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# EVERY CLOSED SMOOTH $n$-MANIFOLD EMBEDS ISOMETRICALLY IN $\mathbb{R}^{n(n+1)/2+28}$

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Abstract. We prove that every closed smooth Riemannian $n$-manifold admits a smooth isometric embedding into $\mathbb{R}^{s_n+28}$, where $s_n = n(n+1)/2$. This improves the previous upper bound $s_n + \max\{n+5,\, 2n\}$ to a dimension-independent surplus of 28.

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## 1. Introduction

Let $(M, g)$ be a smooth Riemannian manifold of dimension $n$. An isometric embedding of $(M, g)$ into $\mathbb{R}^N$ is a smooth embedding $f : M \to \mathbb{R}^N$ satisfying

$$\langle df_x(u), df_x(v)\rangle = g_x(u, v) \qquad (x \in M,\ u, v \in T_xM). \tag{1.1}$$

In local coordinates this is a system of

$$s_n = \frac{n(n+1)}{2}$$

equations, one for each independent metric coefficient, and $N = s_n$ is the smallest dimension in which one can expect solvability for every metric (Remark 1.2). We call $N - s_n$ the *surplus*. Nash [Nas56] proved that every closed Riemannian manifold embeds isometrically in a Euclidean space, and Günther [Gün89a, Gün91] obtained the sharpest classical bound

$$N = \max\left\{\tfrac{n(n+5)}{2},\ \tfrac{n(n+3)}{2} + 5\right\} = s_n + \max\{2n,\ n + 5\}.$$

In these results the surplus grows linearly with $n$. The main result of this paper replaces it by a constant.

**Theorem 1.1.** *Every closed smooth Riemannian manifold $(M^n, g)$ admits a smooth isometric embedding into $\mathbb{R}^{s_n+28}$.*

No orientability, parallelizability, connectedness or Euler characteristic assumption is made. For $n \le 14$ the theorem follows from the classical bound, since $\max\{2n, n+5\} \le 28$ exactly when $n \le 14$; its content is the range $n \ge 15$. The seven-label construction described below is used for $n \ge 16$, and the case $n = 15$ uses a nowhere-zero vector field, which exists on every closed odd-dimensional manifold (§20).

*Remark* 1.2 (The universal dimension $s_n$ cannot be lowered). Let $N < s_n$. Then some smooth metric on the torus $\mathbb{T}^n$ admits no $C^\infty$ isometric immersion into $\mathbb{R}^N$. Indeed, fix $k \ge 0$ and consider maps $F$ defined near $0 \in \mathbb{R}^n$ with $F(0) = 0$. The $k$-jet at 0 of $(dF)^T dF$ depends polynomially on the $(k+1)$-jet of $F$. The $(k+1)$-jets form a space of dimension at most $N\binom{n+k+1}{n}$, and the $k$-jets of symmetric matrices a space of dimension $s_n\binom{n+k}{n}$. Since $\binom{n+k+1}{n} = \frac{n+k+1}{k+1}\binom{n+k}{n}$, the first is smaller than the second as soon as $(s_n - N)(k+1) > Nn$, which holds for large $k$. For such $k$ the image of the jet map has measure zero, so it misses a jet with positive definite constant term. Its Taylor polynomial is a positive definite metric near 0; gluing it to the flat metric by a cutoff equal to 1 near 0 and supported where the polynomial metric is positive definite gives a smooth metric on $\mathbb{T}^n$, and an isometric immersion of it would realize the omitted jet.

**Background.** A map is *free* if its first and second derivatives $\partial_i F$, $\partial_i\partial_j F$ are linearly independent. For a free map the linearized metric operator has an algebraic right inverse, and this is the mechanism of the classical proofs [Nas56, Gün89a]. It costs a surplus linear in $n$ twice over. First, a free map needs $N \ge n + s_n$, simply to accommodate its columns. Second, a free map exists only if there is a bundle monomorphism $J(TM) =$

$TM \oplus \operatorname{Sym}^2 TM \to \varepsilon^N$, and for general $M$ the characteristic classes of $J(TM)$ obstruct such a monomorphism unless $N - (n + s_n)$ is comparable to $n$.

**The idea of the proof.** We replace freeness by a weaker condition. A covector $\xi$ is *good* for an immersion $F$ if the second fundamental form of $F$ is injective on $\operatorname{Sym}^2(\ker \xi)$, and $F$ is *phase regular* if every point has a good covector; no continuity of the good covector is required. For phase-regular maps an exact metric correction holds as soon as $N \geq s_n + 1$ (Theorem 6.1), with all derivatives in transverse parameters, uniformly over atlases with any number of charts, and for polynomially degenerating phase margins (Theorem 6.3). The global problem is then reduced, by composing with fast phases on products of Hopf spheres (Theorem 7.1), to the construction of one exactly isometric phase-regular map, a *selector*, on a model space.

A genuinely free structure is needed only on a subbundle of $TM$ of rank $m = n - 6$, whose complement in the target has rank comparable to $n$, so that general position applies. The remaining six tangent directions are carried by an exactly isometric free map of a product of six Hopf spheres into $\mathbb{R}^{28}$ (Theorem 5.1), and the mixed second derivatives between the two blocks are produced by a normal tensor $T : E \otimes \mathbb{R}^7 \to W$ with seven labels. Its coefficients come from a four-dimensional space of $3 \times 3$ matrices, the complement of an orthogonal product configuration of Tiles type from the theory of unextendible product bases [DT01]. The tensor must satisfy a flat alternating Gram identity, two regularity conditions and the bound

$$\dim_{\mathbb{C}} \mathcal{B}_T \leq 4$$

on the complex family $\mathcal{B}_T$ of five-dimensional label planes $H \subset \mathbb{C}^7$ with $\ker T_{\mathbb{C}} \cap (E_{\mathbb{C}} \otimes H) \neq 0$. This bound is proved by two incidence counts, one over the region where the six active directions split off (Theorem 11.6) and one over the corank-one locus of a generic six-frame (Proposition 12.1). It controls the characteristic variety of the mixed rows of the metric equation, which leads to a periodic differential right inverse (Proposition 17.3). The bound $4 = k - 2$, with $k = 6$ active variables, is critical: the raw characteristic locus then has dimension at most $k$, and Lagrangian components are excluded by a Hessian identity and polynomial jet interpolation on the Hopf spheres.

The dimension count is as follows. For $n = m + 6$ the principal model consists of a free block in $\mathbb{R}^{j_m}$, $j_m = m + s_m$, a seven-frame of normal vectors, and a residual normal bundle of rank $5(m + 1) + 10$, for a total of

$$D_P = j_m + 7 + 5(m + 1) + 10 = s_n + 1.$$

The old map lies in a hyperplane of dimension $D_P - 1$. The six active directions are carried by the free Hopf block in $\mathbb{R}^{j_6+1} = \mathbb{R}^{28}$, whose unit cofactor normal plays the role of the remaining model coordinate, so that

$$N = (D_P - 1) + (j_6 + 1) = s_n + 28.$$

Section 21 shows that 28 is the least surplus compatible with the full complex seven-label condition and these two identities. This is a limitation of the construction, not an obstruction to isometric embeddings in smaller dimension.

**Organization of the paper.** Section 2 describes the proof step by step and Section 3 fixes notation. Section 4 proves the foliated, parametric and relative free realization theorem, Section 5 constructs exact free Hopf blocks, and Section 6 proves the exact metric correction. Section 7 passes from selectors to isometric immersions and then to embeddings. Section 8 constructs the source bundles and their complete boundary classes. Sections 9–12 construct the seven-label tensor: the coefficient model (§9), the incidence estimates (§10), the exterior count (§11) and the core count (§12). Sections 13 and 14 prepare the active block, the protected collar and the horizontal frame, and Sections 15–17 construct the periodic inverse. Section 18 completes the metric, Section 19 absorbs the surviving line, and Section 20 proves Theorem 1.1 in all dimensions. Section 21 proves the lower bound within the construction. Appendix A specifies the finite exact computations used in the proofs.

## 2. Outline of the proof

Throughout, $\gamma(F) = (dF)^T dF$ denotes the induced metric. We fix $n \geq 16$ and the parameters

(2.1)
$$q = 7, \qquad k = 6, \qquad m = n-6, \qquad M = n-5, \qquad r = n-4, \qquad c = 5, \qquad C_0 = 10;$$

the cases $n \leq 15$ are treated in §20.

**Step 1: Free realization.** Free maps realizing a prescribed complete formal class with prescribed leaf metric exist in the foliated, parametric and relative setting when the target exceeds $j_d$ by at least five (Theorem 4.1). The class is realized by the higher-order ample theorem, the map is shortened along the leaves, and the deficit is added by a supported metric primitive with parameters built on Günther's construction.

**Step 2: Exact free Hopf blocks.** On a product of $d$ Hopf spheres there is an exactly isometric free map into $\mathbb{R}^{j_d+1}$, one dimension more than the number of its columns, independently of the sphere sizes (Theorem 5.1). The induction adds one sphere at a time with an orthogonal matrix primitive whose orbit moments are prescribed exactly, and removes the remaining error by an exact metric correction on a torus bundle.

**Step 3: Exact metric correction.** For a phase-regular map in $\mathbb{R}^N$, $N \geq s_n+1$, every nearby positive metric is induced by a nearby map (Theorem 6.1). Positive defects are produced by the quadratic part of high-frequency normal waves in good phases, and the oscillatory errors are removed by finite differential formulas; one smooth iteration converges in every norm, including transverse derivatives. A quantitative form allows phase margins and reference bounds that degenerate polynomially in a small parameter (Theorem 6.3).

**Step 4: Selectors and embeddings.** A selector is an exactly isometric phase-regular map on $B \times (S^{2C-1})^s$, isometric along the Hopf directions with coefficients in a bundle $W \subset B \times \mathbb{R}^s$. Composing a selector with fast Hopf phases and correcting the small metric error by Step 3 gives an isometric immersion of every closed $(M, g)$ with $TM \cong$

$f^*W$ (Theorem 7.1). For $N \geq s_n + 2$ exactly isometric families of probes remove double points (Theorem 7.4).

**Step 5: Source bundles.** A generic six-frame in $TM$ has corank at most one, along a core of dimension at most five, where it determines a possibly nonorientable line (Proposition 8.3). Two source transports identify the complete boundary classes of the exterior and the core (Proposition 8.2).

**Step 6: The seven-label tensor.** For symmetric operators on $E$ the tensor $T : E \otimes \mathbb{R}^7 \to W$ of §9 has a flat alternating Gram, by the partial-transposition invariance of the Tiles projector. Symmetric evaluation bounds the cokernels in the incidence counts (§10). The exterior count, with old rank $M \geq 9$ and reserve $C_0 = 10$, and the core count, with old rank $r \geq 12$ and reserve $c = 5$, give parameters with $\dim_{\mathbb{C}} \mathcal{B}_T \leq 4$ and the regularity used later (Theorem 11.6, Proposition 12.1).

**Step 7: Active block, collar and frames.** The active free block shares its unit cofactor normal with one model coordinate (Proposition 13.4). The activity is selected before the tensor is perturbed, a normalized collar scaling keeps the boundary data, and the actual normal frame is made horizontal by an $h$-principle for an infinitesimally invertible relation (§14).

**Step 8: The periodic mixed inverse.** With the tensor fixed, the mixed rows of the metric equation lead to a periodic operator on the active Hopf source. Its adjoint presents a $\mathscr{D}$-module whose characteristic variety is controlled by the bad-label set. Relative involutivity [Wu22, Theorem 2.1], the exclusion of Lagrangian components for a polynomial activity, and one killing point on every orbit show that the module vanishes, which gives a differential right inverse depending smoothly on all parameters (Proposition 17.3).

**Step 9: Completion and the surviving line.** The selector is completed to any finite precision by the periodic inverse and the free pairing inverses (Proposition 18.1) and made exact by Theorem 6.3. Over the core, the surviving line is absorbed by a twisted normal construction in the same target (Theorem 19.4).

**Logical dependence.** Steps 1 and 2 are independent of each other and of Step 3. Step 3 is used in Steps 4, 7 and 9, Step 1 in Step 9 and in the cases $n \leq 15$, and Step 2 in Step 7. Steps 6 and 8 use no analytic input from Steps 1–4.

## 3. Notation and conventions

All manifolds, bundles, metrics, and maps are smooth unless another regularity is specified. A closed manifold is compact and has no boundary. We write

$$s_d = \frac{d(d+1)}{2}, \qquad j_d = d + s_d, \qquad J(E) = E \oplus \operatorname{Sym}^2 E.$$

The trivial real bundle of rank $a$ is denoted by $\varepsilon^a$. Inner products identify duals when no ambiguity can result. Symmetric powers carry their induced inner products; changing their normalization changes only positive constants in singular value estimates.

Let $E \subset TX$ be an integrable distribution. A torsion-free connection along its leaves defines the *complete column map* of a map $F : X \to \mathbb{R}^N$ by

$$J_E F(v, S) = d_E F(v) + \mathrm{Hess}_E\, F(S).$$

The map is *$E$-free* if this bundle map is injective. Changing the leaf connection adds first columns to second columns; the resulting triangular operation preserves injectivity and gives a homotopy through injections. The *complete free class* is the homotopy class of the injection $J_E F$, including its first columns. A prescribed formal free class is a given bundle monomorphism $J(E) \to \varepsilon^N$. Relative classes agree with the actual complete jet on a neighborhood of the prescribed closed set.

For an integrable rank-$n$ distribution $D$, $n \geq 2$, put

$$\gamma_D(F) = d_D F^* d_D F, \qquad B_F = \Pi_{d_D F(D)^\perp}\, \mathrm{Hess}_D\, F.$$

The second expression is independent of the leaf connection. A $D$-immersion is *phase regular* if at every point there is a nonzero covector $\xi \in D^*$ such that $B_F|_{\mathrm{Sym}^2 \ker \xi}$ is injective. Such a covector is called *good*. With fixed auxiliary tangent norms, the *best-phase margin* is

$$\mu(F, x) = \max_{|\xi|=1} \sigma_{\min}\Big(B_F(x)|_{\mathrm{Sym}^2 \ker \xi}\Big).$$

It is a continuous positive function for a phase-regular map. No continuous choice of a maximizing covector is required. Norms of a partial map include all source derivatives, including slow parameters and directions transverse to the leaf distribution, unless a fiber norm is indicated.

**Theorem 3.1** (Parametric relative open ample theorem)**.** *Let a first-order differential relation for sections of a smooth fibration be open and ample in its principal affine slices. Then a formal solution is homotopic through formal solutions to a holonomic solution. The statement holds for compact parameter families and relative to a neighborhood where the family is already holonomic.*

We use this theorem in the form of [EM02, Theorem 18.4.1]. Each application below checks the principal affine slices. The corresponding higher-order ample theorem is used only in the free-realization argument, with the slice check in Lemma 4.2; see [Spr98, Chapter VIII, Theorem 8.12].

The foundational results are proved in the next four sections. Theorem 4.1 realizes a complete formal free class with its prescribed leaf metric and actual relative data. Theorem 5.1 constructs exact free Hopf blocks in $\mathbb{R}^{j_d+1}$. Theorems 6.1 and 6.3 give exact metric correction, including all parameter derivatives and polynomially degenerating phase margins. Theorems 7.1 and 7.4 pass from a bundle selector to an isometric immersion and then to an embedding. Their dimension bounds are checked at each use.

A *bundle selector* is defined as follows. For a Euclidean subbundle $W \subset B \times \mathbb{R}^s$, let $Y_j$ be the period-one Hopf generators on $(S^{2h-1})^s$ and let

$$Y_w = \sum_{j=1}^{s} w_j Y_j \qquad (w \in W_b).$$

These vectors define an integrable distribution on $B \times (S^{2h-1})^s$, since their coefficients are constant along each Hopf product. A selector is an actual map whose Gram matrix on this distribution is the given metric of $W$ and whose best-phase margin is positive. The coefficient bundle, its inclusion, and the number of sphere factors are fixed before the sphere dimension is chosen.

## 4. Relative realization of free leaf metrics

Let $D \subset TX$ be an integrable distribution of rank $d$ and put $j_d = d + d(d+1)/2$. A torsion-free connection adapted to $D$ identifies the complete leaf derivative of $f : X \to \mathbb{R}^N$ with

$$J_D f : D \oplus \operatorname{Sym}^2 D \longrightarrow \varepsilon^N, \qquad J_D f(v, S) = d_D f(v) + \operatorname{Hess}_D f(S).$$

We call $f$ *$D$-free* when this map is injective. Changing the adapted connection adds first columns to second columns and therefore preserves freeness and the complete formal class. Write $\gamma_D(f) = (d_D f)^* d_D f$.

In this section we show that a free map with prescribed leaf metric exists in every complete formal free class, relative to prescribed data and in families. The proof combines the higher-order open ample theorem [Spr98, Chapter VIII, Theorem 8.12 and its relative extension construction] with a parametrized form of Günther's local metric modification; the threshold $N \geq j_d + 5$ is that of [Gün89a, Satz 3.3]. Subsection 4.2 proves the local modification, with its dependence on transverse variables, and the last subsection proves that the complete class is preserved.

**Theorem 4.1** (Relative free realization)**.** *Let $\pi : X \to B$ be a smooth submersion of compact smooth manifolds, with $\partial X = \pi^{-1}(\partial B)$ when a boundary is present, and let $D \subset \ker d\pi$ be integrable of rank $d \geq 1$. Suppose $N \geq j_d + 5$, and let $h$ be a positive smooth metric on $D$. Let $A \subset B$ be closed and contain $\partial B$. On $\pi^{-1}(U)$, for a neighborhood $U$ of $A$, prescribe a smooth $D$-free map $f_A$ with $\gamma_D(f_A) = h$.*

*Suppose that a bundle monomorphism*

$$J : D \oplus \operatorname{Sym}^2 D \longrightarrow \varepsilon^N$$

*agrees with $J_D f_A$ near $\pi^{-1}(A)$. Then there is a smooth $D$-free map $g : X \to \mathbb{R}^N$ such that*

$$\gamma_D(g) = h, \qquad g = f_A \quad \text{near } \pi^{-1}(A),$$

*and $J_D g$ is homotopic to $J$ through bundle monomorphisms, relative to a neighborhood of $\pi^{-1}(A)$.*

*If $J = J_D f$ for an actual global $D$-free map $f$ extending the prescribed data, $f$ and $g$ can be joined by an actual smooth path of $D$-free maps with the same relative condition. These conclusions hold for compact smooth families without increasing $N$; prescribed family data are retained wherever they are already free and induce the prescribed metric.*

4.1. **Realizing the complete formal free class.**

**Lemma 4.2** (Holonomic free columns)**.** *Let $D \subset TX$ be integrable of rank $d$, let $X$ be compact, and suppose $N \geq j_d + 1$. A monomorphism $J : D \oplus \operatorname{Sym}^2 D \to \varepsilon^N$ that agrees with the complete derivative of a smooth map $f$ near a closed set $K$ is homotopic, relative to a neighborhood of $K$, to $J_D F$ for a smooth $D$-free map $F$. One may require $F = f$ near $K$ and arbitrary $C^0$ approximation to $f$. If the first component of $J$ is $d_D f$ on all of $X$, one may require $C^1$ approximation. When $X$ has boundary, $K$ contains a boundary collar.*

*Proof.* Choose an adapted torsion-free connection and a complement $TX = D \oplus C$. Such a connection is obtained by combining the coordinate connections of foliation charts with a partition of unity: torsion-freeness and preservation of $D$ in its own directions are affine conditions. Write $J = (j_1, j_2)$ and extend its first component to a formal full derivative by

$$L(v + c) = j_1(v) + df(c), \qquad v \in D,\ c \in C.$$

Its value component is $f$. A formal covariant Hessian is freely specifiable as a symmetric tensor on $TX$; extend $j_2$ by the actual Hessian of $f$ on summands with an argument in $C$. These are full formal jets with the prescribed actual jets near $K$.

Consider the open first-order relation of derivatives injective on $D$. In a principal affine slice the derivative varies by $\zeta \otimes u$, $u \in \mathbb{R}^N$. If $\zeta|_D = 0$, the slice is either empty or unrestricted. Otherwise choose a basis $v_1, \ldots, v_d$ of $D$ with $\zeta(v_i) = 0$ for $i < d$ and $\zeta(v_d) = 1$. Only the last first column varies. In a nonempty slice its forbidden values form an affine subspace of codimension $N - d + 1 \geq 2$. Its complement is connected and its convex hull is the whole affine space. The relation is therefore ample in every full-source principal direction.

The relative first-order open ample theorem [EM02, Theorem 18.4.1] gives a formal first-jet homotopy $(f_t, L_t)$ to the derivative of a map $f_1$, preserving injectivity on $D$ and the prescribed neighborhood. Relative smoothing makes this homotopy smooth and stationary near its endpoints; openness preserves its rank conditions. Put $j_{1,t} = L_t|_D$, let $j_{1,t}^+$ be its smooth metric left inverse, and solve

$$\dot{T}_t = (\dot{j}_{1,t}) j_{1,t}^+ T_t, \qquad T_0 = \mathrm{Id}\,.$$

The solution is invertible and satisfies $T_t j_1 = j_{1,t}$ by uniqueness. It is the identity on the protected neighborhood. Consequently $(j_{1,t}, T_t j_2)$ remains a monomorphism throughout. At the endpoint its first component is $d_D f_1$.

Now consider the open second-order relation of complete free leaf columns. Its principal variation fixes the full first jet and adds $\zeta^2 \otimes u$ to the covariant Hessian. If $\zeta|_D = 0$, the slice is again empty or unrestricted. Otherwise, in the basis just used, only the $(v_d, v_d)$ Hessian column changes. A nonempty slice fixes $j_d - 1$ independent other columns; its forbidden affine subspace has codimension $N - j_d + 1 \geq 2$. Thus this relation is ample in every principal direction, including directions with transverse components.

We use the higher-order theorem only in its $C^1$-dense form with holonomic lower jet [Spr98, Chapter VIII, Theorem 8.12 and its relative extension construction]: if

$\mathcal{R} \subset J^2(X, \mathbb{R}^N)$ is open and ample in every principal direction, then every section $\alpha$ of $\mathcal{R}$ whose 1-jet component is holonomic, $\alpha = (j^1 f_1, \mathsf{H})$, is homotopic through sections of $\mathcal{R}$, relative to a neighborhood where it is already holonomic, to a holonomic section $j^2 F$, with $j^1 F$ arbitrarily $C^0$-close to $j^1 f_1$. The holonomy of the lower jet is needed: for formal sections without it the second-order statement fails in general. The relation $\partial_\theta f > 0$ in $J^2(S^1, \mathbb{R})$ has unrestricted top-order slices and formal sections, but no solution. The first-order part is the relation already treated. Since $N \geq j_d$, an injective $L|_D$ extends to injective complete columns by choosing $\mathsf{H}|_{\mathrm{Sym}^2 D}$ with values in a complement of $L(D)$, which has dimension $N - d \geq j_d - d = \dim \mathrm{Sym}^2 D$. Hence the projection of the complete-column relation to $J^1$ is the partial-immersion relation $\{L : L|_D \text{ injective}\}$. Its slices were checked above, and it has the actual solution $j^1 f_1$. The principal subspaces are intrinsic: the fiber of $J^2 \to J^1$ is affine over $\mathrm{Hom}(\mathrm{Sym}^2 TX, \mathbb{R}^N)$, and $\mathsf{H} \mapsto \mathsf{H} + \zeta^2 \otimes u$ does not depend on the adapted connection. Hence the slice computation above gives the ampleness in coordinate directions required in [Spr98, Chapter VIII].

Apply this to $\alpha = (j^1 f_1, \mathsf{H})$, where $\mathsf{H} = T_1 j_2$ on $\mathrm{Sym}^2 D$ and $\mathsf{H}$ is the actual Hessian of $f_1$ on pairs with an argument in $C$. The section lies in the relation because $(d_D f_1, T_1 j_2)$ is a monomorphism. Near $K$ one has $f_1 = f$, $T_1 = \mathrm{Id}$ and $j_2$ equal to the leaf Hessian of $f$, so $\alpha = j^2 f$ there. The theorem gives a homotopy $\alpha_t$ of formal free second jets, relative to the protected neighborhood, ending at $j^2 F$, with $F$ arbitrarily close to $f_1$ in $C^1$, hence to $f$ in $C^0$. The complete columns of $\alpha_t$ give a path of monomorphisms from $(d_D f_1, T_1 j_2)$ to $J_D F$, and concatenation with $(j_{1,t}, T_t j_2)$ proves the homotopy assertion. A relative $C^2$ smoothing keeps the open freeness condition and its complete class. If $j_1 = d_D f$ initially, omit the first-order step and apply this argument directly with lower jet $j^1 f$. Then $F$ is $C^1$-close to $f$. On a manifold with boundary, work on the interior while fixing the boundary collar. The principal directions above are directions of the full source, including the parameters of a compact family; for parameter directions $\zeta|_D = 0$ and the slice is unrestricted. The family and transverse regularity statements therefore follow by applying the theorem on the total space. □

4.2. **A supported metric primitive with parameters.** In this subsection $B \subset \mathbb{R}^d$ is an open ball, $\mathsf{P}$ is a compact convex parameter box, and all maps are smooth on a neighborhood of $\mathsf{P} \times \overline{B}$. Derivatives $\partial_i$ are taken in $x \in B$. For a map $F(p, x)$ let $M_F$ be the $N$ by $j_d$ matrix of the columns $\partial_i F$ $(1 \leq i \leq d)$ and $\partial_i \partial_j F$ $(1 \leq i \leq j \leq d)$, and write

$$\lambda_F(V) = M_F^t V = \Big( \langle \partial_i F, V \rangle ; \langle \partial_i \partial_j F, V \rangle \Big) \in \mathbb{R}^{j_d}$$

for the pairings of $V \in \mathbb{R}^N$. If $F$ is free in $x$, its algebraic pairing inverse is

$$\mathcal{E}_F = M_F (M_F^t M_F)^{-1}, \qquad \lambda_F \circ \mathcal{E}_F = \mathrm{Id}\,.$$

We write $\mathcal{E}_F(h, f)$ when the first and second pairings are prescribed separately. Fix $0 < \alpha < 1$, and let $\| \cdot \|_r$ be the $C^{r,\alpha}(\overline{B})$ norm, applied entrywise to vectors and matrices. On the convex set $\overline{B}$ one has $\| \cdot \|_2 \leq C \| \cdot \|_{C^3}$.

**Lemma 4.3** (Supported quadratic correction)**.** *Let $\chi, \zeta \in C_c^\infty(B)$ with $\zeta = 1$ near $\operatorname{supp}\chi$. There is $K_0 > 0$, depending only on $B, \chi, \zeta, \alpha, d, N$, with the following property. Let $P(p,x)$ be free in $x$, and let $e(p,x)$ be a smooth family of symmetric $d$ by $d$ matrices whose $x$-support lies in the interior of $\{\chi = 1\}$. Put $\eta(p) = \|\mathcal{E}_{P(p)}\|_2$ and $\varrho(p) = \|\mathcal{E}_{P(p)}(0, e(p))\|_2$, and suppose*

$$K_0\,\eta(p)\,\varrho(p) \le \tfrac{1}{4} \qquad (p \in \mathsf{P}). \tag{4.1}$$

*Then there is a jointly smooth $v$ with $\|v(p)\|_2 \le \varrho(p)$, with $v(p,\cdot) = 0$ whenever $e(p,\cdot) = 0$, and with*

$$\gamma_x(P + \chi v) = \gamma_x(P) + e.$$

*Proof.* Let $T = \mathbb{R}^d/\ell\mathbb{Z}^d$ with $\ell$ so large that $\overline{B}$ lies in one fundamental domain; functions supported in $B$ are regarded as functions on $T$. Put $L = \Delta - 1$ and $W = L^{-1}$. The periodic Schauder estimate gives bounded maps $W : C^{r,\alpha}(T) \to C^{r+2,\alpha}(T)$ for all $r \ge 0$. The operator $W$ is a Fourier multiplier; it commutes with $x$-derivatives, and on smooth families it commutes with parameter derivatives. For $z, z' \in C^{2,\alpha}(T, \mathbb{R}^N)$ define symmetric bilinear expressions

$$\begin{aligned} n_i(z,z') &= -\tfrac{1}{2} W\Big[\langle \partial_i z, \Delta z'\rangle + \langle \partial_i z', \Delta z\rangle\Big], \\ m_{ij}(z,z') &= \tfrac{1}{2} W\Big[\sum_k \Big(\langle \partial_i\partial_k z, \partial_j\partial_k z'\rangle + \langle \partial_i\partial_k z', \partial_j\partial_k z\rangle\Big) - \langle \Delta z, \partial_i\partial_j z'\rangle - \langle \Delta z', \partial_i\partial_j z\rangle \\ &\qquad - \tfrac{1}{2}\Big(\langle \partial_i z, \partial_j z'\rangle + \langle \partial_i z', \partial_j z\rangle\Big)\Big]. \end{aligned}$$

Their $C^{2,\alpha}(T)$ norms are at most $C\|z\|\,\|z'\|$, with the $C^{2,\alpha}(T)$ norms on the right. Write $n_i(z) = n_i(z,z)$ and $m_{ij}(z) = m_{ij}(z,z)$. The Leibniz rule gives

$$\Delta\langle \partial_i z, \partial_j z\rangle = \partial_i\langle \partial_j z, \Delta z\rangle + \partial_j\langle \partial_i z, \Delta z\rangle - 2\langle \Delta z, \partial_i\partial_j z\rangle + 2\sum_k \langle \partial_i\partial_k z, \partial_j\partial_k z\rangle.$$

Hence $L$ maps $\partial_i n_j(z) + \partial_j n_i(z) - 2m_{ij}(z)$ and $-\langle \partial_i z, \partial_j z\rangle$ to the same distribution. Since $L$ is injective on distributions on $T$,

$$\partial_i n_j(z) + \partial_j n_i(z) - 2m_{ij}(z) = -\langle \partial_i z, \partial_j z\rangle. \tag{4.2}$$

For $v \in C^{2,\alpha}(\overline{B}, \mathbb{R}^N)$ put $z = \zeta v$, extended by zero, and

$$\begin{aligned} A_i(v) &= \chi\, n_i(z) - \tfrac{1}{2}\partial_i\chi\,|v|^2, \\ B_{ij}(v) &= \chi\, m_{ij}(z) + \partial_i\chi\, n_j(z) + \partial_j\chi\, n_i(z) - \tfrac{1}{2}\partial_i\partial_j\chi\,|v|^2, \\ \Phi_p(v) &= \mathcal{E}_{P(p)}\Big(A(v),\, -\tfrac{1}{2}e(p) + B(v)\Big). \end{aligned}$$

Suppose $v = \Phi_p(v)$ and put $U = \chi v$. Then $\langle \partial_i P, v\rangle = A_i$ and $\langle \partial_i\partial_j P, v\rangle = -\frac{1}{2}e_{ij} + B_{ij}$. Since $\langle \partial_i P, \partial_j U\rangle = \partial_j\langle \partial_i P, U\rangle - \langle \partial_i\partial_j P, U\rangle$,

$$\gamma(P+U)_{ij} - \gamma(P)_{ij} = \partial_j\langle \partial_i P, U\rangle + \partial_i\langle \partial_j P, U\rangle - 2\langle \partial_i\partial_j P, U\rangle + \langle \partial_i U, \partial_j U\rangle.$$

Here $\langle \partial_i P, U\rangle = \chi^2 n_i - \frac{1}{4}\partial_i(\chi^2)|v|^2$ and

$$\langle \partial_i U, \partial_j U\rangle = \chi^2\langle \partial_i v, \partial_j v\rangle + \tfrac{1}{4}\Big(\partial_i(\chi^2)\partial_j|v|^2 + \partial_j(\chi^2)\partial_i|v|^2\Big) + \partial_i\chi\,\partial_j\chi\,|v|^2.$$

The first derivatives of $|v|^2$ cancel, and

$$\gamma(P+U)_{ij}-\gamma(P)_{ij} = \chi e_{ij}+\chi\Big[\chi\Big(\partial_i n_j+\partial_j n_i+\langle\partial_i v,\partial_j v\rangle\Big)+2\partial_i\chi\, n_j+2\partial_j\chi\, n_i-\partial_i\partial_j\chi\,|v|^2-2B_{ij}\Big].$$

On $\operatorname{supp}\chi$ one has $z=v$ near each point, so (4.2) replaces the first parenthesis by $2m_{ij}$. The bracket is then zero by the definition of $B_{ij}$. Finally $\chi e = e$.

Write $\Phi_p(v) = -\frac12\mathcal{E}_{P(p)}(0,e(p))+\mathfrak{B}_p(v,v)$ with $\mathfrak{B}_p$ symmetric bilinear. The Schauder bound and the algebra property of $C^{2,\alpha}$ give $\|\mathfrak{B}_p(v,w)\|_2 \le K_0\eta(p)\|v\|_2\|w\|_2$, with $K_0$ as stated. By (4.1), $\Phi_p$ maps the closed ball of radius $\varrho(p)$ into itself and has Lipschitz constant at most $2K_0\eta\varrho \le \frac12$ there. Put $v_0=0$ and $v_{k+1}=\Phi_p(v_k)$. The iterates converge in $C^{2,\alpha}$, uniformly in $p$, to a fixed point $v(p)$ with $\|v(p)\|_2\le\varrho(p)$. If $e(p,\cdot)=0$, every iterate vanishes.

Each $v_k$ is jointly smooth on $\mathsf{P}\times\overline{B}$. Let $\partial^\gamma$ be a derivative in $(p,x)$. Since $W$ commutes with $\partial^\gamma$,

$$\partial^\gamma v_{k+1} = 2\mathfrak{B}_p(v_k,\partial^\gamma v_k)+\mathfrak{r}_{\gamma,k},$$

where $\mathfrak{r}_{\gamma,k}$ is a finite sum of derivatives of $\mathcal{E}_P(0,e)$ and of bilinear expressions of the same type in $\partial^{\gamma_1}v_k,\partial^{\gamma_2}v_k$ with $|\gamma_1|,|\gamma_2|<|\gamma|$; their coefficients are derivatives of $\mathcal{E}_P,\chi,\zeta$. Since $2K_0\eta\|v_k\|_2\le\frac12$, induction on $|\gamma|$ bounds $\|\partial^\gamma v_k(p)\|_2$ uniformly in $k$ and $p$. Thus for every $r$ the sequence is bounded in $C^{r+1}(\mathsf{P}\times\overline{B})$. It converges uniformly, so by the Arzelà–Ascoli theorem its limit $v$ lies in $C^r$ for every $r$. □

This is Günther's quadratic perturbation identity. In [Gün89a, §5] it is solved with the Dirichlet inverse of the ball and a correction $a^2v$ (Lemma 5.3 and Satz 5.5 there, with the same smallness form [Gün89a, (5.14)–(5.16)]). The periodic resolvent and the first-power cutoff used above give the same conclusion.

Let $F$ be free in $x$. Let $\Pi_U$ be the orthogonal projection onto

$$U_F = \operatorname{span}\{\partial_i F\ (1\le i\le d),\ \partial_k\partial_l F\ (2\le k\le l\le d)\},$$

the span of the columns not containing $\partial_1$ twice or in a first column; put $\Pi_H=\mathrm{Id}-\Pi_U$ and $H=U_F^{\perp}$. Let $L_F$ be the orthogonal complement of all free columns. Put

$$e_j = \Pi_H\partial_1\partial_j F \qquad (1\le j\le d), \qquad E=\operatorname{span}\{e_1,\dots,e_d\}.$$

A relation $\sum c_j e_j = 0$ would put $\sum c_j\partial_1\partial_j F$ in $U_F$; by freeness the $e_j$ are linearly independent. Since each $e_j$ lies in the column span, $H = E\oplus L_F$ orthogonally. The projection defining $e_j$ omits the columns $\partial_1\partial_j F$; projecting away all free columns would make every $e_j$ vanish. This is the subspace $\mathfrak{F}_x$ of [Gün89a, proof of Satz 3.3].

**Lemma 4.4** (Oscillating normal pair)**.** *Suppose $N\ge j_d+5$. There are smooth orthonormal fields $u,v$ on $\mathsf{P}\times\overline{B}$ with values in $H$ such that, with*

$$c_t = \cos t\,u+\sin t\,v, \qquad \tau_t=\partial_t c_t = -\sin t\,u+\cos t\,v,$$

*the following hold at every point.*

(i) *The free columns of $F$, together with $u$ and $v$, are linearly independent.*

(ii) *For every $t\in\mathbb{R}$ and $s\ge0$, the $d+1$ vectors $e_j+s\,\Pi_H\partial_j\tau_t$ $(2\le j\le d)$, $u$, $v$ are linearly independent.*

*Proof.* The bundle $L_F$ has rank at least five and is the image of the smooth projection $\mathrm{Id} - M_F(M_F^t M_F)^{-1} M_F^t$. Choose an orthonormal 5-frame of $L_F$ at one point of the convex set $\mathsf{P} \times \overline{B}$ and transport it along segments from that point by the connection given by this projection. This gives smooth orthonormal sections $f_1, \dots, f_5$ of $L_F$. If $d = 1$ take $u = f_4$, $v = f_5$; (1) holds because $L_F$ is orthogonal to the columns, and (2) is vacuous.

Let $d \geq 2$, $m = d - 1$, and write $x' = (x_2, \dots, x_d)$. We first construct a $2\pi$-periodic immersion $g = (g_1, \dots, g_{m+1}) : \mathbb{R}^m \to \mathbb{R}^{m+1}$ with

$$\partial_{y_i} g_l = 0 \qquad (l < i). \tag{4.3}$$

Put $g^{(1)}(y_1) = (\cos y_1, 2 + \sin y_1)$. If $g^{(k-1)} = (g_1, \dots, g_k)$ is a periodic immersion in $y_1, \dots, y_{k-1}$ with $g_k \geq 1$, put

$$g^{(k)} = \Big(g_1, \dots, g_{k-1},\ g_k \cos y_k,\ A_k + g_k \sin y_k\Big), \qquad A_k = 1 + \max g_k.$$

The last coordinate is at least one. If $\sum_{i<k} \beta_i \partial_i g^{(k)} + \beta_k \partial_k g^{(k)} = 0$, the last two coordinates give $(\sum_{i<k} \beta_i \partial_i g_k)(\cos y_k, \sin y_k) + \beta_k g_k(-\sin y_k, \cos y_k) = 0$; hence $\beta_k = 0$ and then all $\beta_i = 0$. Thus $g^{(k)}$ is an immersion. Its coordinate of index $l \leq k$ depends only on $y_1, \dots, y_l$, which is (4.3). Take $g = g^{(m)}$. This is the triangularity condition [Gün89a, (3.2)].

Let $b_i = e_{i+1}$ for $1 \leq i \leq m$, and $b_{m+i} = f_i$ for $i = 1, 2, 3$. For $0 < \delta \leq 1$ put

$$u_0 = f_4 + \delta \sum_{l=1}^{m+1} g_l(x'/\delta^2)\, b_{l+1}, \qquad v_0 = f_5 + \delta \sum_{l=1}^{m+1} g_l(x'/\delta^2)\, b_{l+2},$$

and let $u, v$ be the Gram–Schmidt orthonormalization of $u_0, v_0$. All vectors involved lie in $H$, and $f_4, f_5$ are orthogonal to every $b_i$. Hence $|u_0|^2 = 1 + O(\delta^2)$ and $\langle u_0, v_0 \rangle = O(\delta^2)$, while the $x$-derivatives of these two functions are $O(1)$. Consequently $u = f_4 + O(\delta)$, $v = f_5 + O(\delta)$, and for $2 \leq j \leq d$,

$$\Pi_H \partial_j \tau_t = \delta^{-1} \sum_{l=1}^{m+1} \partial_{y_{j-1}} g_l(x'/\delta^2) \left(-\sin t\, b_{l+1} + \cos t\, b_{l+2}\right) + R_j. \tag{4.4}$$

Here, and below, $O(\cdot)$ and the bound $|R_j| \leq A$ are uniform in $(p, x, t)$ and $\delta \in (0, 1]$. Statement (1) follows for small $\delta$, since the columns together with $f_4, f_5$ are uniformly independent.

For (2), work with coefficients in the basis $b_1, \dots, b_{m+3}, f_4, f_5$; its Gram matrix and inverse are uniformly bounded. For $\nu \in [0, 1]$, $t \in \mathbb{R}$ and $y \in \mathbb{R}^m$ define $\Lambda = \Lambda_{\nu,t,y} : \mathbb{R}^m \times \mathbb{R}^2 \to \mathbb{R}^{m+3} \times \mathbb{R}^2$ by

$$\Lambda(c, w) = \Big((1 - \nu) \sum_i c_i \varepsilon_i + \nu\, S_t\, Dg(y) c,\ w\Big), \qquad S_t \varepsilon_l = -\sin t\, \varepsilon_{l+1} + \cos t\, \varepsilon_{l+2},$$

where $\varepsilon_l$ is the standard basis and $S_t$ maps $\mathbb{R}^{m+1}$ to $\mathbb{R}^{m+3}$. By (4.3), the $i$th column of $S_t Dg$ has nonzero entries only in rows $k \geq i + 1$. Hence for $\nu < 1$ the first component of $\Lambda$ is lower triangular in $c$ with diagonal $1 - \nu$, and $\Lambda$ is injective. For $\nu = 1$, $Dg(y)$ is injective, and so is $S_t$: for $\xi \in \mathbb{R}^{m+1}$ with first nonzero coordinate $\xi_l$, if $\sin t \neq 0$

row $l+1$ of $S_t\xi$ is $-\sin t\,\xi_l \neq 0$, and if $\sin t = 0$ the map is a shift. Compactness of $[0,1]\times(\mathbb{R}/2\pi)\times(\mathbb{R}/2\pi)^m$ gives $\kappa>0$ with $|\Lambda(c,w)|\geq\kappa(|c|+|w|)$.

Let $s\geq 0$, $\mu = s/\delta$ and $\nu = \mu/(1+\mu)$. By (4.4),

$$\sum_{j=2}^{d} c_j\Big(e_j + s\Pi_H\partial_j\tau_t\Big) + w_1u + w_2v = X + \sum_j c_j sR_j + w_1(u-f_4) + w_2(v-f_5),$$

where the coefficient vector of $X$ is $(1+\mu)\Lambda_{\nu,t,y}\Big(c, w/(1+\mu)\Big)$ with $y = x'/\delta^2$. Therefore $|X|\geq\kappa'((1+\mu)|c|+|w|)$, while the remaining terms have norm at most $\mu\delta A|c|+C\delta|w|$. For small $\delta$ the sum is nonzero unless $c=0$ and $w=0$. Fix such a $\delta$. □

Lemma 4.4 is the variant, for the directions $x_2,\dots,x_d$ and $s\geq 0$, of [Gün89a, Lemma 3.1].

Fix $u,v$ as in Lemma 4.4. Let $\Pi_{uv}$ be the orthogonal projection onto $\{u,v\}^\perp$. By Lemma 4.4(1), $M_F^t\Pi_{uv}M_F$ is invertible, and

$$\mathcal{R} = \Pi_{uv}M_F(M_F^t\Pi_{uv}M_F)^{-1}$$

is a smooth right inverse of $\lambda_F$ with values orthogonal to $u$ and $v$. We use two-scale functions $h(p,x,t)$, $2\pi$-periodic in $t$, and put $h^\epsilon(p,x) = h(p,x,x_1/\epsilon)$. With $D_i^\epsilon = \partial_i + \epsilon^{-1}\delta_{i1}\partial_t$ one has $\partial_i(h^\epsilon) = (D_i^\epsilon h)^\epsilon$. Let $\langle h\rangle$ be the $t$-mean, and for $\langle h\rangle = 0$ let $\mathcal{I}h$ be the primitive of $h$ in $t$ with zero mean. Both operators are linear, act fiberwise in $(p,x)$, and commute with $(p,x)$-derivatives.

**Lemma 4.5** (Two-scale expansion)**.** *Let $a(p,x)$ be smooth with $x$-support in a fixed compact $K_a\subset B$, and let $K\geq 2$. There are smooth two-scale maps $V_1,\dots,V_K$ with the following properties.*

(i) *$V_1 = a^2c_t + Z_1$, where $Z_1$ is independent of $t$ and orthogonal to $u,v$; for $m\geq 2$, $V_m$ is orthogonal to $u,v$.*

(ii) *$V_m(p,x,t) = 0$ for $x\notin\operatorname{supp}a(p,\cdot)$, and $V_m(p,\cdot,\cdot) = 0$ whenever $a(p,\cdot)\equiv 0$.*

(iii) *$|\partial_tV_2| + |\partial_t^2V_2|\leq C|a|$.*

(iv) *There are smooth two-scale $\mathfrak{G}_m$, $K\leq m\leq 2K$, such that $P_\epsilon = F + \sum_{m=1}^K\epsilon^mV_m^\epsilon$ satisfies*

$$\gamma_x(P_\epsilon) = \gamma_x(F) + a^4dx_1^2 + \sum_{m=K}^{2K}\epsilon^m\mathfrak{G}_m^\epsilon.$$

*Proof.* For a two-scale $V$ write $\lambda_F(V) = (\alpha;\beta)$. Since $\langle\partial_iF, D_j^\epsilon V\rangle = D_j^\epsilon\langle\partial_iF,V\rangle - \langle\partial_i\partial_jF,V\rangle$,

$$\langle D_i^\epsilon(F+V), D_j^\epsilon(F+V)\rangle - \langle\partial_iF,\partial_jF\rangle = D_j^\epsilon\alpha_i + D_i^\epsilon\alpha_j - 2\beta_{ij} + \langle D_i^\epsilon V, D_j^\epsilon V\rangle. \tag{4.5}$$

Insert $V = \sum_{m=1}^K\epsilon^mV_m$, with $\lambda_F(V_m) = (\alpha_m;\beta_m)$, $V_0 = 0$, $V_{K+1} = 0$. Put $D_{i,a} = \partial_iV_a + \delta_{i1}\partial_tV_{a+1}$. The coefficient of $\epsilon^m$ in (4.5) is

$$\mathfrak{G}_{m,ij} = \partial_j\alpha_{i,m} + \partial_i\alpha_{j,m} + \delta_{j1}\partial_t\alpha_{i,m+1} + \delta_{i1}\partial_t\alpha_{j,m+1} - 2\beta_{ij,m} + \sum_{a+b=m}\langle D_{i,a}, D_{j,b}\rangle,$$

and these vanish for $m > 2K$. It suffices to arrange that $\mathfrak{G}_0$ has the single nonzero entry $\mathfrak{G}_{0,11} = a^4$ and that $\mathfrak{G}_m = 0$ for $1 \le m \le K-1$; then (4) holds with $\mathfrak{G}_m$ for $m \ge K$.

We construct $V_m$ and $\alpha_{m+1}$ recursively, starting from $\alpha_1 = 0$. Note $\lambda_F(c_t)$ has only the entries $\langle \partial_1\partial_j F, c_t\rangle$, which have zero mean. At step $m \ge 1$, $V_1, \dots, V_{m-1}$ and $\alpha_m$ are known. Put

$$\beta'_{ij,m} = \tfrac{1}{2}\Big(\partial_j\alpha_{i,m} + \partial_i\alpha_{j,m} + \sum_{a+b=m,\ a,b\ge 1} \langle \partial_i V_a, \partial_j V_b\rangle\Big) \qquad (2 \le i \le j \le d),$$

let $V'_m = \mathcal{R}(\alpha_m; \beta'_m) + \delta_{m1} a^2 c_t$, with the entries $\beta_{1j}$ of the argument set to zero, and put

$$S_{1j} = \partial_j\alpha_{1,m} + \partial_1\alpha_{j,m} + \sum_{a=0}^{m-1} \langle D_{1,a}, \partial_j V_{m-a}\rangle, \qquad S_{11} = 2\partial_1\alpha_{1,m} + \sum_{a+b=m} \langle D_{1,a}, D_{1,b}\rangle,$$

for $2 \le j \le d$. In $S$ the term $\partial_t V_{m+1}$ occurs only paired with $D_{1,0} = \partial_t V_1 = a^2\tau_t$, and this pairing vanishes for every map orthogonal to $u, v$; we omit it. Let $S'$ denote $S$ computed with $V'_m$ in place of $V_m$. Define the $t$-independent $\widehat{\beta}_{1j,m} = \frac{1}{2}\langle S'_{1j}\rangle$ $(1 \le j \le d)$, and

$$V_m = V'_m + \mathcal{R}(0; \widehat{\beta}_m),$$

where $\widehat{\beta}_m$ has only the entries $\widehat{\beta}_{1j,m}$. Then $S - S'$ equals $\langle a^2\tau_t, \partial_j \mathcal{R}(0; \widehat{\beta}_m)\rangle$, respectively twice that with $j = 1$. Its mean is zero, because $\tau_t$ has zero mean and the other factor is independent of $t$. Hence $\langle S_{1j}\rangle = 2\widehat{\beta}_{1j,m} = 2\langle \beta_{1j,m}\rangle$. Put

$$\alpha_{j,m+1} = \mathcal{I}\Big(2\beta_{1j,m} - S_{1j}\Big) \quad (j \ge 2), \qquad \alpha_{1,m+1} = \tfrac{1}{2}\mathcal{I}\Big(2\beta_{11,m} - S_{11}\Big).$$

By construction $\lambda_F(V_m) = (\alpha_m; \beta_m)$ with $\beta_{ij,m} = \beta'_{ij,m}$ for $i, j \ge 2$; also $\alpha_1 = 0$ and $Z_1 = \mathcal{R}(0; \widehat{\beta}_1)$ is independent of $t$.

Now $\alpha_1 = 0$ and $D_{i,0} = \delta_{i1} a^2\tau_t$, so the only nonzero entry of $\mathfrak{G}_0$ is $a^4|\tau_t|^2 = a^4$. For $1 \le m \le K-1$, the entries with $i, j \ge 2$ vanish by the definition of $\beta'_m$; the entries $(1, j)$ and $(1, 1)$ equal $S_{1j} + \partial_t\alpha_{j,m+1} - 2\beta_{1j,m}$ and $S_{11} + 2\partial_t\alpha_{1,m+1} - 2\beta_{11,m}$, which vanish by the definition of $\alpha_{m+1}$. This proves (4), and (1) holds by construction. Every quantity is obtained from $a$ by differentiation, multiplication by smooth functions of $(p, x, t)$, $\langle\cdot\rangle$, $\mathcal{I}$ and products, with no term independent of $a$; this gives (2). For (3), $\partial_t\alpha_2$ is $a^2$ times a smooth function, so is $\alpha_2$, and $\partial\alpha_2$ and $\partial V_1$ are $a$ times smooth functions. Thus $\beta'_2$ is $a$ times a smooth function, and $\partial_t^r V_2 = \mathcal{R}(\partial_t^r\alpha_2; \partial_t^r\beta'_2)$ for $r = 1, 2$. □

The construction involves no division by the amplitude: the increment is written as $a^4 dx_1^2$, and the right inverse $\mathcal{R}$ has image orthogonal to the oscillation plane. Günther's finite construction [Gün89a, Satz 3.3, Satz 4.1, (4.2)–(4.5)] instead uses pairing systems against the oscillating columns, which lead to quotients by $a^2$; these do not occur in Lemma 4.5.

**Lemma 4.6** (Finite primitive estimates)**.** *In Lemma 4.5, put* $e_\epsilon = \gamma_x(F) + a^4 dx_1^2 - \gamma_x(P_\epsilon)$. *There are* $\epsilon_0, c_0 > 0$ *and constants* $C_r$ *such that, for* $0 < \epsilon \le \epsilon_0$ *and all* $p$,

$$\det(M_{P_\epsilon}^t M_{P_\epsilon}) \ge c_0, \qquad \|\partial_x^\gamma M_{P_\epsilon}\|_{C^0} \le C_{|\gamma|}\epsilon^{-1-|\gamma|}, \tag{4.6}$$

$$\|P_\epsilon - F\|_{C^0} \le C_0\epsilon, \qquad \|e_\epsilon\|_{C^3} \le C_3\epsilon^{K-3}. \tag{4.7}$$

*Proof.* The last three bounds follow from $\|h^\epsilon\|_{C^r} \le C_r\epsilon^{-r}\|h\|_{C^r}$ and Lemma 4.5(4). Using $\partial_t V_1 = a^2\tau_t$, $\partial_t^2 V_1 = -a^2 c_t$ and the $t$-independence of $Z_1$, the columns of $P_\epsilon$ are

$$\partial_i P_\epsilon = \partial_i F + \delta_{i1}a^2\tau + O(\epsilon), \qquad \partial_1\partial_j P_\epsilon = \partial_1\partial_j F + \partial_j(a^2\tau) + O(\epsilon)\ (j \ge 2),$$
$$\partial_k\partial_l P_\epsilon = \partial_k\partial_l F + O(\epsilon)\ (k,l \ge 2), \qquad \partial_1^2 P_\epsilon = -\epsilon^{-1}a^2 c + g + O(\epsilon),$$

evaluated at $t = x_1/\epsilon$, where $g = \partial_1^2 F + 2\partial_1(a^2\tau) + \partial_t^2 V_2$. Let $\mathcal{C}$ be the list of all columns except $\partial_1^2 P_\epsilon$, and $\mathcal{C}_0$ the same list without the $O(\epsilon)$ terms.

We claim that $\mathcal{C}_0$ and $c$ are linearly independent at every $(p,x,t)$. Suppose a combination with coefficients $x_i$ for the first columns, $y_j$ for $\partial_1\partial_j F + \partial_j(a^2\tau)$, $z_{kl}$, and $w$ for $c$ vanishes. Apply $\Pi_H$; it kills $U_F$ and fixes $\tau, c \in H$. One obtains

$$\sum_{j\ge 2} y_j\big(e_j + a^2\Pi_H\partial_j\tau\big) + \big(x_1a^2 + \textstyle\sum_j y_j\partial_j(a^2)\big)\tau + wc = 0.$$

By Lemma 4.4(2) with $s = a^2$, $y = 0$, $w = 0$ and $x_1a^2 = 0$. The remaining relation lies in $U_F$, and freeness of $F$ gives $x = 0$, $z = 0$. The Gram determinants of $\mathcal{C}_0$ and of $\mathcal{C}_0 \cup \{c\}$ are continuous and positive on the compact set of $(p,x,t)$, hence bounded below. Similarly the free columns of $F$ and $c$ are uniformly independent by Lemma 4.4(1).

The Gram determinant of all columns equals $\det\mathrm{Gram}(\mathcal{C}) \cdot \mathrm{dist}(\partial_1^2 P_\epsilon, \mathrm{span}\,\mathcal{C})^2$, and the first factor is bounded below for small $\epsilon$. Where $a^2 \ge \epsilon^{1/2}$, the distance is at least $\epsilon^{-1/2}\,\mathrm{dist}(c, \mathrm{span}\,\mathcal{C}) - C \ge 1$. Where $a^2 \le \epsilon^{1/2}$, one has $|a| \le \epsilon^{1/4}$; by Lemma 4.5(3), $g = \partial_1^2 F + O(\epsilon^{1/4})$ and $\mathcal{C}$ is within $O(\epsilon^{1/4})$ of the corresponding columns of $F$. The distance is then at least $\mathrm{dist}(\partial_1^2 F, \mathrm{span}(\mathcal{C}\cup\{c\})) - C\epsilon^{1/4}$, which is bounded below. Both cases give $c_0$. □

Set $b = 8j_d + 2$. In Lemma 4.6, the Gram matrix $G$ of $P_\epsilon$ has $C^r$ norms $O(\epsilon^{-2-r})$, its adjugate $O(\epsilon^{-2(j_d-1)-r})$, and $\det G \ge c_0$ gives $\|1/\det G\|_{C^3} = O(\epsilon^{-6j_d-3})$. Hence

$$\|\mathcal{E}_{P_\epsilon}\|_2 \le C\epsilon^{-b}. \tag{4.8}$$

Since $\det G \ge c_0$ and the largest singular value of $M_{P_\epsilon}$ is $O(\epsilon^{-1})$, its least singular value is at least $c\,\epsilon^{j_d-1}$.

**Lemma 4.7** (Parametrized supported primitive)**.** *Let* $\mathsf{P}$ *be a compact convex parameter box and* $B_x$ *a coordinate ball. Let* $F(p,x)$ *be smooth on a neighborhood of* $\mathsf{P} \times \overline{B}_x$ *and free in the* $x$ *directions. Suppose* $N \ge j_d + 5$, *and let* $a(p,x)$ *be smooth with leaf support in a fixed compact subset of* $B_x$. *There is a jointly smooth map* $G$ *such that*

$$\gamma_x(G) = \gamma_x(F) + a^4 dx_1^2,$$

*with* $G$ *free in* $x$, *equal to* $F$ *outside a fixed larger compact leaf support, and arbitrarily close to* $F$ *in* $C^0$. *At a parameter where* $a(p,\cdot) \equiv 0$, *the construction gives* $G(p,\cdot) =$

*$F(p,\cdot)$. In particular it extends by $F$ across a parameter boundary collar on which $a$ vanishes.*

*For a fixed input $F$ it can give a smooth free path $G_s$, $0 \le s \le 1$, from $F$ to such a $G$, with metric $\gamma_x(F) + s^4 a^4 dx_1^2$ and the same support condition.*

*Proof.* Write $B = B_x$ and let $K_a$ contain all leaf supports of $a$. Choose $\chi, \zeta \in C_c^\infty(B)$ with $\chi = 1$ near $K_a$ and $\zeta = 1$ near $\operatorname{supp}\chi$, and let $K_0$ be the constant of Lemma 4.3. Fix $u, v$ by Lemma 4.4, take $K = 2b+4$ in Lemma 4.5, and let $P_\epsilon$, $e_\epsilon$ be as in Lemma 4.6. The $x$-support of $e_\epsilon$ lies in $K_a$. By (4.7) and (4.8), uniformly in $p$,

$$\varrho \le C\epsilon^{-b}\epsilon^{K-3} = C\epsilon^{b+1}, \qquad K_0\eta\varrho \le C\epsilon^{-2b}\epsilon^{K-3} = C\epsilon.$$

Choose $\epsilon$ so small that (4.1) holds, that $C\epsilon^{b+1}$ is smaller than the least singular value $c\epsilon^{j_d-1}$ of $M_{P_\epsilon}$ divided by the constant in $\|\chi v\|_{C^2} \le C\|v\|_2$, and that $C_0\epsilon$ is below a prescribed $C^0$ tolerance. Lemma 4.3 gives a jointly smooth $v$, and

$$G = P_\epsilon + \chi v$$

satisfies $\gamma_x(G) = \gamma_x(P_\epsilon) + e_\epsilon = \gamma_x(F) + a^4 dx_1^2$. It is free, because the change of its columns is smaller than the least singular value of $M_{P_\epsilon}$. It equals $F$ outside $\operatorname{supp}\chi$. If $a(p,\cdot) \equiv 0$ then $P_\epsilon(p,\cdot) = F(p,\cdot)$ and $e_\epsilon(p,\cdot) = 0$ by Lemma 4.5(2), so $v(p,\cdot) = 0$ and $G(p,\cdot) = F(p,\cdot)$. Hence $G$ extends by $F$ across the spatial boundary and across each zero-amplitude parameter collar.

For the path replace $a$ by $sa$ and include $s \in [0,1]$ in the parameter box. All constants above are uniform on the enlarged box. At $s = 0$ the output is $F$, which gives the actual free path. A smooth reparametrization makes it stationary at its endpoints when paths are concatenated. □

### 4.3. **Global metric completion and complete classes.**

**Lemma 4.8** (Finite positive decomposition)**.** *For a smooth positive metric $\Delta$ on $D$ over a compact source, there are finitely many foliation boxes, supported smooth amplitudes $a_\nu$, and noncritical linear leaf phases $\phi_\nu$ in those boxes such that*

$$\Delta = \sum_\nu a_\nu^4 (d_D\phi_\nu)^2.$$

*For a tensor $\beta^4\Delta$ with smooth $\beta$, the amplitudes can all be multiplied by $\beta$.*

*Proof.* At the center of a small foliation box choose a constant coframe $\theta_1, \dots, \theta_d$ with $\Delta_0 = \sum_i \theta_i^2$. For $d \ge 2$, choose $\tau > 0$ with $2\tau(d-1) < 1$. The tensors $\theta_i^2$ and $(\theta_i \pm \theta_j)^2$ span the symmetric tensors and express $\Delta_0$ with positive coefficients $1 - 2\tau(d-1)$ and $\tau$, respectively. A fixed right inverse of this linear spanning map gives smooth positive coefficients $c_{\ell j}$ for nearby $\Delta$. For $d = 1$ one coefficient suffices. Each constant covector is $d_D\phi_{\ell j}$ for a linear noncritical leaf phase.

Choose finitely many supported smooth bumps $\psi_\ell$ subordinate to these boxes, with $S = \sum_\ell \psi_\ell^4 > 0$. Then

$$a_{\ell j} = \psi_\ell S^{-1/4} c_{\ell j}^{1/4}$$

gives the identity. The coefficients are positive near each bump support, so zero extension is smooth; in particular no root of a vanishing nonnegative function is taken.

A finite refinement and a constant linear leaf coordinate change put each phase in the form $x_1$, with its support inside a contractible coordinate product. Multiplying the amplitudes by $\beta$ proves the last assertion. □

*Proof of Theorem 4.1.* Suppose first that $A$ is nonempty and not all of $B$. Shrink its prescribed neighborhood and choose nested neighborhoods $V_0, V_1, V_2, V_3$ of $A$, each closure contained in the next, such that $J$ equals the actual complete jet of $f_A$ near $\pi^{-1}(\overline{V}_3)$. A base cutoff extends $f_A$ to a global smooth map $f$ agreeing with it there. Lemma 4.2 gives a global free map $F$ in the specified complete class, with the same equality. If an actual global free input was supplied, take it as $F$ and omit this step.

Choose $M \geq 1$ with $\gamma_D(F) \leq Mh$, and a constant $\lambda_0 > 0$ with $\lambda_0^2 M < 1/2$. Set $\beta_0 = (1-\lambda_0^2)^{1/4}$. Choose a smooth $\beta : B \to [0, \beta_0]$ that is zero near $\overline{V}_1$ and is $\beta_0$ outside $V_2$. Put

$$\lambda = (1 - (\beta \circ \pi)^4)^{1/2}, \qquad F_0 = \lambda F.$$

Because $D \subset \ker d\pi$, all first and second leaf columns are multiplied by $\lambda > 0$. Thus $F_0$ is free, agrees with $f_A$ near the protected set, and is joined to $F$ by the actual free path $((1-t) + t\lambda)F$.

Where $\beta \circ \pi \neq 0$, define

$$\Delta = \frac{h - \lambda^2 \gamma_D(F)}{(\beta \circ \pi)^4},$$

and on its zero set put $\Delta = h$. This is a smooth positive metric. Wherever $\beta$ is not its constant outer value, one has $F = f_A$ and the quotient is identically $h$. Elsewhere the numerator is at least $h/2$. Consequently

$$h - \gamma_D(F_0) = (\beta \circ \pi)^4 \Delta. \tag{4.9}$$

Apply Lemma 4.8 using a foliation cover subordinate to the open cover

$$\pi^{-1}(V_1), \qquad X \setminus \pi^{-1}(\overline{V}_0).$$

The amplitudes in boxes of the first type are zero. Every nonzero primitive is supported away from a fixed neighborhood of $\pi^{-1}(A)$, and there are only finitely many primitives.

Apply Lemma 4.7 successively. Each input is the actual free output of the preceding step, and its metric increases by the corresponding fixed summand in (4.9). These summands add up to the deficit. The resulting $g$ is smooth in all variables, free, induces $h$, and equals $f_A$ on a common protected neighborhood. Concatenating the actual free paths of the primitive lemma, with stationary endpoint reparametrizations, preserves the complete class. Together with the scaling path and the formal homotopy from Lemma 4.2, this proves the assertion about $J_D g$. For an actual initial free map all these paths are paths of actual free maps.

If $A = \varnothing$, use the holonomic lemma with any value map, a sufficiently small positive constant scaling, and the same finite metric completion. If $A = B$, the prescribed map already solves the problem. Boundary collars are in the protected region, so all modifications are interior.

For a compact smooth family, include its parameter manifold in the slow base and in the full source. The distribution remains vertical and integrable with rank $d$. The

principal-slice counts in Lemma 4.2 do not involve source dimension, and Lemma 4.7 includes all additional variables in its compact parameter product. Thus the proof gives one jointly smooth family with the same target dimension and relative equality on prescribed parameter collars. Products with boundary have corners only in protected collars; one performs the interior construction while keeping those collars fixed. □

*Remark* 4.9. The relative condition requires the prescribed maps to induce $h$. The amplitude homotopies of Lemma 4.7 preserve the complete free class and are stationary on every protected parameter subfamily. They also give the asserted paths of actual free maps.

## 5. Exact free maps on products of Hopf spheres

In this section we construct exact free maps on products of Hopf spheres, by induction on the number of factors.

Write $s_d = d(d+1)/2$ and $j_d = d+s_d$. The Hopf action on $S^{2h-1} \subset \mathbb{C}^h$ is $R_t z = e^{2\pi i t} z$, with period-one generator $X$. For a function on a Hopf sphere, possibly with additional parameters, write

$$\overline{a}(z) = \int_0^1 a(R_t z)\, dt, \qquad I_X a(z) = \int_0^1 t\, a(R_t z)\, dt \quad \text{if } \overline{a} = 0.$$

Integration by parts gives $X I_X a = a$ and $\overline{I_X a} = 0$. These definitions require no choice of origin on an orbit. An orbit-invariant function is called *basic*.

**Theorem 5.1** (Exact free Hopf blocks)**.** *Let $d \geq 1$ and $h_1, \dots, h_d \geq 2$. On*

$$\mathcal{X}_d = \prod_{i=1}^{d} S^{2h_i-1}$$

*let $X_1, \dots, X_d$ be the commuting period-one Hopf generators. There is a smooth map*

$$H_d : \mathcal{X}_d \longrightarrow \mathbb{R}^{j_d+1}$$

*such that $\langle X_i H_d, X_j H_d \rangle = \delta_{ij}$ and all $j_d$ vectors*

$$X_i H_d, \qquad X_i X_j H_d \quad (1 \leq i \leq j \leq d)$$

*are independent at every point. Their orthogonal complement has a global smooth unit section. The map is defined on the full sphere product, so its restrictions to coordinate subsphere products are restrictions of a single map. The derivative bounds may depend on the fixed sphere dimensions.*

The proof uses the first-order open ample theorem [EM02, Theorem 18.4.1] for two auxiliary immersions. The exact metric step rests on the periodic quadratic identity underlying Günther's method [Gün89b, Gün91]. The section does not depend on the global isometric realization or metric-repair theorems.

### 5.1. **Auxiliary immersions and normalized plane curves.**

**Lemma 5.2** (Immersion along a framed distribution)**.** *If a smooth manifold has a framed rank-b distribution, it admits a smooth map to $\mathbb{R}^{b+1}$ whose differential is injective on that distribution.*

*Proof.* A formal derivative sends the given frame to the first $b$ coordinate vectors; extend it to the tangent bundle using any complementary distribution. The relation requiring this restricted derivative to be injective is open and ample. A principal affine derivative slice fixes the derivative on a hyperplane. If that hyperplane contains the distribution, a nonempty allowed slice is the whole slice. Otherwise it fixes a $(b-1)$-plane in the distribution. On a nonempty allowed slice the fixed images of that plane are independent, and the excluded free derivative is an affine $(b-1)$-plane in $\mathbb{R}^{b+1}$. Its complement is connected and its convex hull is the whole slice, since the excluded plane has codimension two. The open ample theorem for sections of a fibration therefore gives the required smooth map, also when the source is closed. □

**Lemma 5.3** (A reflection-symmetric planar selector)**.** *For $h \geq 2$ there is a smooth map $f : S^{2h-1} \to \mathbb{R}^2$, real analytic on each Hopf orbit, such that*

$$|Xf| = 1, \qquad f(-z) = \operatorname{diag}(1,-1)f(z), \qquad \overline{f_1 X f_2 - f_2 X f_1} = 0.$$

*Write $v = Xf = (s,t)$, $u = Jv = (-t,s)$, and $Xv = \kappa u$. Then $\kappa$ is analytic and not identically zero on any orbit, has both signs on every orbit, and has only finitely many zeros there.*

*Proof.* Over $\mathbb{R}P^{2h-1} = S^{2h-1}/\{z \sim -z\}$, consider the flat rank-two bundle associated to the reflection $\operatorname{diag}(1,-1)$. The Hopf field descends to the quotient. For sections $g$ the relation $\nabla_X g \neq 0$ is open ample: in a principal slice not fixing $X$, its excluded set is one point in a two-dimensional affine space. There is a formal solution with value zero and derivative $\beta \otimes e_1$, where $\beta(X) = 1$ and $e_1$ is the invariant first coordinate. The same open ample theorem gives a section whose lift satisfies $g(-z) = \operatorname{diag}(1,-1)g(z)$ and $Xg \neq 0$.

Approximate this lift in $C^1$ by a finite Fourier sum for the circle action. The Fourier coefficients are global smooth functions; pairing opposite frequencies preserves reality, and the finite sum preserves the reflection symmetry. Smoothness and integration by parts in the circle variable give $C^1$ convergence. Choose the sum close enough that its orbit derivative remains nonzero, and still call it $g$.

Set $a = |Xg|$, $\ell = \overline{a} > 0$, and $q = a/\ell - 1$. The orbit-preserving map

$$\Psi(z) = R_{I_X q(z)} z$$

has derivative $1 + X I_X q = a/\ell > 0$ along every orbit and has degree one on each orbit, so it is a global smooth diffeomorphism. It commutes with the half-period involution because $a$ and $I_X q$ do. The map $f = (g \circ \Psi^{-1})/\ell$ has unit orbit speed and the required reflection symmetry. Each orbit reparametrization is analytic with positive derivative and analytic inverse, so $f$ is analytic on every orbit. The signed area integrand changes sign under the half-period involution, so it has zero mean.

The sphere is simply connected, so the circle-valued map $v$ has a smooth real angle lift $\beta$. Thus $\kappa = X\beta$ has zero mean. If $\kappa$ vanished identically on an orbit, $v$ would be a constant unit vector there, contradicting $\overline{Xf} = 0$. Analyticity and zero mean now give finitely many zeros and both signs. □

**Lemma 5.4** (A free space-curve selector)**.** *There is a smooth map $c : S^{2h-1} \to \mathbb{R}^3$ with $|Xc| = 1$ and with $Xc, X^2c$ independent everywhere.*

*Proof.* For the planar map above, put $g = (f, |f|^2/2)$. Its two derivatives are

$$Xg = (v, \langle f, v\rangle), \qquad X^2 g = (Xv, 1 + \langle f, Xv\rangle).$$

If $Xv \neq 0$, their planar components are independent. If $Xv = 0$, the second vector is $(0, 0, 1)$, still independent of the first. Apply the same orbitwise speed normalization to $g$. Reparametrization and basic positive target scaling multiply the first derivative by a nonzero scalar and replace the second by a nonzero multiple of itself plus a multiple of the first, so independence is preserved. The normalized cross product supplies a smooth unit binormal. □

## 5.2. **Orthogonal matrix primitives with prescribed moments.**

**Lemma 5.5** (Uniform orbitwise oscillation)**.** *For the map in Lemma 5.3 and every fixed smooth amplitude $a$,*

$$\overline{ae^{iTt}} \longrightarrow 0$$

*uniformly over the Hopf quotient as $T \to +\infty$.*

*Proof.* The function $t$ is nonconstant on every orbit. A constant component of the unit vector $(s, t)$ would have zero mean, hence be zero, and continuity would make the other component a constant unit vector, contradicting $\overline{v} = 0$. Consequently the analytic function $Xt$ has a zero set of orbit measure zero. Choose cutoffs decreasing to its zero-set indicator, equal to one on $|Xt| \leq \delta$ and supported in $|Xt| < 2\delta$. Their averages are continuous on the compact quotient and decrease pointwise to zero, so they converge uniformly to zero. On their complements, integration by parts using $Xe^{iTt} = iT(Xt)e^{iTt}$ bounds the integral by $O_\delta(T^{-1})$. We first choose $\delta$ small and then $T$ large. The same proof applies to any finite list of amplitudes. □

**Lemma 5.6** (Orthogonal primitive and scalar moments)**.** *For every $h \geq 2$ and $p \geq 2$, there are smooth maps on $S^{2h-1}$*

$$f : \ S^{2h-1} \to \mathbb{R}^2, \qquad P : \ S^{2h-1} \to \mathrm{Mat}_{p,p}(\mathbb{R}), \qquad \eta : \ S^{2h-1} \to \mathbb{R}$$

*with the planar properties of Lemma 5.3, such that, for $Q = XP$,*

$$Q \in \mathrm{SO}(p), \qquad \overline{Q} = 0, \qquad \overline{P^t Q} = 0, \tag{5.1}$$

$$XQ = 0 \text{ on } \{\kappa = 0\}, \qquad \overline{\eta} = 0, \qquad \overline{\eta^2 Q^t XQ} = 0. \tag{5.2}$$

*The function $\eta - |f|^2/2$ is flat on $\{\kappa = 0\}$; in particular $X^2\eta = 1$ there.*

*Proof.* We construct the three-dimensional matrix, then the matrices of every dimension, and finally the scalar. Put $b = \langle f, u\rangle$ and

$$\rho_0 = \begin{cases} e^{-1/\kappa^2}, & \kappa \neq 0, \\ 0, & \kappa = 0. \end{cases}$$

This is smooth, positive off the curvature-zero set, and flat on that set. The two functions $1, s$ are independent on every orbit, and so are $t, b$. For the latter pair, a relation $b = ct$ would make $f + (c, 0)$ parallel to $v$ everywhere. At a maximum of $|f + (c, 0)|^2$ its derivative is zero; the parallel vector must then vanish, so the maximum would be zero, contradicting $|Xf| = 1$. A relation with zero coefficient of $b$ would force $t \equiv 0$, already excluded. Since the curvature zeros are finite on each orbit, these independence statements remain true after restriction to $\kappa \neq 0$.

*The three-dimensional matrix.* Regard $v, u$ as planar vectors in $\mathbb{R}^3$, and set

$$Q_3(\theta) = \Big[v,\ e_3 \cos\theta - u \sin\theta, -e_3 \sin\theta - u \cos\theta\Big].$$

Its columns form a positively oriented orthonormal frame. For $\theta_0 = Tt$ consider the four constraints

$$G(\theta) = \overline{(\cos\theta,\ t\sin\theta, s\cos\theta,\ b\sin\theta)}$$

and their derivative vector

$$F_T = (-\sin\theta_0,\ t\cos\theta_0, -s\sin\theta_0,\ b\cos\theta_0).$$

Use the four variations $h_a = \rho_0 (F_T)_a$, with coefficients $\alpha_a$ constant on each orbit. The derivative at zero of $\alpha \mapsto G(\theta_0 + \sum_a \alpha_a h_a)$ is $\overline{\rho_0 F_T F_T^t}$. By Lemma 5.5 it converges uniformly, after permuting coordinates, to the direct sum of one half the weighted Gram matrices of $(1, s)$ and $(t, b)$. Both are positive definite, with uniformly positive smallest eigenvalues on the compact quotient. Also $G(\theta_0) \to 0$ uniformly, and the second derivatives in $\alpha$ are uniformly bounded since the values of the variations are bounded independently of $T$.

For all sufficiently large $T$, the finite-dimensional implicit function theorem therefore gives a unique small solution $\alpha_T$, tending uniformly to zero. Uniformity follows by multiplying the equation by the inverse of its derivative at zero and applying the contraction theorem on a ball of radius a fixed multiple of $|G(\theta_0)|$, on which the derivative remains invertible. The solutions are smooth basic functions by the ordinary implicit function theorem and uniqueness. Put $\theta = \theta_0 + \sum_a (\alpha_T)_a h_a$. Then

$$\|\theta - Tt\|_{C^0} + T^{-1}\|X(\theta - Tt)\|_{C^0} \longrightarrow 0. \tag{5.3}$$

Here $X\alpha_T = 0$ and $Xh_a = O(T + 1)$.

Under the half-period involution, $s$ is even and $t, b$ are odd. Each $h_a$ is odd and flat on $\kappa = 0$. Consequently $\theta(-z) = -\theta(z)$, and $X\theta = 0$ on that zero set. The reflection identity is

$$Q_3(-z) = RQ_3(z)D, \qquad R = \operatorname{diag}(1, -1, 1), \quad D = \operatorname{diag}(1, 1, -1).$$

Reflection kills all means of the last two columns except their first and third, respectively second, components. These three remaining means are the first three constraints in

$G = 0$. The first column has mean zero because it equals $Xf$. Therefore $\overline{Q_3} = 0$ and $P_3 = I_X Q_3$ is defined globally. The matrix $\Omega = \overline{P_3^t Q_3}$ is skew by integrating $X(P_3^t P_3)$. Its reflection law $\Omega = D\Omega D$ kills entries $13, 23$ in ordinary one-based matrix notation. The remaining entry is $-\overline{b \sin\theta} = 0$: the first column of $P_3$ is $f - \overline{f}$, and the second column of $Q_3$ has mean zero. Thus $\overline{P_3^t Q_3} = 0$. Finally $Xv = Xu = X\theta = 0$ on $\kappa = 0$, hence $XQ_3 = 0$ there.

*The required positive cone.* If necessary, we increase $T$ before fixing these matrices. We claim that the values on each orbit, away from $\kappa = 0$, of

$$a_T = (\kappa \sin\theta,\ \kappa\cos\theta, T^{-1}X\theta,\ \kappa) \tag{5.4}$$

positively span $\mathbb{R}^4$. On a fixed orbit the real angle $\beta$ of $v$ is a nonconstant periodic analytic function. Choose a regular level strictly between its minimum and maximum, away from angles where either sine or cosine vanishes. The periodic function crosses that level with both signs of derivative. Hence there are two inverse branches over a common small interval of angle values, with $\kappa$ of opposite signs and $|\kappa|, |\cos\beta|$ bounded below. The branches and their bounds persist on a quotient neighborhood.

Choose two interior angle values with different cosines $c_1, c_2$. For each of them and for large $T$, the monotone function $T\sin\beta$ attains the phases $0, \pi/2, \pi, 3\pi/2$ modulo $2\pi$ at angle values within $O(T^{-1})$ of the chosen value. Use the same angle values on the two inverse branches. Before the correction in (5.3), the corresponding normalized vectors $a_T/|\kappa|$ approach the sixteen vectors

$$\pm(\sin\varphi, \cos\varphi, c_a, 1), \qquad a = 1, 2, \quad \varphi = 0, \pi/2, \pi, 3\pi/2.$$

They span $\mathbb{R}^4$: phase differences span the first two coordinates, and phase sums give the independent vectors $(0, 0, c_1, 1)$, $(0, 0, c_2, 1)$. Their symmetric convex hull contains a neighborhood of zero. By (5.3) the correction changes these vectors by $o(1)$, so this property persists. A finite collection of the quotient neighborhoods covers the compact base; choose one $T$ above their finitely many thresholds. This proves the claim uniformly, without a global choice of inverse branches.

*Every matrix dimension.* In dimension two set

$$P_2 = f_1 I + f_2 J, \qquad Q_2 = sI + tJ.$$

Then $XP_2 = Q_2 \in \mathrm{SO}(2)$, $\overline{Q_2} = 0$, and

$$\overline{P_2^t Q_2} = \overline{\langle f, v\rangle} I + \overline{f_1 t - f_2 s} J = 0.$$

Also $Q_2^t X Q_2 = \kappa J$ and $XQ_2 = 0$ where $\kappa = 0$. For even $p$, take repeated two-by-two blocks; for odd $p \geq 3$, take one three-by-three block followed by two-by-two blocks. This gives (5.1) and the first assertion of (5.2).

Let $A = Q^t X Q$, viewed in its fixed coefficient space $V \subset \mathfrak{so}(p)$. For even $p$ this is the common repeated $J$-line; for $p = 3$ it is $\mathfrak{so}(3)$; for odd $p \geq 5$ it is $\mathfrak{so}(3)$ plus the common repeated $J$-line. The independent entries of $Q_3^t X Q_3$ are $\kappa\sin\theta, \kappa\cos\theta, -X\theta$. Thus the values of $A$, restricted to $\kappa \neq 0$, positively span $V$ on every orbit. For even $p$ use the two signs of $\kappa$; for odd $p$ use (5.4), deleting its fourth coordinate when $p = 3$.

Fixed invertible changes and rescalings of coefficient coordinates preserve this property. All matrices and their derivative bounds are now fixed.

*Smooth positive densities with prescribed vector moment.* Identify $V$ with $\mathbb{R}^r$. For every smooth basic vector $y$ there is a nonnegative smooth function $\rho$, positive where $\kappa \neq 0$ and flat where $\kappa = 0$, with

$$\overline{A\rho} = y, \qquad \sqrt{\rho} \text{ smooth and flat on } \{\kappa = 0\}.$$

The construction is finite-dimensional. Choose finitely many global smooth functions $h_a$, including $h_0 = 1$, such that $v_a = \overline{A\rho_0^2 h_a^2}$ positively span $\mathbb{R}^r$ at every base point. Locally choose narrow bumps about finitely many orbit points whose $A$-values positively span; their weighted integrals remain positive spanning when the bumps are sufficiently narrow. Multiply by base cutoffs supported in a local trivialization and positive on a smaller neighborhood. A finite cover gives the global list. Adding its other vectors and $h_0$ preserves positive spanning.

Minimize over $\lambda \in \mathbb{R}^r$ the function

$$\Phi(\lambda) = \sum_a e^{\langle \lambda, v_a \rangle} - \langle \lambda, y \rangle.$$

Its Hessian is positive definite, and compactness and positive spanning give a uniform $c > 0$ such that $\max_a \langle e, v_a \rangle \geq c$ for every unit $e$. Thus $\Phi(\lambda) \geq e^{c|\lambda|} - |\lambda||y|$, so it has a unique minimizer, which depends smoothly on the base point by the implicit function theorem and uniqueness. Its equation is $\sum_a e^{\langle \lambda, v_a \rangle} v_a = y$. Therefore

$$\rho = \rho_0^2 \sum_a e^{\langle \lambda, v_a \rangle} h_a^2$$

has the required moment. The sum is everywhere strictly positive because $h_0 = 1$, so its square root has the asserted regularity.

*The final scalar.* Let $\eta_* = |f|^2/2$, and set

$$\eta_b = \eta_* - \rho_0 \frac{\overline{\eta_*}}{\overline{\rho_0}}.$$

The denominator is positive on the compact quotient. Then $\overline{\eta_b} = 0$, and $\eta_b - \eta_*$ is flat on $\kappa = 0$. Apply the density construction to $y = -\overline{A\eta_b^2}$, and define, for a second large frequency,

$$\eta_L = \eta_b + \sqrt{2\rho}\cos(Lt).$$

Lemma 5.5 gives $\overline{\eta_L} \to 0$ and $\overline{A\eta_L^2} \to 0$ uniformly. To make both moments vanish, we use the $(r+1)$ variations given by the components of $\rho_0 f_L$, where $f_L = (1, 2A\eta_L)$, with basic coefficients. The derivative matrix of these constraints at zero is $\overline{\rho_0 f_L f_L^t}$. Its limit has quadratic form

$$\overline{\rho_0 (c + 2\eta_b \langle b', A \rangle)^2} + 4\overline{\rho_0 \rho \langle b', A \rangle^2}, \qquad (c, b') \in \mathbb{R} \times \mathbb{R}^r.$$

It is positive definite: the second term is positive when $b' \neq 0$ because the $A$-values span $V$ and the weights are positive off $\kappa = 0$; when $b' = 0, c \neq 0$, the first term is positive. The eigenvalue lower bound is uniform by compactness. The second derivatives of the constraints in the coefficients are uniformly bounded in $L$, since all values of $\eta_L$ and the variations are uniformly bounded. For a fixed sufficiently large $L$, the same uniform

implicit function argument gives a smooth basic correction with both moments zero. Every correction of $\eta_*$ is flat on $\kappa = 0$. Finally

$$X^2\eta_* = |Xf|^2 + \langle f, X^2 f\rangle = 1 \quad \text{where } \kappa = 0.$$

This completes the proof. □

5.3. **A normalized exact metric correction on a torus bundle.**

**Lemma 5.7** (Quadratic correction with a singularly scaled inverse)**.** *Let $\mathcal{X}$ be a compact principal $\mathbb{T}^b$-bundle with commuting period-one generators $D_1, \dots, D_b$. Let $P_\epsilon : \mathcal{X} \to \mathbb{R}^N$ be a smooth family with*

$$\gamma(P_\epsilon)_{ab} := \langle D_a P_\epsilon, D_b P_\epsilon\rangle = \delta_{ab} + \epsilon^3 (S_3)_{ab} + \epsilon^4 (S_4)_{ab},$$

*where $S_3, S_4$ are smooth, also at $\epsilon = 0$. For $\epsilon \neq 0$, suppose that the full pairing map*

$$z \longmapsto \Big(\langle D_a P_\epsilon, z\rangle, \langle D_a D_b P_\epsilon, z\rangle\Big)_{a,\, a\le b}$$

*has a smooth right inverse $C_\epsilon$, and that $\widetilde{C}_\epsilon = \epsilon C_\epsilon$ extends smoothly through zero. Then, on one sufficiently small interval in $\epsilon$, there is a jointly smooth correction $U_\epsilon$ such that*

$$\gamma(P_\epsilon + U_\epsilon) = I_b, \qquad \|U_\epsilon\|_{C^2(\text{each fiber})} = O(\epsilon^2)$$

*uniformly over the base. The inverse need not have its image in the span of the original columns.*

*Proof.* On each period-one torus put $L = \sum_a D_a^2 - 1$ and $W = L^{-1}$. The periodic Schauder estimate gives a bounded map $W : c^{0,\alpha} \to c^{2,\alpha}$, where $0 < \alpha < 1$ and $c^{r,\alpha}$ denotes the closure of smooth functions in the Hölder norm. The operator commutes with derivatives and translations. All torus fibers have the same periods, so the operator and its bound are independent of the base point. Since the transition functions are translations, it defines a global fiber operator. This is the fixed periodic resolvent used in the classical quadratic correction [Gün89b, Gün91].

For target-valued functions $v, w$, define symmetric bilinear data

(5.5)

$$Q_a(v,w) = \tfrac{1}{2} W\Big[\langle Lv, D_a w\rangle + \langle Lw, D_a v\rangle\Big],$$

$$Q_{ab}(v,w) = W\Big[\sum_c \Big(\langle D_a D_c v, D_b D_c w\rangle + \langle D_a D_c w, D_b D_c v\rangle\Big)$$

(5.6)
$$+ \tfrac{1}{2}\Big(\langle D_a v, D_b w\rangle + \langle D_a w, D_b v\rangle\Big) - \langle Lv, D_a D_b w\rangle - \langle Lw, D_a D_b v\rangle\Big].$$

They are bounded bilinear maps from two copies of $c^{2,\alpha}$ to $c^{2,\alpha}$: every product inside $W$ uses at most two derivatives, and the resolvent gains two derivatives. For $v = w$, applying $L$ and using the product rule and the commutation of the $D_a$, we obtain

$$D_a Q_b(v,v) + D_b Q_a(v,v) + Q_{ab}(v,v) = \langle D_a v, D_b v\rangle. \tag{5.7}$$

Indeed, the derivatives of $\langle Lv, D_b v\rangle$ provide the third-derivative terms; the other terms in (5.6) cancel $2\langle Lv, D_a D_b v\rangle$ and supply $2\sum_c \langle D_a D_c v, D_b D_c v\rangle + \langle D_a v, D_b v\rangle$. The result

is $L\langle D_a v, D_b v\rangle$, and injectivity of $L$ proves the identity. Smooth approximation extends it to the little Hölder space.

Set $Q_{\mathrm{data}} = (-Q_a, Q_{ab}/2)$, using the same ordering as the pairing map. Solve on each fiber

$$v = \widetilde{C}_\epsilon\Big(0, (\epsilon S_3 + \epsilon^2 S_4)/2\Big) + \widetilde{C}_\epsilon Q_{\mathrm{data}}(v, v). \tag{5.8}$$

The smooth multiplication operators $\widetilde{C}_\epsilon$ have uniformly bounded norms on $c^{2,\alpha}$. Thus on a fixed small ball the fixed-point map is a contraction, uniformly over the compact base and small $\epsilon$. The forcing is $O(\epsilon)$, so the unique solution satisfies $\|v\|_{c^{2,\alpha}} = O(|\epsilon|)$.

Set $U_\epsilon = \epsilon v$. The exact right-inverse property gives

$$\begin{aligned}\langle D_a P_\epsilon, U_\epsilon\rangle &= -Q_a(U_\epsilon, U_\epsilon),\\ \langle D_a D_b P_\epsilon, U_\epsilon\rangle &= \tfrac{1}{2}(\epsilon^3 S_3 + \epsilon^4 S_4)_{ab} + \tfrac{1}{2} Q_{ab}(U_\epsilon, U_\epsilon).\end{aligned}$$

Substitution in the identity

$$\gamma(P+U)_{ab} - \gamma(P)_{ab} = D_a\langle D_b P, U\rangle + D_b\langle D_a P, U\rangle - 2\langle D_a D_b P, U\rangle + \langle D_a U, D_b U\rangle$$

and then (5.7) cancel the metric error.

It remains to prove joint smoothness; the base may have large dimension. In a local bundle chart the quadratic operator is one fixed torus operator. Its linearization in the unknown in (5.8) is

$$I - 2\widetilde{C}_\epsilon Q_{\mathrm{data}}(v, \cdot),$$

invertible by the same uniform Neumann bound. The Banach implicit function theorem gives smooth dependence into $c^{2,\alpha}$ on all transverse parameters and on $\epsilon$, on this same interval. For the vertical variables, translate the smooth coefficients and forcing by a torus parameter; they depend smoothly on that parameter into $c^{2,\alpha}$. The resulting solution depends smoothly on it by the same theorem; uniqueness and translation invariance of $Q_{\mathrm{data}}$ identify it with the translate of $v$. Hence all vertical derivatives exist in $c^{2,\alpha}$, together with all mixed transverse derivatives, and the solution is jointly smooth. Uniqueness glues it across the translation transition functions. Increasing the derivative order imposes no new smallness condition. □

### 5.4. **The rank-one-complement induction.**

*Proof of Theorem 5.1.* For $d = 1$, use Lemma 5.4; its target is $\mathbb{R}^3 = \mathbb{R}^{j_1+1}$, and its binormal supplies the complementary unit vector.

Suppose the result holds for $m$, and write $H = H_m$, with old Hopf fields $Y_1, \dots, Y_m$. Let $\nu$ be the oriented unit complement to its ordered first and symmetric second columns. On the new sphere use Lemma 5.6 with $p = m+1$, and retain its notation $f, P, Q, \eta, v, \kappa$. Put

$$F_0 = (H, f, 0_m), \qquad D = j_m + 1 + 2 + m = s_m + 2m + 3 = j_{m+1} + 1. \tag{5.9}$$

This map has exact partial Gram $I_{m+1}$.

Inside the old target and the $m$ new constant coordinates, the orthogonal complement to the old free columns is a rank-$(m+1)$ bundle $E$ with orthonormal frame $T_y = (\nu, e_1, \dots, e_m)$. Its induced connection along each $Y_i$ is zero, since the new vectors are

constant and orthogonal to the old target, and $\langle \nu, Y_i \nu \rangle = 0$. Thus projection of $Y_i(T_y a)$ to $E$ has coordinates $Y_i a$. The components of $Y_i \nu$ in the old free-column span remain in the full derivatives, although their pairings with vectors in $E$ vanish.

By Lemma 5.2, choose $A : \mathcal{X}_m \to \mathbb{R}^{m+1}$ with independent columns $Y_i A$. Let $n$ be their oriented unit cofactor normal. For a small positive constant $\delta$, to be fixed before any metric-correction parameter, set

$$Z(y,z) = T_y\Big(P(z)A(y) + \delta\eta(z)Q(z)n(y)\Big). \tag{5.10}$$

The field $Z$ is perpendicular to every first column and every pure old or pure new second column of $F_0$; the mixed second columns of $F_0$ are zero. Hence the linearized metric satisfies $D\gamma_{F_0} Z = 0$ in every row.

*Every mixed period is zero.* Since $XZ \in E$, only the $E$-projection of $Y_i Z$ contributes to their pairing. With $C = \delta\eta Q n$, the pairing is, in coordinates,

$$\langle Y_i Z, XZ \rangle = \langle PY_iA + Y_iC,\ QA + XC \rangle.$$

The mean of the term quadratic in $A$ is $(Y_iA)^t \overline{P^tQ} A = 0$. Integration by parts gives

$$\overline{\langle PY_iA, XC\rangle} = -\delta\overline{\eta}\,\langle Y_iA, n\rangle = 0, \qquad \overline{\langle Y_iC, QA\rangle} = \delta\overline{\eta}\,\langle Y_in, A\rangle = 0.$$

The first equality also vanishes pointwise after integration by parts because $Y_iA \perp n$; the second uses the separate condition $\overline{\eta} = 0$. The remaining term is

$$\langle Y_iC, XC\rangle = \delta^2\eta X\eta\,\langle Y_in, n\rangle + \delta^2\eta^2 (Y_in)^t Q^t X Q n.$$

Its first summand is zero pointwise, and the second has zero mean by (5.2). Consequently

$$\overline{\langle Y_iZ, XZ\rangle} = 0 \quad \text{for every old point and every new orbit.} \tag{5.11}$$

*Mixed rank and the pure new column.* The $E$-coordinates of $Y_iXZ$ are

$$M_i = QY_iA + \delta\Big((X\eta)Q + \eta XQ\Big)Y_in.$$

For sufficiently small $\delta$, these $m$ columns are independent everywhere, with a uniform margin. Let $n_\delta$ be the projection of $Qn$ to their orthogonal complement. The projection formula $M(M^tM)^{-1}M^t$ gives $n_\delta = Qn + O(\delta)$ uniformly.

At $\kappa = 0$, one has $XQ = 0$ and $X^2\eta = 1$, so

$$X^2(PA + \delta\eta Qn) = \delta\Big(Q + \eta X^2 Q\Big)n.$$

Twice differentiating $Q^tQ = I$ shows that $Q^tX^2Q$ is skew at these points. It follows that

$$\langle Qn, X^2(PA + \delta\eta Qn)\rangle = \delta.$$

The second vector has norm $O(\delta)$ on that compact zero set. Replacing $Qn$ by $n_\delta$ changes the pairing by $O(\delta^2)$. Fix a positive $\delta$ small enough that mixed independence holds and this pairing is at least $\delta/2$ on the zero set. Compactness supplies a fixed $d_0 > 0$ for which it is at least $\delta/4$ throughout $|\kappa| < d_0$. Thus the $m$ mixed columns and the pure new second column add $m+1$ independent $E$-directions on this region.

*Cancellation through second order.* Set $e = |XZ|^2$ and $k_2 = \overline{\kappa^2} > 0$; the latter is basic with a positive minimum. Define

$$b = \frac{\kappa\overline{e}}{2k_2}, \qquad u = I_X(-e/2 + \kappa b), \qquad w_i = I_X\Big(-\langle Y_i Z, XZ\rangle - Y_i u\Big).$$

The first primitive has zero-mean input by construction, and the others by (5.11), since $Y_i$ commutes with averaging and $\overline{u} = 0$. In the old free-column span choose the unique vector $W_H$ with

$$\langle Y_i H, W_H\rangle = w_i, \qquad \langle Y_i Y_j H, W_H\rangle = \tfrac{1}{2}\Big(Y_i w_j + Y_j w_i + \langle Y_i Z, Y_j Z\rangle\Big).$$

The induction hypothesis provides the smooth pairing inverse. Put $W_{\mathrm{pl}} = uv + bJv$ in the new plane and $W = (W_H, W_{\mathrm{pl}}, 0_m)$. The pure old rows of $D\gamma_{F_0}W$ are $-\langle Y_i Z, Y_j Z\rangle$; the mixed rows are $Xw_i + Y_i u = -\langle Y_i Z, XZ\rangle$; and the pure new row is $2Xu - 2\kappa b = -e$. Therefore $D\gamma_{F_0}W = -\gamma(Z)$ in every row.

The actual polynomial family

$$P_\epsilon = F_0 + \epsilon Z + \epsilon^2 W$$

has the exact expansion

$$\gamma(P_\epsilon) = I + \epsilon^3 S_3 + \epsilon^4 S_4, \tag{5.12}$$

$$(S_3)_{ab} = \langle D_a Z, D_b W\rangle + \langle D_a W, D_b Z\rangle,$$
$$(S_4)_{ab} = \langle D_a W, D_b W\rangle,$$

where $D_a$ runs through $Y_1, \dots, Y_m, X$. All its coefficients are fixed smooth functions on the compact product.

*A globally normalized pairing inverse.* Near $|\kappa| < d_0$, let $\Pi$ be projection away from the planar normal $Jv$, acting as the identity in the other target blocks. Project all first and second columns of $P_\epsilon$ by $\Pi$, divide each mixed second column and the pure $XX$ column by $\epsilon$, and leave the other columns unscaled. Call the resulting $D$-by-$(D-1)$ matrix $T_{\mathrm{near}}$. It extends smoothly through $\epsilon = 0$: the mixed base columns vanish, and $\Pi(\kappa Jv) = 0$ identically. At zero its unscaled columns are the old free columns and $v$. Its remaining columns have the $m+1$ independent $E$-projections established above. Thus its columns are independent, with a positive margin, on any slightly smaller closed region.

The matrix $T_{\mathrm{near}}(T_{\mathrm{near}}^t T_{\mathrm{near}})^{-1}$, applied to pairing data whose normalized entries have also been divided by $\epsilon$, is an exact right inverse $C_{\mathrm{near}}$ for the original pairing map. Its image lies in $\operatorname{im}\Pi$, so pairing with an original column equals pairing with its projection. Multiplying the inverse by $\epsilon$ turns every data multiplier into $\epsilon$ or one. Hence $\epsilon C_{\mathrm{near}}$ extends smoothly through zero.

On $|\kappa| > d_0/3$, use the unprojected columns and divide only the mixed second columns by $\epsilon$. At zero, the old free columns, $v$, and $\kappa Jv$ are independent, and the mixed columns add their independent $E$-projections. The same formula gives $C_{\mathrm{far}}$ and a smooth extension of $\epsilon C_{\mathrm{far}}$.

Choose a fixed smooth cutoff equal to one for $|\kappa| \le d_0/2$ and zero for $|\kappa| \ge 3d_0/4$. Use the near inverse on $|\kappa| \le 7d_0/8$ and the far inverse on $|\kappa| \ge d_0/3$. Their cutoff-weighted sum is an exact right inverse $C_\epsilon$, because the right-inverse identities are linear and the

weights sum to one. The products extend by zero outside their buffered domains. On one common small interval, $\epsilon C_\epsilon$ is smooth in all variables through zero.

For a fixed sufficiently small positive $\epsilon$, Lemma 5.7 now gives a smooth exact map $H_{m+1} = P_\epsilon + U_\epsilon$. The correction $U_\epsilon$ is $O(\epsilon^2)$ in the vertical $C^2$ norm, so it changes the unscaled normalized columns by $O(\epsilon^2)$ and the divided columns by $O(\epsilon)$, and the positive margins in the two regions persist. All $j_{m+1} = D-1$ first and symmetric second columns are independent, and their Gram is exactly $I_{m+1}$. Their oriented unit cofactor supplies the global complementary unit vector. This completes the induction. □

*Remark* 5.8 (The role of the extra coordinate). The curvature-zero set in this construction cannot be discarded. On a simply connected Hopf sphere a unit planar tangent has a real angle lift, so its signed curvature has zero orbit average and must vanish on every orbit. Hence there is no free planar selector. The space-curve lift handles the initial case, and the pure new second column in the induction replaces the planar curvature at its zeros. Orthogonal products alone would have zero mixed second columns; the displacement (5.10) produces these columns, and its moment identities permit exact correction. The Gauss equation is not imposed separately; it holds for the actual exactly isometric map just constructed.

## 6. Exact metric correction with full source regularity

We prove the metric correction used later. We need its quantitative form because the phase margin in the finite completion tends to zero. The proof uses finite differential formulas followed by one smooth iteration; smooth dependence on transverse variables is part of the construction.

Let $D \subset TX$ be an integrable rank-$n$ distribution on a compact smooth manifold. We first take $X$ closed. In a foliated chart write $(y, z)$, where $D$ is spanned by $\partial_{y_1}, \dots, \partial_{y_n}$. Set

$$\gamma_D(F)_{ij} = \langle F_i, F_j \rangle, \qquad \Pi_F = I - d_D F\, \gamma_D(F)^{-1} (d_D F)^*, \qquad B_{ij} = \Pi_F F_{ij}.$$

Throughout this section $i, j$ denote leaf derivatives. The full source need not be immersed. Under a foliated change of coordinates the extra terms in $F_{ij}$ are first leaf derivatives, which $\Pi_F$ kills; hence $B$ is a well-defined tensor on $D$. The normal bundle is $d_D F(D)^\perp$. For $\xi \neq 0$ the transverse pairing

$$\beta_\xi : d_D F(D)^\perp \longrightarrow \mathrm{Sym}^2(\ker \xi)^*, \qquad \beta_\xi(v)(S) = \langle B(S), v \rangle$$

is onto if and only if $\xi$ is a good phase covector. Its kernel has dimension $N - s_n$, which is positive for $N \geq s_n + 1$.

All $C^m$ norms below contain derivatives in both $y$ and $z$. We use the maximum of the chart norms rather than their sum. An admissible atlas has fixed inner coordinate balls with larger buffers, controlled transition maps and inverse maps, and a bounded degree for the intersection graph of enlarged supports. The number of charts is unrestricted.

**Theorem 6.1** (Uniform exact metric correction). *Fix $n \geq 2$, $\dim X \geq n$, $N \geq s_n + 1$, an admissible atlas class, sequences of reference-map and transition derivative bounds, leaf ellipticity bounds, and a positive lower bound $c_0$ for the best-phase margin. Given a*

*sufficiently small full $C^2$ radius $\rho > 0$, there are a finite reference order $J_*$ and a number $\varepsilon_* > 0$ with the following property. If $F_0$ is a smooth $D$-immersion with the specified reference bounds through $J_*$ and best-phase margin at least $c_0$, then every smooth positive partial metric $h$ with*

$$\|h - \gamma_D(F_0)\|_{C^{440}} < \varepsilon_*$$

*is induced by a smooth $D$-immersion $F$ in the same $\mathbb{R}^N$, with $\|F - F_0\|_{C^2} < \rho$ and best-phase margin at least $c_0/2$. The initial constants use only finitely many of the prescribed reference and atlas bounds and are independent of the number of charts. The solution is the limit of a single sequence, which converges in every full source smooth norm. Higher derivatives of an individual input affect only finite waiting times within that sequence.*

6.1. **Positive phases and supported finite corrections.** We first record the geometric preparation. In a leaf frame put $q = n(n-1)/2$ and

$$P_\xi = |\xi|^2 I - \xi\xi^t, \qquad D_B(\xi) = \sum \Big(q\text{-minors of } B \circ \operatorname{Sym}^2 P_\xi\Big)^2.$$

This nonnegative homogeneous polynomial has degree $8q$ and is positive exactly on good nonzero covectors. At a phase-regular jet it is not identically zero. A polynomial vanishing on an open set is zero, as follows by successive restriction to coordinate lines. Hence good covectors form an open dense set.

A positive metric has a positive rank-one square basis with good directions. For a convenient basis, identify $\mathbb{R}^n$ with the zero-sum hyperplane in $\mathbb{R}^{n+1}$ and use the edge roots $(e_i - e_j)/\sqrt{n+1}$. Their squares sum to the identity. They are linearly independent: a relation gives a graph Laplacian that vanishes both on that hyperplane and on the constant vector, so all its off-diagonal coefficients vanish. Apply a square root of the metric and a generic orthogonal rotation. Each of the finitely many directions is good because the bad rotations have empty interior. The inverse of the square basis and its positive coefficients persist on a neighborhood. Extend the directions as differentials of coordinate-linear leaf phases, and shrink the full chart, including the transverse variables.

Choose covering bumps and divide them by the square root of the sum of their squares. This gives cutoffs $\chi_\nu$ and pointwise linear functionals $\ell_\nu$ such that, for tensors $A$ in a fixed neighborhood of $h_0 = \gamma_D(F_0)$,

$$A = \sum_\nu b_\nu(A)^2\, d_D\phi_\nu^2, \qquad b_\nu(A) = \chi_\nu \sqrt{\ell_\nu(A)}. \tag{6.1}$$

The square-root arguments have a positive lower bound near their cutoff supports. We obtain a local unit vector $n_\nu \in \ker \beta_{d\phi_\nu}$ by projecting a fixed ambient coordinate vector and normalizing; one of these projections has length at least $1/\sqrt{N}$ at a center. The projection is

$$\Pi_F - \beta^*(\beta\beta^*)^{-1}\beta\Pi_F.$$

It depends smoothly on the actual partial two-jet on fixed margins.

Multiply successive phase labels by sufficiently separated positive constants and divide their square coefficients by the squares of those constants. This preserves (6.1).

Every interacting sum or difference, divided by its larger weight, is close to the dominant good covector. Thus the primary phases, their doubles, and all interacting sums and differences are noncritical and good. Each needed phase $\psi$ has a finite adapted foliated cover with $y_1 = \psi$: keep $z$ and the other leaf coordinates fixed and use a leaf partial derivative bounded away from zero. By strict monotonicity this change of coordinates is injective on a sufficiently small rectangle, and differentiating the inverse identity bounds the derivatives of its inverse. All constructions persist in one $C^2$ neighborhood of $F_0$.

For a tensor or coefficient $a$ write

$$|a|_{m,\ell} = \sum_{j=0}^{m} \ell^j \|a\|_{C^j}.$$

For a map, the partial two-jet norm also includes its value and its first and second leaf derivatives, before the full observation derivatives are taken. Each leaf derivative costs at most $\ell^{-1}$ at the next weighted order. Products and smooth matrix operations on fixed positive margins obey the corresponding weighted product and chain estimates.

**Lemma 6.2** (A finite positive increment)**.** *Fix the preceding phase data, inner and outer positive-decomposition neighborhoods, and weighted bounds $D_j$ for the partial two-jet of $F$ and for the input tensor $H$. Assume that $F$ lies in the prepared partial two-jet domain and $H$ lies in the inner positive neighborhood. Let $Q \geq 1$ be an integer. For $0 < \delta \leq \tau \leq \ell \leq 1$ and $\eta = \tau/\ell$ sufficiently small there is a real smooth displacement $U$, given by a fixed finite differential formula, such that for every full observation order $m$,*

$$|U|_{m,\tau} \leq C_m \delta\tau, \tag{6.2}$$

$$|\gamma_D(F+U) - \gamma_D(F) - \delta^2 H|_{m,\tau} \leq C_m \left(\delta\eta^Q + \frac{\delta^3}{\tau}\right). \tag{6.3}$$

*The smallness condition uses only finitely many $D_j$, independently of $m$, and each $C_m$ uses finitely many further bounds. The formula is smooth in auxiliary parameters, and every summand is supported in a primary cutoff support or in a pairwise intersection of such supports.*

*Proof.* We write out the finite solver, including the coefficients of its mixed rows. Complexify the coefficient spaces. In coordinates with $y_1 = \psi$, write an ambient amplitude as

$$Z = d_D F(g^{-1}u) + v, \qquad g = \gamma_D(F), \quad v \perp d_D F(D).$$

The Gram identity gives

$$F_{ij} = \Gamma^k_{ij} F_k + B_{ij}, \qquad (D\gamma_D)_F Z_{ij} = \partial_i u_j + \partial_j u_i - 2\Gamma^k_{ij} u_k - 2\langle B_{ij}, v\rangle.$$

Conjugation by $e^{i\psi/\tau}$ adds $(i/\tau)(\delta_{i1}u_j + \delta_{j1}u_i)$. For $a, b \geq 2$ put

$$T(u)_{ab} = \tfrac{1}{2}(\partial_a u_b + \partial_b u_a) - \Gamma^k_{ab} u_k, \qquad J = \beta_\psi^*(\beta_\psi \beta_\psi^*)^{-1}.$$

For a mode target $h$ and a chosen $v_f \in \ker \beta_\psi$, set

$$v = v_f + JT(u) - \tfrac{1}{2} J h_H.$$

This solves all transverse rows exactly. The remaining equation is $(i/\tau+L)u = k$, where

$$\begin{aligned}(Lu)_1 &= \partial_1 u_1 - \Gamma^k_{11} u_k - \langle B_{11}, JT(u)\rangle,\\ k_1 &= \tfrac{1}{2} h_{11} + \langle B_{11}, v_f\rangle - \tfrac{1}{2}\langle B_{11}, Jh_H\rangle,\\ (Lu)_a &= \partial_1 u_a + \partial_a u_1 - 2\Gamma^k_{1a} u_k - 2\langle B_{1a}, JT(u)\rangle,\\ k_a &= h_{1a} + 2\langle B_{1a}, v_f\rangle - \langle B_{1a}, Jh_H\rangle.\end{aligned}$$

Note that the last coefficient is $-1$, whereas in the divided 11 row it is $-1/2$. A residual $e$ in this tangent system gives the residual $2e_1$ in the 11 metric row, $e_a$ in the $1a$ row, and zero in every transverse row.

For $p \geq 1$ use the finite Neumann polynomial

$$u_p = \frac{\tau}{i} \sum_{l=0}^{p-1} \left(-\frac{\tau}{i} L\right)^l k. \tag{6.4}$$

Its residual is exactly $(-1)^{p-1}(\tau/i)^p L^p k$. Since $L$ has leaf order one, this is bounded by $C_m \eta^p$ times finitely many higher weighted norms of $h, v_f$. The reconstruction has a fixed finite derivative loss. If $h = 0$, it also gives

$$Z = v_f + O_{m,\ell}(\eta\, v_f),$$

where the notation means the preceding weighted estimate with a finite higher input order. All operations are differential and preserve supports. When an adapted phase needs more than one chart, split the amplitudes by subordinate cutoffs; the leading free amplitudes recombine because these cutoffs sum to one.

Take $p = Q + 2$. For a trial tensor $A$ use homogeneous primary modes with

$$v_{f,\nu} = \sqrt{2}\,\delta\tau\, b_\nu(A) n_\nu, \qquad U_1(A) = \sum_\nu \Re(e^{i\phi_\nu/\tau} Z_\nu(A)).$$

Then $|U_1|_{m,\tau} \leq C_m \delta\tau$, $|(D\gamma_D)_F U_1|_{m,\tau} \leq C_m \delta\tau\eta^p$, and $|d_D U_1|_{m,\tau} \leq C_m\delta$. Before the exponentials are inserted, the leading first-derivative coefficient is $i\sqrt{2}\delta b_\nu n_\nu d_D\phi_\nu$; all remaining coefficients carry a factor $\eta$.

Expand the exact quadratic tensor $\gamma_D(U_1)$. Its formal zero-phase coefficient is

$$\delta^2\Big(A + \mathcal{T}(A)\Big), \qquad \mathcal{T}(A) = O_{m,\ell}(\eta).$$

This is the extraction of a coefficient from a finite oscillatory expression; no integration or ergodic averaging is involved. The only zero pairs are a primary label and its conjugate. The leading mean follows from (6.1). Every other mean term contains one of the coefficient errors just estimated. The product rule and the positive lower bound in the square roots also give

$$|\mathcal{T}(A) - \mathcal{T}(\bar{A})|_{m,\ell} \leq C_m \eta\, |A - \bar{A}|_{m+l,\ell} \tag{6.5}$$

for a fixed finite loss $l$. This can be checked term by term: after the leading $A - \bar{A}$ cancels, every remaining product contains a factor $(\tau/i)L$, the term $JT(u_p)$, or a slow derivative multiplied by $\tau$, and each of these contributes $\tau/\ell$.

Define $A_0 = H$ and $A_{j+1} = H - \mathcal{T}(A_j)$ for $0 \leq j < Q$. These are finitely many substitutions rather than an infinite inverse series. Working backward through their

finite derivative orders, we bound the weighted norms required at each step by finitely many earlier ones. Since the initial tensor lies in the inner positive neighborhood, taking $\eta$ small keeps all $A_j$ in the outer one. This domain check uses a fixed finite list and does not depend on the observation order. Applying (6.5) repeatedly gives

$$A_Q + \mathcal{T}(A_Q) - H = O_{m,\ell}(\eta^{Q+1}).$$

Once the finite formulas are defined, their higher estimates require no further smallness condition.

Each nonzero quadratic coefficient is $O_{m,\ell}(\delta^2)$ and carries one of the prepared doubled or sum/difference phases. Solve its negative by (6.4), now with $v_f = 0$, and let $U_2$ be the sum of the real corrections. The reconstructed normal term $-Jh_H/2$ has size $O(\delta^2)$, and

$$|U_2|_{m,\tau} \le C_m\delta^2, \qquad |(D\gamma_D)_F U_2 + (\gamma_D(U_1) - \text{its mean})|_{m,\tau} \le C_m\delta^2\eta^p.$$

Set $U = U_1 + U_2$. The exact quadratic expansion of $\gamma_D$ leaves the errors

$$C_m\left(\delta\tau\eta^p + \delta^2\eta^{Q+1} + \delta^2\eta^p + \frac{\delta^3}{\tau} + \frac{\delta^4}{\tau^2}\right).$$

Since $\delta \le \tau$ and $p = Q + 2$, these imply (6.2)–(6.3). All estimates use full observation derivatives. These derivatives commute with the finite formulas by the product rule, and derivatives of an exponential in any source direction cost at most one $\tau^{-1}$ per order. This also proves the assertion about parameters. □

6.2. **Smoothing at the true map and one smooth limit.** For each integer $r$ choose a compactly supported smooth convolution kernel with integral one and vanishing moments of degrees 1 through $r$. Such a kernel is a linear combination of $r + 1$ distinct dilates of one integral-one kernel: the equations for the coefficients are the nonsingular Vandermonde equations for the dilation powers $0, \dots, r$. A multivariate moment scales according to its total degree, so the same equations cancel every required moment. Localize in buffered foliated charts, convolve in all $(y, z)$ coordinates, multiply by secondary cutoffs equal to one near the localized supports, and sum. This defines a smoothing operator $S_\ell$ on maps and partial symmetric tensors.

Let $\mathcal{J}_D^2 G$ denote the value and the first two leaf derivatives in these charts, and write

$$W_m(t; G, H) = \sum_{j=0}^{m} t^j\Big(\|\mathcal{J}_D^2 G\|_{C^j} + \|H\|_{C^j}\Big).$$

Under a foliated transition the transverse coordinate is independent of the leaf coordinate. Consequently two leaf derivatives of a localized convolution can be assigned to the input using only $\mathcal{J}_D^2 G$, rather than two arbitrary full derivatives. Further observation derivatives are assigned to the kernel, and derivatives of cutoffs and transition coefficients use lower partial jet terms. The product rule and Taylor's formula with the cancelled moments therefore give, if $W_r(t; G, H) \le B$ and $0 < \tau \le \ell \le t \le 1$,

$$|\mathcal{J}_D^2 S_\ell G|_{m,\ell} + |S_\ell H|_{m,\ell} \le C_{m,r} B, \tag{6.6}$$

$$|\mathcal{J}_D^2(G - S_\ell G)|_{m,\tau} + |H - S_\ell H|_{m,\tau} \le C_{m,r}\left[B(\ell/t)^r + W_m(t; G, H)(\tau/t)^r\right]. \tag{6.7}$$

Indeed, an observation order $j \le r$ contributes at most $CBt^{-r}\ell^{r-j}\tau^j \le CB(\ell/t)^r$. For $j > r$, the boundedness of smoothing in the inherited $C^j$ norm gives the second term, because $(\tau/t)^j \le (\tau/t)^r$. The same arguments prove the estimates for bundle tensors once the fixed frame derivatives are included. The estimates involve all transverse derivatives.

Fix $h_0 = \gamma_D(F_0)$ and a $C^2$ neighborhood on which the prepared phase data are valid. Make its radius $\rho_1$ smaller than $\rho/2$ and small enough to retain the phase margin $c_0/2$. Choose $B$ greater than four times $\|\mathcal{J}_D^2 G\|_{C^0} + \|h_0\|_{C^0}$ throughout that neighborhood. At an integer amplitude stage $k \ge 10$ use

(6.8)
$$\delta = t^k, \qquad \ell = t^{11/10}, \qquad \tau = t^{6/5}, \qquad r = Q = 40(k+1), \qquad H = \delta^{-2}(h - \gamma_D(G)).$$

Assume $W_r(t; G, H) \le B$ and that $H$ lies in a smaller positive-decomposition neighborhood of $h_0$. The next reserve $\delta_+$ will be either $\tau^k$ or $\tau^{k+1}$. Apply Lemma 6.2 at the smoothed map $\widetilde{G} = S_\ell G$, with

$$H_{\mathrm{tar}} = S_\ell H - (\delta_+/\delta)^2 h_0.$$

For $t$ sufficiently small, the smoothing tail keeps $\widetilde{G}$ in the same geometric domain and $H_{\mathrm{tar}}$ in the inner positive neighborhood. All finite weighted inputs are bounded by (6.6). Thus the finite formula gives an increment $U$ with constants depending on $k, B$ and the fixed phase data, but not on uncontrolled higher norms of the current map or target.

Update the true map, $G_+ = G + U$, and define $H_+ = \delta_+^{-2}(h - \gamma_D(G_+))$. With the polarized metric term

$$\mathrm{Cross}_D(V, W)_{ij} = \langle V_i, W_j\rangle + \langle V_j, W_i\rangle$$

we have the exact identity

$$\delta_+^2(H_+ - h_0) = \delta^2(H - S_\ell H) - E - \mathrm{Cross}_D(G - S_\ell G, U), \tag{6.9}$$

where $E = \gamma_D(\widetilde{G} + U) - \gamma_D(\widetilde{G}) - \delta^2 H_{\mathrm{tar}}$. In particular, no square of the smoothing defect remains. Full derivatives of this identity are estimated by the ordinary product rule.

For the smaller choice $\delta_+ = \tau^{k+1}$, the six powers of $t$ obtained from the finite error and the two smoothing tails, after division by $\delta_+^2$, are

$$\frac{13k+8}{5}, \quad \frac{3k-18}{5}, \quad \frac{18k+8}{5}, \quad \frac{38k+28}{5}, \quad \frac{13k+8}{5}, \quad \frac{33k+28}{5}.$$

For example, the quadratic interaction gives $3k - 6/5 - 12(k+1)/5 = (3k-18)/5$, and the linear finite-accuracy term gives $k + Q/10 - 12(k+1)/5 = (13k+8)/5$. The remaining terms are, respectively, $\delta^2 B(\ell/t)^r$, $\delta^2 W_m(\tau/t)^r$, $\delta B(\ell/t)^r$, and $\delta W_m(\tau/t)^r$. All powers are positive for $k \ge 10$, with minimum $12/5$. The larger reserve requires a weaker division. Hence the same increment satisfies, for every full order $m$,

$$|H_+ - h_0|_{m,\tau} \le C_m(k) t^{1/5}(1 + W_m(t; G, H)), \tag{6.10}$$

$$W_m(\tau; G_+, H_+) \le \|\mathcal{J}_D^2 G\|_{C^0} + \|h_0\|_{C^0} + C_m(k) t^{1/5}(1 + W_m(t; G, H)). \tag{6.11}$$

For the second estimate, the inherited positive-order map terms gain $\tau/t = t^{1/5}$, the partial two-jet of the increment is $O_m(\delta/\tau)$, and the positive-order terms of the fixed $h_0$ gain a factor $\tau$. The coarse exponent $1/5$ is smaller than every exponent needed above.

Choose a finite stage threshold so that the geometric domain, the positive cone, and the controlled $W_r$ bound are preserved for both choices of reserve. At that stage also impose

$$\|U\|_{C^{\lfloor k/2 \rfloor}} \le \rho_1 t/8. \tag{6.12}$$

This is possible because $k + (6/5)(1 - \lfloor k/2 \rfloor) > 1$. The bound controls full source derivatives, not only leaf derivatives.

For the initialization take $k = 10$ and first choose $t_0$ small for the stage thresholds, for the finite reference norms through order 442 and any further finite orders used by the fixed phase formulas, for the scaling distance, and so that $\sum_{j\ge 0} t_0^{(6/5)^j} < 1$. Put

$$\delta_0 = t_0^{10}, \qquad G_0 = \sqrt{1 - \delta_0^2}\, F_0, \qquad H_0 = h_0 + \delta_0^{-2}(h - h_0).$$

This is the normalized defect of the scaled map. After $t_0$ is fixed, choose $\varepsilon_*$ so that the $C^{440}$ bound in the theorem puts $H_0$ in the smaller positive cone and gives $W_{440} \le B$. No higher derivative of $h$ enters this initial choice.

At each step replace $t$ by $t^{6/5}$. Hold $k$ fixed while waiting for the next controlled order $40(k + 2)$. For any fixed $m$, (6.11) has the form

$$x_{j+1} \le B/4 + c_j(1 + x_j), \qquad c_j = C_m(k) t_j^{1/5} \longrightarrow 0.$$

Every $x_j$ is finite. Once $c_j \le 1/4$, the sequence is bounded by a geometric recurrence, and its upper limit is then at most $B/4$. It therefore enters $x_j < B$ after finitely many steps. The finitely many scale thresholds of the next stage also hold eventually. At a step where both conditions hold, choose $\delta_+ = \tau^{k+1}$ in the formula. The recurrence places the new normalized defect in the next controlled bound. Thus every stage is finite and $k \to \infty$.

The initial scaling and (6.12) keep all maps within $\rho_1/4$ of $F_0$ in full $C^2$. For any fixed $m$, eventually $\lfloor k/2 \rfloor \ge m$, so the same sequence is absolutely summable in $C^m$. It converges to a smooth map $F$ in all full source norms. The normalized defects remain bounded in $C^0$ and $\delta = t^k \to 0$, whence $\gamma_D(F) = h$. Positivity makes $F$ a $D$-immersion, and the retained phase neighborhood gives the asserted margin and displacement.

6.3. **Uniformity and transverse parameters.** We now justify the uniform assertions in Theorem 6.1; they do not follow from qualitative openness alone. Bounded partial two-jets, ellipticity, and a positive best-phase margin give a compact jet set. The good positive-square bases above give a finite template cover of this set with uniform margins, inverse-basis bounds, and positive coefficient reserves. A fixed full $C^3$ bound permits a fixed refinement mesh, and the projected coordinate normal fields then have uniform normalization denominators.

The refined enlarged support graph still has bounded degree. Color it with its degree plus one colors, and label a wave by its color and square-basis index. At most a fixed number $L$ of labels occur, independently of the number of charts. Supports with the

same label are disjoint. Give label $a$ the weight $\Lambda^a$ for one sufficiently large fixed $\Lambda$. The transition bounds and phase margins imply that all interacting sums and differences are uniformly good. Their adapted foliated charts have uniform inverse bounds and need only a bounded number of rectangles per overlap. Squared partitions have uniform derivative bounds because only a bounded number of terms meet at any point.

Every expression in Lemma 6.2 consists of a fixed finite number of differentiations, products, positive square roots, inverse matrices on fixed margins, and locally bounded sums. Its constants and its finite backward list of derivative orders therefore depend on the accuracy and the local overlap degree, not on the total number of charts. The same holds for the moment-smoothing estimates. At initial accuracy 440 choose $J_*$ larger than all reference and transition derivative orders in this finite list, in the phase preparation, and in the initialization through 442. The initial scale and error threshold in the preceding proof are then uniform, and higher input derivatives enter only the subsequent finite waits. This completes the proof of Theorem 6.1.

In particular, for a compact parameter manifold $P$ use $X = M \times P$ and $D = TM$. The construction smooths in both position and parameters, while the metric differentiates only in $M$. It yields one jointly smooth exact family rather than a selection of unrelated exact solutions at each parameter. Parameter sets with boundary can be placed in a compact parameter manifold using a slightly larger collar. The leafwise construction also applies over a compact slow base with boundary: extend its smooth data across a slow collar, make them constant in the outer collar, and double. This retains partial immersion and phase margins, since they involve leaf derivatives only. A small target difference extends by a smooth collar extension bounded in each fixed $C^j$ norm, as in Seeley's extension theorem [See64]; take the doubled target to be the extended reference metric plus that difference. Restriction gives the desired correction. We do not require the correction to be fixed on a prescribed source region.

### 6.4. **Polynomially degenerating phase margins.**

**Theorem 6.3** (Polynomial metric correction from finite reference bounds)**.** *Fix $n \geq 2$, $\dim X \geq n$, $N \geq s_n + 1$, an admissible buffered foliated atlas class, and uniform leaf ellipticity bounds for both the reference and target metrics. There is a finite reference order $J_* \geq 442$, independent of the growth exponents and of a later approximation order, with the following property. For every $a, b \geq 0$ and $d_0 > 0$ there is a finite integer $K$ such that smooth $D$-immersions $f_\epsilon$ and positive partial metrics $h_\epsilon$ satisfying*

$$\|f_\epsilon\|_{C^{J_*}} + \|h_\epsilon\|_{C^{J_*}} + \|h_\epsilon^{-1}\|_{C^{J_*}} \leq C\epsilon^{-a}, \tag{6.13}$$

$$\inf_X \max_{|\xi|=1} \sigma_{\min}\Big(B_{f_\epsilon}|_{\mathrm{Sym}^2 \ker \xi}\Big) \geq c\epsilon^b, \tag{6.14}$$

$$\|h_\epsilon - \gamma_D(f_\epsilon)\|_{C^{440}} \leq C'\epsilon^K \tag{6.15}$$

*admit exact smooth solutions $F_\epsilon$ in the same target with*

$$\gamma_D(F_\epsilon) = h_\epsilon, \qquad \|F_\epsilon - f_\epsilon\|_{C^2} \leq \epsilon^{d_0}$$

*for all sufficiently small $\epsilon > 0$. The finite reference bound includes any coefficient fields used to describe $D$ and the partial metric in the specified charts. The exponent $K$*

*depends on the growth exponents and the fixed atlas class, not on the positive prefactors $C, c, C'$, the number of charts, or input norms above $J_*$; these may affect the final smallness threshold and the subsequent finite waits. No smoothness in $\epsilon$ is required.*

*Proof.* We quantify the finite construction just given. A qualitative openness radius alone would not give a power bound. All controlled norms $W_m$ and initial defect norms are measured in the original admissible atlas. The phase-adapted charts are used only in the finite mode formulas, and their derivative losses are included in the power bounds for those formulas. Thus the initial $C^{440}$ error condition requires no additional coordinate-change factor.

First we choose good positive-square directions quantitatively. The polynomial $D_B$ has degree $8q$. At a unit good covector with restricted singular values at least $c\epsilon^b$, it is at least $c_1\epsilon^{2qb}$. Since evaluation on the unit sphere is bounded by the coefficient norm in this fixed-dimensional polynomial space,

$$\|D_B\|_{\mathrm{coeff}} \geq c_2\epsilon^{2qb}.$$

The coefficient-normalized nonnegative homogeneous polynomials of degree $8q$ form a compact set. For each such polynomial and each positive metric in the fixed ellipticity interval, a generic rotation of the edge-root square basis avoids its zero set in all directions simultaneously. Indeed, the direction obtained from a fixed nonzero root, a rotation, and a positive square root of the metric ranges over the sphere after normalization, so the preimage of the zero set is a proper real analytic subset of each component of $O(n)$. For finitely many roots the complement of these preimages is nonempty. The function

$$(D, h) \longmapsto \max_{R\in O(n)} \min_{\nu} D(h^{1/2}Rw_\nu)$$

is continuous, by uniform continuity on compact sets, and positive on the compact parameter space of polynomials and metrics, so it has a positive minimum. The resulting square basis has uniformly bounded inverse and positive coefficients, and $D_B(\alpha_\nu) \geq c_3\epsilon^{2qb}$ for every selected direction.

The other $q-1$ singular values are at most $C\|B\| \leq C\epsilon^{-a}$. Taking a square root of the minor sum and using the equivalence of the fixed exterior-power norms therefore gives

$$\sigma_{\min}\Big(B|_{\mathrm{Sym}^2 \ker \alpha_\nu}\Big) \geq c_4\epsilon^{a_0}, \qquad a_0 = qb + (q-1)a. \tag{6.16}$$

Before phase-frequency rescaling, the tensor neighborhood of the positive decomposition has a radius independent of $\epsilon$, and on it the arguments $\ell_\nu$ of the square roots have a lower bound $c_\ell > 0$, independent of $\epsilon$, near their cutoff supports. Rescaling a phase by a factor $\lambda$ and its coefficient functional by $\lambda^{-2}$ does not change that neighborhood, since $\ell_\nu(A) \geq c_\ell$ is equivalent to $\lambda^{-2}\ell_\nu(A) \geq c_\ell\lambda^{-2}$. It does change the lower bound. For the label weights $\lambda = \epsilon^{-Wl}$ chosen below, with $l \leq L_0$ for the fixed number $L_0$ of labels, the rescaled functionals satisfy only $\lambda^{-2}\ell_\nu \geq c_\ell\epsilon^{2WL_0}$. Their square roots have lower bound $c_\ell^{1/2}\epsilon^{WL_0}$, and by the chain rule every fixed derivative of $\chi_\nu\lambda^{-1}\sqrt{\ell_\nu}$ is bounded by a power of $\epsilon^{-1}$. These powers depend on $W$ and $L_0$ and are among the finite power bounds collected below. We use no uniform lower bound for the rescaled coefficients.

The true normal projection, the transverse Gram matrices and their inverses, and the local free-normal projections now have polynomial bounds through every fixed initial derivative order. This follows from the product rule and $D(Q^{-1}) = -Q^{-1}(DQ)Q^{-1}$, using ellipticity and (6.16) for the denominators. The same holds for the normalized normal fields and the positive square roots. Choose a grid of diameter $\epsilon^{A_1}$, where $A_1$ exceeds the finitely many first-derivative growth and margin exponents. Freeze the selected linear phases at its centers. The choices are made independently at each center, and no continuous maximizing rotation is needed. A projected ambient coordinate vector has length at least $1/\sqrt{N}$ at the center, and refinement keeps its length at least $1/(2\sqrt{N})$ on the cell. The mean value theorem preserves half of the phase margin on enlarged cells. The local enlarged degree is bounded independently of the number of cells, and every fixed cutoff derivative is bounded by a power of $\epsilon^{-1}$.

Color this enlarged graph and give a wave label the weight $\epsilon^{-Wl}$, where the number of labels is fixed. Choose $W$ so that lower frequencies divided by a dominant one are smaller than the angular margin of the good directions. All quadratic interaction phases then remain good. Their adapted leaf-coordinate Jacobians and inverse derivatives have polynomial bounds: use a dominant leaf partial derivative with a power lower bound, and differentiate the inverse identity. Subdivision into smaller rectangles with subordinate cutoffs preserves bounded local degree; even when the number of such rectangles grows, each local sum has bounded multiplicity. Thus the data are valid throughout a full $C^2$ radius

$$r_\epsilon \geq c_5 \epsilon^{A_2} \tag{6.17}$$

for a finite $A_2$. Assigning a new frequency level to each individual chart would not give this conclusion; the fixed color labels keep all exponents finite.

Choose $J_*$ larger than the maximal differential order in the initial $Q = 440$ formula, its finite backward domain checks, the phase preparation, and the initialization through 442. This finite list is determined before $a, b, d_0$ or an approximation precision are chosen. These later choices alter grid sizes and numerical exponents but do not increase any differentiation order in the list. All its operations are finite products, differentiations, smooth matrix inverses with power lower bounds, and positive square roots with power lower bounds. Therefore all initial constants and inverse margins are bounded by $C\epsilon^{-P}$ for some finite exponent $P$.

For the iteration choose

$$B_\epsilon = C_B \epsilon^{-(a+1)}, \qquad \rho_\epsilon = \epsilon^{A_2+d_0+2}.$$

For a sufficiently large fixed $C_B$, this bounds the zero-order partial two-jet and reference tensor with the required strict reserve. For small $\epsilon$ the displacement budget is less than $r_\epsilon/4$ and at most $\epsilon^{d_0}$. We include its reciprocal among the finite power bounds.

Consequently every initial stage condition has the form

$$C\epsilon^{-P_i} t^{\theta_i} \leq c_i \epsilon^{Q_i}, \qquad \theta_i > 0,$$

for a fixed finite list. Choose $\mu$ so large that $\mu\theta_i > P_i + Q_i + 1$ for this list, and also

$$\mu > 2a + 2, \qquad 20\mu - a > A_2 + d_0 + 3.$$

Now set $t_0 = \epsilon^\mu$, $\delta_0 = t_0^{10}$, and

$$G_0 = \sqrt{1-\delta_0^2}\, f_\epsilon, \qquad H_0 = \gamma_D(f_\epsilon) + \delta_0^{-2}(h_\epsilon - \gamma_D(f_\epsilon)).$$

The initial scaling has $C^2$ size $O(\epsilon^{20\mu-a}) = o(\rho_\epsilon)$. For $1 \le j \le 440$ the weighted reference-map terms are $O(\epsilon^{\mu j - a})$ and the reference metric terms are $O(\epsilon^{\mu j - 2a})$, so they tend to zero. Finally choose an integer $K > 20\mu + 1$. The normalized initial metric difference in $C^{440}$ is $O(\epsilon^{K-20\mu})$, so it lies in the fixed tensor neighborhood and satisfies the controlled initial norm. The order of choices is $J_*$, the finite power bounds, $\mu$, $K$, and only then the smallness threshold in $\epsilon$. The positive prefactors affect only that last threshold.

Fix a sufficiently small $\epsilon$. All particular higher derivatives and all phase data are now fixed finite smooth quantities. Apply the true-map iteration of Section 6.2, with the budget $\rho_\epsilon$ in (6.12). Its later constants may depend on those higher derivatives, but the finite-wait argument works at every stage because $C_m(k,\epsilon)t_j^{1/5} \to 0$. Thus one sequence converges in all full source norms to an exact metric solution, within the prescribed phase-valid radius and displacement bound. The bounded-degree arguments are unchanged for varying scaled atlases in the fixed class. This proves the theorem. □

## 7. From selectors to isometric embeddings

In this section we construct an isometric immersion from an exact bundle selector and then separate its double points without changing the metric. A colored partition keeps the number of Hopf coordinates fixed while the source mesh is refined, so the assembly uses a selector on one fixed, sufficiently large Hopf product.

**Theorem 7.1** (Direct assembly of a bundle selector)**.** *Let $n \ge 2$, $N \ge s_n + 1$, and let $W \subset B \times \mathbb{R}^s$ be a rank-$n$ Euclidean subbundle over a closed smooth base. Suppose that for every sufficiently large integer $C$ there is a smooth map*

$$S_C : B \times (S^{2C-1})^s \longrightarrow \mathbb{R}^N$$

*whose Gram matrix on the coefficient distribution determined by $W$ is the given Euclidean metric of $W$, and whose best-phase margin is positive on this compact source. The bundle, the coefficient inclusion, and the number $s$ are fixed before $C$ varies.*

*Let $M$ be a closed smooth $n$-manifold, let $f : M \to B$ be smooth, and fix a bundle isomorphism $TM \cong f^*W$. Then every smooth positive metric $g$ on $M$ is induced by a smooth phase-regular immersion $M \to \mathbb{R}^N$ whose image lies in any prescribed ball about the origin.*

*Proof.* The analytic input is the atlas-uniform ordinary case of Theorem 6.1.

*A fixed coefficient matrix.* There is a smooth map $\phi : M \to \mathbb{R}^d$ with injective derivative for some finite $d$. For example, take a finite coordinate cover and smooth functions $\beta_i$ supported in its charts, with at least one $\beta_i > 0$ at every point, and use all functions $\beta_i$ and $\beta_i x_i^a$ as coordinates of $\phi$. If $d\phi(v) = 0$, a chart with $\beta_i > 0$ gives first $d\beta_i(v) = 0$ and then $\beta_i\, dx_i^a(v) = 0$ for every $a$, so $v = 0$.

Normalize the given isomorphism by a positive square root to obtain an isometry $J : (TM, g) \to f^*W$: if its initial pullback metric is $g_0$, the positive $g_0$-self-adjoint

square root $P$ satisfying $g(v,w) = g_0(Pv, Pw)$ gives $J = J_0 P$. Let $L : TM \to \varepsilon^s$ be $J$ followed by the coefficient inclusion and put $A = d\phi$. With any fixed auxiliary metric, set

$$A^+ = (A^*A)^{-1}A^*, \qquad R_x = L_x A_x^+.$$

These are global smooth bundle maps, and $R_x A_x = L_x$. In particular $L_x^* L_x = g_x$.

*A fixed number of colors.* Fix an auxiliary Riemannian metric on $M$. For each sufficiently large $K$, take a maximal $r/K$-separated net, with $r > 0$ fixed and small; it gives a covering by balls of radius $r/K$. Choose bumps on the balls of radius $2r/K$ and normalize them by the square root of the sum of their squares. The resulting $\chi_i$ satisfy

$$\sum_i \chi_i^2 = 1, \qquad \|\nabla^j \chi_i\|_\infty \le C_j K^j.$$

The constants are uniform in $K$. Disjoint balls of radius $r/(3K)$ around neighboring centers fit inside a ball of radius a fixed multiple of $r/K$, and uniform volume comparison in compact coordinate charts bounds their number. This also bounds the degree of the graph of intersecting closed support balls. We color that graph with a fixed number of colors, so that closed supports of the same color are disjoint, and increase the number of colors, by empty colors if necessary, to an integer $C$ for which $S_C$ is available. We choose this one selector now. Its derivatives and its positive phase margin are independent of all subsequent $K, T$.

*One actual global approximate map.* Let $x_i$ be the center of the $i$th support and put $R_i = R_{x_i}$. For $T \ge K$ define, in slot $\ell$ and color $a$,

$$z_{\ell,a}(x) = \sum_{i \text{ of color } a} \chi_i(x) \exp\Big(2\pi i T (R_i \phi(x))_\ell\Big). \tag{7.1}$$

Each phase in this formula is a global real function, so no lattice rounding is needed. Since supports of the same color are disjoint, $\sum_a |z_{\ell,a}|^2 = 1$ for every $\ell$. Thus $z_\ell(x) \in S^{2C-1}$ and

$$F_{K,T}(x) = S_C(f(x), z_1(x), \dots, z_s(x))$$

is a smooth global map. Its values are bounded by $\sup |S_C|$, so its $C^0$ norm does not grow with $K$ or $T$.

On the support centered at $x_i$,

$$E_i(x) = R_i A_x - L_x = O(K^{-1}).$$

Every positive-order derivative of $E_i$ in coordinates scaled by $T^{-1}$ is $O(T^{-1})$, hence $O(K^{-1})$, since derivatives of $R_i$ vanish and the remaining coefficient maps belong to one fixed smooth family. For a bounded local tangent field $v$, differentiating (7.1) gives

$$T^{-1} v(z_{\ell,a}) = 2\pi i (L_x v)_\ell z_{\ell,a} + \sum_{i \text{ of color } a} \Big(T^{-1} v(\chi_i) + 2\pi i \chi_i (E_i v)_\ell\Big) e^{2\pi i T (R_i\phi)_\ell}.$$

The first term is the standard Hopf action with coefficient $L_x v$, and the remaining terms are $O(K/T + K^{-1})$.

These bounds hold in every fixed derivative order in the $T^{-1}$-scaled charts. Indeed, each derivative of a cutoff contributes $K/T \le 1$, each positive-order derivative of a phase $T(R_i\phi)$ contributes a bounded factor $T^{1-j} R_i \partial^j \phi$, and every differentiated slow

coefficient gains $T^{-1}$. Every error term retains an $E_i$ factor, a differentiated $E_i$, or a factor $K/T$, and only a bounded number of supports meet any point. Repeating the calculation for two leaf derivatives gives, for every fixed $J$, the same error bound in $C^J$ for both the first and second columns. The additional derivatives of local tangent frames and of $f$ carry a factor $T^{-1}$.

Choose $g$-orthonormal local frames. Dropping these errors gives the first and second coefficient derivatives of the fixed selector at the point $(f(x), z(x))$, under the isometry $J_x$. Their first Gram matrix is the identity and their best-phase margin has a fixed positive lower bound. The normal projector is a smooth function of the first columns on a fixed ellipticity interval. Consequently the actual maps have a uniform positive best-phase margin once $K^{-1} + K/T$ is small, and

$$\|\gamma(F_{K,T}) - T^2 g\|_{C^J_{\mathrm{scaled}}} \le C_J(K^{-1} + K/T). \tag{7.2}$$

The tensor norm here uses the scaled coordinate fields, in which the components of $T^2 g$, their inverses, and every fixed derivative are uniformly bounded.

*Uniform atlases and repair.* Use exponential coordinate balls $x = \exp_p(y/T)$ of fixed buffered $y$-radii, with centers in a maximal net of spacing proportional to $T^{-1}$. Compactness gives a positive injectivity radius and uniform coordinate estimates. Transitions and inverse transitions have uniformly bounded derivatives: derivatives of order $j \ge 2$ gain $T^{1-j}$, and first derivatives have bounded condition numbers. The packing argument bounds the enlarged overlap degree. Hence these atlases lie in one class covered by Theorem 6.1.

Take $K = T^{1/2}$. All reference derivatives of $F_{K,T}$ are uniformly bounded in these atlases, its phase and ellipticity margins are fixed, and the error in (7.2) is $O(T^{-1/2})$ in $C^{440}$. Fix a sufficiently small scaled $C^2$ displacement tolerance. The uniform theorem supplies a positive error threshold independent of $T$ and of the number of charts; we choose $T$ large enough to meet it. The repaired map $F^*$ has metric $T^2 g$ and is phase regular, so $F^*/T$ has metric $g$. Its phase margin remains positive because multiplication by a nonzero scalar does not change rank. Finally

$$\|F^*/T\|_\infty \le (\|S_C\|_\infty + \rho)/T,$$

where $\rho$ is the fixed repair tolerance. Taking $T$ sufficiently large gives the stated bound on the image. □

7.1. **Exact families with controlled value and second-form responses.** The passage from immersions to embeddings requires smooth families along which the metric is exactly fixed; the existence of nearby repaired maps does not provide such families. The following lemma uses the finite increments and the full-parameter iteration of the metric-repair proof.

**Lemma 7.2** (Paired exact-metric probes)**.** *Let $n \ge 2$, $N \ge s_n+1$, and let $F_0 : M \to \mathbb{R}^N$ be a smooth phase-regular immersion of a closed manifold. Put $h_0 = \gamma(F_0)$. Supply a finite list of waves*

$$A_\ell = \chi_\ell \nu_\ell \otimes d\psi_\ell^2,$$

*where $\chi_\ell$ is compactly supported in a coordinate chart, $d\psi_\ell$ is nonzero and good on a neighborhood of its support, and $\nu_\ell$ is a smooth normal field perpendicular to $B_{F_0}(\operatorname{Sym}^2 \ker d\psi_\ell)$ there. The fields $\nu_\ell$ may vanish.*

*There are fixed positive weights $\lambda_\ell$ and a fixed ball $|u| < \rho$ with two parameters $u_{\ell,1}, u_{\ell,2}$ for each wave such that, for every sufficiently small $t > 0$, a jointly smooth family $F_{t,u}$ on this ball satisfies $\gamma(F_{t,u}) = h_0$ and is phase regular. Put*

$$\tau = t^{6/5}, \qquad \sigma = t^{44/5}, \qquad a = t^{56/5}, \qquad c_{\ell,1} = \cos(\lambda_\ell \psi_\ell/\tau), \quad c_{\ell,2} = \sin(\lambda_\ell \psi_\ell/\tau).$$

*There are smooth orthogonal identifications of its normal bundles with the normal bundle of $F_0$ such that the identified second fundamental form $\widetilde{B}_{t,u}$ has parameter derivatives*

$$\partial_{u_{\ell,j}} \widetilde{B}_{t,u} = \sigma A_\ell c_{\ell,j} + R^B_{\ell,j}, \tag{7.3}$$

$$\partial_{u_{\ell,j}} F_{t,u} = -a\lambda_\ell^{-2} \chi_\ell \nu_\ell c_{\ell,j} + R^V_{\ell,j}. \tag{7.4}$$

*Uniformly on the fixed parameter ball,*

$$|R^B_{\ell,j}| \le \sigma(\eta_B(t) + C|u|), \qquad |R^V_{\ell,j}| \le a(\eta_V(t) + C|u|),$$

*where $\eta_B(t), \eta_V(t) \to 0$. Moreover,*

$$\|F_{t,u} - F_0\|_{C^2} \le C\sigma, \qquad \|\partial_u F_{t,u}\|_{C^1} \le Ct^{10}, \qquad \|\partial_u F_{t,u}\|_{C^2} \le C\sigma.$$

*All norms are taken in fixed finite charts. The family need not be regular in $t$, need not satisfy $F_{t,0} = F_0$, and need not have prescribed support. The remainders in the two response formulas are global.*

*Proof.* We record the parameter and tail estimates, which are stronger than the qualitative conclusion of metric repair.

*The first finite increment.* Adjoin the given probe phases to the positive-square background phases of Lemma 6.2, and give every probe and every background wave a distinct primary label, even when their unweighted phase functions coincide. Choose successively large positive weights for this finite list. On a common compact support, a sum or difference of two weighted covectors, divided by the larger weight, is as close as desired to the corresponding good covector. Thus primary, doubled, and interacting sum and difference phases are nonzero and good with fixed margins. Scaling $\psi_\ell$ by $\lambda_\ell$ and $\nu_\ell$ by $\lambda_\ell^{-2}$ leaves $A_\ell$ unchanged.

Set

$$\delta = t^{10}, \quad \ell_0 = t^{11/10}, \quad \tau = t^{6/5}, \quad \delta_1 = \tau^{10}, \quad G_0 = \sqrt{1-\delta^2}\, F_0.$$

The homogeneous finite mode solver of Lemma 6.2 is linear in its chosen free normal amplitude. Give its $\ell$th probe the complex amplitude

$$-\delta\tau\, \chi_\ell \lambda_\ell^{-2} \nu_\ell (u_{\ell,1} - i u_{\ell,2}).$$

Its leading real displacement is the corresponding combination of cosine and sine. Its formal quadratic mean, divided by $\delta^2$, is

$$H_{\mathrm{pr},\ell}(u) = \frac{\chi_\ell^2 |\nu_\ell|^2 (u_{\ell,1}^2 + u_{\ell,2}^2)}{2\lambda_\ell^2}\, d\psi_\ell^2.$$

Subtract their sum from $H_{\mathrm{first}} = (1 - (\delta_1/\delta)^2)h_0$ before performing the positive-square background decomposition. For a sufficiently small fixed parameter ball, this tensor remains in its positive decomposition neighborhood for all small $t$.

For the all-parameter correction, place the parameter ball inside a coordinate chart of a compact parameter torus, and compose its coordinate map with a smooth cutoff into the small ball that equals the original coordinates on a smaller ball. Every expression then extends smoothly to the compact parameter space, and the conclusions are restricted back to the smaller ball.

We now describe the dependence of the mean correction on $u$. The formal zero-phase mean is

$$\delta^2 \left[A_{\mathrm{tr}} + H_{\mathrm{pr}}(u) + T_{\mathrm{mean}}(A_{\mathrm{tr}}, u)\right].$$

The last term satisfies the finite difference estimate with factor $\tau/\ell_0$ proved in the finite-increment lemma. Only a primary label and its conjugate contribute to the zero phase. Consequently the background terms are independent of $u$ when $A_{\mathrm{tr}}$ is fixed, and the probe terms are quadratic in $u$. Starting from $A_0 = H_{\mathrm{first}} - H_{\mathrm{pr}}$, the finite substitutions

$$A_{j+1} = H_{\mathrm{first}} - H_{\mathrm{pr}} - T_{\mathrm{mean}}(A_j, u)$$

therefore give $\partial_u A_j = O(|u|)$, uniformly in every required finite weighted slow norm. This also follows by induction: the iterates are even near zero, their second parameter derivatives are bounded by the finite formulas, and integration along the segment from 0 to $u$ bounds their first derivatives. Their positive domains persist by the finite backward derivative estimates of the same lemma.

Take finite accuracy 440 and solve for the nonzero quadratic modes. The first actual increment $U_{0,u}$ and its error satisfy

$$|U_{0,u}|_{r,\tau} \le C_r \delta\tau, \qquad |E_{0,u}|_{r,\tau} \le C_r \left[\delta(\tau/\ell_0)^{440} + \delta^3/\tau\right],$$

where the norms count all spatial and parameter derivatives, and

$$E_{0,u} = \gamma(G_0 + U_{0,u}) - \gamma(G_0) - \delta^2 H_{\mathrm{first}}.$$

Parameter derivatives commute with the spatial mode equations and do not differentiate the oscillatory phases, so these are the finite estimates already proved, with parameters included.

Differentiating the leading probe twice in space yields $\sigma A_\ell c_{\ell,j}$; differentiating its slow coefficients instead costs a factor $\tau/\ell_0$. The background parameter response is $O(|u|\sigma)$ by the even-mean argument, and secondary quadratic corrections have parameter $C^2$ norm $O(\delta^2/\tau^2) = O(\sigma^2)$. The first increment therefore has the required second-derivative response up to $O(\sigma\tau/\ell_0 + \sigma^2 + |u|\sigma)$. Its parameter $C^1$ response is $O(\delta)$. At the level of values the leading response is the principal term in (7.4), with finite errors $O(a\tau/\ell_0 + a|u| + \delta^2)$.

*One joint exact correction with a negligible tail.* Put $G_{1,u} = G_0 + U_{0,u}$ and start the iteration at scale $t_1 = \tau$. Its normalized defect is

$$H_{1,u} = \delta_1^{-2}(h_0 - \gamma(G_{1,u})) = h_0 - \delta_1^{-2} E_{0,u}.$$

After division by $\delta_1^2$, the two error terms displayed above have powers $t^{30}$ and $t^{24/5}$. Thus the controlled initial weighted norms are bounded uniformly on the compact parameter space for all small $t$.

Apply the full-parameter iteration of Section 6.2 on the product of $M$ with this parameter space. The metric operator differentiates only in the $M$ directions, while the smoothing and observation norms include all directions. The iteration is therefore a single joint construction rather than a separate choice of solutions at different parameter values. It gives exact metric $h_0$ and joint smooth convergence.

The finite increment at stage $k_j$ and scale $t_j$ has total-order $r$ bound

$$C_r(k_j)t_j^{k_j+(6/5)(1-r)}.$$

At entry to every stage $k > 10$, add the finitely many waiting conditions $C_r(k)t_j^{k-10} \le C_*$ for $r = 1, 2, 3$. They are eventually satisfied during the preceding stage, so they do not affect the finite termination of each wait. Choose $C_* \ge \max_{r=1,2,3} C_r(10)$. Since $t_{j+1} = t_j^{6/5}$ starts at $t_1 = t^{6/5}$, summing the tail against a geometric majorant gives

$$\|\mathrm{Tail}\|_{C_x^2 C_u^1} \le 2C_* t^{228/25} = o(\sigma), \tag{7.5}$$

$$\|\mathrm{Tail}\|_{C_x^1 C_u^1} \le 2C_* t^{264/25} = o(\delta), \tag{7.6}$$

$$\|\partial_u \mathrm{Tail}\|_{C^0} \le 2C_* t^{12} = o(a). \tag{7.7}$$

These follow from total orders $3, 2, 1$, respectively. The spatial $C^2$ norm of the tail is also $O(t^{264/25}) = o(\sigma)$. All constants are independent of the initial $t$, because every stage uses the same controlled input bound and the same geometric margins; the finite waiting conditions provide this uniformity.

*The true normal form.* For the exact family the Christoffel symbols of $h_0$ are fixed:

$$B_{F_{t,u},ij} = \partial_i \partial_j F_{t,u} - \Gamma_{ij}^k(h_0)\partial_k F_{t,u}.$$

The parameter derivative of the connection term is $O(\delta)$. The maps and their first parameter derivatives are $C^1$-close to $F_0$ at scale $O(\delta)$, so their normal projectors $\Pi_{t,u}$ differ from $\Pi_0$ by $O(\delta)$ and have parameter derivatives of the same size. If $\iota$ is the inclusion of the old normal bundle in the ambient bundle, then

$$R_{t,u} = \Pi_{t,u}\iota(\iota^*\Pi_{t,u}\iota)^{-1/2}$$

is a smooth isometry from the old normal bundle to the new one. Its inverse on the new normal bundle gives the stated identification. Its parameter derivative is $O(\delta)$, so applying and differentiating it changes the second-form response by $o(\sigma)$.

Combining the estimates proves (7.3) with, for example, an error function bounded by a constant times $t^{1/10} + t^{44/5} + t^{6/5} + t^{8/25}$, plus $C|u|$. For (7.4), use $\delta^2/a = \sigma$ and (7.7); the corresponding error is bounded by a constant times $t^{1/10} + t^{44/5} + t^{4/5}$, plus $C|u|$. All of these tend to zero. The bounds on the first increment and on the tail also give the asserted uniform $C^1$ and $C^2$ estimates. Only the principal waves have the prescribed supports; the remainders are global. □

7.2. **The available normal value directions.**

**Lemma 7.3** (A quadratic saturation bound)**.** *Let $V$ be a real $n$-dimensional Euclidean space and $B : \operatorname{Sym}^2 V \to W$ a linear map. Write $K = \ker B$, $d = \dim K$, and suppose the contraction $K \to V$, $Q \mapsto Q\xi$, is injective on a nonempty open cone $\mathcal{O} \subset V^*$. For $\xi \in \mathcal{O}$ put*

$$E_\xi = \Big(B(\operatorname{Sym}^2 \ker \xi)\Big)^\perp.$$

*If $d \geq 1$, then*

$$\dim \operatorname{span}_{\xi\in\mathcal{O}} E_\xi \geq \dim W - \frac{d(d-1)}{2}.$$

*If $d = 0$, these spaces span $W$.*

*Proof.* Choose a basis $Q_1, \dots, Q_d$ of $K$ and let $\mathcal{R}(K)$ be the space of symmetric matrices $P$ satisfying

$$Q_1\xi \wedge \cdots \wedge Q_d\xi \wedge P\xi = 0 \quad \text{for every } \xi.$$

Let $\Omega$ be the open set where the first $d$ vectors are independent; it contains $\mathcal{O}$. Define $a_i(\xi) = Q_i(\xi,\xi)$ and $a = (a_1, \dots, a_d)$. The map $a$ is a submersion on $\Omega$ because $Da_i = 2Q_i\xi$. Choose $\xi_0 \in \Omega$ with $a(\xi_0) \neq 0$; such a point exists, since a nonzero quadratic polynomial cannot vanish on an open set.

For $P \in \mathcal{R}(K)$ put $p_P(\xi) = P(\xi,\xi)$. Its derivative lies in the row span of $Da$ on $\Omega$. In submersion coordinates $(a, z)$ near $\xi_0$ it is therefore independent of $z$, so $p_P = f_P \circ a$ locally. The map $P \mapsto Dp_P(\xi_0)$ on $\mathcal{R}(K)$ has image of dimension $d$: the image lies in the row span of $Da(\xi_0)$ and contains that span because $K \subset \mathcal{R}(K)$. Let $\mathcal{L}_0$ be its kernel and write $J = Da(\xi_0)$. For $P \in \mathcal{L}_0$, the chain rule gives

$$Df_P(a(\xi_0)) = 0, \qquad D^2p_P(\xi_0) = J^*C_PJ, \qquad C_P = D^2f_P(a(\xi_0)).$$

Homogeneity gives $J\xi_0 = 2a(\xi_0) =: h \neq 0$ and $D^2p_P(\xi_0)(\xi_0, \cdot) = Dp_P(\xi_0) = 0$. Since $J$ is surjective, $C_Ph = 0$. The assignment $P \mapsto C_P$ is linear and injective: $J^*C_PJ$ determines $C_P$, and the constant Hessian determines the quadratic polynomial $p_P$. Symmetric forms on $\mathbb{R}^d$ annihilating a fixed nonzero vector form a space of dimension $d(d-1)/2$. Thus

$$\dim \mathcal{R}(K) \leq d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}.$$

The annihilator of the span in the statement is

$$\Big(\operatorname{span}_{\xi\in\mathcal{O}} E_\xi\Big)^\perp = B(\mathcal{R}(K)).$$

Indeed, a vector in the left side belongs to $B(\operatorname{Sym}^2 \ker \xi)$ for every good $\xi$. Choose one preimage $P$. For each such $\xi$ there is $Q \in K$ with $(P - Q)\xi = 0$. The defining wedge polynomial for $\mathcal{R}(K)$ therefore vanishes on an open set, hence identically. Conversely, its vanishing and the independence of the $Q_i\xi$ give such a $Q$ for every $\xi \in \mathcal{O}$. A symmetric form annihilating $\xi$ lies in $\operatorname{Sym}^2 \ker \xi$, which proves the equality. Since $K \subset \mathcal{R}(K)$ is the kernel of $B$, $\dim B(\mathcal{R}(K)) \leq d(d-1)/2$, as required. For $d = 0$, the same argument gives a unique preimage $P$ annihilating every $\xi$ in an open set; hence $P = 0$. □

### 7.3. **Isometric approximation by embeddings.**

**Theorem 7.4** (Isometric approximation by embeddings)**.** *Let $n \geq 2$, $N \geq s_n + 2$, and let $F : M \to \mathbb{R}^N$ be a smooth phase-regular immersion of a closed smooth manifold. For every $\varepsilon > 0$ there is a smooth phase-regular embedding $G : M \to \mathbb{R}^N$ satisfying*

$$G^*\delta = F^*\delta, \qquad \|G - F\|_{C^2} < \varepsilon.$$

*The manifold need not be connected or orientable.*

*Proof.* Put $q = n(n-1)/2 = s_n - n$ and $m = N - n$. Phase regularity gives $\operatorname{rank} B_F \geq q$ everywhere. It is an open $C^2$ condition with a uniform positive best-phase margin on the compact manifold: finitely many local good covectors and their singular-value margins give such a neighborhood.

We use an elementary parameter-avoidance observation. Let $D : P \times Y \to \mathbb{R}^b$ be smooth, where $P \subset \mathbb{R}^p$ is open and $\dim Y = a$. If the joint derivative in source and parameter has rank at least $k > a$ at every relevant zero, then the parameters admitting such a zero form a set of measure zero. At each zero choose $k$ component equations with independent derivatives. Their zero set is locally a manifold of dimension $p + a - k < p$ containing the incidence set. Its projection to $P$ has measure zero: on countably many compact coordinate cubes the projection is Lipschitz, and covering a $d$-cube by $O(h^{-d})$ cubes gives total $p$-volume $O(h^{p-d}) \to 0$ when $d < p$. Countably many charts suffice. The same argument applies to equations for an incidence submanifold in local coordinates.

*Raising the minimal rank of the second fundamental form.* At a rank-$q$ point let $K = \ker B_F$, of dimension $n$, and $U = (\operatorname{im} B_F)^\perp$, of dimension $N - s_n \geq 2$. The normal space to the rank-$q$ stratum is $\operatorname{Hom}(K, U)$, of dimension $n(N - s_n) > n$. To see this, choose an invertible $q$-by-$q$ block and write the remaining rank condition as the vanishing of a Schur complement; its derivative is the $K$-to-$U$ block.

For every good $\xi$ at this point, $B_F(\operatorname{Sym}^2 \ker \xi) = \operatorname{im} B_F$, so its free normal space is $U$. A legal wave $\nu \otimes \xi^2$ induces on the normal space of the rank stratum the map $Q \mapsto Q(\xi, \xi)\nu$. Evaluation at covectors in any open good cone spans $K^*$, since a quadratic polynomial vanishing on an open set is zero. Hence finitely many such waves span $\operatorname{Hom}(K, U)$.

Extend the covectors as coordinate-linear phases on small charts, extend their normals by smooth projection onto the free normal spaces, and use compactly supported cutoffs equal to one on smaller charts. These projections are smooth because the restricted second form has an inverse Gram matrix on a fixed positive singular-value margin. Since the rank-$q$ locus is compact, a finite collection of such charts and waves has a common spanning margin.

Apply Lemma 7.2. After division by $\sigma$, each pair of second-form parameter columns has principal terms $A_\ell \cos\theta_\ell$ and $A_\ell \sin\theta_\ell$, whose contribution to the column Gram matrix is $A_\ell A_\ell^*$, independent of the angle. First shrink the fixed parameter ball and then take $t$ small. The global normalized remainders and the small changes in the Schur coordinates preserve the spanning margin at every actual rank-$q$ incidence near the original locus. Away from this neighborhood the $(q+1)$st singular value has a

positive minimum, which small $C^2$ changes preserve, and no rank below $q$ can occur. The avoidance observation, with source dimension $n$ and normal rank $n(N-s_n) > n$, yields a parameter with no rank-$q$ point. This gives an immersion $F_1$ with the same metric, arbitrarily $C^2$-close to $F$, with $\operatorname{rank} B_{F_1} \geq q+1$ everywhere. If the original locus is empty, take $F_1 = F$.

*Legal normal value directions.* At any point of $F_1$, put $K = \ker B_{F_1}$ and $d = \dim K$. The rank preparation gives $d \leq n-1$. A covector is good if and only if $Q \mapsto Q\xi$ is injective on $K$, so Lemma 7.3 applies and gives

$$\dim \operatorname{span}_{\xi \text{ good}} E_\xi \geq m - \frac{d(d-1)}{2} \geq (q+2) - \frac{(n-1)(n-2)}{2} = n+1.$$

For $d = 0$ the conclusion follows from the last clause of that lemma and the same lower bound on $m$. Choose $n+1$ independent vectors, each in some $E_\xi$, and extend them locally as legal normal waves as above. This requires no global good covector or normal frame.

*Separating the remaining pairs.* There is an open neighborhood $V_{\mathrm{diag}}$ of the diagonal in $M \times M$ on which no map sufficiently $C^1$-close to $F_1$ has coincidences other than at identical points. To see this, take convex source balls on which the projection of $F_1$ to its tangent plane at the center has derivative close to a fixed invertible matrix. The same property holds for nearby maps, and integration along a straight segment gives a uniform lower bound for the displacement. Finitely many smaller balls and a Lebesgue number give the assertion. Put $K_{\mathrm{pair}} = (M \times M) \setminus V_{\mathrm{diag}}$, which is compact and disjoint from the diagonal.

At each old coincidence $(x, y) \in K_{\mathrm{pair}}$ choose disjoint source neighborhoods. Choose $n+1$ independent legal normal vectors at $x$, with cutoffs equal to one near $x$ and supported away from the neighborhood of $y$. These normal vectors and the $n$ tangent vectors of $F_1$ at $x$ are independent, and remain so on a smaller product neighborhood. Finitely many such neighborhoods cover the compact coincidence set and give one finite wave list with uniform margins.

Apply Lemma 7.2 at $F_1$ to this list and consider

$$D_t(u, x, y) = F_{1,t,u}(x) - F_{1,t,u}(y).$$

Divide its parameter columns by $a = t^{56/5}$. On a selected product neighborhood, their principal paired terms at $x$ span the chosen $n+1$ normal directions, and their principal terms at $y$ vanish. The fixed nonzero weights $\lambda_\ell^{-2}$ preserve independence, and the sine-cosine pairing makes the spanning margin independent of every fast angle. The global remainder of the difference is bounded by $2(\eta_V(t) + C|u|)$ and is included in the estimate. The $x$-source columns converge to $dF_{1,x}$. After first shrinking a fixed parameter ball and then taking $t$ small, the joint derivative has rank at least $2n+1$ throughout these neighborhoods. The target dimension allows this, since $s_n + 2 \geq 2n+1$ for $n \geq 2$.

The complement of these neighborhoods in $K_{\mathrm{pair}}$ has a positive old separation margin, and uniform $C^0$ closeness excludes new coincidences there. The avoidance observation, now with source dimension $2n$ and rank $2n+1$, gives a parameter with no coincidence in

$K_{\mathrm{pair}}$. The fixed $C^1$ neighborhood handles $V_{\mathrm{diag}}$. The resulting map is globally injective, including between distinct components.

Each modification preserves the original metric. We divide the $C^2$ tolerance between the rank preparation and the pair separation, while preserving the phase margin and the prepared rank margin. The result is an injective smooth immersion within the asserted distance of $F$, and compactness of $M$ makes it an embedding. Every construction used only local frames, so nonorientable manifolds are included. □

**Corollary 7.5** (Selector assembly in the embedding range)**.** *Under the selector hypotheses of Theorem 7.1, if $N \geq s_n + 2$, then for every closed $M$ with a given isomorphism $TM \cong f^*W$ and every positive smooth metric on $M$ there is a smooth phase-regular isometric embedding in the same $\mathbb{R}^N$, with image arbitrarily small. In particular, this range includes the target $s_n + 28$ used in the main construction.*

*Proof.* Apply Theorem 7.1 and then Theorem 7.4. For a prescribed image radius, assign half the radius to the first map and half to the $C^0$ part of the second approximation. □

## 8. The corank-one source and its complete boundary class

In this section we derive the source constructions needed below from bundle transversality, including the clutching map on first and second columns. No metric realization theorem is used. As before, $s_d = d(d+1)/2$, $j_d = d + s_d$, and $J(E) = E \oplus \operatorname{Sym}^2 E$.

**Lemma 8.1** (Relative rank avoidance)**.** *Let $U$ and $W$ be smooth Euclidean bundles of ranks $u$ and $w$, $w \geq u$, over a compact smooth base of dimension $b$. If $w - u + 1 > b$, there is a smooth isometric injection $U \to W$. It can be chosen relative to an injection prescribed near a closed subset, with equality on a possibly smaller neighborhood of that subset. If $w - u + 1 > b + 1$, any two isometric injections are joined by a homotopy through isometric injections that is constant on collars of the two parameter endpoints. The relative version of this homotopy statement also holds.*

*Proof.* In the space of $w$-by-$u$ matrices, the stratum of matrices of rank $u - \ell$ has codimension $\ell(w-u+\ell)$. Indeed, on a chart with an invertible $(u-\ell)$-minor, the Schur complement is a $(w - u + \ell)$-by-$\ell$ matrix whose vanishing is the rank condition; this also shows that each fixed-rank stratum is smooth. The least codimension among the noninjective strata is $w - u + 1$.

The required transversality can be obtained in a finite-dimensional family. Choose finitely many bundle charts and subordinate cutoffs such that the associated local matrix perturbations span every fiber of $\operatorname{Hom}(U, W)$ on the compact set where perturbation is needed. Adding these perturbations with independent real coefficients makes the evaluation map a submersion there. Sard's theorem, applied to the inverse images of the finitely many rank strata, gives a parameter for which the resulting section is transverse to each stratum. The cutoffs vanish on the prescribed neighborhood, and a sufficiently small perturbation preserves the positive injectivity margin on a compact collar of that neighborhood. A transverse preimage of a stratum whose codimension exceeds the base dimension is empty.

For a homotopy, start with any smooth interpolation of the two bundle maps, constant near the endpoints, and apply the same argument on the cylinder, of dimension $b+1$, relative to its endpoint collars. Finally, replace every injective map $z$ by $z(z^*z)^{-1/2}$. This smooth polar normalization preserves isometric prescribed data and gives all the assertions. □

**Proposition 8.2** (Two source transports and complete extension)**.** *Let $k \geq 2$ be even and $m \geq 1$. Let $P$ be a closed smooth manifold of dimension $q \leq k-1$, let $K \subset P \times \mathbb{R}^k$ be a metric line bundle, and let $Q \to P$ be a Euclidean bundle of rank $m+1$. Put*

$$\mathcal{N} = \mathrm{Hom}(K, Q), \qquad C = S(\mathcal{N}), \qquad d = q+m, \qquad a = k-1, \qquad R = (m+1)a + s_a,$$

*and assume $R > d$ and $j_m > d$. Fix an orthogonal complex structure $J_c$ on $\mathbb{R}^k$ and define*

$$L = J_c K, \quad B_0 = (K \oplus L)^{\perp}, \quad A = K \oplus B_0, \quad H = Q \oplus B_0.$$

*A point $v \in C$ is a unit isometry $K \to Q$. Set $E_v = (vK)^{\perp} \subset Q$ and identify $H|_C = E_v \oplus A$ by using $v$ on the $K$ summand of $A$. There are a rank-$j_m$ Euclidean bundle $J_0 \to P$ and smooth isometries*

$$\kappa_v : J(E_v) \longrightarrow J_0,$$

*constructed from source bundles alone, with the following properties.*

*Define the bundle*

$$S_+ = A \oplus (Q \otimes A) \oplus \mathrm{Sym}^2 A$$

*over $P$. For every Euclidean target bundle $W \to P$ and every pair of monomorphisms*

$$U_0 : J_0 \longrightarrow W, \qquad A_+ : S_+ \longrightarrow W$$

*with orthogonal images, the boundary map*

$$B_v(x+s) = U_0 \kappa_v(x) + A_+(s), \qquad x \in J(E_v), \quad s \in A \oplus (E_v \otimes A) \oplus \mathrm{Sym}^2 A, \tag{8.1}$$

*extends to a smooth monomorphism $J(H) \to W$ over $D(\mathcal{N})$ which equals the radial pullback of (8.1) on a boundary collar. All bundles over the disk in this assertion are pulled back from $P$. If an additional injection $\ell : L \to W$ has image perpendicular to $U_0(J_0) + A_+(S_+)$, the extension of $J(H) \oplus L$ can keep $\ell$ fixed.*

*If $B_{\mathrm{ext}}$ is a compact manifold with boundary $C$ and $E_{\mathrm{ext}} \to B_{\mathrm{ext}}$ is a rank-$m$ bundle identified with $E_v$ on a product boundary collar, there is a smooth rank-$j_m$ bundle on $B_{\mathrm{ext}} \cup_C D(\mathcal{N})$ satisfying*

$$J_{\mathrm{ext}}|_{B_{\mathrm{ext}}} = J(E_{\mathrm{ext}}), \qquad J_{\mathrm{ext}}|_{D(\mathcal{N})} = J_0. \tag{8.2}$$

*Its transition map on $C$ is $\kappa_v$. This bundle, like $\kappa_v$, is chosen before any target data.*

*Proof.* All bundles on $C$ not indexed by $v$ are pulled back from $P$. We construct the two transports in turn.

*First transport.* Let $Z = K \otimes A$, of rank $a$. The map $v$ defines an isometric injection $z_v : Z \to S_+$ into the $Q \otimes A$ summand, with orthogonal complement

$$S_v = A \oplus (E_v \otimes A) \oplus \mathrm{Sym}^2 A.$$

Thus $\operatorname{rank} S_+ = R + a$ and $\operatorname{rank} S_v = R$. By Lemma 8.1, choose an isometric injection $z_0 : Z \to S_+$ over $P$ and write $S_0 = z_0(Z)^\perp$. The two injections $z_v$ and $z_0$ over $C$ have rank gap $R > d = \dim C$, so the same lemma on $C \times [0,1]$ gives an isometric path $z_t$ from $z_v$ to $z_0$, constant near $t = 0, 1$.

We use the following orthogonal lifting formula twice. If $z_t$ is a path of isometric injections, put

$$C_t = z_t^* \dot{z}_t, \qquad \Omega_t = \dot{z}_t z_t^* - z_t \dot{z}_t^* - z_t C_t z_t^*. \tag{8.3}$$

Differentiating $z_t^* z_t = \mathrm{Id}$ shows that $C_t$ and $\Omega_t$ are skew-adjoint and that $\Omega_t z_t = \dot{z}_t$. The solution of $\dot{R}_t = \Omega_t R_t$, $R_0 = \mathrm{Id}$, is therefore orthogonal and satisfies $R_t z_v = z_t$. This is a global linear ordinary differential equation on a compact parameter family, so $R_t$ depends smoothly on all parameters. In particular,

$$\psi_v = R_1|_{S_v} : S_v \longrightarrow S_0$$

is an isometry.

*Second transport.* The splitting $H|_C = E_v \oplus A$ induces

$$J(H)|_C = J(E_v) \oplus S_v.$$

With the usual normalized inclusion of mixed symmetric tensors, this is an orthogonal decomposition. Its summands have ranks $j_m$ and $R$; in particular $\operatorname{rank} J(H) = j_m + R$. Over $P$, rank avoidance gives an isometric injection $i_0 : S_0 \to J(H)$, since the rank gap $j_m$ exceeds $q$. Put $J_0 = i_0(S_0)^\perp$. Over $C$, define $i_v : S_0 \to J(H)$ as $\psi_v^{-1}$ followed by the inclusion of $S_v$. The gap $j_m > d$ gives an isometric homotopy from $i_v$ to $i_0$, and formula (8.3) lifts it to orthogonal maps $T_t$ of $J(H)$ with $T_0 = \mathrm{Id}$ and $T_1 i_v = i_0$. Taking orthogonal complements yields the required isometry

$$\kappa_v = T_1|_{J(E_v)}.$$

Neither transport involves $W$ or a target map.

*Extension of the complete injection.* Starting from (8.1), use the path

$$x + s \longmapsto U_0 \kappa_v(x) + A_+ R_t(s).$$

It consists of injections because its two summands lie in the fixed orthogonal subbundles $U_0(J_0)$ and $A_+(S_+)$. Define an injection independent of $v$ by

$$F_0(y + i_0 z) = U_0 y + A_+ z, \qquad y \in J_0, \quad z \in S_0.$$

The preceding path ends at $F_0 T_1$: on $J(E_v)$ this follows from the definition of $\kappa_v$, and on $S_v$ from $i_v \psi_v = \mathrm{Id}_{S_v}$ and $T_1 i_v = i_0$. Follow this path by the reverse path $F_0 T_t$ from $F_0 T_1$ to $F_0$. Reparametrize both paths to be constant near their endpoints and place their concatenation in a radial annulus of $D(\mathcal{N})$. Keep $B_v$ radially constant on an outer collar and use the pullback of $F_0$ near the zero section. The latter is independent of the angular variable $v$, so the resulting extension is smooth at the zero section. Its image lies in $U_0(J_0) + A_+(S_+)$, which proves the assertion about $\ell$.

Finally, use $\kappa_v$ as the transition isomorphism between $J(E_{\mathrm{ext}})$ and the pullback of $J_0$ on product collars. Gluing bundle charts along this smooth invertible transition gives

the bundle in (8.2), with the stated restrictions. This construction is independent of $U_0, A_+$, and $W$. □

**Proposition 8.3** (A generic frame with an exact normal form)**.** *Let $V \to B$ be a rank-$n$ Euclidean bundle over a closed smooth $n$-manifold. Write $n = m + k$ with $m \geq 1$ and $k \geq 2$ even, and assume*

$$2(m+2) > n, \qquad (m+1)(k-1) + s_{k-1} > n-1, \qquad j_m > n-1. \tag{8.4}$$

*There is a smooth bundle map $\varphi : B \times \mathbb{R}^k \to V$ whose rank defects form a closed smooth corank-one locus $P$ of dimension $k-1$, possibly empty. Its kernel line $K$, cokernel bundle $Q$, and normal bundle have ranks $1$, $m+1$, and $m+1$, respectively, and the derivative identifies its normal bundle canonically with $\mathrm{Hom}(K, Q)$.*

*After a modification of $\varphi$ in a tubular neighborhood that introduces no new rank defects, its expression on a smaller tube is*

$$\mathrm{diag}(A_{\mathrm{iso}}, \sigma),$$

*where $A_{\mathrm{iso}} : K^\perp \to \mathrm{im}(\varphi|_P)$ is a radially pulled-back isometry and $\sigma \in \mathrm{Hom}(K, Q)$ is the actual normal coordinate. The exterior old bundle $E_{\mathrm{ext}} = (\mathrm{im}\,\varphi)^\perp$ has rank $m$ and boundary restriction $E_v$. With $L, B_0, A, H$ as in Proposition 8.2, the line $A_{\mathrm{iso}}L$ is a surviving line in $V$, its orthogonal complement is identified with $H = Q \oplus B_0$, and that proposition supplies the complete transports, the bundle $J_{\mathrm{ext}}$, and the complete boundary extension.*

*Proof.* Apply the finite-dimensional transversality argument of Lemma 8.1 to a section of $\mathrm{Hom}(\varepsilon^k, V)$. The stratum of corank $\ell$ has codimension $\ell(n-k+\ell)$. For $\ell \geq 2$ this is at least $2(m+2) > n$, so these strata are avoided. The corank-one stratum has codimension $m+1$, so its transverse inverse image is the stated $P$. The kernel $K$, image $I$, and orthogonal cokernel $Q$ vary smoothly over $P$. The normal space to the corank-one stratum at a map is canonically $\mathrm{Hom}(K, Q)$, as the Schur complement in the rank formula shows, and the derivative identifies the normal bundle of $P$ with this bundle.

Choose a tubular neighborhood, radial identifications of its domain and target bundles with their restrictions to $P$, and the splittings $K^\perp \oplus K$ and $I \oplus Q$. On a sufficiently small tube the upper-left block $A$ is invertible. Writing the remaining blocks as $b, c, d_0$, the Schur complement

$$\sigma = d_0 - cA^{-1}b$$

is a fiberwise defining coordinate for $P$. Its normal derivative is an isomorphism, so, after shrinking the tube, the map taking a point to its footpoint and $\sigma$ is a diffeomorphism onto a neighborhood of the zero section in $\mathrm{Hom}(K, Q)$. This is the inverse function theorem along a compact submanifold; injectivity follows by taking a common sufficiently small tube. We use $\sigma$ as the fiber coordinate. The exact factorization is

$$\varphi = \begin{pmatrix} 1 & 0 \\ cA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 & A^{-1}b \\ 0 & 1 \end{pmatrix}. \tag{8.5}$$

Multiply the off-diagonal entries of the two shear matrices by a radial cutoff equal to zero on an inner tube and to one near the outer edge. Both shears remain invertible, so the only rank defect is still $\sigma = 0$.

Let $A_{\mathrm{iso}}$ be the polar isometry of $A$ on $P$, pulled back radially. On $P$ the operator $A_{\mathrm{iso}}^* A$ is positive definite, and on a smaller tube its symmetric part remains positive definite. Consequently every convex combination of $A$ and $A_{\mathrm{iso}}$ is invertible: after multiplication by $A_{\mathrm{iso}}^*$, its symmetric part is a convex combination of positive definite operators. Another radial cutoff therefore changes $A$ to $A_{\mathrm{iso}}$ on the inner tube, keeps the old map near the outer edge, and introduces no new rank defect. This proves the exact normal form.

Since $J_c$ is orthogonal and $J_c^2 = -\mathrm{Id}$, $J_c K$ is orthogonal to $K$ and canonically isomorphic to it; no orientation or unit section of $K$ is needed. The isometry $A_{\mathrm{iso}}$ identifies $K^\perp = L \oplus B_0$ with $I$, and hence identifies $V$ on the inner tube with $L \oplus (Q \oplus B_0)$. At a boundary direction $v$, the span of the frame is $A_{\mathrm{iso}}(K^\perp) \oplus vK$, with complement $E_v$. Polar normalization of the frame on the exterior gives the same orthogonal identification, including a product collar. On this collar, use radial metric connections preserving the orthogonal summands $E_v$, $A$, and $L$, and write $I_{r,r'}$ for their radial isometric transports. Before fixing the coefficient model, choose a smooth function $\rho(r)$ equal to the boundary value on a smaller collar and equal to $r$ outside a larger one, and replace each coefficient injection $C_r$ there by $C_{\rho(r)} I_{r,\rho(r)}$. Make the same pullback choice for the complete formal data, using $I \oplus \mathrm{Sym}^2 I$ on complete columns, and take the leaf connection to be a radial pullback. These choices preserve all boundary values, injection ranks, the protected hyperplane, and the given fiber metric, and they are homotopic to the original choices relative to the boundary and the complement of the collar. Thus the coefficient data are product data on a fixed smaller collar. Here $q = k - 1$ and $\dim S(\mathrm{Hom}(K, Q)) = n - 1$, so the last two inequalities of (8.4) are the two hypotheses of Proposition 8.2. If $P$ is empty, the exterior is all of $B$ and no core extension is required. □

**Lemma 8.4** (Isometric coefficient extension)**.** *In the setting of Proposition 8.2, fix an isometric injection $I_Q : Q \to \varepsilon^b$ over $P$. The boundary coefficient injection*

$$H = E_v \oplus vK \oplus B_0 \longrightarrow \varepsilon^b \oplus \varepsilon^k,$$

*equal to $I_Q$ on $E_v$, $v^*$ on $vK$, and the inclusion on $B_0$, extends isometrically over $D(\mathcal{N})$ with a radial boundary collar. The extension can keep the surviving $L$ coefficient as the unchanged inclusion in $\varepsilon^k$.*

*Proof.* For $0 \le t \le \pi/2$, $q_1 \in Q$, and $b_1 \in B_0$, define on the sphere bundle

$$C_t(q_1 + b_1) = \big(I_Q(P_{E_v} q_1 + \cos t\, P_{vK} q_1), \sin t\, v^* q_1 + b_1\big). \tag{8.6}$$

The two terms in the second coordinate lie in the perpendicular subbundles $K$ and $B_0$ of $\varepsilon^k$, so the squared norm is

$$|P_{E_v} q_1|^2 + \cos^2 t\, |P_{vK} q_1|^2 + \sin^2 t\, |P_{vK} q_1|^2 + |b_1|^2 = |q_1|^2 + |b_1|^2.$$

By polarization, $C_t$ preserves inner products. At $t = \pi/2$ it is the stated boundary map, and at $t = 0$ it is $(I_Q q_1, b_1)$, independent of $v$. Let $t$ be a smooth radial function equal to

zero near the zero section and to $\pi/2$ on a boundary collar. Formula (8.6) then gives a smooth isometric extension over the disk. Its second component lies in $K \oplus B_0$, which is perpendicular to $L$, so appending the unchanged inclusion of $L$ preserves isometry. The formulas involve only $v, v^*$ and orthogonal projectors, and hence remain valid under sign changes of the possibly nonorientable line $K$. □

Propositions 8.2 and 8.3 provide the source data for the global construction. The extension they give is an extension of a complete formal injection. To apply a relative exact free realization, the actual incoming complete jet must first be connected to (8.1) by a homotopy through injections. This homotopy is supplied by the protected activity and the complete-class arguments below, and is not constructed here for an arbitrary actual boundary map.

## 9. A flat coefficient model with seven labels

This section constructs the coefficient model of the exterior construction and proves its ordinary and curvature regularity. We work with a Euclidean vector bundle $E$ of rank $M$; its base dimension is specified in each avoidance argument. All incidence dimensions are computed after complexification, while the actual bundle maps and metrics are real. We write $\mathrm{Sym}(E)$ for selfadjoint endomorphisms and use algebraic transposes after complexification. A subspace of $E_{\mathbb{C}}$ need not be nondegenerate for its restricted bilinear form.

9.1. **The coefficient space and its flat Gram tensor.** Let $e_0, e_1, e_2$ be the standard basis of $\mathbb{R}^3$, and put

$$\begin{aligned} p_1 &= e_0 \otimes (e_0 - e_1), & p_2 &= (e_0 - e_1) \otimes e_2, \\ p_3 &= e_2 \otimes (e_1 - e_2), & p_4 &= (e_1 - e_2) \otimes e_0, \\ p_5 &= (e_0 + e_1 + e_2) \otimes (e_0 + e_1 + e_2). \end{aligned}$$

These five tensors are mutually orthogonal. Their orthogonal complement, identified with a space of 3 by 3 matrices, is

$$\mathcal{S} = \left\{ C(u,v,w,z) = \begin{pmatrix} u & u & v \\ w & -2(u+v+w+z) & v \\ w & z & z \end{pmatrix} : u,v,w,z \in \mathbb{R} \right\}. \tag{9.1}$$

The orthogonal projection onto $\mathcal{S}$ is

$$P = I - \frac{1}{2}\sum_{i=1}^{4} p_i p_i^t - \frac{1}{9} p_5 p_5^t.$$

Choose an isometry $\mathcal{S} \simeq \mathbb{R}^4$ and let $L : \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}^4$ be the resulting quotient map, so that $L^*L = P$. Write $L_{i\rho b}$ for its entries and $B_\rho = L(e_\rho \otimes -)$. The identity and each product projector are invariant under partial transpose. Consequently,

$$\sum_i L_{i\rho b} L_{i\sigma c} = \sum_i L_{i\rho c} L_{i\sigma b}. \tag{9.2}$$

This is the coefficient flatness identity.

**Lemma 9.1** (The coefficient seed)**.** *The complexification of $\mathcal{S}$ has dimension four and contains no nonzero rank-one matrix. For each nonzero $\lambda \in \mathbb{C}^3$, the evaluation $C \mapsto C\lambda$ has rank at least two. It has rank two at exactly six projective directions,*

$$[1:-1:0],\ [0:0:1],\ [0:1:-1],\ [1:0:0],\ [1:1:1],\ [-2:1:-2], \tag{9.3}$$

*and has rank three elsewhere. There are exactly six constant-row-space coefficient pencils and exactly six constant-column-space coefficient pencils; all are real.*

**Lemma 9.2** (The determinant cubic)**.** *The surface $\Sigma = \{[C] \in \mathbb{P}\mathcal{S}_{\mathbb{C}} : \det C = 0\}$ is a smooth cubic surface. In particular, its set of projective lines is finite.*

*Proof.* In the coordinate order $(u, v, w, z)$ its equation is

$$f = -2u^2z - 3uvz - 3uwz - 2uz^2 + 3uvw + 2v^2w + 2vw^2 + 3vwz. \tag{9.4}$$

Smoothness has the following finite exact certificate. List monomials in ascending lexicographic exponent order. Form the 56 by 80 integer matrix whose columns are the coefficients of each of the four partial derivatives of $f$ multiplied by each degree-three monomial, in degree five. Its square minor on zero-based columns

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,$$
$$20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32, 34, 35, 37, 38, 39, 40, 41,$$
$$42, 43, 44, 45, 47, 48, 49, 55, 58, 59, 60, 61, 62, 63, 64, 65, 69, 79$$

is 152546 modulo the prime 1000003. All entries are determined by (9.4), so Gaussian elimination in this finite field is an exact rational-rank certificate. Hence the Jacobian ideal contains every degree-five monomial over $\mathbb{Q}$, so its generators have no common nonzero complex zero, and the surface is smooth. This also proves that $\mathcal{S}_{\mathbb{C}}$ has no nonzero rank-one matrix: all cofactors, and hence all four partial derivatives of $f$, vanish at such a matrix.

By adjunction, a line on a smooth cubic surface has self-intersection $-1$, and its normal bundle in the surface is $\mathcal{O}_{\mathbb{P}^1}(-1)$. Hence its embedded infinitesimal deformation space is zero, and the projective scheme parametrizing lines is zero dimensional, hence finite. □

*Proof of Lemma 9.1.* It remains to prove the assertions about evaluations and pencils. For $a = (x, y, z)$ and $\lambda = (p, q, r)$, the equations $a^tC\lambda = 0$ for every $C \in \mathcal{S}$ are

$$x(p+q) = 2yq, \quad r(x+y) = 2yq, \quad z(q+r) = 2yq, \quad p(y+z) = 2yq.$$

If $yq = 0$, the nonzero projective tensor solutions are $e_0 \otimes (1, -1, 0)$, $(1, -1, 0) \otimes e_2$, $e_2 \otimes (0, 1, -1)$, and $(0, 1, -1) \otimes e_0$. If $yq \neq 0$, normalize $y = q = 1$. Elimination gives $p^2 + p - 2 = 0$ and the two solutions $(1, 1, 1) \otimes (1, 1, 1)$ and $(-2, 1, -2) \otimes (-2, 1, -2)$. The eliminated denominators are nonzero by the displayed equations, so the list is exhaustive.

An evaluation of rank at most one would have a kernel of dimension at least three inside 3 by 2 matrices. Its projective plane would meet the rank-one Segre variety $\mathbb{P}^2 \times \mathbb{P}^1$ in $\mathbb{P}^5$, contrary to the first assertion. Therefore the six drops have rank exactly two, and each gives a two-dimensional kernel pencil with a fixed row space. Conversely,

every constant-row-space pencil has a common right kernel and occurs in this list. The transpose argument gives the constant-column-space pencils, and all factors in the list are real. □

9.2. **Graphs, labels, and regularity.** For $A_1, A_2, A_3 \in \mathrm{Sym}(E)$ define

$$S_{ib} = \sum_{\rho=1}^{3} L_{i\rho b} A_\rho, \qquad J_A(Y) = \left( -\sum_{b=1}^{3} S_{ib} y_b \right)_{i=1}^{4} \oplus Y, \quad Y = (y_1, y_2, y_3). \tag{9.5}$$

Let $V$ have rank $M + C$, let $D : E \otimes \mathbb{R}^3 \to V$, and put $K = J_A(\ker D) \subset E \otimes \mathbb{R}^7$. Its complex bad-label set is

$$\mathcal{B}_K = \{H \in \mathrm{Gr}_5(\mathbb{C}^7) : K_{\mathbb{C}} \cap (E_{\mathbb{C}} \otimes H) \neq 0\}. \tag{9.6}$$

Equivalently, one may use the normal two-plane $U = H^\perp$, of complex parameter dimension ten. The weak condition we require is $\dim_{\mathbb{C}} \mathcal{B}_K \leq 4$.

The corresponding seven maps $T_a : E \to E^4 \oplus V$ are the four canonical inclusions and the three maps $(S_{ib})_i \oplus D_b$. Their ordinary operator is

$$\mathcal{L}_T : (E^4 \oplus V)^7 \longrightarrow E \otimes \Lambda^2 \mathbb{R}^7, \qquad (v_a) \longmapsto (T_a^* v_b - T_b^* v_a)_{a<b}.$$

The curvature tensor is $\Omega(T)_{ab} = T_a^* T_b - T_b^* T_a$. For the top blocks, (9.2) gives

$$\Phi(S)_{bc} = \sum_i [S_{ib}, S_{ic}] = 0.$$

**Lemma 9.3** (Full coefficient evaluation)**.** *The map*

$$\mathrm{Sym}_3^3 \longrightarrow \mathrm{Mat}_{4,3}, \qquad (H_1, H_2, H_3) \longmapsto \sum_\rho B_\rho H_\rho$$

*is onto. The map*

$$\mathfrak{so}(3)^3 \longrightarrow \mathrm{Mat}_{4,3}, \qquad (X_1, X_2, X_3) \longmapsto \sum_\rho B_\rho X_\rho$$

*is injective. In particular, for any two distinct coefficient basis directions, the map* $(X, Y) \mapsto B_\rho X + B_\sigma Y$ *on* $\mathfrak{so}(3)^2$ *is injective.*

*Proof.* Use the integer coefficient basis in the order $(u, v, z, w)$ before Gram normalization. Order the symmetric basis by $(00), (01), (02), (11), (12), (22)$, using entries one in both positions off the diagonal. Flatten the 4 by 3 output row by row. The square minor of the evaluation matrix on zero-based columns $0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 13, 14$ has determinant 20. For the skew statements, order the skew basis by $(01), (02), (12)$ and use zero-based output-row indices. The twelve-by-nine matrix of the triple map, with columns ordered first by $\rho = 0, 1, 2$ and then by the skew basis, has the nine-row minor on rows $0, 1, 2, 3, 4, 5, 6, 7, 10$ of determinant $-4$. The pairs $(0, 1), (0, 2), (1, 2)$ also have nonzero six-row minors on $(0, 1, 2, 3, 4, 5)$, $(0, 2, 3, 6, 7, 10)$, and $(0, 2, 3, 6, 7, 10)$, of respective determinants $-4, -1, -4$. Invertible Gram normalization of the four output rows preserves all these ranks. □

**Lemma 9.4** (Ordinary and curvature regularity)**.** *Let* $A = (A_1, A_2, A_3) \in \mathrm{Sym}(E)^3$ *and let* $S = S(A)$, $S_{ib} = \sum_\rho L_{i\rho b} A_\rho$, *be as in* (9.5).

(i) *The real bad locus of symmetric triples with a common nonzero commutator-kernel vector is closed and has codimension at least $2M-3$. Outside this locus the seven-map ordinary operator is onto for every $D$.*

(ii) *The selfadjoint curvature differential*

$$D\Phi_S : \mathrm{Sym}(E)^{4\times 3} \longrightarrow \Lambda^2\mathbb{R}^3 \otimes \mathfrak{so}(E), \qquad (\dot S_{ib}) \longmapsto \Big(\sum_i \big([\dot S_{ib}, S_{ic}] + [S_{ib}, \dot S_{ic}]\big)\Big)_{b<c},$$

*is onto if and only if $A_1, A_2, A_3$ have no nonzero common skew-adjoint commutant. The triples for which it fails form a closed semialgebraic set of codimension at least $2M$. A triple whose first two operators have no nonzero common skew-adjoint commutant lies outside this set; in particular so does a regular pair with the third coefficient operator set to zero. Surjectivity of $D\Phi_{S'}$ persists for all selfadjoint arrays $S'$ near $S$.*

*Proof.* For the first count, fix a line $\mathbb{R}v$, put $d = M-1$, and write $A_i = \left(\begin{smallmatrix} a_i & b_i^t \\ b_i & C_i \end{smallmatrix}\right)$. The equations $[A_i, A_j]v = 0$ are $C_ib_j - C_jb_i = a_ib_j - a_jb_i$. If the three $b_i$ span an $h$-plane, their rank condition costs $(d-h)(3-h)$ equations. For $h = 0, 1, 2, 3$, the linear equations in the $C_i$ have respective ranks $0, 2d, 3d-1, 3d-1$. For $h = 3$ their only relation is the cyclic symmetry relation among the three outputs; the smaller cases follow by one- and two-column symmetric evaluation. After returning the $d$ parameters of the line, the four codimension bounds are $2d, 3d-2, 3d-3, 2d-1$, each at least $2M-3$. The locus is the projection of a closed set along the unit sphere of $E$, hence closed.

An adjoint witness for $\mathcal{L}_T$ is a skew-label array $z_{ab}$ with $\sum_b T_b z_{ab} = 0$. These equations take values in $E^4 \oplus V$, and their $E^4$ component is the witness system for $D = 0$, so it suffices to treat $D = 0$. The last-label equations determine the top–last entries of the witness, and the top-label equations determine its top–top entries. Skew symmetry of the latter leaves

$$\sum_{b<c} ([S_{jb}, S_{ic}] + [S_{ib}, S_{jc}])z_{bc} = 0 \qquad (1 \le i \le j \le 4).$$

The coefficients of $[A_\rho, A_\sigma]z_{bc}$ form a 10 by 9 matrix. With lexicographic pair order and the integer basis above, its first nine rows have determinant 448. Hence every $z_{bc}$ belongs to the common kernel of the three commutators, proving ordinary regularity.

For the curvature statement, pair $D\Phi_S(\dot S)$ with a skew array $z_{bc} = -z_{cb}$ of skew-adjoint operators by $\sum_{b<c} \mathrm{tr}(z_{bc}^t Y_{bc})$. Since $\mathrm{tr}(z[X,Y]) = \mathrm{tr}(X[Y,z])$ and both arrays are skew in $(b,c)$, the pairing equals $-\sum_{i,b} \mathrm{tr}(\dot S_{ib} W_{ib})$, where

$$W_{ib} = \sum_c [S_{ic}, z_{bc}] = \sum_\rho \sum_c L_{i\rho c}[A_\rho, z_{bc}]$$

is selfadjoint. As $\dot S$ is arbitrary, $z$ is an adjoint witness exactly when $W = 0$. Put $(X_\rho)_{cb} = [A_\rho, z_{bc}]$; then $X_\rho \in \mathfrak{so}(3) \otimes \mathrm{Sym}(E)$ and $W = \sum_\rho B_\rho X_\rho$. The triple injectivity of Lemma 9.3, tensored with $\mathrm{Sym}(E)$, gives $X_\rho = 0$. Hence every $z_{bc}$ is a common skew-adjoint commutant of $A_1, A_2, A_3$, and the witness vanishes when no nonzero one exists. Conversely, a nonzero common skew-adjoint commutant $K$ gives the witness $z_{01} = -z_{10} = K$, with all other entries zero.

A nonzero common skew commutant has a nonzero frequency space of real dimension $2s$ carrying an orthogonal complex structure. The parameters for that space and structure have dimension $2s(M-2s)+s(s-1)$. Each symmetric operator preserving it and commuting with the complex structure has codimension $2s(M-2s)+s(s+1)$. The projected pair locus therefore has codimension $s(2M-3s+3)\geq 2M$. The triple locus lies in the product of this pair locus with the third factor, so its codimension is at least $2M$. It is the projection of a closed semialgebraic set along the unit sphere of $\mathfrak{so}(E)$, hence closed and semialgebraic. Surjectivity is open. □

## 10. Symmetric evaluation and the incidence estimates

This section bounds the cokernels of symmetric evaluation without nondegeneracy assumptions and derives from these bounds the incidence estimates used in the exterior count.

### 10.1. **Cokernels without nondegeneracy assumptions.**

**Lemma 10.1** (Symmetric evaluation)**.** *If $T\subset \mathbb{C}^R\otimes W$ has dimension $t$ and $\dim W=h$, evaluation of $R$ symmetric old operators on $T$ has cokernel dimension at most*

$$\chi(t,h)=\left\lfloor \frac{t}{h}\right\rfloor \binom{h}{2}+\binom{t-h\lfloor t/h\rfloor}{2}. \tag{10.1}$$

*In particular this is at most $\binom{t}{2}$.*

*Proof.* The cokernel is the tensor intersection $(T\otimes W)\cap(\mathbb{C}^R\otimes\Lambda^2 W)$; indeed, by alternating symmetry, once one old factor lies in $W$ so does the other. Choose a diagonal one-parameter subgroup on the coefficient and old factors with distinct weights on all coordinate tensors. The Grassmann limit is a coordinate subspace $\bigoplus_\rho e_\rho\otimes W_\rho$. Intersection dimension is upper semicontinuous, so the desired dimension is at most $\sum_\rho\binom{\dim W_\rho}{2}$. Given total dimension $t$ and each summand dimension at most $h$, convexity gives (10.1). The argument applies equally to an isotropic $W$. □

Let $B_1,\dots,B_R:\mathbb{C}^u\to\mathbb{C}^h$ be jointly injective and define

$$\mathcal{Z}=\{Z\in \mathrm{Mat}_{h,u}: B_\rho Z^t \text{ is alternating for every } \rho\}.$$

Let $r$ be its maximum matrix rank. If $r>0$, compatible base changes put a maximum-rank witness in diagonal form and give

$$B_\rho=\begin{pmatrix} K_\rho & D_\rho\\ 0 & F_\rho\end{pmatrix},\qquad Z=\begin{pmatrix} A & U\\ V & 0\end{pmatrix},\qquad K_\rho^t=-K_\rho. \tag{10.2}$$

The zero block follows from maximality of the rank in every pencil. The $K_\rho$ are jointly injective, and their images span the first $r$-plane. The off-diagonal equations determine $V$ from $U$.

**Lemma 10.2** (Cokernel bounds)**.** *No nonzero member of $\mathcal{Z}$ has rank one. For maximum rank $r \geq 2$,*

$$\dim \mathcal{Z} \leq ru - \frac{r(r+1)}{2}, \tag{10.3}$$

$$\dim \mathcal{Z} \leq r(u-2) + 1. \tag{10.4}$$

*Furthermore, on putting $K_r = \binom{r}{2}$,*

$$\dim \mathcal{Z} \leq \Phi_R(r,h,u) := \max_{0\leq b\leq r} \min\{b\max(r,u-r) + K_r, (r-b)\max(r,h-r) + RK_r\}. \tag{10.5}$$

*Proof.* A rank-one alternating matrix is zero; hence a rank-one witness would contradict joint injectivity. In (10.2), the homogeneous $A$-fiber for fixed $U$ has dimension at most $\binom{r}{2}$ by Lemma 10.1, giving $r(u-r) + \binom{r}{2}$. For (10.4), one nonzero $K_\rho$ of rank $s \geq 2$ makes the symmetric part of $K_\rho A^t$ have rank $sr - s(s-1)/2 \geq 2r-1$ as a linear function of $A$.

Let $b, c$ be the maximum ranks of the $U, V$ block spaces. A generic matrix attains both, and a block minor gives rank at least $b + c$; thus $b + c \leq r$. Flanders' bound for rectangular matrix spaces [Fla62] gives dimensions at most $b\max(r, u-r)$ and $c\max(r, h-r)$ for these two projected spaces. The kernel of projection onto $U$ has dimension at most $K_r$. In the kernel of projection onto $V$, all outputs lie in the first old $r$-plane. The injective map $Z \mapsto (B_\rho Z^t)_\rho$ takes it into $R$ copies of that plane's alternating tensors, giving dimension at most $RK_r$. Taking both bounds and $c \leq r - b$ proves (10.5). □

When $R = 4$, $h \geq u$ and $1 \leq r \leq u/2$, set $x = u - r$, $y = h - r$. A weighted average of the two projection bounds gives

$$\dim \mathcal{Z} \leq F_{h,u}(r) := \frac{rxy}{x+y} + \binom{r}{2}\frac{4x+y}{x+y}. \tag{10.6}$$

This function is increasing on the stated interval, since

$$\left(\frac{rxy}{x+y}\right)' \geq \frac{x^2(y-x)}{(x+y)^2} \geq 0,$$

while $\kappa = (4x+y)/(x+y)$ satisfies $\kappa \geq 1$ and $\kappa' \geq -3/(2r)$, giving $(\binom{r}{2}\kappa)' \geq (r+1)/4 > 0$.

**Lemma 10.3** (A single coefficient evaluation)**.** *Suppose every witness in $\mathcal{Z}$ has rank at most $2L$. For a coefficient combination $B_d$ put $a = \dim\ker B_d$. Then*

$$\dim \mathcal{Z} \leq \binom{a}{2} + \max\left\{\binom{2L+1}{2}, \frac{L(2(u-a)-L-1)}{2}\right\}. \tag{10.7}$$

*Proof.* The image of $Z \mapsto B_d Z^t$ lies in $\Lambda^2(\operatorname{im} B_d)$ and has upper rank $2L$. The alternating bounded-rank theorem [SP16, Theorem 1.3] gives the maximum in (10.7). If $2L \geq u - a$, its first term bounds the whole alternating space. In the kernel,

$\operatorname{im} Z^{t} \subset \ker B_{d}$. The joint coefficient input restricted to that kernel has dimension $a$, so Lemma 10.1 bounds this kernel by $\binom{a}{2}$. Rank-nullity proves the assertion. □

10.2. **Dominating tuples and overlap dimensions.** We permit an independent positive metric coefficient $A_{0}$ in the exterior count. A normal label $(z,\xi)\in\mathbb{C}^{4}\oplus\mathbb{C}^{3}$ then has augmented coefficient matrix $(C_{z};-\xi)$, and its equation is $\sum_{\rho}C_{z,\rho b}A_{\rho}y_{b}-\sum_{b}\xi_{b}A_{0}y_{b}=0$. Congruence by $A_{0}^{-1/2}$ returns the standard metric. Thus this device adds an algebraic symmetric parameter but no target coordinate. The same bound applies to the three top-operator coefficients before normalization. In the core row-expansion arguments the metric coefficient is retained as a fourth augmented direction, as specified in the core section and in Lemma 10.7.

Take an irreducible component of the incidence of $([X],U)$ with $[X]\in\mathbb{P}K$ and $U$ annihilating $X$, whose normal-label image has dimension $d$. Choose a projective tuple of length $\ell$ spanning its matrix span together with its complex conjugate. The allocation of tuple entries between the two components is fixed on a finite stratum. Independence is an open condition in the product, whose fiber has dimension at least $\ell d$. All subsequent ranks are generic ranks on a dominating stratum of this product. The qualification is needed, since the $\ell d$ parameters cannot be returned after the product is replaced by an arbitrary smaller subvariety.

For real data the total column spans of the component and its conjugate are complexifications of real spaces. Their Euclidean forms are nondegenerate. This justifies the orthogonal block coordinates in the disjoint count below; the individual witness spaces need not be nondegenerate.

Write

$$r_{i}=\operatorname{rank}Y_{i}=3-\gamma_{i},\quad l_{i}=\dim(U_{i}\cap\mathbb{C}^{3}),\quad t_{i}=2-l_{i},\quad t=\sum_{i}t_{i},\tag{10.8}$$

$$e=\sum_{i}(\gamma_{i}-l_{i}),\quad h=\dim\sum_{i}\operatorname{col}Y_{i},\quad \delta=\sum_{i}r_{i}-h,\quad \tau=\delta+e.$$

Let $E_{i}$ be abstract copies of the last-column spaces, $E_{\mathrm{abs}}=\bigoplus_{i}E_{i}$, and let $N\subset E_{\mathrm{abs}}$ be the relation space of dimension $\delta$. The abstract coefficient input has dimension $t$. Its maps into $E_{\mathrm{abs}}$ are denoted by $B_{\rho}$. Let $J$ be the kernel of their joint map after quotienting by $N$, and set

$$j=\dim J,\quad W_{J}=\sum_{\rho}B_{\rho}(J),\quad w=\dim W_{J},\quad u=t-j,\quad L=\tau-j.$$

**Lemma 10.4** (Weighted coefficient expansion)**.** *Outside the exceptional normal families described below, restriction through every last matrix is injective on the coefficient input and*

$$\dim\sum_{\rho}B_{\rho}(J')\geq\frac{3}{2}\dim J'-e\tag{10.9}$$

*for every input subspace $J'$. Consequently $w\geq 3j/2-e$, $3j\leq 2\tau$, and every symmetric-evaluation cokernel witness has rank at most $2\tau-3j=2L-j$.*

*Proof.* For a full-last block a one-dimensional coefficient input has output dimension at least two, and a two-dimensional input has joint output dimension at least three. For a rank-two last block the one-dimensional quotient can lose at most one, charged by $\gamma_i - l_i$. When $\gamma_i = l_i = 1$, exclusion of $\mathcal{E}_1$ ensures that the remaining one-dimensional input still has output rank two, so no loss is charged in that case. The inequality for a direct sum follows inductively by projection onto the first summand and then intersection with the remaining summands. Since $W_J \subset N$, this proves the first two consequences. For a cokernel witness of rank $s$, skewness puts the coefficient images of its $s$-dimensional transpose image in its $s$-dimensional output. Lift the input back across $J$ and apply (10.9). Quotienting by $N$ loses at most $\delta$ dimensions, so $3(j+s)/2 - e - \delta \le s$. □

For a fixed last matrix the normal-plane parameter cost is

(10.10)
$$d_i = (2-l_i)5 + l_i(\gamma_i - l_i), \qquad \Delta_i = 10 - d_i, \qquad r_i\gamma_i - \Delta_i = (3-\gamma_i)\gamma_i - l_i(5 - \gamma_i + l_i).$$

Indeed, one chooses the pure-last plane in the last kernel, its projected top plane, and its lift. The respective costs are $l_i(\gamma_i - l_i)$, $(2-l_i)(4-(2-l_i))$, and $(2-l_i)(3-l_i)$.

**Proposition 10.5** (Raw and complementary incidence bounds)**.** *Put $\Lambda = C - 10 + d + 1$. On a dominating positive-overlap stratum the projected bad-data codimension is at least*

$$LM + \ell\Lambda - \sum_i (r_i\gamma_i - \Delta_i) - (\delta - w)h - j(t-j) - \dim \mathcal{Z}. \tag{10.11}$$

*If the normal contractions are onto, a second bound on the same stratum is $\ell\Lambda$.*

*Proof.* Choose $J$ at cost $j(t-j)$. Its abstract output $W_J$ is fixed; choosing $N$ containing it costs $(\delta - w)h$. The quotient injection $E_{\mathrm{abs}}/N \to E$ costs $Mh$. The abstract last matrices, modulo their block general-linear changes and their individual projectivizations, cost $\sum_i r_i\gamma_i - \ell$. Thus the total last-matrix cost is $Mh + (\delta - w)h + \sum_i r_i\gamma_i - \ell + j(t-j)$. The independent symmetric equations cost $(t-j)M - \dim \mathcal{Z}$. The residual equations $DY_i = 0$ cost $\ell(M + C)$ because the $Y_i$ are independent. Subtract the normal cost $\sum_i d_i$ and return the fiber dimension $\ell d$. The identity $t + \ell - h = \delta + e$ gives (10.11).

For the complementary estimate fix the top operators and normals. An onto normal contraction from $E^3$ to $E^2$ has projective kernel dimension $M - 1$. Hence the tuple parameters cost at most $\ell(M - 1 + 10)$. The same independent residual equations and fiber return give $\ell(M + C) - \ell(M + 9) + \ell d = \ell\Lambda$. Both counts refer to the same dominating stratum. □

**Proposition 10.6** (Disjoint input spaces)**.** *If $\tau = 0$, let $g$ be the dimension of the total old-column span of the graphs. Then the bad-data codimension is at least*

$$(5\ell - g)(M - g) + \ell\Lambda, \qquad g \le 5\ell - 1 \tag{10.12}$$

*when the label image is positive dimensional. If each graph has rank four, one may instead use $g \le 4\ell$.*

*Proof.* The last-column spaces are disjoint and $\gamma_i = l_i$. For the positive metric gauge, fix the total column plane $G$ and restrict $A_0$ to be block diagonal for $G \oplus G^{\perp}$. Normalization preserves $G$ and maps this augmented incidence submersively onto the ordinary

one. The inside-$G$ metric block remains free, while the off-diagonal top equations are unchanged. Hence the augmented coefficient evaluation may be used in this count. Choose their containing old $g$-plane at cost $g(M-g)$ and their projective last matrices at cost $hg+\sum_i r_i\gamma_i-\ell$. Off that plane, free rectangular symmetric blocks impose $4\ell(M-g)$ independent top-column equations. Within it, normal evaluation costs $tg$: Lemma 10.4 gives zero cokernel at $\tau=j=0$. Include the residual equations and the normal costs as before. Since $t+\ell=h$, the result is (10.12) minus $\sum_i(r_i\gamma_i-\Delta_i)$, a nonpositive correction.

If all graphs have rank five and $g=5\ell$, their column spaces are disjoint. A linear combination then has rank at most five only when its active summands have the same row space. The rank-five locus in their projective span is a finite union of coordinate projective subspaces with constant labels. An irreducible component with positive-dimensional label image cannot lie there. This proves the strict inequality. Rank-four graphs give the alternative directly. □

**Lemma 10.7** (Saturated single overlap)**.** *If all last matrices have rank three, every normal has top rank two, $j=0$, and $\delta=1$, a nonzero cokernel implies that one tuple top pencil is a line on $\Sigma$. This holds for both the three-coefficient family and the augmented four-coefficient family. Thus, on a component not contained in that pencil locus, the cokernel is zero.*

*Proof.* Its maximum witness rank is two. In (10.2) all the 2 by 2 skew blocks are scalar multiples of one invertible alternating form. If $W$ is the two-dimensional transpose input, the actual coefficient maps on $W$ have the form $\lambda_\rho T$ with $T$ injective. Lift across the relation line $N=\mathbb{C}n$:

$$\widetilde{B}_\rho|_W=\lambda_\rho\widetilde{T}+n\mu_\rho.$$

For each block $i$, let $p_i:W\to U_i$ be its input projection, $n_i$ the block projection of $n$, and $\widetilde{T}_i$ that of $\widetilde{T}$. For every $w\in W$, its coefficient matrix in this block is the sum $\lambda\otimes\widetilde{T}_i(w)+\mu(w)\otimes n_i$, of rank at most two. If some $p_i$ has rank two, its whole augmented normal pencil, hence its top pencil, has rank at most two, as required.

Suppose all nonzero $p_i$ have rank one. For such a block $n_i\neq 0$, since otherwise each nonzero coefficient matrix in the block would have rank one. For $k\in\ker p_i$, the displayed tensor is zero, so $\mu(k)\in\mathbb{C}\lambda$. The maps $p_i$ are jointly injective; thus two nonzero ones have distinct kernel lines in $W$. Consequently the induced map $W\to\mathbb{C}^R/\mathbb{C}\lambda$ from $\mu$ is zero. All block coefficient matrices now have rank at most one, again a contradiction. Therefore a rank-two block exists. A generic tuple from a component not contained in this real-defined closed pencil locus avoids it in every entry, which proves the final assertion. □

## 11. The exterior tensor construction with reserve ten

This section proves that at reserve ten the parameters of the seven-label family can be chosen, relative to prescribed data near a closed set, so that the weak bad-label bound, ordinary regularity, and surjectivity of the curvature differential hold everywhere.

11.1. **Exceptional normals and initial rank conditions.** A normal $U \in \mathrm{Gr}_2(\mathbb{C}^7)$ has pure-last rank $l = \dim(U \cap \mathbb{C}^3)$. Let $\mathcal{E}_1$ be the top-rank-one normals whose top matrix belongs to $\Sigma$ and whose pure-last line lies in its row plane. Its dimension is at most $2 + 1 + 2 = 5$. Let $\mathcal{P}_0$ be the normals whose top pencil is one of the six constant-row-space pencils and whose two last lifts lie in its common row plane. Its dimension is four. Outside these families, two augmented coefficient inputs have joint row dimension at least three. Restriction through a last matrix of rank at least two is injective on $\mathcal{S}$, since its kernel would otherwise be a rank-one coefficient matrix. These facts give the block expansion used in Lemma 10.4.

**Lemma 11.1** (Constant-column pencils)**.** *At reserve $C$ the incidence of a bad-label component of dimension at least five whose generic top pencil has constant column space has codimension at least $M + C$, after excluding a locus of codimension $2M$ in the symmetric parameters.*

*Proof.* For one of the six real pencils, real changes of coefficient, pencil, and last-coordinate bases put its contraction in the form

$$\begin{pmatrix} A - q_{11}I & B - q_{21}I & -q_{31}I \\ -q_{12}I & A - q_{22}I & B - q_{32}I \end{pmatrix}.$$

To obtain this form, remove the common zero coefficient row. A 2 by 3 pencil of constant rank two has a quadratic right kernel with independent coordinates; otherwise a regular square block would drop rank somewhere on $\mathbb{P}^1$. The two linear syzygies of $(s^2, st, t^2)$ give the displayed form.

For independent old vectors $v, w$, the map $(A, B) \mapsto (Av, Bv + Aw, Bw)$ is onto. Indeed, prescribing $(x, y, z)$ and putting $h = Aw$ leaves just $\langle v, h\rangle = \langle w, x\rangle$ and $\langle w, h\rangle = \langle w, y\rangle - \langle v, z\rangle$, two independent equations even for isotropic spans. A projective rank-two adjoint witness and the six lifts therefore give codimension at least $3M - (2M - 1) - 6 = M - 5$.

The locus where $A, B$ have a common eigenspace of dimension at least two has codimension $2M$: fixing the plane and two eigenvalues costs $2(2M - 1)$ equations and $2(M - 2) + 2$ parameters. Away from it there are finitely many common eigenlines. A rank-one witness $(\alpha v, \beta v)$ forces $Av = av$, $Bv = bv$, and three independent equations on the six lifts. Allowing $[\alpha : \beta]$ gives a label set of dimension at most four. Thus a component of dimension $d \geq 5$ generically has a rank-two adjoint witness. On the exact adjoint-corank-$s$ stratum, return its projective witness fiber $s - 1$; residual failure costs $\max(C - s + 1, 0)$; the label fiber returns $d$. The result is

$$M - 5 + (s - 1) + \max(C - s + 1, 0) + d \geq M + C.$$

□

**Lemma 11.2** (Singular rectangular pencils)**.** *For $1 \leq d \leq M$, pairs of $M$ by $d$ complex matrices $(P, Q)$ whose pencil $P + tQ$ has deficient column rank for every $t$ have codimension at least $2M - d + 1$.*

*Proof.* Choose a nonzero polynomial kernel vector of smallest degree, $x(t) = x_0 + \cdots + x_j t^j$, with $x_0 \neq 0$. Its coefficients are independent. Indeed, a first dependence $x_h = \sum_{i<h} a_i x_i$ gives the shorter chain $y_k = x_k - \sum_{a=1}^{k} a_{h-a} x_{k-a}$, $0 \leq k < h$, satisfying $Py_0 = 0$, $Py_k = -Qy_{k-1}$ and $Qy_{h-1} = 0$. Thus $j < d$. A fixed coefficient frame imposes $M(j+2)$ independent linear equations, whereas frames modulo scale have $d(j+1)-1$ parameters. The difference is $M(j+2) - d(j+1) + 1 \geq 2M - d + 1$. There are finitely many possible minimal degrees. □

**Lemma 11.3** (Exceptional lifts)**.** *The parameter locus on which the actual bad labels in $\mathcal{E}_1$ have dimension five has codimension at least $M + C - 2$.*

*Proof.* Fix the top direction and pure-last line $(z, k)$, at a cost of three parameters. Choose last coordinates with $K_z = \operatorname{span}(e_0, e_1)$ and $k = \mathbb{C}e_0$. A last matrix killing $k$ is $(w, Y_2) \in E^2$. On the lift line $\xi = e_2 + te_1$ its remaining normal equation is $A_b w - Y_2 - tw = 0$. The map $(w, Y_2) \mapsto (A_b w - Y_2, -w)$ is an invertible linear map of $E^2$. Let the restriction $D' : E^2 \to V$ have target corank $s$ and kernel dimension $d = M - C + s$. Its rank-stratum cost is $sd$. A framed kernel therefore gives an unrestricted pair of $M$ by $d$ matrices for the displayed pencil.

If $0 < d \leq M$, Lemma 11.2 adds $2M - d + 1 = M + C - s + 1$ equations when every lift on that line is bad. The sum is $M + C + 1 + s(d-1) \geq M + C + 1$. If $d = 0$ there is no bad lift. If $d > M$, then $s \geq C + 1$ and the rank-stratum cost alone is at least $(C+1)(M+1) > M + C + 1$. Returning $(z, k)$ gives $M + C - 2$.

A five-dimensional bad-label component inside the five-dimensional $\mathcal{E}_1$ fills one of its irreducible components. For generic $(z, k)$ every lift in that component is then bad, so the chosen pencil is bad. The preceding exclusion rules this out. The projective boundary and lower-dimensional coefficient strata already have dimension at most four. A finite coefficient-chart cover makes the construction algebraic. □

**Lemma 11.4** (The preliminary exterior exclusions)**.** *For $C = 10$ and $M \geq 9$, parameters can be chosen so that any bad-label component of dimension $d \geq 5$ has generic onto normal contraction, last rank two or three, and graph rank four or five. For rank four its dominating tuple length satisfies $\ell \geq d - 1$; for rank five it satisfies $\ell \geq d + 1$.*

*Proof.* The counts below, including the exceptional ones, specify the part of the parameter space that is removed. The exact normal parameter dimension is ten. Outside $\mathcal{E}_1$, $\mathcal{P}_0$, and the constant-column pencils, two adjoint witnesses with total old-column dimension $h \leq 4$ have coefficient span at least $h + 2$. For $h = 4$ this comes from two disjoint two-column spaces; for $h = 3$ the constant-rank Kronecker pencil supplies $3 + 2$ coefficient dimensions: after basis changes its two witnesses are $e_1 \otimes u_1 + e_2 \otimes u_2$ and $e_2 \otimes u_1 + e_3 \otimes u_2$. If the joint coefficient rank is $t_0 \geq 3$, their two coefficient spaces intersect in at most $t_0 - 2$ dimensions, since the intersection is supported on the middle old line. Their sum has dimension at least $t_0 + 2 \geq 5$. For $h = 2$ a rank-one pencil member and projection away from its old line supply $2 + 2$; for $h = 1$ a failure to supply three would be a constant-column pencil. The same augmented argument handles top rank one. Symmetric evaluation minus the witness-plane cost gives

$$(t - h)M + (h - 2)^2 - \chi(t, h) \geq 2M - 3.$$

For a single old-rank-two adjoint witness the coefficient span has dimension three and its cokernel is at most one. A nonzero cokernel contains a coefficient line tensored with the old two-plane, which forces the augmented top pencil onto the determinant surface. There are only finitely many such projective pencils, so their two lifts cost six, not ten. This gives $(3M-1)-(2M-1)-6 = M-6$. In the zero-cokernel case it gives $3M-(2M-1)-10 = M-9$. For an old-rank-one witness, coefficient rank at least three is stronger; at rank two its top direction lies in $\Sigma$, returning one parameter and again giving $M-9$. The lower-dimensional top-rank-one normal stratum is no worse.

At exact corank one, residual failure and a label fiber of dimension $d \geq 5$ thus give $M-9+C+5 = M+C-4$. At exact corank $s \geq 2$, return the Grassmannian witness fiber $2(s-2)$ in the two-witness estimate; the result is

$$2M-3-10+2(s-2)+\max(C-s+1,0)+5.$$

At $C = 10$ this has minimum $2M+1$. Constant-column pencils are covered by Lemma 11.1.

Lemma 11.3 supplies the bound $M+C-2$ on $\mathcal{E}_1$. The family $\mathcal{P}_0$ already has the allowed dimension four.

For rank-one last matrices $Y = v \otimes \lambda$, the maximal graph row space is $\operatorname{im} L_\lambda \oplus \mathbb{C}\lambda$. At ordinary $\lambda$ it has dimension four, and its supporting normal family has dimension two; allowing $\lambda$ adds two. At one of the six exceptional directions its row rank is three, and the supporting normal family is $\mathrm{Gr}_2(\mathbb{C}^4)$, also of dimension four. These maximal-rank graphs are harmless. A rank drop costs $M+C-4$ at an ordinary direction and $M+C-1$ at an exceptional direction.

For last rank $t = 2, 3$, graph rank at most $\rho$ has codimension at least

$$(M-\rho)(4+t-\rho)-\varepsilon_t+M+C-t(M+3-t)+1, \qquad \varepsilon_2 = 3, \quad \varepsilon_3 = 0. \tag{11.1}$$

To see this, choose a projective rank-$t$ last matrix, choose its containing old $\rho$-plane, and impose the four top columns modulo that plane. The full-rank evaluation loss is zero by Lemma 9.3; at rank two it is at most $3 \dim \Lambda^2 \mathbb{C}^2 = 3$. The residual equation then gives (11.1). For $\rho = 3$ the two bounds are $2M+C-13$ and $2M+C-11$.

Thus the relevant lower bounds at $C = 10$ are

| failure | codimension lower bound |
|---|---|
| ordinary/exceptional rank-one graph drop | $M+6,\ M+9$ |
| exceptional lift pencil | $M+8$ |
| last-rank-two/three low graph rank | $2M-3,\ 2M-1$ |
| actual nononto contraction | $M+6,\ 2M+1$ |
| constant-column pencil | $M+10$ |
| ordinary and curvature regularity | $2M-3,\ 2M$ |

Every number exceeds $M+5$ when $M \geq 9$. A rank-four graph has a two-dimensional normal-label fiber, giving $\ell - 1 \geq d-2$. A rank-five graph has a unique label, giving $\ell - 1 \geq d$. □

## 11.2. **The single-evaluation improvement.**

**Lemma 11.5** (The pure full-last overlap strata)**.** *At reserve ten, every positive-overlap stratum with $\gamma_i = l_i = j = 0$ and $\ell \geq 4$ has codimension greater than $M + 5$ whenever $M \geq 6\ell - 5$.*

*Proof.* Here $\delta = L$, $h = 3\ell - L$, $u = 2\ell$, and $E_{\mathrm{abs}} = \bigoplus_i E_i$ with $\dim E_i = 3$. Choose one coefficient direction $d$ that evaluates each two-dimensional normal input injectively into its last block. If all evaluations of a block had rank at most one, their linear matrix space would have a common image or common kernel. The former contradicts minimum coefficient rank two, the latter joint injectivity. Thus each block has a nonzero quadratic minor in $d$. The product of chosen minors is a nonzero polynomial of degree at most $2\ell$ in each of four coordinates. The fixed grid $\{0, 1, \ldots, 2\ell\}^4$ meets its nonzero locus, by iterated one-variable interpolation. This is a finite choice, with no continuous parameter cost.

Fix the normal data and that direction before choosing the relation plane $N$. The unquotiented evaluation has an image $I$ of dimension $2\ell$ in the $3\ell$-dimensional last direct sum. Stratify by $a = \dim(N \cap I) = \dim \ker B_d$. If $1 \leq L \leq \ell$, this stratum has codimension

$$a(\ell - L + a) \tag{11.2}$$

in $\mathrm{Gr}_L(\mathbb{C}^{3\ell})$. This cost is additional: the raw relation expense is $Lh$ and $j = e = w = 0$, so the coefficient-kernel conditions remove none of it. The quotient injection is unaffected, and further open conditions cannot increase this stratum's dimension.

At $M = 6\ell - 5$ the raw excess over $M + 5$, before its cokernel subtraction, is $3L\ell + L^2 - 5L$. Apply Lemma 10.3 and add (11.2). Subtracting the two possible branches gives, respectively,

$$E_1 = L(3\ell - L - 6) + a(\ell - L) + \frac{a(a+1)}{2} > 0, \tag{11.3}$$

$$E_2 = L(\ell - 3) + \frac{3L(L-1)}{2} + a\ell + \frac{a(a+1)}{2} > 0. \tag{11.4}$$

These inequalities hold for $\ell \geq 4$, $1 \leq L \leq \ell$, and $0 \leq a \leq L$. If $L > \ell$, the crude cokernel cap $\binom{2\ell}{2}$ gives excess at least $2\ell^2 + \ell - 4$, attained at $L = \ell + 1$. The excess has slope $L - 1 \geq 0$ in $M$, which covers the remaining range. □

## 11.3. **All tuple lengths and all old ranks.**

**Theorem 11.6** (Reserve-ten exterior avoidance)**.** *Let $E$ have rank $M \geq 9$ over a compact smooth manifold $X$, possibly with boundary, of dimension at most $M + 5$, and let $V$ have rank $M + 10$. Call parameters $(A, D)$ at $x \in X$, with $A \in \mathrm{Sym}(E_x)^3$ and $D : E_x \otimes \mathbb{R}^3 \to V_x$,* good *if the structured seven-label family satisfies the weak bound $\dim_{\mathbb{C}} \mathcal{B}_K \leq 4$, ordinary regularity, and surjectivity of the curvature differential $D\Phi_{S(A)}$ of Lemma 9.4. Let $X_0 \subset X$ be closed, possibly empty, and let $(A^0, D^0)$ be a smooth good family on an open neighborhood of $X_0$. Then there is a smooth good family over $X$ that coincides with $(A^0, D^0)$ on a neighborhood of $X_0$. The only hypothesis on $(A^0, D^0)$ is goodness. Positive residual scaling and sufficiently small normalized curvature corrections preserve the selected compact open activity conditions.*

*Proof.* After Lemma 11.4, a forbidden component has $d \geq 5$ and an onto generic contraction. Thus $\Lambda = d+1 \geq 6$ and $\ell \geq 4$. The complementary bound excludes every length with $6\ell > M+5$. It remains to work at $M \geq 6\ell - 5$. Since (10.11) minus $M+5$ has slope $L-1 \geq 0$, it suffices to examine the endpoint $M = 6\ell - 5$.

*Long tuples.* Suppose $\ell \geq 16$. Generic last and pure-last ranks are constant on the irreducible component and its conjugate, leaving $(\gamma, l) = (0,0), (1,0), (1,1)$. For the first two cases set $\epsilon = 0, 1$ respectively. Then

$$h = 3\ell - L - j, \quad u = 2\ell - j, \quad r_* = 2L - j,$$

and $\delta - w \leq L - j/2$. The raw excess before subtracting the cokernel is at least

$$3L\ell - \frac{j\ell}{2} + L^2 + \frac{Lj}{2} + \frac{j^2}{2} - 5L - 2\epsilon\ell. \tag{11.5}$$

For $L \leq \ell/2$, the hypotheses of (10.6) hold, and monotonicity gives

$$\dim \mathcal{Z} \leq \frac{6}{5}(2L-j)(\ell - L) + \frac{11}{10}(2L-j)(2L-j-1).$$

When $r_* \leq 1$ the cokernel is zero, so the same inequality is harmless. The resulting excess is at least

$$L\left(\frac{3}{5}\ell - L - \frac{14}{5}\right) + j\left(\frac{7}{10}\ell + \frac{37}{10}L - \frac{3}{5}j - \frac{11}{10}\right) - 2\epsilon\ell. \tag{11.6}$$

The quadratic bound (10.3), valid for $L \leq \ell$, alternatively gives

$$L(3L - \ell - 4) + j\left(\frac{3}{2}\ell + \frac{1}{2}L - \frac{1}{2}\right) - 2\epsilon\ell. \tag{11.7}$$

For $\epsilon = 0$ the $j$ coefficients are positive. If $3L > \ell + 4$, use (11.7); otherwise $L \leq (\ell+4)/3 < (3\ell - 14)/5$, and use (11.6). For $\epsilon = 1$, $L + j \geq \ell$ and $j \leq 2L$ give $\ell/3 \leq L \leq \ell/2$, $j \geq \ell/2$. The first term in (11.6) is at least $\ell^2/20 - 7\ell/5$, and its $j$ coefficient is at least $23\ell/15 - 11/10$. Hence the excess is at least $\ell(49\ell - 237)/60 > 0$. For $\ell/2 < L \leq \ell$, (11.7) is strictly greater than $\ell^2/4 - 4\ell \geq 0$. For $L > \ell$, use $\dim \mathcal{Z} \leq 2\ell^2 - \ell$ in (11.5); the result is at least $2\ell^2 - 6\ell > 0$.

For $(\gamma, l) = (1,1)$ one has $h = 2\ell - L - j$, $u = \ell - j$, and normal loss $-3\ell$. If $L \leq \ell/2$, the quadratic cap gives excess at least

$$(2L + j + 3)\ell + 3L^2 + \tfrac{1}{2}Lj - 4L - \tfrac{1}{2}j > 0.$$

If $L > \ell/2$, the crude cap $\binom{\ell}{2}$ gives a value strictly greater than $7\ell^2/4 + \ell$. This treats all positive-overlap long tuples.

*The finite short range.* For $4 \leq \ell \leq 15$ enumerate

$$\begin{gathered}(\gamma, l) = (0,0), (1,0), (1,1), \quad 0 \leq \delta < (3-\gamma)\ell, \quad 0 \leq j \leq (2-l)\ell, \\ e = (\gamma - l)\ell, \quad t = (2-l)\ell, \quad h = (3-\gamma)\ell - \delta, \quad \tau = \delta + e, \\ L = \tau - j \geq 1, \quad 3j \leq 2\tau, \quad w = \max\{0, \lceil (3j - 2e)/2 \rceil\} \leq \delta.\end{gathered}$$

The family $\gamma = l = j = 0$ is already proved by Lemma 11.5. For every other row put $u = t - j$, $r_* = 2\tau - 3j$, and bound its cokernel by

$$\max_{2\le r\le \min(r_*,h,u)} \min\left\{ ru - \frac{r(r+1)}{2},\ r(u-2)+1,\ \Phi_4(r,h,u)\right\}, \tag{11.8}$$

where an empty maximum is zero. Use the exact normal loss $(3-\gamma)\gamma - l(5-\gamma+l)$ in (10.11). This is a finite integer calculation, with the following row totals and minima:

| $\ell$ | direct rows | symbolic rows | minimum direct excess |
|---|---|---|---|
| 4 | 108 | 11 | 7 |
| 5 | 170 | 14 | 9 |
| 6 | 246 | 17 | 11 |
| 7 | 335 | 20 | 13 |
| 8 | 439 | 23 | 15 |
| 9 | 556 | 26 | 17 |
| 10 | 687 | 29 | 19 |
| 11 | 832 | 32 | 21 |
| 12 | 991 | 35 | 23 |
| 13 | 1163 | 38 | 25 |
| 14 | 1350 | 41 | 27 |
| 15 | 1550 | 44 | 29 |

There are 8427 direct rows and 330 symbolic rows. The enumeration formula specifies every operation, and the accompanying certificate uses only integer arithmetic. Every row has nonnegative slope in $M$, so these finite checks cover the short-length range for every old rank $M$.

*Zero overlap.* For generic graph rank five, $\ell \ge 6$. If $5\ell \le M$, (10.12) is at least $M + \ell + 1 > M + 5$; otherwise integrality gives $6\ell > M + 5$. For graph rank four, the only remaining range is $M \ge 6\ell - 5 > 4\ell$. The excess from (10.12) is at least

$$\ell(M - 4\ell) + 6\ell - M - 5 \ge \ell(2\ell - 5) > 0.$$

This treats every length and every rank stratum.

*Smooth choices and curvature.* The complex incidence strata and their projected fiber-dimension conditions are defined over the reals. Their complex codimension bounds imply at least the same real codimension on real parameters. In the disjoint count, the nondegenerate open tuple stratum contains the component-and-conjugate spanning tuples and retains their full returned fiber dimension.

Let $\mathcal{P}_x = \mathrm{Sym}(E_x)^3 \oplus \mathrm{Hom}(E_x \otimes \mathbb{R}^3, V_x)$, and let $\mathcal{N}_x \subset \mathcal{P}_x$ be the set of parameters that are not good. It is the union of three sets. The incidence set $\{\dim_{\mathbb{C}} \mathcal{B}_K \ge 5\}$ is closed, since projectivity of the bad-label incidence makes the fiber dimension upper semicontinuous. It is the real part of a constructible set, hence semialgebraic. By Lemma 11.4 and the preceding counts it lies in a finite union of semialgebraic sets of codimension greater than $M + 5$, so its own codimension exceeds $M + 5$. The ordinary failure set is closed, since surjectivity is open. By Lemma 9.4(1) it lies in a locus of codimension at least $2M - 3 > M + 5$. The curvature failure set is closed of codimension

at least $2M > M+5$, by Lemma 9.4(2). All three sets are invariant under $O(E_x)\times O(V_x)$, since $L$ is fixed. Over a chart $U$ with orthonormal frames, therefore, $\mathcal{N}|_U = U \times \mathcal{N}_0$ for one closed semialgebraic set $\mathcal{N}_0$ of codimension greater than $M + 5 \geq \dim X$. A semialgebraic set is a finite union of smooth submanifolds of no larger dimension. Thus $\mathcal{N}$ is covered by finitely many submanifolds of open subsets of $\mathcal{P}$, each of codimension greater than $\dim X$.

Choose closed neighborhoods $N_1 \subset \operatorname{int} N_2$ of $X_0$ with $N_2$ inside the domain of $(A^0, D^0)$. Choose a cutoff equal to one on $N_2$ and supported in that domain, and let $p$ be the product of the cutoff with $(A^0, D^0)$, extended by zero. Choose a smooth $\psi \geq 0$ with $\psi^{-1}(0) = N_1$, and smooth sections $\sigma_1, \dots, \sigma_K$ of $\mathcal{P}$ spanning every fiber. Put $p_c = p+\psi \sum_k c_k\sigma_k$. On $(X\setminus N_1)\times\mathbb{R}^K$ and on $(\partial X\setminus N_1)\times\mathbb{R}^K$ the map $(x, c) \mapsto p_c(x)$ is a submersion to $\mathcal{P}$. By the parametric transversality theorem, for almost every $c$ the section $p_c$ is transverse on both sets to each of the finitely many submanifolds covering $\mathcal{N}$. Their codimension exceeds the dimension, so $p_c$ misses $\mathcal{N}$ on $X \setminus N_1$. On $N_1$ it equals $(A^0, D^0)$, which is good. Hence $(A, D) = p_c$ is a smooth good family over $X$ that coincides with $(A^0, D^0)$ on $\operatorname{int} N_1 \supset X_0$. For $X_0 = \emptyset$ this is the absolute statement.

Put $S = S(A)$. By goodness and compactness, $D\Phi_S$ is onto on $X$ with a uniformly bounded right inverse. Replace $D$ by $\eta D$, $\eta > 0$; this leaves its kernel fixed and introduces only the last–last curvature error $\eta^2\Omega(D)$. The surjective differential $D\Phi_S$ corrects that error by $S_\eta = S+O(\eta^2)$. After normalizing the residual target by $\eta^{-1}$, the graphs converge to the chosen graph. Projectivity of the bad-label incidence makes its fiber-dimension upper bound open; in particular the fixed compact activity and boundary homotopy margins survive. Block elimination shows $D\Omega_T$ is onto: first prescribe top–top entries, then top–last entries using skew parts, and last–last entries through $D\Phi_{S_\eta}$. Scaling the whole map and applying the finite-dimensional implicit-function theorem similarly accommodates a prescribed bounded background curvature. These are tensor corrections; the realization and assembly theorems proved above pass from them to an actual isometric map. □

## 12. The core tensor construction with reserve five

The exterior construction must be supplemented by data on a core whose dimension is bounded independently of the dimension of the manifold. In this section we construct such data on a core of dimension at most five. The required improvement concerns the actual symmetric matrix parameters; it does not follow by treating the columns of a structured graph as independent.

We use the Tiles coefficient space of Lemma 9.1. Put $A = \mathbb{C}^3$, let $P = \mathbb{C}^4$ denote its coefficient space, and write $E = \mathbb{C}^r$. The label space is $P \oplus A$. We denote by $X(Y)$ the structured graph of a last matrix $Y : A \to E$; it is an actual kernel graph if and only if $DY = 0$, where

$$D : E \otimes A^* \longrightarrow V, \qquad \dim V = r + c.$$

Signs in the top columns play no role in the counts. In the coefficient calculation we permit the positive metric gauge: the last columns are then $A_0Y$, where $A_0$ is positive definite, and there are three independent symmetric top operators. Normalizing the

metric returns the original family without changing its target dimension. Whenever a row-expansion argument uses the last lifts, we retain the four augmented coefficient directions, including $A_0$.

We work with normal two-planes instead of their annihilating five-planes. Thus

$$\mathcal{B} = \{U \in \mathrm{Gr}_2(P \oplus A) : X(Y)|_U = 0 \text{ for some } 0 \neq Y \in \ker D\}.$$

For a real normal three-plane $Z$, the corresponding Schubert fiber is $\mathcal{B} \cap \mathrm{Gr}_2(Z_{\mathbb{C}})$. We call it *complete* when it equals that projective plane.

**Proposition 12.1** (Core tensor family)**.** *Let $r \geq 12$, $c = 5$, and let the slow parameter space have dimension at most five. In the structured real Tiles family the following finite-dimensional conditions can be imposed simultaneously:*

(i) *the complex bad-normal set has dimension at most four;*

(ii) *the Schubert fiber over every real normal three-plane is finite or complete;*

(iii) *the prescribed standard activity is safe;*

(iv) *ordinary regularity and the full selfadjoint curvature differential are surjective, as are the residual ordinary operator and the top curvature differential;*

(v) *the ordinary and curvature conditions required on every rank-$(r-1)$ compression hold simultaneously.*

*The relative version used below fixes the regular collar constructed in Section 14. Normalization and preservation of the complete boundary class are established in Lemma 14.2.*

12.1. **A rank-two symmetric-evaluation lemma.** The first estimate sharpens a graph-rank exclusion at the endpoint. All duals and transposes in its proof are algebraic; in particular, an isotropic complex old subspace causes no exception.

**Lemma 12.2.** *Let $S \subset \mathrm{Mat}_{3\times 3}(\mathbb{C})$ have dimension four and contain no nonzero matrix of rank one. If $Y : \mathbb{C}^3 \to E$ has rank two, the four-column symmetric evaluation associated with $S$ and $Y$ has cokernel dimension at most one.*

*Proof.* Every nonzero left evaluation $C \mapsto a^t C$ on $S$ has rank at least two. Otherwise its kernel has dimension at least three and, after changing rows, lies in $\mathrm{Mat}_{2\times 3}$. By the projective dimension theorem its projective plane meets the rank-one variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, a contradiction. The same holds for right evaluation.

Put $W = \operatorname{im} Y$. The restricted coefficient map $S \to \mathbb{C}^3 \otimes W$ is injective, since a kernel element has all rows in the one-dimensional last kernel and hence has rank at most one. Denote its four-dimensional image by $T$. Symmetric duality identifies the cokernel with

$$(T \otimes W) \cap (\mathbb{C}^3 \otimes \Lambda^2 W).$$

Because $\dim W = 2$, its dimension is the dimension of

$$K = \{a \in \mathbb{C}^3 : a \otimes W \subset T\}.$$

If $\dim K \geq 2$, then $T = K_0 \otimes W$ for a two-plane $K_0 \subset \mathbb{C}^3$. A nonzero covector annihilating $K_0$ would send every original seed matrix into the one-dimensional kernel of $Y$, contradicting the left evaluation bound. Therefore $\dim K \leq 1$. □

For last rank three the symmetric evaluation is onto by Lemma 9.3. If the full graph has rank at most $\rho$, choose a $\rho$-plane containing $\operatorname{im} Y$. For last rank $t = 2, 3$, the resulting projected incidence codimension is at least

$$(r-\rho)(4+t-\rho) - \epsilon_t + r + c - t(r+3-t) + 1, \qquad \epsilon_2 = 1, \quad \epsilon_3 = 0. \tag{12.1}$$

Here the extra containing plane costs $(\rho - t)(r - \rho)$, the four-column evaluation outside it costs at least $4(r-\rho) - \epsilon_t$, a projective last matrix costs $t(r+3-t)-1$, and $DY = 0$ costs $r + c$. At $\rho = 4$ the bounds are $r + c - 10$ and $r + c - 11$, which equal seven and six at $(r, c) = (12, 5)$.

12.2. **Preparations and exceptional normal planes.** For a last direction $\lambda \in \mathbb{P}(A)$, write $D_\lambda(v) = D(v \otimes \lambda)$. Two distinct directions with nonzero kernels have projected incidence codimension $2c-2$: select two independent projective rank-one matrices, each costing $r+1$ parameters, and impose their independent residual equations, of total rank $2(r + c)$. A two-dimensional kernel at one direction costs $2(c + 2) - 2$, and failure of injectivity at one fixed direction costs $c + 1$. At $c = 5$ all these quantities exceed five. Consequently there is at most one bad last direction at each slow point, its kernel is a line, and all six exceptional last directions of the Tiles seed can be made injective.

At an ordinary direction, a rank-one graph has maximal rank four. Its rank-drop incidence has codimension $r + c - 4$: the residual rank-one incidence costs $c - 1$, and the drop of the three independent top evaluations modulo the last vector costs $r - 3$. Thus the only remaining rank-one graph contributes a complete plane $\mathrm{Gr}_2(W_\lambda)$, for a normal three-plane $W_\lambda$. Its intersection with $\mathrm{Gr}_2(Z)$ is empty, a point, or that whole plane. By (12.1), every other bad graph has rank five and determines a unique bad normal.

We retain the notation $\mathcal{E}_1$ and $\mathcal{P}_0$ from the exterior coefficient analysis. The set $\mathcal{E}_1$ consists of normals with a one-dimensional top projection $z$ on the determinant surface, a pure-last line $k$ in the two-dimensional row space of $C_z$, and a last lift. Its dimension is at most $2 + 1 + 2 = 5$. The locus $\mathcal{P}_0$ consists of normals whose top projection is one of the six constant-row pencils and whose lifts both lie in its fixed row two-plane; its dimension is four.

We treat the constant-column pencils before counting actual ordinary normals. Their canonical operator has two independent symmetric coefficients. By the symmetric evaluations in the exterior proof, a common eigenline costs $r - 1$ and a rank-two adjoint costs $r - 5$. Both costs exceed five at $r = 12$. Outside the two exceptional normal loci and these column pencils, the projected single-adjoint bound is $r - 9$. Requiring an actual residual witness on the corank-one stratum adds $c$ conditions. The actual nononto-normal bound is therefore

$$r + c - 9. \tag{12.2}$$

For exact adjoint corank $s \geq 2$, the two-witness bound, normal parameters, witness-fiber return and residual equations give

$$2r + 2s - 17 + \max(c + 1 - s, 0) \geq 2r + c - 14. \tag{12.3}$$

Both bounds exceed five, the dimension of the core. Hence the contraction is onto at every actual ordinary bad normal; surjectivity at an arbitrary normal is not claimed.

**Lemma 12.3.** *The preceding preparations allow* $\dim_{\mathbb{C}} \mathcal{B} \leq 4$.

*Proof.* The rank-one graph planes have dimension two and $\mathcal{P}_0$ has dimension four. On the generic lift stratum of $\mathcal{E}_1$, the last two-vector equation has an identity coefficient. For a fixed normal, its top equation and residual equation have projected codimension $c+1$. A five-dimensional fiber returns the five normal parameters, and the codimension is still $c + 1 = 6$. The remaining lift stratum has dimension at most four.

A forbidden component of dimension $d \geq 5$ therefore has generically onto contractions and rank-five graphs with unique normal labels. A dominating independent tuple spanning this component and its conjugate has length $\ell \geq d + 1$. The complementary residual count of the exterior argument is

$$\ell(c - 10 + d + 1) = \ell(c + d - 9) \geq 6.$$

This exceeds the core dimension and excludes the component. □

We next give the exceptional Schubert counts. They do not improve as $r$ increases and hence matter at reserve five. A proper fiber cannot contain a nonzero actual graph killing $Z$. Consequently the restriction to $Z$ is injective on the span of its supporting actual graphs, and distinct normals require independent graph witnesses.

For $\mathcal{E}_1$, write $B_z = \operatorname{row} C_z$. Over a bad rank-one direction there is no additional actual witness unless it kills $B_z$: the independent rank-one witness costs $r-1$, a second projective matrix killing its last line costs $2r-1$, the two residual equations cost $2(r+c)$, the remaining top equation costs $r$, and the normal parameters cost five. The difference is $2c - 3 > 5$.

For two independent witnesses with nonzero restrictions to $B_z$, the combined top-equation and last-parameter cost is at least the cost of $2r-1$ top equations. If their old evaluation vectors are independent, this is symmetric evaluation on two independent vectors. If they are proportional, their parameters lose $r - 1$, while one top equation still costs $r$. Together with the residual equations this gives $2c + 1$ before we allow the common plane $Z$.

If $Z$ has top rank one, put $B = Z \cap A$. When $B \neq B_z$, its parameters cost five and its exceptional normals form a pencil; two prescribed points give $2c - 4$. When $B = B_z$, the common-plane parameters cost three. A component with fixed pure-last line adds one parameter and gives $2c - 3$. When that line varies, two general choices and the return of the two-dimensional actual choice fiber give the same bound. If $Z$ has top rank two, its top projective line lies on the smooth determinant cubic, which contains finitely many lines. The common last line is determined on a nonconstant-row pencil; on a constant-row pencil it costs one parameter. The lifts cost four. Thus the common-plane family costs at most five, and the two-point bound is again $2c - 4$. At

a pure-last normal, the supporting last matrix has rank at most one. Its contribution is therefore one of the complete rank-one graph planes treated above, and it creates no additional proper positive-dimensional component. Since the minimum of these bounds is six, a proper exceptional component is excluded.

**Lemma 12.4** (Constant-row two-point evaluation)**.** *For each constant-row Tiles pencil, its restriction to the common row two-plane is equivalent to the symmetric two-by-two coefficient space. If two independent old-vector pairs $(x_i, y_i)$, $i = 1, 2$, have total span of dimension $h$, their two-point symmetric evaluation has coefficient dimension four for $h \geq 2$ and three for $h = 1$. The respective cokernel bounds for $h = 4, 3, 2, 1$ are $1, 3, 2, 0$.*

*Proof.* We use the canonical rational kernel bases specified in Section A.3. For the six exceptional directions, the annihilators of the coefficient hyperplanes in $\mathrm{Mat}_{2\times 2}$, read row by row, are

| $\lambda$ | $H$ | $\det H$ |
|---|---|---|
| $(1, -1, 0)$ | $(7/2, 1, -1, 1)$ | $9/2$ |
| $(0, 0, 1)$ | $(1, -1, 0, 1)$ | $1$ |
| $(0, 1, -1)$ | $(-1, -2/7, 2/7, 1)$ | $-45/49$ |
| $(1, 0, 0)$ | $(1, 0, -1, 1)$ | $1$ |
| $(1, 1, 1)$ | $(2, 1, -1, 1)$ | $3$ |
| $(-2, 1, -2)$ | $(1/2, 1, -1, 1)$ | $3/2$. |

They are obtained by substituting the four seed matrices, restricting to $\lambda^{\perp}$, and solving the three coefficient equations. Every displayed annihilator is invertible. Invertible changes of the two bases carry it to the standard alternating matrix, so its annihilator hyperplane becomes the symmetric matrices.

In this form the four coefficient vectors are $e_1 x_i + e_2 y_i$ and $e_2 x_i + e_3 y_i$. A coefficient relation of rank one would make the two old-vector pairs dependent. A rank-two relation can be put into the form $x_1 = 0,\ y_1 + x_2 = 0,\ y_2 = 0$, which forces $h = 1$. This proves the stated coefficient dimensions. For $h = 4$, the dual equations are

$$\operatorname{sym} \sum_i x_i \otimes \alpha_i = 0, \qquad \operatorname{sym} \sum_i (y_i \otimes \alpha_i + x_i \otimes \beta_i) = 0, \qquad \operatorname{sym} \sum_i y_i \otimes \beta_i = 0.$$

The first and last equations give $\alpha_1 = t x_2, \alpha_2 = -t x_1$ and $\beta_1 = u y_2, \beta_2 = -u y_1$, and the middle equation forces $t = u$. Hence the cokernel has dimension one. For $h = 3, 2, 1$ the general symmetric-evaluation estimate gives $\chi(4, 3) = 3$, $\chi(4, 2) = 2$, and $\chi(3, 1) = 0$. All calculations are tensorial and remain valid on isotropic spans. □

For a proper curve in $\mathcal{P}_0$, the common $Z$ has the fixed top pencil, a last line inside its row plane, and lifts inside its row plane modulo that line. Its parameter cost is $1 + 2 = 3$. Choose two points by fixing two distinct values of a nonconstant affine lift coordinate; one lift parameter remains at each point. The restrictions of the two witnesses to the row plane are independent, since the complementary exceptional last line is residual-injective. If their four old columns span an $h$-plane, their parameter space loses $(4 - h)(r - h)$. Lemma 12.4, two independent residual equations, the three

common-plane parameters, and the two remaining lift parameters give

(12.4)
| $h$ | 4 | 3 | 2 | 1 |
|---|---|---|---|---|
| codimension | $2c-4$ | $r+2c-9$ | $2r+2c-9$ | $2r+2c-6$. |

For $h \geq 2$, for example, this is $2c-3-\chi(4,h)+(4-h)(r-h)$, with the sharper loss one at $h=4$. Every bound is greater than five. The curve need not attain every fixed affine value: we use a countable dense collection of pairs, and a nonconstant algebraic map misses only finitely many generic values.

12.3. **The common-plane incidence and its finite rows.** After the exceptional reductions, a remaining proper Schubert component has ordinary rank-five graphs. Let $\ell$ be the dimension of the span of the component and its conjugate, and choose a generic independent tuple from them. The complementary estimate with the common three-plane charged once is $c\ell-12$, because $\dim \mathrm{Gr}_3(\mathbb{C}^7)=12$. For $c=5$ it excludes every $\ell \geq 4$. It remains to consider $\ell = 2, 3$.

Write $\gamma \in \{0,1\}$ for the last corank and $l$ for the pure-last normal rank, and use the generic ranks

$$\begin{aligned} 0 \leq l \leq \gamma, \qquad e=\ell(\gamma-l), \qquad h=\ell(3-\gamma)-\delta, \\ u=\ell(2-l)-j, \qquad L=\delta+e-j, \qquad w=\max\{0,\lceil 3j/2-e\rceil\}. \end{aligned} \tag{12.5}$$

Here $\delta$ is the dimension of the relation space among the last images, $j$ is the joint coefficient-kernel dimension, and $w$ is a lower bound for its unquotiented row span. The admissible positive-overlap rows satisfy

$$0 \leq \delta \leq (\ell-1)(3-\gamma), \quad 0 \leq j \leq \ell(2-l), \quad L \geq 1, \quad j \leq 2L, \quad w \leq \delta. \tag{12.6}$$

Put $R=\min(2L-j,u,h)$. The skew-cokernel bound permits

$$\chi = 0 \quad (R \leq 1), \qquad \chi = Ru - \frac{R(R+1)}{2} \quad (R \geq 2). \tag{12.7}$$

The raw projected incidence codimension is at least

$$Lr + c\ell - 12 - \ell\gamma(3-\gamma) + 2l - (\delta-w)h - ju - \chi. \tag{12.8}$$

We count parameters, including the projection fiber. The last matrices and their coefficient kernel cost at most

$$rh + (\delta-w)h + \ell(3-\gamma)\gamma - \ell + ju.$$

There are $ru-\chi$ independent symmetric equations and $\ell(r+c)$ independent residual equations. The normal planes inside $Z$ cost $2\ell$, and the choice of $Z$ costs twelve. Each dominating choice lies on a positive-dimensional component, so its actual projection fiber returns at least $\ell$ parameters. When $l=1$, the pure-last condition saves two common-plane parameters. Subtracting gives (12.8). The dominating stratum is needed here: the fiber return fails for an arbitrary tuple subvariety.

At $(r, c) = (12, 5)$, the only raw rows at most five are

(12.9)

| $\ell$ | $\gamma$ | $l$ | $\delta$ | $j$ | $L$ | $h$ | $u$ | $\chi$ | raw |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 0 | 1 | 0 | 1 | 5 | 4 | 5 | 0 |
| 2 | 1 | 0 | 0 | 1 | 1 | 4 | 3 | 0 | 3 |
| 2 | 1 | 0 | 1 | 2 | 1 | 3 | 2 | 0 | 2 |
| 2 | 1 | 1 | 1 | 0 | 1 | 3 | 2 | 1 | 4 |
| 3 | 0 | 0 | 1 | 0 | 1 | 8 | 6 | 9 | −2 |
| 3 | 0 | 0 | 2 | 0 | 2 | 7 | 6 | 14 | −1 |
| 3 | 1 | 0 | 0 | 2 | 1 | 6 | 4 | 0 | 1 |
| 3 | 1 | 0 | 1 | 2 | 2 | 5 | 4 | 5 | 3 |
| 3 | 1 | 1 | 1 | 0 | 1 | 5 | 3 | 3 | 3. |

There are seventeen pair rows and forty-seven triple rows in (12.6). The endpoint certificate enumerates all of them; the table lists every row that needs a geometric refinement.

12.4. **Pencils, conics, and one-relation saturation.** The following facts concern the restriction of the span of actual graphs to $Z$. Properness makes this restriction injective. Every graph from the component restricts to a rank-one map, and the kernels vary with the normal planes.

**Lemma 12.5.** *If a proper component and its conjugate span a two-dimensional restriction space, its maps have common image. If their total span has dimension three, the component is a nonsingular conic whose image vectors span a two-plane. Thus three general generators satisfy one output relation with all coefficients nonzero, at a projective parameter cost of two.*

*Proof.* A line of rank-one matrices has common image or common kernel; otherwise independent image vectors and independent covectors produce a rank-two generic sum. Varying kernels exclude the second alternative.

Suppose the component spans a projective plane. The rank-one locus is cut out by quadratic minors. If all these minors vanish on the plane, a linear space of rank-one maps has common image or common kernel. A common kernel contradicts varying labels; a common image identifies the three-dimensional space with all covectors on $Z$, making the fiber complete. Therefore some quadratic minor does not vanish on the plane. An irreducible nondegenerate plane curve contained in its zero set is a conic. Both Segre projections of the conic are nonconstant. Since the hyperplane degree is two, their degrees are $(1, 1)$, and its image vectors span a two-plane. Three general image vectors have a unique projective dependence with all coefficients nonzero.

Finally, a line component and a different conjugate line cannot span a proper projective plane. The lines intersect, and their nonzero intersection map forces their two common image lines to coincide. Their span then consists of one image vector times all covectors on $Z$, again making the fiber complete. □

Lemma 10.7 applies to the present four augmented coefficient directions, including the metric coefficient $A_0$. Its proof uses the actual one-dimensional relation space, so

no coefficient direction is deleted and no reduction from four augmented directions to three is made.

On a dominating tuple stratum, a generic block lying in the finite set of cubic pencils forces its irreducible normal component into one such pencil. The common $Z$ then has top rank two, since top rank three would determine a unique two-plane in $Z$ with that fixed top projection. Consequently $Z \cap A$ is a line. The reality of $Z$ also identifies the fixed top plane of the conjugate component.

12.5. **Coefficient-kernel exclusions and the corank-one rows.** If $\gamma = 1, l = 0, \delta = 0$, the last images are disjoint. Since the seed contains no rank-one matrices, its restriction through each rank-two last map is injective. Therefore $j = 0$, and all rows with $j > 0$ in this case are impossible.

If $\gamma = 1, l = 0, \delta = 1, j = 2$, let $N$ be the relation line and $J$ the two-dimensional coefficient kernel. Choose a block with nonzero projection of $N$. The projection of $J$ to that normal input is injective: a zero component would make every coefficient row in $N$ vanish, and joint injectivity would then kill the vector. Hence it is an isomorphism. The augmented rows of the whole normal pencil lie in one line in the restricted last image. Restoring its last kernel places all top and lift rows in a fixed two-plane in $A$. This is the locus $\mathcal{P}_0$, already excluded. The argument allows zero projections of $N$ on other blocks.

Consider next $\gamma = l = 1, \delta = 1, j = 0$. Each old coefficient input is a line with output span two. For any $s$-dimensional subspace of their direct sum, projection to one block followed by passage to its kernel shows that the unquotiented row span has dimension at least $2s$. The one-dimensional relation reduces this by at most one. A rank-$s$ skew witness would require $2s - 1 \leq s$. Rank-one witnesses are impossible, so the cokernel vanishes. At length three the raw value rises from three to six.

At length two it rises from four to five. If $Z \cap A$ is a line, the last kernels coincide, saving two parameters relative to two independent points of $\mathbb{P}(A)$. This saving is separate from the common-plane saving already charged: the diagonal kernel choice and a common $Z$ meeting $A$ may be counted independently, which only overcounts. If $D_Z = Z \cap A$ has dimension two, the proper restriction pencil has common image, so its nonzero images of $D_Z$ coincide. The last relation line lies in $(D_Z/k_1) \oplus (D_Z/k_2)$, a fixed two-plane in the four-dimensional abstract last sum. This saves two parameters in its relation-line fiber, and no additional common-plane saving is charged in this branch. Both cases give seven. Intersection dimension three would contradict $l = 1$.

12.6. **Free rectangular blocks and the remaining positive overlaps.** For full last rank and $l = j = 0$, put

$$\widetilde{H} = \bigoplus_{i=1}^{\ell} E_i, \qquad \dim E_i = 3, \qquad H = \widetilde{H}/N, \qquad \dim H = 3\ell - \delta.$$

The total last span of a component and its conjugate is real and therefore nondegenerate for the positive old metric. Write $E = H \oplus Q$ and hold $A_0$ fixed. The three top symmetric operators have arbitrary rectangular blocks $H \to Q$, independent of their

symmetric blocks inside $H$. Last matrices whose images lie in $H$ do not sample the transposed blocks.

For fixed $Z$ and fixed last matrices, let

$$F : \bigoplus_i Z \longrightarrow \mathbb{C}^3 \otimes H$$

be the expanded top coefficient map. If it is injective, the rectangular blocks prescribe all $Q$-valued evaluations on $\bigoplus_i Z$ independently. After the old normal equations on $\bigoplus_i U_i$, the new outputs $\pi_Q X_i z_i$, with chart vectors $z_i \in Z \setminus U_i$, are independent arbitrary vectors in $Q$. Contributions of $A_0$ are affine right-hand sides. The old skew allowance concerns the symmetric blocks inside $H$, so the new output conditions impose additional codimension in the correct actual parameter fiber. The full graph-image span is not fixed here.

*Two generators and one relation.* The raw value is zero and $h = 5$. If $Z \cap A = 0$, the map $F$ is injective: a kernel vector would put all unquotiented rows of a nonzero block seed matrix in the projection of the relation line, giving rank at most one. Its top coefficient must therefore vanish, and transversality of $Z$ makes the whole block vector vanish. By Lemma 12.5 the two new $Q$-outputs must be proportional. The rank-at-most-one locus in $\mathrm{Mat}_{(r-5)\times 2}$ has codimension $r - 6$, which equals six at the endpoint. We retain the old allowance $\chi = 5$.

If $Z \cap A = \mathbb{C}k$, this common-plane position has codimension two, which has not been charged before because $l = 0$. The nonzero vectors $Y_1 k, Y_2 k$ are proportional. Hence the relation line lies in their fixed abstract two-plane inside $\mathbb{C}^6$, saving four further parameters. The two costs occur successively, first in $Z$ and then in the relation fiber at fixed $Z$, and together give six. A larger last intersection is incompatible with $U_i \cap A = 0$.

*Three generators and one relation.* The raw value is $-2$ and the old allowance is nine. A nonzero skew witness has rank at most two. Lemma 10.7 therefore either removes the cokernel, giving seven, or forces the component into a fixed cubic pencil. In the latter case $Z \cap A = \mathbb{C}k$. By the conic conclusion of Lemma 12.5, the three nonzero vectors $X_i k = A_0 Y_i k$ are dependent. Since $A_0$ is invertible, the relation line lies in the fixed three-plane $\bigoplus_i \mathbb{C}(Y_i k) \subset \mathbb{C}^9$. Its position saves six parameters in $\mathbb{P}^8$, and the last-line condition on $Z$ saves two. Keeping the old allowance nine gives $-2 + 6 + 2 = 6$. We add no separate single-evaluation position saving here.

*Three generators and two relations.* Now $h = 7$, $u = 6$, and the raw value is $-1$ with allowance fourteen. The relation plane varies in $\mathrm{Gr}_2(\mathbb{C}^9)$, of dimension fourteen. There are three exhaustive branches.

First suppose $Z \cap A = 0$ and $F$ is injective. Choose a coefficient direction whose evaluation on every old normal input is injective before quotient. If the quotient evaluation has kernel dimension $a = 0, 1, 2$, its relation-plane position costs $a(a + 1)$. The one-evaluation skew estimate from the exterior proof gives the respective cokernel caps $10, 10, 11$; after the position saving their maximum is ten. The three new $Q$-outputs are arbitrary. The conic relation forces their matrix to have rank at most two, a condition

of codimension $r-9$. At $r=12$ the total bound is

$$-1+14-10+3=6.$$

The relation-position cost concerns only last and normal data, while the three extra equations concern independent rectangular operator blocks over each such stratum.

Second suppose $Z\cap A=0$ and $F$ is not injective. A projective kernel vector belongs to $\mathbb{P}(\bigoplus_i Z)=\mathbb{P}^8$. Its unquotiented row span lies in $N$ and, since some block seed matrix is nonzero, has dimension at least two. It therefore equals $N$ and determines that relation plane uniquely. The bad relation locus has dimension at most eight, hence codimension at least six. In this branch we do not also charge the preceding relation-position saving. The uniform one-evaluation cap before that saving is eleven, giving

$$-1+14-11+6=8.$$

Finally suppose $Z\cap A=\mathbb{C}k$. Its position saves two. By the conic, the three nonzero vectors $Y_ik$ are dependent, so $N$ meets their fixed abstract three-plane. The condition $\dim(N\cap\mathbb{C}^3)\geq 1$ has Schubert codimension five in $\mathrm{Gr}_2(\mathbb{C}^9)$. At fixed $Z$ this is a new relation-position condition. Retaining the old allowance fourteen gives $-1+2+5=6$, and no competing relation-position cost is added. These three branches settle the last row requiring a full-last refinement.

12.7. **Zero overlap: independent shared-output equations.** At zero overlap the last images are disjoint, and $\gamma=l$. If $\gamma=1$, the generic last kernels of the component and its conjugate coincide for $\ell\leq 3$, because a combination of last matrices with disjoint old images has last rank two only when its active summands share the same row plane. A positive-dimensional component in each distinct kernel group would therefore require at least two independent generators. Equating the kernels saves $2(\ell-1)$, and requiring $Z$ to contain their line saves two. This cancels the raw last-kernel loss $2\ell$. For either last rank the disjoint-graph estimate is therefore

$$(5\ell-g)(r-g)+5\ell-12,\qquad 5\leq g\leq\min(r,5\ell-1), \tag{12.10}$$

where $g$ is the dimension of the full graph-image span. The strict upper bound $g<5\ell$ holds because independent full graph images would make every irreducible rank-five combination have a constant normal label.

Lemma 12.5 gives the extra equation

$$\sum_{i=1}^{\ell}\theta_iX_iz_i=0,\qquad \theta_i\neq 0, \tag{12.11}$$

at projective coefficient cost $\ell-1$. We prove that it adds $g$ independent equations in the symmetric parameter fiber; arbitrary graph-column freedom is not assumed.

In full last rank, $Z\cap A=0$. Otherwise the images of a nonzero pure-last vector under the $\ell$ graphs would be nonzero vectors in disjoint old last spaces, contradicting the common image line for $\ell=2$ or the common image two-plane for $\ell=3$. In last rank two the same argument gives $Z\cap A=k$, the common last kernel. We work on $Z$ or $Z/k$, respectively.

On each expanded coefficient block every subspace has row span of dimension at least its own dimension. On every nonzero subspace of an old normal input the expansion is strict. For full last rank these statements follow from minimum seed rank two and the exclusion of $\mathcal{P}_0$. A three-dimensional subspace of the seed cannot have rows in a fixed two-plane, since its projective plane in $\mathbb{P}\operatorname{Mat}_{3\times 2}$ would meet the rank-one Segre variety. For last rank two, injectivity of the restricted seed gives weak expansion, and the exclusion of $\mathcal{E}_1$ gives strict expansion on the old normal line.

Adjoin to the old direct sum one shared scalar input, projecting to $\theta_i z_i$ in every block, and call the resulting input $V'$. Let $B_\rho$ be its augmented coefficient maps. Suppose a nonzero dual symmetric-evaluation witness $\zeta$ exists. Set $J = \operatorname{im}\zeta^t$, $W = \operatorname{im}\zeta$, and $s = \dim J = \dim W$. Skewness gives

$$\sum_\rho B_\rho(J) \subset W, \qquad \dim \sum_\rho \mathbb{C}B_\rho j \le s-1 \quad (0 \ne j \in J). \tag{12.12}$$

For the second assertion, write $j = \zeta^t\alpha$. Then $\alpha$ does not vanish on $W$, while $\alpha(B_\rho\zeta^t\alpha) = 0$. This argument pairs vectors with covectors and requires no nondegeneracy of the restriction of the old metric to $W$.

If a block projection of $J$ has a nonzero kernel $K$, the shared scalar vanishes on $K$, so this kernel lies in the other old normal inputs. Projection followed by kernel induction gives strict row expansion there, of at least $\dim K + 1$. The chosen block projection gives weak expansion of at least its own dimension. Counting that output projection and its kernel gives total row dimension at least $s+1$, contradicting (12.12). This count cannot be replaced by adding the dimensions of two correlated output projections.

Every block projection of $J$ is therefore injective. In full last rank, $s \le 3$, and minimum seed rank two excludes $s \le 2$. If $s = 3$, a block projection fills $Z$, whose top coefficient space is three-dimensional. This space contains an invertible matrix, since a smooth cubic surface contains no projective plane. Its three independent coefficient outputs contradict the bound $s - 1 = 2$. In last rank two, $s \le 2$. Joint injectivity excludes $s = 1$, while for $s = 2$ we project to the old normal line. Its coefficient rank is two outside $\mathcal{E}_1$, again contradicting (12.12). Thus the combined symmetric evaluation is onto.

Fix the real $g$-dimensional full graph span and choose the positive metric gauge block diagonal with respect to it and its orthogonal complement. The off-diagonal equations place all graph images in that span. Its inside symmetric blocks remain free, and the preceding surjectivity applies there. Equation (12.11) therefore adds exactly $g$ independent equations after the old normal equations. Returning its $\ell - 1$ projective coefficients improves (12.10) by $g - (\ell - 1)$.

At $r = 12$ the final bounds are

$$(10-g)(12-g) + g - 3 > 5 \qquad (5 \le g \le 9), \tag{12.13}$$

$$(15-g)(12-g) + g + 1 > 5 \qquad (5 \le g \le 12). \tag{12.14}$$

Their minima are nine and thirteen. For larger $r$, every fixed row increases. The newly allowed triple values $g = 13, 14$ satisfy $r \ge g$ and have bound at least $g + 1 > 5$. The value $g = 15$ does not occur for a proper dominating tuple.

12.8. **Completion of the core proposition.** The finite endpoint enumeration of (12.6), using only the refinements just proved, gives

| $\ell$ | all rows | impossible | retained | minimum positive-overlap bound |
|---|---|---|---|---|
| 2 | 17 | 2 | 15 | 6 |
| 3 | 47 | 3 | 44 | 6. |

Together with (12.13)–(12.14) this excludes every proper positive-dimensional Schubert fiber. Each positive-overlap bound has positive coefficient $L$ in $r$, and all added output costs are constant or increasing. The exceptional and preliminary bounds also persist for all $r \geq 12$.

The endpoint margins are collected below. The listed numbers are projected codimensions, that is, codimensions after the witnesses are allowed to vary.

| excluded failure at $(r, c) = (12, 5)$ | codimension lower bound |
|---|---|
| two distinct bad last directions | 8 |
| higher kernel at one last direction | 12 |
| one fixed exceptional last direction | 6 |
| rank-one graph rank drop | 13 |
| rank-two / rank-three graph rank at most four | 7 / 6 |
| column-pencil common eigenline / rank-two adjoint | 11 / 7 |
| actual ordinary corank one / higher corank | 8 / 15 |
| additional exceptional graph above a bad last direction | 7 |
| proper exceptional Schubert fiber | 6 |
| ordinary Schubert tuple of length at least four | 8 |

Choose a real unit vector $\eta \in \mathbb{R}^7$ whose top coefficient $C_\eta$ is invertible, and fix an isometry $I : \mathbb{R}^6 \to \eta^\perp$. For the prescribed rank-five subbundle $A \subset \varepsilon^6$, the normal two-plane of $I(A)$ contains $\eta$ and varies only over the slow base. We verify injectivity on $E \otimes I(A)$ for this formal standard activity.

For a fixed such normal plane, an old-rank-two adjoint witness has three independent coefficient inputs. A nonzero symmetric-evaluation cokernel would force its whole top normal pencil onto the determinant cubic, which is impossible because the pencil contains $C_\eta$. There are therefore $3r$ equations and $2r - 1$ projective witness parameters, giving the bound $r + 1$. For an old-rank-one witness, the top coefficient has rank three except at finitely many zeros of the nonzero determinant polynomial on the pencil. A rank-two zero costs no coefficient-direction parameter; its $2r$ equations and $r - 1$ old-vector parameters again give $r + 1$. At a rank-three direction the witness equations have rank $3r$, and the parameters are the $r - 1$ old-vector parameters and one coefficient direction in the normal pencil. These $3r$ equations and $r$ parameters give the stronger bound $2r \geq r + 1$.

A pure-last direction cannot support a nonzero witness, because the metric coefficient $A_0$ is invertible.

Once the contraction is onto, its kernel has dimension $r$. Restriction of the free residual map to this kernel is onto the space of all maps from that kernel to the rank-$(r + c)$ residual target. Its noninjective locus has codimension $c + 1 = 6 > 5$. The

fixed-normal failure bound $r+1>5$ and this residual bound are therefore avoided over the whole slow core. This proves that the prescribed formal standard activity is safe, before the later actual activity is chosen.

Finally, Lemma 9.4 applies to the Tiles family at old rank $r$: its ordinary and curvature failure loci are closed, of codimensions at least $2r-3$ and $2r$. The residual ordinary adjoint is a nonzero skew-label witness of label rank $s=2$ or 3. Fixing its label image and projective coefficients, its equations on the free residual maps have rank $s(r+c)$, and its parameters cost at most $r\binom{s}{2}+s(3-s)-1$. Thus residual ordinary failure has codimension at least

$$\min_{2\le s\le 3}\left\{s(r+c)-r\binom{s}{2}-s(3-s)+1\right\}=\min(r+2c-1,3c+1)\ge 16>5.$$

It too can be avoided simultaneously. Every compression has old rank $r-1$. To include all hyperplanes, we parametrize them by $\mathrm{Gr}_{r-1}(E)$, of dimension $r-1$; together with the five-dimensional slow core the joint parameter dimension is at most $5+(r-1)=r+4$. For each fixed real hyperplane $H\subset E$, the restriction map

$$\mathrm{Sym}(E)^3\longrightarrow\mathrm{Sym}(H)^3,\qquad (A_\rho)_\rho\longmapsto(P_HA_\rho|_H)_\rho,$$

is surjective: extend each selfadjoint operator on $H$ by zero on $H^\perp$. Lemma 9.4 therefore applies on every fiber with old rank $r-1$. Ordinary failure costs $2r-5$, and curvature-pair failure costs $2r-2$. At $r=12$ these are $19>16$ and $22>16$; their excesses are $r-9$ and $r-6$, respectively, and remain positive for every $r\ge 12$. For fixed $H$, the triples whose compression to $H$ lies in a failure locus of Lemma 9.4 form the preimage of that closed locus under the surjective restriction map. The union over $H\in\mathrm{Gr}_{r-1}(E)$ is the projection of a closed set along a compact Grassmannian. It is therefore closed, of codimension at least $(2r-2)-(r-1)=r-1$ for curvature and $(2r-5)-(r-1)=r-4$ for the ordinary operator; both exceed five. We apply the relative genericity principle to the full symmetric triple and the residual maps, as in the proof of Theorem 11.6, with base dimension at most five, and avoid these closed loci, the full-rank failure loci above, and the ordinary and incidence bad loci counted above. All required conditions then hold for the same parameters, relative to any collar on which they already hold. The finite-or-complete property is used to select the compact activity homotopy before the normalized small corrections. These later corrections retain its open injection and regularity margins; the finite-or-complete condition itself need not be open under arbitrary perturbations. This proves Proposition 12.1.

## 13. The active block and the shared normal coordinate

Theorem 5.1 provides the metric and free columns of the active block. In this section we prove the additional complete-class statement needed on the core, without adding a coordinate.

**Lemma 13.1** (High homotopy groups of the orthogonal frame group)**.** *For $D\ge 2$, the group $\pi_i(\mathrm{SO}(D))$ is finite whenever $i>2D-3$.*

*Proof.* For $D = 2$ the group is a circle and its higher homotopy groups vanish. For the induction step we use the fiber bundle $\mathrm{SO}(D-1) \to \mathrm{SO}(D) \to S^{D-1}$. By Serre's finiteness theorem [Ser53], $\pi_i(S^{D-1})$ is finite for $i > 2D-3$. This range lies above $2(D-1)-3$, so the left group in the homotopy exact sequence is finite by induction. The middle group is an extension of subquotients of these two finite groups, hence finite. □

**Lemma 13.2** (Coordinate powers kill maps to a finite Postnikov range)**.** *Let $X = (S^{2c-1})^k$, based at $(e_1, \dots, e_1)$, and let $Y$ be a path-connected space such that $\pi_{j(2c-1)}(Y)$ is finite for $1 \le j \le k$. For every continuous map $F : X \to Y$ there is a positive integer $K$ such that $F \circ P_K$ is nullhomotopic, where $P_K$ is the product of*

$$p_K : S^{2c-1} \to S^{2c-1}, \qquad p_K(z) = \frac{(z_1^K, \dots, z_c^K)}{(\sum_\alpha |z_\alpha|^{2K})^{1/2}}.$$

*Proof.* The map $p_K$ is smooth, fixes $e_1$, and has degree $K^c$: the degree is the number $K^c$ of inverse images of a regular value with all coordinates nonzero under the corresponding complex polynomial map, and every local sign is positive. Moreover $p_K p_L = p_{KL}$.

Put $d = 2c-1$. The product CW structure of $X$ has cells only in dimensions $0, d, 2d, \dots, kd$. The based product map $P_K$ preserves every skeleton and induces multiplication by $K^{cj}$ on each $jd$-cell of the quotient $X^{(jd)}/X^{((j-1)d)}$. We first homotope the map at the zero cell to a fixed base point of $Y$. Suppose inductively that its restriction to $X^{((j-1)d)}$ is constant. Then its restriction to $X^{(jd)}$ factors through a wedge of $jd$-spheres. Choose $K_j$ divisible by the exponent of the finite group $\pi_{jd}(Y)$. After precomposition with $P_{K_j}$ all these sphere classes vanish, so the restriction is nullhomotopic relative to the lower skeleton. By the homotopy extension property this homotopy extends to $X$, and we obtain a representative constant on $X^{(jd)}$. We repeat this for $j = 1, \dots, k$. The final map is nullhomotopic on all of $X$, and its class is that of $F \circ P_K$, with $K = K_1 \cdots K_k$. Throughout, we track homotopies of maps on the whole product; the complete class is not replaced by the classes on the individual sphere factors. □

**Proposition 13.3** (Exact active free block with one surplus coordinate)**.** *Put $D = j_k + 1$ and let the common sphere size satisfy $c \ge D$. There is an actual smooth map*

$$b : (S^{2c-1})^k \longrightarrow \mathbb{R}^{j_k+1}$$

*with exact standard partial Gram and independent complete first and symmetric second Hopf columns, whose complete column map is homotopic through injections to the standard constant injection $\mathbb{R}^{j_k} \hookrightarrow \mathbb{R}^{j_k+1}$. All coordinate-subsphere restrictions are literal restrictions of this one map. The integer $K$ may depend on $c$.*

*Proof.* Let $H$ be the actual free exact unit-Gram map from Theorem 5.1. The ordered $j_k$ free columns have a canonical oriented unit cofactor normal in $\mathbb{R}^D$. Applying Gram–Schmidt to the columns and appending that normal, we obtain a smooth complete frame map $F : X \to \mathrm{SO}(D)$. Since $2c-1 \ge 2D-1 > 2D-3$, every homotopy group of $\mathrm{SO}(D)$ in the cell dimensions of $X$ is finite. The preceding lemma gives an integer $K$ with $F \circ P_K$ nullhomotopic.

The coordinate powers intertwine each Hopf flow with its $K$-fold speedup. Thus, for the standard generators $Y_i$,

$$Y_i(H \circ P_K) = K(Y_i H) \circ P_K, \qquad Y_i Y_j (H \circ P_K) = K^2 (Y_i Y_j H) \circ P_K.$$

Define $b = K^{-1} H \circ P_K$. Its first columns are the composed first columns of $H$, so its partial Gram is the identity. Its second columns are $K$ times the composed second columns of $H$, so the full columns remain independent. Multiplying the second column group by a path of positive scalars from $K$ to 1 gives a homotopy through injections from its complete column map to that of $H$ composed with $P_K$. Orthogonalization and its inverse are likewise homotopies through injections. The nullhomotopy of $F \circ P_K$ therefore supplies the required complete homotopy to the standard frame. Continuous frame homotopies may be smoothed relative to endpoint collars; by compactness and openness of injectivity the smoothing preserves their rank. □

For $k = 6$ the target dimension is $j_6 + 1 = 28$, so every common sphere size $c \geq 28$ lies in the required topological range. The integer $K$ and all derivative bounds may depend on this fixed sphere size; the assembly does not require a uniform choice across different sphere sizes.

**Proposition 13.4** (One shared normal coordinate)**.** *Let the principal model have target $V_0 \oplus \mathbb{R}e$ of dimension $D_P$, and let its old free map $H(x)$ take values in $V_0$, of dimension $D_P - 1$. Suppose that the complete old class can be realized in $V_0$ and that all normal-frame realizations and their curvature equations hold in the model target $V_0 \oplus \mathbb{R}e$. Let $b(y) : X_A \to \mathbb{R}^{j_k+1}$ be the actual exact active-free map in the standard complete class and let $\eta(y)$ be its unit cofactor normal. Then the normal-frame perturbation, periodic mixed inverse, finite metric completion, phase protection and complete boundary class can all be realized in the physical target*

$$V_0 \oplus \mathbb{R}^{j_k+1},$$

*whose dimension is $D_P + j_k$, one less than the unshared sum.*

*Proof.* Put $F_0(x, y) = (H(x), b(y))$ and define an isometric fiber map

$$\Phi_y : V_0 \oplus \mathbb{R}e \longrightarrow V_0 \oplus \mathbb{R}^{j_k+1}, \qquad \Phi_y(v + te) = (v, t\eta(y)).$$

It fixes $V_0$, is independent of the old fast variable $x$, and satisfies

$$\langle \Phi u, \partial_i \Phi v \rangle = 0. \tag{13.1}$$

This follows from $\langle \eta, \partial_i \eta \rangle = 0$. Moreover, since $\eta$ is perpendicular to the complete active jet,

$$\langle \partial_i \eta, b_j \rangle = -\langle \eta, b_{ij} \rangle = 0. \tag{13.2}$$

The active free-jet complement is a line, so $\partial_i \eta$ belongs to the span of the complete active columns. Together with (13.2), this shows that all nonconstant parts of $\Phi$ are pure active second-column components.

Let $\nu = (\nu_1, \ldots, \nu_q)$ be the model horizontal frame, let $a(y)$ be the activity, and set

$$Z = \Phi_y(\nu(x) a(y)).$$

The model frame is perpendicular to $J_E H$. Equations (13.1)–(13.2), together with this orthogonality, show that $Z$ pairs to zero with every first and pure second column of $F_0$. Its mixed base second columns are zero. Consequently $D\Gamma_{F_0} Z = 0$ in every metric row. For an old derivative $\partial_\alpha$ and an active derivative $\partial_i$,

$$\partial_\alpha Z = \Phi(\partial_\alpha \nu\, a), \qquad \partial_i Z = \Phi(\nu\, \partial_i a) + (\partial_i \Phi)(\nu a).$$

The second summand is perpendicular to the image of $\Phi$, while the first gives the original horizontal-frame pairing. Hence $\Gamma(Z)_{\alpha i} = 0$, as in the unshared construction. The pure active quadratic row changes, but the active free pairing inverse cancels it in the second-order correction.

For a model residual normal field $v(x, y)$ we use the physical field $\widehat{v} = \Phi_y v$. It is orthogonal to the complete jet of $F_0$, since $\Phi$ identifies the model old free-normal bundle with the orthogonal complement of the old and active pure complete-column blocks. Its old derivatives have no $\partial_\alpha \Phi$ term, and (13.1) cancels every new term in the mixed quadratic cross expression. Thus

$$\begin{aligned} \sigma(Z, \widehat{v})_{\alpha i} &= \langle \partial_\alpha(\nu a), \partial_i v\rangle + \langle \nu \partial_i a, \partial_\alpha v\rangle \\ &= \partial_i \langle \partial_\alpha \nu\, a, v\rangle - 2\langle U(\partial_\alpha, \partial_i a), v\rangle. \end{aligned}$$

This is the mixed recurrence used in Proposition 18.1, so the periodic operator $B^* v + d_A f$ applies to the same model coefficient bundle. Its unknowns may depend on active variables; (13.1) still cancels their new frame-derivative terms.

At every higher step we change $W_{r-1}$ only by such a field $\widehat{v}$. The subsequent correction $W_r$ may lie in the direct sum of the old and active complete-column spans, where its prescribed first and second pairings are solvable by independent algebraic inverses. It need not lie in the image of $\Phi$. All its effects are included in the next known error coefficient, whereas the freedom used to cancel that coefficient is again a field $\widehat{v}$. Thus the induction constructs all the finite coefficients required in Proposition 18.1.

For the phase calculation, the extra term $(\partial_i \Phi)(\partial_\alpha \nu\, a)$ in a mixed second column belongs to the complete active-column span by (13.2). It need not belong to the smaller second-column span of the selected active hyperplane, so we do not eliminate using only those columns. Instead we project onto the image under $\Phi$ of the model old normal bundle. This kills every complete active column and leaves $\Phi U(\mathrm{Id}_E \otimes d_A a)$ after eliminating old components. Its protected injectivity proves independence of the actual mixed columns together with the pure columns. It also allows a formal homotopy multiplying all the extra active components by a scalar from one to zero, while the projected mixed map is held fixed. Thus all the original phase margins and protected complete classes persist. Theorem 6.3 applies in the resulting target, whose dimension still exceeds $s_n + 1$.

Finally we use the full orthogonal frame homotopy of $b$, including $\eta$, to the standard frame, and carry $\Phi$ along this homotopy. Throughout it the active complete columns are orthogonal to the image of the model normal bundle. At the standard endpoint $\Phi$ is one fixed isometric inclusion, so the slow complete boundary injection $U_0 \kappa_v \oplus A_+$ of (8.1) is obtained in the physical target. The complete source transports of Proposition 8.2

and the relative hyperplane fill therefore apply with no new class assumption. This proves the proposition. □

## 14. Protected activity, collars, and actual normal frames

In this section we select the protected activity, join the core and exterior tensors along a collar, and realize the formal normal frame by an actual horizontal frame. The two tensor constructions have different base dimensions. The exterior tensor is chosen over an $n$-dimensional slow base, and the tensor on the core over $P$, whose dimension is at most five. Once the slow choices are made, they are pulled back to the whole Hopf source. General position is never applied as though this larger source had dimension five.

14.1. **Selecting the activity before perturbing the tensor.** On the prepared core let $A = K \oplus B_0$ be the rank-five subbundle of the trivial active bundle $\varepsilon^6$. The core construction gives a tensor $T$ with a safe standard injection on $Q^+ \otimes A$, where $Q^+ = Q \oplus \varepsilon$. For each real four-plane $Z \subset \mathbb{R}^7$, its bad five-plane fiber through $Z$ is either finite or the whole projective plane of five-planes containing $Z$.

**Lemma 14.1** (Protected activity)**.** *There is a partial immersion $a_0$ from the six active Hopf factors to $\mathbb{R}^7$, independent of the old fast variables, and a formal derivative homotopy*

$$F_s : \mathbb{R}^6 \longrightarrow \mathbb{R}^7, \qquad F_0 = I, \quad F_1 = d_A a_0,$$

*such that $T$ is injective on $Q^+ \otimes F_s(A)$ for every core point, active point, and $s \in [0,1]$. The activity and its formal class extend over the exterior relative to an inner collar on which they are unchanged.*

*Proof.* Consider the injective linear maps $F : \mathbb{R}^6 \to \mathbb{R}^7$ for which $T|_{Q^+ \otimes F(A)}$ is injective. This open relation contains $I$; we show that it is ample. Fix the derivative on a principal hyperplane $C \subset \mathbb{R}^6$. If $C = A$, the protected image is fixed, and the immersion condition excludes an affine subspace of codimension two. Otherwise $Z = F(A \cap C)$ has dimension four. A nonempty admissible slice cannot have a whole bad fiber through $Z$, since its existing safe five-plane belongs to that fiber; hence the bad fiber is finite. Each bad five-plane excludes an affine five-plane in the seven-dimensional free derivative variable, and immersion excludes at most one further affine subspace of codimension two. The complement is connected and its convex hull is the whole space. If a principal hyperplane of the full source contains all active directions, the relation imposes no varying condition in that slice. Hence the relation is ample.

Apply Theorem 3.1 with the core variables as parameters. The resulting activity and homotopy have a positive compact injection margin. Pull them radially to an inner collar. Extend the underlying map by a cutoff, and extend its formal derivative by running $F_s$ backwards to $I$ on a larger collar. The ordinary partial-immersion relation is open and ample, so its relative realization gives the exterior extension. Later perturbations of the activity may be taken arbitrarily small in full $C^1$; straight-line interpolation of their derivatives adds a short protected homotopy. □

The finite-or-complete fiber property need not be open, and it is used only in this lemma. Subsequent tensor perturbations preserve the compact family of injective restrictions already selected.

14.2. **Normalized positive collar scaling.**

**Lemma 14.2** (Collar compatibility in the counted complement)**.** *Suppose the algebraic tensor families on the core and the exterior have the ordinary, selfadjoint, and curvature regularities and the relative rank-avoidance properties proved in the preceding sections. With $r = M + 1$, $c = 5$, and $C_0 = 10 = c + 5$, they determine compatible curvature-prescribed tensors that coincide on a protected collar and retain the activity family of Lemma 14.1. No additional target coordinate is needed.*

*Proof.* The counted normal bundle of the whole core has rank $5r + 5$. Choose its four top copies of $Q^+$; their complement $V_Q$ has rank $r+5$. On the boundary $Q^+ = E_v^+ \oplus vK$, so the normal bundle splits as

$$(E_v^+)^4 \oplus V_0 \oplus V_Q, \qquad V_0 = (vK)^4.$$

Extend the four top copies of $E_v^+$ to the exterior. Their rank-failure codimension is $M + 11 > n = M + 5$, so relative general position supplies the extension, and polar normalization makes it isometric. Let $V_E$ denote its orthogonal complement. On the boundary

$$V_E = V_0 \oplus V_Q, \qquad \operatorname{rank} V_E = 4 + (r + 5) = M + 10.$$

The summand $V_0$ is used only on the collar; we do not extend it globally.

Compress the universally flat selfadjoint top array of the whole core to $E_v^+$. Its omitted rows give a map $D_0$ to $V_0$, and its retained top array is denoted by $S^E$. Orthogonality and the two flat identities give $\Omega(D_0) = 0$. The total residual on the collar is $D_{\text{full}} = D_0 \oplus D_Q$. Its kernel is the restriction of the kernel on the whole core, so the required weak-label bound persists. Since compression is linear in the symmetric operators, $S^E = S(A^E)$ with $A_\rho^E = P_{E_v^+} A_\rho|_{E_v^+}$, a triple of rank $r - 1 = M$. By item (5) of Proposition 12.1 this compression avoids the ordinary and curvature failure loci of Lemma 9.4. By that lemma the ordinary operator with residual $D_{\text{full}}$ is onto, and $D\Phi_{S^E}$ is onto. Thus $(A^E, D_{\text{full}})$ is good, in the sense of Theorem 11.6, on an open collar. Apply that theorem to this $V_E$-valued residual and its top arrays, with $X_0$ a closed collar inside the open one. The resulting exterior family coincides with $(A^E, D_{\text{full}})$ near $X_0$.

Choose a slow cutoff $\eta$, equal to zero on the inner collar and to one on a larger collar, and a small positive scalar $\delta$. On the collar use the physical residual

$$\delta^\eta D_0 \oplus \delta D_Q;$$

outside use $\delta D_{\text{full}}$. The two formulas agree where $\eta = 1$. Normalize by the bundle automorphism that acts on the collar as the identity on the top copies, as $\delta^{-\eta}$ on $V_0$, and as $\delta^{-1}$ on $V_Q$, and outside as $\delta^{-1}$ on $V_E$. The normalized residual is then the fixed good residual $D_{\text{full}}$ everywhere. Since both physical scales are positive, the kernel is not enlarged.

First apply the selfadjoint top inverse on the whole core to remove its $O(\delta^2)$ last-last flatness error. The resulting changes in its compressed top rows and omitted rows are

$O(\delta^2)$. Extend them radially, and insert the change of the omitted rows as $\delta^\eta\zeta\Delta D_0$, where $\zeta$ becomes zero only after $\eta = 1$. Its normalized size is $O(\delta^2)$. The change in the collar flatness equation is still $O(\delta^2)$: the corrected omitted rows have alternating Gram $O(\delta^2)$, and their rescaling multiplies that Gram by the factor $\delta^{2\eta} - 1$, whose absolute value is at most one. The exterior selfadjoint inverse removes the remaining error and vanishes on the protected inner band.

Hence the normalized correction tends to zero. Choose $\delta$ after the protected compact activity family has been fixed; then its positive injection margin, all ordinary regularities, and the exterior weak bound survive. In any prescribed finite norm the additional cutoff losses are $O(\delta^2|\log\delta|^a)$ and also tend to zero.

Transport the target data through the free homotopy of the old map, and fix this positive $\delta$. Multiply the flat tensor by $\lambda$ and solve the prescribed connection-curvature equation at size $\lambda^{-2}$ with the full curvature inverse, first on $Q^+$ and then on the exterior relative to the protected collar. Rescaling back gives the required actual curvature. For $\lambda$ sufficiently large every normalized open margin is preserved. The corrections are small homotopies and preserve the collar class. The construction introduces neither an extra top copy nor a separately extended $V_0$. □

14.3. **The fixed-base horizontal step.** Let $X$ be the compact old source, $E \subset TX$ its integrable coefficient distribution of rank $m$, and $H$ the old $E$-free map. We may assume that $H$, $E$, and all data below are defined on an open manifold $X' \supset X$. Put $V = (\operatorname{im} J_E H)^\perp$, with the target metric and the normal connection along $E$,

$$\nabla_u w = P_V(\partial_u w) \qquad (u \in E,\ w \in \Gamma(V)).$$

It is a metric partial connection along $E$, with leafwise curvature $R^V(u,u') = [\nabla_u, \nabla_{u'}] - \nabla_{[u,u']}$.

*Microextension.* Let $Y = X' \times \mathbb{R}$, with projection pr to $X'$ and coordinate $t$. Put $E^+ = (d\,\mathrm{pr})^{-1}(E) = E \oplus \mathbb{R}\partial_t$, an integrable distribution of rank $m+1$, and pull $V$ and $\nabla$ back by pr. Then $\nabla$ is a metric partial connection along $E^+$, $\nabla_{\partial_t} = \partial_t$ in pulled-back frames, and $R^V(\partial_t, \cdot) = 0$. In a foliated chart we use leaf coordinates $\sigma = (\sigma^0, \dots, \sigma^m)$, $\sigma^0 = t$, spanning $E^+$, and transverse coordinates $z$. Greek indices run over leaf coordinates. Multi-indices $I, J, K, L$ run over *all* chart coordinates, and $I + i$ raises the $i$-th entry by one.

*Frames.* A frame is a section $\nu$ of the Stiefel bundle $\mathrm{St}(V) \to Y$ of orthonormal seven-frames, viewed as an isometry $\nu : \mathbb{R}^7 \to V$. For $B_1, B_2 \in \operatorname{Hom}(\mathbb{R}^7, V_p)$ let $B_1^* B_2 \in \operatorname{End}(\mathbb{R}^7)$ be the matrix $(\langle B_1\epsilon_a, B_2\epsilon_b\rangle)$. Put $W_\nu = V \ominus \nu(\mathbb{R}^7)$ and, for $u \in E^+$,

$$\Theta(\nu)(u) = \nu^*\nabla_u\nu, \qquad U(\nu)_u = P_{W_\nu}\nabla_u\nu \in \operatorname{Hom}(\mathbb{R}^7, W_\nu).$$

Thus $\Theta(\nu)_{ab} = \theta_{ab} = \langle \nu_a, \nabla\nu_b\rangle$, and $\nabla_u\nu = \nu\,\Theta(\nu)(u) + U(\nu)_u$. The frame is horizontal along $E$ exactly when $\Theta(\nu)|_E = 0$.

*Pointwise algebra.* For a frame $\nu$ at $p$ and $T \in \operatorname{Hom}(E_p^+ \otimes \mathbb{R}^7, W_\nu)$ write $T_u = T(u\otimes\cdot)$ and $T_a = T(\cdot\otimes\epsilon_a)$. Identify $\Lambda^2\mathbb{R}^7$ with $\mathfrak{so}(7)$. The ordinary operator and the alternating Gram are, as after (9.5) but over $E^+$,

$$\mathcal{L}_T : \operatorname{Hom}(\mathbb{R}^7, W_\nu) \to (E_p^+)^* \otimes \mathfrak{so}(7), \quad \mathcal{L}_T(v)(u) = T_u^* v - v^* T_u, \qquad \Omega(T)(u,u') = T_u^* T_{u'} - T_{u'}^* T_u,$$

so that $\mathcal{L}_T(v)_{ab} = T_a^* v_b - T_b^* v_a$ and $\Omega(T)_{ab} = T_a^* T_b - T_b^* T_a$. Call $T$ *regular* if $\mathcal{L}_T$ is onto, and then put $R_T = \mathcal{L}_T^*(\mathcal{L}_T \mathcal{L}_T^*)^{-1}$. It depends smoothly on $T$ in the open regular set. The pair $(\nu, T)$ is *compatible* if $\Omega(T) + \nu^* R_p^V \nu = 0$ on $\Lambda^2 E_p^+$.

Let $\mathcal{P} \to X'$ be the bundle of pairs $(\nu, T)$ as above, with $E_x^+ = E_x \oplus \mathbb{R}\partial_t$. Pull it back to $Y$. Let $G_x$ be the group of $g \in GL(E_x^+)$ with $g(u) - u \in \mathbb{R}\partial_t$ for $u \in E_x$ and $g(\partial_t) \in \mathbb{R}^\times \partial_t$. It acts by $(\nu, T) \mapsto (\nu, T \circ (g \otimes \mathrm{Id}))$. An open set $\mathcal{O} \subset \mathcal{P}$ is *admissible* if it consists of regular pairs and is $G$-invariant. The following identities hold for every $g \in GL(E_x^+)$:

$$\mathcal{L}_{T\circ(g\otimes \mathrm{Id})}(v)(u) = \mathcal{L}_T(v)(gu), \qquad \ker(T \circ (g \otimes \mathrm{Id})) = (g^{-1} \otimes \mathrm{Id}) \ker T.$$

Hence regularity, the weak-label bound for $\ker T$, and injectivity of $T$ on $E^+ \otimes \Pi$ for $\Pi$ in a prescribed compact family are $G$-invariant. These are the ordinary surjectivity, weak-label, and whole-core protected injection conditions on $Q^+ \otimes F_s(A)$ of Lemma 14.1.

**Lemma 14.3** (Linearization)**.** *Let $\nu$ be a frame and $w \in \Gamma(\mathrm{Hom}(\mathbb{R}^7, W_\nu))$, a vertical tangent field of $\mathrm{St}(V)$ along $\nu$. Then*

$$(D\Theta)_\nu(w) = -\mathcal{L}_{U(\nu)}(w).$$

*This expression contains no derivative of $w$. Consequently, wherever $U(\nu)$ is regular,*

$$M_\nu(g) = -R_{U(\nu)}(g) \qquad (g \in \Gamma((E^+)^* \otimes \mathfrak{so}(7)))$$

*is a right inverse of $(D\Theta)_\nu$. It is of order zero in $g$, has values in $\Gamma(\nu^* T^{\mathrm{vert}} \mathrm{St}(V))$, and has coefficients depending smoothly on the 1-jet of $\nu$.*

*Proof.* Since $\nu^* w = 0$, the curve $\nu_s = (\nu + sw)(1 + s^2 w^* w)^{-1/2}$ consists of frames and has velocity $w$. Hence $(D\Theta)_\nu(w)(u) = w^* \nabla_u \nu + \nu^* \nabla_u w$. As $\nu^* w \equiv 0$ and $\nabla$ is metric, $\nu^* \nabla_u w = -(\nabla_u \nu)^* w$. As $w$ takes values in $W_\nu$, only $U(\nu)_u$ contributes to either term. The result is $w^* U(\nu)_u - U(\nu)_u^* w = -\mathcal{L}_{U(\nu)}(w)(u)$. □

For a section $\nu$ of $\mathrm{Hom}(\mathbb{R}^7, V)$, not necessarily a frame, define $\Theta(\nu)(u) = \nu^* \nabla_u \nu \in \mathrm{End}(\mathbb{R}^7)$ and

$$\Psi(\nu)(u, u') = (\nabla_{u'} \nu)^* \nabla_u \nu - (\nabla_u \nu)^* \nabla_{u'} \nu - \nu^* R^V(u, u') \nu.$$

Let $d_{E^+}$ denote the leafwise exterior derivative.

**Lemma 14.4** (Compatibility identities)**.** *For every section $\nu$ of $\mathrm{Hom}(\mathbb{R}^7, V)$,*

$$\Psi(\nu) = -d_{E^+}\Theta(\nu), \qquad d_{E^+}\Psi(\nu) = 0, \qquad \Theta(\nu) + \Theta(\nu)^* = d_{E^+}(\nu^* \nu).$$

*At a point where $\nu$ is a frame and $\Theta(\nu) = 0$, one has $\nabla \nu = U(\nu)$ and $\Psi(\nu) = -(\Omega(U(\nu)) + \nu^* R^V \nu)$.*

*Proof.* In leaf coordinates write $\Theta_\alpha = \Theta(\nu)(\partial_\alpha)$ and $\Psi_{\alpha\beta} = \Psi(\nu)(\partial_\alpha, \partial_\beta)$. Since $\nabla$ is metric, $\partial_\beta \Theta_\alpha = (\nabla_\beta \nu)^* \nabla_\alpha \nu + \nu^* \nabla_\beta \nabla_\alpha \nu$. Antisymmetrizing and using $[\nabla_\alpha, \nabla_\beta] = R^V(\partial_\alpha, \partial_\beta)$ gives $\partial_\beta \Theta_\alpha - \partial_\alpha \Theta_\beta = \Psi_{\alpha\beta}$, which is the first identity. The second follows from $d_{E^+}^2 = 0$, and the third from $\partial_\alpha(\nu^* \nu) = (\nabla_\alpha \nu)^* \nu + \nu^* \nabla_\alpha \nu$. The last assertion follows from $\nabla_u \nu = \nu \Theta(u) + U(\nu)_u$. □

In particular, a horizontal frame on $Y$ satisfies the connection-curvature equation $\Omega(U(\nu)) + \nu^* R^V \nu = 0$ on $\Lambda^2 E^+$.

*Jet relations.* Let $\mathcal{O}$ be admissible, and let $A^1 \subset J^1(\mathrm{St}(V))$ be the open set of 1-jets whose pair $(\nu, U(\nu))$ lies in $\mathcal{O}$. For $k \geq 0$ let $Z_k \subset J^{k+1}(\mathrm{St}(V))$ be the set of $(k+1)$-jets at points $p \in Y$ that lie over $A^1$ and satisfy

$$j_p^k \Theta = 0 \quad \text{and} \quad j_p^k \Psi = 0.$$

This is well defined because $\Theta$ and $\Psi$ are of first order. A point of $Z_0$ is a compatible pair $(\nu, T) \in \mathcal{O}$, together with the leaf derivatives $\nabla_u \nu = T_u$ and arbitrary transverse first derivatives. For $\alpha \geq 1$ put

$$\mathcal{R}^\alpha = \{\, j_p^{\alpha+1}\nu : \ j_p^\alpha \Theta(\nu) = 0, \ j_p^1 \nu \in A^1 \,\}.$$

Then $Z_\alpha \subset \mathcal{R}^\alpha$. The 1-jet projection of $\mathcal{R}^\alpha$ lies in $Z_0$, since $\Psi = -d_{E^+}\Theta$ is determined by $j^1\Theta$.

**Lemma 14.5** (Prolongation)**.** *For every $k \geq 1$ the projection $Z_k \to Z_{k-1}$ is onto. Its fibres are affine subspaces, of constant dimension, of the fibres of $J^{k+1} \to J^k$, and it has a smooth section. Consequently every continuous section of $Z_0$ over a subset $S \subset Y$ lifts to a continuous section of $Z_\alpha$ for every $\alpha \geq 1$. If such a lift is prescribed over $S \cap N$, where $N$ is a neighborhood of a closed set $C$, then the lift may be chosen to agree with it near $C$.*

*Proof.* Fix $p$ and a $k$-jet $j \in Z_{k-1}$ at $p$. Use a foliated chart and an orthonormal frame of $V$ near $p$. Sections of $\mathrm{Hom}(\mathbb{R}^7, V)$ become matrix-valued maps, and $\nabla_\alpha = \partial_\alpha + A_\alpha$ with $A_\alpha$ skew. A $(k+1)$-jet with $k$-jet $j$ is given by its top coefficients $\hat\nu_I = \partial^I \nu(p)$, $|I| = k+1$. Indexing by multi-indices records the symmetry of mixed partial derivatives, and every array $(\hat\nu_I)$ is realized by a polynomial map. The jets of sections of $\mathrm{St}(V)$ are those with $\partial^I(\nu^*\nu)(p) = 0$ for $|I| = k+1$. Let $\nu = \nu(p)$, $T = U(\nu)(p)$, $W = W_\nu$, all determined by $j$, and decompose

$$\hat\nu_I = \nu\, \omega_I + w_I, \qquad \omega_I \in \mathrm{End}(\mathbb{R}^7), \quad w_I \in \mathrm{Hom}(\mathbb{R}^7, W).$$

Since $k+1 \geq 2$, the chain rule shows that the coefficients of order $k+1$ enter linearly in $\partial^I$ of a zero-order expression with $|I| = k+1$, and in $\partial^J$ of a first-order expression with $|J| = k$. Since $\Theta(p) = 0$, we have $\nabla\nu(p) = T$. Since $\nu^*\nu = 1$ and $\nu^* w_I = 0 = T_u^* \nu$, one obtains, on all matrix jets with $k$-jet $j$,

$$\partial^I(\nu^*\nu) = \omega_I + \omega_I^* + c_I, \tag{i}$$

$$\partial^J \Theta_\alpha = \omega_{J+\alpha} + e_{J,\alpha}, \tag{ii}$$

$$\partial^K \Psi_{\alpha\beta} = \mathcal{L}_T(w_{K+\alpha})(\partial_\beta) - \mathcal{L}_T(w_{K+\beta})(\partial_\alpha) + \psi_{K,\alpha\beta}, \tag{iii}$$

for $|I| = k+1$ and $|J| = |K| = k$. Here $c_I, e_{J,\alpha}, \psi_{K,\alpha\beta}$ are the values at the jet with $k$-jet $j$ and $\hat\nu = 0$, so they depend only on $j$. In (iii) the linear terms are $T^*_{\partial_\beta}\hat\nu_{K+\alpha} - \hat\nu^*_{K+\alpha} T_{\partial_\beta} - (\alpha \leftrightarrow \beta)$, and the parts $\nu\omega$ drop out because $T^*\nu = 0$. Since $j \in Z_{k-1}$, a jet over $j$ lies in $Z_k$ exactly when the right sides of (i)–(iii) vanish.

*The $\omega$-system.* By the third identity of Lemma 14.4, $\partial^J\Theta_\alpha + (\partial^J\Theta_\alpha)^* = \partial^{J+\alpha}(\nu^*\nu)$ on all matrix jets. So the symmetric part of (ii) is (i) for $I = J+\alpha$. Suppose $I$ contains

the leaf indices $\alpha \neq \beta$, and put $L = I - \alpha - \beta$. The two equations (ii) indexed by $(L+\beta, \alpha)$ and $(L+\alpha, \beta)$ prescribe the same $\omega_I$. By the first identity of Lemma 14.4, the difference of their constant terms is

$$e_{L+\beta,\alpha} - e_{L+\alpha,\beta} = \partial^{L+\beta}\Theta_\alpha - \partial^{L+\alpha}\Theta_\beta = \partial^L \Psi_{\alpha\beta},$$

a function of $j$ since $|L| = k-1$. It vanishes because $j \in Z_{k-1}$. Hence (i) and (ii) determine $\omega_I$ uniquely and consistently when $I$ contains a leaf index. When $I$ contains no leaf index they prescribe its symmetric part and leave its skew part free. They do not involve $w$.

*The $w$-system.* Let $\xi = (\xi^\alpha, \xi^\mu)$ be polynomial variables dual to the chart coordinates, and let $d_\xi$ be the exterior derivative in the leaf variables $\xi^\alpha$ alone. Identify an array $(\tau_I)_{|I|=d}$ of leaf $q$-covectors with values in $\mathfrak{so}(7)$ with the homogeneous polynomial leaf $q$-form $\tau = \sum_I \tau_I\, \xi^I / I!$. Then

$$(d_\xi \tau)_K(\partial_\alpha, \partial_\beta) = \tau_{K+\alpha}(\partial_\beta) - \tau_{K+\beta}(\partial_\alpha) \quad (q=1), \qquad (d_\xi\tau)_{K'}(\partial_\alpha, \partial_\beta, \partial_\gamma) = \sum_{\text{cyc}} \tau_{K'+\gamma}(\partial_\alpha, \partial_\beta) \quad (q=2).$$

Put $\Lambda(w) = \sum_{|I|=k+1} \mathcal{L}_T(w_I)\, \xi^I / I!$ and $\psi = \sum_{|K|=k} \psi_K\, \xi^K / K!$, where $\psi_K(\partial_\alpha, \partial_\beta) = \psi_{K,\alpha\beta}$. The vanishing of (iii) is then the single equation

$$d_\xi \Lambda(w) = -\psi. \tag{14.1}$$

All relations between the equations (iii) for different $(K, \alpha, \beta)$ are contained in this form.

The right side is closed. Indeed, by Lemma 14.4, $\sum_{\text{cyc}} \partial_\gamma \Psi_{\alpha\beta} = 0$ for every matrix-valued map. Hence $\sum_{\text{cyc}} \partial^{K'+\gamma}\Psi_{\alpha\beta}$, $|K'| = k-1$, vanishes at every $(k+1)$-jet of matrix maps. Evaluating at the jet with $k$-jet $j$ and $\hat{\nu} = 0$ gives

$$d_\xi \psi = 0. \tag{14.2}$$

Let $\rho = \sum_\alpha \xi^\alpha \partial_{\xi^\alpha}$. On leaf $q$-forms whose coefficients have degree $i$ in $\xi^\alpha$ and arbitrary degree in $\xi^\mu$, the operator $d_\xi \iota_\rho + \iota_\rho d_\xi$ acts as multiplication by $i+q$. Both $d_\xi$ and $\iota_\rho$ preserve $i+q$. For $q \geq 1$ put $h\tau = \sum_i (i+q)^{-1} \iota_\rho \tau^{(i)}$, where $\tau^{(i)}$ is the part of degree $i$ in $\xi^\alpha$. Then $d_\xi h + h d_\xi = \mathrm{Id}$ on forms of positive degree, and $h$ raises the total degree by one. By (14.2), $\kappa = -h\psi$ is a homogeneous polynomial leaf 1-form of degree $k+1$ with $d_\xi \kappa = -\psi$. Put $w_I = R_T(\kappa_I)$. Then $\mathcal{L}_T(w_I) = \kappa_I$, so $\Lambda(w) = \kappa$ and (14.1) holds. Only the surjectivity of $\mathcal{L}_T$ is used.

*Structure of the fibres.* The data above define a jet $s_k(j) \in Z_k$ over $j$. For $\omega_I$ with no leaf index take the symmetric solution $-c_I/2$. Since $c, e, \psi$ are polynomial in $j$ and $R_T$ is smooth on regular tensors, $s_k$ is smooth. The fibre over $j$ is $s_k(j)$ plus a linear space, which is the sum of two parts:

- the $\omega$ with $\omega_I + \omega_I^* = 0$ for all $I$ and $\omega_I = 0$ when $I$ has a leaf index;
- the kernel of $w \mapsto d_\xi \Lambda(w)$.

The latter map is onto the space of closed polynomial leaf 2-forms of degree $k$, by the argument just given. That space does not depend on $j$, so the dimension of the fibre is constant. For the last assertions lift inductively with $s_k$. To obtain the relative statement, replace the lift at each stage by the fibrewise affine combination $\phi\, \hat{s} + (1-\phi)\, s_k$,

where $\hat{s}$ is the prescribed lift and $\phi$ is a cutoff equal to one near $C$; the neighborhoods shrink finitely many times. □

*Remark* 14.6. The unknown is a section over the fixed base $Y$; neither $H$ nor $(V, \nabla)$ is varied. Hence the top coefficients of a jet of order $k+1$ range over an affine space modelled on $\mathrm{Sym}^{k+1} T_p^* Y \otimes T_\nu^{\mathrm{vert}}$, where

$$T_\nu^{\mathrm{vert}} = T_\nu \mathrm{St}(V_p) = \nu \cdot \mathfrak{so}(7) \oplus \mathrm{Hom}(\mathbb{R}^7, W_\nu).$$

The first summand consists of the internal rotations, on which $\theta|_{\mathrm{Vert}}$ is the identity of $\mathfrak{so}(7)$. The second consists of the normal variations, on which the contraction of $d\theta$ with the leaf derivative is $\mathcal{L}_T$. At top order the two systems decouple. The compatibility of the $\omega$-system is the previous-order condition $\partial^L \Psi = 0$; this is why $\Psi$ is included in $Z_k$. The compatibility of the $w$-system is (14.2). No ordering of the multi-indices is used. The base and transverse coordinates enter only through the constants $c, e, \psi$, which depend on lower jets.

**Lemma 14.7** (Horizontal realization on the counted microextension)**.** *Let $\mathcal{O} \subset \mathcal{P}$ be admissible. Let $(\nu^\circ, T^\circ)$ be a continuous section over $X'$ of compatible pairs with values in $\mathcal{O}$, that is, a formal orthonormal seven-frame in $V$ on the microextension, satisfying its connection-curvature equation, with $\mathcal{L}_{T^\circ}$ onto. Then there is a smooth frame $\nu$ on a neighborhood of $X \times 0$ in $Y$ with the following properties:*

- *$\Theta(\nu) = 0$ and $(\nu, U(\nu)) \in \mathcal{O}$;*
- *$(\nu, U(\nu))|_{X\times 0}$ is homotopic to $(\nu^\circ, T^\circ)|_X$ through continuous sections of compatible pairs in $\mathcal{O}$;*
- *the frame $\nu_0 = \nu|_{X\times 0}$ is horizontal along $E$, with $U(\nu_0) = U(\nu)|_E$;*
- *the old map $H$ is unchanged.*

*Relative version: let $C \subset X$ be closed, and let $\nu'$ be a horizontal frame near $C \times 0$ with $(\nu', U(\nu')) \in \mathcal{O}$ and $(\nu^\circ, T^\circ) = (\nu', U(\nu'))|_{t=0}$ near $C$. Then one may take $\nu = \nu'$ near $C \times 0$, with the homotopy constant near $C$.*

Thus every open $G$-invariant condition imposed on the whole extended domain holds for the actual frame. This includes the protected family of Lemma 14.1. The actual tensor $U(\nu_0)$ is joined to the formal one within $\mathcal{O}$, but need not coincide with it.

*Proof. Infinitesimal inversion.* Take the fibration $\mathfrak{X} = \mathrm{St}(V) \to Y$, the vector bundle $(E^+)^* \otimes \mathfrak{so}(7)$, and the first-order operator $\Theta$. It is $\mathfrak{so}(7)$-valued on frames by Lemma 14.4. By Lemma 14.3, $\Theta$ is infinitesimally invertible over the open set $A^1$, with order $r = 1$, inversion of order $s = 0$, and defect $d = 1$. Fix $\alpha \geq \max\{d+s, 2r+2s\} = 2$. By [BD23, Theorem 2.12], which is [Gro86, pp. 119–120], the sheaf $\Phi$ over $Y$ of germs of smooth $A^1$-regular solutions of $\Theta(\nu) = 0$ is microflexible. Moreover, the jet map $J^{\alpha+1}$ from $\Phi$ to the sheaf of continuous sections of $\mathcal{R}^\alpha$ is a local weak homotopy equivalence.

*Sharp motion.* Let $\delta(x,t) = (x, \varphi(x,t))$ with $\partial_t \varphi \neq 0$ be a local diffeomorphism. These maps form the fibre-preserving pseudogroup of [BD23, Example 2.8(1)], which sharply moves $X' \times 0$. Since $\mathrm{pr} \circ \delta = \mathrm{pr}$, $\delta$ preserves $\mathrm{St}(V)$, the pulled-back connection,

and $E^+ = (d\,\mathrm{pr})^{-1}E$. Hence $\nu \circ \delta$ is a frame, with

$$\Theta(\nu \circ \delta)_p(u) = \Theta(\nu)_{\delta(p)}(d\delta_p u), \qquad U(\nu \circ \delta)_p = U(\nu)_{\delta(p)} \circ (d\delta_p \otimes \mathrm{Id}).$$

On $E^+$, $d\delta_p$ is the element $u \mapsto u + u(\varphi)\partial_t$, $\partial_t \mapsto \partial_t\varphi\,\partial_t$ of $G_x$. Since $\mathcal{O}$ is pulled back from $X'$ and $G$-invariant, $\Phi$ is invariant. By [BD23, Theorem 2.7] ([Gro86, p. 78]), $\Phi_0 = \Phi|_{X'\times 0}$ is flexible.

*Restriction to $t = 0$.* Let $\Phi_\flat$ be the sheaf over $X'$ of continuous sections of $\mathcal{R}^\alpha|_{X'\times 0}$. It is flexible, being a sheaf of sections of a subset of a fibre bundle. All the structures are invariant under the translations $\tau_c$ in $t$. So restriction from germs along $X' \times 0$ of sections of $\mathcal{R}^\alpha$ to $\Phi_\flat$ is a homotopy equivalence over every open set: its inverse is the $t$-invariant extension, and the contraction is $s \mapsto \tau_{(1-\lambda)t}\, s(x, \lambda t)$. The stalks of $\Phi_0$ at $x$ are those of $\Phi$ at $(x, 0)$. Hence the composite $\Phi_0 \to \Phi_\flat$ is a local weak homotopy equivalence of flexible sheaves. By the sheaf homomorphism theorem [BD23, Theorem 2.6] ([Gro86, p. 77]), it is a weak homotopy equivalence over $X$. The same holds relative to $C$: the restriction maps to $\mathrm{Op}\, C$ are Serre fibrations, and the homotopy sequences show that the induced map of fibres is a weak homotopy equivalence.

*The formal section.* Fix a complement $F$ of $E$ in $TX'$ and a metric connection $\nabla'$ on $V$ extending $\nabla$ along $E$, and pull both back to $Y$. At $(x, 0)$ define the 1-jet with value $\nu^\circ(x)$ by $\nabla'_u\nu = T^\circ_u$ for $u \in E^+$ and $\nabla'_f\nu = 0$ for $f \in F$. This is a jet of a section of $\mathrm{St}(V)$, since $\nu^{\circ *}T^\circ = 0$. It has $\Theta = 0$ and pair $(\nu^\circ, T^\circ)$, so it lies in $Z_0$. In the relative case replace the transverse part by $\phi\,\nabla'_f\nu'$, where $\phi$ is a cutoff equal to one near $C$. This changes no condition defining $Z_0$. Lemma 14.5 lifts this section of $Z_0$ over $X' \times 0$ to a continuous section $s$ of $Z_\alpha \subset \mathcal{R}^\alpha$. In the relative case $s$ equals $j^{\alpha+1}\nu'$ near $C \times 0$; this jet lies in $Z_\alpha$ because $\Theta(\nu') \equiv 0$ implies $\Psi(\nu') \equiv 0$.

*Conclusion.* Surjectivity on $\pi_0$ of the fibre map gives an $A^1$-regular solution $\nu$ near $X \times 0$, equal to $\nu'$ near $C \times 0$. It also gives a homotopy in $\Phi_\flat(X)$ from $s$ to $j^{\alpha+1}\nu|_{X\times 0}$ that is constant near $C$. Its projection to 1-jets lies in $Z_0$, because $\alpha \geq 1$. It is therefore a homotopy through compatible pairs in $\mathcal{O}$ from $(\nu^\circ, T^\circ)$ to $(\nu, U(\nu))|_{X\times 0}$.

*Restriction to the old source.* On $X' \times 0$ the connection is the original one along $E \subset E^+$. Hence $\Theta(\nu_0) = \Theta(\nu)|_E = 0$ and $U(\nu_0) = U(\nu)|_E$. Restriction from $E^+$ to $E$ preserves the following open conditions:

- regularity, since $\mathcal{L}_{T|_E}$ is $\mathcal{L}_T$ followed by restriction;
- the weak-label bound, since $\ker T|_{E\otimes\mathbb{R}^7} = \ker T \cap (E \otimes \mathbb{R}^7)$, so the bad-label set can only shrink;
- injectivity on $E \otimes \Pi$.

Only $\nu$ was constructed, so $H$ is unchanged. The auxiliary missing core line is an algebraic derivative variable. It is carried in the normal complement and imposes no additional differential equation. □

The argument relies on the formal-to-actual theorem for an infinitesimally invertible operator and on the sharp-motion theorem. Invariance under the shears $(x, t) \mapsto (x, \varphi(x, t))$ requires the open conditions to be imposed on all of $E^+$; conditions imposed only on a smaller principal source slice would not be preserved.

## 15. An activity with a periodic mixed differential inverse

Let $k = q - 1 \geq 2$. The active source is a product of $k$ Hopf spheres with its free commuting $\mathbb{T}^k$ action. Let $a$ be an actual partial immersion into $\mathbb{R}^q$, independent of the old fast variables; it may depend smoothly on the slow base. For an actual horizontal frame put

$$W = (\operatorname{im} J_E H + \operatorname{span} \nu)^\perp, \qquad U_a(u) = P_W \nabla_u \nu_a, \qquad B_y = U \circ (\mathrm{Id}_E \otimes d_A a).$$

The operator to be inverted on the compact active source is

$$\mathcal{Q}(v, f) = B_y^* v + d_A f, \qquad v \in W, \quad f \in E^*. \tag{15.1}$$

The old bundles $E, W$, their local identifications, and $U$ are pulled back from the old source and are independent of the active variables. The moving physical isometry in Proposition 13.4 identifies its mixed equation with this fixed old-bundle model.

In this section we construct a differential right inverse on the periodic source, with coefficients depending smoothly on all the old parameters. The critical inequality is

$$\dim_{\mathbb{C}} \mathcal{B}_U \leq q - 3 = k - 2.$$

We first eliminate formal adjoint solutions at selected points, and then eliminate the nonzero characteristic variety throughout each active orbit.

15.1. **A finite-jet killing test.** Fix real vector spaces $E, W$ and a linear map $X \mapsto UX$ from $E \otimes \mathbb{R}^q$ to $W$, with kernel $K$. Write $e_u : E \otimes \mathbb{R}^q \to E$ for contraction by a label covector $u$. All spaces are complexified when convenient. We assume

(H) $\dim \mathcal{B}_K \leq q - 3$, and the ordinary frame adjoint has zero kernel.

The second condition says that every skew tensor in $E \otimes \Lambda^2 \mathbb{C}^q$ whose label columns lie in $K_{\mathbb{C}}$ is zero; this holds over $\mathbb{C}$ because it holds over $\mathbb{R}$. The incidence

$$\{(H, [\nu]) : H \in \mathcal{B}_K, \ [\nu] \in \mathbb{P}(H^\perp)\}$$

has dimension at most $q - 2 < q - 1$. We call a real covector $\nu \neq 0$ *good* if $[\nu]$ is not in its projection. For good $\nu$ put $K_\nu = \ker(e_\nu|_K)$. Then $e_\xi$ is injective on $K_\nu$ for every complex covector $\xi \notin \mathbb{C}\nu$; otherwise a nonzero tensor in $K_\nu$ would be supported on $(\nu, \xi)^\perp$, a bad label plane incident to $\nu$.

*Graph charts.* Fix an orthonormal frame of $\mathbb{R}^q$ with coordinates $(y_1, \dots, y_k, z)$, coordinate covectors $u_1, \dots, u_k, n$, and $e_i = e_{u_i}$. A leaf germ transverse to the $z$-axis is a graph $z = f(y)$ with conormal $\nu(y) = n - \sum_i f_i(y) u_i$, $f_i = \partial_i f$. We write the adjoint unknown as a densitized ambient flux $X \in E \otimes \mathbb{C}^q$. Its equations are

$$UX = 0, \qquad e_\nu X = e_n X - \sum_i f_i e_i X = 0, \qquad \mathcal{D}X := \sum_i e_i \partial_i X = 0. \tag{15.2}$$

Write $\mathcal{P}_f X$ for the triple on the left. We keep the ambient unknown, so no frame of $K$ is chosen, and $e_n$ need not be surjective.

*Prolongation matrices.* By Taylor expansion we identify $m$-jets at 0 with polynomials of degree at most $m$. For $N \geq 0$ define

(15.3)

$$P_N : J_0^{N+1}(E\otimes\mathbb{C}^q) \longrightarrow J_0^{N+1}(W)\oplus J_0^{N+1}(E)\oplus J_0^N(E), \qquad j_0^{N+1}X \longmapsto \Big(j_0^{N+1}(UX),\, j_0^{N+1}(e_\nu X),\, j_0^N(\mathcal{D}X)\Big)$$

The unknowns are the $\partial^\alpha X(0)$ with $|\alpha| \leq N+1$. The rows are $\partial^\beta(UX)(0)$ and $\partial^\beta(e_\nu X)(0)$ for $|\beta| \leq N+1$, and $\partial^\beta(\mathcal{D}X)(0)$ for $|\beta| \leq N$. By the Leibniz formula

$$\partial^\beta(f_i e_i X)(0) = \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma f_i(0)\, e_i \partial^{\beta-\gamma} X(0).$$

Hence each row involves only derivatives of $X$ of order at most $N+1$, and $P_N$ is well defined. Its entries are linear in $U$ and affine in the jet $j_0^{N+2}f$; the value $f(0)$ does not enter. The same formula at a point $y$ defines $P_N(y)$ from $j_y^{N+2}f$. For every holomorphic $X$,

$$P_N(y)\, j_y^{N+1}X = \Big(j_y^{N+1}(UX),\, j_y^{N+1}(e_\nu X),\, j_y^N(\mathcal{D}X)\Big).$$

**Definition 15.1.** The data $(U, f)$ are *killing at $y$ to order $N$* if $P_N(y)$ is injective, that is, has full column rank.

A formal solution of (15.2) at $y$ is a power series $X$ with $\mathcal{P}_f X = 0$ formally. Its $(N+1)$-jet lies in $\ker P_N(y)$ for every $N$. In general $\ker P_N(y)$ is larger than the image of the formal solutions, because it contains truncated jets that admit no extension. The definition requires the truncated kernel itself to vanish. This condition is stronger than the absence of formal solutions, and it is open.

**Lemma 15.2** (Killing gives a local left inverse)**.** *Let $U$ and $f$ depend holomorphically on parameters $\theta$, and let $P_N$ be injective at $(y_0, \theta_0)$. Then $P_N(y, \theta)$ is injective on a neighborhood $\Omega$ of $(y_0, \theta_0)$. On $\Omega$ there is a differential operator $L$ in $y$, of order at most $N+1$, with holomorphic coefficients, such that $L\mathcal{P}_f = \mathrm{Id}$. In particular the relative module presented by (15.2) vanishes on $\Omega$, and the system has no nonzero formal solution at any point of $\Omega$.*

*Proof.* Choose rows $I$ with $\det(P_N)_I(y_0, \theta_0) \neq 0$, and shrink $\Omega$ so that this minor does not vanish there. Put $\Lambda = (P_N)_I^{-1} \circ \mathrm{pr}_I$. Then $\Lambda P_N = \mathrm{Id}$, and the entries of $\Lambda$ are rational in the entries of $P_N$ with denominator this minor. Let $\Lambda_0$ be the block of $\Lambda$ that returns the value $X(y)$. By the identity preceding the definition,

$$X(y) = \Lambda_0(y, \theta)\Big(j_y^{N+1}(UX),\, j_y^{N+1}(e_\nu X),\, j_y^N(\mathcal{D}X)\Big) =: (L\mathcal{P}_f X)(y)$$

for all polynomial, hence all holomorphic or formal, $X$. The identity $L\mathcal{P}_f = \mathrm{Id}$ expresses each generator of the presented module as a differential combination of the relations. □

**Lemma 15.3** (Finite-jet killing)**.** *Assume* (H) *and fix a graph chart. Let $c \in \mathbb{R}^k$ be such that $\nu = n - \sum_i c_i u_i$ is good. Then there exist $N$ and a real jet $j_0^{N+2}f$ with $f(0) = 0$ and $df(0) = c$ at which $P_N$ is injective. Consequently, for each $N' \geq N$ the set*

$$F_{\leq N'}(U) = \{j_0^{N'+2}f :\ P_M \text{ is not injective for every } M \leq N'\}$$

*is a proper real algebraic subset of the jet fiber. It is contained in the zero set of one maximal minor $\Phi$ of some $P_M$, $M \le N'$. This $\Phi$ is a polynomial in $(U, j_0^{N'+2} f)$ and is not identically zero in the jet variables at this $U$. Injectivity of $P_M$ is an open condition in $(U, j_0^{M+2} f)$.*

*Proof. Step 1: detecting syzygies.* Fix $c$ and good $\nu$. For $\xi \in \mathbb{C}^k \setminus 0$ write $e_\xi = \sum_i \xi_i e_i$. Since $\xi \notin \mathbb{C}\nu$, the map

$$M_\xi : K_{\mathbb{C}} \to E \oplus E, \qquad M_\xi Y = (e_\nu Y, e_\xi Y)$$

is injective. Define $\widehat{T}_\xi : K_\nu \to \operatorname{coker} M_\xi$ by $X \mapsto [(e_\xi X, 0)]$. We claim that these maps have zero joint kernel. Let $X$ lie in every $\ker \widehat{T}_\xi$. The unique $Y_\xi \in K_{\mathbb{C}}$ with $e_\nu Y_\xi = e_\xi X$ and $e_\xi Y_\xi = 0$ is given locally by Cramer's rule with a nonvanishing maximal minor of $M_\xi$. It is therefore holomorphic on $\mathbb{C}^k \setminus 0$ and homogeneous of degree one. Since $k \ge 2$, Hartogs extension makes it linear: $Y_\xi = \sum_i \xi_i Y_i$ with $Y_i \in K_{\mathbb{C}}$. Define label columns $R_\nu = X$ and $R_{u_i} = -Y_i$. Then $e_\nu R_\nu = 0$ and $e_i R_\nu = e_i X = -e_\nu R_{u_i}$. Comparing the coefficient of $\xi_i \xi_j$ in $e_\xi Y_\xi = 0$ gives $e_i R_{u_j} = -e_j R_{u_i}$. Thus $R$ is a skew tensor with columns in $K_{\mathbb{C}}$. By (H), $R = 0$, and so $X = 0$.

Let $Z$ be the $\mathbb{C}[\zeta]$-module of pairs $(\lambda_1, \lambda_2)$ of $E^*$-valued polynomials with $\lambda_1(\zeta) e_\nu + \lambda_2(\zeta) e_\zeta = 0$ on $K_{\mathbb{C}}$. Give $\lambda_1$ weight $\ell$ and $\lambda_2$ weight $\ell - 1$. The weighted homogeneous parts of a syzygy are then syzygies, so $Z$ has finitely many homogeneous generators $(\lambda_1^{(s)}, \lambda_2^{(s)})$, of weights $\ell_s \ge 0$. For $\xi \ne 0$, Cramer's rule near $\xi$ gives a rational basis of the left null space of $M_\xi$ whose denominators do not vanish at $\xi$. Clearing the denominators shows that the generators evaluated at $\xi$ span this null space. Since $(e_\xi X, 0)$ pairs with $(\lambda_1, \lambda_2)$ to $\lambda_1 e_\xi X$,

$$\ker \widehat{T}_\xi = \{X \in K_\nu : \lambda_1^{(s)}(\xi) e_\xi X = 0 \text{ for all } s\}.$$

These conditions are polynomial in $\xi$. If they hold for all real $\xi \ne 0$, they hold for all complex $\xi$, and then $X = 0$. Since descending chains of subspaces of $K_\nu$ are finite, there are distinct real $\xi_1, \dots, \xi_J \ne 0$ such that

$$X \longmapsto \Big(\lambda_1^{(s)}(\xi_j) e_{\xi_j} X\Big)_{s,j} \quad \text{is injective on } K_\nu.$$

*Step 2: the flat truncated kernel.* Let $f^0 = c \cdot y$, so $\nu(y) \equiv \nu$, and let $P_N^0$ be the corresponding matrix. Write a jet as $X = \sum_{m \le N+1} X_m$ with $X_m$ homogeneous of degree $m$. The rows of $P_N^0$ are $UX_m$ and $e_\nu X_m$ in degree $m \le N+1$, and $\mathcal{D}X_m$ in degree $m - 1 \le N$. All these rows lie in the truncation range. Hence

$$\ker P_N^0 = \bigoplus_{m \le N+1} V_m, \qquad V_m = \{X \in \mathbb{C}[y]_m \otimes K_\nu : \mathcal{D}X = 0\}. \tag{15.4}$$

We show that $V_m = 0$ for $m > d$, for some $d$. Let $S = \mathbb{C}[\xi]$. The pairing $\langle p \otimes \lambda, X \rangle = \lambda\Big((p(\partial)X)(0)\Big)$ between $S_m \otimes K_\nu^*$ and $\mathbb{C}[y]_m \otimes K_\nu$ is perfect, and $\langle \xi_i \varphi, X \rangle = \langle \varphi, \partial_i X \rangle$. For $\mu \in E^*$ put $\sigma_\mu(\xi) = \mu \circ e_\xi|_{K_\nu} \in S_1 \otimes K_\nu^*$. Then $\langle p\sigma_\mu, X \rangle = \langle p \otimes \mu, \mathcal{D}X \rangle$. So $V_m$ is the annihilator of $I_m$, where $I$ is the submodule generated by the $\sigma_\mu$, and $\dim V_m = \dim N_m$ for $N = (S \otimes K_\nu^*)/I$. The fiber of $N$ at $\xi$ is the cokernel of $\mu \mapsto \sigma_\mu(\xi)$, which is dual to $\ker(e_\xi|_{K_\nu})$; it vanishes for $\xi \ne 0$. By Nakayama's lemma and the Nullstellensatz,

$(\xi)^{d+1}N = 0$ for some $d$. Since $N$ is generated in degree zero, $N_m = 0$ for $m > d$. Put $V = \bigoplus_{m\le d} V_m$. This is the space of all polynomial, equivalently all formal, solutions at the flat graph. By (15.4),

$$\ker P_N^0 = V \qquad (N \ge d-1).$$

Thus at the flat graph, and only there, the truncated kernels and the formal solutions coincide.

*Step 3: exponential perturbation at a finite order.* Put

$$f_1 = \sum_j \Big(e^{\xi_j\cdot y} - 1 - \xi_j\cdot y\Big), \qquad f_t = c\cdot y + t f_1.$$

Then $j_0^1 f_t = j_0^1 f^0$ and $P_N(t) = P_N^0 + tA$ exactly, where

$$AX = \Big(0,\, -j_0^{N+1}(g_X),\, 0\Big), \qquad g_X = \sum_i \partial_i f_1\, e_i X = \sum_j \Big(e^{\xi_j\cdot y} - 1\Big) e_{\xi_j} X.$$

Let $\mathcal{W}$ be the span of the functions $e^{\xi_j\cdot y} y^\alpha$ and $y^\alpha$ with $|\alpha| \le d$; it is finite dimensional. The kernels of its Taylor truncations to order $m$ decrease with $m$, and their intersection is zero by analyticity. Choose $m_0$ such that truncation to order $m_0$ is injective on $\mathcal{W}$, and put $N_1 = \max\{d-1,\ m_0 + \max_s \ell_s - 1\}$.

We claim that for $N \ge N_1$ the induced map $V \to \operatorname{coker} P_N^0$, $X_0 \mapsto [AX_0]$, is injective. Suppose $X_0 \in V$ and $P_N^0 X_1 = -AX_0$ with $\deg X_1 \le N+1$. Since the rows cover all degrees of $X_1$, we get $UX_1 = 0$, $\mathcal{D}X_1 = 0$, and $e_\nu X_1 \equiv g_{X_0}$ modulo $(y)^{N+2}$. As $X_1$ is $K_{\mathbb{C}}$-valued and the operators have constant coefficients, the syzygies give

$$\lambda_1^{(s)}(\partial)(e_\nu X_1) = -\lambda_2^{(s)}(\partial)(\mathcal{D}X_1) = 0.$$

Hence $F_s := \lambda_1^{(s)}(\partial) g_{X_0}$ vanishes modulo $(y)^{N+2-\ell_s}$, so its Taylor coefficients vanish through order $m_0$. Using $\lambda(\partial)(e^{\xi\cdot y}h) = e^{\xi\cdot y}\lambda(\partial+\xi)h$, we find

$$F_s = \sum_j e^{\xi_j\cdot y}\, \lambda_1^{(s)}(\partial+\xi_j)(e_{\xi_j}X_0) - \lambda_1^{(s)}(\partial)\Big(\sum_j e_{\xi_j}X_0\Big) \in \mathcal{W}. \tag{15.5}$$

Therefore $F_s = 0$. The frequencies $\xi_1,\dots,\xi_J, 0$ are distinct, and exponentials with distinct frequencies are independent over polynomials. Hence $\lambda_1^{(s)}(\partial+\xi_j)(e_{\xi_j}X_0) = 0$ for all $s, j$. If $X_0 \ne 0$, let $X_0^{\mathrm{top}}$ be its top homogeneous part, of degree $m$. Every term of $\lambda_1^{(s)}(\partial+\xi_j)$ other than $\lambda_1^{(s)}(\xi_j)$ lowers degree, so the degree-$m$ part of the identity is $\lambda_1^{(s)}(\xi_j) e_{\xi_j} X_0^{\mathrm{top}} = 0$. Applied to each $K_\nu$-valued monomial coefficient of $X_0^{\mathrm{top}}$, the detecting rows of Step 1 force that coefficient to vanish, a contradiction. This proves the claim.

*Step 4: full column rank.* Fix $N \ge N_1$. Choose a complement $V'$ of $V = \ker P_N^0$ and a complement $C$ of $\operatorname{im} P_N^0$. In block form from $V' \oplus V$ to $\operatorname{im} P_N^0 \oplus C$,

$$P_N(t) = \begin{pmatrix} B + tA_{11} & tA_{12} \\ tA_{21} & tA_{22} \end{pmatrix}.$$

Here $B$ is invertible and $A_{22} : V \to C$ is the injective map of Step 3. If $(x', x) \in \ker P_N(t)$ and $t$ is small, the first row gives $x' = -t(B + tA_{11})^{-1} A_{12} x$. The second row then gives

$t\big(A_{22} - tA_{21}(B + tA_{11})^{-1}A_{12}\big)x = 0$. For small $t \neq 0$ the bracket is injective, so $x = 0$ and $x' = 0$. The maximal minors of $P_N(t)$ are polynomials in $t$, not all identically zero, so $P_N(t)$ is injective for all but finitely many real $t$. The jet $j_0^{N+2} f_t$ is real and has first jet $c$.

*Step 5: the algebraic consequences.* The failure set of $P_M$ is the common zero set of its maximal minors, which are polynomials in $(U, j_0^{M+2} f)$. So $F_{\leq N'}(U)$ is real algebraic. It omits the jet found above, so it is proper. Since the jet fiber is an affine space, some maximal minor $\Phi$ of some $P_M$, $M \leq N'$, is not identically zero in the jet variables, and $F_{\leq N'}(U) \subset \{\Phi = 0\}$. A nonzero minor stays nonzero under perturbation, which gives openness. □

*Uniformity and killing points.* Let $\mathcal{U}$ be a compact set of real matrices satisfying (H). We use the finitely many coordinate frames of $\mathbb{R}^q$. At every point some frame has normal component at least $q^{-1/2}$ along its $z$-axis, and so gives a graph chart. For $U_0 \in \mathcal{U}$ the lemma gives one order $N$ and one real jet at which $P_N$ is injective. By openness, the same jet and the same $N$ serve on a neighborhood of $U_0$. Compactness gives one order $N_*$ such that $F_{\leq N_*}(U)$ is proper in every chart for every $U \in \mathcal{U}$. A point of an active leaf is a *killing point* for $U$ if, in a chart of this kind, $P_M$ is injective there for some $M \leq N_*$; here $P_M$ is computed from the graph representation of the activity. By Lemma 15.2, the module presented by (15.2) then vanishes near the point, for all nearby complex values of $y$, of the matrix entries and of the activity coefficients.

The graph jet $j_0^{N_*+2} f$ is a rational function of the activity jet, whose denominators are powers of the Jacobian of the chart projection and are nonzero on the chart. The map from activity jets to graph jets at a fixed point is a submersion, as one sees by varying the activity along the $z$-axis. Hence failure is contained in a proper algebraic subset of the activity jet fiber. After the chart denominators are cleared, it lies in the zero set of a polynomial $\Phi$ in the vertical jet variables, smooth in the old parameters and not identically zero at each of them. This is the form used in the orbitwise transversality argument below. The argument is unchanged when the dimension of $K$ varies, since the ambient unknown $X$ and the rows $UX = 0$ are part of (15.3).

## 15.2. **Finite complex Gauss germs.**

**Lemma 15.4** (Finite Gauss germs)**.** *An actual partial immersion on the active Hopf source can be perturbed in any prescribed $C^1$ neighborhood so that, after finite analytic approximation in the active variables, every complexified leafwise Gauss-map germ at a real point is finite. This can be done jointly with finitely many open finite-jet conditions and compact smooth slow parameters.*

*Proof.* For $f \in \mathbb{C}[[y_1, \dots, y_k]]$ let $\mu(f)$ be the possibly infinite colength of its Jacobian ideal. Let $Z_N$ be the closed algebraic set of $(N+1)$-jets for which

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[[y]]}{(\partial f) + (y)^{N+1}} \geq N + 1.$$

This condition is equivalent to $\mu(f) \geq N + 1$ for every extension of the jet. Indeed, until the maximal-ideal filtration of a local algebra ends, each successive quotient has

positive dimension, and if the filtration stabilizes, Nakayama's lemma makes that power zero. This proves the equivalence and the nested inverse-image containment of the $Z_N$.

Their codimensions tend to infinity. Otherwise these nondecreasing integer codimensions would stabilize at some $c$. A codimension-$c$ component at the next jet level would have to be the whole affine jet cylinder over a codimension-$c$ component at the preceding level. There are finitely many minimal components at the first stable level, so one of them would persist as a full cylinder to every order. But every finite Taylor polynomial $T$ has an extension

$$T + \sum_i y_i^D$$

with isolated critical points, for $D > \deg T$. The leading gradient terms $D y_i^{D-1}$ have no common projective zero, so the gradient zero scheme is finite in affine space. This contradicts the existence of an infinite-colength cylinder.

For each real unit height vector $u$, the map from activity jets to the scalar jet of $\langle a, u\rangle$ is a submersion; vary the activity by $u\,\delta f$. Choose $N$ so that the preceding codimension exceeds the source dimension, the slow parameter dimension, and $k$ for the height parameter. Jet transversality then gives every critical height jet finite Jacobian colength. This is detected by open finite-order rank conditions, and compactness gives one finite order and a common tolerance.

Approximation by polynomials in the ambient sphere coordinates, with smooth coefficients in the slow variables, preserves these conditions and partial immersion. In a graph chart the Gauss map is represented by $df$. The fiber over the central normal is the critical scheme of the corresponding height function. Its finite colength makes the complex Gauss germ finite. The germ may branch; it need not be a local diffeomorphism. □

15.3. **A killing point on every active torus orbit.** Choose distinct $\vartheta_1, \dots, \vartheta_r \in \mathbb{T}^k$, where $r$ exceeds the dimension of the whole old–active parameter product. Since the action is free, their translates of any active point are distinct, and activity jets there can be varied independently. For a fixed old matrix $U$, simultaneous failure of the finite-jet test at all translates has codimension at least $r$. Multijet transversality therefore gives an activity for which every active orbit contains a killing point for every old parameter.

The same argument applies to a smooth old family. At a zero of a nonzero polynomial in the vertical jet variables, we stratify by a minimal-order nonzero vertical derivative. On that stratum the preceding derivative vanishes and has nonzero vertical differential, so the stratum lies in a smooth hypersurface. The fixed degree gives finitely many derivative choices on each matrix chart. A single such hypersurface need not contain the whole local polynomial zero set. At distinct translates the corresponding vertical jet variables, and hence these hypersurface differentials, are independent. The resulting finite collection of smooth containing submanifolds has codimension at least the number of translates. Parametric transversality, with the whole smooth old family included in the base dimension, then gives the required avoidance.

Both the orbit-killing property and the finite-Gauss property are open on their compact real parameter products. They can therefore be imposed after horizontal actualization, inside the arbitrarily small $C^1$ neighborhood allowed by the protected core homotopy. One final finite polynomial approximation in active sphere coordinates preserves them. Its degree may depend on the fixed source and on $n$; this does not affect the target count.

## 16. Eliminating critical characteristic components

16.1. **A necessary Hessian identity.** Let $\mathcal{B} \subset \mathrm{Gr}_2(\mathbb{C}^{k+1})$ be an algebraic family of projective lines in $\mathbb{P}^k$. In a fixed affine chart put

$$I_{\mathcal{B}} = \{(p, \ell) : p \in \ell,\ \ell \in \mathcal{B}\}, \qquad S = \bigcup_{\ell \in \mathcal{B}} \ell.$$

For $f : \mathbb{C}^k \to \mathbb{C}$, write $g = df$. A line $\ell$ through $p = g(y)$ gives a direction $[\xi] \in \mathbb{P}(T_p\mathbb{C}^k)$, which the coordinate pairing identifies with a projective covector at $y$. Let $\Sigma_f \subset T^*\mathbb{C}^k \setminus 0$ be the conic locus obtained in this way. Up to an overall sign, it is the raw characteristic locus of the graph-normal construction.

**Lemma 16.1** (Hessian obstruction to a raw Lagrangian branch)**.** *Assume that $g$ is finite and $\dim \mathcal{B} \le k-2$. Let a $k$-dimensional irreducible component of $\Sigma_f$ be Lagrangian. Then it lies over an irreducible component $B$ of $\mathcal{B}$ of dimension $k-2$. Write $S_B$ for the union of the lines in $B$. At a generic point of this component, either* $\det \mathrm{Hess}\, f = 0$ *identically on its base projection, or, at a nonramified point $y$,*

$$n^t \,\mathrm{Hess}\, f(y)\, n' = 0 \quad \text{for all } n, n' \in N^*_{g(y)} S_B. \tag{16.1}$$

*Proof.* The projectivized component has dimension $k-1$. Its finite map to the line incidence must dominate a component $I_B$ of the same dimension, since lower-dimensional line families cannot contribute. Consequently its base projection dominates $S_B$. Moreover $S_B$ has positive dimension, since it contains the lines being parametrized, and $\dim S_B \le k-1$.

At a nonramified point, $g$ is a local biholomorphism and the base projection is an open part of $Y = g^{-1}(S_B)$. A conic Lagrangian whose generic base is $Y$ is generically its conormal bundle: the symplectic form forces its covectors to annihilate $TY$, and the dimensions of the conic fibers then agree. Thus its covectors fill

$$N^*_y Y = (\mathrm{Hess}\, f(y))^t N^*_{g(y)} S_B.$$

On the other hand, each raw direction is tangent to a line lying in $S_B$, so it lies in $T_{g(y)} S_B$. Pairing the preceding equality with every normal to $S_B$ and using the symmetry of the Hessian gives (16.1). If the component has no nonramified generic point, the determinant alternative holds instead. □

At a smooth point of $S_B$, choose any nonzero rational normal vector $n(p)$ on a dense algebraic open subset. A necessary condition along the whole raw Lagrangian component is then

$$\det A \left( n(p)^t A\, n(p) \right) = 0, \qquad A = \mathrm{Hess}\, f(y), \quad p = df(y). \tag{16.2}$$

For fixed $p$, this is a nonzero polynomial in the symmetric matrix $A$. Neither a positive definite form nor a nonisotropic complex normal is needed for this: the rank-one symmetric tensor $n \otimes n$ is nonzero whenever $n \neq 0$.

16.2. **Arbitrarily large codimension in polynomial coefficients.**

**Proposition 16.2** (Fixed-family polynomial avoidance)**.** *Fix an algebraic $\mathcal{B}$ as above with $\dim \mathcal{B} \leq k-2$. Let $L \geq 1$. In the affine space $\mathcal{P}_D$ of polynomials of degree at most $D$, where $D \geq 3L-1$, the following set has Zariski closure of codimension at least $L$:*

$$\{f : df \text{ is finite and } \Sigma_f \text{ has a } k\text{-dimensional Lagrangian component}\}. \tag{16.3}$$

*The same conclusion holds after prescribing any fixed finite jet at one point, if $D$ is increased sufficiently. In particular, generic real polynomial perturbations avoid* (16.3) *inside any specified $C^1$ neighborhood on a compact set.*

*Proof.* There are finitely many top-dimensional components $B$ to consider. We restrict to one of their dense smooth incidence charts and a dense smooth chart of $S_B$ carrying the normal $n$ used in (16.2).

Take $L$ pairs $(p_i, \ell_i) \in I_B$ and $L$ distinct points $y_i$. The universal equations

$$df(y_i) = p_i, \qquad 1 \leq i \leq L$$

impose $kL$ independent affine conditions on polynomial coefficients, because arbitrary two-jets can be interpolated at $L$ distinct points by polynomials of degree $3L-1$. For each $i$ one may take the product of the cubes of separating linear forms for the points $y_j$, $j \neq i$, multiplied by a quadratic Taylor polynomial whose jet compensates for the nonzero product at $y_i$.

Thus the unrestricted $L$-point incidence has dimension

$$\dim \mathcal{P}_D + L(k-1).$$

After these first derivatives have been fixed, the $L$ Hessians are still independent, and (16.2) imposes at least one further condition on each of them. The resulting constructible incidence therefore has dimension at most

$$\dim \mathcal{P}_D + L(k-2). \tag{16.4}$$

Every polynomial in (16.3) has a fiber of dimension at least $L(k-1)$ in this incidence: choose $L$ general points on the projectivized Lagrangian component. Their base points can be taken distinct because the base projection has positive dimension. The component dominates $I_B$, so all points can be taken in the dense charts just chosen. The fiber-dimension theorem applied to (16.4) gives codimension at least $L$ for the parameter set.

For a prescribed jet of order $R$ at $y_0$, we exclude $y_i = y_0$ and multiply the $i$th interpolation factor by a separating linear form vanishing at $y_0$ to order $R+1$. Degree $R+3L$ suffices to retain independent two-jets at the $y_i$ while leaving the prescribed jet unchanged. Finite gradient maps exist in this affine coefficient space: extend the prescribed Taylor polynomial by sufficiently high pure powers of every coordinate. Their highest gradient terms have no common projective zero, and the nonvanishing of the

corresponding homogeneous resultant defines a nonempty Zariski open set of finite gradient maps. Finally, real coefficients are Zariski dense in the complex coefficient space, so a proper algebraic exception can be avoided by an arbitrarily small real perturbation. □

*Parameter dependence.* In the proposition the bad line family is fixed, and the proof uses that particular algebraic incidence family. The bound can be made independent of the degree of $\mathcal{B}$ because no degree estimate for the bad set is used. Uniformity in a varying family and in all Hopf coordinate germs requires an additional algebraic-family and openness argument; the parameter-family statement needed for the application is proved in Section 17.

## 16.3. **A sufficient relative-module criterion.**

**Lemma 16.3** (No Lagrangian raw fibers implies zero-section support)**.** *Let $X \to P$ be a smooth analytic family of relative dimension $k$, and let $\mathcal{M}$ be a coherent relative differential module. Suppose a closed conic analytic set $\Sigma \subset T^*(X/P)$ contains its characteristic variety. In a parameter neighborhood assume:*

(i) *every fiber $\Sigma_p \setminus 0$ has dimension at most $k$;*

(ii) *no irreducible $k$-dimensional component of any $\Sigma_p \setminus 0$ is Lagrangian.*

*Then the characteristic variety of $\mathcal{M}$ is contained in the zero section in that neighborhood.*

*Proof.* The relative characteristic variety is involutive by relative Gabber involutivity. Suppose it had an irreducible component off the zero section. Choose a smooth point at which the projection of this component to the parameter space has locally constant maximal rank. The projection is locally a submersion onto its smooth local image, and its smooth fibers are coisotropic in the vertical symplectic cotangent space. The last assertion follows from involutivity: any covector on the vertical tangent space annihilating the tangent to the fiber extends to a covector annihilating the total tangent, so its vertical Hamiltonian vector belongs to that tangent and to the vertical tangent space.

The fiber has dimension at least $k$ by coisotropy and at most $k$ by the assumed raw bound. Hence it is Lagrangian and is an open part of a $k$-dimensional raw fiber component, contradicting the second assumption. Analytic continuation of the symplectic restriction across the smooth locus justifies passing from that open part to the component. □

We use relative involutivity in the form recalled in [Wu22, Theorem 2.1]. A special fiber of an involutive relative variety need not be involutive; for this reason the proof uses a generic smooth fiber over the local image of one characteristic component.

## 17. Polynomial activities on Hopf sources

17.1. **Algebraic orbit parameters and independent jets.** The complexification of a real Hopf sphere $S^{2c-1}$ can be written

$$X_c = \{(z, w) \in \mathbb{C}^c \times \mathbb{C}^c : w^t z = 1\}.$$

The complexified Hopf action is $t \cdot (z, w) = (tz, t^{-1}w)$. Its orbit quotient is the affine variety

$$Q_c = \{P \in \mathrm{Mat}_c(\mathbb{C}) : P^2 = P,\ \mathrm{tr}\, P = 1\}, \qquad (z, w) \longmapsto zw^t. \tag{17.1}$$

Indeed, every such idempotent has rank one, and two factorizations $P = zw^t$ with $w^t z = 1$ differ by a unique nonzero scalar. Choosing a nonzero matrix entry gives algebraic local sections, so the map is an algebraic principal $\mathbb{C}^*$-bundle. The quotient is smooth of dimension $2c - 2$; this also follows from its description as the orbit of $\mathrm{diag}(1, 0, \ldots, 0)$ under conjugation by $\mathrm{GL}_c$.

For a product of $k$ active spheres let $X$ and $Q$ be the corresponding products. Every complex active leaf is an algebraic torus $(\mathbb{C}^*)^k$, and $t = \dim Q$ is finite. The real Hopf source is the compact locus $w = \overline{z}$. Here and below "real coefficients" refers to polynomials in the original real coordinates, equivalently to the real form in $(z, w)$ fixed by simultaneous conjugation and exchange of $z, w$, and not to unrelated real coefficients in the two complex coordinate lists.

**Lemma 17.1** (Ambient interpolation of leafwise two-jets)**.** *For $L$ distinct complex points of $X$, scalar ambient polynomials of degree at most $3L - 1$ realize arbitrary two-jets along the active leaves at these points. The same holds componentwise for maps to $\mathbb{C}^{k+1}$.*

*Proof.* Choose an ambient linear function separating the $L$ points. For the $i$th point, the product of the cubes of the separating factors for the other $L-1$ points has degree $3L-3$, vanishes to order three at those points, and is nonzero at the $i$th point. Multiplying it by an appropriate ambient quadratic Taylor polynomial compensates for its two-jet at the $i$th point and gives the desired ambient two-jet. Summing over $i$ gives simultaneous interpolation of ambient two-jets. Since each leaf is immersed in the ambient affine space, the restriction map from ambient two-jets to leafwise two-jets is surjective. □

17.2. **Varying bad line families without extra parameters.** Let $P$ be a finite-type complex parameter space of dimension $p$ and let

$$\mathcal{B} \subset P \times \mathrm{Gr}_2(\mathbb{C}^{k+1})$$

be a closed algebraic family, projective over $P$, with $\dim \mathcal{B}_u \le k - 2$ for all $u \in P$. In the application $P$ is the open locus of matrix parameters satisfying the ordinary adjoint surjectivity and this weak-label bound. Upper semicontinuity applies because the bad-label incidence is projective.

We use a finite stratification of $P$ and finite covers of its strata on which the geometric irreducible components of the fibers of $\mathcal{B}$ are separately parametrized. This does not add parameter dimension. We recall the standard spread-out construction. At the

generic point of an irreducible stratum, all the finitely many geometric components are defined over a finite extension of its function field. Normalize a nonempty affine open in that extension. After shrinking, take the closures of the components; by generic flatness and geometric irreducibility on an open set, their fibers are geometrically irreducible, cover every fiber, and remain distinct components. Repeat on the complement, whose dimension is smaller. Noetherian induction terminates. The specialization and geometric-irreducibility results used are [Sta, Tags 054V and 0559].

For each top-dimensional component $B_u$, its projective line incidence $I_{B_u}$ has dimension $k-1$. Its proper image $S_{B_u}$ in $\mathbb{P}^k$ is the union of its lines. Shrinking and stratifying again gives a dense smooth locus in every $S_{B_u}$, with a relative conormal bundle of positive rank. Finite affine charts give nonzero rational normal sections $n(u,p)$. These choices introduce finite covers and charts, but no new continuous choice of hypersurface or normal family.

17.3. **A global algebraic exceptional set of activity coefficients.** Let $\mathcal{A}_D$ be the vector space of $(k+1)$-tuples of ambient polynomials of degree at most $D$, restricted to $X$. We retain redundant ambient coefficients, which is harmless. Write $N_D = \dim \mathcal{A}_D$.

**Proposition 17.2** (Uniform coefficient exceptional set)**.** *For every $L \geq 1$ and $D \geq 3L-1$, there is a complex algebraic subset $\mathcal{E}_D \subset \mathcal{A}_D$ of codimension at least*

$$L - p - t \tag{17.2}$$

*with the following property. If $a \notin \mathcal{E}_D$, then for every $u \in P$ and every complex active orbit, at every point where $a$ is a partial immersion and its Gauss germ is finite, the associated raw characteristic germ has no $k$-dimensional Lagrangian component. The set may be replaced by its union with its complex conjugate without changing the bound.*

*Proof.* Use one of the finitely many component and normal charts above, one algebraic orbit chart in $Q$, and one of the $k+1$ graph projection charts for the target. Take $L$ distinct points $x_i$ on an orbit and $L$ incidence points $(p_i, \ell_i) \in I_{B_u}$, and impose that the Gauss normal of $a$ at $x_i$ equals $p_i$.

On the partial-immersion locus, the map from the first derivative of $a$ to its projective normal is a submersion onto $\mathbb{P}^k$. Thus the normal constraints impose $kL$ independent equations. After the first derivatives have been fixed, the graph Hessians at the $L$ points are arbitrary and independent by Lemma 17.1. The necessary equation

$$\det A_i \, n(u,p_i)^t A_i n(u,p_i) = 0 \tag{17.3}$$

imposes at least one additional independent condition at each point. All expressions are algebraic after clearing the denominators of the chosen charts. This is an incidence in coefficient space, not a statement about generic unrestricted tensors.

The resulting $L$-point incidence has dimension at most

$$N_D + p + t + L(k-2). \tag{17.4}$$

If a tuple $(u, \mathrm{orbit}, a)$ has a Lagrangian raw germ at a point where the Gauss germ is finite, analytic continuation in its algebraic irreducible component makes that component Lagrangian. Its projectivization has dimension $k-1$, maps generically finitely

onto a top-dimensional $I_{B_u}$, and satisfies (17.3) on a dense open subset, as in the preceding Hessian lemma. Thus the fiber of the incidence (17.4) over this tuple contains an $L(k-1)$-dimensional family of point tuples. The points may be taken distinct because the base projection has positive dimension.

The locus of tuples whose incidence fiber has dimension at least $L(k-1)$ is constructible, and the fiber-dimension theorem bounds its dimension by $N_D + p + t - L$. Its image in $\mathcal{A}_D$ and the Zariski closure of that image have no larger dimension. Taking the finite union over all the charts and component strata proves (17.2).

Only Gauss germs near the point in question need be finite. A Lagrangian analytic branch there is an open analytic part of an algebraic component of the same dimension. Its symplectic identity extends over that component, and the finite germ makes its map to $I_{B_u}$ generically finite. Therefore other components, or remote points where the Gauss map is not finite, cause no omission in the count. □

The algebraic closure in this proposition gives openness: its complement is a single Zariski open set of coefficient space. Separate nonzero Hessian conditions at isolated points would not by themselves give openness.

17.4. **Smooth slow coefficients and the protected class.** Let the slow parameter manifold be compact of dimension $b$. Choose $L > p + t + b$ and then a finite degree $D \geq 3L - 1$ large enough for the required analytic approximation. For a smooth polynomial-coefficient family $a_0 : B_{\mathrm{slow}} \to \mathcal{A}_D(\mathbb{R})$, arbitrarily small constant coefficient translations $v \in \mathcal{A}_D(\mathbb{R})$ achieve

$$a_0(x) + v \notin \mathcal{E}_D \quad \text{for every } x. \tag{17.5}$$

Indeed, the real algebraic exceptional set has real codimension greater than $b$. We stratify it and apply parametric transversality to $(x, v) \mapsto a_0(x) + v$, which is a submersion in $v$. Compactness then gives a positive margin from the closed exceptional set. The translation can be taken as small as desired in any prescribed finite norm on the real compact Hopf source, and it is independent of every old fast variable.

The choices are made in the following order, so the Hopf source size does not increase. First fix any common sphere size allowed by the source construction and the active-frame class argument; this fixes $X$, $Q$, and $t$. Next choose $L$, the polynomial degree, and then the differential-inverse order. Theorem 7.1 permits the actual activity and its finite derivative orders to depend on the fixed sphere size, while its slow coefficient bundles and number of factors remain fixed. The later expansion order and repair frequencies are chosen afterwards; they require no new Hopf factors and no larger common sphere size.

In the order of construction, the horizontal frame has already retained the compact protected activity homotopy before this analytic perturbation is made. An arbitrarily small $C^1$ perturbation appends a short straight derivative homotopy, which preserves its injection margin and complete class. A new polynomial activity need not agree literally with the old one on an open active collar; agreement on collars is provided by the relative clauses of the preceding tensor and frame constructions. The polynomial

approximation therefore imposes no incompatible analytic continuation condition on an active collar.

17.5. **The critical periodic inverse.**

**Proposition 17.3** (Critical weak-label periodic inverse)**.** *Let $U$ be the actual derivative tensor of a horizontal normal frame, with surjective ordinary frame operator. Let the active source be a product of $k = q-1$ Hopf spheres and fix a compact protected homotopy of its first-column injection. Assume*

$$\dim \mathcal{B}_U \le q - 3.$$

*Then the activity can be chosen in an arbitrarily small protected $C^1$ neighborhood so that the periodic mixed operator has a global smooth finite-order differential right inverse, depending smoothly on all old and slow parameters.*

*Proof.* We first apply Lemma 15.3. In choosing its initial good graph normal we use only that the union of the bad normal lines has dimension $\dim \mathcal{B}_U + 1 \le k - 1 < k$. Hence a real good normal exists and is dense, and the finite-type flat system, detecting rows, exponential perturbation, and compact finite-jet test of Lemma 15.3 apply. The orbitwise multijet argument in Section 15 supplies a killing point on every active torus orbit, and Lemma 15.4 supplies finite complex Gauss germs at every real point. Both conditions are open in a fixed sufficiently high finite norm on the relevant compact parameter products.

We approximate in that norm by an ambient polynomial activity with smooth slow coefficients. Enlarging the degree as necessary, we use (17.5) to remove the algebraic exceptional set by an arbitrarily small constant polynomial perturbation. This preserves the finite-Gauss and orbit-killing conditions, partial immersion, old-fast independence, and the protected complete class.

We treat matrix entries, activity coefficients and transverse complex Hopf coordinates as independent complex parameters, and differentiate only in the active variables. Near each relevant real parameter, the finite Gauss property and the projective bad-label dimension bound give raw characteristic fiber dimension at most $k$. Proposition 17.2 and the open coefficient condition exclude every $k$-dimensional Lagrangian raw fiber component throughout that neighborhood. The relative no-Lagrangian criterion therefore forces the completed characteristic variety into the zero section.

The relative module here is the module presented by the formal adjoint equations. In graph coordinates, with densitized ambient flux $X$, they read

$$UX = 0, \qquad e_n X - \sum_i f_i e_i X = 0, \qquad \sum_i e_i \partial_i X = 0.$$

Passing to these coordinates is an invertible change of unknowns and density. The entries of $U$, the finitely many activity coefficients, and the transverse Hopf coordinates are independent complex parameters; the actual smooth family is substituted only after the inverse is found.

A good filtration and zero-section characteristic support imply coherence over the full holomorphic function sheaf: locally the cotangent generators act nilpotently on the

associated graded module, so finitely many graded pieces generate it over the function ring. The relative differential action gives an integrable connection. On a fixed parameter fiber a coherent module with a holomorphic connection is locally free, and its rank is constant on a connected active leaf. Lemma 15.3 makes that module zero near a killing point, so it vanishes along the whole active orbit. Ordinary Nakayama for the coherent function module then makes the relative module zero near every relevant real parameter point.

Vanishing of the presented module expresses its identity matrix as a finite sum of differential multiples of the adjoint equations. This is a differential left inverse with holomorphic parameter coefficients. Smooth dependence comes from this parameter-level step, not from a separate choice of inverses on individual fibers. We pull those coefficients back by the actual smooth parameter family and take formal adjoints. Choose a finite subordinate partition of unity $\{\chi_i\}$ and local right inverses $R_i$. The formula $R(h) = \sum_i R_i(\chi_i h)$ is globally defined because differential operators preserve support, and $\mathcal{Q}R(h) = \sum_i \chi_i h = h$. Compactness bounds the order of $R$ and each fixed coefficient norm. All operations take place on the original periodic source. □

## 18. Finite metric completion and the complete core class

This section completes the old free map and the active block to a polynomial family with arbitrarily small metric error, repairs it exactly, and identifies the resulting boundary class on the core.

Let $H$ be the actual old free map, $b$ the exact active free block, and $\nu$ the actual horizontal normal frame. Proposition 13.4 places these in the physical space $V_0 \oplus \mathbb{R}^{28}$ by means of the isometries $\Phi_y$. Write

$$F_0 = (H, b), \quad Z = \Phi_y(\nu a), \quad F_\epsilon = F_0 + \epsilon Z + \sum_{j=2}^{P} \epsilon^j W_j.$$

The source and all coefficients are fixed before $\epsilon$ is chosen. The zeroth-order active metric is the full prescribed active metric, not that metric minus $\Gamma_A(a)$.

**Proposition 18.1** (Arbitrary finite precision)**.** *For every finite $P$, the coefficients can be chosen on the whole original source so that*

$$\Gamma_{E\oplus A}(F_\epsilon) - (h_E \oplus h_A) = O(\epsilon^{P+1})$$

*in every fixed full derivative norm. The true best-phase margin is at least $c\epsilon$ for small $\epsilon$. On the protected core collar the specified rank-$(n-1)$ bundle $E_v \oplus A$ is free, with the prescribed complete class.*

*Proof.* The field $Z$ is perpendicular to the complete pure old and active columns, hence $D\Gamma_{F_0} Z = 0$. Horizontality and Proposition 13.4 also give a zero mixed row in $\Gamma(Z)$. The two independent pure free-block pairing inverses choose $W_2$ with $D\Gamma_{F_0} W_2 = -\Gamma(Z)$.

Suppose the metric is correct through order $r-1$, $r \geq 3$, and write $e^{[r]}$ for its next coefficient. A change $W_{r-1} \mapsto W_{r-1} + \Phi_y v$, with $v$ in the residual normal bundle, leaves

the preceding equation unchanged. Its new mixed contribution at order $r$ is

$$d_A T - 2B^* v, \qquad T_\alpha = \left\langle \sum_a a_a \nabla_{e_\alpha} \nu_a, v \right\rangle, \quad B = U \circ (\mathrm{Id}_E \otimes d_A a).$$

We use the periodic differential inverse of Proposition 17.3 to solve

$$B^* v + d_A f = \tfrac{1}{2} e^{[r]}_{\mathrm{mixed}}.$$

The new mixed error is then $d_A(T+2f)$. Prescribe $\langle H_\alpha, W_r \rangle = -T_\alpha - 2f_\alpha$ and zero active first pairings. The second pairings in the pure old and active free blocks then cancel their remaining symmetric rows; these are global algebraic inverses. All other effects of changing $W_{r-1}$ start after degree $r$, since $2r-2 > r$. This proves the induction on the compact periodic source; it involves no choice of origin and no indefinite integration on a Hopf circle.

At an active point, the five-planes in $\operatorname{im} d_A a \subset \mathbb{R}^7$ form a Grassmannian of dimension five, which a complex bad-label set of dimension at most four cannot cover. Choose a good real plane; if all real planes were bad, their Zariski density would give the excluded coverage. Project away the pure complete blocks and divide the mixed columns by $\epsilon$. They converge to the injective restriction of $U(\mathrm{Id}_E \otimes d_A a)$ on this good plane. Since the pure blocks are independent, the complete hyperplane jet has a positive normalized margin. Compactness, with the maximum over phases, gives a uniform margin $c\epsilon$ for the true normal second form. On the core we use the already protected plane instead of selecting a maximizing one; the compact activity homotopy gives the specified complete class. □

Before applying the polynomial correction theorem, we use the product slow collar fixed in the source preparation. The coefficient bundle $E \oplus A$, its metric, the protected hyperplane, and the leaf connection have radial pullback components there. Let $\mathcal{R}$ be a fixed slow retraction, constant in the radial variable on an inner collar and the identity outside a slightly larger one, and apply it to the whole finite polynomial family: replace $F_\epsilon$ by $F_\epsilon \circ \mathcal{R}$. In product coefficient coordinates $d\mathcal{R}(Y) = Y \circ \mathcal{R}$ for every leaf field $Y$, so the complete leaf jets and the metric error of the new family are the corresponding pullbacks. All full derivative estimates lose only constants independent of $\epsilon$. The $c\epsilon$ phase margin and the protected augmented complete class persist, the latter also along the retraction homotopy. The resulting family and prescribed partial metric have literal product collars. We double this fixed slow source and its data, apply Theorem 6.3 on the closed double, and restrict to the original exterior. In particular, no extension across a collar whose width depends on $\epsilon$ is needed.

To apply that theorem, first choose its entrance exponent with phase exponent one and a full $C^2$ displacement exponent strictly greater than one. Its reference finite derivatives are bounded independently of $\epsilon$ for each fixed finite construction. Choose $P$ above the resulting entrance exponent, construct all coefficients, and then take $\epsilon$ small enough. Exact repair gives an exterior selector in the same target with $C^2$ displacement $o(\epsilon)$. The phase and normalized complete collar injections persist. Constant prefactors may depend on $P$; since they are fixed before $\epsilon$ is selected, the choices are not circular.

If a literal product collar is required after correction, precompose once more with a fixed retraction supported inside the collar where the reference family is radially constant. This preserves the exact prescribed metric and changes the $C^2$ distance from that reference only by a fixed factor, so the distance remains $o(\epsilon)$.

18.1. **The actual complete boundary class.** On the core sphere boundary the normalized limiting hyperplane injection is

$$J_{E_v}H \ \oplus \ J_A b \ \oplus T_Q(\mathrm{Id}_{E_v} \otimes d_A a). \tag{18.1}$$

This expression is obtained by triangular elimination of pure components and multiplication of mixed columns by the positive scalar $\epsilon^{-1}$; both operations are homotopies through injections. First reverse the protected activity homotopy while holding the actual whole-core tensor fixed. Next reverse the supplied horizontal-frame homotopy while holding the standard activity fixed. Transport the old complete class to its slow injection and contract the complete active class, including its cofactor normal, by Proposition 13.3. The whole mixed block stays injective throughout. The endpoint is

$$U_0\kappa_v \oplus A_+|_{J(A)\oplus(E_v\otimes A)}.$$

This is the complete injection in Proposition 8.2. The whole-$Q$ derivative enters only as algebraic class data; we make no claim about an actual derivative of the old map in the missing $Q/E_v$ direction.

We also retain the old $L$ first column. In the active standard block it is independent of $J(A)$ and perpendicular to the other standard blocks, so the same homotopy extends the augmented injection $J(H) \oplus L$. Lemma 8.4 extends the coefficient injection of $H$ across the core disk while retaining the separate $L$ coefficient. The resulting coefficient distribution on the Hopf source is vertical and integrable, since all its coefficients depend only on the slow base. Place the complete boundary homotopy in an inner radial annulus and keep the actual complete jet fixed on the protected outer collar. After removing the prospective new line inputs, apply the relative free realization Theorem 4.1 to the hyperplane $H$ with its actual incoming complete collar jet. This gives an exact $H$-free fill. Literal radial collars can be arranged by slow retractions, which preserve all leaf derivatives and metrics. The exterior is unchanged on the outer annulus, and its full metric remains exact there.

**Lemma 18.2** (Two normal line copies after the fill)**.** *Suppose the physical target is* $\mathbb{R}^{s_n+C}$ *and* $C-1 > \dim P$. *After this relative fill there is an isometric copy of* $2L$ *perpendicular to the complete actual* $H$ *jet and to a unit augmented line* $e(L)$. *This line can be chosen to equal the actual old unit* $L$ *first derivative on the peeling annulus.*

*Proof.* Since $j_{n-1} = s_n - 1$, the slow complement to the extended augmented injection $J(H) \oplus L$ has rank $C$. The noninjective locus for a map from $2L$ to this complement has codimension $C-1 > \dim P$. General position over $P$ therefore supplies a monomorphism, and polar normalization makes it isometric. Pull these slow data back to the fast source and transport them through the retained complete free homotopy. Rank avoidance is applied only on the slow base, not on the much larger fast source.

On the fixed collar we choose the transported normal representative to equal the nonzero complete-normal projection of the old $L$ first column, multiplying its unit representative by the positive norm of that projection. To recover the actual unit first column, add its component in the complete $H$ span, extend that component by a cutoff into the tube, and keep the complete-normal component nonzero. The added component is perpendicular to the first $H$ columns, since the old metric is exact on the annulus, and it may be extended inside that subbundle. Normalize the resulting nonzero line. The selected $2L$ is perpendicular to both components, so this modification preserves its orthogonality and gives the required actual annular unit line. All steps are invariant under sign transitions of $L$. □

At $C = 28$ the rank-failure codimension is $27 > 5$. Theorem 19.4 applies to these data and supplies the final exact selector in the same physical target.

## 19. Absorbing the surviving line in the same target

We construct a supported normal loop for an arbitrary metric line bundle on the corank-one core. The intermediate twisted source is used only over the core tube. Signed integer transport then produces a single global map on ordinary Hopf inputs, to which one final nonrelative repair is applied.

19.1. **A contact family with line-bundle coefficients.** Let $L$ be a metric real line bundle over a compact closed base $P$. For $C \geq 2$ put

$$Z_C(L) = S(\varepsilon^C \oplus L^C), \qquad A(z) = -z.$$

Its unoriented Hopf distribution is canonically the pullback of $L$. In a unit frame $\ell$ of $L$, identify a point with $z = u + iv$; changing $\ell$ to $-\ell$ changes the local coordinate $z$ to $\bar{z}$ and the period-one generator $X_\ell$ to $-X_\ell$. Orbit averaging is independent of orientation. The zero-mean primitive $I_X$ changes line parity: using $L^2 \cong \varepsilon$, it sends scalar functions to $L$-valued functions and $L$-valued functions to scalars.

For a zero-mean function in a local unit frame, the primitive is

$$I_X h(z) = \int_0^1 t h(R_t z)\, dt.$$

It satisfies $X I_X h = h$ and $\overline{I_X h} = 0$. Reversing the orientation changes its sign, which gives the stated line parity.

**Lemma 19.1** (Twisted finite strict contact families)**.** *For every finite $d$ there is a small parameter neighborhood of the zero section of*

$$\varepsilon^d \oplus L^d \oplus L \quad \textit{with variables } (q, p, \tau)$$

*and a smooth family of sections $a(q, p, \tau, z)$ of $2L$ over $Z_C(L)$ such that, in every unit frame of $L$,*

$$|X_\ell a| = 1, \qquad \overline{a} = 0, \qquad \overline{a \wedge X_\ell a} = 0, \tag{19.1}$$

$$\overline{\langle d_{q,p,\tau} a, X_\ell a \rangle} = d\tau - \sum_{j=1}^{d} p_j dq_j. \tag{19.2}$$

*Equation* (19.2) *is an identity of $L$-valued one-forms. The parameter dependence is smooth over the compact quotient by the Hopf circles. On the Legendrian family $p = 0, \tau = 0$, the $q$ derivatives are $2L$-valued and the $p$ derivatives are ordinary $\mathbb{R}^2$-valued; these parities are part of the assertion.*

*Proof. A global unit-speed zero-area base selector.* Pass to the quotient $Z_C(L)/\{1, A\}$ and let $\gamma_A$ be the sign line of its double cover. On this quotient use the flat metric rank-two bundle

$$L \oplus (L \otimes \gamma_A).$$

The Hopf line descends and is still isomorphic to $L$. The open relation that the covariant derivative along this line be nonzero is ample, since its nontrivial principal slices are punctured real planes. It has a global formal section with zero value whose derivative from the Hopf line into the first $L$ factor is the identity. Theorem 3.1 supplies an actual section.

Lift it to $Z_C(L)$ and use the canonical trivialization of the pulled-back sign line. The lift is a section of $2L$ satisfying

$$a(Az) = Ra(z), \qquad R = \operatorname{diag}(1, -1), \qquad |X_\ell a| > 0. \tag{19.3}$$

Intrinsic arclength normalization is independent of the local orientation: speed and mean length are scalars, and the primitive of the normalized speed defect is $L$-valued, as required for an orbit shift. Normalization preserves (19.3), as does subtraction of the orbit mean. We obtain a centered unit-speed base selector. Its signed-area integrand changes sign under $A$, so its orbit area is zero. The construction requires no equivariant convex integration at fixed points of complex conjugation, since the only quotient used is the free half-period quotient.

Write locally $v = X_\ell a$ and $n = Jv$. Both are ordinary $\mathbb{R}^2$-valued functions, unchanged when the unit frame of $L$ changes. Put $X_\ell v = \kappa n$; then $\kappa$ is $L$-valued. On every Hopf fiber sphere the unit velocity map has zero turning number, since that sphere is simply connected. Consequently $\bar{\kappa} = 0$. The function $k_2 = \overline{\kappa^2}$ is a positive scalar basic function with a positive compact minimum: if the curvature vanished on an orbit, its unit tangent would be constant, contradicting the zero mean of the derivative of a closed curve.

*Uniform contact tangent pairs with the correct parity.* Choose a smooth scalar cutoff $\eta = \eta(v_1)$ supported where $v_1 > \delta > 0$ and positive where $v_1 > 2\delta$, with $2\delta$ below the minimum, over all orbits, of $\max_{\text{orbit}} v_1$. That minimum is positive, because a closed unit-speed planar curve cannot have $v_1 \le 0$ everywhere, nor can $v_1$ vanish identically. Moreover,

$$w_0 = \overline{a_2^2 \eta^2 v_1} > 0 \tag{19.4}$$

on every orbit, hence uniformly. Indeed, if $a_2$ vanished on the nonempty open set $v_1 > 2\delta$, then there $v_2 = Xa_2 = 0$ and unit speed would force $v_1 = 1$. Such a set cannot have a boundary at $v_1 = 2\delta < 1$, and it cannot be the whole closed orbit. This proves the strict positivity in (19.4).

For distinct positive integers $c_j$ and large $K$ use

$$h_{q,j} = a_2 \eta \cos(K c_j a_1), \qquad h_{p,j} = a_2 \eta \sin(K c_j a_1). \tag{19.5}$$

The first is $L$-valued and the second scalar. Both are odd under $A$, because $a_1$ and $v_1$ are even and $a_2$ is odd. For either choice let

$$\beta_h = h - \bar{h} - \kappa\frac{\overline{\kappa h}}{k_2}, \qquad \alpha_h = I_X(\kappa\beta_h), \qquad u_h = \alpha_h v + \beta_h n - \overline{\alpha_h v + \beta_h n}. \tag{19.6}$$

For $h_q$ both $\alpha_h$ and $\beta_h$ are $L$-valued, and for $h_p$ both are scalar, so these formulas produce the stated $q$ and $p$ tangent parities. They satisfy $\bar{\beta}_h = \overline{\kappa\beta_h} = 0$, and hence define unit-speed tangent vectors with $\overline{\langle u_h, v\rangle} = 0$.

All corrections preserve the half-period symmetry: $\kappa$ and $\beta_h$ are odd under $A$, $\alpha_h$ is even, $v(Az) = Rv(z)$, and $n(Az) = -Rn(z)$, so $u_h(Az) = Ru_h(z)$. In particular every finite linear combination of the allowed tangent vectors preserves (19.3), and therefore preserves zero signed area exactly, not only to first order.

On the support of $\eta$, $X_\ell a_1 = v_1 > \delta$. Repeated integration by parts in the Hopf field, whose phase derivative is bounded below on the support of $\eta$, gives, uniformly on the compact quotient,

$$\overline{\langle u_{q,j}, X_\ell u_{p,j}\rangle} = \frac{Kc_j}{2}w_0 + o(K). \tag{19.7}$$

Pairings between different frequencies, and pairings of the same type, are $o(K)$. These estimates are independent of the unit frame: changing it changes $a_1$ and $X$ together, and the two parameter parities compensate. For the orbit one-form $\lambda = \overline{\langle da, X_\ell a\rangle}$, the exterior derivative on two commuting tangent variations is

$$d\lambda(u_q, u_p) = -2\overline{\langle u_q, X_\ell u_p\rangle}.$$

Divide both vectors of the $j$th pair by $\sqrt{Kc_j w_0}$ and reverse the sign of its $p$ vector. Equation (19.7) then makes $d\lambda$ close to the canonical $L$-valued form $\sum_j dq_j \wedge dp_j$ on $\varepsilon^d \oplus L^d$. The sign reversal respects all line parities and the half-period symmetry. Relabel these normalized vectors as $u_{q,j}, u_{p,j}$.

A smooth basic bundle automorphism makes this skew form exactly canonical. One intrinsic construction joins it to the canonical form by a short linear path $\Omega_t$ and solves the linear matrix transport with $-\frac{1}{2}\Omega_t^{-1}\dot{\Omega}_t$; the two $L$ factors cancel, so this is a globally defined ordinary bundle endomorphism. The construction also shows that the normalization commutes with every sign transition of $L$.

*Integrating the finite family and normalizing the contact form.* Form the actual family $g_{q,p} = a + \sum q_j u_{q,j} + \sum p_j u_{p,j}$. The products $p_j u_{p,j}$ are $2L$-valued, so this family is global. It satisfies (19.3), has nonzero Hopf derivative for sufficiently small parameters, and has zero area by the half-period symmetry. Intrinsic speed normalization, division by orbit length, and centering preserve these properties and leave its tangent vectors at zero unchanged. Append an orbit shift $\tau \in L$. The resulting finite parameter one-form is $d\tau + \lambda(q,p)$, where $\lambda$ is $L$-valued and $d\lambda$ is symplectic near zero on the parameter fibers.

Apply the ordinary finite-dimensional radial-primitive Moser argument fiberwise to make $d\lambda$ canonical, and remove the resulting closed difference in $\lambda$ by a change of $\tau$. The argument is global on the parameter bundle: the path of $L$-valued forms, its radial primitive, and the inverse form defining the Moser vector field are invariant under

the simultaneous transitions $p \mapsto -p$, $\tau \mapsto -\tau$, and $\lambda \mapsto -\lambda$. Compactness gives one common small parameter neighborhood. This proves (19.2), with all dependence smooth on the whole twisted sphere bundle, without trivializing any nontrivial line bundle. □

19.2. **The geometric data for absorption.** Let $X_0$ be the completed old Hopf source over the closed slow base. On a core tube assume a metric splitting $V = H \oplus L$, where $L$ is a metric line bundle and $\operatorname{rank} H = n - 1$. We work in a local unit frame of $L$ and track the sign changes below. Write $Y_i$ for local commuting coefficient fields of $H$, and $Y_0$ for the old coefficient field of that unit section. All their coefficients depend only on the slow base, so they commute in every such local frame.

Suppose the completed old map $F_0 : X_0 \to \mathbb{R}^N$ is exact and phase regular off the tube, is $H$-free and $H$-isometric throughout the tube, and remains the original exact rank-$n$ map on its peeling annulus. Let $\sigma$ be a slow function equal to zero on an outer neighborhood and to $\pi/2$ on the inner filled region, and put

$$\chi = \sin\sigma, \qquad c = \cos\sigma.$$

The support of $\chi$ lies inside the tube. Choose a second slow cutoff $\beta$, supported in a larger tube and equal to one on a neighborhood of $\operatorname{supp}\chi$. These slow cutoffs and their collar supports are fixed before the common Hopf sphere size is varied.

We require the following augmented normal data, supplied by the complete boundary-class argument. There is a smooth unit field $e$ orthogonal to the first $H$ columns, with $J_H F_0 \oplus e$ injective, and an isometric bundle map $\nu : 2L \to \varepsilon^N$, represented locally by two orthonormal fields $\nu_1, \nu_2$ orthogonal to all columns of $J_H F_0 \oplus e$. On the region where $c \neq 0$ and $\chi$ may vary, $e = Y_0 F_0$ and the old rank-$n$ map is still exact. Thus, wherever it is used,

$$cY_0F_0 = ce. \tag{19.8}$$

Only this actual augmented jet and its two actual normal fields are required. Lemma 18.2 constructs them from the standard complete class by rank avoidance on the slow core and transport through the complete boundary homotopy.

For the finite construction we use the twisted sphere bundle $Z_C(L)$. Instead of imposing an isometric coefficient embedding at this intermediate stage, we use the fixed coefficient map with $H$ unchanged and line coefficient

$$cY_0 + \beta X_{\text{new}}. \tag{19.9}$$

It is smooth and injective everywhere. On the wave region $\beta = 1$; on its complement where $\beta$ changes, $\chi = 0, c = 1$ and all maps below are independent of the twisted sphere. The coefficient metric is uniformly comparable to the original one, and no division by $\sin\sigma$ occurs.

Choose an actual smooth map $j : X_0|_{\text{tube}} \to \mathbb{R}^d$ whose derivative on $H$ is injective; for instance, an ordinary embedding of this compact source into a sufficiently large Euclidean space. Rescale its image into the small $q$-ball of Lemma 19.1. The parameter

dimension $d$ is internal to the curve family; it is neither a physical target dimension nor an additional Hopf slot. Put

$$a(x,z) = a(j(x),0,0,z), \quad v = Xa, \quad n_a = Jv, \quad Z = \nu a = \nu_1 a_1 + \nu_2 a_2. \tag{19.10}$$

The profile is centered and has zero area. Since the skew connection matrix $\nu^* d\nu$ takes values in $\mathfrak{so}(2)$, its contribution to an orbit mean is proportional to signed area. Equation (19.2) therefore gives, for every old derivative $Y$,

$$\overline{\langle YZ, XZ\rangle} = 0, \qquad \overline{Z} = 0, \qquad |XZ| = 1. \tag{19.11}$$

19.3. **Arbitrary finite periodic completion, including the cutoff.**

**Proposition 19.2** (Finite completion in the existing target)**.** *For every finite integer $L \geq 2$ there are fixed smooth coefficient maps $F_1, \dots, F_L$ on the old source fibered with the twisted sphere, taking values in the same $\mathbb{R}^N$, such that*

$$F_1 = \chi Z, \qquad F_r \in \chi C^\infty, \qquad \overline{F_r} \in \chi^2 C^\infty, \tag{19.12}$$

*and the actual maps*

$$F_\epsilon^{[L]}(x,z) = F_0(x) + \sum_{r=1}^{L} \epsilon^r F_r(x, P_T z), \qquad \epsilon = T^{-1}, \quad T \in \mathbb{N}, \tag{19.13}$$

*have partial metric error $O(\epsilon^L)$ before observation derivatives. Here $P_T$ is the normalized coordinate power map, so $(P_T)_* X = TX$. In any fixed full source atlas, for every fixed observation order $J$,*

$$\|\Gamma_D(F_\epsilon^{[L]}) - g\|_{C^J} \leq C_{J,L}\epsilon^{L-J}, \qquad \|F_\epsilon^{[L]}\|_{C^J} \leq C'_{J,L}(1+\epsilon^{1-J}). \tag{19.14}$$

*The best-phase margin is bounded below independently of small $\epsilon$. Every coefficient vanishes outside the wave tube.*

*Proof.* It suffices first to work with formal powers of $\epsilon$, treating the new field in (19.9) on the wave region as $D_0 = cY_0 + \epsilon^{-1}X$. The leading line tangent is

$$w = ce + \chi\nu v.$$

It is a unit vector orthogonal to the first $H$ columns, because $c^2 + \chi^2 = 1$ and $e \perp \nu$. The leading metric is therefore exactly $g$. Also $F_1$ is orthogonal to the complete $H$ jet, so its linear pure-$H$ metric variation is zero.

At order $r \geq 1$ the mixed, line, and next pure-$H$ coefficients have the triangular form

$$E_{r,i}^{\mathrm{mix}} = X\langle Y_i F_0, F_{r+1}\rangle + C_{r,i}, \tag{19.15}$$

$$E_r^{\mathrm{line}} = 2\langle w, XF_{r+1}\rangle + D_r, \tag{19.16}$$

$$E_{r+1}^H = \mathcal{L}_H F_{r+1} + Q_{r+1}^H, \tag{19.17}$$

where $C_r, D_r, Q_{r+1}^H$ depend only on the coefficients through $F_r$. Here

$$(\mathcal{L}_H W)_{ij} = Y_i\langle Y_j F_0, W\rangle + Y_j\langle Y_i F_0, W\rangle - 2\langle Y_i Y_j F_0, W\rangle.$$

These formulas follow by expanding the quadratic metric; the negative power occurs only in the displayed $X$ derivative.

We retain the ideals in (19.12) throughout. They imply

$$C_r, D_r \in \chi C^\infty, \qquad \overline{C_r}, \overline{D_r} \in \chi^2 C^\infty, \qquad Q^H_{r+1} \in \chi^2 C^\infty. \tag{19.18}$$

Indeed, all quadratic terms contain two factors in $\chi C^\infty$. The only linear terms involve fixed old derivatives of $F_r$, and their averages contain $\overline{F_r} \in \chi^2 C^\infty$. All cutoffs are slow, so old leaf derivatives and $X$ preserve these ideals and commute with orbit averaging. These are identities in smooth ideals, not only pointwise big-O estimates.

At the initial step $r = 1$, the mixed mean vanishes without a correction:

$$C_{1,i} = -2c\chi\langle Y_i e, Z\rangle + \chi^2\langle Y_i Z, XZ\rangle,$$

whose average is zero by (19.11). Likewise

$$D_1 = 2c^2\chi\langle e, Y_0 Z\rangle + 2c\chi^2\langle XZ, Y_0 Z\rangle$$

has zero mean, by centering and (19.11).

For $r \geq 2$ we first kill the mixed mean by changing $F_r$ in a leading-kernel direction. Write $b_j = \partial_{p_j} a$ on the Legendrian $q$-family. For a basic section $\zeta(x, [z]) \in L^d$ put

$$V_r = \chi\nu \sum_j b_j \zeta_j. \tag{19.19}$$

It is orthogonal to $J_H F_0 \oplus e$, is centered, and satisfies $\langle w, XV_r\rangle = 0$ by the unit-speed derivative identity in Lemma 19.1. Thus adding it to $F_r$ preserves all previously canceled rows, including the pure-$H$ row at order $r$ and the mixed and line rows at order $r - 1$.

Its change to the mixed mean at order $r$ is

$$\delta\overline{C_{r,i}} = -\chi^2\, Y_i j \cdot \zeta. \tag{19.20}$$

To verify this, note that the terms involving the old line tangent are $-2c\langle Y_i e, V_r\rangle$, whose mean is zero because $V_r$ is centered. The remaining terms are $\chi$ times

$$\overline{\langle Y_i Z, XV_r\rangle + \langle Y_i V_r, XZ\rangle}.$$

The normal-frame connection contributions are derivatives of the zero signed-area tensor and hence vanish. What remains is the variation of the pulled-back contact one-form (19.2), which at $p = 0, \tau = 0$ is $-\chi\, Y_i j \cdot \zeta$. Derivatives of $\zeta$ are multiplied by the vanishing coefficient of $dp$ in (19.2), so none survives. This proves (19.20).

Since $d_H j$ is injective, its adjoint maps $\mathbb{R}^d$ onto $H^*$ and has a smooth algebraic right inverse. Divide the known mixed mean by its smooth factor $\chi^2$ in (19.18), and choose $\zeta$ by that right inverse so that the new mixed mean is zero. The division takes place in the smooth ideal and therefore introduces no singularity at $\chi = 0$. The new coefficient $F_r + V_r$ still satisfies both ideals in (19.12). From now on we use this updated $F_r$ and recompute the known line coefficient $D_r$ and the next pure-$H$ coefficient $Q^H_{r+1}$; (19.18) still holds, including any quadratic terms created when $r = 2$.

Set the first $H$ pairings for $F_{r+1}$ to

$$\alpha_i = -I_X C_{r,i}. \tag{19.21}$$

They lie in $\chi C^\infty$ and have zero mean. Put $h = -D_r/2$ and write $\bar{h} = \chi^2 h_0$ smoothly. For the current unit planar loop let $Xv = \kappa n_a$ and $k_2 = \overline{\kappa^2} > 0$. Define

$$b = -\kappa h_0/k_2, \qquad u = I_X(h + \chi^2 \kappa b), \qquad W_{\text{line}} = uw + \chi b \nu n_a. \tag{19.22}$$

The primitive is defined because its input has zero mean, and the formula involves no division by $\chi$. Since $Xw = \chi\kappa\nu n_a$,

$$\langle w, XW_{\text{line}} \rangle = Xu - \chi^2 \kappa b = h.$$

This solves (19.16). The vector $W_{\text{line}}$ is orthogonal to every first $H$ column. It lies in $\chi C^\infty$, and its mean lies in $\chi^2 C^\infty$ because $\bar{u} = 0$, $u \in \chi C^\infty$, and $\overline{\kappa n_a} = \overline{Xv} = 0$.

Use the algebraic pairing inverse of the augmented injection $J_H F_0 \oplus e$ to choose $W_H$ in its column span with

$$\begin{aligned} \langle Y_i F_0, W_H \rangle &= \alpha_i, \\ \langle e, W_H \rangle &= 0, \\ \langle Y_i Y_j F_0, W_H \rangle &= \tfrac{1}{2}(Y_i \alpha_j + Y_j \alpha_i + Q^H_{r+1,ij}) - \langle Y_i Y_j F_0, W_{\text{line}} \rangle. \end{aligned}$$

Then $F_{r+1} = W_H + W_{\text{line}}$ solves (19.15)–(19.17). Indeed, $W_H \perp \nu$ and $\langle e, W_H \rangle = 0$, so $\langle w, XW_H \rangle = 0$, and the pure-$H$ pairing formula gives $\mathcal{L}_H F_{r+1} = -Q^H_{r+1}$. The pairing inverse is independent of the twisted sphere. Its prescribed first pairings have zero mean, and its prescribed second pairings have mean in $\chi^2 C^\infty$. Consequently $W_H$, and therefore $F_{r+1}$, satisfies both ideals in (19.12). This completes the finite induction.

After constructing the coefficients through $F_L$, all pure-$H$ coefficients through order $L$ and all mixed and line coefficients through order $L-1$ vanish, so the remaining metric polynomial starts at order $\epsilon^L$. The normalized power map has full derivative bounds $\|P_T\|_{C^J} \le C_J(1+T)^J$. The chain rule applied to the finitely many fixed coefficients proves (19.14). The coefficient is chosen after the observation order is fixed.

Finally, the first $H$ columns and second $H$ columns differ from those of $F_0$ by $O(\epsilon)$ in value, while the line tangent is $w + O(\epsilon)$. At $\chi = 0$ the augmented old injection $J_H F_0 \oplus e$ protects the phase. At $\chi > 0$ the line has a nonzero component in the normal plane $\nu$, and at $\chi = 1$ it lies in that plane. Thus $J_H F_0$ together with $w$ is injective everywhere, with a uniform compact margin. Projection off the full first tangent span preserves injectivity of the normal second form on $\operatorname{Sym}^2 H$, and the $O(\epsilon)$ perturbation retains a fixed positive margin. Off the tube the original phase-regular map is unchanged. The large pure-line second derivative does not enter this phase check. □

*Independence of the local orientation.* Under a change of unit frame of $L$, the functions $a$, $\partial_q a$ are $2L$-valued, while $v$, $Jv$, and $\partial_p a$ are ordinary two-vectors. The curvature $\kappa$ and the old line columns $e, w$ are $L$-valued. The mixed error $C_r$ belongs to $H^* \otimes L$, and its primitive $-I_X C_r$ belongs to $H^*$. The contact parameter $\zeta$ belongs to $L^d$, and in the line solve $b$ and $u$ belong to $L$. Consequently $V_r$, $uw$, $\chi b \nu Jv$, and every coefficient $F_r$ are ordinary ambient vectors, and all formulas in the finite induction descend over the nontrivial line. The plane connection is $\mathfrak{so}(2)$-valued with transition $-I_2$; its orbit mean is removed by the zero signed-area identity. This proves the finite completion for arbitrary metric lines.

19.4. **Signed transport to ordinary Hopf factors.** The intermediate twisted sphere need not extend outside the core tube, and we eliminate it before exact repair. Choose the fixed inclusion $I_L : L \hookrightarrow \varepsilon^6$ provided by the active bundle and add six ordinary Hopf factors with coefficient $\beta I_L P_L$; keep the old $H$ coefficient and $c$ times the old $L$ coefficient. This gives a fixed, uniformly injective coefficient map $A_0$, since $\beta = 1$ on the support of $\chi$ and $c = 1$ where $\beta$ can decrease to zero.

Fix the original sphere size $h$. A finite colored mesh of the slow core at scale $T^{-1}$ can be chosen with a fixed number $a_0$ of colors, disjoint supports within each color, and a squared partition of unity $\sum_a \rho_a^2 = 1$. For example, fixed coordinate grids with uniformly enlarged boxes have bounded intersection degree, and a fixed finite atlas together with a coloring of this bounded-degree graph provides the construction. In fully scaled slow coordinates all cutoff derivatives are uniformly bounded. Use unit line frames $\ell_a$ from one fixed trivializing atlas and put $l_a = I_L \ell_a \in S^5$. Round $T l_a$ at the center of its mesh box to an integer row $Q_a \in \mathbb{Z}^6$.

For an integer row $Q$, the tensor product of normalized coordinate power maps, with conjugate powers for negative entries and a fixed unit vector for zero entries, is a unit map $\Phi_Q$ satisfying

$$X_i \Phi_Q = 2\pi i Q_i \Phi_Q, \qquad \|\Phi_Q\|_{C^j} \le C_j (1 + |Q|)^j.$$

The polynomial derivative estimate follows by putting $m_z = \max |z_i| \ge h^{-1/2}$. A derivative of order $a$ of the numerator of a normalized power, divided by its norm, is at most

$$C_a q^a \frac{|z_i|^{q-a}}{\|(z_1^q, \dots, z_h^q)\|} \le C_a h^{a/2} q^a.$$

Differentiating the normalization gives the stated bound inductively; the finitely many $q < a$ are absorbed into the constant.

In each color block form

$$\sum_{a \text{ of that color}} \rho_a \Big( \operatorname{Re} \Phi_{Q_a}, \ell_a \operatorname{Im} \Phi_{Q_a} \Big).$$

The disjoint supports and the squared partition produce a unit map

$$W_T : P \times (S^{2h-1})^6 \longrightarrow Z_{h_*}(L), \qquad h_* = a_0 h^6.$$

The intermediate size $h_*$ is independent of $T$. Changing a local line orientation changes both the signed frequency row and the complex structure, and preserves the ideal derivative.

On a mesh support the rounding estimate is $Q_a/T - l_a = O(T^{-1})$. For scaled slow coordinates $u = T(b - b_a)$, every nonzero multi-index satisfies

$$\partial_u^\alpha (Q_a/T - l_a) = -T^{-|\alpha|} \partial_b^\alpha l_a.$$

Thus the error is $O(T^{-1})$ in every fixed full scaled norm, including cutoffs and all projective variables. The frequency rows are constant on each support and are never differentiated as rounded functions of $b$. The line field determined by $I_L$, divided by $T$, acts on $W_T$ as the twisted Hopf field up to that error, through every fixed leaf-jet order.

Construct the contact family and all finite coefficients once on $Z_{h_*}(L)$. Put $\epsilon = K_0^{-1}$, where $K_0$ is an integer coil frequency, and let $P^{\mathrm{tw}}_{K_0}$ be the normalized coordinate power on that twisted sphere; it is well defined because coordinate powers commute with conjugation. Simultaneously apply $P_T$ to every old Hopf factor. The actual map on the ordinary source is

$$G_{\epsilon,T} = F_0 \circ P_T^{\mathrm{old}} + \sum_{r=1}^{L} \epsilon^r F_r\Big(P_T^{\mathrm{old}} x, P^{\mathrm{tw}}_{K_0} W_T\Big). \tag{19.23}$$

Extend the sum by zero outside the tube. Smoothness follows from $F_r \in \chi C^\infty$; no local exact solutions are pasted together.

Scale every old, new, projective and slow source coordinate by $T^{-1}$. For the possibly nonisometric $A_0$, use the images of an orthonormal $V$ frame as leaf columns and complete them by an orthonormal complementary frame. The resulting chart matrices and their inverses have uniformly bounded condition numbers. Their transition and inverse-transition derivatives and enlarged overlap degrees are uniformly bounded. In particular these are atlases to which Theorem 6.3 applies, regardless of mesh cardinality. The chain rule gives, for every fixed $J$,

$$\|G_{\epsilon,T}\|_{C^J_{\mathrm{scaled}}} \le C_{J,L}(1+\epsilon^{1-J}), \tag{19.24}$$

$$\|(\Gamma_D(G_{\epsilon,T}) - T^2 g)(Y/T, Y/T)\|_{C^J_{\mathrm{scaled}}} \le C_{J,L}\epsilon^{L-J} + C'_{J,L}\epsilon^{-B_J}/T. \tag{19.25}$$

Here $B_J$ is finite; its value is immaterial because $T$ is chosen after $\epsilon$. The reference exponent $1-J$ does not grow with $L$: after the coefficients are fixed, the worst chain-rule factor is $\epsilon K_0^J$. Comparison of first and second leaf jets also preserves the fixed positive phase margin for sufficiently large $T$.

**Lemma 19.3** (Changing an injective coefficient model)**.** *Let $A_0 : V \to \varepsilon^{s_0}$ be a fixed injective coefficient map, and suppose its ordinary Hopf models have exact phase-regular selectors inducing the prescribed metric of $V$ for every sufficiently large common sphere size. Let $A_1 : V \to \varepsilon^{s_1}$ be a fixed isometric coefficient map. Then the $A_1$ models have exact phase-regular selectors in the same Euclidean target.*

*Proof.* The smooth rectangular matrix $C = A_0 A_1^*$ satisfies $CA_1 = A_0$. For a fixed final sphere size $h$, apply the preceding colored mesh construction to each row of $C$, rounding $TC$ on each support. The signed tensor maps, in color blocks, send the $s_1$ original spheres to $s_0$ ordinary spheres of one intermediate size $a_0 h^{s_1}$ independent of $T$. Choose the selector on this intermediate source before choosing $T$.

The identity $CA_1 = A_0$ gives the ideal first and second leaf jets of the composed selector in the scaled frame $Y/T$. By the calculation above, the rounding error is $O(T^{-1})$ in every fixed full scaled norm. All reference bounds, phase margins, chart constants and ellipticity bounds are uniform in $T$; they may depend on the fixed intermediate selector. Apply Theorem 6.1 with these fixed bounds and sufficiently large $T$, and divide the repaired map by $T$. Its metric is the prescribed metric of $V$, and all outputs lie in the original target. The factor counts and coefficient maps were fixed before $h$ varied, as required for assembly. □

**Theorem 19.4** (Supported absorption of an arbitrary metric line)**.** *Suppose the old completed selector and its augmented injection $J_H F_0 \oplus e(L)$ have the collar properties above, and that its complete normal complement contains an isometric copy of $2L$. If $N \geq s_n+1$, the last line is realized in the same $\mathbb{R}^N$, whether or not $L$ is orientable. The result is one smooth selector on ordinary Hopf factors, with exact metric and positive compact phase margin.*

*Proof.* Fix the finite reference order $J_*$ and the repair exponents from Theorem 6.3 before choosing the finite completion order. Use reference growth exponent $J_*+1$, phase exponent zero, and any fixed positive displacement exponent to obtain the required error exponent $K$. Choose $L > K + 440$ and construct all finite coefficients on the intermediate twisted sphere of fixed size. Choose a sufficiently small reciprocal integer $\epsilon = K_0^{-1}$. Finally take $T$ larger than a sufficiently high power of $\epsilon^{-1}$ so that the second term of (19.25), through order 440, is $O(\epsilon^K)$ and the phase comparisons hold.

Equations (19.24)–(19.25) satisfy all hypotheses of the uniform repair statement. One exact repair on the whole ordinary source gives $\Gamma_D(G^*) = T^2 g$. Dividing by $T$ gives the desired exact selector for $A_0$, and Lemma 19.3 returns it to an isometric coefficient model. No additional target coordinate is used. The order of choices is: repair orders and exponents, finite precision, coil frequency, transport frequency, and one final exact repair. Global tails of that final repair are allowed because no further relative gluing is performed. □

## 20. The global target and all source dimensions

In this section we prove Theorem 1.1 in all source dimensions.

*Proof in dimensions at least sixteen.* Fix $n \geq 16$ and put

$$q = 7, \quad k = 6, \quad m = n - 6, \quad M = n - 5, \quad r = n - 4, \qquad c = 5, \quad C_0 = 10.$$

We choose a finite coefficient embedding of $TM$ and all slow source data before varying the common Hopf sphere size. In this proof we write $M^n$ for the manifold when it could be confused with the exterior rank $M$. The prepared-source inequalities are

$$2(m+2) > m+6, \qquad 5(m+1)+15 > m+5, \qquad j_m > m+5.$$

They hold for $m \geq 10$. Proposition 8.3 gives a core of dimension at most five, with the possibly nonorientable line $L = J_c K$. Since the activity $k = 6$ is even, there is no Euler obstruction.

The principal algebraic model has dimension

$$D_P = j_m + 7 + 5(m+1) + 10 = j_m + 5m + 22 = s_n + 1. \tag{20.1}$$

Place the old free map in a fixed physical hyperplane of dimension $D_P - 1$. The first rank-failure codimension for its complete injection is $D_P - j_m = 5m + 22 > n$, and a seven-frame in its complement has first rank-failure codimension $5m + 15 > n$. These choices concern the slow complete bundle $J_{\mathrm{ext}}$ fixed by the source transports. The surplus over $j_m$ exceeds five, so Theorem 4.1 gives the old actual map in that complete class.

To apply the absolute form of Theorem 4.1 on a closed source, we choose product slow collars before making this first free realization. The exact normal form identifies the exterior bundle on its inner collar with $E_v$ by fiberwise isometries; we use a metric connection for the radial identifications. Consequently the prescribed fiber metric has constant radial components in these identifications, and the given metric on $V$ is unchanged. Choose the coefficient injection and the complete formal injection to be radial pullbacks on a smaller collar, retaining their prescribed boundary values. Double the exterior slow base, these bundles, and the formal data across that product collar. The resulting Hopf source is a compact submersion over a closed base, so the absolute realization theorem applies. Restrict the exact free map to the exterior. Finally, compose in the slow variable with a radial retraction that is constant on an inner collar and the identity outside a larger product collar. Since the coefficient fields are radial pullbacks there, their leaf derivatives commute with this retraction. Its effect on the leaf metric is pullback by the stated fiberwise isometries, so the metric remains the prescribed one. The complete columns undergo the corresponding pullback homotopy and remain injective. This gives an actual product collar with the same target dimension and complete formal class.

The counted normal ranks are $5r + 5$ on the core and $5M + 10$ on the exterior. The core construction of Proposition 12.1 is valid for $r \geq 12$, and the exterior construction of Theorem 11.6 for $M \geq 9$. In our range $r \geq 12$ and $M \geq 11$. After the four top copies the residual ranks are $r + 5$ and $M + 10$. The exact boundary identity

$$4 + (r + 5) = M + 10$$

accounts for all four omitted-line rows. The top-copy inclusion failure codimensions are $r + 6 > 5$ on the core and $M + 11 > n$ on the exterior.

Lemmas 14.1–14.7 provide the compatible actual frame and activity, with complete collars retained. The critical weak bound is $\dim_{\mathbb{C}} \mathcal{B} \leq 4 = q - 3 = k - 2$, so Proposition 17.3 applies. The active block has dimension $j_6 + 1 = 28$, and its cofactor normal shares one virtual principal coordinate. The physical target is therefore

$$\begin{aligned} N &= (D_P - 1) + (j_6 + 1) = D_P + 27 \\ &= j_m + 5m + 49 = s_n + 28. \end{aligned}$$

The shared coordinate is saved without deleting any algebraic normal row.

Finite completion and repair produce the exact exterior selector. The complete boundary homotopy supplies the relative $H$-free fill; its rank is $j_{n-1} = s_n - 1$, so its target surplus is 29, above the threshold in Theorem 4.1. The augmented complete bundle has rank $s_n$ and complement rank 28. Lemma 18.2 and Theorem 19.4 absorb the surviving line without adding a physical coordinate. This yields an exact phase-regular selector on the original ordinary coefficient model for every sufficiently large fixed common sphere size. Applying Corollary 7.5 with base $M^n$, coefficient bundle $TM$, and identity classifying map gives the desired embedding.

At the endpoint $n = 16$ we have $m = 10$, $M = 11$, $r = 12$, $D_P = 137$, and $N = 164 = s_{16} + 28$. The source inequalities are $24 > 16$, $70 > 15$, and $65 > 15$; the old complete and seven-frame failure codimensions are 72 and 65. The boundary matching

is $4 + 17 = 21$, and the final two-line codimension is $27 > 5$. Thus every inequality is strict at $n = 16$. □

## 20.1. **The dimensions below sixteen.**

**Proposition 20.1** (Two lower-dimensional constructions)**.** *Every closed Riemannian $n$-manifold, $n \geq 2$, has an embedding with*

$$N \leq s_n + n + \max(n, 5).$$

*If every component has a nowhere-zero tangent field, then*

$$N \leq s_n + \max(n, 5) + 1.$$

*Proof.* The bundle $J(TM)$ has rank $j_n$. The first noninjectivity stratum for a map to a trivial bundle of rank $j_n + \max(n, 5)$ has codimension $\max(n, 5) + 1 > n$, so general position gives a monomorphism. The target is at least $j_n + 5$, and Theorem 4.1, applied with base a point and distribution $TM$, gives an exact free isometric immersion. The same-target upgrade in Theorem 7.4 gives an embedding, since the target is at least $s_n + 2$ in every case.

For the second bound, normalize the given tangent field and write $TM = H \oplus \varepsilon^1$, with $\operatorname{rank} H = n - 1$. Choose a coefficient model for $H$ and one separate Hopf coefficient for the line. General position embeds $J(H)$ in the trivial bundle of rank $j_{n-1} + \max(n, 5)$ over the original slow base, and Theorem 4.1 supplies its exact free selector. Lemma 19.1 with a one-point base, $L = \varepsilon^1$, $C = h \geq 2$, and parameters $q = p = \tau = 0$ gives a map $b_h : S^{2h-1} \to \mathbb{R}^2$ with $|Xb_h| = 1$. We use it as the selector for the separate line input and append its two orthogonal target coordinates. The metric is the exact orthogonal sum, and the complete $H$ columns together with the first column of the new line are independent. Hence the normal second fundamental form is injective on $\operatorname{Sym}^2 H$ with positive compact margin, and Corollary 7.5 gives

$$j_{n-1} + \max(n, 5) + 2 = s_n + \max(n, 5) + 1.$$

The distribution $H$ on the original manifold need not be integrable, since its coefficient distribution on the Hopf source is vertical and integrable. □

*Completion of the proof of Theorem 1.1.* For $2 \leq n \leq 14$, the first bound in Proposition 20.1 has surplus at most 28. Every connected closed odd-dimensional manifold, orientable or not, has Euler characteristic zero. Its tangent Euler obstruction with orientation coefficients therefore vanishes, so it has a nowhere-zero tangent field. By the second bound, dimension 15 has surplus 16. A closed one-manifold is a finite disjoint union of metric circles; planar circles of the prescribed lengths, translated apart, give its isometric embedding. A closed zero-manifold is finite and embeds in a line. Padding these targets if necessary covers $n = 0, 1$. Together with the construction in dimensions at least sixteen, this proves the theorem. □

20.2. **Quantifiers and scope.** The construction first fixes the slow bundles, their complete classes, and the number of Hopf factors. For a fixed sufficiently large common sphere size, it then chooses the active free block, protected activity homotopy, normalized collar scale, and curvature scale. Horizontal realization precedes the finite polynomial activity perturbation and the periodic differential inverse. The repair exponent is fixed before the finite metric precision, and the finite coefficients before the small amplitude. For line absorption, the finite precision precedes the coil frequency, which precedes the signed transport frequency and the final repair. A source sphere is never enlarged after it has been fixed.

The theorem concerns closed manifolds. An extension to noncompact manifolds would require separate exhaustion and uniformity arguments. The number 28 is a universal upper bound on the surplus in the closed case; we do not know whether it is optimal or whether a different construction gives a smaller constant.

## 21. The lower bound within the seven-label construction

This section proves a lower bound for the full complex weak-label condition imposed on the auxiliary whole-core tensor. It is not a lower bound for isometric embeddings, and it does not apply to an interface which protects only the activity used by the construction.

**Proposition 21.1** (Universal weak-label dimension)**.** *Let $E$ have complex dimension $r$, let $q \geq 3$, and suppose $0 \leq c < 2r$ and $c \leq 2q-5$. For any linear map*

$$T : E \otimes \mathbb{C}^q \longrightarrow W, \qquad \dim W \leq (q-2)r + c,$$

*its bad-label set*

$$\mathcal{B}_T = \{H \in \mathrm{Gr}_{q-2}(\mathbb{C}^q) : \ker T \cap (E \otimes H) \neq 0\}$$

*satisfies*

$$\dim \mathcal{B}_T \geq 2q - 5 - c.$$

*Proof.* Choose $K \subset \ker T$ of dimension $k = 2r - c$. Write $V = E \otimes \mathbb{C}^q$, of dimension $qr$, and let $\mathcal{G} = \mathrm{Gr}_k(V)$. Consider the closed incidence

$$\mathcal{I} = \{(K', H) \in \mathcal{G} \times \mathrm{Gr}_{q-2}(\mathbb{C}^q) : K' \cap (E \otimes H) \neq 0\}.$$

For fixed $H$, the subspace $E \otimes H$ has dimension $(q-2)r$. The locus of $k$-planes meeting it nontrivially is an irreducible Schubert variety of codimension $c+1$ in $\mathcal{G}$. Indeed, choosing an intersection line and then a $k$-plane containing that line gives dimension

$$((q-2)r - 1) + (k-1)(qr - k),$$

and a generic such $k$-plane meets that subspace exactly in that line. Subtracting from $k(qr-k)$ gives $c+1$. Local trivializations over the label Grassmannian show that $\mathcal{I}$ is irreducible and

$$\dim \mathcal{I} = \dim \mathcal{G} + 2(q-2) - (c+1).$$

The projection $\mathcal{I} \to \mathcal{G}$ is surjective. If $r \leq q-2$, every nonzero tensor in $K'$ has rank at most $q-2$ and is therefore supported on some such $H$. If $r > q-2$, the projective variety of $r$-by-$q$ matrices of rank at most $q-2$ has codimension $2(r-q+2)$ in $\mathbb{P}(V)$. By

the projective dimension theorem, its intersection with $\mathbb{P}(K')$ has projective dimension at least

$$(2r - c - 1) - 2(r - q + 2) = 2q - 5 - c \geq 0$$

and is therefore nonempty.

Since the projection is proper and dominant, its generic fiber has dimension $2q-5-c$, and by upper semicontinuity of fiber dimension every fiber has at least that dimension. The fiber over $K$ is contained in $\mathcal{B}_T$, which proves the proposition. □

**Corollary 21.2** (Critical reserve and target dimension)**.** *Suppose $q \geq 3$, $2r > q - 3$, and the complex bad-label set of the core tensor is required to satisfy $\dim \mathcal{B}_T \leq q - 3$. Assume the two dimension identities*

$$C_0 = c + q - 2, \qquad N - s_n = C_0 + 3q - 3.$$

*Then $c \geq q - 2$ and $N - s_n \geq 5q - 7$. For $q = 7$ the least possible surplus under these requirements is 28.*

*Proof.* If $c \leq q - 3$, then $c < 2r$ and $c \leq 2q - 5$, and the preceding proposition gives $\dim \mathcal{B}_T \geq 2q - 5 - c \geq q - 2$, a contradiction. Thus $c \geq q - 2$, and substitution gives $N - s_n = c + 4q - 5 \geq 5q - 7$. □

The corollary concerns the full complex label condition and the specified target decomposition. It is not a lower bound on the Euclidean dimension of an isometric embedding. It does not show that six-label constructions fail, or that a construction protecting only selected active planes obeys the same bound. It shows that increasing the old rank under these requirements cannot improve 28.

21.1. **Further directions.** The theorem treats closed manifolds with arbitrary smooth positive metrics. The preceding bound identifies the source of the constant within the present construction. Improving it requires changing the full complex label condition, the coefficient tensor, or the source and target decomposition. Extending the result to noncompact manifolds requires an exhaustion argument with compatible quantitative control. These questions remain open.

## Appendix A. Exact algebraic calculations

This appendix specifies the finite calculations used in the paper. The matrices, polynomial coefficients, integer ranges, and reduction rules are given in the body of the paper and in the procedures below. All ranks and determinants are computed over $\mathbb{Q}$, except for the modular cubic certificate identified below. All arithmetic is exact.

Five Python programs reproduce these calculations. Their roles are as follows.

(i) `seed_certificate.py` forms the rational orthogonal projection defining the Tiles space, checks its partial-transpose identity, excludes all thirty-two product-vector factor branches, checks the six exceptional contractions, and computes the displayed symmetric-evaluation and ordinary-operator minors. It also checks the triple and the three pairwise skew-evaluation minors.

(ii) `cubic_certificate.py` expands the determinant of the seed matrix and forms the degree-five Jacobian multiplication matrix. It checks rank 56 modulo 1000003 and the specified nonzero minor. Full rank modulo a prime implies full rational rank, so the check certifies the displayed algebraic smoothness proof over $\mathbb{C}$, rather than sampling points on the cubic.

(iii) `exterior_certificate.py` enumerates every admissible short-tuple row for $4 \le \ell \le 15$. It checks 8427 rows directly; the remaining 330 rows are covered by the uniform single-evaluation lemma in the text. It also checks that the old-rank slopes are nonnegative. The estimates for all larger tuple lengths are the symbolic inequalities in the proof of Theorem 11.6.

(iv) `core_certificate.py` enumerates all seventeen pair rows and forty-seven triple rows at $(r, c) = (12, 5)$. After the geometrically proved exclusions and corrections, the positive-overlap minima are both six. The zero-overlap minima are nine and thirteen. The proof in the text shows that the inequalities persist for unbounded $r$, including the graph ranks thirteen and fourteen that become available.

(v) `pencil_certificate.py` uses rational row reduction to form the six constant-row coefficient pencils and checks that their two-by-two annihilator matrices are nonsingular. Their determinants are $9/2, 1, -45/49, 1, 3, 3/2$, in the order used in the core section.

These checks presuppose the parameter spaces, projection fibers, and independent equation counts established in the text. In particular, the added codimensions in the core program are inputs from the geometric arguments, and a successful run verifies their integer consequences. The independence of the equations and the geometric realization arguments are proved in the preceding sections.

A.1. **Exact arithmetic conventions and the seed matrices.** We specify the finite calculations independently of any implementation. All matrix indices start at zero. A matrix is flattened row by row, and tuples of indices are ordered lexicographically in ascending order. Rational computations use exact fractions. Determinants can be evaluated by Gaussian elimination: in each successive column choose the first available nonzero pivot, exchange rows when necessary, and eliminate below it. The determinant is the product of the pivots times the signs of the row exchanges; a missing pivot gives zero. The same procedure is used over the stated finite field, with division by nonzero field elements.

For all seed computations, the ordered coordinate basis is $(u, v, z, w)$, and the four integer matrices are

$$M_0 = \partial_u C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \partial_v C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$M_2 = \partial_z C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_3 = \partial_w C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A.1}$$

The integer slices $\widetilde{B}_\rho \in \mathrm{Mat}_{4,3}$ are given by

$$(\widetilde{B}_\rho)_{ib} = (M_i)_{\rho b}, \qquad 0 \le i < 4, \quad 0 \le \rho, b < 3.$$

These are the slices before the Gram normalization in the seed construction. That normalization is an invertible change of the four output coordinates and preserves the ranks being checked.

Let $E_{bc}$ denote the elementary three-by-three matrix. We use the symmetric basis

$$E_{00},\ E_{01}+E_{10},\ E_{02}+E_{20},\ E_{11},\ E_{12}+E_{21},\ E_{22}$$

and the skew basis $E_{01}-E_{10}, E_{02}-E_{20}, E_{12}-E_{21}$. The twelve-by-eighteen symmetric evaluation matrix has columns $\mathrm{vec}(\widetilde{B}_\rho S)$, ordered first by $\rho = 0, 1, 2$ and then by the displayed symmetric basis. Its minor on columns $0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 13, 14$ is 20. The twelve-by-nine triple skew evaluation matrix has columns $\mathrm{vec}(\widetilde{B}_\rho K)$, ordered first by $\rho = 0, 1, 2$ and then by the displayed skew basis. Its minor on rows $0, 1, 2, 3, 4, 5, 6, 7, 10$ has determinant $-4$. For a pair $\rho < \sigma$, the twelve-by-six skew evaluation matrix has columns $\mathrm{vec}(\widetilde{B}_a K)$, ordered first by $a = \rho, \sigma$ and then by the displayed skew basis. The minors are

| $(\rho, \sigma)$ | selected rows | determinant |
|---|---|---|
| $(0, 1)$ | $0, 1, 2, 3, 4, 5$ | $-4$ |
| $(0, 2)$ | $0, 2, 3, 6, 7, 10$ | $-1$ |
| $(1, 2)$ | $0, 2, 3, 6, 7, 10$ | $-4$. |

These are the skew-evaluation assertions of Lemma 9.3.

For the ordinary operator in Lemma 9.4, the ten-by-nine coefficient matrix has rows $(i, j)$, $0 \le i \le j < 4$, and columns ordered first by $(\rho, \sigma)$, $0 \le \rho < \sigma < 3$, then by $(b, c)$, $0 \le b < c < 3$. Its entries are

$$\begin{aligned} Q_{ij}^{\rho\sigma,bc} = {} & (\widetilde{B}_\rho)_{jb}(\widetilde{B}_\sigma)_{ic} - (\widetilde{B}_\sigma)_{jb}(\widetilde{B}_\rho)_{ic} \\ & + (\widetilde{B}_\rho)_{ib}(\widetilde{B}_\sigma)_{jc} - (\widetilde{B}_\sigma)_{ib}(\widetilde{B}_\rho)_{jc}. \end{aligned} \tag{A.2}$$

The determinant on its first nine rows is 448. All these are determinants of specified integer matrices.

A.2. **The Jacobian multiplication matrix.** The polynomial coordinates in this computation are $x = (u, v, w, z)$, as in (9.4); this order differs from the seed basis $(u, v, z, w)$ in (A.1), and no permutation is applied to the polynomial. Write $f = \sum_\nu f_\nu x^\nu$, with unlisted coefficients zero, and let

$$\mathcal{M}_d = \{\alpha \in \mathbb{Z}^4_{\ge 0} : \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = d\},$$

ordered lexicographically. The rows of the Jacobian multiplication matrix are indexed by $\beta \in \mathcal{M}_5$, and its columns are ordered first by $\rho = 0, 1, 2, 3$, then by $\alpha \in \mathcal{M}_3$. The entry, namely the coefficient of $x^\beta$ in $x^\alpha \partial_\rho f$, is

$$A_{\beta,(\rho,\alpha)} = \begin{cases} (\beta_\rho - \alpha_\rho + 1) f_{\beta-\alpha+e_\rho}, & \beta - \alpha \in \mathbb{Z}^4_{\ge 0}, \\ 0, & \text{otherwise.} \end{cases} \tag{A.3}$$

There are 56 rows and 80 columns. All rows together with the fifty-six columns listed in the proof of Lemma 9.2 give a determinant of 152546 modulo 1000003. The modulus is prime, as can be checked by trial division up to 1000. Hence the corresponding integer determinant is nonzero. The implication from this rank calculation to smoothness and to the absence of rank-one seed matrices is proved in that lemma.

A.3. **Canonical bases for the six constant-row pencils.** For a rational matrix $T$, let $\mathcal{N}(T)$ be its ordered canonical nullspace basis, defined as follows. Compute the reduced row echelon form, list the free columns in increasing order, and for each such column set its free coordinate to one and all other free coordinates to zero. The pivot coordinates are then the negatives of the corresponding entries in the reduced matrix. This rule fixes both the basis and its order.

For each of the six real vectors $\lambda$ in (9.3), use the matrices (A.1) to form

$$E_\lambda = (M_0\lambda\ M_1\lambda\ M_2\lambda\ M_3\lambda), \qquad (v_0, v_1) = \mathcal{N}(E_\lambda), \qquad C_i = \sum_{a=0}^{3} (v_i)_a M_a.$$

Here $E_\lambda$ has rank two. Let the columns of the three-by-two matrix $B$ be $\mathcal{N}(\lambda^t)$, and put

$$R = B(B^tB)^{-1}.$$

The matrix $B^tB$ is invertible, since $B$ is real and has full column rank. This step concerns only the six displayed real directions, not Gram matrices on arbitrary complex subspaces. Define the three-by-four rational matrix

$$(D_\lambda)_{\rho,\bullet} = \Big((C_0R)_{\rho 0}, (C_0R)_{\rho 1}, (C_1R)_{\rho 0}, (C_1R)_{\rho 1}\Big), \qquad 0 \le \rho < 3. \tag{A.4}$$

Each $D_\lambda$ has rank three, so its canonical nullspace consists of one vector $h$. Reshape it row by row as $H = \left(\begin{smallmatrix} h_0 & h_1 \\ h_2 & h_3 \end{smallmatrix}\right)$. This procedure gives the six four-tuples in Lemma 12.4, in the order printed there, and the determinants

$$9/2, \quad 1, \quad -45/49, \quad 1, \quad 3, \quad 3/2.$$

The three rows in (A.4) are the row-by-row coordinates of the coefficient operators on the pencil and its common row plane, and the equation $D_\lambda h = 0$ says that $H$ annihilates their hyperplane under the entrywise pairing. Invertibility of $H$ is unchanged by invertible changes of either two-dimensional basis; the canonical choices above also fix the numerical entries in the table.

A.4. **The exterior short-tuple enumeration.** For nonnegative integers $R, h, u$, define the following finite integer function, with an empty maximum interpreted as zero:

$$\begin{aligned} K_q &= q(q-1)/2, \\ \phi(q,h,u) &= \max_{0\le b\le q} \min\{b\max(q, u-q) + K_q, (q-b)\max(q, h-q) + 4K_q\}, \\ \mathfrak{c}(R,h,u) &= \max_{2\le q\le \min(R,h,u)} \min\{qu - q(q+1)/2,\ q(u-2)+1,\ \phi(q,h,u)\}. \end{aligned}$$

This is the cap in (11.8); its three terms are proved in Lemma 10.2. The following pseudocode specifies all loops, filters, and arithmetic for the short range. Intervals include both endpoints, and `ceil` denotes the integer ceiling.

```
for ell in 4,...,15:
    M = 6*ell - 5
    direct = symbolic = 0
    minimum = +infinity
    for (gamma,l) in [(0,0),(1,0),(1,1)]:
        e = (gamma-l)*ell;  t = (2-l)*ell
        for delta in 0,...,(3-gamma)*ell-1:
            h = (3-gamma)*ell-delta;  tau = delta+e
            for j in 0,...,t:
                L = tau-j;  u = t-j
                w = max(0,ceil((3*j-2*e)/2))
                if L<1 or 3*j>2*tau or w>delta: continue
                assert L-1 >= 0
                if gamma==0 and l==0 and j==0:
                    symbolic = symbolic+1
                    continue
                a = (3-gamma)*gamma-l*(5-gamma+l)
                b = L*M+6*ell-ell*a-(delta-w)*h-j*u
                b = b-cap(2*tau-3*j,h,u)
                excess = b-M-5
                assert excess > 0
                minimum = min(minimum,excess)
                direct = direct+1
    record(ell,direct,symbolic,minimum)
```

Here `cap` is the function $\mathfrak{c}$ defined above. The outputs are the twelve rows printed in the proof of Theorem 11.6: the direct and symbolic totals are 8427 and 330, and the minimum at each length is $2\ell-1$. The symbolic rows are covered for every length by Lemma 11.5; the long-length estimates and the nonnegative old-rank slopes in the theorem cover the unbounded ranges. The finite enumeration involves no extrapolation in either variable.

A.5. **The core endpoint enumeration.** The row generator is (12.5)–(12.6): take $\ell = 2, 3$, $(\gamma, l) = (0,0), (1,0), (1,1)$, $0 \le \delta \le (\ell-1)(3-\gamma)$, and $0 \le j \le \ell(2-l)$. Define

$$e = \ell(\gamma - l), \quad h = \ell(3-\gamma) - \delta, \quad u = \ell(2-l) - j, \quad L = \delta + e - j, \quad w = \max\{0, \lceil (3j-2e)/2 \rceil\},$$

and keep the rows with $L \ge 1$, $j \le 2L$, and $w \le \delta$. Put

$$R = \min(2L - j, u, h), \qquad \chi = \begin{cases} 0, & R \le 1, \\ Ru - R(R+1)/2, & R \ge 2, \end{cases}$$

and let $b$ denote the raw bound

$$b = Lr + c\ell - 12 - \ell\gamma(3-\gamma) + 2l - (\delta - w)h - ju - \chi.$$

Write $k = (\gamma, l, \delta, j)$. In the following decision table the first applicable rule supplies the revised bound $b'$. The last column gives the geometric proofs, including independence of the added equations, that justify each rule.

| Condition | Revised bound or exclusion | Proof location |
|---|---|---|
| $\gamma = 1, l = 0, \delta = 0, j > 0$ | Exclude: direct-sum injectivity. | Section 12.5; Lemma 9.1. |
| $k = (1, 0, 1, 2)$ | Exclude: the $\mathcal{P}_0$ locus. | Section 12.5; (12.4). |
| $\ell = 2,\ k = (0, 0, 1, 0)$ | $b' = b + \min\{r - 6, 6\}$. | Two generators and one relation. |
| $\ell = 3,\ k = (0, 0, 1, 0)$ | $b' = b + \min\{\chi, 8\}$. | Three generators and one relation; Lemma 10.7. |
| $\ell = 3,\ k = (0, 0, 2, 0)$ | $b' = b + \min\{\chi + r - 19,\ \chi - 5,\ 7\}$. | Three generators and two relations. |
| $k = (1, 1, 1, 0)$ | $b' = b + \chi + 2$ if $\ell = 2$; $b' = b + \chi$ if $\ell = 3$. | Section 12.5. |
| Every other row | $b' = b$. | (12.8). |

In the three-branch rule the three increments are $\chi + r - 19 = (\chi - 10) + (r - 9)$, $\chi - 5 = (\chi - 11) + 6$, and $7 = 2 + 5$, respectively. Thus the table records the separate alternatives proved in the text; their codimensions are not added to one another. The endpoint algorithm is:

```
set r=12 and c=5
for ell in [2,3]:
    form every row in the stated finite ranges
    discard rows failing L>=1, j<=2*L, w<=delta
    count the remaining rows before applying the decision table
    apply the first matching decision-table rule to each row
    count exclusions and record all retained revised bounds
    assert every retained revised bound is greater than 5
    record the smallest retained revised bound
```

It gives, respectively, $17, 47$ rows before the decision table, $2, 3$ exclusions, $15, 44$ retained rows, and minimum 6 in both cases. For zero overlap, evaluate (12.13) for every integer $5 \le g \le 9$, and (12.14) for every integer $5 \le g \le 12$. The two minima are 9 and 13. The proof of Proposition 12.1 treats separately the new values $g = 13, 14$ that occur when the old rank increases, and proves persistence of the positive-overlap bounds by symbolic arguments.

These procedures give the arithmetic content of the finite calculations. Their use in the dimension estimates rests on the incidence and equation-independence proofs in the body of the paper.

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Generative AI was used in developing the proofs. The finite calculations are specified in Appendix A and can be reproduced with the accompanying programs.

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