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EQUIVALENCE OF THE FOUR-VARIABLE HESSIAN AND PLANAR JACOBIAN CONJECTURES

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Abstract. We prove that the four-variable Hessian conjecture and the planar Jacobian conjecture are equivalent over the complex numbers.

1. Introduction

Kraus [13] stated the planar Jacobian assertion in 1884. As Rodríguez Díaz [18, Section 2.1] explains, the argument breaks down at infinity, where a coordinate function has a pole and its derivative at the point in question is undefined. In 1939 Keller [12] asked whether a polynomial transformation with integer coefficients and Jacobian determinant one has a polynomial inverse. He also established the birational case [12, Satz 3], in which a rational inverse is already given. For an integer $n \geq 1$, write $x = (x_1, \dots, x_n)^{\mathsf{T}}$ for the coordinate column on $\mathbb{C}^n$. For a polynomial map $G = (G_1, \dots, G_n) \colon \mathbb{C}^n \to \mathbb{C}^n$, its Jacobian matrix is $DG = (\partial G_i/\partial x_j)_{1 \leq i,j \leq n}$. Call G a Keller map if $\det DG$ is a nonzero constant. The Jacobian conjecture JC_n asserts that every Keller map of $\mathbb{C}^n$ is a polynomial automorphism, that is, it has a two-sided polynomial inverse.

Bass, Connell and Wright [3] reduced the family of Jacobian conjectures to maps $x + H(x)$ with H cubic homogeneous, allowing the dimension to increase. De Bondt and van den Essen [8] further reduced this family to the case in which DH is symmetric. The symmetry identity $\partial H_i/\partial x_j = \partial H_j/\partial x_i$ gives

$$\frac{\partial}{\partial x_j}\left(\frac{1}{4}\sum_{i=1}^{n} x_i H_i(x)\right) = \frac{1}{4}\left(H_j(x) + \sum_{i=1}^{n} x_i \frac{\partial H_j}{\partial x_i}(x)\right) = H_j(x),$$

by homogeneity of H_j. The symmetric reduction therefore concerns gradient maps in a dimension that may exceed that of the original map.

For a polynomial f on $\mathbb{C}^n$, let $\nabla f = (\partial f/\partial x_i)_{i=1}^n$ be its gradient column and let $\operatorname{Hess} f = (\partial^2 f/\partial x_i \partial x_j)_{i,j=1}^n$. The Hessian conjecture HC_n is the restriction of JC_n to $G = \nabla f$. Meng [14] formulated the conjecture through the formal Legendre transform. De Bondt and van den Essen formulated it independently in terms of gradient maps [1, Remark 4.5]. Subtract the

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affine part of f so that $\nabla f(0) = 0$. This changes the gradient by a constant translation. For $\det \operatorname{Hess} f \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, the linear term of ∇f is $\operatorname{Hess} f(0)x$ with invertible matrix. Write $p = (p_1, \dots, p_n)^\mathsf{T}$ for the target coordinate column. The formal inverse $X(p)$ starts with $\operatorname{Hess} f(0)^{-1}p$. At each total degree $k \geq 2$, the degree-k part of $\nabla f(X(p)) = p$ is linear in the degree-k part of X, with coefficient matrix $\operatorname{Hess} f(0)$. All other terms are determined at lower degrees because the nonlinear part of ∇f has degree at least two in its inputs. This recursion gives a unique right inverse. A left inverse is obtained by the same degree comparison for $X(\nabla f(x)) = x$: composition of each homogeneous unknown with $\operatorname{Hess} f(0)x$ is an invertible linear operation. Associativity makes the two inverses equal.

Meng's formal Legendre transform is

$$f^*(p) = p^\mathsf{T} X(p) - f(X(p)).$$

For $1 \leq i \leq n$, differentiation gives

$$\frac{\partial f^*}{\partial p_i} = X_i(p) + \sum_{j=1}^{n} \left(p_j - \frac{\partial f}{\partial x_j}(X(p)) \right) \frac{\partial X_j}{\partial p_i}(p) = X_i(p).$$

Thus f^* is polynomial exactly when X is polynomial: one direction follows by differentiation and the other by the formula for f^*. A polynomial X satisfies both inverse identities as polynomial identities. Since $D(\nabla f) = \operatorname{Hess} f$, this recovers the gradient formulation. The doubling construction in Lemma 1.2 gives $\mathrm{HC}_{2n} \Rightarrow \mathrm{JC}_n$, while the gradient formulation gives $\mathrm{JC}_n \Rightarrow \mathrm{HC}_n$.

Dillen [10] proved the Hessian conjecture in dimension two, and de Bondt [5] proved it in dimension three. De Bondt's proof uses a linear change of variables and a positive weighting whose leading form has nonzero Hessian determinant. In July 2026, Alpöge announced a noninjective Keller map in dimension three. Its Jacobian determinant is -2 and three distinct points have the same image [2, 17]. Adjoining identity coordinates preserves both the determinant and the collision, so the example refutes JC_n for every $n \geq 3$. Meng and Yang [15, Theorem 6.1] used doubling followed by a one-variable partial Legendre transform to obtain a degree-fourteen polynomial in five variables with Hessian determinant 128 and a noninjective gradient. Adjoining a term $x_{n+1}^2/2$ adds an identity gradient coordinate and a Hessian block of determinant one, so this example refutes HC_n for every $n \geq 5$. The one-variable assertions follow from the fact that a polynomial with nonzero constant derivative is affine. The two remaining dimension-specific assertions are therefore JC_2 and HC_4.

For HC_4, Ni [16, Theorem 1.2] treats quartic potentials by splitting the highest homogeneous part according to its Hessian rank. Adamus and Hajto [1, Theorem 4.17 and Remark 4.5] develop a Legendre-potential criterion for invertibility and distinguish the known implication $\mathrm{HC}_4 \Rightarrow \mathrm{JC}_2$ from its converse. The converse follows from the following description of four-variable potentials.

Theorem 1.1. *Let $f \in \mathbb{C}[x, y, z, w]$ have nonzero constant Hessian determinant. Either ∇f is a polynomial automorphism or, in some constant linear coordinates, f has the form*

$$f = R(x, y) + zA(x, y) + wB(x, y), \qquad \det D(A, B) \in \mathbb{C}^{\times}.$$

Here $R, A, B \in \mathbb{C}[x, y]$ and $D(A, B)$ is the Jacobian matrix with respect to x, y. Consequently HC_4 *and* JC_2 *are equivalent.*

For a homogeneous polynomial of degree at least two in four variables, the Gordan–Noether theorem gives a nonzero constant vector in the Hessian kernel when the Hessian determinant vanishes [11, 4]. The polynomial $y^2 + x^2z + x^3w$, for example, is multiplied by t^6 under the substitution $(x, y, z, w) \mapsto (tx, t^3y, t^4z, t^3w)$, so it is homogeneous for weights $(1, 3, 4, 3)$. Its Hessian is singular because the z, w rows are proportional over $\mathbb{C}(x)$. Its gradient components $2xz + 3x^2w, 2y, x^2, x^3$ vanish at the origin and are linearly independent over $\mathbb{C}$. A constant vector in the Hessian kernel would give a linear combination of these components that is constant, hence zero, so that vector must vanish. A translation direction of a Zariski closed set $Z \subseteq \mathbb{C}^4$ is a nonzero vector d such that $a + td \in Z$ for all $a \in Z$ and $t \in \mathbb{C}$. In this example, fixing $x \neq 0$ lets the first two gradient components vary freely, with the last two fixed at (x^2, x^3). The gradient-image closure is the product of $\mathbb{C}^2$ with the closure of the curve parametrized by (x^2, x^3), and translations in the first two coordinates preserve it. De Bondt asked for such a direction for the gradient-image hypersurface of an arbitrary four-variable polynomial with Hessian rank three [7, Section 6, Problem 1]. Theorem 2.3 supplies the direction, and Corollary 2.4 answers the relation question in all singular ranks. De Bondt's low-rank classification [6, Corollary 7.3] then determines the possible leading forms as weights change.

Lemma 1.2. *Let $b = (b_1, \dots, b_n)^{\mathsf{T}}$ and $v = (v_1, \dots, v_n)^{\mathsf{T}}$ be two blocks of n variables. For a polynomial $R(b)$ and a polynomial map $G\colon \mathbb{C}^n \to \mathbb{C}^n$, set $f(b, v) = R(b) + v^{\mathsf{T}}G(b)$. Then*

$$\det \operatorname{Hess} f = (-1)^n (\det DG)^2.$$

If this determinant is a nonzero constant, a polynomial change of source coordinates and a permutation of the target coordinates carry ∇f to the map $(b, v) \mapsto (G(b), v)$. In this case ∇f is a polynomial automorphism if and only if G is.

Proof. With $\operatorname{Hess}_b f$ denoting the matrix of second derivatives in b with v fixed, the two blocks of ∇f and the Hessian are

$$\nabla f(b, v) = \big(\nabla R(b) + DG(b)^{\mathsf{T}}v,\, G(b)\big), \qquad \operatorname{Hess} f = \begin{pmatrix} \operatorname{Hess}_b f & DG^{\mathsf{T}} \\ DG & 0 \end{pmatrix}.$$

Moving the last n rows above the first n uses n^2 row transpositions. The resulting block triangular matrix has diagonal blocks DG, DG^{T}, giving the determinant formula.

If $\det \operatorname{Hess} f$ is a nonzero constant, so is $\det DG$, since its square is a unit in $\mathbb{C}[b_1, \dots, b_n]$. Use the source coordinate $v' = \nabla R(b) + DG(b)^{\mathsf{T}} v$ in place of v, fixing b. The inverse substitution is

$$v = (DG(b)^{\mathsf{T}})^{-1}\big(v' - \nabla R(b)\big),$$

which is polynomial by the adjugate formula for the inverse matrix. The gradient in these source coordinates sends (b, v') to $(v', G(b))$. Exchanging its output blocks gives $(b, v') \mapsto (G(b), v')$. If G is invertible, the inverse of this product is $(b, v') \mapsto (G^{-1}(b), v')$. Conversely, the second block of any inverse of the product is v', since the product fixes that block. Restricting both composition identities to $v' = 0$ shows that the first block of the inverse on that slice is a two-sided inverse of G. □

2. Translation directions of singular Hessians

Ranks of polynomial and rational matrices are taken over the rational function field in the source coordinates. We write $X = (X_1, \dots, X_m)^{\mathsf{T}}$, $\partial_j = \partial/\partial X_j$, and e_j for the j-th standard coordinate vector. Subscripts on functions denote partial derivatives. The dot product is the complex bilinear pairing $a \cdot b = a^{\mathsf{T}} b$. Closures of algebraic images are taken in the Zariski topology. A translation direction of an algebraic set $Y \subseteq \mathbb{C}^m$ is a constant vector d such that $Y + \tau d = Y$ for every $\tau \in \mathbb{C}$.

Theorem 2.1. *Let $V \colon \mathbb{C}^3 \dashrightarrow \mathbb{C}^3$ be rational with nilpotent DV. Suppose $H = \operatorname{Hess} f$ for a polynomial $f \in \mathbb{C}[X_1, X_2, X_3]$, $\det H \neq 0$ as a polynomial, and $H\,DV$ is symmetric. Then the image of V is contained in an affine plane.*

Proof. If DV has rank zero, V is constant. Suppose its rank is one. By the characteristic-zero rank criterion the image closure is a curve. At a point where V is defined and its differential has rank one, a line in a direction outside the kernel has nonconstant restriction. Such a restriction is dominant onto the image curve. Its function field thus embeds in a one-variable rational function field. By Lüroth's theorem the image function field is generated by one rational function $t = p/q$ of the source coordinates, with coprime polynomials p, q and $q \neq 0$. Write $V = A(t)$, where $A(T)$ is a vector of rational functions in an indeterminate T.

If the image is contained in no affine plane, the three components of $A'(T)$ are linearly independent over $\mathbb{C}$. The trace identity is $A'(t) \cdot \nabla t = 0$. It implies

$$A'(T) \cdot \nabla(p - Tq) = 0 \quad \text{in } \mathbb{C}(T)[X_1, X_2, X_3].$$

Indeed $q \nabla t = \nabla(p - Tq)|_{T=t}$, where the gradient on the right holds T fixed. After clearing the denominators of $A'(T)$, the resulting polynomial vanishes at $T = p/q$, so it is divisible by $p - Tq$ over the rational function field in the source coordinates. This divisor is primitive as a polynomial in T over the source polynomial ring, since p, q are coprime. Gauss's lemma gives divisibility in the polynomial ring. Differentiation lowers the total degree in

the source variables, and multiplication by a polynomial in T alone does not raise it. The differentiated expression therefore has smaller source degree than $p - Tq$ and must vanish.

Let $M(T)$ be the matrix whose rows are the coefficient vectors of the source monomials in $\nabla(p - Tq)$. Its entries are affine in T, and its rank is at most two. If its rank is two, the cross product of two independent rows gives a polynomial kernel vector of degree at most two. The independence of the components of A' implies independence of the three constant coefficient vectors of this polynomial vector. Conjugating V by the linear map that sends these three coefficient vectors to the standard coordinate vectors makes the kernel vector $(1, T, T^2)^{\mathsf{T}}$. In coordinates x, y, z the coefficient identities are

$$p_x = 0, \qquad p_y = q_x, \qquad p_z = q_y, \qquad q_z = 0.$$

The four identities give $p = P(y) + B(z)$, $q = xP'(y) + C(y)$, and $B'(z) = xP''(y) + C'(y)$. Comparison of the coefficient of x gives $P'' = 0$, and the remaining equality makes B' and C' constant. Thus $p = ay + bz + c$, $q = ax + by + d$, and $\nabla(p - Tq)$ is constant in the source variables. This contradicts rank two for M.

The matrix M has rank one, since rank zero would make p, q and hence t constant. Its row space defines a nonzero linear form $L(T)$ in the source variables over $\mathbb{C}(T)$. Derivatives of $p - Tq$ along the kernel of $L(T)$ vanish, so $p - Tq$ is a polynomial in $L(T)$. If its source degree is $D \geq 2$, its top homogeneous part is $c(T)L(T)^D$ with $c(T) \neq 0$. This part is affine in T, so its second derivative in T is zero. Dividing that derivative by L^{D-2} and reducing modulo L gives $D(D-1)c(T)(L')^2 = 0 \bmod L$. Since a nonzero linear form is prime, L' is proportional to L, and all ratios of nonzero coefficients of L have zero T-derivative. They are therefore constant. Then p, q are both polynomials in a single constant linear form, obtained by evaluating $p - Tq$ at two general constants T. The same conclusion holds for $D = 1$ if their linear parts are proportional. In either case the identity $A'(T) \cdot \nabla(p - Tq) = 0$ gives a fixed nonzero linear relation on A'. Thus p, q are affine with independent linear parts. Use $x = p$, $y = q$ as the first two new source coordinates and conjugate V by the same affine change. The Jacobian changes by conjugation and the Hessian by inverse congruence, which preserve nilpotence and Hessian symmetry.

Now $t = x/y$, and $\nabla t = (1, -t, 0)^{\mathsf{T}}/y$ in the trace identity gives $A'_1 = tA'_2$. Independence gives $A'_2 \neq 0$. Put $k = A'_3/A'_2$ and $F(t, y, z) = f(ty, y, z)$. Symmetry of the outer product $H\,DV = HA'(t)\nabla t^{\mathsf{T}}$ makes $HA'(t)$ proportional to ∇t. Thus $H(t, 1, k)^{\mathsf{T}}$ is proportional to $(1, -t, 0)^{\mathsf{T}}$. Contracting with $(t, 1, 0)$ and $(0, 0, 1)$ gives, by the chain rule at fixed t,

$$\partial_y(F_y + kF_z) = \partial_z(F_y + kF_z) = 0.$$

In particular $F_y + kF_z$ depends only on t.

The coefficient of $y^j z^\ell$ in F has t-degree at most j, since substituting $x = ty$ sends $x^a y^b z^\ell$ to $t^a y^{a+b} z^\ell$. Since $\det H \neq 0$, we have $F \neq 0$. Write

$F = \sum_{\ell=0}^{N} F_\ell(t,y)z^\ell$, with $N = \deg_z F$. For $N \geq 1$, the coefficient of z^N in $F_y + kF_z$ is $\partial_y F_N = 0$. The degree restriction at $j = 0$ makes $F_N = c$ a nonzero constant. For $N \geq 2$, the coefficient of z^{N-1} is $\partial_y F_{N-1} + Nc\,k = 0$. Hence the coefficient of yz^{N-1} in F is $-Nc\,k$, so $k = at + b$ for constants a, b. Then $A_3' = aA_1' + bA_2'$, contradicting independence. For $N \leq 1$, the only possible term involving z is a constant multiple of z, and the condition on $F_y + kF_z$ makes F_0 affine in y. The same degree restriction gives $F = (at + b)y + cz + d$ with constant coefficients, where $c = 0$ if $N = 0$. This makes f affine, contrary to $\det H \neq 0$.

Suppose now that $N = DV$ has rank two. At a point where N has rank two and H is invertible, use a constant conjugation to make N the Jordan block J with $J_{12} = J_{23} = 1$ and all other entries zero. The identity $HJ = J^{\mathsf{T}}H$ gives

$$H = \begin{pmatrix} 0 & 0 & a \\ 0 & a & b \\ a & b & c \end{pmatrix}, \qquad a \neq 0.$$

A further constant change preserving J makes H the antidiagonal identity at this point. Write $H = R(aI + bJ + cJ^2)$, where I is the identity and R the antidiagonal identity. Put $U = (bJ + cJ^2)/a$, so $U^3 = 0$. The change given by the congruence matrix $a^{-1/2}(I - U/2 + 3U^2/8)$ has this effect, since the square of $I - U/2 + 3U^2/8$ multiplied by $I + U$ is I, and $RJ = J^{\mathsf{T}}R$. Write $K_{i,j\ell} = \partial_j\partial_\ell V_i$ and $T_{ij\ell} = \partial_i\partial_j\partial_\ell f$, evaluated at this point. At the chosen point the derivatives of $\operatorname{tr}(N^3), \operatorname{tr}(N^2), \operatorname{tr}(N)$, divided by $3, 2, 1$, respectively, give

$$K_{3,1\ell} = 0, \qquad K_{2,1\ell} + K_{3,2\ell} = 0, \qquad K_{1,1\ell} + K_{2,2\ell} + K_{3,3\ell} = 0.$$

The $(1,2)$ and $(1,3)$ symmetry equations for $\partial_\ell(HN) = (\partial_\ell H)J + R\,\partial_\ell N$ at the chosen point give

$$T_{11\ell} + K_{3,2\ell} = K_{2,1\ell}, \qquad T_{12\ell} + K_{3,3\ell} = K_{1,1\ell}.$$

The second trace equation makes the first identity $T_{11\ell} = -2K_{3,2\ell}$. The first trace equation gives $K_{3,31} = K_{3,13} = 0$. Thus $T_{112} = T_{121}$ gives $K_{1,11} = -2K_{3,22}$. The second trace equation gives $K_{2,21} = K_{2,12} = -K_{3,22}$. Substituting into the third at $\ell = 1$ yields $K_{1,11} = K_{3,22}$. In particular $K_{3,22} = 0$. Let ω be a local nonzero one-form annihilating $\operatorname{im} N$, with value dX_3 at this point. Differentiating $\omega N = 0$ gives

$$\partial_\ell\omega_1 = -K_{3,2\ell}, \qquad \partial_\ell\omega_2 = -K_{3,3\ell}.$$

The coefficient of $dX_1 \wedge dX_2 \wedge dX_3$ in $\omega \wedge d\omega$ at the point is $-K_{3,31} + K_{3,22} = 0$. Thus the rank-two distribution $\operatorname{im} N$ is involutive on the regular locus.

Assume the image of V is not planar. Locally write $V = (u, v, w(u,v))$, where u, v are two algebraically independent components of V and w is algebraic over $\mathbb{C}(u,v)$. Let $p = w_u$, $q = w_v$; derivatives in u, v extend uniquely to the image function field in characteristic zero. Write M for the 3×2 matrix with columns $(1,0,p)^{\mathsf{T}}, (0,1,q)^{\mathsf{T}}$, and let $J_0 = D(u,v)$.

Since $N = MJ_0$ is nilpotent of rank two in dimension three, N^2 has rank one. The matrix M is injective and J_0 is surjective over the source function field. Hence $N^2 = M(J_0M)J_0$ gives $\operatorname{rank}(J_0M) = 1$, and $N^3 = 0$ gives $(J_0M)^2 = 0$. A nonzero nilpotent two-by-two matrix has a nonzero off-diagonal entry. Exchange the first two source and target coordinates if needed to put that entry in position $(1,2)$. Call it b and write the $(1,1)$ entry as $-b\beta$. Trace and determinant zero give

$$J_0M = b\begin{pmatrix} -\beta & 1 \\ -\beta^2 & \beta \end{pmatrix}, \qquad b \neq 0.$$

The columns of M define vector fields $L_1 = \partial_1 + p\partial_3$ and $L_2 = \partial_2 + q\partial_3$. Their bracket is $(L_1(q) - L_2(p))\partial_3$. The two columns of J_0M give $(L_1(u), L_1(v)) = (-b\beta, -b\beta^2)$ and $(L_2(u), L_2(v)) = (b, b\beta)$. The chain rule then gives

$$[L_1, L_2] = -b(w_{uu} + 2\beta w_{uv} + \beta^2 w_{vv})\partial_3.$$

Since $\partial_3 \notin \operatorname{im} N$, involutivity makes the coefficient vanish. If all three second derivatives vanished, w_u, w_v would have zero differential and would be constant. Indeed, if an element of the image field has both partial derivatives zero, differentiating its monic minimal polynomial over $\mathbb{C}(u,v)$ gives a relation of smaller degree for each partial derivation. Minimality forces both derivatives of every coefficient to vanish, so the coefficients are in $\mathbb{C}$ and the element itself is in $\mathbb{C}$. Subtracting the corresponding linear function of u, v from w would then give a constant. This contradicts nonplanarity. Hence β is algebraic over the image function field. On each component of the geometric generic fiber, p, q, β are constant. Here the geometric generic fiber means the fiber after extending the image function field to its algebraic closure. Its tangent line is $\ker N$. Since $N^2 = M(J_0M)J_0$ has rank one and image equal to $\ker N$, that kernel is generated by

$$(1, \beta, p + q\beta)^{\mathsf{T}}.$$

Two independent linear forms vanishing on this vector are constant on the component, because their differentials vanish there in characteristic zero. The closure of the component is therefore an affine line.

Let B be the closure of the image of the rational map sending a source point X to the projective closure of $X + \ker N(X)$ in the Grassmannian of lines in $\mathbb{P}^3$. The fibers of this assignment are contained in their assigned lines and contain a dense open subset of those lines at general points, by the description of the geometric generic fibers of V. Hence B is an irreducible surface. On a nonempty open subset of B the restrictions of V to the lines are defined and constant, and the lines meet the rank-two regular locus. To see openness after discarding a proper closed set, trivialize the family of lines on a Grassmannian chart. For each component of V, constancy on a line is the proportionality of the coefficient vectors of the numerator and denominator as polynomials in the line parameter, with denominator not identically zero. The proportionality identities hold at the generic line. The omitted subfamily has dimension at most one, so its incidence has dimension

at most two. At a general point of the affine chart each remaining line through the point has tangent direction $\ker N$ there. Thus there is exactly one family line through a general point.

Over a resolution of B, the incidence space is a smooth $\mathbb{P}^1$-bundle. Its evaluation map to $\mathbb{P}^3$ is proper and birational. The ramification divisor is the divisor of the determinant of the differential, a section of the canonical line bundle of the incidence space tensored with the pullback of the inverse canonical line bundle of $\mathbb{P}^3$. On a general incidence fiber these canonical bundles have degrees -2 and -4, respectively, so the ramification divisor has degree two there. Over the generic point of a divisor of $\mathbb{P}^3$, the evaluation map has zero-dimensional fiber. Otherwise the inverse image of the divisor would have dimension at least three, contradicting dominance and irreducibility of the incidence space. By upper semicontinuity of fiber dimension, the map is proper and quasi-finite over an open neighborhood of that generic point, hence finite there. A finite birational morphism to a normal variety is an isomorphism. The image of each ramification component therefore has dimension at most one. If one component maps to a point, its intersection with each incidence fiber contains at most one point because evaluation embeds each fiber as its corresponding line. Thus the two-dimensional component dominates the two-dimensional base, and all general family lines pass through its image. Otherwise the focal locus, defined as the scheme-theoretic image of the ramification divisor, has one-dimensional support.

De Poi's classification [9, Theorem 0.1] gives three possibilities with a one-dimensional focal locus. The lines are the secants of a twisted cubic, the pencils in the planes through a fixed line with their centers on that line, or the joins of a fixed line and a curve. We consider these families and the families through one point. In each case $\ker N$ is orthogonal to $\operatorname{im} N$ for the bilinear pairing $a^{\mathsf{T}} H b$, since N is self-adjoint for that pairing. Also $\ker N = \operatorname{im} N^2 \subseteq \operatorname{im} N$. In particular $d^{\mathsf{T}} H d = 0$ for every $d \in \ker N$.

Two generically finite endpoints. For a secant family or a join family whose two endpoint loci are not contained in the plane at infinity, write $X = (1-\lambda)A(s)+\lambda B(t)$, $r = B(t)-A(s)$, and $V(X) = W(s,t)$, where s, t are local parameters on the endpoint curves. The derivatives of X with respect to s, t, λ are $(1-\lambda)A', \lambda B', r$, so dominance gives generic independence of A', B', r. In this frame the columns of N are the coordinates of $W_s/(1-\lambda), W_t/\lambda, 0$. Its trace is the sum of the A' coefficient of W_s divided by $1-\lambda$ and the B' coefficient of W_t divided by λ. Multiplying by $\lambda(1-\lambda)$ shows that these coefficients both vanish, since they depend only on s, t. Thus $W_s - \alpha B'$ and $W_t - \gamma A'$ are proportional to r, with α, γ depending on s, t. The sum of the three principal two-by-two minors is $-\alpha\gamma/(\lambda(1-\lambda))$, which vanishes by nilpotence. Since W_s, W_t are independent generically, exactly one of α, γ is nonzero generically. Take $\alpha = 0$, so $\operatorname{im} N = \operatorname{span}(A', r)$. Let n be a normal to this plane with respect to the usual bilinear dot product. Then Hr is proportional to n. Integrating along the line and using polynomiality at the

finite endpoint gives

$$\nabla f(X) = \nabla f(A) + n\,R(\lambda), \qquad f(X) = f(A) + \lambda \nabla f(A) \cdot r.$$

Here R may depend on s, t. The second identity follows by integrating $\partial_\lambda f = \nabla f(X) \cdot r = \nabla f(A) \cdot r$. Differentiating it in t at fixed s, λ gives $\lambda(\nabla f(X) - \nabla f(A)) \cdot B' = 0$. Since n annihilates precisely $\operatorname{span}(A', r)$, we have $n \cdot B' \neq 0$. It follows that $R = 0$. This makes the gradient on a dominant family depend only on s, contradicting its rank three. For $\gamma = 0$ the same calculation uses the finite endpoint $B(t)$ and differentiation in s.

Lines through a point. If the point c is finite, the direction $X - c$ is H-isotropic. The identity $(X-c)^{\mathsf{T}} H(X)(X-c) = 0$ multiplies the degree-j part of the expansion of f about c by $j(j-1)$. Thus f is affine, a contradiction.

For a point at infinity choose the common line direction to be e_3. Then $V = (A(x,y), B(x,y), C(x,y))$, and H-isotropy gives $f = zP(x,y) + Q(x,y)$. The polynomial P is nonconstant since H is nonsingular. The two-dimensional Jacobian of (A, B) is nilpotent. If (A, B) is constant the desired relation follows. Otherwise it has rank one and $(A, B) = (a(t), b(t))$ for a nonconstant rational function t by Lüroth's theorem as above. Let $r = (a'(t), b'(t))^{\mathsf{T}}$. The planar Jacobian is $r(\nabla t)^{\mathsf{T}}$. Its trace is $(\nabla t)^{\mathsf{T}} r = 0$. Since $He_3 = (P_x, P_y, 0)^{\mathsf{T}}$ annihilates $\operatorname{im} N$, also $(\nabla P)^{\mathsf{T}} r = 0$. Therefore $dt \wedge dP = 0$, as r is nonzero. By the characteristic-zero rank criterion t is algebraic over $\mathbb{C}(P)$. On each component of a general level of P, t is constant by algebraicity. The tangent direction is then the constant direction r. A linear form annihilating r has zero differential on the component, so the component is dense in an affine line. Since P is polynomial, each such line lies in its level. Lines in distinct levels cannot meet, so they are parallel. Comparing with a line in a different level shows that the direction is common to all components of general levels. A constant linear form annihilating this direction therefore annihilates (a', b'), giving a constant affine relation between A and B.

Pencils with centers on a fixed line. For a fixed line not contained in the plane at infinity, a general center is finite, with the constant center case already treated. The restriction of f to each general pencil plane is affine along each line through its center by H-isotropy. Expansion at the center makes f affine on the entire plane. Expanding instead at a fixed finite point common to these planes shows every homogeneous part of degree at least two vanishes on their union, which is dense. This contradicts nonsingularity.

If the fixed line is at infinity, write the parallel planes and their pencils as

$$X = (\lambda, q + a(t)\lambda, t), \qquad V(X) = W(t, q),$$

where t specifies the plane, q specifies its affine line, and $a(t)$ its common slope. A constant a gives parallel lines. Otherwise $q = y - a(z)x$ and $t = z$ give

$$\operatorname{tr} N = -aW_{1,q} + W_{2,q} + W_{3,t} - \lambda a' W_{3,q}.$$

The coefficient of λ vanishes, so $a' \neq 0$ implies $W_{3,q} = 0$. Now the third component of V depends only on $t = z$, and its z-derivative is an eigenvalue of N by the shape of the bottom row. Nilpotence makes that derivative zero, so V_3 is constant.

Joins with an endpoint locus at infinity. If the fixed line is at infinity and the curve has finite general point, the lines through each such point have all directions in a fixed two-dimensional vector plane. By H-isotropy, the restriction of f to the corresponding affine plane is affine. As these planes sweep a dense set, $f = \alpha(z)x + \beta(z)y + \gamma(z)$ in suitable affine coordinates. Its Hessian is singular.

Suppose the fixed line has a finite point and the curve lies at infinity. Put the affine part of the fixed line on the z-axis. Choose the first coordinate so that it is nonzero at the general point of the infinite curve. The infinite endpoint then has coordinates $[1 : t : q(t) : 0]$, and the finite endpoint is $(0, 0, s)$. The general join has affine coordinates $X = (\lambda, t\lambda, s + \lambda q(t))$. The parameter t is nonconstant, since otherwise the joins would all lie in the plane $y = tx$. The function $q(t)$ may be algebraic over $\mathbb{C}(t)$. Write $V(X) = W(s, t) = (a, ta + b, c)$ with a, b, c functions of s, t. The two parameter differentials are $dt = (dy - t\,dx)/\lambda$ and $ds = dz - q\,dx - \lambda q'dt$. The trace of N is $(a + b_t)/\lambda + c_s - qa_s - q'b_s$, so $b_t = -a$ and $c_s = qa_s + q'b_s$. If $b_s = 0$ identically, then $a_s = c_s = 0$, contrary to rank two. The Jacobian N is the product of the matrix with columns W_s, W_t and the matrix with rows ds, dt. Applying ds and dt to these columns gives the reversed product

$$\begin{pmatrix} c_s - qa_s - q'b_s & c_t - qa_t - q'(a + b_t) \\ b_s/\lambda & (a + b_t)/\lambda \end{pmatrix} = \begin{pmatrix} 0 & c_t - qa_t \\ b_s/\lambda & 0 \end{pmatrix}.$$

The reversed product is nilpotent, since its fourth power has $N^3 = 0$ between the same two factors. Its determinant thus vanishes. As $b_s \neq 0$, this gives $c_t = qa_t$. Differentiating $c_s = qa_s + q'b_s$ in t and $c_t = qa_t$ in s yields $q'a_s + q''b_s + q'b_{st} = 0$. The relation $b_t = -a$ cancels the first and third terms, so $q'' = 0$. Differentiating the monic minimal polynomial of q' over $\mathbb{C}(t)$ forces its coefficients to have zero derivative, by minimality. Hence q' is constant, and the same argument for $q - tq'$ shows that q is affine in t. The covector $(tq' - q, -q', 1)$ is therefore constant and nonzero. Its products with W_s, W_t are $c_s - qa_s - q'b_s$ and $c_t - qa_t - q'(a + b_t)$, respectively, both zero. It gives a constant affine relation on V. If both endpoint loci lie at infinity, the joins remain in that plane and cannot dominate the affine chart. □

Lemma 2.2. *Let $h \in \mathbb{C}[X_1, X_2, X_3, w]$ have Hessian rank three, and suppose $A(X) = \mathrm{Hess}_X h(X, 0)$ is nonsingular over $\mathbb{C}(X_1, X_2, X_3)$, where Hess_X differentiates in X_1, X_2, X_3 alone. Set*

$$v(X) = A(X)^{-1} \nabla_X h_w(X, 0).$$

Then

$$\nabla_X h(X - wv(X), w) = \nabla_X h(X, 0).$$

Moreover Dv is nilpotent and $A\,Dv$ is symmetric.

Proof. The gradient-image closure is an irreducible hypersurface, because the dimension of the image of a polynomial map in characteristic zero equals the generic rank of its differential. Let R be an irreducible defining polynomial of this hypersurface and put $Q = \nabla R(\nabla h)$. The relation $R(\nabla h) = 0$ implies $\mathrm{Hess}(h)Q = 0$. For $Z = (X_1, X_2, X_3, w)$, the derivation $D_Q = \sum_{i=1}^{4} Q_i\, \partial/\partial Z_i$ annihilates every component of ∇h. By the chain rule it also annihilates every component of Q. Inductively, for any polynomial g and integer $j \geq 1$,

$$D_Q^j g = \sum_{i_1,\ldots,i_j=1}^{4} Q_{i_1} \cdots Q_{i_j}\, \frac{\partial^j g}{\partial Z_{i_1} \cdots \partial Z_{i_j}},$$

since D_Q acts only on the derivative of g in each summand. The coefficient of τ^j in the polynomial $g(Z + \tau Q(Z))$ is therefore $D_Q^j g/j!$. Taking g to be each component of ∇h gives $\nabla h(Z + \tau Q(Z)) = \nabla h(Z)$.

The slice $w = 0$ maps densely into the hypersurface, since the first three gradient coordinates have Jacobian A there. Thus $Q(X, 0) \neq 0$ generically. On the slice the first three Hessian rows are $(A\ \nabla_X h_w(X, 0))$, so their kernel is spanned by $(-v, 1)$. The fourth row leaves this kernel unchanged because the full Hessian has rank three there. Hence $Q(X, 0) = Q_4(X, 0)(-v(X), 1)$, where Q_4 is its fourth component. Substitution of $\tau = w/Q_4(X, 0)$ gives the claimed characteristic identity. Differentiating it in X yields

$$\mathrm{Hess}_X\, h(X - wv(X), w)(I - wDv) = A(X),$$

where I is the three-by-three identity. Both matrices on the left have entries in $\mathbb{C}(X)[w]$. Their determinants have nonzero product independent of w, so each determinant is constant in w. At $w = 0$ the second determinant is one, giving $\det(I - wDv) = 1$. Thus the characteristic polynomial of Dv is T^3, and Cayley–Hamilton gives $(Dv)^3 = 0$. Finally, differentiating $Av = \nabla_X h_w(X, 0)$ in X gives ADv as the difference of $\mathrm{Hess}_X\, h_w(X, 0)$ and the matrix with (i, j) entry $\sum_{k=1}^{3} v_k \partial_j A_{ik}$. The latter is symmetric because $\partial_j A_{ik} = \partial_i A_{jk}$. □

Theorem 2.3. *If $h \in \mathbb{C}[X_1, X_2, X_3, X_4]$ has Hessian rank three, then the closure of its gradient image has a nonzero translation direction.*

Proof. Choose a constant linear change and a constant slice as in Lemma 2.2. Such a choice exists because a symmetric rank-three matrix at a regular point restricts nondegenerately to a three-plane. Translate the remaining coordinate so the slice is $w = 0$. Apply Theorem 2.1 to v, using the Hessian of $h(X, 0)$ as the symmetrizing matrix. There are a constant $b \in \mathbb{C}^3 \setminus \{0\}$ and $d \in \mathbb{C}$ with $b \cdot v = d$.

As in Lemma 2.2, let R be an irreducible equation of the gradient-image closure and $Q = \nabla R(\nabla h)$. On the slice $w = 0$, the kernel direction $(-v, 1)$ gives $(b, d) \cdot Q = 0$. The slice has dense image in $R = 0$, so the derivative of R in direction (b, d) vanishes on $R = 0$. Irreducibility makes that derivative

divisible by R, and the degree bound forces it to vanish identically. Thus $R(Y+\tau(b,d)) = R(Y)$ for every target point Y and scalar τ. An affine source change $Z = SZ' + a$ sends the gradient image by the invertible target map $Y \mapsto S^{\mathsf{T}}Y$, so existence of a nonzero translation direction also holds in the original coordinates. □

Corollary 2.4. *Let $h \in \mathbb{C}[X_1, X_2, X_3, X_4]$ have identically zero Hessian determinant. There exist a nonzero polynomial $R \in \mathbb{C}[Y_1, Y_2, Y_3, Y_4]$ and a nonzero $d \in \mathbb{C}^4$ such that $R(\nabla h) = 0$ and $D_d R = 0$, where $D_d R = \sum_{i=1}^4 d_i \partial R/\partial Y_i$. Thus Problem 1 of* [7, Section 6] *has an affirmative answer.*

Proof. For rank three, take an irreducible equation R of the gradient-image closure and the nonzero translation direction d from Theorem 2.3. Translation invariance implies that $D_d R$ vanishes on $R = 0$, hence is divisible by R. Its degree forces $D_d R = 0$. If the rank is at most two, the Jacobian of the first three gradient coordinates has rank at most two, so those coordinates are algebraically dependent in characteristic zero. A nonzero relation R in Y_1, Y_2, Y_3 then satisfies $D_d R = 0$ for $d = (0,0,0,1)^{\mathsf{T}}$. □

3. Weighted leading forms

Fix $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ with nonzero constant Hessian determinant. All Hessian ranks are taken over the rational function field in the source variables unless a specialization is specified. For rational weights $w_1, \ldots, w_4$, the weight of $\prod_{i=1}^4 x_i^{a_i}$ is $\sum_{i=1}^4 a_i w_i$. The leading form is the sum of the terms of maximal weight. Let E be this maximal weight for f and put

$$W = 4E - 2\sum_{i=1}^{4} w_i.$$

In position (i,j) of the Hessian, differentiating twice lowers the weight of every surviving monomial by $w_i + w_j$. Hence each product in the determinant expansion has weight at most W, and its component of weight W comes from the leading form. Since the full determinant is a nonzero constant, $W \geq 0$. For $W > 0$ the leading form has zero Hessian determinant, and for $W = 0$ it has the same Hessian determinant as the full polynomial.

Along an affine path of weights, each monomial weight is affine in the path parameter. We choose paths on which the monomials of the current leading form stay tied, and call the first parameter at which another monomial joins them a *wall*. The finite monomial support gives a strict gap until that parameter. At a wall, a small further step in a specified direction selects the terms of greatest slope from the tied terms, since all others have a strict gap.

Lemma 3.1. *If all weights are positive and the leading form has nonzero constant Hessian determinant, the gradient of the full polynomial is a polynomial automorphism.*

Proof. Let P be the leading form and $A = \operatorname{Hess} P(0)$. This matrix is invertible. Both A and A^{-1} have entries supported on pairs (i, j) with $w_i + w_j = E$, because A identifies each weight block with its complementary block. Thus component i of $A^{-1}\nabla f$ has weight at most w_i. The part of its linear terms in variables of weight w_i is x_i: the leading form contributes $A^{-1}AX = X$, where $X = (x_1, \dots, x_4)^{\mathsf{T}}$, and the lower-weight terms contribute only weights less than w_i. Any nonlinear monomial of weight at most w_i uses only variables of weight less than w_i, since every factor has positive weight. Successive substitution in increasing order of the weights gives the polynomial inverse. □

An *apex direction* is a vector $d \in \mathbb{C}^4$ such that the Zariski closure of the gradient image is invariant under translation by λd for all $\lambda \in \mathbb{C}$. The apex directions form a vector space. A nonzero such direction is a projective image apex in the terminology of [6, Theorem 7.2 and Corollary 7.3]: the condition that each polynomial relation on the gradient still vanishes after translation by λd is equivalent to invariance of the closure. The *complementary source space* of an apex space is its annihilator under the source-target pairing $(d, v) \mapsto d^{\mathsf{T}} v$.

For Hessian rank three, de Bondt's classification gives the following possibilities. If the apex space is one-dimensional, align it with a target coordinate axis. Over the rational function field of the corresponding source variable, the potential is a polynomial in a linear form of the other three variables plus an affine polynomial in them. The Hessian restricted to the complementary source space has rank one. Rank zero would give full Hessian rank at most two, since only one row and column are added. If the apex space has dimension two, the potential is jointly affine in the two complementary source variables. If it has dimension three, there is a one-dimensional constant source kernel. For Hessian rank two, either there is a two-dimensional constant source kernel or the apex space is one-dimensional and the potential is jointly affine in the three complementary source variables.

The gradient of a form of weight E is equivariant for source scaling with weights w_i and target scaling with weights $E - w_i$. Here and below, rational weights may be cleared of denominators to define the diagonal action of $\mathbb{C}^\times$. The equal-weight blocks on source and target coincide. The apex space is invariant under the target scaling, and the constant Hessian kernel under source scaling. Both spaces therefore split into their intersections with the equal-weight blocks. Subtracting an affine polynomial translates the gradient and does not change its Hessian.

Let P be a wall form and H the leading form just before the wall. Use the old weights from a point on the preceding open segment, with E the old weight of H. Then $P - H$ consists of terms of weight less than E. Write adj for the classical adjugate. Each product in entry (i, j) of $\operatorname{adj}(\operatorname{Hess} P)$ has old weight at most $3E - 2\sum_k w_k + w_i + w_j$, because the deleted row and column omit weights w_j, w_i. Its component at that bound is the corresponding entry for H. For constant vectors u, v, the Hessian minor on u, v is the

determinant of the two-by-two matrix of the Hessian bilinear form in those vectors. If u, v are homogeneous for the old weights, the same comparison shows that the old minor survives as the maximal-weight part of the new one. If a constant vector d has a unique nonzero coordinate of greatest old weight and the corresponding principal cofactor of $\operatorname{Hess} H$ is nonzero, then $d^{\mathsf{T}} \operatorname{adj}(\operatorname{Hess} P) d \neq 0$: its term of greatest old weight comes from that diagonal entry alone.

When $\operatorname{Hess} P$ has rank three, the gradient image closure is an irreducible hypersurface. Its reduced equation has nonzero gradient on the smooth locus. The image of ∇P is dense, so this normal, evaluated at ∇P, is nonzero over the rational function field. Differentiating the equation shows that the normal spans $\ker \operatorname{Hess} P$. The adjugate is a nonzero scalar multiple of the outer product of this normal with itself. Since a translation direction is tangent to the hypersurface, its adjugate contraction vanishes.

Under a constant invertible change from the column X of source coordinates to $X' = SX$, the gradient at X' is $S^{-\mathsf{T}} \nabla f(S^{-1} X')$. Thus an apex d becomes $S^{-\mathsf{T}} d$. Suppose the source and target coordinates are split into corresponding base and fiber blocks, and the apex plane is the graph of a matrix U from the target base block to its fiber block. The source shear $\text{base}' = \text{base} + U^{\mathsf{T}} \text{fiber}$ sends this graph to the target base block. At a wall it preserves weights because the graph is invariant under diagonal target scaling. In the applications a variable added to another at a crossing has strictly smaller old weight. Substituting the inverse shear preserves the old leading form, since every term in the expansion involving a replacement variable has smaller old weight than the original term.

Let A, B be nonzero weighted homogeneous polynomials in two variables of positive weights, of respective weighted degrees $p, q > 0$. If $dA \wedge dB = 0$, contracting this equality with the weighted Euler field gives $pA\, dB = qB\, dA$. Clear the weight denominators to take p, q integral. Then $d(A^q / B^p) = 0$, so $A^q / B^p \in \mathbb{C}^\times$. For each irreducible factor, its multiplicities in A, B are in the ratio $p : q$. Dividing that ratio by $\gcd(p, q)$ gives $A = aS^r$, $B = bS^s$ for nonzero constants a, b, positive integers r, s, and a common polynomial S. The equality $A = aS^r$ forces S to be weighted homogeneous by comparing its smallest and largest weights. Call S genuinely binary if it is not a polynomial in a nonzero constant linear form. Equivalently, $D_v S$ is nonzero for every nonzero constant base vector v, where D_v denotes differentiation in the constant direction v. In fact a vanishing derivative makes S independent of the coordinate in direction v. If S is a polynomial in a linear form and the base weights differ, that form is proportional to a coordinate. Otherwise its highest ordinary power contains monomials of different weights and the same ordinary degree.

In the one-apex form, let x be the source variable corresponding to the apex, of nonzero weight w_x. Call it the active variable, and call the three remaining variables $Y = (Y_1, Y_2, Y_3)^{\mathsf{T}}$ passive. Passive degree means the total degree in Y_1, Y_2, Y_3. The part of passive degree at least two is a linear

combination over $\mathbb{C}(x)$ of the powers L^k for $k \geq 2$, where $L = \ell(x)^{\mathsf{T}} Y$ for a coefficient column $\ell(x) \in \mathbb{C}(x)^3$. We call L the phase. A phase variable is a passive variable with nonzero coefficient in L. When the passive Hessian has rank one, normalize a nonzero entry ℓ_i to 1. The passive Hessian is the second derivative in L times $\ell\ell^{\mathsf{T}}$, so the ratio of its (i,j) to its (i,i) entry is ℓ_j. This ratio is homogeneous of weight $w_{Y_i} - w_{Y_j}$, where w_{Y_k} denotes the weight of Y_k. A nonzero homogeneous rational function of x has no zero or pole in $\mathbb{C}^\times$, since scaling would give infinitely many. Thus ℓ_j is zero or a constant times $x^{(w_{Y_i} - w_{Y_j})/w_x}$ with integral exponent. For $k \geq 2$, the coefficient of L^k is the coefficient of Y_i^k in the restriction obtained by setting the other passive variables to zero, so it is polynomial in x.

3.1. **Two fiber variables with a common binary polynomial.**

Lemma 3.2. *Suppose the leading form is $R(x,y) + zA(x,y) + wB(x,y)$ with $A = aS^r$ and $B = bS^s$ for nonzero constants a, b, positive integers r, s, and a genuinely binary polynomial S. The weights of x, y are positive, the weights of z, w are distinct, and $W > 0$. Then after a constant linear change f is jointly affine in two variables, with a Keller map as its pair of coefficients.*

Proof. Raise both fiber weights at unit speed, fixing the base weights. After a small step the leading form is $H = zA + wB$. The value of W is constant. Define

$$U = S_y^2 S_{xx} - 2S_x S_y S_{xy} + S_x^2 S_{yy}.$$

If $U = 0$, the tangent direction is constant on each irreducible component of a regular generic level of S. On the chart $S_y \neq 0$, the derivative of S_x/S_y along $S_y\partial_x - S_x\partial_y$ equals U/S_y^2. A rational function on a complex curve with zero differential is constant. On a component with $S_y = 0$, the chart $S_x \neq 0$ gives the same conclusion with the roles of x, y reversed. A constant tangent direction forces a linear form annihilating that direction to be constant on the component. Its closure is then a line contained in the level. Lines in different levels are disjoint and hence parallel. Comparing each component with a different level gives a fixed direction along all generic components. The derivative of S in that direction vanishes on a dense set, so S is a polynomial in a constant linear form. This contradiction gives $U \neq 0$.

Write A', A'', B', B'' for the first and second derivatives of the one-variable polynomials aT^r, bT^s at $T = S$. The base Hessian of H is

$$(zA' + wB') \operatorname{Hess} S + (zA'' + wB'')(\nabla S)(\nabla S)^{\mathsf{T}}.$$

The cofactor complementary to z is the determinant of this two-by-two block bordered by $B'\nabla S$ and a zero diagonal entry. Expanding along the last row and column evaluates it as the negative of the base Hessian on $B'(S_y, -S_x)^{\mathsf{T}}$. The outer-product term vanishes on this vector, so the cofactor is $-(B')^2(zA' + wB')U$. The cofactor complementary to w is $-(A')^2(zA' + wB')U$. Both are nonzero. The minors on v and one of the fiber axes are $-(D_v A)^2$ and $-(D_v B)^2$, respectively, for any nonzero

constant base vector v. Since $D_vA = A'D_vS$ and $D_vB = B'D_vS$, both minors are nonzero.

At a first wall the two cofactors survive, so the new form has rank three. Its apex space exists by Theorem 2.3. It has zero intersection with the fiber plane: this intersection is weight invariant, and the two distinct fiber weights and nonzero cofactors exclude both fiber axes. If the apex space were a line, its weight block would contain at most one fiber axis. Choose the other fiber and an old-weight-homogeneous base direction annihilated by the apex. Such a base direction exists because the two base weights are either equal, or the apex has at most one base component. The surviving mixed minor in these directions contradicts complementary rank at most one. The apex space is therefore a graph over the base plane.

The transpose graph shear puts the new form into jointly affine form in the two fibers. At a wall a distance $\tau > 0$ along this segment, a monomial of total fiber degree j tied at the wall has old weight $E+(1-j)\tau$. In particular its fiber-free part vanishes and its fiber-linear part in the old coordinates is exactly $zA+wB$. The inverse shear fixes the two fibers and is the identity on the base at $z = w = 0$. Substituting it into a form with zero fiber-free part leaves the fiber-linear part unchanged. Thus the shear restores H exactly, and the weights can continue increasing at the same speeds. Every genuine wall requires a base/fiber weight equality. The base weights stay fixed, so the four base/fiber pairs yield at most four walls. Beyond the last wall, a term of total fiber degree at least two would overtake H. The full polynomial is therefore jointly affine in the fibers. Lemma 1.2 gives the required planar coefficient map. □

3.2. **A phase involving three variables.**

Lemma 3.3. *Under the standing hypothesis on f, its leading form cannot be*

$$a(x)(Y_1 + bx^rY_2 + cx^sY_3)^m$$

with a nonzero polynomial $a(x)$, $m \geq 2$, $bc \neq 0$, distinct exponents $0, r, s \in \mathbb{Z}$, active weight $w_x = 1$, three distinct passive weights greater than one, and $W > 0$. The expression is understood to be polynomial even if some phase coefficients are Laurent monomials.

Proof. For a polynomial $F(x, L)$ over $\mathbb{C}(x)$ in the passive linear form $L = \ell(x)^{\mathsf{T}}Y$, the passive Hessian is $F_{LL}\ell\ell^{\mathsf{T}}$. The mixed Hessian row is $(F_{xL} + F_{LL}\partial_x L)\ell^{\mathsf{T}}+F_L(\ell')^{\mathsf{T}}$, where a prime denotes differentiation in x. To compute the principal cofactor deleting a passive variable, keep the other two passive indices i, j. Expansion along the active row and column yields $-F_{LL}$ times the square of the mixed row evaluated on $(\ell_j, -\ell_i)^{\mathsf{T}}$. Thus this cofactor equals $-F_{LL}F_L^2(\ell_i'\ell_j - \ell_j'\ell_i)^2$. For coefficients proportional to $1, x^r, x^s$ with distinct exponents all three cofactors are nonzero.

Raise all passive weights at unit speed. The leading weight has slope m and W has positive slope $4m - 6$. At the first new wall the cofactors

survive. The wall form has rank three, and its apex space is the target x axis, since the distinct weights and surviving cofactors exclude all other axes and Theorem 2.3 makes the apex space nonzero. A term of passive degree j tied at the wall a distance $\tau > 0$ from the preceding weights has old weight $E + (m - j)\tau$. Thus degrees below m are absent and the degree-m part is the old form. In the complementary rank-one representation its pure Y_1^m coefficient is nonzero. With the Y_1 coefficient of the linear form normalized to one, the ratios of the $Y_1^{m-1}Y_j$ coefficients to m times the Y_1^m coefficient recover the other two phase coefficients. They are exactly bx^r, cx^s. The highest power at a genuine wall therefore has degree greater than m. A small common raise retains that power with the same phase. The passive degree of the full polynomial bounds this strictly increasing sequence. The final degree equals the maximal passive degree D of the full polynomial, since a term of larger passive degree would overtake the final form.

Let N_D be the sum of the terms of passive degree D. Its Hessian determinant is zero by weights $(0, 1, 1, 1)$, for which $W = 4D - 6 > 0$. At any finite point on the final segment, the leading form of N_D is the same as that of f. Its three nonzero passive cofactors survive the addition of the lower-weight terms of N_D. Each nonzero passive vector has a unique nonzero coordinate of greatest weight on that segment, so its adjugate contraction is nonzero. Under the weights $(0, 1, 1, 1)$, the apex space splits into its active and passive intersections. The passive intersection vanishes, and Theorem 2.3 gives the active axis as the apex space. It follows that $N_D = a(x)(l(x)^{\mathsf{T}}Y)^D$ for a rational coefficient vector l. Its components are linearly independent over $\mathbb{C}$: otherwise a constant passive kernel would make the rank-three gradient image a hyperplane with three independent translations.

The Wronskian, the determinant with rows $l^{\mathsf{T}}, (l')^{\mathsf{T}}, (l'')^{\mathsf{T}}$ with primes denoting derivatives in x, is nonzero. To see this, at any point where they are regular, use constant row operations on the three independent Taylor series to obtain distinct leading orders n_1, n_2, n_3. The coefficient of order $n_1+n_2+n_3-3$ in their Wronskian is the product of their nonzero leading coefficients and the determinant with rows 1, n_i, and $n_i(n_i - 1)$ indexed by columns $i = 1, 2, 3$. This Vandermonde determinant is nonzero. Choose x_0 where the original coefficients have no poles, $a(x_0) \neq 0$, and $l(x_0), l'(x_0), l''(x_0)$ are independent. Translate $x = x_0 + \xi$ and take the three passive coordinates

$$u = l(x_0)^{\mathsf{T}}Y, \qquad v = l'(x_0)^{\mathsf{T}}Y, \qquad t = \tfrac{1}{2}l''(x_0)^{\mathsf{T}}Y.$$

In these coordinates the Taylor series of $l(x_0 + \xi)^{\mathsf{T}}Y$ has initial terms $u + \xi v + \xi^2 t$, and every remaining coefficient is of order at least three in ξ. Thus a monomial $\xi^k u^i v^j t^h$ of N_D satisfies $k \geq j + 2h$. Its terms with equality give

$$P_0 = a(x_0)(u + \xi v + \xi^2 t)^D.$$

Subtract the affine part of the full polynomial and assign weights $(-\eta, 1, 1+\eta, 1+2\eta)$ to (ξ, u, v, t), with $\eta > 0$. The weight of $\xi^k u^i v^j t^h$ is $i + j + h + \eta(j + 2h - k)$. For passive degree $d = i + j + h$ with $2 \leq d < D$, positive

slope $j + 2h - k$ is an integer at least one, so the monomial reaches weight D at some $\eta \leq D - 2$. For passive degree one the only term of slope two is the removed constant multiple of t, and slope at most one reaches weight D no sooner than $\eta = D - 1$. Passive degree zero has nonpositive slope.

If such a wall occurs, $W = 4D - 6 - 4\eta \geq 2$. In the wall form, the terms of passive degree D are exactly P_0. The three passive cofactors of P_0 survive because each product in a given cofactor drops passive degree by the same total number of passive differentiations. Replacing any factor from P_0 by a derivative of a term of lower passive degree strictly lowers the degree of the product. The three cofactors are nonzero by the formula above, and all weights are distinct. Again the unique apex is ξ. The passive degree-D part fixes the rank-one phase to $u + \xi v + \xi^2 t$. A lower power of degree j would have coefficient a constant multiple of $\xi^{(j-D)/\eta}$. Since this is the coefficient of u^j for $j \geq 2$, polynomiality excludes it. Degrees at most one have already been excluded by the wall comparison. This contradicts a genuine wall. Consequently at $\xi = 0$ every monomial of passive degree at least two involves only u. The passive Hessian on this hyperplane has rank at most one, and adjoining the active row and column gives rank at most three. This contradicts the unit determinant. □

3.3. **One or two phase variables.** An inverse or a two-fiber reduction means either a polynomial inverse to the full gradient or a representation $f = R(x, y) + zA(x, y) + wB(x, y)$ after a constant linear source change, with $\det D(A, B) \in \mathbb{C}^\times$.

Lemma 3.4. *Suppose the leading form is*

$$H = ax^\ell (w + x^A y)^m + bx^p z, \qquad ab \neq 0, \quad \ell \geq 0, \quad A, p \geq 1, \quad m \geq 2,$$

where the exponents are integers. Assume $w_x = 1$, the other weights exceed one, and $W > 0$. Then ∇f is a polynomial automorphism and $p = 1$.

Proof. Raise the weights of y, w at unit speed and that of z at speed m. The leading weight has slope m, so W has slope $2m - 4 \geq 0$. For a form $F(x, L) + B(x)z$ with $L = w + r(x)y$, primes denoting derivatives in x, the vector

$$n = (0, -B', F_L r', rB')^{\mathsf{T}}$$

is in the Hessian kernel in coordinate order (x, y, z, w). The passive rows annihilate it because their passive contributions involve only the phase direction $(r, 0, 1)$ and the x component of n is zero. In the active row the difference between the xy entry and r times the xw entry is $F_L r'$, while the xz entry is B'. These terms cancel on n. The cofactor deleting y is $-F_{LL}(B')^2$ by expansion in its z row. Since it is nonzero, the Hessian has rank three, and its adjugate is $-F_{LL} nn^{\mathsf{T}}$ by comparison of the yy entry. With $r = x^A$, $B = bx^p$, and $F = ax^\ell L^m$, all three passive cofactors are nonzero. Before a wall the old passive weights can be taken distinct, since both differences of the z weight from a phase weight have slope $m - 1 > 0$.

The old contraction criterion excludes every passive apex even at a repeated new weight. The active weight is strictly smaller than each passive weight, so weight invariance separates its axis from the passive block. Theorem 2.3 therefore makes the apex space the target x axis.

Let γ, δ be the weights of z, w at the wall, and let E be its leading weight. The pure w^m term allows normalization of the coefficient of w in the passive linear form. For a monomial in x, y, w of passive degree j at the wall a distance $\tau > 0$ from the old weights, the old weight is $E_{\text{old}} + (m - j)\tau$. Consequently the restriction to $z = 0$ has no terms of passive degree below m, and its degree-m part agrees with $ax^{\ell}(w + x^A y)^m$. Comparing the pure w^m and the yw^{m-1} coefficients fixes the coefficient of y in the phase. By the homogeneous ratio formula, the resulting phase is

$$w + x^A y + kx^{\delta-\gamma} z.$$

Every phase power of degree $j \geq 2$ has coefficient proportional to $x^{E-j\delta}$, by its pure w^j term, and only powers $j \geq m$ occur. The coefficient of a term linear in z alone has old weight E_{old} when multiplied by z, because z has slope m. Thus the affine part is exactly $bx^p z$.

If the new z coefficient is nonzero and its exponent differs from $0, A$, take the highest power, with degree M, by raising all three phase variables at unit speed for a small step. The affine z term then drops, and Lemma 3.3 gives a contradiction. If the exponent is 0 or A, remove it by $w' = w + kz$ or $y' = y + kz$, respectively. Each shear preserves the old leading form, since the weight of z crosses the weight of the affected phase variable from below. Restart the path with the highest phase power M and with z speed M. At every wall either $M > m$ or one of the two shears occurs. The degree in the passive variables bounds M, and each difference of the z weight from a phase weight strictly increases on all segments. There are at most two shears.

After the last wall the path can be extended indefinitely. Each monomial then satisfies

$$\deg_y + \deg_w + m \deg_z \leq m,$$

where $\deg_y, \deg_w, \deg_z$ in this inequality are the individual monomial exponents. Thus the full potential is $F(x, y, w) + zB(x)$, and the coefficient bx^p survives in B. Expanding its Hessian determinant along the z row and column gives $-(B')^2 \det \operatorname{Hess}_{y,w} F$. It follows that B' and $\det \operatorname{Hess}_{y,w} F$ are nonzero constants, and $p = 1$. The output f_z determines x by an affine inverse. The two outputs F_y, F_w have a polynomial inverse in y, w over $\mathbb{C}(x)$ by the Hessian theorem in two variables over a field of characteristic zero [5]. Subtract their values at $(y, w) = (0, 0)$ to obtain a map with zero constant term. Its linear coefficient matrix is invertible over $\mathbb{C}[x]$ because its determinant is the constant binary Hessian determinant. To construct its formal inverse, at each total degree in the two output variables solve for the unknown homogeneous part by applying this inverse matrix to the error determined by lower degrees. This constructs a unique formal inverse

over $\mathbb{C}[x]$. Over $\mathbb{C}(x)$ it equals the Taylor expansion at the origin of the polynomial inverse of the shifted map. It therefore truncates, and restoring the output shift gives a polynomial inverse over $\mathbb{C}[x]$. Finally $f_x = F_x + B'z$ determines z polynomially. □

Lemma 3.5. *Suppose the leading form is $ax^\ell y^m + bx^p z + cx^q w$ with nonzero coefficients and integer exponents $\ell \geq 0$, $m \geq 2$, $p > q \geq 1$. Assume $w_x = 1$, the other weights exceed one, and $W > 0$. Then there is an inverse or a two-fiber reduction.*

Proof. Write β for the weight of y and put $E = \ell + m\beta$. The fiber weights are $E - p, E - q$. Raise β at unit speed and both fiber weights at speed m. Then $W = 2(p + q - 1 - \beta)$, with zero at $\beta = p + q - 1$. For a wall at distance τ along the segment, a tied monomial $x^i y^j z^k w^\ell$ has old weight $E_{\text{old}} + [m - j - m(k + \ell)]\tau$, where E_{old} is the leading weight at the start of the segment. At a wall before that value, the nonzero old principal cofactors deleting z, w survive. Deleting z gives the determinant in (x, y, w), which expands along the w row to $-H_{yy}(cqx^{q-1})^2$, where H denotes the old leading form. Deleting w similarly gives $-H_{yy}(bpx^{p-1})^2$. Thus the wall form has rank three and has a nonzero apex space by Theorem 2.3. This space has zero intersection with the fiber plane, whose weights are distinct. If it is a line other than the target x axis, its weight block contains y and at most one fiber axis. The minor in x and the other fiber survives and is nonzero, contrary to complementary rank at most one.

If the apex space has dimension two, it is a graph over (x, y). Only y can mix with a fiber at a weight equality. After the transpose shear, the new form is $R(x, y) + zG(x, y) + wK(x, y)$, with $\det D(G, K) = 0$. The coefficients have distinct degrees p, q. On $y = 0$ they equal bx^p, cx^q, since the corresponding terms have old weight E_{old} and the shear preserves the old leading form. If the common polynomial is genuinely binary, Lemma 3.2 applies. Otherwise it is a power of x, since the base weights differ and the old x powers are nonzero. This fixes both coefficients exactly. The old-weight formula with $k = \ell = 0$ excludes degrees in y below m and fixes the degree-m term. Retain the largest y-degree M and both fiber terms by taking slopes $1, M, M$ for y, z, w. If $M = m$, there is a genuine wall only when the graph shear is nontrivial.

If the apex space is the target x axis, the complementary Hessian has rank one and the phase is

$$y + dx^{\beta-(E-p)}z + ex^{\beta-(E-q)}w$$

at the wall. The y^m term permits normalization of the coefficient of y to one. Setting $z = w = 0$ and comparing old weights excludes all powers below m, including affine terms in y and the active-only term. The monomials linear in a single fiber alone have old weight E_{old} since each fiber has slope m. This retains exactly the affine partners $bx^p z, cx^q w$. If a fiber occurs in the phase with exponent zero, add the corresponding constant multiple of that

fiber to y as a new coordinate. This shear preserves the old leading form since the fiber weight crosses the weight of y from below. Three remaining nonzero phase entries have distinct exponents, and choosing their highest power by a small common raise contradicts Lemma 3.3. With two entries, raise the two phase weights at speed one and the absent variable at the speed of the highest power. That selects the largest power and the absent variable's affine partner, to which Lemma 3.4 applies after naming the phase variables by their increasing weights. With one phase entry, restart at the slopes $1, M, M$.

All restarts use finite support and a sufficiently small positive step. Both fiber weights minus β strictly increase on every segment, so at most two shears occur. Every other continuation increases m to a higher passive degree. The finite sequence therefore exits by a previous lemma or reaches $W = 0$ with positive weights. Lemma 3.1 gives the inverse there. □

Lemma 3.6. *Suppose a leading form has Hessian rank three, $W > 0$, and a one-dimensional apex space. Assume the weights are positive and the apex lies in a smallest-weight block. Then there is an inverse or a two-fiber reduction.*

Proof. A constant change within that block makes the apex the active coordinate x, and we scale the weights to make its weight one. The complementary rank-one form has a phase linear in the passive variables over $\mathbb{C}(x)$. Normalize one nonzero phase coefficient to one. The others are constant multiples of Laurent monomials in x, by weight invariance of the passive Hessian image. Constant changes within the passive weight blocks leave at most one phase entry per block. A variable absent from the phase has a nonconstant affine coefficient. Otherwise it gives a constant source kernel and a hyperplane image, contradicting uniqueness of the apex. Two absent variables cannot have equal weights because their affine coefficients would be proportional monomials.

Let $m \geq 2$ be the degree in the phase. Raise its variable weights at speed one and each absent variable at speed m. For a small step the leading form is the highest phase power together with the affine terms in the absent variables. All passive weights now exceed one. With three phase variables Lemma 3.3 gives a contradiction. With two phase variables, normalize the coefficient of the larger-weight variable to one. The other coefficient is a nonzero constant times a positive integral power of x, since its exponent is the difference of their weights. Absorb the constant by rescaling that variable and apply Lemma 3.4. With one phase variable, the other two variables have nonconstant affine coefficients of distinct positive integral exponents, so Lemma 3.5 applies. In each case the coefficient of the pure power of the greatest-weight phase variable is a polynomial in x, which gives the nonnegative exponent in the required normal form. □

Lemma 3.7. *Suppose the leading form is $\sum_{i=1}^{3} b_i x^{p_i} Y_i$, with $b_i \neq 0$, integers $p_1 > p_2 > p_3 \geq 1$, $w_x = 1$, passive weights greater than one, and $W > 0$. Then there is an inverse or a two-fiber reduction.*

Proof. Raise the three passive weights at unit speed. The leading weight has slope one and W has slope -2. Before $W = 0$ the old minors in (x, Y_i) survive, since each is $-(b_i p_i x^{p_i - 1})^2 \neq 0$. The passive weights have the form $E - p_i$ and stay distinct. A constant source kernel cannot occur, since it would contain a weight axis and each axis is excluded by an old mixed minor. A rank-two wall therefore has one apex. It must be x, because the complementary variables otherwise include an old mixed minor. The rank-two affine form then fixes all three partners by weight, and an active-only term would have exceeded the old leading weight. This cannot give a genuine wall.

At a rank-three wall the apex space has dimension one or two by Theorem 2.3 and the exclusion of a constant kernel. A single apex again has to be x, so Lemma 3.6 applies. A two-apex plane must contain x because the complementary pair otherwise has a surviving mixed minor. The potential then is jointly affine in two fibers with coefficient degrees two of the p_i. Their common polynomial either is genuinely binary, giving Lemma 3.2, or is a power of x by the old nonzero x powers. In the latter case both coefficients are unchanged. A genuinely new term must be nonlinear in the remaining passive variable, since an active-only term would have exceeded the old degree and the old form fixes all passive-linear terms. Retaining its highest power and the two affine partners by the paired raise gives Lemma 3.5. If the first wall occurs at $W = 0$, or if no wall occurs by then, the leading form has nonzero constant Hessian determinant and the weights are positive. Lemma 3.1 applies. □

3.4. **A pure power of a lower-weight variable.**

Lemma 3.8. *Suppose the leading form is $ay^M + bx^p z + cx^q w$, with $abc \neq 0$ and integers $0 < q < p < M$. Its weights on x, y, z, w are $1, \beta, M\beta - p, M\beta - q$, respectively, with $p/M \leq \beta < 1$. Then there is an inverse or a two-fiber reduction.*

Proof. Increase β at unit speed, fixing the weight of x and keeping the three terms tied. The determinant weight is $2(p + q - 1 - \beta)$. The old-weight formula from Lemma 3.5 applies with $m = M$. At a wall before $W = 0$, the two fiber cofactors survive by the same computation. They give rank three and exclude an intersection of the apex space with the fiber plane. Theorem 2.3 gives an apex space of dimension one or two. The weight of w is greater than β, since $(M-1)\beta - q \geq p - q - p/M > 0$. It is greater than 1 after any positive step because $M\beta - q \geq p - q \geq 1$ initially. The two fiber weights differ by $p - q > 0$, so the only possible equalities are

$$M\beta - p = \beta, \qquad M\beta - p = 1, \qquad \beta = 1.$$

The corresponding values of β are $p/(M-1)$, $(p+1)/M$, and 1. They are in increasing order if $p < M-1$ and coincide if $p = M-1$. Values at or below the initial β do not produce a new wall.

Suppose the apex space is a line. Its generator has zero w component because the weight of w exceeds the other three at the wall and the fiber axis is excluded. It has a nonzero x component, since otherwise its annihilator contains the x, w axes, whose old restricted minor is nonzero. If its vector, scaled to have x coordinate 1, is $(1, \lambda, \mu, 0)^{\mathsf{T}}$, the source shear $x' = x + \lambda y + \mu z$ with the other coordinates fixed aligns the target apex with x. The coefficients λ, μ can be nonzero only for variables whose weights equal 1 at the wall. They had strictly smaller old weight. At the simultaneous crossing the old order is x, y, z in decreasing weight. This alignment preserves the old leading form. Complementary rank one gives a phase in y, z, w with nonzero y coefficient. Its highest power is exactly M by the retained term ay^M and the weight bound. Normalize the coefficient of y to one. The coefficients of $y^{M-1}z$ and $y^{M-1}w$ show that the two remaining phase coefficients must be actual monomials in x of exponents $p-(M-1)\beta$ and $q-(M-1)\beta$, respectively, or zero. Indeed $p-(M-1)\beta \le p/M < 1$, and $q-(M-1)\beta \le q-p+p/M < 0$. Thus only a constant z coefficient can occur, at the first listed equality, and it is removed by a constant shear of y by z that again preserves the old form. Each lower power y^j on the wall would have old weight $M\beta_{\mathrm{old}} + (M-j)(\beta - \beta_{\mathrm{old}}) > M\beta_{\mathrm{old}}$. The affine terms in z, w are fixed by the old comparison. The form after the shear is therefore the original leading form. Its gradient image closure is the product of $\mathbb{C}^2$ with the closure of the curve (bx^p, cx^q): for each $x \neq 0$, the x-output varies freely with the fibers and the y-output varies freely with y. Hence there are at least two independent apex directions, contradicting the one-dimensional apex space.

If the apex space is a plane, it is transverse to the fiber plane. The graph shear puts the form into $R(x,y) + zG(x,y) + wK(x,y)$. It preserves the old leading form since it only mixes a fiber of smaller old weight into a base variable. The coefficients of degrees p, q have vanishing Jacobian by Lemma 1.2 and $W > 0$. They therefore have a common weighted homogeneous polynomial. Their restrictions to $y = 0$ are bx^p, cx^q by the old-weight formula for the monomials linear in one fiber and independent of y. If it is genuinely binary, apply Lemma 3.2. Otherwise the old nonzero powers of x make it a power of x for unequal base weights. At $\beta = 1$ it may be a power of $x + \lambda y$, rectified by an additional old-weight-preserving shear. This recovers both old affine partners. The same comparison of y powers fixes $R = ay^M$: higher powers exceed the new weight, and lower powers exceed the old weight. Hence every continuing wall is a constant shear at one of the three displayed equalities, with the same leading form afterward. There are finitely many such walls. If there is no earlier exit, the path reaches $\beta = p + q - 1$ with positive weights and $W = 0$. At that parameter the full leading form has the original nonzero constant Hessian determinant,

including if another monomial ties there. Lemma 3.1 gives a polynomial inverse. □

4. The highest homogeneous part

Proposition 4.1. *Let $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ have degree at least three and nonzero constant Hessian determinant. Then either ∇f is a polynomial automorphism or, in constant linear coordinates x, y, z, w, the polynomial has the form $R(x, y) + zA(x, y) + wB(x, y)$ with $\det D(A, B) \in \mathbb{C}^\times$.*

Proof. Subtract the affine part of f. Restoring it translates the gradient and, in the affine-in-z, w form, adds constants to A, B and an affine polynomial to R. For an invertible linear change L, the identity $\nabla(f \circ L) = L^\mathsf{T}(\nabla f) \circ L$ shows that gradient invertibility is invariant under the change.

Write $D \geq 3$ for the total degree and f_D for the sum of the monomials of degree D. The degree-$4(D-2)$ component of $\det \operatorname{Hess} f$ is $\det \operatorname{Hess} f_D$, hence it vanishes. Some second derivative of f_D is nonzero because f_D has degree $D \geq 3$ in characteristic zero. By [6, Corollary 7.3(iii)], a constant linear change makes f_D depend on one, two or three coordinates, according to its Hessian rank, with a nonsingular Hessian in those coordinates.

Use positive rational weights and denote the weight of a variable t by w_t. Each path stops at the first wall or at $W = 0$, whichever comes first. At $W = 0$, including a tie, Lemma 3.1 gives a polynomial inverse. A constant vector in the Hessian kernel of a leading form gives a constant directional derivative. This derivative is zero because the affine terms were removed. Conversely, for a rank-three leading form, such a nonzero vector gives a linear relation on its gradient. The image closure then has dimension three and lies in a hyperplane, so it is that hyperplane and its apex space has dimension three.

By Theorem 2.3, a rank-three leading form has a nonzero apex space. One, two or three apices means that this space has the indicated dimension. With two apices, neither affine fiber coefficient is constant and the two are not proportional, since either case gives a constant kernel. By Lemma 1.2 the coefficient pair has zero Jacobian. Its coefficients are thus nonzero constant multiples of positive powers of a common weighted homogeneous polynomial, as shown in Section 3. Equal coefficient degrees would give proportional coefficients, so the fiber weights are distinct.

We use the following weight change when the face is $G(x, y)+c_1x^pz+c_2x^qw$ with $p > q \geq 1$ and $c_1c_2 \neq 0$. Suppose x has minimum weight and the largest exponent m of y in G is at least two. Increase the weight of y at speed one and those of z, w at speed m, fixing the weight of x. The maximal slope among the monomials of the face is m. The monomials with that slope are

$$c_0x^\ell y^m + c_1x^pz + c_2x^qw,$$

where $c_0 \neq 0$ and ℓ is a nonnegative integer, since homogeneity fixes the x-exponent of a term with y-exponent m. There are finitely many monomials

outside the face, all with strict weight gaps. A sufficiently small positive step therefore selects the displayed form, keeps $W > 0$, and makes the other weights strictly greater than that of x. After dividing the weights by that of x, Lemma 3.5 applies.

Highest part of rank three.

Write $f_D = h(x, y, z)$ with nonsingular ternary Hessian. Keep the weights of x, y, z equal to one and raise that of w. There is a monomial involving w, since otherwise $\det \operatorname{Hess} f = 0$. If the exponent of w in such a monomial is one, the monomial has another factor of weight at least one and reaches weight D by w-weight $D-1$. For exponent at least two it reaches weight D by w-weight $D/2$. Thus the first new face P is reached at some $\alpha \in (1, D-1]$. Since $P|_{w=0} = h$, the ternary Hessian minor has nonzero restriction at $w = 0$. Also $W = 4D - 2(3+\alpha) \geq 2D - 4 > 0$, so $\det \operatorname{Hess} P = 0$. Thus P has Hessian rank three.

The apex space splits between the ternary block and the w axis. A single apex on the w axis would make the complementary Hessian have rank at most one, contradicting the restriction to h. Thus a single apex has minimum weight and Lemma 3.6 applies. If two apices include the w axis, their complementary source plane lies in the ternary block. Then h is affine in two coordinates of that block. The two affine coefficients are homogeneous of degree $D-1$ in the one remaining coordinate, hence proportional. This would give a constant kernel for the ternary Hessian. So the two apices lie in the ternary block. Align them with x, y and write

$$P = G(x, y) + zA(x, y) + wB(x, y).$$

If the common polynomial of A, B is not a polynomial in a constant linear form, apply Lemma 3.2. Otherwise, a constant change of the equal-weight coordinates x, y makes both coefficients powers of x. If $G_{yy} = 0$, then P is jointly affine in y, z, w. Its Hessian vanishes on their principal block, and adjoining the x row and column gives rank at most two. Thus $G_{yy} \neq 0$, and the weight change keeping the two power coefficients applies. Finally three apices would give a constant kernel line. The ternary Hessian of h excludes a line in its block, and $P_w \neq 0$ excludes the w axis.

Highest part of rank two.

Write $f_D = h(s, t)$ with nonsingular binary Hessian. Keep the weights of s, t equal to one and increase those of z, u together. A monomial of total degree at least two involving z or u meets weight D by the common weight $D-1$, by the same exponent bound as above. Such a monomial exists by the unit Hessian condition. At $D-1$ the determinant weight is zero, so assume the first new face P occurs at a common weight $\beta \in (1, D-1)$. The face has rank two or three and $P(s, t, 0, 0) = h(s, t)$.

Rank two at the first face. A two-dimensional constant kernel would lie entirely in the z, u block by the old binary Hessian. The kernel would be the whole z, u block, so P would be independent of z, u, contrary to the new

terms at the wall. In the one-apex case the apex lies in the s,t block, since an apex in the z,u block would have the nonsingular binary Hessian in its complementary space. Align the apex with s. The normal form is affine in t,z,u. Its t coefficient is a nonzero multiple of s^{D-1}, by the binary Hessian. Its z,u coefficients are proportional because they have the same weighted degree in s. A constant change in z,u gives

$$P = as^D + bts^{D-1} + czs^{D-\beta}, \qquad bc \neq 0. \tag{1}$$

Here $D-\beta$ is a positive integer. The variable u is unused.

Rank three at the first face. If there is a single apex, it belongs to the s,t block, since the complementary rank bound excludes the z,u block. Apply Lemma 3.6. If there are two apices, they cannot span the z,u block, since that would give P affine in s,t. They cannot span the s,t block either: its two affine coefficients in z,u would have equal weighted degree. Aligning the mixed apex plane with s,z gives

$$P = G(s,z) + tA(s,z) + uB(s,z).$$

Here $A(s,0)$ is a nonzero multiple of s^{D-1} by the binary Hessian. The coefficients have degrees $D-1 > D-\beta$. If their common polynomial belongs to $\mathbb{C}[L]$ for a constant linear form L, weighted homogeneity and its nonzero restriction to $z=0$ make it a power of s up to a nonzero scalar. In this case $G_{zz} \neq 0$, since otherwise P is jointly affine in t,z,u and has Hessian rank at most two. Apply the weight change keeping the two power coefficients. If the common polynomial is not of this type, apply Lemma 3.2. If there are three apices, the constant kernel line lies in the z,u block. Align the kernel with u, so that $P = H(s,t,z)$ with nonsingular ternary Hessian.

Increasing the unused weight. For either $H(s,t,z)$ or (1), increase only the weight of u, starting at β. Its first new face Q occurs at a weight $\gamma \in (\beta, D-1]$, by the exponent bound and the unit Hessian condition. The determinant weight is

$$4D - 2(2+\beta+\gamma) \geq 2(D-\beta-1) > 0.$$

The terms independent of u retain their weights, so $Q|_{u=0}$ is the preceding face. Some monomial involving u reaches the face at the wall, and therefore $Q_u \neq 0$.

If this restriction is H, the rank remains three and the u cofactor is nonzero. Thus the apex space has zero u component, since the u axis is its own weight block and the cofactor excludes that axis. Three apices are impossible because a constant kernel line would have to be the u axis. A unique apex in the s,t block is handled by Lemma 3.6. A unique z apex would give a single nonlinear phase over $\mathbb{C}(z)$ in s,t,u. At least one of the s,t phase coefficients is nonzero. Otherwise $Q|_{u=0} = H$ is affine in s,t, which makes $\operatorname{Hess} h$ zero. Since $w_s = w_t = 1$, any ratio of two nonzero s,t phase coefficients is a rational function of z of weight zero and therefore constant, by the phase coefficient formula in Section 3. After normalizing one such coefficient, the phase is $L(s,t) + a(z)u$, with L a constant linear

form. Choose a second constant linear form M so that L, M are coordinates on the s, t plane. Setting $u = 0$ gives

$$H = \Phi(z, L) + C(z)M$$

over $\mathbb{C}(z)$. Here $\Phi(z, L)$ and $C(z)$ are polynomials, since they are respectively the restriction at $M = 0$ and the derivative with respect to M of the polynomial H. The weighted degree $D - 1 > 0$ of C gives $C(0) = 0$. Hence $h = \Phi(0, L)$ has Hessian rank at most one, a contradiction.

With two apices in the H case, their plane is the s, t block or is spanned by z and one line in that block. For the first plane the two affine coefficients depend on s, t and have respective degrees $D - \beta > D - \gamma > 0$. A common polynomial in a constant linear form can be made a power of s by a constant base change. Then the pure base term has a nonzero tt derivative, because H has nonsingular Hessian. For the second plane align the base line with s. The affine coefficients in t, u have degrees $D - 1 > D - \gamma > 0$, and the coefficient of t restricts to a nonzero multiple of s^{D-1} at $z = 0$. A common polynomial in a constant linear form is therefore a power of s. The pure base term now has a nonzero zz derivative by the ternary Hessian of H. The weight change keeping the two power coefficients applies in both cases. A common polynomial that is not a polynomial in a constant linear form gives Lemma 3.2.

It remains to handle (1) at this new face. Write $p = D-1$ and $q = D-\beta$. The old mixed Hessian minors on (s, t) and (s, z) are respectively

$$-(bps^{p-1})^2, \qquad -(cqs^{q-1})^2.$$

They survive by restriction to $u = 0$. The old face has no constant kernel on its s, t, z variables: its s derivative contains nonzero t, z terms, and the other derivatives are nonzero multiples of distinct powers of s. Thus a constant kernel of Q, decomposed into the new weight blocks, could only be the u axis. This excludes the rank-two case with two apices and the rank-three case with three apices.

For a single apex at either rank, the z, u axes are excluded by the (s, t) minor. Indeed, the Hessian on the complementary space has rank at most one at rank three, and is zero at rank two. For an apex $d_s e_s + d_t e_t$ in the base block, where e_s, e_t, e_z are coordinate vectors, its annihilator contains $d_t e_s - d_s e_t$ and e_z. The old second derivative in the z direction is zero and the mixed derivative on these directions is $d_t cqs^{q-1}$. The restricted minor is therefore $-(d_t cqs^{q-1})^2$, so $d_t = 0$. If the rank is three, apply Lemma 3.6. If the rank is two, the normal form gives

$$Q = as^D + bts^p + czs^q + dus^{D-\gamma}, \qquad d \neq 0.$$

The exponent $D - \gamma$ is a positive integer smaller than q. A small common increase of the three weights other than that of s selects the three linear partners, whose slopes are all one, while as^D has slope zero. Lemma 3.7 applies.

Suppose Q has rank three and two apices. Their plane splits between the s,t block and the z,u axes, and must contain the s axis. Indeed, the plane spanned by z,u is excluded by the (s,t) minor. For a plane spanned by $d_se_s+d_te_t$ and e_u, the preceding restricted minor forces $d_t=0$. For a plane spanned by $d_se_s+d_te_t$ and e_z, the second derivative of the old face along $d_te_s-d_se_t$ has t coefficient $d_t^2bp(p-1)s^{p-2}$. This must vanish by joint affineness in the complementary plane, so $d_t=0$. The restrictions used here are derivatives in the old variables and commute with setting $u=0$.

If the apex plane is the s,t block, write $Q=G(s,t)+cs^qz+uB(s,t)$, where the old restriction fixes $G(s,t)=as^D+bts^p$. The zero Jacobian of the coefficient pair gives $cqs^{q-1}B_t=0$, so B depends only on s. This makes Q affine in t,z,u, with Hessian rank at most two. For the s,z plane, write $Q=G(s,z)+bs^pt+uB(s,z)$, where $G(s,z)=as^D+czs^q$. Now $bps^{p-1}B_z=0$, so the same rank bound holds. For the s,u plane the coefficients of t,z restrict at $u=0$ to bs^p,cs^q. If their common polynomial is a polynomial in a constant linear form, it is a power of s, and the pure base term must have u-degree at least two by rank three. The unary weight change applies. If their common polynomial does not belong to $\mathbb{C}[L]$ for any constant linear form L, Lemma 3.2 applies.

Highest part of rank one.

Write $f_D=a_0s^D$ with $a_0\neq 0$. Keep s at weight one and initially increase the other three weights together. We first examine any rank-three face P of weight D containing a_0s^D, with all other weights greater than one and positive determinant weight.

A single apex of minimum weight is handled by Lemma 3.6. If instead the unique apex is a heavier coordinate t after a constant change within its weight block, the complementary normal form over $\mathbb{C}(t)$ has a phase with a nonzero s coefficient because P contains a_0s^D. Normalize the phase to $s+a(t)u+b(t)v$. Setting $u=v=0$ shows that its highest power is D and has coefficient a_0: a larger exponent of s would exceed the weight D. The coefficients of $s^{D-1}u,s^{D-1}v$ are $Da_0a(t),Da_0b(t)$. They are polynomials in t of weights $1-w_u,1-w_v<0$. Since $w_t>0$, both vanish. Then $P=G(s,t)+uA(t)+vB(t)$. The gradient image closure lies in the product of the s,t target plane with the image closure of $t\mapsto(A(t),B(t))$. The latter is irreducible of dimension at most one. Since the gradient image closure has dimension three, it equals the product. This gives at least two apices, excluding the heavy single-apex case.

A plane of two apices contains the s axis: by weight invariance, otherwise it lies in the heavy block and its complementary space contains s, forcing $P_{ss}=0$. Align the plane with s,t and write

$$P=G(s,t)+uA(s,t)+vB(s,t).$$

If the common polynomial of A,B is not a polynomial in a constant linear form, apply Lemma 3.2. Otherwise the unequal weights of s,t make the common polynomial a power of one of them. A power of s gives the weight

change above, since $G_{tt} = 0$ would make the face affine in t, u, v and force rank at most two. For a power of t, order the fibers so the coefficients are $c_1 t^p, c_2 t^q$ with $p > q \geq 1$. Increase the weight of s at speed one and those of u, v at speed D, leaving the t weight fixed. In $G(s,t)$ only $a_0 s^D$ has s-exponent D, and none has a higher exponent because the old weight of s is one and the old weight of t is positive. Thus the slope-D terms are exactly $a_0 s^D + c_1 t^p u + c_2 t^q v$. Divide all weights by the t weight. The new s weight β is less than one for a sufficiently small step. Since the selected terms have weight $D\beta$, the two fiber weights are $D\beta - p, D\beta - q > 0$. In particular $0 < q < p < D$ and $p/D < \beta < 1$, so Lemma 3.8 applies with t as its weight-one variable.

If the common weight of the other three variables is β, the determinant weight along the initial path is $4D - 2(1 + 3\beta)$. Thus the endpoint occurs at $\beta = (2D - 1)/3$ if there is no earlier wall. At an earlier wall $P_{ss} \neq 0$ because of the term $a_0 s^D$. In rank one, [6, Corollary 7.3(i),(ii)] gives a three-dimensional constant kernel. This kernel splits along the weight blocks and cannot contain s, so it is the whole heavy block. That contradicts the terms involving heavy variables at a wall. In rank two with a two-dimensional constant kernel, the same weight argument places that kernel in the heavy block. A linear change within the block puts the kernel on two coordinate axes, so P is independent of those variables. In rank two with a single apex, the apex is s, since any heavy apex gives $P_{ss} = 0$ by the complementary affine form. Then the three heavy variables occur affinely, with coefficients proportional to the same power of s by their equal weights. A linear change within their block makes P independent of two of them. Rank three with one or two apices was treated above.

If the rank-three face has three apices, its constant kernel lies in the heavy block by the s^D term. A linear change within that block makes the form $H(s,t,u)$, with nonsingular ternary Hessian and no dependence on v, whose weight is maximal. More generally, whenever such a ternary form occurs, raise only the weight of v. The determinant weight decreases at speed two because the weight D of the leading form is fixed. At a wall before $W = 0$, the cofactor complementary to v restricts at $v = 0$ to the nonzero determinant of the ternary Hessian. Thus the new Hessian has rank three. Its constant kernel, if nonzero, would have to be the v axis by weight invariance and the nonsingular ternary restriction. This is impossible since v occurs. The form therefore has one or two apices.

For the rank-two face with two unused variables, write $H(s,t)$ with nonsingular binary Hessian. The unused u, v have equal weights at least as large as that of t. Increase both weights at unit speed. The determinant weight decreases at speed four. At a wall before $W = 0$, the restriction at $u = v = 0$ is H, so the new Hessian has rank at least two. Any constant kernel splits by the new weights and is contained in the u, v block, by the nonsingular binary Hessian of H. A two-dimensional kernel would be the entire block, contrary to the new terms. A rank-three face with three apices has its kernel

in the u, v block by the binary Hessian, and a linear change in that block gives the ternary case just treated. A rank-three face with fewer apices was handled above. Finally a rank-two face with one apex has apex s, since the s^D term excludes all heavy apices. Its u, v coefficients are proportional by their equal weight. A constant linear change in u, v gives the face

$$a_0 s^D + bts^p + cus^q, \qquad bc \neq 0, \quad p > q \geq 1.$$

The restriction $H(s,t) = a_0 s^D + bts^p$ has Hessian determinant $-b^2 p^2 s^{2p-2} \neq 0$, so $b \neq 0$ and $p \geq 1$. A monomial involving u, v reaches the face, so $c \neq 0$. The exponents $p = D - w_t$ and $q = D - w_u$ are integers, with $p > q$ since $w_u > w_t$, and $q > 0$ because the affine part was removed. The weight of v equals that of u.

Raise only the weight of v. The determinant weight decreases at speed two. At a wall before $W = 0$ the four weights are distinct. The old face has only v as a constant kernel direction, since its s derivative has nonzero t, u terms and its t, u derivatives are independent powers of s. The new face involves v, so no constant kernel can survive. Its Hessian rank is two or three by the surviving (s,t) minor and the positive determinant weight. Rank three gives one of the cases already handled. Rank two gives the unique apex s, again by the s^D term. The new v coefficient is a nonzero multiple of s^r with $r = D - w_v$. Homogeneity and the absence of affine terms give $r \in \mathbb{Z}_{>0}$, and $w_v > w_u$ gives $r < q$. A small common increase of the three weights other than that of s now selects the three linear partners, all with slope one, while $a_0 s^D$ has slope zero. Apply Lemma 3.7. □

5. Proof of the main theorem

Proof of Theorem 1.1. If f is quadratic, its gradient is affine with invertible linear part $\operatorname{Hess} f$. The determinant condition excludes degrees below two, and Proposition 4.1 gives the two alternatives for degree at least three. Under JC_2, the map (A, B) in the second alternative has a polynomial inverse. Lemma 1.2, applied with $n = 2$, then gives a polynomial inverse of the gradient. This proves $\mathrm{JC}_2 \Rightarrow \mathrm{HC}_4$.

Conversely, given a planar Keller map $G = (A, B)$, apply Lemma 1.2 to $f = zA + wB$. Under HC_4, the gradient is a polynomial automorphism, so the lemma gives a polynomial inverse of G. □

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