Contracting a pendant edge can increase the mean subtree order

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Abstract

Cambie, Jooken and Wagner conjectured that contracting an edge of a connected graph decreases the mean subtree order by at least 1/3, as it does for trees. We construct unicyclic graphs of order n and maximum degree 3 in which contracting a pendant edge increases the mean subtree order by $n/450+O(1)$, disproving the conjecture. In this family the bound first fails at 481 vertices and contraction first increases the mean at 631 vertices.

1 Introduction

Throughout this paper, we consider finite, simple and connected graphs. A subtree of a graph G is a subgraph of G that is a tree, not necessarily induced (single vertices included). Let $N(G)$ be the number of subtrees of G and $R(G)$ the total order of all subtrees of G, that is, the sum of their orders (numbers of vertices). The mean subtree order of G is $\mu(G) = R(G)/N(G)$, and the local mean $\mu(G, v)$ at a vertex v is the average order of the subtrees containing v. For an edge e, G/e denotes the graph obtained from G by contracting e.

The mean subtree order of trees was introduced by Jamison [2], who later conjectured [3, Conj. 5.6] that $\mu(T) - \mu(T/e) \geq 1/3$ for every edge e of every tree T. Luo, Xu, Wagner and Wang [4] proved this for pendant edges and Wang [5] in general. Cambie, Jooken and Wagner [1, Conj. 1.8] conjectured that $\mu(G) - \mu(G/e) \geq 1/3$ for every edge e of every connected graph G, with equality if and only if G is a path. We show that the conjecture fails even for pendant edges of unicyclic graphs with maximum degree 3.

For an integer $t \geq 1$, let H_t be the graph obtained from the cycle labelled $0, 1, \ldots, 8t-1$ in cyclic order by joining one end of a path on t new vertices to vertex 0, and one end of another such path to vertex $2t$. We call 0 and $2t$ the attachment vertices and the two paths the attached paths. Let v be the vertex t, and let G_t be obtained from H_t by adding a new vertex w adjacent to v. Both graphs are unicyclic with maximum degree 3, G_t has order $n = 10t + 1$, and $G_t/vw = H_t$ because w is a leaf.

Theorem 1.1. *As $t \to \infty$,*

$$\mu(G_t) - \mu(G_t/vw) = -\frac{t}{45} + O(1) = -\frac{n}{450} + O(1).$$

The smallest t with $\mu(G_t) - \mu(G_t/vw) < 1/3$ is $t = 48$, and the smallest t with $\mu(G_t) < \mu(G_t/vw)$ is $t = 63$, where $n = 631$ and

$$\mu(G_{63}) - \mu(G_{63}/vw) = -\frac{4293783638}{49101085318287}.$$

The graph H_t is a variant of the graph $DB^{\star 1}_{n,w}$ in the proof of [1, Prop. 5.1], with the attachment vertices at distance $2t$ instead of 2 and paths in place of sets of leaves. For sufficiently large t, the local mean $\mu(H_t, v)$ lies below $\mu(H_t)$ by an amount linear in t, which cannot happen in a tree [2]. Adding w then lowers the mean, so contracting vw raises it.

2 Proof of Theorem 1.1

Lemma 2.1. *Let M be a set of $j \geq 1$ vertices of the cycle C_m, and let $d_1, \ldots, d_j$ be the numbers of edges on the arcs of C_m between consecutive vertices of M (so $d_1 = m$ if $j = 1$). The subtrees of C_m are its paths, the m spanning paths being distinct. The number Q and the total order S of the subtrees of C_m containing M are*

$$Q = m + \sum_{i=1}^{j} \binom{d_i}{2}, \qquad S = mQ - \sum_{i=1}^{j} \binom{d_i + 1}{3}.$$

For $M = \emptyset$, $Q = m^2$ and $S = m^2(m+1)/2$.

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Proof. A subtree of C_m containing M is one of the m spanning paths, or it omits a nonempty interval of vertices not in M, which lies inside one arc. An arc of length d_i has $d_i - 1$ interior vertices, hence $\binom{d_i}{2}$ such intervals, and their numbers of vertices sum to $\sum_{\ell=1}^{d_i-1} \ell(d_i - \ell) = \binom{d_i+1}{3}$. For $M = \emptyset$ there are m paths of each order $\ell < m$ and m spanning paths. □

Proof of Theorem 1.1. Let $N_t = N(H_t)$ and $R_t = R(H_t)$, and let A_t be the number and B_t the total order of the subtrees of H_t containing v. The subtrees of G_t are the subtrees of H_t, the single vertex w, and the subtrees of H_t containing v extended by the edge vw. Hence

$$N(G_t) = N_t + A_t + 1, \qquad R(G_t) = R_t + B_t + A_t + 1. \tag{1}$$

A subtree of H_t meeting the cycle intersects it in a path, since the attached paths cannot connect distinct cycle vertices. We call it the cycle path of the subtree. The subtree is determined by its cycle path together with an initial segment, possibly empty, of the attached path at each attachment vertex on the cycle path. There are $t+1$ choices for each segment, with $t/2$ vertices on average, so a cycle path through h attachment vertices yields $(t+1)^h$ subtrees, with $ht/2$ added vertices on average.

For a set X of cycle vertices, let Q_X and S_X be the number and total order of cycle paths containing X. Let $U = \{0, 2t\}$. For a cycle path P through h attachment vertices, expand $(t+1)^h = \sum_{X \subseteq U \cap V(P)} t^{|X|}$. Summing over P gives

$$\begin{aligned} N_t &= \sum_{X \subseteq U} t^{|X|} Q_X + t(t+1), \\ R_t &= \sum_{X \subseteq U} t^{|X|} \Big(S_X + \tfrac{1}{2}(t+1)|X|Q_X \Big) + \tfrac{1}{3}t(t+1)(t+2). \end{aligned} \tag{2}$$

The last terms account for the subtrees contained in the attached paths, and the second term in R_t is justified by the identity

$$\sum_{X \subseteq U \cap V(P)} |X| t^{|X|} = ht(t+1)^{h-1}.$$

Replacing Q_X, S_X by $Q_{X \cup \{v\}}, S_{X \cup \{v\}}$ and omitting the last terms gives A_t, B_t.

Lemma 2.1 with $m = 8t$ gives the following values.

X	arc lengths	Q_X	$3S_X$
$\emptyset$	none	$64t^2$	$768t^3 + 96t^2$
$\{0\}, \{2t\}, \{v\}$	$(8t)$	$32t^2 + 4t$	$512t^3 + 96t^2 + 4t$
$\{0, 2t\}$	$(2t, 6t)$	$20t^2 + 4t$	$368t^3 + 96t^2 + 4t$
$\{0, v\}, \{2t, v\}$	$(t, 7t)$	$25t^2 + 4t$	$428t^3 + 96t^2 + 4t$
$\{0, 2t, v\}$	$(t, t, 6t)$	$19t^2 + 4t$	$347t^3 + 96t^2 + 4t$

Substituting into (2), for instance $N_t = Q_\emptyset + 2tQ_{\{0\}} + t^2 Q_{\{0,2t\}} + t(t+1)$, and expanding,

$$\begin{aligned} N_t &= 20t^4 + 68t^3 + 73t^2 + t, & 3R_t &= 428t^5 + 1288t^4 + 1085t^3 + 119t^2 + 2t, \\ A_t &= 19t^4 + 54t^3 + 40t^2 + 4t, & 3B_t &= 404t^5 + 1096t^4 + 807t^3 + 116t^2 + 4t. \end{aligned}$$

These values were also verified by computer for $t \le 6$, by enumerating the connected vertex sets of H_t and G_t and counting the spanning trees of each induced subgraph.

These polynomials and (1) give $\mu(H_t) = \frac{107}{15}t + O(1)$ and $\mu(G_t) = \frac{64}{9}t + O(1)$. Their difference is $-t/45 + O(1) = -n/450 + O(1)$. They also give $\mu(H_t, v) - \mu(H_t) = B_t/A_t - R_t/N_t = -13t/285 + O(1)$, as claimed above.

Evaluating the polynomials for $1 \le t \le 63$ shows that $\mu(G_t) - \mu(H_t) \ge 1/3$ for $t \le 47$, $\mu(G_{48}) - \mu(H_{48}) < 1/3$, and $\mu(G_t) > \mu(H_t)$ for $t \le 62$. At $t = 63$ the polynomials give $N(H_{63}) = 332352216$, $R(H_{63}) = 148441475868$, $N(G_{63}) = 645320026$ and $R(G_{63}) = 288225062749$, and the difference of means is the fraction in Theorem 1.1. □

References

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