# Quartic Fekete points on every real simplex

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**Abstract**

We prove that on every real simplex Bos's quartic interpolation nodes uniquely maximize the absolute Vandermonde determinant, up to ordering, among all configurations of the prescribed size. In dimension four the squared Euclidean norm of the cardinal vector exceeds one at the centroid. We bound its excess by products of the cardinal polynomials for tetrahedral face centers. These products also measure correlations between columns of the evaluation matrix, which give the required loss in the determinant. An elementary consequence of the same corrected squared-norm estimate is Bos's fourth-power bound in every dimension, with equality exactly at the nodes. The corrected estimate follows from cubic perturbations, reduction to at most two distinct positive coordinate values, and coefficient monotonicity in their multiplicities. All polynomial bounds are given explicitly. The common base case is the tetrahedral Fejér bound previously stated by Panda and Pal in equivalent optimal-design form. We give an independent proof of that bound and determine all its equality points.

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## 1 The extremal configuration

The choice of interpolation nodes determines how an error in one datum affects the interpolant elsewhere. Let $\ell_i$ be the cardinal polynomial that is one at the $i$th node and zero at the others. An error $\varepsilon$ at that node changes the interpolated value at $x$ by $\varepsilon\ell_i(x)$. There is a determinant interpretation of the same factor: replacing the $i$th node by $x$ multiplies the Vandermonde determinant by $\ell_i(x)$. Hence on a *Fekete set*, one that maximizes the absolute determinant, each $|\ell_i(x)|$ is at most one throughout the domain [4, Remark 0.8].

On $[-1,1]$ the absolute determinant in the monomial basis is simply $\prod_{i<j}|y_i - y_j|$. For degree $n \geq 1$ its maximizing nodes are the endpoints and the interior zeros of $P_n'$, where $P_n$ is the Legendre polynomial of degree $n$ [15, §2]. Fejér proved that at these nodes the stronger inequality $\sum_i \ell_i(x)^2 \leq 1$ holds on the whole interval [6, pp. 264–265]. This squared sum has a statistical meaning: with independent errors of common variance $\sigma^2$ at the nodes, the variance of the interpolated value at $x$ is $\sigma^2 \sum_i \ell_i(x)^2$.

On a simplex, Bos constructed families of nodes that agree with the lower-dimensional construction on each face [1, 4]. For quadratics the vertices and edge midpoints satisfy the Fejér inequality in every dimension [4, Proposition 2.1]. For higher degrees, Taylor, Wingate and Vincent computed candidate Fekete sets on the triangle through degree 19 [15]. Here we fix degree four and allow the dimension to vary. We prove that Bos's nodes uniquely maximize the determinant even among configurations with no prescribed symmetry or face structure. Section 6 examines which steps extend in the polynomial degree and identifies the interior determinant to maximize on a triangle with prescribed boundary nodes.

An invertible affine map preserves the extremal problem, so we use barycentric coordinates on a real $d$-simplex and write

$$m = d + 1, \qquad s = \sum_{i=1}^{m} x_i, \qquad \Delta_d = \{x \in \mathbb{R}^m : x_i \geq 0,\ s = 1\}, \qquad N = \binom{m+3}{4}.$$

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Let $\mathcal{P}_4(\Delta_d)$ be the $N$-dimensional space of real polynomials of total degree at most four on $\Delta_d$. For a basis $b = (b_1, \dots, b_N)$ of this space, let $V_b(Y) = (b_j(y_i))_{i,j=1}^N$ for $Y = (y_1, \dots, y_N) \in \Delta_d^N$. A change of basis multiplies $|\det V_b(Y)|$ by a constant independent of $Y$.

Put $r = \sqrt{5}$, $\eta = \sqrt{3/7}$, $\alpha = (4 + r)/11$ and $\beta = (7 - r)/22$. Bos's set $\mathcal{F}_m$ is obtained from the following representatives:

$$(1), \quad (1/2, 1/2), \quad \big((1 + \eta)/2, (1 - \eta)/2\big), \quad (\alpha, \beta, \beta), \quad (1/4, 1/4, 1/4, 1/4). \tag{1.1}$$

Retain the vectors of length at most $m$, append zeros to length $m$, and take all coordinate permutations. The vertices, edge nodes, triangular face nodes, and tetrahedral face centers give $m + 3\binom{m}{2} + 3\binom{m}{3} + \binom{m}{4} = N$ points in all.

**Theorem 1.1** (Quartic Fekete theorem)**.** *For every $d \geq 1$, every basis $b$ of $\mathcal{P}_4(\Delta_d)$, and every $Y \in \Delta_d^N$,*

$$|\det V_b(Y)| \leq |\det V_b(\mathcal{F}_{d+1})|,$$

*where the nodes on the right may be ordered arbitrarily. Equality holds if and only if $Y$ is an ordering of $\mathcal{F}_{d+1}$.*

The maximality assertion is Bos's Conjecture 4.3 [4, p. 285]. For $z \in \mathcal{F}_m$ write $L_{m,z}$ for the quartic cardinal polynomial, equal to one at $z$ and zero at every other node. Lemma 2.1 proves unisolvence. In this basis the Vandermonde matrix at the nodes is the identity. A sufficient condition for determinant maximality is the Fejér bound $S_m(x) = \sum_{z \in \mathcal{F}_m} L_{m,z}(x)^2 \leq 1$, studied in several variables by Bos [2]. Indeed, Hadamard's determinant inequality bounds the absolute determinant of a square matrix by the product of the Euclidean lengths of its rows [8, pp. 505–506]. For the matrix $M = (L_{m,z}(y_i))_{i,z}$ this gives

$$|\det M| \leq \prod_{i=1}^{N} \sqrt{S_m(y_i)}.$$

For nonzero rows, equality in Hadamard's inequality holds precisely when the rows are pairwise orthogonal. If $S_m \leq 1$ throughout the simplex, the nodal determinant is maximal, and $|\det M| < 1$ whenever some $S_m(y_i) < 1$. Bos proved the quartic bound on the triangle [1] and posed the tetrahedral case as Conjecture 4.2 [4, p. 285]. Panda and Pal subsequently stated the equivalent four-ingredient design bound [12, §3, Remark 2.1].

**Theorem 1.2** (Tetrahedral Fejér bound and its equality cases)**.** *On $\Delta_3$ one has*

$$S_4(x) = \sum_{z \in \mathcal{F}_4} L_{4,z}(x)^2 \leq 1. \tag{1.2}$$

*Equality holds precisely at the 35 points of $\mathcal{F}_4$.*

In the mixture model a design is a probability measure $\mu$, and $D$-optimality means maximizing the determinant of its information matrix $G_\mu = \int b(x)^{\mathsf{T}} b(x)\, d\mu(x)$, with $b$ viewed as a row. For equal masses on $N$ points $Y$, its determinant is $N^{-N} |\det V_b(Y)|^2$. Thus Fekete maximality compares these equally weighted configurations, while $D$-optimality compares all probability measures [4, Propositions 0.5–0.6]. Panda and Pal prove the three-ingredient case in Theorem 3.1 of [12]. In the four-ingredient case they specify equal masses $1/35$ at precisely the nodes $\mathcal{F}_4$. Their edge parameter is $(7 - \sqrt{21})/14 = (1 - \eta)/2$ and their face parameters are

$\beta, \beta, 1-2\beta = \alpha$. They state $D$-optimality for this design and omit its proof [12, §3, Remark 2.1]. In the cardinal basis its information matrix is $I_{35}/35$. The Kiefer–Wolfowitz equivalence theorem says that a design with nonsingular $G_\mu$ is $D$-optimal precisely when its sensitivity $b(x)G_\mu^{-1}b(x)^{\mathsf{T}}$ is at most $N$ at every $x$ [10]. Here the sensitivity is $35S_4(x)$ and $N = 35$, giving exactly (1.2). Section 4 gives an independent proof and determines the equality set, from which uniqueness of the optimal measure follows.

The first obstruction to the same determinant argument occurs in dimension four. At the centroid $c = (1/5, \dots, 1/5)$, Bos's calculation [4, §4] gives

$$S_5(c) = \frac{9298842 + 80243\sqrt{5}}{9453125} > 1. \tag{1.3}$$

Thus the uniform design on these 70 nodes fails the equivalence criterion in the five-ingredient case proposed in [12, Remark 2.2 and Table 2]. Theorem 1.1 gives a unique maximizer among all 70-point configurations. The proof bounds the excess $S_m - 1$ pointwise by products of distinct face-center cardinals. The same products force a loss in the determinant of their column Gram matrix (Section 2). Section 3 proves the pointwise bound in all dimensions by cubic perturbations and uniform algebraic inequalities in the coordinate multiplicities.

Bos also conjectured a uniform pointwise bound for the fourth powers.

**Theorem 1.3** (Fourth-power bound)**.** *For every $d \geq 1$, $m = d + 1$, and $x \in \Delta_d$,*

$$\sum_{z \in \mathcal{F}_m} L_{m,z}(x)^4 \leq 1.$$

*Equality holds if and only if $x \in \mathcal{F}_m$.*

This is Bos's Conjecture 4.6 [4, p. 286], with its equality cases. Section 5 derives it from the same corrected squared-norm bound used in the determinant proof. Retaining the center cross terms in $S_m^2$ absorbs the correction and gives all equality cases. The fourth-power bound alone does not give the determinant estimate. For instance, the matrix of order three with diagonal $-1/2$ and off-diagonal $3/4$ has row sums one and fourth-power row sums $89/128$, yet its determinant is $25/16$.

## 2 From cardinal polynomials to determinants

### 2.1 Restriction to faces and the row bound

We represent each polynomial of degree at most four on $s = 1$ by its unique homogeneous quartic on $\mathbb{R}^m$. The monomials $x^\nu$ with $|\nu| = 4$ form a basis. We use the same notation $L_{m,z}$ for the homogeneous representative.

**Lemma 2.1.** *The set $\mathcal{F}_m$ is unisolvent. On a coordinate face the cardinals for nodes on that face restrict to its cardinal basis, and the other cardinals restrict to zero.*

*Proof.* Suppose a homogeneous quartic vanishes at every node. Its restriction to an edge vanishes, since the edge contains five distinct nodes. On a triangular face the three coordinates divide the restricted polynomial, so their product divides it. The remaining linear factor vanishes at the three interior nodes. These are noncollinear because $\alpha \neq \beta$, and therefore the restriction vanishes on the whole triangle. On a tetrahedral face all four coordinates divide

the restriction, which is therefore a constant times their product. The centroid makes that constant zero. These divisibility statements use homogeneity to extend vanishing on a face to its coordinate subspace.

Every quartic monomial uses at most four coordinates. Its coefficient can therefore be read on a coordinate subspace where the polynomial vanishes by the preceding argument. All coefficients are zero. The node count in Section 1 now gives unisolvence. On a face the nodal values determine each restriction uniquely, and homogeneity extends this rule to its entire coordinate subspace. □

For a four-element set $J$, Bos gives the cardinal at the center of the corresponding face as $q_J(x) = 256 \prod_{i\in J} x_i$ [4, Proposition 4.4]. Its nodal values also follow by direct substitution. Put

$$S_m = \sum_{z\in\mathcal{F}_m} L_{m,z}^2, \qquad G_m = s^8 - S_m, \qquad E_m = \sum_{J\neq K} q_J^3 q_K^3,$$

where the last sum is over *ordered* pairs of four-element sets. The center products in $E_m$ vanish whenever at most four coordinates are positive. At other points they bound the excess of $S_m$ above one and enter the determinant estimate with the opposite sign.

**Lemma 2.2.** *For $x \in \Delta_{m-1}$, $\sum_J q_J(x)^2 \le 1$. Equality holds precisely at tetrahedral face centers.*

For $m \ge 4$ this is a special case of the uniform-profile theorem of Orlitsky, Santhanam, Viswanathan and Zhang [11, Theorem 14]. In eight independent draws with outcome probabilities $x_i$, the probability of the pattern 11223344 (four distinct outcomes, each drawn twice in the indicated order) is $4!\sum_{|J|=4}\prod_{i\in J} x_i^2$. Their theorem gives the uniform distribution on four outcomes as maximizer, since its support-size criterion holds already at four, $(5/4)^8/5 > 1$. We give a direct proof in barycentric coordinates.

*Proof.* For $m < 4$ the sum is empty. For $m \ge 4$ it equals $4^8 e_4(x_1^2, \dots, x_m^2)$, where $e_k$ denotes the $k$th elementary symmetric polynomial. If its maximum is positive, at least four coordinates are positive there. Fix two positive coordinates $a, b$, their sum $v$, and the remaining coordinates $w$. With $u = ab$ the unscaled expression is $e_4(w^2) + (v^2 - 2u)e_3(w^2) + u^2 e_2(w^2)$. It is strictly convex on $0 \le u \le v^2/4$ since $e_2(w^2) > 0$. Unequal positive $a, b$ would give an interior maximum. Thus the $k \ge 4$ positive coordinates all equal $1/k$. The resulting value is $c_k = \binom{k}{4}(4/k)^8$, with $c_4 = 1$ and $c_{k+1}/c_k = k^8/((k-3)(k+1)^7) < 1$. The ratio bound is immediate at $k = 4, 5$ and follows for $k \ge 6$ from $(1 - 3/k)(1 + 1/k)^7 \ge (1 - 3/k)(1 + 7/k) > 1$. Thus equality requires four equal positive coordinates, and each such point gives equality. □

**Lemma 2.3.** *On $\Delta_{m-1}$ one has $S_m \le 1 + E_m/16$. The inequality is strict away from $\mathcal{F}_m$.*

Section 3 proves the lemma by cubic perturbations and explicit algebraic inequalities, uniformly in the dimension. On faces of dimension at most three there is at most one nonzero $q_J$, and the lemma reduces by restriction to the tetrahedral Fejér bound.

## 2.2 The 70-point case

There are five four-element sets when $m = 5$. Set $T(x) = \sum_{J\neq K} q_J(x) q_K(x)$. Cauchy–Schwarz gives $T = (\sum_J q_J)^2 - \sum_J q_J^2 \le 4\sum_J q_J^2 \le 4$ by Lemma 2.2. Also $E_m \le T^2/4$. Indeed, each product $q_J q_K$ is at most $1/2$, and the sum of the squares of the ordered products is at most $T^2/2$ because every unordered product occurs twice. Lemma 2.3 gives $S_m \le 1 + T^2/64$.

For an arbitrary 70-point configuration let $M$ denote its evaluation matrix in the cardinal basis, with the five center columns last. Take $R = \max_i T(y_i)$ and $t = R/32$. Use the inner product on row vectors given by $A_t = \operatorname{diag}(I_{65}, K_t)$, where $K_t$ has diagonal one and off-diagonal $-t$. The $i$th diagonal entry of $MA_tM^{\mathsf{T}}$ is

$$S_m(y_i) - tT(y_i) \le 1 + \frac{T(y_i)(T(y_i) - 2R)}{64} \le 1.$$

For a row attaining $R$, the middle expression equals $1 - 16t^2$. The eigenvalues of $K_t$ are $1 + t$ with multiplicity four and $1 - 4t$. As $0 \le t \le 1/8$,

$$\det K_t = (1 + t)^4(1 - 4t) = 1 - t^2(10 + 20t + 15t^2 + 4t^3) \ge 1 - 13t^2.$$

Thus $A_t$ is positive definite. In its Gram-matrix form, Hadamard's inequality gives $\det G \le \prod_i G_{ii}$ for every positive semidefinite $G$ [8, p. 505]. Applying it to $G = MA_tM^{\mathsf{T}}$ gives $|\det M|^2 \det K_t = \det(MA_tM^{\mathsf{T}}) \le 1 - 16t^2$, proving $|\det M| < 1$ if $R > 0$. If $R = 0$, every point has a zero coordinate, since all five $q_J$ are positive at every interior point. By face restriction and Theorem 1.2, all rows then have norm at most one. A point outside $\mathcal{F}_5$ gives a row of norm strictly below one, so the product of the row norms is less than one. Equality therefore requires 70 distinct nodes.

### 2.3 A determinant estimate in arbitrary dimension

For an $n \times n$ real matrix $M = [U\ Q]$ let $q_{ij}$ be the entries of $Q$, with $p$ columns, and write $M_{i*}$ for row $i$. The sums over $j \ne k$ below count ordered pairs of columns of $Q$.

**Lemma 2.4.** *Suppose $q_{ij} \ge 0$ and $\sum_j q_{ij}^2 \le 1$ for every row. If $M$ is nonsingular, then*

$$\log|\det M|^2 \le \sum_i \left( \|M_{i*}\|_2^2 - 1 - \frac{1}{16} \sum_{j \ne k} q_{ij}^3 q_{ik}^3 \right). \tag{2.1}$$

*Proof.* For a positive definite $k \times k$ matrix $X$, summing $\log v \le v - 1$ over its eigenvalues gives $\log\det X \le \operatorname{tr} X - k$. This proves the case $p = 0$ by taking $X = M^{\mathsf{T}}M$. For $p > 0$, $H = Q^{\mathsf{T}}Q$ is positive definite since $M$ is nonsingular. Put $D(H) = \operatorname{tr} H - p - \log\det H$. This is Stein's covariance loss with the identity as reference matrix [9, eq. (72)]. Let $\Pi = QH^{-1}Q^{\mathsf{T}}$ be the orthogonal projection onto the columns of $Q$. Eliminating the $H$ block from $M^{\mathsf{T}}M$ leaves the Schur complement $U^{\mathsf{T}}U - U^{\mathsf{T}}QH^{-1}Q^{\mathsf{T}}U = U^{\mathsf{T}}(I - \Pi)U$. The block determinant formula [8, eq. (0.8.5.1)] gives $|\det M|^2 = \det H\ \det(U^{\mathsf{T}}(I - \Pi)U)$, with the empty determinant equal to one if $p = n$. Projecting the columns of $U$ with $I - \Pi$ can only shorten them, so the trace bound on the Schur complement gives

$$\log|\det M|^2 \le \operatorname{tr}(U^{\mathsf{T}}U) - (n - p) + \log\det H = \sum_i (\|M_{i*}\|_2^2 - 1) - D(H).$$

Write $h_j = H_{jj} > 0$ and $C_{jk} = H_{jk}/\sqrt{h_j h_k}$. The matrix $C$ is positive definite with diagonal one. Normalizing the columns of $Q$ gives the identity

$$D(H) = \sum_j (h_j - 1 - \log h_j) - \log\det C. \tag{2.2}$$

The scalar inequality gives $h_j - 1 - \log h_j \geq 0$. Hadamard's inequality gives $\det C \leq 1$ because $C_{jj} = 1$. We use these two nonnegative terms to bound sums over disjoint sets of ordered pairs.

For $j \neq k$ let $w_{jk} = \sum_i q_{ij}^2 q_{ik}^2$ and $B_{jk} = w_{jk}/\sqrt{h_j h_k}$. Set $w_{jj} = B_{jj} = 0$. The row hypothesis gives $\sum_k w_{jk} = \sum_i q_{ij}^2 \sum_{k \neq j} q_{ik}^2 \leq h_j$. For every $v \in \mathbb{R}^p$,

$$|v^{\mathsf{T}} B v| \leq \sum_{j<k} w_{jk}(v_j^2/h_j + v_k^2/h_k) \leq \|v\|_2^2.$$

Thus the spectrum of $B$ lies in $[-1, 1]$. Also $0 \leq B_{jk} \leq C_{jk}$ entrywise because $0 \leq q_{ij} q_{ik} \leq 1$. Hence $\operatorname{tr}(B^2) \leq \operatorname{tr}(BC)$ by summing the entrywise products. For every positive definite $T$, applying $\log\det X \leq \operatorname{tr} X - p$ to $X = C^{1/2} T C^{1/2}$ gives

$$-\log\det C \geq p - \operatorname{tr}(CT) + \log\det T.$$

Equality holds at $T = C^{-1}$, giving the variational formula for the log determinant [5, Example 3.23]. Choose $T = I - B/2$, which is positive definite by the spectral bound on $B$. Since $\operatorname{tr} C = p$, $\operatorname{tr} B = 0$, and $\log(1-u) \geq -u - u^2$ for $|u| \leq 1/2$, we obtain

$$-\log\det C \geq \tfrac{1}{2}\operatorname{tr}(BC) + \log\det(I - B/2) \geq \tfrac{1}{2}\operatorname{tr}(BC) - \tfrac{1}{4}\operatorname{tr}(B^2) \geq \tfrac{1}{4}\operatorname{tr}(BC). \tag{2.3}$$

Write $\sum_i \sum_{j \neq k} q_{ij}^3 q_{ik}^3 = E_{\mathrm{lo}} + E_{\mathrm{hi}}$. In $E_{\mathrm{lo}}$ both $h_j, h_k$ are at most two. For each such pair,

$$B_{jk} C_{jk} = \frac{(\sum_i q_{ij}^2 q_{ik}^2)(\sum_i q_{ij} q_{ik})}{h_j h_k} \geq \tfrac{1}{4} \sum_i q_{ij}^3 q_{ik}^3.$$

All cross terms are nonnegative. Summing and using (2.3) gives $-\log\det C \geq E_{\mathrm{lo}}/16$.

For $E_{\mathrm{hi}}$, use $v - 1 - \log v \geq v/8$ for $v \geq 2$, by differentiation and $\log 2 < 3/4$. For each row and $j$,

$$q_{ij} \sum_{k \neq j} q_{ik}^3 \leq q_{ij}(1 - q_{ij}^2)^{3/2} \leq \tfrac{1}{2}.$$

The indicator that $h_j > 2$ or $h_k > 2$ is at most $\mathbf{1}_{h_j > 2} + \mathbf{1}_{h_k > 2}$. By symmetry in $j, k$,

$$E_{\mathrm{hi}} \leq 2 \sum_{j : h_j > 2} \sum_i q_{ij}^3 \sum_{k \neq j} q_{ik}^3 \leq \sum_{j : h_j > 2} h_j \leq 8 \sum_j (h_j - 1 - \log h_j).$$

Substitution into (2.2) gives $D(H) \geq E_{\mathrm{lo}}/16 + E_{\mathrm{hi}}/8 \geq (E_{\mathrm{lo}} + E_{\mathrm{hi}})/16$, as required. □

*Proof of Theorem 1.1.* Fix any $Y = (y_1, \ldots, y_N)$ and set $M_{iz} = L_{m,z}(y_i)$. Interpolation of each $b_j$ gives $V_b(Y) = M V_b(\mathcal{F}_m)$. For $m \leq 4$, Theorem 1.2, face restriction and Hadamard's inequality give $|\det M| \leq 1$, strictly if any point lies outside $\mathcal{F}_m$. For $m \geq 5$ take $Q$ to consist of the tetrahedral face-center columns. Singular matrices have zero determinant. For any nonsingular matrix the row hypothesis of Lemma 2.4 follows from Lemma 2.2. Lemma 2.3 makes every summand in (2.1) nonpositive, strictly so for any point outside $\mathcal{F}_m$. Thus $|\det M| \leq 1$. The nodal configuration gives the identity matrix. Equality requires distinct nodal rows, hence precisely the prescribed set. □

## 3 The corrected squared-norm bound

Homogenize the assertion of Lemma 2.3 by setting

$$\Phi_m = s^{16}G_m + E_m/16, \qquad D_m = \sum_{i=1}^{m} \partial_{x_i}, \qquad R_m = sD_mG_m + 16mG_m.$$

The degree of $\Phi_m$ is 24 and that of $R_m$ is eight. For $x \geq 0$ with $s > 0$, homogeneity gives

$$\Phi_m(x) = s^{24}\big[1 - S_m(x/s) + E_m(x/s)/16\big].$$

Thus Lemma 2.3 is equivalent to nonnegativity of $\Phi_m$ on the orthant together with its stated equality set on $s = 1$. If $x_m = 0$, the cardinal restriction rule gives $G_m(x', 0) = G_{m-1}(x')$, and every term of $E_m$ using the last coordinate vanishes. In particular $\Phi_m(x', 0) = \Phi_{m-1}(x')$. We prove strict positivity in the open orthant for $m \geq 5$ and recover the remaining points by face restriction. Our induction follows segments from a facet to the centroid after normalization to $s = 1$. Bos used the same segments and raised the degree of a polynomial to obtain nonnegative coefficients. This bounded the partial sum omitting the tetrahedral face centers in dimensions $3 \leq d \leq 7$ [4, proof of Proposition 4.4]. Here we control the derivative of the corrected polynomial $\Phi_m$ along the segment.

**The input polynomial.** For a nonnegative integer vector $v$ let part$(v)$ be its positive entries in decreasing order. Symmetry and face restriction show that the coefficient of $x^v$ in $363G_m$, for $|v| = 8$, depends only on $\lambda = \text{part}(v)$. Table 1 records this ordinary monomial coefficient $g_\lambda = a + b\sqrt{5}$. Multiplicities are accounted for by summing over distinct monomials, with no additional factor on an individual coefficient.

Table 1: The 22 coefficients of $363G_m$. Digits concatenate the parts of $\lambda$. Each pair represents $a + b\sqrt{5}$.

| $\lambda$ | $a$ | $b$ | $\lambda$ | $a$ | $b$ |
|---|---|---|---|---|---|
| 8 | 0 | 0 | 41111 | 25065 | −174285 |
| 71 | 7260 | 0 | 332 | 4016166 | −246554 |
| 62 | −87120 | 0 | 3311 | 547658 | 607874 |
| 611 | −14421 | −1881 | 3221 | 2858902 | −208510 |
| 53 | 348480 | 0 | 32111 | 2810280 | −23430 |
| 521 | 211772 | 70004 | 311111 | 3671370 | 421110 |
| 5111 | 285549 | 35541 | 2222 | −33498204 | 732492 |
| 44 | −537240 | 0 | 22211 | −4607622 | −66090 |
| 431 | 192445 | −115995 | 221111 | 956922 | −383826 |
| 422 | −4545497 | 313841 | 2111111 | 6116790 | −396270 |
| 4211 | −701079 | −141739 | 11111111 | 14636160 | 0 |

The explicit cardinals in Appendix A give the table by squaring and collecting monomials. The two non-midpoint edge cardinals are conjugate under $\sqrt{21} \mapsto -\sqrt{21}$, so their squared sum lies in $\mathbb{Q}(\sqrt{5})$. This construction uses the five cardinal formulas and face restriction in every dimension.

We also need the octic in elementary symmetric functions. Write $e_j = e_j(x)$, $e_0 = 1$, and $e_1 = s$. Let

$$\mathcal{H}(s, e_2, e_3, e_4) = s^8 F(e_2/s^2, e_3/s^3, e_4/s^4),$$

where $F$ is the explicit tetrahedral polynomial in (4.2); the right side is polynomial after cancellation. The stable polynomial $\mathcal{G}$ representing $G_m$ for every $m$ is

$$\begin{aligned}363(\mathcal{G}-\mathcal{H}) &= (-3885987+728963r)s^3e_5\\ &\quad+(17210374-2817998r)se_2e_5+(19321502+3933518r)e_3e_5\\ &\quad+(-8761898+3618862r)s^2e_6+(-4646668-8052364r)e_2e_6\\ &\quad+(7995164+1003412r)se_7+(-10666112+2914432r)e_8. \end{aligned}\tag{3.1}$$

Equation (3.1) follows by expanding the elementary functions and comparing with Table 1. Set $e_j = 0$ for $j > m$ when evaluating in $m$ coordinates. Since $D_m e_j = (m-j+1)e_{j-1}$, define

$$\mathcal{R}_m = s\sum_{j=1}^{8}(m-j+1)e_{j-1}\frac{\partial\mathcal{G}}{\partial e_j}+16m\mathcal{G}, \qquad \mathcal{B}=\partial_m\mathcal{R}_m. \tag{3.2}$$

Substitution of the actual elementary functions gives $R_m$. When a coordinate vector is an argument of $\mathcal{R}_m$ or $\mathcal{B}$, its elementary functions are understood to be substituted. The formula is affine in the formal parameter $m$. On a face, this parameter remains the ambient number of coordinates; it is not replaced by the size of the positive support.

**Varying three coordinates.** Take three distinct positive coordinates with sum $u$, pairwise sum $v$ and product $k$. They are the roots of $z^3-uz^2+vz-k$. Choose disjoint intervals about the roots, each with positive endpoints at which the cubic has opposite signs. These signs persist for all sufficiently small changes of $k$, so the perturbed cubic has one root in each interval. The intervals can be arbitrarily small. Thus we can vary $k$ in either direction, keeping $u, v$ and all other coordinates fixed, within the same face. This is the real-root perturbation used by Riener [14, Corollary 3.2(4) and proof of Proposition 3.3].

**Five barycentric coordinates.** The correction in this case has the useful factorization

$$E_5/16 = 2^{45}(x_1x_2x_3x_4x_5)^3e_3(x_1^3,\dots,x_5^3). \tag{3.3}$$

Indeed, two different four-element sets omit two different coordinates; each unordered pair occurs twice in $E_5$.

**Lemma 3.1.** *The polynomial $\Phi_5$ is positive at every point of the open orthant.*

*Proof.* Work on $s = 1$. We first exclude a local minimum having three distinct positive coordinates. Hold the other two coordinates fixed, and write their sum and product as $a, b$. For the selected three coordinates write $u, v, k$ for their sum, pairwise sum and product. Thus $u = 1-a$. We vary $k$ in the two-sided interval just constructed.

Under the variation $e_2$ is fixed, while the derivatives of $e_3, e_4, e_5$ with respect to $k$ are $1, a, b$. The terms quadratic in these variables in $G_5$, obtained from Table 1, are

$$A(e_2)e_3^2+Be_3e_4+Ce_4^2+De_3e_5,$$

where $A, B, C$ are given in (4.2) and $D = (19321502+3933518r)/363$. All remaining terms are constant or linear along the variation. Consequently $G_5''/2 = A(e_2)+aB+a^2C+bD$. Since $A' > 0$, $A'+D > 0$, $e_2 = v+au+b \le u^2/3+au+b$, and $b \le a^2/4$, we obtain

$$\begin{aligned}\tfrac{1}{2}G_5'' &\le \frac{-100806811+14769605r}{2178}a^2+\frac{17876302-8609402r}{1089}a+\frac{-376852+90950r}{99}\\ &< -31000a^2-1200a-1700.\end{aligned}$$

The last bounds use $223/100 < r < 9/4$.

It remains to control the second derivative of the correction. Set $d_3 = a^3 - 3ab \geq 0$. The sum of cubes and the sum of the pairwise products of cubes of the selected triple are respectively $u^3 - 3uv + 3k$ and $v^3 - 3uvk + 3k^2$. Equation (3.3) gives

$$(E_5/16)'' = 2^{45}b^3\big[30k^4 + d_3(60k^3 - 36uvk^2 + 6v^3k) + b^3(36k^2 - 18uvk + 6u^3k)\big].$$

Use $uv \geq 9k$, $v \leq u^2/3$ and $k \leq u^3/27$. The term $d_3(60k^3 - 36uvk^2)$ is nonpositive, and $36k^2 - 18uvk + 6u^3k \leq -126k^2 + 6u^3k \leq u^6/14$. Moreover $b^3 \leq a^6/64$, $b^6 \leq a^{12}/4096$, and $b^3d_3 \leq a^9/256$. For the last bound, the function $b^3(a^3 - 3ab)$ increases for $0 \leq b \leq a^2/4$. It follows that

$$(E_5/16)'' \leq 2^{45}\left[\frac{10a^6u^{12}}{64 \cdot 177147} + \frac{2a^9u^9}{256 \cdot 243} + \frac{a^{12}u^6}{4096 \cdot 14}\right] < 39000a^2$$

when $a > 0$; the derivative is zero when $a = 0$. To check the numerical bound, divide by $a^2$ and maximize each product $a^j(1-a)^{16-j}$ at $a = j/16$. The three resulting upper bounds are

$$3840, \qquad \frac{1312993546389}{67108864} < 19566, \qquad \frac{7119140625}{458752} < 15519,$$

whose sum is less than 39000. Thus in all cases

$$\Phi_5'' < -23000a^2 - 2400a - 3400 < 0. \tag{3.4}$$

Every boundary point satisfies $\Phi_5 \geq 0$ by face restriction and Theorem 1.2. A negative global minimum on the compact simplex would therefore be interior. By (3.4), its coordinates could have at most two distinct values. Such a point, including the all-equal case, is a permutation of $(\underbrace{A, \ldots, A}_{p}, \underbrace{A+T, \ldots, A+T}_{5-p})$ with $A > 0$, $T \geq 0$, and $1 \leq p \leq 4$. Appendix B gives explicit positive coefficient bounds for all four binary forms, contradicting a negative minimum. This proves nonnegativity everywhere. An interior zero would now be a local minimum and would be excluded by the same argument. Homogeneity extends strict positivity to the open orthant. □

**A two-value reduction for the induction polynomial.**

**Lemma 3.2.** *For $m \geq 6$, every local minimum of $R_m$ relative to the interior of any face of $\Delta_{m-1}$ has at most two distinct positive coordinate values.*

*Proof.* Choose three distinct positive coordinates, if they exist. Keep their sum $u$ and pairwise sum $v$ fixed and vary their product $k$, as in Lemma 3.1. This gives a two-sided curve in the same face. Let $a, b$ be the elementary functions of degrees one and two in the other coordinates. On $s = 1$, $u = 1 - a$ and $e_2 = v + au + b$ is fixed, while the derivatives of $e_3, e_4, e_5, \ldots$ are $1, a, b, \ldots$. Only the terms containing $e_3^2, e_3e_4, e_4^2, e_3e_5$ in (3.2) contribute a second derivative. Their coefficients have the form $A_*(e_2), B_*, C_*, D_*$. Thus

$$\tfrac{1}{2}R_m'' = A_*(e_2) + aB_* + a^2C_* + bD_*.$$

Here $A'_* > 0$ and $A'_* + D_* > 0$. Using $e_2 \le (1-a)^2/3 + a(1-a) + b$ and $b \le a^2/2$ gives

$$\begin{aligned}\tfrac{1}{2}R''_m \le\ & \frac{-462502064 + 160573264r}{1089} m a^2 \\ & + \left(\frac{246084376 - 126288176r}{1089} m + \frac{29150056 - 23606360r}{363}\right) a \\ & + \frac{-56740394 + 9398698r}{1089} m + \frac{-17876302 + 8609402r}{363} \\ <\ & -90000ma^2 - (30000m + 60000)a - 32000m + 5000 < 0.\end{aligned} \tag{3.5}$$

The coefficient bounds use $223/100 < r < 9/4$; the last inequality holds for $m \ge 6$. A strictly negative second derivative contradicts a local minimum on the admissible curve. □

Write $P \ge Q$ when $P - Q$ has nonnegative coefficients in the displayed polynomial variables. Such an inequality implies $P \ge Q$ when those variables are nonnegative. In a vector argument, $A^p$ will mean $p$ repeated entries equal to $A$.

**One larger coordinate.**

**Lemma 3.3.** *Let $p \ge 1$ and $m \ge \max(6, p+1)$. For $A, T \ge 0$ with $A + T > 0$,*

$$R_m(A^p, A+T, 0^{m-p-1}) > 0.$$

*Proof.* The coefficient bounds in Appendix C give

$$\begin{aligned}\frac{363}{p}\mathcal{R}_{p+1}(A^p, A+T) &\ge 10^5 pAT^3\Psi(pA, T) + 8712A^8 + 7260T^8 && (p \ge 5),\\ 363\mathcal{B}(A^p, A+T) &\ge 10^5 pAT^3\Psi(pA, T) + 8712A^8 + 7260T^8 && (p \ge 3),\end{aligned} \tag{3.6}$$

where

$$\Psi(Z, T) = Z^2(5Z^2 - 7ZT + 3T^2) + T^2(7Z^2 - ZT + T^2).$$

Both quadratic forms are positive definite, with discriminants $-11$ and $-27$. The right sides of (3.6) are therefore positive. For the remaining multiplicities, put $\sigma = (p+1)A + T$. We multiply by a power of this positive linear form to test the coefficients, as in Pólya's theorem [13]. Appendix C checks the exponent three and gives

$$\begin{aligned}\sigma^3\mathcal{R}_6(A^p, A+T) &\ge 4\sigma^{11} && (p = 1, 2, 3, 4),\\ \sigma^3\mathcal{B}(A^p, A+T) &\ge 6\sigma^{11} && (p = 1, 2).\end{aligned} \tag{3.7}$$

For $p \ge 5$, use $\mathcal{R}_m = \mathcal{R}_{p+1} + (m-p-1)\mathcal{B}$. For $p \le 4$, use $\mathcal{R}_m = \mathcal{R}_6 + (m-6)\mathcal{B}$. The preceding positive bounds prove both cases. □

**At least two larger coordinates.** We interpolate the multiplicities by polynomials. For real parameters $n, q$ define

$$E_j(n, q; A, T) = \sum_{h=0}^{j} \binom{q}{h}\binom{n-h}{j-h} A^{j-h}T^h, \qquad E_0 = 1. \tag{3.8}$$

The binomial coefficients here are falling-factorial polynomials. For integers $n \geq q \geq 0$, $E_j$ is the elementary function on $(A^{n-q}, (A+T)^q)$. This follows by choosing the $h$ factors contributing $T$ from the $q$ larger coordinates, and the remaining $j-h$ factors contributing $A$ from the other $n-h$ coordinates. Set

$$C(n,q;A,T) = 363\mathcal{R}_n(E_1,\dots,E_8), \qquad B(n,q;A,T) = 363\mathcal{B}(E_1,\dots,E_8). \tag{3.9}$$

The notation on the right means substitution into the elementary variables of (3.2). In particular, if $p+q = n \leq m$,

$$363R_m(A^p, (A+T)^q, 0^{m-n}) = C(n,q;A,T) + (m-n)B(n,q;A,T). \tag{3.10}$$

There are two different derivatives in this argument. The derivative defining $\mathcal{B}$ in (3.2) holds the elementary variables fixed. In contrast, $\partial_n C$ and $\partial_n B$ below differentiate the complete polynomials after the substitution (3.8), keeping $q, A, T$ fixed. Thus all dependence of the elementary functions on $n$ is included.

**Lemma 3.4.** *For integers $p \geq 1$, $q \geq 2$, $m \geq \max(6, p+q)$, and $A, T \geq 0$ with $A+T > 0$, the left side of* (3.10) *is positive.*

*Proof.* Appendix C proves the following coefficient bounds. For $u, v \geq 0$, put $Z = (1+u)A + (1+v)T$. Then

$$\begin{aligned} \partial_n C(6+u, 2+v; A, T) &\geq 4000Z^8, \\ \partial_n B(6+u, 2+v; A, T) &\geq 69696AZ^7. \end{aligned} \tag{3.11}$$

The coefficient order is in $u, v, A, T$. For fixed $q \geq 2$ and nonnegative $A, T$, both $C(n,q;A,T)$ and $B(n,q;A,T)$ are consequently nondecreasing in the real variable $n$ on $n \geq 6$. More explicitly, for $n \geq n_0 \geq 6$ and $P = C$ or $B$,

$$P(n,q;A,T) - P(n_0,q;A,T) = \int_{n_0}^{n} \partial_n P(t,q;A,T)\,dt \geq 0. \tag{3.12}$$

Only the polynomial extension in (3.8) is used at noninteger $t$; no simplex of noninteger dimension is invoked.

The boundary family $n = q+1$ satisfies, for $w \geq 0$,

$$\begin{aligned} C(6+w, 5+w; A, T) &\geq 8712(1+w)^9(A+T)^8, \\ B(6+w, 5+w; A, T) &\geq 8712(1+w)^8(A+T)^8. \end{aligned} \tag{3.13}$$

Both right sides are positive. Finally, for

$$(n,q) \in \{(3,2),(4,2),(4,3),(5,2),(5,3),(5,4),(6,2),(6,3),(6,4)\},$$

the explicit bounds in the appendix give

$$\begin{aligned} C(n,q;A,T) + (6-n)B(n,q;A,T) &\geq 2000(nA+qT)^8, \\ B(n,q;A,T) &\geq 2000(nA+qT)^8. \end{aligned} \tag{3.14}$$

Let $n = p+q$. If $n \leq 5$, write the right side of (3.10) as $C + (6-n)B + (m-6)B$ and apply (3.14). Suppose $n \geq 6$. If $q = 2,3,4$, take $n_0 = 6$ in (3.12) and use (3.14). If $q \geq 5$, then $p \geq 1$ implies $n \geq q+1 \geq 6$; take $n_0 = q+1$ and use (3.13) with $w = q-5$. In this integration $q$ remains fixed: the boundary substitution $n = 6+w$, $q = 5+w$ only supplies the initial value. Both initial values are strictly positive whenever $A+T > 0$, including $A = 0$ or $T = 0$. By (3.12) they stay positive even if a derivative vanishes. Hence $C, B > 0$, and $m \geq n$ makes (3.10) positive. This treats both unbounded multiplicities and the ambient dimension. □

We can now conclude that $R_m(x) > 0$ for every $x \geq 0$, $x \neq 0$, and $m \geq 6$. Normalize to $s = 1$ and take a minimizer. It lies in the relative interior of the face determined by its positive coordinates, so Lemma 3.2 leaves at most two distinct positive values. A support of size one gives $R_m(1, 0, \ldots, 0) = 20(m-1) > 0$. All other possibilities are covered by Lemmas 3.3 and 3.4; equal positive coordinates correspond to $T = 0$. Homogeneity removes the normalization $s = 1$.

*Completion of the proof of Lemma 2.3.* For $m \leq 4$ the face restriction and Theorem 1.2 give $\Phi_m \geq 0$, with equality on $s = 1$ exactly at the nodes. The five-coordinate argument in Lemma 3.1 gives the next case. For $m \geq 6$ we have

$$D_m \Phi_m = s^{15} R_m + \tfrac{1}{16} D_m E_m > 0 \qquad (0 \neq x \geq 0),$$

since $E_m$ has nonnegative monomial coefficients in the original variables. A zero coordinate reduces the assertion to $m - 1$ coordinates. For a point with all coordinates positive, put $t = \min_i x_i$, $\mathbf{1} = (1, \ldots, 1)$, and $y = x - t\mathbf{1}$. The face identity gives $\Phi_m(y) \geq 0$ by induction, with $\Phi_m(0) = 0$ if all coordinates of $x$ coincide. For $0 < v \leq t$ the point $y + v\mathbf{1}$ is nonzero, so

$$\Phi_m(x) - \Phi_m(y) = \int_0^t D_m \Phi_m(y + v\mathbf{1})\, dv > 0.$$

Deleting zero coordinates reduces a point on $s = 1$ to the positive orthant on its support. Supports of size at least five give strict positivity by the preceding argument. On smaller supports the tetrahedral equality characterization gives exactly the nodes. □

# 4 The tetrahedral Fejér bound

We now prove Theorem 1.2, the common base case for the two higher-dimensional results. Timofte's half-degree principle tests a symmetric octic on the nonnegative orthant at points with at most four distinct nonzero coordinates [16, Corollary 2.1(1)]. In four variables this leaves the whole tetrahedron. For the specific octic $G_4$, varying the product of three distinct positive coordinates while fixing their sum and their pairwise sum gives strictly negative curvature, as the Hessian below shows.

## 4.1 The octic in elementary symmetric functions

Enumerate $\mathcal{F}_4 = \{\xi_1, \ldots, \xi_{35}\}$ and put $L_j = L_{4,\xi_j}$. Let $v(x) = (x^\nu)_{|\nu|=4}$ be a row vector, $V$ the matrix with rows $v(\xi_i)$, and $W = V^{-1}$. The entries of $v(x)W$ are the homogeneous cardinal polynomials. Define the homogeneous deficit

$$H(x) := G_4(x) = s^8 - v(x) W W^{\mathsf{T}} v(x)^{\mathsf{T}}. \tag{4.1}$$

On $\Delta_3$ this is $1 - \sum_j L_j^2$. Permuting the coordinates permutes the nodes and hence the cardinal polynomials, so $H$ is symmetric. We will also evaluate (4.1) at unnormalized coordinates. Its sign is unchanged by positive rescaling.

Let $e_k$ denote the $k$th elementary symmetric polynomial in $x$, and write $e = e_2$, $p = e_3$, $q = e_4$. On $s = 1$ we have $H(x) = F(e, p, q)$ with

$$F(e, p, q) = U(e) + P(e)p + R(e)q + A(e)p^2 + Bpq + Cq^2. \tag{4.2}$$

The coefficients in (4.2) are

$$\begin{aligned} U(e) &= 20e - 360e^2 + 2100e^3 - 3920e^4, \\ P(e) &= \frac{4623 - 57r}{11} + \frac{-125333 + 7219r}{33}e + \left(\frac{29764}{3} - 924r\right)e^2, \\ R(e) &= \frac{16458 - 5230r}{11} + \frac{4491922 - 33070r}{363}e + \frac{-1742572 + 221332r}{33}e^2, \\ A(e) &= \frac{-136228 + 14134r}{11} + \frac{77200 - 3296r}{3}e, \\ B &= \frac{2845034 - 2736862r}{363}, \qquad C = \frac{-17739344 + 1312048r}{363}. \end{aligned}$$

Equation (4.2) follows by substituting the cardinals in Appendix A into (4.1). Equivalently, homogenize each monomial $e^i p^j q^k$ in (4.2) by multiplying it by $s^{8-2i-3j-4k}$, expand the elementary functions, and compare with the entries of Table 1 supported on at most four coordinates. The paired edge formulas explain the cancellation of $\sqrt{21}$.

**Lemma 4.1.** *For every $0 \le e \le 3/8$, the Hessian of $F$ in $(p, q)$ at fixed $e$,*

$$M(e) = \begin{pmatrix} 2A(e) & B \\ B & 2C \end{pmatrix},$$

*is negative definite. This interval contains $e_2(\Delta_3)$.*

*Proof.* Since $2e = 1 - \sum_i x_i^2$ on $\Delta_3$ and $\sum_i x_i^2 \ge 1/4$, we have $0 \le e \le 3/8$. The slope of $A$ is positive, $C < 0$, and $A(3/8) = (-30078 + 9602r)/11 < 0$. The determinant $4A(e)C - B^2$ therefore decreases with $e$. At the right endpoint it equals

$$\det M(3/8) = \frac{8(4149878507601 - 1515033492901r)}{131769} > 0.$$

The signs follow from $2 < r < 9/4$. The two leading principal minors now give negative definiteness. □

## 4.2 A minimum has at most two distinct positive coordinates

**Lemma 4.2.** *Every local minimum of $H$ in the interior of $\Delta_3$ has at most two distinct coordinate values. If the restriction of $H$ to a triangular face has a local minimum in its relative interior, at least two of the three positive coordinates coincide.*

*Proof.* Suppose $x \in \Delta_3$ has three pairwise distinct positive coordinates $a, b, c$. Write $d$ for the fourth coordinate and, after a permutation, take $x = (a, b, c, d)$. Hold $d$ fixed. For both signs of $t$ sufficiently close to zero, the polynomial

$$K(z) - t, \qquad K(z) = (z - a)(z - b)(z - c),$$

has three distinct positive roots $a(t), b(t), c(t)$ tending to $a, b, c$, respectively, by the sign-change argument for three simple roots. The sum of these roots stays $a + b + c$, the negative of the cubic's quadratic coefficient. Thus $x(t) = (a(t), b(t), c(t), d)$ belongs to $\Delta_3$ and $x(t) \to x$.

Let $e, p, q$ be the invariants at $x$. The polynomial with roots the coordinates of $x(t)$ is

$$(z - d)(K(z) - t) = z^4 - z^3 + ez^2 - (p + t)z + (q + dt). \tag{4.3}$$

In particular, $e_2$ stays fixed and $H(x(t)) = F(e, p + t, q + dt)$. This is a quadratic in $t$. Using its Hessian from Lemma 4.1, for every sufficiently small $t > 0$ we obtain

$$H(x(t)) + H(x(-t)) - 2H(x) = t^2(1, d)M(e)(1, d)^{\mathsf{T}} < 0. \tag{4.4}$$

At least one of the two nearby points has smaller deficit than $x$.

Every interior point with three or four distinct values contains such a triple. In particular, at $(u, u, v, w)$ choose $a = u$, $b = v$, $c = w$ and $d = u$. Although the full quartic has a repeated root, the three roots of $K$ are simple. For a face with three distinct positive coordinates take $d = 0$. The two nearby points then lie on that same face, so (4.4) also excludes a relative minimum there. □

### 4.3 The remaining one-parameter families

By Lemma 4.2, it suffices to study binary homogeneous restrictions of $H$. We give their factorizations in nonnegative variables $a, b$. Each identity can be obtained by substituting into the homogeneous form of (4.2).

**Edges and triangular faces.** On an edge we have

$$H(a, b, 0, 0) = 20ab(a - b)^2(a^2 - 5ab + b^2)^2. \tag{4.5}$$

When $a + b = 1$, the quadratic factor is $1 - 7ab$, whose roots give $a = (1 \pm \eta)/2$. Together with the endpoints and midpoint these are the five zeros on the edge.

For a face point with a repeated coordinate the factorization is

$$H(a, a, b, 0) = ab\left(b - \frac{3 + r}{2}a\right)^2 T(a, b). \tag{4.6}$$

The remaining quartic can be written as

$$T(a, b) = 40(b^2 - \lambda ab + a^2)^2 + a^2(A_0b^2 + B_0ab + C_0a^2). \tag{4.7}$$

The coefficients in this decomposition are

$$\lambda = \frac{4397 - 383r}{880}, \qquad A_0 = \frac{23382299 + 5935673r}{29040},$$
$$B_0 = \frac{-229907 + 88485r}{33}, \quad C_0 = \frac{316632 - 138752r}{33}.$$

Here $A_0, C_0 > 0$, and

$$4A_0C_0 - B_0^2 = \frac{-4238873363448 + 1896536114072r}{59895} > 0.$$

For these signs it suffices to use $559/250 < r < 9/4$, which follows by squaring. Thus the quadratic form in (4.7) is positive definite. When $a = 0$, the first square is $40b^4$, so $T(a, b) > 0$ whenever $(a, b) \neq (0, 0)$.

Each triangular face is compact. Its minimum occurs on an edge or, by Lemma 4.2, at a repeated-coordinate point. Equations (4.5) and (4.6) prove nonnegativity on the whole boundary. Any zero in a face interior is therefore a relative minimum. Up to permutation it has the form $(a, a, b, 0)$ with $a, b > 0$. Solving $b = (3 + r)a/2$ and $2a + b = 1$ gives $a = \beta$, $b = \alpha$.

**Two pairs and a triple.** An interior point with at most two coordinate values has multiplicities $2+2$ or $3+1$, allowing $a=b$ in either expression. For two pairs, direct substitution gives

$$H(a,a,b,b) = ab(a-b)^2\big[c(a-b)^4 + d\,ab(a-b)^2 + f\,a^2b^2\big], \tag{4.8}$$

where

$$c = \frac{144384-17408r}{33}, \quad d = \frac{-433948+285060r}{363}, \quad f = \frac{2147592+1441384r}{363}.$$

All three coefficients are positive by the same bounds on $r$. Thus (4.8) is strictly positive for distinct positive $a, b$.

For a triple the analogous factorization is

$$H(a,a,a,b) = a(a-b)^2 Q(a,b). \tag{4.9}$$

Here $Q$ is the binary quintic given by

$$\begin{aligned} 121Q(a,b) = {} & (264682-26576r)a^5 + (3606167+2531r)a^4b \\ & + (-3940095+238179r)a^3b^2 + (685905+148093r)a^2b^3 \\ & - (87021+1881r)ab^4 + 7260b^5. \end{aligned} \tag{4.10}$$

The mixed signs in (4.10) can be resolved by bounding $g(t) = Q(t,1-t)$ on subintervals of $0 \le t = a/(a+b) \le 1$. We use the Bernstein coefficient bounds described by Garloff [7, §§1.1–1.2]. For each interval $[\ell, h]$ in Table 2, expand $g(\ell + (h-\ell)z) = \sum_{k=0}^{5} d_k z^k$. Its Bernstein coefficients on that interval are

$$b_j = \sum_{k=0}^{j} d_k \frac{\binom{j}{k}}{\binom{5}{k}}, \qquad g(\ell+(h-\ell)z) = \sum_{j=0}^{5} b_j \binom{5}{j} z^j (1-z)^{5-j}. \tag{4.11}$$

For a coefficient $b_j = u + vr$ with $u, v \in \mathbb{Q}$, a rational lower bound is $u + 559v/250$ if $v \ge 0$ and $u + 9v/4$ if $v < 0$. The table gives the integer parts of these bounds. For example, on the first interval $b_1 = (1989 - 171r)/220 \ge 6417/880 > 7$.

Table 2: Integer lower bounds for the Bernstein coefficients of $Q(t, 1-t)$.

| Interval | $b_0$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ |
|---|---|---|---|---|---|---|
| $[0, 1/4]$ | 60 | 7 | 29 | 35 | 23 | 17 |
| $[1/4, 1/2]$ | 17 | 11 | 11 | 40 | 148 | 346 |
| $[1/2, 1]$ | 346 | 743 | 1499 | 2704 | 3831 | 1693 |

The Bernstein basis functions are nonnegative and sum to one on $0 \le z \le 1$. It follows that $g > 0$ throughout $[0,1]$. By homogeneity $Q(a,b) = (a+b)^5 g(a/(a+b)) > 0$ for all nonnegative $a, b$ with $a + b > 0$. Thus (4.9) is strictly positive at every interior point of this family with $a \ne b$.

*Proof of Theorem 1.2.* The continuous function $H$ attains its minimum on $\Delta_3$. The boundary calculation excludes a negative boundary minimum. An interior minimum has at most two distinct coordinate values by Lemma 4.2, and (4.8) and (4.9) exclude a negative value there. This proves (1.2).

Every zero of $H$ is now a global minimum. In the interior the two factorizations force all four coordinates to coincide, giving the centroid. On a face interior the zeros are the permutations of $(\alpha, \beta, \beta, 0)$, and (4.5) gives precisely the five nodes on each edge. These exhaust the orbits in (1.1). Conversely, the cardinal conditions give equality at every listed point. □

**Corollary 4.3.** *The uniform probability measure on $\mathcal{F}_4$ is the unique quartic D-optimal design on $\Delta_3$.*

*Proof.* Work in the cardinal basis $L = (L_1, \dots, L_{35})^{\mathsf{T}}$ and put $G_\mu = \int LL^{\mathsf{T}}\, d\mu$. By Theorem 1.2, $\operatorname{tr} G_\mu = \int \sum_j L_j^2\, d\mu \le 1$. The arithmetic–geometric mean inequality for the eigenvalues of $G_\mu$ gives

$$\det G_\mu \le (\operatorname{tr} G_\mu/35)^{35} \le 35^{-35}.$$

The uniform measure on $\mathcal{F}_4$ attains the bound since its matrix is $I/35$. Equality forces $\operatorname{tr} G_\mu = 1$ and every eigenvalue to equal $1/35$. The trace condition and the equality set in Theorem 1.2 force $\mu$ to be supported on $\mathcal{F}_4$. In that case $G_\mu$ is diagonal with the node masses on its diagonal, so all masses are $1/35$. □

# 5 The fourth-power bound and local squared norms

The corrected squared-norm estimate also proves the fourth-power theorem. The cross terms between distinct tetrahedral face-center cardinals absorb the correction in Lemma 2.3.

*Proof of Theorem 1.3.* For $x \in \Delta_{m-1}$ set

$$A_m(x) = \sum_{J \ne K} q_J(x)^2 q_K(x)^2,$$

with ordered pairs as in the definition of $E_m$. By Lemma 2.2,

$$0 \le A_m \le \left(\sum_J q_J^2\right)^2 \le 1, \qquad q_J q_K \le \frac{q_J^2 + q_K^2}{2} \le \frac{1}{2} \quad (J \ne K).$$

Consequently $E_m \le A_m/2$. Expanding the square of $S_m$ and retaining only the cross terms between center cardinals gives

$$\sum_{z \in \mathcal{F}_m} L_{m,z}^4 \le S_m^2 - A_m.$$

Lemma 2.3 now yields

$$\begin{aligned}\sum_{z \in \mathcal{F}_m} L_{m,z}^4 &\le \left(1 + \frac{E_m}{16}\right)^2 - A_m \\ &\le \left(1 + \frac{A_m}{32}\right)^2 - A_m = 1 - A_m\left(\frac{15}{16} - \frac{A_m}{1024}\right) \le 1.\end{aligned}$$

If $x \notin \mathcal{F}_m$, Lemma 2.3 is strict. Since $S_m \ge 0$ and $1 + E_m/16 > 0$, the first inequality in this display is then strict. The cardinal conditions give equality at each node. The argument also covers $m < 4$, when both center sums are empty. □

## 5.1 The squared norm near a node

Although $S_5$ exceeds one at the centroid of $\Delta_4$, its behavior near each interpolation node persists in every dimension.

**Corollary 5.1.** *For every $m \ge 2$ and $z \in \mathcal{F}_m$ there is a relative neighborhood of $z$ in $\Delta_{m-1}$ in which $S_m(x) < 1$ for $x \ne z$.*

*Proof.* For $m \le 4$ the assertion follows from Theorem 1.2 and face restriction. Suppose $m \ge 5$, let $I = \operatorname{supp} z$, and take $j \notin I$. The cardinal conditions and $s(z) = 1$ give

$$\partial_j G_m(z) = 8 - 2\partial_j L_{m,z}(z). \tag{5.1}$$

For nodes with $|I| \le 3$, restrict to the coordinate subspace indexed by $I \cup \{j\}$. The derivative is therefore determined by a tetrahedral cardinal polynomial. For $|I| = 4$ the cardinal is $q_I$, which is independent of $x_j$. The values in Table 3 follow.

Table 3: Derivatives in a coordinate outside the support of a node. Here $r = \sqrt{5}$ and $G_m = s^8 - S_m$.

| Node type | $\partial_j L_{m,z}(z)$ | $\partial_j G_m(z)$ |
|---|---|---|
| Vertex | $-6$ | 20 |
| Edge midpoint | $(-150 + 34r)/33$ | $(564 - 68r)/33$ |
| Other edge node | $(-57 - 17r)/33$ | $(378 + 34r)/33$ |
| Triangular face node | $-1$ | 10 |
| Tetrahedral face center | 0 | 8 |

For completeness, the values can also be obtained directly from Table 1. If the positive coordinates of $z$ are $z_1, \dots, z_k$, then

$$\partial_j G_m(z) = \frac{1}{363} \sum_{\substack{\nu \in \mathbb{Z}^k_{\ge 0} \\ |\nu| = 7}} g_{\mathrm{part}(\nu,1)} z_1^{\nu_1} \cdots z_k^{\nu_k}.$$

The five substitutions are the representatives in (1.1). All the derivatives in the last column of Table 3 are positive, using $\sqrt{5} < 9/4$ for the midpoint. By continuity there is a ball about $z$ in which $\partial_j G_m > 0$ for every $j \notin I$.

For $x$ in that ball with $x \ge 0$, let $x_I$ be obtained by setting the coordinates outside $I$ to zero. The segment $[x_I, x]$ stays in the ball. Integration gives $G_m(x) > G_m(x_I) \ge 0$ whenever $x \ne x_I$. The last inequality follows by homogeneity from Theorem 1.2 on the face with support $I$. On that face and $s = 1$ there are finitely many nodal equality points. Shrinking the ball to exclude the other nodes proves the assertion. □

## 6 Extension to higher polynomial degree

Fix a simplex and increase the degree from four to $n$. The determinant estimate of Lemma 2.4 still applies. For a unisolvent set $X$ with cardinals $\ell_z$, put $S_X = \sum_{z \in X} \ell_z^2$. A sufficient condition for $X$ to be Fekete is that a fixed subset $q_1, \dots, q_p$ of the cardinals satisfy, throughout the simplex,

$$q_j \ge 0, \qquad \sum_j q_j^2 \le 1, \qquad S_X \le 1 + \tfrac{1}{16} \sum_{j \ne k} q_j^3 q_k^3. \tag{6.1}$$

Indeed, applying the lemma to the matrix $(\ell_z(y_i))_{i,z}$ bounds the determinant ratio for every competing configuration. The issue in higher degree is how large a squared row norm this estimate can absorb. If $a_j = q_j^2$ and $h = \sum_j a_j \le 1$, then

$$\sum_{j \ne k} (a_j a_k)^{3/2} \le \sum_j a_j^2 (h - a_j) \le h^3/4.$$

The first inequality uses $\sqrt{a_j a_k} \le (a_j + a_k)/2$ and the second uses $a_j(h - a_j) \le h^2/4$. Consequently (6.1) requires $S_X \le 65/64$.

**An obstruction on the quintic triangle.** At the centroid $c$ of the triangle Bos reports $S_X(c) \approx 1.2246$ for his 21 nodes [4, Numerical Example 5.2, p. 289]. One quintic already gives an exact separation. On $\sum x_i = 1$ take $f = 27x_1x_2x_3(15e_2 - 4)$. It vanishes at the 15 boundary nodes and $f(c) = 1$. The six interior nodes form two three-point orbits of $(t, t, 1-2t)$, with $0.1480 < u < 0.1481$ and $0.4208 < v < 0.4209$ as their parameters (Appendix D). On such an orbit $f = 27t^2(1-2t)[1-5(1-3t)^2]$. Bounding the two factors separately on the displayed intervals gives $|f| < 1/4$ on the first orbit and $|f| < 1/2$ on the second. Interpolation at $c$ and Cauchy–Schwarz give

$$1 \le S_X(c) \sum_{z \in X} f(z)^2 < S_X(c)\, 3\big((1/4)^2 + (1/2)^2\big).$$

Thus $S_X(c) > 16/15 > 65/64$. Every choice of columns in (6.1) encounters the same obstruction.

**Two steps that depend on the degree.** For a theorem uniform over simplices, the quartic face-center bound survives in degree five. For $m \ge n$ and $n \ge 2$, consider the natural products $q_J = n^n \prod_{i \in J} x_i$, $|J| = n$. For $n = 5$ they are the cardinals at the centers of Bos's four-dimensional faces [4, §5, p. 291]. In general,

$$\max_{x \in \Delta_{m-1}} \sum_J q_J(x)^2 = \max_{n \le k \le m} \binom{k}{n} (n/k)^{2n}.$$

The related maximization over all support sizes is the uniform-profile theorem [11, Theorem 14]. Here the sum is $n^{2n} e_n(x_1^2, \dots, x_m^2)$. At a positive maximum, choose two positive coordinates and fix their sum and all other coordinates $w$. As in Lemma 2.2, the expression is strictly convex in their product: the coefficient of its square is $n^{2n} e_{n-2}(w^2) > 0$. Hence all $k$ positive coordinates are equal. For $n = 5$ the ratio of successive terms on the right is $1/g(1/k)$ with $g(t) = (1-4t)(1+t)^9$. Its derivative has the sign of $1-8t$, and $g(1/5) > g(0) = 1$. The maximum is therefore one, attained at $k = 5$. For $n \ge 6$, already $k = n+1$ gives $(n+1)(n/(n+1))^{2n} > 1$. At $n = 6$ this is $6^{12}/7^{11} > 1$, and for $n \ge 7$ the value exceeds $(n+1)e^{-2} > 1$. Thus these products first violate the uniform bound in degree six, on a six-dimensional simplex.

The reduction to two coordinate values also uses the degree. Along the variation in Section 3, the sum and pairwise sum of three coordinates are fixed. A symmetric polynomial of degree $2n$ has degree at most $\lfloor 2n/3 \rfloor$ in their product. For the quartic squared norm the dependence is at most quadratic. At degree five the term $e_1 e_3^3$ is already possible, so the curvature argument would require control of a cubic term.

**A fixed triangle in degrees five and six.** There is another way to approach higher degrees on the same domain. In degree $n \ge 3$, prescribe the three vertices and $n-1$ further distinct points in the relative interior of each edge. The $3n$ boundary nodes leave $r_n = \binom{n-1}{2}$ points $Z$ in the interior. A polynomial of degree at most $n$ vanishing at the boundary nodes has $n+1$ zeros on each edge and hence vanishes on all three edges, so it is $wp$ with $w = x_1x_2x_3$ and $\deg p \le n-3$. Bos studied this separation of boundary and interior nodes in [3] and applied it to the quintic triangle in [4, §5, pp. 287–288]. With the boundary fixed, the quantity to maximize over $Z$ is

$$\left(\prod_{z \in Z} w(z)\right) \left|\det\big(p_j(z)\big)_{z \in Z,\, 1 \le j \le r_n}\right|, \tag{6.2}$$

where the $p_j$ form any basis of the polynomials of degree at most $n-3$. To see this, complete the functions $wp_j$ to a degree-$n$ basis. Put the boundary rows first and the $wp_j$ columns last. The upper right block vanishes, leaving the product of the two diagonal determinants.

For quintics, (6.2) is a weighted quadratic determinant at six points. Bos computed it for two symmetric triples. Appendix D proves that his choice uniquely maximizes it among permutation-invariant sets of six interior points. For sextics the corresponding problem is a weighted cubic determinant at ten points. An unrestricted quintic proof along this route would have to maximize (6.2) over arbitrary positions of the six interior points and then allow the number and locations of boundary nodes to vary. The unrestricted quintic case remains open here.

# A Explicit homogeneous cardinal polynomials

Bos gives barycentric formulas for the quartic cardinals [1] and lists them by node type in [4, §4]. We use homogeneous expressions in the coordinates on and off the support of a node. The nodal evaluations below verify their coefficients. For a fixed support $I$, take all sums below over its complement and put

$$\sigma_j = \sum_{i\notin I} x_i^j, \qquad \varepsilon_j = e_j((x_i)_{i\notin I}), \qquad M_{21} = \sigma_1\sigma_2 - \sigma_3.$$

Empty sums and positive-degree elementary functions of an empty set are zero. The notation $a, b, c$ denotes coordinates in $I$.

At a vertex, with $I = \{i\}$ and $a = x_i$, the cardinal is

$$\begin{aligned} 132L_i = {} & 132a^4 - 792a^3\sigma_1 + 792a^2\sigma_2 + (1566 + 342r)a^2\varepsilon_2 \\ & - 132a\sigma_3 - (480 + 120r)aM_{21} - (1554 + 306r)a\varepsilon_3. \end{aligned} \tag{A.1}$$

For the midpoint on an edge with support $I = \{i, j\}$, write $a = x_i, b = x_j$. Then

$$\begin{aligned} 132L_{ij}^{\mathrm{mid}} = {} & ab\big[-704(a^2+b^2) + 3520ab + (-2400 + 544r)(a+b)\sigma_1 \\ & + (2528 - 864r)\sigma_2 + (2432 - 448r)\varepsilon_2\big]. \end{aligned} \tag{A.2}$$

For the other two edge nodes, put $h = \sqrt{21}$ and

$$\begin{aligned} U &= 1078(a-b)^2 - (441 + 245r)(a+b)\sigma_1 + (98 + 294r)\sigma_2 + (1568 + 392r)\varepsilon_2, \\ V &= 154(a^2 - b^2) - (385 + 77r)(a-b)\sigma_1. \end{aligned}$$

Their cardinals are

$$132L_{ij}^{\pm} = ab(U \pm hV). \tag{A.3}$$

The plus sign belongs to the node $((7+h)/14, (7-h)/14)$ in the ordered coordinates $(i, j)$, and the minus sign belongs to its reversal. In particular their squared sum is $2a^2b^2(U^2 + 21V^2)/132^2$.

Two details matter when comparing the edge formulas with those printed in [4, §4, p. 284]. On $s = 1$, the printed midpoint expression has value $(14r - 43)/33$ at the midpoint. Adding $76 - 14r$ inside its braces gives (A.2). In the oriented-edge expression, replacing the repeated $\lambda_i$ in the second linear term by $\lambda_j$ gives $L_{ij}^-$ in (A.3) for $a = \lambda_i, b = \lambda_j$.

For a triangular node with value $\alpha$ at $i$ and values $\beta$ at $j, k$, let $I = \{i, j, k\}$ and $(a, b, c) = (x_i, x_j, x_k)$. Then

$$132L_{i|jk} = abc\big[(330 + 4818r)a + (2244 - 2376r)(b + c) - (4818 + 66r)\sigma_1\big]. \tag{A.4}$$

Finally, the cardinal at a tetrahedral face center is

$$L_I = 256 \prod_{i \in I} x_i \qquad (|I| = 4). \tag{A.5}$$

Each displayed polynomial contains the product of the coordinates in its support. It therefore vanishes at nodes whose supports do not contain that support. On the containing supports, substitution of the five patterns in (1.1) gives one at its own node and zero at all the others. For example, on an edge the vertex expression is $a(a - b)((a + b)^2 - 7ab)$, and (A.3) has value one at its assigned non-midpoint node because $ab = 1/7$ and $a - b = \pm\sqrt{21}/7$. At every tetrahedral center containing the support, the bracket in each of (A.2)–(A.4) vanishes. The remaining triangular substitutions use $\alpha + 2\beta = 1$, $\alpha = (4 + r)/11$, $\beta = (7 - r)/22$, and $r^2 = 5$. Unisolvence identifies these polynomials with the cardinal basis.

Squaring (A.1)–(A.5) and summing over supports gives Table 1 and (3.1). Only support sizes at most eight can occur. The cancellation in the squared pair (A.3) explains why the resulting octic has coefficients in $\mathbb{Q}(\sqrt{5})$.

# B The four binary forms in five coordinates

This appendix supplies the finite algebra in Lemma 3.1. All coefficients and bounds needed for the argument are given here. For $1 \le p \le 4$, put $q = 5 - p$ and define

$$H_p(z) = 363G_5(\underbrace{1, \dots, 1}_{p}, \underbrace{z, \dots, z}_{q}) = \sum_{j=0}^{8} h_{p,j} z^j.$$

Table 4 gives $H_1, H_2$ directly from Table 1. Symmetry and homogeneity give $H_3(z) = z^8 H_2(1/z)$ and $H_4(z) = z^8 H_1(1/z)$; these are polynomial identities, including at $z = 0$ after cancellation.

Table 4: Coefficients $h_{p,j} = a_{p,j} + b_{p,j}\sqrt{5}$ of the two octics.

| $j$ | $a_{1,j}$ | $b_{1,j}$ | $a_{2,j}$ | $b_{2,j}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 29040 | 0 | 2382336 | −287232 |
| 2 | −435006 | −11286 | −15413304 | 2738556 |
| 3 | 5077380 | 982212 | 62019474 | −5862582 |
| 4 | −35500485 | −1384047 | −101452455 | 6537219 |
| 5 | 104017092 | 348060 | 89414028 | −6274488 |
| 6 | −118452912 | −1138584 | −53890698 | 1690512 |
| 7 | 47579136 | 0 | 18460818 | 334098 |
| 8 | 0 | 0 | 794046 | −79728 |

For $A, T \ge 0$ write the degree-24 binary form as

$$\Phi_5(A^p, (A + T)^q) = \sum_{j=0}^{24} \binom{24}{j} \beta_{p,j} (5A)^{24-j} (qT)^j, \tag{B.1}$$

where powers inside the argument denote repeated coordinates. Here is an explicit formula for every coefficient. Set

$$g_{p,\ell} = \sum_{h=\ell}^{8} \binom{h}{\ell} h_{p,h}.$$

The coefficient $c_{p,j} = \binom{24}{j} 5^{24-j} q^j \beta_{p,j}$ is

$$\begin{aligned} c_{p,j} = \frac{1}{363} \sum_{\substack{0\le \ell \le 8 \\ 0 \le j-\ell \le 16}} \binom{16}{j-\ell} 5^{16-j+\ell} q^{j-\ell} g_{p,\ell} \\ + 2^{45} \sum_{\ell=0}^{3} \binom{p}{\ell} \binom{q}{3-\ell} \binom{3q+9-3\ell}{j}. \end{aligned} \tag{B.2}$$

Binomial coefficients outside their usual nonnegative range are zero. The first sum expands $(5+qT)^{16} H_p(1+T)/363$. The second follows from (3.3), since

$$E_5(1^p, (1+T)^q)/16 = 2^{45} \sum_{\ell=0}^{3} \binom{p}{\ell} \binom{q}{3-\ell} (1+T)^{3q+9-3\ell}.$$

Table 5: Integer lower bounds for $1000\beta_{p,j}$. Each entry applies to every index in the indicated range.

| $j$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ |
|---|---|---|---|---|
| 0, 1 | 3 | 3 | 3 | 3 |
| 2–5 | 4 | 6 | 9 | 21 |
| 6–9 | 22 | 50 | 91 | 103 |
| 10–13 | 65 | 153 | 213 | 193 |
| 14–17 | 125 | 319 | 309 | 453 |
| 18–21 | 144 | 513 | 416 | 575 |
| 22, 23 | 66 | 425 | 312 | 666 |
| 24 | 0 | 258 | 0 | 0 |

Substitution in (B.2) gives the rational lower bounds in Table 5. They use only $2236/1000 < \sqrt{5} < 2237/1000$: in each coefficient $a + b\sqrt{5}$, replace the radical by $2236/1000$ for $b \ge 0$ and by $2237/1000$ for $b < 0$. Thus every coefficient in (B.1) is nonnegative, and $\beta_{p,0} \ge 3/1000$ for every $p$. In particular

$$\Phi_5(A^p, (A+T)^q) \ge \frac{3}{1000}(5A)^{24} > 0 \qquad (A > 0,\ T \ge 0),$$

as used in Lemma 3.1.

# C Uniform algebra for the two-value families

This appendix proves the coefficient bounds used in Lemmas 3.3 and 3.4. All input polynomials are specified by (4.2), (3.1), and (3.2). The comparisons are finite rational inequalities after using

$$a + br \ge \ell(a + br) := \begin{cases} a + 2236b/1000, & b \ge 0, \\ a + 2237b/1000, & b < 0. \end{cases} \tag{C.1}$$

Here $2236/1000 < r < 2237/1000$, as follows by squaring. For coefficient comparisons we apply $\ell$ separately to each coefficient after collecting it in $\mathbb{Q}(r)$. Equivalently, $\ell(a+br)$ is the minimum of the two rational endpoint values. We do not apply $\ell$ separately to factors or treat it as a ring homomorphism.

## C.1 A direct formula for every coefficient

We give a second way to obtain the binary coefficients, directly from Table 1. It also specifies the finite arithmetic in this appendix without a multivariate elementary-function expansion. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, put $d_\lambda = \prod_{j\ge1} m_j(\lambda)!$, where $m_j(\lambda)$ is the multiplicity of the part $j$, and write $(z)_{\underline{h}} = \prod_{j=0}^{h-1}(z-j)$. Define the nonnegative integers $\kappa_{\lambda;i,h}$ and the polynomials $M_{\lambda,i}$ by

$$\begin{aligned} \kappa_{\lambda;i,h} &= [t^i z^h] \prod_{a=1}^{l} \big(1 + z((1+t)^{\lambda_a} - 1)\big), \\ M_{\lambda,i}(n,q) &= \frac{1}{d_\lambda} \sum_{h=0}^{l} \kappa_{\lambda;i,h} (q)_{\underline{h}} (n-h)_{\underline{l-h}}. \end{aligned} \tag{C.2}$$

Indices outside their nonnegative ranges give zero. If $m_\lambda$ denotes the sum of distinct monomials with exponent partition $\lambda$, then

$$m_\lambda(A^{n-q}, (A+T)^q) = \sum_{i=0}^{|\lambda|} M_{\lambda,i}(n,q) A^{|\lambda|-i} T^i. \tag{C.3}$$

To prove this identity, first label the $l$ parts. Choose the $h$ parts that contribute a positive power of $T$; the binomial coefficients of those powers are counted by $\kappa$. Their coordinate indices can be assigned in $(q)_{\underline{h}}$ ways. The other labeled parts can use any of the remaining $n-h$ coordinates, in $(n-h)_{\underline{l-h}}$ ways. Dividing by $d_\lambda$ removes the labelings of equal parts. This proves (C.3) for integer $n \ge q \ge 0$ and specifies its polynomial extension.

With $g_\lambda$ from Table 1, set

$$H_i = \sum_{\lambda\vdash 8} g_\lambda M_{\lambda,i}, \qquad J_i = \sum_{\mu\vdash 7} g_{\mu\cup(1)} M_{\mu,i}. \tag{C.4}$$

The appended partition is sorted, and $H_i = 0$ outside $0 \le i \le 8$, $J_i = 0$ outside $0 \le i \le 7$. Write the normalized binary coefficients as

$$C = \sum_{i=0}^{8} \binom{8}{i} \widehat{C}_i(n,q) A^{8-i} T^i, \qquad B = \sum_{i=0}^{8} \binom{8}{i} \widehat{B}_i(n,q) A^{8-i} T^i. \tag{C.5}$$

Then the explicit formulas are

$$\boxed{\begin{aligned} \widehat{C}_i &= \frac{(24-i)nH_i + (9-i)qH_{i-1}}{\binom{8}{i}}, \\ \widehat{B}_i &= \frac{16H_i + nJ_i + qJ_{i-1}}{\binom{8}{i}}. \end{aligned}} \tag{C.6}$$

Indeed, $H(A,T)=\sum_i H_i A^{8-i}T^i$ is $363G_n$ on the two-value family. If one adds a coordinate $y$, the coefficient of $y$ in $363G_{n+1}$ is $J(A,T)=\sum_i J_i A^{7-i}T^i$: only partitions with a part equal to one contribute. A common increase in the $n$ displayed coordinates differentiates $H$ with respect to $A$ while holding $T$ fixed. Each of the $m-n$ zero coordinates contributes $J$. Consequently, with $s=nA+qT$,

$$363R_m = s\partial_A H + 16nH + (m-n)(sJ+16H).$$

Comparing with (3.10) and collecting $T^i$ proves (C.6) for integer $n\ge q\ge 0$. It is also an identity in the formal parameters: for each nonnegative integer $q$ it holds for infinitely many $n$, and then each coefficient in $n$ agrees for infinitely many $q$.

For clarity, the integers in (C.2) can be generated by the short recurrence

$$\kappa_{\varnothing;0,0}=1, \qquad \kappa_{\lambda\cup(a);i,h}=\kappa_{\lambda;i,h}+\sum_{b=1}^{a}\binom{a}{b}\kappa_{\lambda;i-b,h-1}.$$

All other initial entries are zero. Thus every comparison below uses only this integer recurrence, falling factorials of order at most eight, and the two rational endpoints in (C.1).

## C.2 One larger coordinate

Define

$$\sum_{i=0}^{8} c_i(p)T^i = \frac{363}{p}\mathcal{R}_{p+1}(1^p,1+T), \qquad \sum_{i=0}^{8} d_i(p)T^i = 363\mathcal{B}(1^p,1+T).$$

These polynomials are obtained by the substitution

$$e_j=\binom{p}{j}+(1+T)\binom{p}{j-1}, \qquad 1\le j\le 8. \tag{C.7}$$

Equivalently, (C.6) gives

$$c_i(p)=\frac{\binom{8}{i}}{p}\widehat{C}_i(p+1,1), \qquad d_i(p)=\binom{8}{i}\widehat{B}_i(p+1,1).$$

The numerator defining $c_i$ vanishes at $p=0$, since $\mathcal{R}_1(1+T)=0$, and is therefore divisible by $p$. Both $c_i, d_i$ have degree at most $8-i$ in $p$. Table 6 specifies targets $\lambda_i(p)$ and positive integers $\rho_i$.

Table 6: Targets and residual bounds for the one-larger-coordinate family.

| $i$ | $\lambda_i(p)$ | $\rho_i$ |
|---|---|---|
| 0 | 0 | 8712 |
| 1 | 0 | 69696 |
| 2 | 0 | 180143 |
| 3 | $500000p^5$ | 350859 |
| 4 | $-700000p^4$ | 588215 |
| 5 | $1000000p^3$ | 95119 |
| 6 | $-100000p^2$ | 114867 |
| 7 | $100000p$ | 4791 |
| 8 | 0 | 7260 |

Substitution of (C.7) into (3.2), followed by collection in $T$ and expansion in $w$, gives

$$\begin{aligned} c_i(5+w) - \lambda_i(5+w) &\geq \rho_i(1+w)^{8-i}, \\ d_i(3+w) - \lambda_i(3+w) &\geq \rho_i(1+w)^{8-i}. \end{aligned} \tag{C.8}$$

More explicitly, for each $0 \leq j \leq 8-i$, applying $\ell$ to the coefficient of $w^j$ on either left side gives at least $\rho_i\binom{8-i}{j}$. This gives all coefficient comparisons in (C.8). For example, $c_8 = d_8 = 7260$, while

$$c_7(p) = (108999 - 1881r)p + 65241 + 1881r.$$

The other rows follow by the same falling-factorial expansion in (C.7); no coefficient of degree above $8-i$ occurs.

The five nonzero targets give

$$\sum_i \lambda_i(p) A^{8-i} T^i = 10^5 pAT^3\left[5(pA)^4 - 7(pA)^3 T + 10(pA)^2 T^2 - pAT^3 + T^4\right].$$

The bracket is $\Psi(pA, T)$ from (3.6). The residual constant and leading coefficients in (C.8) are at least 8712 and 7260, respectively. Dropping the other nonnegative residuals proves (3.6).

For (3.7), use the same substitution with the indicated fixed $p$ and multiply by $((p+1)A+T)^3$. Normalize the coefficient of $A^{11-j}T^j$ by $\binom{11}{j}(p+1)^{11-j}$. For $\mathcal{R}_6$ at $p = 1, 2, 3, 4$, the lower bounds from (C.1), for every $0 \leq j \leq 11$, are respectively

$$\frac{16791}{1000}, \quad \frac{6359}{1000}, \quad \frac{4339}{1000}, \quad \frac{5690}{1000}.$$

For $\mathcal{B}$ at $p = 1, 2$ they are 7485/1000 and 6187/1000. These bounds exceed four and six, respectively, proving (3.7).

### C.3 Two or more larger coordinates

Use the normalized coefficients in (C.5). The substitution (3.8) gives degree at most $9-i$ in $n$ and at most $i$ in $q$ for $\widehat{C}_i$; for $\widehat{B}_i$ the corresponding bounds are $8-i$ and $i$. Indeed, a term contributing $T^h$ from $E_j$ has degree at most $j-h$ in $n$ and $h$ in $q$. Weighted homogeneity of (3.2) then gives these bounds, including the one additional factor of $n$ in $C$.

The following coefficient inequalities give the full arithmetic needed for (3.11) and (3.13):

$$\begin{aligned} \ell\Big([u^a v^b]\,\partial_n \widehat{C}_i(6+u, 2+v)\Big) &\geq \mu_i \binom{8-i}{a}\binom{i}{b}, && {\scriptstyle 0\leq i\leq 8,\ 0\leq a\leq 8-i,\ 0\leq b\leq i}, \\ \ell\Big([u^a v^b]\,\partial_n \widehat{B}_i(6+u, 2+v)\Big) &\geq \nu_i \binom{7-i}{a}\binom{i}{b}, && {\scriptstyle 0\leq i\leq 7,\ 0\leq a\leq 7-i,\ 0\leq b\leq i}, \\ \ell\Big([w^a]\widehat{C}_i(6+w, 5+w)\Big) &\geq 8712\binom{9}{a}, && 0 \leq i \leq 8,\ 0 \leq a \leq 9, \\ \ell\Big([w^a]\widehat{B}_i(6+w, 5+w)\Big) &\geq 8712\binom{8}{a}, && 0 \leq i \leq 8,\ 0 \leq a \leq 8. \end{aligned} \tag{C.9}$$

Here $[u^a v^b]$ and $[w^a]$ denote ordinary coefficient extraction, and the constants are given in Table 7. Also $\partial_n \widehat{B}_8 = 0$. Each bound in (C.9) follows by inserting (C.2)–(C.4) into (C.6), differentiating where indicated, expanding the shifted falling factorials, and applying (C.1) to

Table 7: Rational lower bounds in (C.9). For $i \le 7$, $\mu_i = 8712(9-i)$ and $\nu_i = 8712(8-i)$.

| $i$ | $\mu_i$ | $\nu_i$ | $i$ | $\mu_i$ | $\nu_i$ |
|---|---|---|---|---|---|
| 0 | 78408 | 69696 | 5 | 34848 | 26136 |
| 1 | 69696 | 60984 | 6 | 26136 | 17424 |
| 2 | 60984 | 52272 | 7 | 17424 | 8712 |
| 3 | 52272 | 43560 | 8 | 4101 | 0 |
| 4 | 43560 | 34848 | | | |

each collected coefficient. This gives respectively 165, 120, 90, and 81 rational comparisons in the four lines. The indicated degree bounds include every coefficient. For the boundary substitutions, the total degrees are at most nine and eight, respectively.

For example, $\partial_n \widehat{C}_8(n,q) = \widehat{B}_8(q)$ is independent of $n$. The lower bounds for the coefficient of $v^a$ in $\widehat{B}_8(2+v)$, divided by $\binom{8}{a}$, for $a = 0,\dots,8$, are

$$(1159865, 1542714, 622638, 119284, 7399, 4101, 7668, 9449, 8712).$$

This gives the exceptional value $\mu_8 = 4101$. The remaining rows use the stated linear formulas for $\mu_i, \nu_i$. More precisely, for $i \le 7$ the normalized minimum in the first line of (C.9) occurs at $(a,b) = (8-i,i)$ and equals $\mu_i$. In the second line it occurs at $(a,b) = (7-i,i)$ and equals $\nu_i$. The minima in the boundary lines occur at $a = 9$ and $a = 8$, respectively, and equal 8712 for every $i$. In the exceptional row the smallest normalized lower bound occurs at $a = 0$, $b = 5$. Its exact calculation from (C.6) is

$$\frac{[v^5]\widehat{B}_8(2+v)}{\binom{8}{5}} = \frac{6980759}{672} - \frac{629525}{224}r, \qquad \ell\left(\frac{6980759}{672} - \frac{629525}{224}r\right) = \frac{15748667}{3840} > 4101.$$

Summing the first line of (C.9) over $a, b$ gives

$$\partial_n \widehat{C}_i(6+u, 2+v) \ge \mu_i (1+u)^{8-i}(1+v)^i.$$

Since every $\mu_i \ge 4000$, substitution into (C.5) proves the first inequality in (3.11). For the second, use $\nu_i\binom{8}{i} = 69696\binom{7}{i}$ for $i \le 7$, and $\partial_n \widehat{B}_8 = 0$. The last two lines of (C.9), followed by the binomial theorem in $A, T$, give (3.13).

For the small supports put $C_6 = C + (6-n)B$. Table 8 bounds each coefficient of $C_6$ and $B$ from below after division by the corresponding coefficient of $(nA+qT)^8$. The entries follow from (C.6) and (C.1); all exceed 2000, proving (3.14).

Table 8: Lower bounds for all normalized coefficients and listed $q$ at each support size.

| $n$ | $q$ | $C_6$ | $B$ |
|---|---|---|---|
| 3 | 2 | 4695 | 2584 |
| 4 | 2,3 | 5808 | 2904 |
| 5 | 2,3,4 | 2065 | 2181 |
| 6 | 2,3,4 | 3360 | 2270 |

# D The symmetric quintic triangle

Write $O_t$ for the orbit of $(t, t, 1-2t)$. In Bos's construction [4, §5, p. 288], $u, v$ are the roots in $(0.1480, 0.1481)$ and $(0.4208, 0.4209)$ of the polynomial

$$P(t) = 2737800t^{10} - 8660340t^9 + 11981963t^8 - 9523289t^7 + 4800577t^6 \\ - 1598001t^5 + 354286t^4 - 51415t^3 + 4649t^2 - 235t + 5.$$

All ten roots are isolated by the alternating signs of $P(10^{-4}j)$ at $j = 0$, 1000, 1480, 1481, 2550, 3000, 4208, 4209, 4580, 5000, 6000.

**Proposition D.1.** *Among 21-point configurations consisting of the three vertices, four further points on each edge, and six interior points invariant as a set under all permutations of the vertices, Bos's quintic nodes uniquely maximize the absolute Vandermonde determinant, up to ordering.*

*Proof.* For the boundary factor preceding (6.2), use the three vertex coordinates and the four functions $x_i x_j x_i^k (x_i + x_j)^{3-k}$, $0 \le k \le 3$, on each edge $ij$. Write $\tau_1, \dots, \tau_4$ for the $x_i$-coordinates of its four interior edge nodes. The edge block has absolute determinant $\prod_a \tau_a(1-\tau_a) \prod_{a<b} |\tau_a - \tau_b|$. This univariate Vandermonde product with endpoints 0, 1 is uniquely maximized by the degree-five Gauss–Lobatto set [15, §2].

An interior permutation orbit has size one, three or six, with only the centroid giving size one. On a single six-point orbit $e_2$ is constant, so the weighted quadratic determinant vanishes. Every nonsingular symmetric choice therefore has the form $O_s \cup O_t$, $0 < s < t < 1/2$. Write $w_t = t^2(1-2t)$. Bos's factorization [4, §5, pp. 287–288] gives (6.2) as a fixed positive constant times

$$K(s,t) = \left| [w_s w_t (t-s)]^3 (1-3s)^2 (1-3t)^2 (2-3s-3t) \right|.$$

Set $\sigma = s + t$ and $\tau = st$. The boundary of the closed parameter triangle has $K = 0$. At a positive maximum the stationary equations are $G(s,t) = G(t,s) = 0$, where

$$G(s,t) = -\tfrac{1}{6} s(1-2s)(s-t)(3s-1)(3\sigma-2)\, \partial_s \log K(s,t).$$

Taking their difference divided by $s - t$, then substituting in their sum, gives

$$\tau = \frac{((45\sigma - 59)\sigma + 27)\sigma - 5}{84\sigma - 55}, \\ J(\sigma) = 52650\sigma^5 - 166545\sigma^4 + 207899\sigma^3 - 127879\sigma^2 + 38700\sigma - 4600 = 0.$$

The numerator in the first line is $3205/197568$ at $\sigma = 55/84$. Substitution in the sum after multiplication by $(84\sigma - 55)^2$ gives $-2(3\sigma - 2)J(\sigma)$, and $3\sigma = 2$ gives $K = 0$. The value to compare is

$$K^2 = (\sigma^2 - 4\tau)^3 \tau^{12} (1 - 2\sigma + 4\tau)^6 (1 - 3\sigma + 9\tau)^4 (3\sigma - 2)^2.$$

The intervals $10^{-6}(j, j+1)$ for $j = 398396, 568845, 650697, 705430, 839877$ each have opposite endpoint signs for $J$ and account for all its roots. Substitution in the formulas for $\tau$ and $K^2$ gives $10^{28} K^2 < 1/16$ in the first, fourth and fifth intervals. In the third, $(1-2s)(1-2t) = 1 - 2\sigma + 4\tau < 0$, so this interval is inadmissible. Since $10^{28} K(3/20, 21/50)^2 > 1$, the maximum occurs in the second interval. There $x^2 - \sigma x + \tau$ has signs $+, -, -, +$ at $0.1480, 0.1481, 0.4208, 0.4209$, respectively. Eliminating $s$ from the two stationary polynomials gives

$$\operatorname{Res}_s\big(G(s,t), G(t,s)\big) = 3t(2t-1)(3t-1)^4 P(t).$$

At a positive maximum none of the factors preceding $P(t)$ vanishes. Thus $P(t) = 0$ and, by symmetry, $P(s) = 0$. The isolating intervals for Bos's decic now identify the unique maximizing pair as $(u, v)$. □

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