The accompanying computations check its formulas and normalization, together with the denominator and dilation identities. They use exact integer and rational arithmetic.

For the Gram entries, the residue formula and the moment integral provide two calculations. With every fraction reduced, put

$$\Delta = \prod_{\alpha = v/d \in \Lambda} d^2, \qquad P_*(x) = \prod_{v/d \in \Lambda \setminus \mathcal{A}} (d^2 x + v^2).$$

Since $\mathcal{A} \subseteq \Lambda$,

$$\frac{\prod_{\alpha \in \Lambda}(x + \alpha^2)}{Q(x)} = \frac{P_*(x)}{\Delta}.$$

For $0 \le j < S$, the elementary beta integral gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y^{2j}}{(y^2+4)^S}\, dy = 2^{2j-4S+2} \frac{(2j)!(2S-2j-2)!}{j!(S-j-1)!(S-1)!}. \tag{A.1}$$

Thus $M_{b,c}/C_1$ is σ/Δ^m times the moment functional in (A.1) applied to $x^a P_*(x)^m h_b(x) h_c(x)$. Its degree is at most $m(DL/2 + N) - a < S$, so all the moments converge.

The checker uses $m = 1, 3, 5$, $B = 1, 2, 3, 4$, $r = 3$ and $S = 10mD + 1$ for the seed calculations. For complete prime dilations it uses $(m, B) = (1, 1), (3, 1), (5, 1), (3, 2)$ with the same rule for S, and primes $17, 19$. These smaller parameters test identities; decay is checked at the theorem's displayed endpoints.

Exact check	Number passed
Differentiated-kernel evaluations	120
Gram entries, residues versus moments	90
Positive leading principal minors	30
Integer seed-vector identities	12
Selected-pole congruences	800
Other-pole valuation bounds	6800
Constant congruences	14
Grouped integrality, seed and prime indices	6564

Run `checks/checks.py` for these calculations and the rational inequalities in Section 8. Run `checks/endpoints.py` for the five individual intervals and a direct all-m check at $B = 1$, $S = 67m$. Both programs use only the Python standard library and should be run without `-O`, since their assertions perform the checks.

A.2 Independent audit

An independent mathematical audit reported on 5 November 2026 found no substantive mathematical error or gap in the underlying plain-text proof, identified by its hash in the

accompanying audit record. The audit covered the general arguments for arbitrary allowed parameters and independently reproduced the finite checks without running or importing the supplied programs. It also checked a prime dilation with seed index $\ell = 3$, prime 17 and index $n = 51$. The audit was not formal verification or certification of priority.

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