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AN EXPLICIT PRIME GAP UNDER THE RIEMANN HYPOTHESIS WITH AN ELEMENTARY PRIME POWER WINDOW AND AN UPDATED ZERO COUNT

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Abstract. Dudek proved that if the Riemann hypothesis is true, then for every real $x \geq 2$ there is a prime in the interval $(x-(4/\pi)\sqrt{x}\log x, x]$. We give a proof of this statement in which each cited input is stated explicitly or replaced. With $h = (2/\pi)\sqrt{x}\log x$, the prime power contribution $\psi - \theta$ is needed only across the window $(x-h, x+h]$, and there it is bounded by an elementary count of squares and higher powers, without the Riemann hypothesis and without the bounds of Rosser and Schoenfeld. The sum over the nontrivial zeros is bounded from the explicit zero count of Hasanalizade, Shen and Wong for $T \geq 70$, by trading slack between the head and the tail of the sum at the split point $T = 2x/h$. The range $2 \leq x < 67001$ is settled by a short table of prime gaps. The Riemann hypothesis is open and this note makes no progress on it.

1. Introduction

We cite A. W. Dudek, *On the Riemann hypothesis and the difference between primes* [Dudek], by its journal version: all theorem, lemma, equation and page numbers of [Dudek] below are those of Int. J. Number Theory 11 (2015), 771–778. The main result of [Dudek] is the following.

Theorem A (Dudek [Dudek, Theorem 1.2, p. 772])**.** *Suppose the Riemann hypothesis is true. Then there is a prime in the interval*

$$\left(x - \frac{4}{\pi}\sqrt{x}\log x,\ x\right]$$

for all $x \geq 2$.

This improves the constant $8/5$ of Ramaré and Saouter [RaSa, Theorem 1] to $4/\pi$. Later work went further: Dudek, Grenié and Molteni [DGM, Theorem 1.1] proved a prime in a symmetric interval of total length $(1+4/\log x)\sqrt{x}\log x$, and Carneiro, Milinovich and Soundararajan [CMS, Theorem 5] proved, under the Riemann hypothesis, that for $x \geq 4$ there is a prime in $[x, x + (22/25)\sqrt{x}\log x]$. Theorem A is therefore not the best known constant. In [Dudek], Theorem 1.1, p. 771, which Dudek quotes from Cramér, gives a constant $c > 0$ with $\pi(x + c\sqrt{x}\log x) - \pi(x) > \sqrt{x}$ for all $x \geq 2$, and Theorem 1.3, p. 772, gives a prime in $(x - (1+\varepsilon)\sqrt{x}\log x, x]$ for all sufficiently large x.

This note gives a proof of Theorem A that follows Dudek's argument and states every explicit input. Write

$$h(x) = \frac{2}{\pi}\sqrt{x}\log x, \qquad H(x) = 2h(x) = \frac{4}{\pi}\sqrt{x}\log x.$$

Theorem 1.1. *Suppose the Riemann hypothesis is true. Then for every real $x \geq 2$ there is a prime p with $x - H(x) < p \leq x$. The proof uses, besides the Riemann hypothesis, only the explicit zero count* (1) *of Hasanalizade, Shen and Wong* [HSW, Corollary 1.2]*, and the form of* [Dudek, Lemma 2.1] *stated in Section 4.*

The mathematical content of this note is two observations. The proof uses $\psi - \theta$ only across a short window, where the prime power mass is $O(\log^2 x)$ by counting (Section 2). The zero sum can be closed with a zero count whose constant is too large for separate head and tail bounds, by trading slack at the split point (Section 3). Neither is claimed as new. Section 5 describes the verification of these results and where the code is.

Outline of Dudek's proof. Section 2 of [Dudek] uses the weight $w(n) = 1 - |n - x|/h$ on the open interval $(x - h, x + h)$ together with

$$\psi_1(x) = \sum_{n \leq x} (x - n)\Lambda(n) = \int_2^x \psi(t)\, dt,$$

see [Dudek, p. 773]. The argument has five steps.

(1) **Explicit formula for ψ_1,** [Dudek, Lemma 2.1, p. 773]**.** For $x > 0$, x not an integer,

$$\psi_1(x) = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\log(2\pi) + \varepsilon(x), \qquad |\varepsilon(x)| < \frac{12}{5},$$

display (2.2), where ρ runs over the nontrivial zeros of the Riemann zeta function. The proof bounds a sum over zeros by $2^{3/2}\sum_\rho 1/|\rho|^2$ and quotes the value of this sum from Davenport [Da].

(2) **Weighted prime-power sum,** [Dudek, Lemma 2.2, p. 774]**.** For $x > 0$ and $h > 0$,

$$\sum_{x-h<n<x+h} \Lambda(n)w(n) = h - \frac{1}{h}\Sigma + \varepsilon(h), \qquad |\varepsilon(h)| < \frac{48}{5h},$$

where

$$\Sigma = \sum_\rho \frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)}.$$

(3) **Splitting the sum over zeros.** Writing $\rho = 1/2 + i\gamma$, the sum Σ is split at $|\gamma| = \alpha x/h$. For the head, [Dudek, (2.4), p. 775] uses the zero count

$$N(T) < \frac{T\log T}{2\pi} \qquad \text{for } T > 15.$$

This bound follows from Corollary 1 of Trudgian [Tr], which states that for $T \geq e$

$$\left| N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{7}{8} \right| \leq 0.112\log T + 0.278\log\log T + 2.510 + \frac{0.2}{T},$$

and which gives $N(T) < T\log T/(2\pi)$ for all $T \geq 8.55$. For the tail, [Dudek, p. 775] uses Lemma 1 (ii) of Skewes [Sk].

(4) **Passing from Λ to primes.** Dudek uses Theorem 14 and equation (5.5) of Rosser and Schoenfeld [RS], [Dudek, p. 775]:

$$0.98\sqrt{x} < \psi(x) - \theta(x) < 1.11\sqrt{x} + 3x^{1/3} \qquad \text{for } x \geq 121.$$

(5) **Positivity and the threshold.** With $\alpha = 2$ and $h = h(x)$, the first display of [Dudek, p. 776] states that $\sum_{x-h<p\le x+h} \log p$ exceeds

$$\mathrm{rhs}(x) = h - \frac{2}{\pi}\left(\frac{x}{\sqrt{x-h}} + \frac{(x+h)^{3/2}}{x}\right)\log\left(\frac{2x}{h}\right) - 1.11\sqrt{x+h} - 3(x+h)^{1/3} + 0.98\sqrt{x-h} - \frac{48}{5h},$$

which is positive for all $x \ge 65000$. Replacing $x+h$ by x gives Theorem A for

$$x \ge 65000 + \frac{2}{\pi}\sqrt{65000}\log 65000 \approx 66798.7,$$

at the end of Section 2.2 of [Dudek], p. 776. The remaining values of x were checked there by a Mathematica computation.

Table 1 lists how the inputs of the printed proof are used here. Four cited inputs are replaced: the zero count of [Tr], the bound of [RS], the tail bound of [Sk] and the value from [Da]. The zero count of [HSW] is a choice of input; this note does not examine whether the input of [Tr] also suffices for the argument below.

TABLE 1. Inputs of the printed proof of [Dudek] and of the proof given here.

Component	Printed proof [Dudek]	This note
Zero hypothesis	Riemann hypothesis	Riemann hypothesis
Zero count	[Tr, Corollary 1], as (2.4), $T > 15$	[HSW, Corollary 1.2], as (1), $T \ge 70$
Prime power step	[RS], for $x \ge 121$	Proposition 2.1, by counting
ψ_1 error bound	$\lvert\varepsilon(x)\rvert < 12/5$	$\lvert\varepsilon(y)\rvert \le 2.393$, all $y > 0$
Sum $\sum_\rho 1/\lvert\rho\rvert^2$	Value from Davenport [Da]	Not used
Tail of $\sum 1/\gamma^2$	Lemma 1 (ii) of Skewes [Sk]	From (1), step 2 of Section 3
Large range	$x \ge 65000$, $X \gtrsim 66798.7$	$x \ge 65000$, $X \ge 66799$
Small range	Mathematica, no table printed	Table 2, $2 \le x < 67001$

2. The prime power window

The proof of Theorem A uses the difference $\psi - \theta$ only through its increment across the window $(x-h, x+h]$. The two-sided bound of Rosser and Schoenfeld [RS] implies the following window bound, which is therefore all that Dudek's argument requires; we prove it directly.

Proposition 2.1. *For every real $x \ge 65000$, with $h = h(x) = (2/\pi)\sqrt{x}\log x$,*

$$\big(\psi(x+h)-\theta(x+h)\big)-\big(\psi(x-h)-\theta(x-h)\big) < 1.11\sqrt{x+h}+3(x+h)^{1/3}-0.98\sqrt{x-h}.$$

The proof goes through the following bound on the mass of the window.

Lemma 2.2. *Let $1 \le a \le b$ be real numbers with $b^{1/3} < a^{1/3} + 1$. Then*

$$\big(\psi(b)-\theta(b)\big)-\big(\psi(a)-\theta(a)\big) \le \frac{\log b}{2}\big(\sqrt{b}-\sqrt{a}+1\big) + \frac{(\log b)^2}{3\log 2}.$$

Proof. Since $\psi(y) = \sum_{n \le y} \Lambda(n)$ and $\theta(y) = \sum_{p \le y} \log p$, the difference $\psi(y) - \theta(y)$ is the sum of $\Lambda(n)$ over non-prime integers $n \le y$. Consequently, the left side of the lemma is precisely the von Mangoldt mass of the prime powers p^k with $k \ge 2$ lying in the interval $(a, b]$.

First, the non-prime von Mangoldt mass over a natural-number interval $(A, B]$ with $A \le B$ satisfies

$$\sum_{\substack{A < n \le B \\ n \text{ not prime}}} \Lambda(n) \le \sum_{2 \le k \le \log_2 B} \frac{\log B}{k} \cdot \#\{m \in \mathbb{N} : A < m^k \le B\}.$$

Indeed, every non-prime $n \in (A, B]$ with $\Lambda(n) \ne 0$ is of the form $n = p^k$ for a prime p and an integer $k \ge 2$ satisfying $k \le \log_2 B$. For such n, its von Mangoldt weight satisfies

$$\Lambda(n) = \log p = \frac{\log(p^k)}{k} \le \frac{\log B}{k},$$

and for each fixed exponent k, the map $n \mapsto p$ is injective.

Second, for real numbers $0 \le a \le b$, the number of k-th powers in the integer interval $(\lfloor a \rfloor, \lfloor b \rfloor]$ satisfies

$$\#\{m \in \mathbb{N} : \lfloor a \rfloor < m^k \le \lfloor b \rfloor\} \le b^{1/k} - a^{1/k} + 1.$$

Applying these two bounds separates the prime powers in $(a, b]$ into squares, $k = 2$, and higher powers, $3 \le k \le \log_2 b$:

- For $k = 2$, there are at most $\sqrt{b} - \sqrt{a} + 1$ squares in $(a, b]$, each carrying weight $\log p \le \frac{1}{2} \log b$.
- For each $3 \le k \le \log_2 b$, the bound $b^{1/k} < a^{1/k} + 1$ follows from the case $k = 3$, since $b^{1/k} - a^{1/k}$ decreases as k increases when $b \ge a \ge 1$; so there is at most one k-th power in $(a, b]$, carrying weight $\log p \le \frac{1}{k} \log b \le \frac{1}{3} \log b$.

There are at most $\log b / \log 2$ exponents k, which gives the last term. □

Proof of Proposition 2.1. Let $h = h(x)$, $a = x - h$, $b = x + h$, $u = x^{1/6}$, $L = \log x$ and $M = \log b$. For all $x \ge 65000$,

$$u \ge 6.34, \qquad L \le 1.5u + 2.32, \qquad L \ge 11, \qquad h \le 0.63662\, u^3 L.$$

These give $h \le x/10$, so $0.9x \le a$ and $b \le 1.1x$. Hence $\sqrt{a} \ge 0.948\, u^3$, and $(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) = 2h$ gives $\sqrt{b} - \sqrt{a} \le 0.6716\, L$. Also $a^{1/3} \ge 0.965\, u^2$ gives $b < (a^{1/3} + 1)^3$, so Lemma 2.2 applies. Further $L \le M \le L + 0.1 \le 1.5u + 2.42$. Since $\sqrt{a} \le u^3 \le \sqrt{b}$ and $u^2 \le b^{1/3}$, the target satisfies

$$1.11\sqrt{b} + 3b^{1/3} - 0.98\sqrt{a} \ge 0.13\, u^3 + 3u^2.$$

By Lemma 2.2 and $3 \log 2 \ge 2.0794$, the window mass is at most

$$\frac{M}{2}(0.6716\, M + 1) + \frac{M^2}{2.0794} \le 0.8168\, M^2 + 0.5\, M.$$

Substituting $M \le 1.5u + 2.42$, the proof closes with the polynomial inequality

$$0.8168\,(1.5u + 2.42)^2 + 0.5\,(1.5u + 2.42) < 0.13\, u^3 + 3u^2 \qquad \text{for } u \ge 6.34.$$

Indeed the left side is at most $1.8378\, u^2 + 6.681\, u + 5.994$, so the difference of the two sides is at least $0.13\, u^3 + 1.1622\, u^2 - 6.681\, u - 5.994$, which exceeds 31 at $u = 6.34$ and is increasing for $u \ge 2$. □

3. The zero sum from the Hasanalizade, Shen and Wong count

Let $N(T)$ be the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $0 < \beta < 1$ and $0 < \gamma \le T$, counted with multiplicity. Hasanalizade, Shen and Wong [HSW, Corollary 1.2, p. 221, display (1.5)] prove that for $T \ge e$

$$|N(T) - M(T)| \le R(T),$$

where

$$M(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right), \qquad R(T) = 0.1038\log T + 0.2573\log\log T + 9.3675.$$

By the symmetry of the zeros, it follows that for $70 \le T \le t$

$$\begin{aligned} \#\{\rho : |\gamma| \le T\} &\le 2\big(M(T) + R(T)\big) + \tfrac{7}{2}, \\ \#\{\rho : T < |\gamma| \le t\} &\le 2\big(M(t) + R(t)\big) - 2\big(M(T) - R(T)\big) + \tfrac{7}{2}, \end{aligned} \tag{1}$$

with multiplicity. The constant $7/2$ is not needed; keeping it makes the constants below match those of Section 5. Only (1) is used below.

Proposition 3.1. *Let (γ_i) be any family of real numbers satisfying* (1) *in place of the ordinates of the zeros, and let Σ be the sum of* [Dudek, Lemma 2.2] *over $\rho_i = 1/2 + i\gamma_i$. For every $x \ge 65000$, with $h = h(x)$,*

$$|\Sigma| \le \left(\frac{2xh}{\pi\sqrt{x-h}} + \frac{4h(x+h)^{3/2}}{2\pi x}\right)\log\left(\frac{2x}{h}\right).$$

This proposition takes no Riemann hypothesis. The hypothesis enters only when the family (γ_i) is taken to be the ordinates of the nontrivial zeros, so that $\rho_i = 1/2 + i\gamma_i$ are the zeros themselves.

The bound has the same form as [Dudek, (2.5) and p. 776, $\alpha = 2$]. The route differs. From (1), with its constant 9.3675, the separate head and tail estimates that would give this form do not both hold near $T = 70$: the tail inequality

$$\sum_{|\gamma|>T} \frac{1}{\gamma^2} \le \frac{\log T}{\pi T}$$

needs about $T \ge 170$, while at the split point $T = 2x/h$ with $x \ge 65000$ we only know $T \ge 70$.

Proof of Proposition 3.1. Keep the split at $T = 2x/h$ and trade slack between head and tail in four steps.

(1) **Head slack.** For all $T \ge 70$, the first inequality of (1) implies

$$\#\{|\gamma| \le T\} \le \frac{T\log T}{\pi} - \big(0.89\,T - 22.25 - 0.502\log T\big),$$

so the head count has a positive slack of about $0.89\,T$ below the main term $(T\log T)/\pi$. Indeed,

$$2M(T) = \frac{T\log T}{\pi} - \frac{\log(2\pi)+1}{\pi}\,T, \qquad \frac{\log(2\pi)+1}{\pi} \ge 0.89,$$

and for $T \ge 70$ one has $R(T) \le 0.251\log T + 9.375$, so $2R(T) + 7/2 \le 0.502\log T + 22.25$. Write $S(T) = 0.89\,T - 22.25 - 0.502\log T$ for this slack.

(2) **Tail with excess.** For $T \geq 70$, partial summation against the second inequality of (1) gives

$$\sum_{|\gamma|>T} \frac{1}{\gamma^2} \leq \frac{\log T}{\pi T} + \frac{E(T)}{T^2}, \qquad E(T) = 1.004 \log T + 41.26 - 0.25\, T,$$

and the excess $E(T)$ is positive until about $T = 187$. For $T \geq 70$ one has $S(T) \geq 0$ and $1.1\, E(T) \leq S(T)$.

(3) **Comparison of head and tail weights.** Let

$$a = h^2 (x-h)^{-1/2} \qquad \text{and} \qquad B = 4(x+h)^{3/2}$$

denote the head weight and tail weight, respectively. If $35h \leq x$, which is equivalent to $T = 2x/h \geq 70$, then

$$B \leq 1.1\, a\, T^2.$$

(4) **Combined split bound.** Let D_i denote the second-difference term

$$D_i = \frac{(x+h)^{\rho_i+1} - 2x^{\rho_i+1} + (x-h)^{\rho_i+1}}{\rho_i(\rho_i+1)}$$

for the i-th zero. Each head term has $\|D_i\| \leq a$, by the integral representation of [Dudek, p. 774], and each tail term has $\|D_i\| \leq B/\gamma_i^2$, as in [Dudek, p. 775]. The head count and the tail bound give

$$\sum_i \|D_i\| \leq \left(\frac{T \log T}{\pi} - S(T) \right) a + B \left(\frac{\log T}{\pi T} + \frac{E(T)}{T^2} \right).$$

Using $B \leq 1.1\, a\, T^2$ and $1.1\, E(T) \leq S(T)$, the excess from the tail sum is absorbed into the slack of the head count, which yields

$$\sum_i \|D_i\| \leq \frac{T \log T}{\pi}\, a + \frac{B \log T}{\pi T}.$$

Since $|\Sigma| \leq \sum_i \|D_i\|$, substituting $T = 2x/h$, $a = h^2(x-h)^{-1/2}$, and $B = 4(x+h)^{3/2}$, this upper bound equals the right side of Proposition 3.1.

□

4. Completion of the proof

- **Lemmas 2.1 and 2.2 of** [Dudek]**.** We use Lemma 2.1 of [Dudek] in the form

$$\left| \psi_1(y) - \left(\frac{y^2}{2} - Z(y) - y \log(2\pi) \right) \right| \leq 2.393 \qquad (y > 0),$$

where $Z(y)$ is the zero sum for $y > 1$, integer y allowed. The bound is below the printed $12/5 = 2.4$, and its proof does not use the value of $\sum_\rho 1/|\rho|^2$; it is part of the checked development of Section 5. For $y \leq 1$ only the identity is needed, and the proof below uses Z only for $y > 1$. Lemma 2.2 of [Dudek] follows by taking the second difference across $x-h, x, x+h$, with error at most $48/(5h)$; we use it for $h < x$, which holds for $x \geq 65000$.

- **Primes in the window for $x \geq 65000$.** Let $h = h(x)$. Since $35h \leq x$, the split point $T = 2x/h$ is at least 70. As $0 \leq w \leq 1$ and $\Lambda \geq 0$, Lemma 2.2 of [Dudek] and Proposition 3.1 give

$$\psi(x+h) - \psi(x-h) > h - \frac{2}{\pi}\left(\frac{x}{\sqrt{x-h}} + \frac{(x+h)^{3/2}}{x}\right)\log\left(\frac{2x}{h}\right) - \frac{48}{5h},$$

  and subtracting the bound of Proposition 2.1 gives $\sum_{x-h<p\leq x+h} \log p > \mathrm{rhs}(x)$. By the monotonicity argument of [Dudek, p. 776], made explicit by elementary bounds on each term, $\mathrm{rhs}(x) > 0$ for every real $x \geq 65000$, so there is a prime in $(x-h, x+h]$.
- **Large range.** Let $X \geq 66799$. The map $x \mapsto x + h(x)$ is continuous and increasing, and $65000 + h(65000) \leq 66799$, so there is $x \geq 65000$ with $x + h(x) = X$. A prime p in $(x - h(x), x + h(x)]$ satisfies $p \leq X$ and $p > X - 2h(x) = X - H(x) \geq X - H(X)$, as H is increasing.
- **Small range.** For every integer n with $2 \leq n \leq 67000$ there is a prime $p \leq n$ with $n - p < g$, where g is the gap of the row of Table 2 that contains n; the largest values of $n - p$ in the six rows are $0, 1, 1, 3, 13, 71$. Let x be real with $2 \leq x < 67001$, put $n = \lfloor x \rfloor$, and take this prime p. Then $p \leq n \leq x$, and since $n - p \leq g - 1$ and $x < n + 1$,

$$x - p < n + 1 - p \leq g.$$

  If the row of n starts at t, then $g \leq H(t) \leq H(x)$ because $t \leq x$ and H is increasing, so $x - H(x) < p \leq x$. This uses no Riemann hypothesis.

Since $[2, 67001)$ and $[66799, \infty)$ overlap, Theorem 1.1 follows.

TABLE 2. Prime gap data for $2 \leq x < 67001$. Each integer n in a row has a prime $p \leq n$ with $n - p < g$.

Integers n	Gap g	$H(t)$ at the first $n = t$, about
2	1	1.25
3 to 4	2	2.42
5 to 6	2	4.58
7 to 22	4	6.56
23 to 199	14	19.15
200 to 67000	72	95.40

Cited inputs. Table 3 lists every cited input of the printed proof and of the proof here, with the page of the source that was read.

5. VERIFICATION

The main results of this note are checked in Lean 4 with Mathlib [Mathlib], version `v4.35.0-rc3`, in the repository `https://github.com/sthubbar/goldbach-lean`, directory `entries/dudek2014`, commit `da5cf9edad7896290771dfcab8feee2906399e3d` [Repo], registered as Palomar entry `PALOMAR-2026-11-01-000028`, version 1 [Palomar].

Theorem 1.1 is

`LeanLab.PNTPort.ExplicitPsi1.Trivial.theorem2_of_RH_hsw`,

TABLE 3. Cited inputs: source and page, printed statement, and use.

Input	Page	Printed statement	Use
Corollary 1.2 of [HSW]	p. 221, display 1.5	$\lvert N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e}\rvert \le 0.1038\log T + 0.2573\log\log T + 9.3675$ for $T \ge e$	Used for $T \ge 70$, as (1)
Corollary 1 of [Tr]	Section 1; used in [Dudek], p. 775	$\lvert N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{7}{8}\rvert \le 0.112\log T + 0.278\log\log T + 2.510 + 0.2/T$ for $T \ge e$, giving $N(T) < T\log T/(2\pi)$ for $T \ge 8.55$	Replaced by (1)
Theorem 14 and (5.5) of [RS]	pp. 71, 77; used in [Dudek], p. 775	$0.98\sqrt{x} < \psi(x) - \theta(x)$ for $x \ge 121$; $\psi(x) - \theta(x) < 1.11\sqrt{x} + 3x^{1/3}$ for $x > 0$	Replaced by Proposition 2.1
Lemma 1 (ii) of [Sk]	p. 50; used in [Dudek], p. 775	$\sum_{\gamma \ge T} 1/\gamma^2 < \log T/(2\pi T)$ for $T \ge \gamma_1$	Replaced, from (1)
Value of $\sum_\rho 1/\lvert\rho\rvert^2$, [Da]	pp. 81–82; used in [Dudek], p. 773	$\gamma + 2 - \log 4\pi$ under the Riemann hypothesis	Not used

which proves `PrimeInDudekInterval x` for all real $x \ge 2$, with `dudekH` the function H. It takes two hypotheses, neither declared as an axiom. The first, `RiemannHypothesis`, is the Riemann hypothesis as stated in Mathlib. The second, `HSWN70 gZ`, is (1) for the ordinates of the nontrivial zeros counted with multiplicity, written with main term $M + 7/8$ and remainder $R + 7/8$; it is taken from Corollary 1.2 of Hasanalizade, Shen and Wong [HSW] and is not proved in Lean, and the step from that corollary to (1) is argued by hand in Section 3. Proposition 2.1, Lemma 2.2, Proposition 3.1, the form of Lemma 2.1 of [Dudek] with 2.393, the positivity of rhs and the small range are `window_holds`, `mass_le`, `sigmaBd_of_hsw`, `lemma4_Zpatch`, `rhs_pos` and `small_range`; the last evaluates Table 2 in the kernel. Under the same two hypotheses the repository also proves Theorem 1.1 of [Dudek] with $c = 10^8$ and Theorem 1.3 of [Dudek], as `theorem1_of_RH_hsw` and `theorem3_of_RH_hsw`; their proofs are not given in this note.

Code availability. In `entries/dudek2014`, run `lake exe cache get`, `lake build` and `lake comparator --config=comparator-dudek2014.json`; the comparator checks against `ChallengeDudek2014.lean`. The axioms used are `propext`, `Quot.sound` and `Classical.choice`, with no `sorry`. Sixteen files under `LeanLab/PNTPort/StrongPNT` are from `PrimeNumberTheoremAnd` [PNT], commit `c39a751132c88b6e8080b74c74023fd95b3d8be0`, Apache-2.0, credited to its contributors.

Steve Hubbard designed the research program and directed this work. AI tools, Claude and Gemini, were used to write the Lean code and draft the text. The author takes full responsibility for the content.

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