ERA STATISTICS FOR THE VACUUM BIANCHI IX FLOW

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Abstract. We prove the Gauss statistical law for eras of the exact vacuum Bianchi IX flow with zero cosmological constant. Eras are defined by transverse events along the solution, and their lengths count curvature transitions. Along almost every trajectory, the era phases have the Gauss distribution and every finite block of era lengths has its Gauss frequency. The tail decays inversely with era length; the limiting truncated mean grows logarithmically with the cutoff, while the full mean diverges. One invariant Borel exceptional set is null on every smooth transverse section and in the four-dimensional constraint manifold. The proof combines estimates for exact return branches with absolute continuity of their coding measure. Transverse contraction controls the accumulated phase error within each era, uniformly over unbounded era lengths and arbitrarily unequal small curvature variables. A density with strictly expanding section flux then forces almost every recurrent orbit to enter a statistical set on a small section.

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1. Introduction

The vacuum Bianchi IX equations describe a spatially homogeneous universe whose anisotropic geometry undergoes repeated transitions as a singularity is approached. In the description of Belinskii, Khalatnikov and Lifshitz [1], these transitions are organized into eras. Each era contains a finite number of epochs, and the succession of eras is governed by continued fractions. The statistical theory developed by Khalatnikov, Lifshitz, Khanin, Shchur and Sinai [7] predicts that the inverse era parameter has density

$$\frac{1}{\log 2}\frac{1}{1+x}, \qquad 0 < x < 1.$$

Establishing this law for the differential equations requires control of both individual trajectories and the measure of the initial data that produce them.

We prove this statistical law for eras defined by transverse events of the exact vacuum Bianchi IX flow. The exceptional initial data form one invariant Borel set that is null on every smooth transverse section and in the four-dimensional phase space. The result includes the frequencies of all finite blocks of era lengths, their tail distribution, and the divergence of their arithmetic mean. Thus arbitrarily long eras occur along a typical trajectory, and their contribution prevents the mean number of transitions per era from settling to a finite value. Joint frequencies also retain correlations between successive eras. Theorem 2.4 gives the precise statement after the event convention has been specified.

The main difficulty occurs near the three Taub points of the Kasner circle. An era may contain arbitrarily many transitions. The two small curvature variables may have arbitrarily different sizes, and the time between successive transitions need not remain bounded. Estimates on a fixed compact part of the Kasner circle therefore do not control the statistical problem. Moreover, closeness to the Mixmaster attractor does not determine how initial volume is distributed among the possible symbolic itineraries. Both issues enter the proof.

1.1. **Relation to earlier work.**

Ringström's attractor theorem [9] established the approach of generic Bianchi IX solutions to the union of the vacuum type I and type II states. Brehm [4] developed estimates near the Taub spaces and proved further results on attraction and particle horizons. We use the consequence of his IX classification that every positive-wall solution outside the Taub subspaces has vanishing pairwise curvature products. This input is stated in Proposition 2.2.

The passage from attraction to asymptotic statistics has been articulated by Heinzle and Uggla [6]. Their formulation separates the dynamics of the Kasner map from assertions about the exact Bianchi IX equations. Liebscher, Härterich, Webster and Georgi [8] constructed families of solutions approaching a periodic heteroclinic cycle, with a method extending to chains uniformly separated from the Taub points. Béguin [3] obtained nonperiodic shadowing families for Kasner points in closed forward-invariant sets without periodic points. Reiterer and Trubowitz [10] treated chains with unbounded continued fraction digits subject to a growth condition. Béguin and Dutilleul [2] subsequently obtained a precise geometric description of shadowing families and proved that their union has positive four-dimensional Lebesgue measure. Their Theorems 1.5 and 1.9 establish this measure statement for unions of shadowing families over sets of Kasner parameters of positive measure.

Our result determines era statistics on a set of full transverse measure in the exact IX phase space. The proof estimates the distribution of actual era itineraries directly. The local comparison with continued fractions is combined with a global argument that brings almost every orbit into a statistical set on a fixed section. This also makes it possible to choose the exceptional set independently of the initial section and the observation function.

The sampling convention matters. Damour and Lecian [5] analyze invariant measures and correlations for cosmological billiards, including the effects of retaining or identifying the spatial axis labels. Here each era receives equal weight, and the recorded scalar quantities are unchanged by an overall renaming of the axes. We define these quantities at specified crossings of the exact solution, so the statistical statement includes the existence and ordering of the events being counted.

1.2. **The mechanism of the proof.** The proof has three main tasks: transfer Gauss statistics to perturbed returns, construct complete era returns with estimates uniform near Taub, and bring almost every initial condition into their statistical domain. The first task separates comparison of observations from control of the measure of initial data.

For the comparison, if an era has length k and terminal phase y, its initial phase is close to $g_k(y) = (k + y)^{-1}$. Two consecutive ideal inverse branches contract by at least a factor of four. Consequently, an error tending to zero in the inverse recurrence produces an asymptotically identical Gauss orbit. Section 3.1 establishes this comparison while keeping the digits fixed. Continuous observations can then be transferred without crossing the discontinuities of the forward Gauss map.

To control the measure of the coding, we use suitable curves of small transverse amplitude. The exact era branches have summable distortion and summable losses of parameter length. For every finite word, the measure of the initial points carrying that word is bounded by a constant times the length of its Gauss cylinder. This bound passes to Borel sets of codes and makes the coding measure absolutely continuous. Birkhoff's theorem, first applied to the countable family of cylinders, then supplies one typical set for all the desired observations. Sections 3.2 and 3.3 develop these measure estimates and the statistical transfer.

Sections 4–7 give the local estimates. They are organized around the loss of transverse amplitude through each curvature core. That loss controls the phase error and its derivative. Summing the losses over an era avoids a bound that grows with the number of cores. A terminal connection produces a smaller amplitude for the next era, so the estimates can be iterated even when the era lengths are unbounded. Section 8 combines the passages into actual era returns and applies the statistical transfer.

To reach general initial data, we use radial families on small sections near the Taub points, constructed in Sections 9–11. Their returns reach the local statistical construction. Angular families complete the coverage near zero shear in Section 13. The points left outside these entrance selections have finite flux for a density equivalent to the natural section density. The first return strictly increases this flux. Section 12 proves that almost every recurrent point eventually leaves the excluded set.

Recurrence follows from a separate use of the local statistics. They exclude almost every trajectory that eventually stays away from all Taub points. The attractor input and deep Taub visits then give the section crossings needed for capture. Transport of the resulting null sets along the flow produces one invariant exceptional set, valid on every transverse section. These global steps and the final proof of Theorem 2.4 occupy Section 14.

All results and proofs in this article have been formalized in Lean 4; see Data availability.

2. The exact flow and the statistical theorem

2.1. **The normalized vacuum equations.** We use the normalization of Brehm [4, Section 2.2]. The variable t increases towards the past singularity on the expanding branch. Write $p = \Sigma_+$ and $s = \Sigma_-$ for the normalized shear variables, and N_1, N_2, N_3 for the

curvature variables. Set

$$Q = N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1),$$
$$S^2 = p^2 + s^2, \qquad G = S^2 + Q,$$
$$\mathcal{M}_{+++} = \{(p, s, N_1, N_2, N_3) \in \mathbb{R}^5 : G = 1,\ N_i > 0\}. \tag{2.1}$$

The equations are

$$\begin{aligned} p' &= Qp - 2N_1^2 + N_2^2 + N_3^2 + N_1N_2 - 2N_2N_3 + N_1N_3, \\ s' &= Qs + \sqrt{3}(-N_2^2 + N_3^2 + N_1N_2 - N_1N_3), \\ N_1' &= (-S^2 + 2p)N_1, \\ N_2' &= (-S^2 - p + \sqrt{3}s)N_2, \\ N_3' &= (-S^2 - p - \sqrt{3}s)N_3. \end{aligned} \tag{2.2}$$

These are the Hubble-normalized equations for vacuum Bianchi IX with zero cosmological constant. The reduction from the spacetime equations to this system is described in [4, Appendix A.4]. We denote its maximal flow by ϕ_t and the maximal interval through q by $I(q)$.

Lemma 2.1 (Forward existence)**.** *The flow preserves $\mathcal{M}_{+++}$. Every $q \in \mathcal{M}_{+++}$ has $[0, \infty) \subset I(q)$. For $t \geq 0$,*

$$0 < N_i(\phi_t q) \leq N_i(q)e^t. \tag{2.3}$$

Proof. Differentiating G using (2.2) gives $G' = 0$. Each curvature equation has the form $N_i' = \lambda_i N_i$, so

$$N_i(t) = N_i(0) \exp\left(\int_0^t \lambda_i(u)\, du\right)$$

on the interval of existence. Positivity is preserved. The three rates can be written

$$\begin{aligned} \lambda_1 &= 1 - (p-1)^2 - s^2, \\ \lambda_2 &= 1 - (p + 1/2)^2 - (s - \sqrt{3}/2)^2, \\ \lambda_3 &= 1 - (p + 1/2)^2 - (s + \sqrt{3}/2)^2. \end{aligned}$$

Thus $\lambda_i \leq 1$, which proves (2.3) as long as the solution exists. On every bounded positive time interval the curvature variables are bounded. The constraint gives

$$p^2 + s^2 = 1 - Q \leq 1 + 2(N_1N_2 + N_2N_3 + N_3N_1),$$

so the shear variables are bounded as well. A solution of the smooth polynomial vector field cannot have a finite maximal endpoint while remaining in a bounded subset of $\mathbb{R}^5$. This proves forward existence. □

The Kasner circle is the zero-curvature boundary $N_1 = N_2 = N_3 = 0$, $p^2 + s^2 = 1$. Its three Taub points are

$$T_1 = (-1, 0), \qquad T_2 = (1/2, -\sqrt{3}/2), \qquad T_3 = (1/2, \sqrt{3}/2).$$

When regarded as points of $\mathbb{R}^5$, their three curvature coordinates are zero. Axis permutations act simultaneously on (N_1, N_2, N_3) and on the shear components

$$\sigma_1 = -2p, \qquad \sigma_2 = p - \sqrt{3}s, \qquad \sigma_3 = p + \sqrt{3}s.$$

Two generators, which will also specify the event convention, are

$$\begin{aligned} \mathsf{S}(p, s, N_1, N_2, N_3) &= (p, -s, N_1, N_3, N_2), \\ \mathsf{C}(p, s, N_1, N_2, N_3) &= \left(\frac{-p + \sqrt{3}s}{2}, \frac{-\sqrt{3}p - s}{2}, N_2, N_3, N_1\right). \end{aligned} \tag{2.4}$$

Both preserve the constraint and commute with the flow, as follows by substitution in (2.2).

The first Taub subspace is $\mathcal{T}_1 = \{q \in \mathcal{M}_{+++} : s = 0,\ N_2 = N_3\}$; the other two are obtained by cyclic permutation. Put $\mathcal{T} = \bigcup_{i=1}^{3} \mathcal{T}_i$ and

$$\delta_i = 2\sqrt{N_j N_k}, \qquad \{i,j,k\} = \{1,2,3\}.$$

Proposition 2.2 (Decay of the curvature products)**.** *For every $q \in \mathcal{M}_{+++} \setminus \mathcal{T}$ and each $i = 1, 2, 3$,*

$$\delta_i(\phi_t q) \longrightarrow 0 \qquad (t \longrightarrow \infty). \tag{2.5}$$

In particular, $N_j(\phi_t q)N_k(\phi_t q) \to 0$ whenever $j \neq k$.

Proof. Brehm's global IX classification [4, Theorem 3, p. 45] places every initial point outside $\mathcal{T}$ in the attracting class. After a finite time, his local attractor theorem applies. Equation (6.2c) of that theorem [4, Theorem 2, p. 44] is (2.5). A finite shift of time does not alter this limit. The last assertion follows from $\delta_i^2 = 4N_jN_k$. □

2.2. **Coordinates for a curvature core.** In an ordered axis frame with $A = 1 + p > 0$, define

$$M = \frac{\sqrt{3}s}{A}, \qquad (U,V,W) = \frac{\sqrt{3}}{A}(N_1,N_2,N_3), \qquad d\tau = A\,dt. \tag{2.6}$$

The variable τ is used for local estimates. All arrival times in the main theorem refer to t. The constraint yields the inverse formula

$$\begin{aligned} D &= 3 + M^2 + U^2 + (V-W)^2 - 2U(V+W), \qquad A = \frac{6}{D}, \\ (p, s, N_1, N_2, N_3) &= \left(A - 1, \frac{AM}{\sqrt{3}}, \frac{AU}{\sqrt{3}}, \frac{AV}{\sqrt{3}}, \frac{AW}{\sqrt{3}}\right). \end{aligned} \tag{2.7}$$

The inverse has $G = 1$ whenever $D \neq 0$, and $D > 0$ corresponds to $A > 0$. For example, substituting (2.6) into D gives $D = 6/(1+p)$ directly from $G = 1$.

Write $C = U(V + W - U)$. Quotient differentiation in (2.2), followed by division by A, gives

$$\begin{aligned} \dot{M} &= -V^2 + W^2 + U(V-W) - MC, \\ \dot{U} &= \tfrac{1}{2}(1 - M^2 - (V-W)^2 + U^2)U, \\ \dot{V} &= (M - 1 - C)V, \\ \dot{W} &= (-M - 1 - C)W. \end{aligned} \tag{2.8}$$

Dots denote τ derivatives throughout the paper. These identities use the constraint in the curvature equations; the equation for M follows without it. The time change is increasing on the coordinate domain, so it preserves the order and crossing direction of its events.

Two identities explain the usefulness of these coordinates. If $E = (M-1)^2 + V^2$, then

$$\begin{aligned} \dot{E} &= -2CE + 2(M-1)(U-W)^2, \\ \frac{d}{d\tau}(VW) &= -2(1+C)VW. \end{aligned} \tag{2.9}$$

They follow by differentiating with (2.8). On the single-wall boundary $U = W = 0$, the first identity makes E constant. For positive small U, W, it identifies the terms that change the core radius. The later estimates retain these terms and control their accumulated effect through an entire era.

We write $\mathrm{D}_t = d/dt$ for differentiation in the original normalized time and use δ for tangent variations. The local estimates also use an increasing angle or a growing wall as an independent variable. The derivative conventions are summarized here:

Independent variable	Derivative convention
Normalized time t	D_t or d/dt; primes in (2.2)
Projective time τ	An overdot
Core angle or local wall clock	A prime after the clock is specified
Radial family parameter ρ	A prime along the specified family

2.3. **Events and complete eras.** We now define the observations whose statistics are studied. A core starts at an upward crossing of $V = 1$ and ends at its first subsequent return to $V = 1$, with downward crossing there. Between these two events $V > 1$ and the projective coordinate domain remains valid. At the exit set

$$R = \sqrt{(M-1)^2 + 1}.$$

There are two kinds of connection to the next core. If $R > 4$, change the ordered frame by S; in projective coordinates this replaces (M, U, V, W) by $(-M, U, W, V)$. If $2 < R < 4$, change the frame by C in (2.4). In either case the next entrance is the first subsequent $V = 1$ event in the new frame, and its crossing is upward. We call the first connection a reflection and the second a terminal connection.

Definition 2.3 (Complete era)**.** A complete era begins at the core entrance following a terminal connection. It consists of successive cores joined by reflections, up to and including the first core followed by a terminal connection. Its length K is the number of cores it contains. At its initial entrance define

$$\kappa = \sqrt{(M-1)^2 + 1}, \qquad \widehat{u} = \kappa/2, \qquad x = 2/\kappa. \tag{2.10}$$

The coordinates (κ, u, w) at an entrance record its radius and its two small walls. Here u and w are the entrance values of U and W, while the era parameter is $\widehat{u} = \kappa/2$. The statistics concern K and $x = 1/\widehat{u}$. The wall amplitudes u, w control the errors in their return law. All these radii and phases are dimensionless coordinates of the normalized state.

The existence assertion below includes every event in this definition. The constructed tail avoids $R = 2, 4$, and its first era has a preceding terminal connection on the same trajectory. At each constructed entrance the projective state has the form

$$(M, U, V, W) = \left(1 + \sqrt{\kappa^2 - 1}, u, 1, w\right), \qquad \kappa > 2, \quad u, w > 0. \tag{2.11}$$

The identity $K = \lfloor \widehat{u} \rfloor$ will be a consequence of the return estimates; K is defined by counting events. Figure 1 shows the ordering of these events.

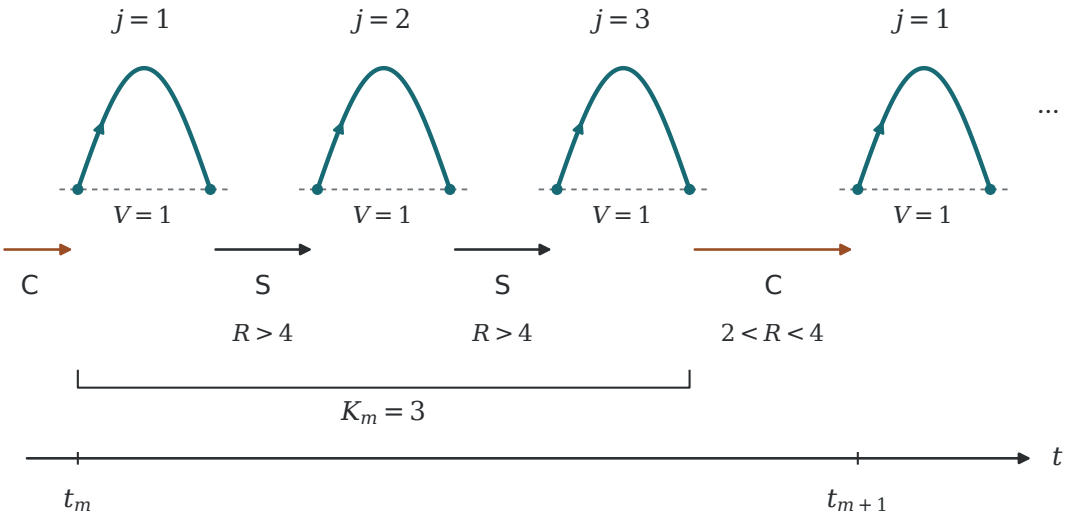


FIGURE 1. An era of three cores and the beginning of the next era. Each schematic profile uses its own ordered frame, with entrance and exit marked at $V = 1$. The connections use S for $R > 4$ and C for $2 < R < 4$. A terminal connection precedes t_m. The bracket counts the cores of the era, and t_{m+1} marks the following era entrance. Spacing along t indicates event order.

2.4. **The statistical law.** A smooth transverse section is a three-dimensional smooth submanifold of $\mathcal{M}_{+++}$ whose tangent space is complementary to the flow direction. A set is null on such a section if its intersection with the section has measure zero for every smooth positive density. Equivalently, its inverse image has three-dimensional Lebesgue measure zero in every local section chart. We use the corresponding coordinate definition for null sets in the four-dimensional manifold $\mathcal{M}_{+++}$.

Let ν be the probability measure on $(0, 1)$ given by

$$d\nu(x) = \frac{dx}{\log 2\,(1 + x)}. \tag{2.12}$$

For a positive integer k put $g_k(x) = (k + x)^{-1}$. For a finite word $\mathbf{b} = (b_0, \ldots, b_{r-1})$ of positive integers, its Gauss cylinder is the open interval

$$I_{\mathbf{b}} = g_{b_0} \circ \cdots \circ g_{b_{r-1}}((0, 1)).$$

On irrational points this is precisely the set with the indicated initial continued fraction digits. The rational points have zero ν measure.

Theorem 2.4 (Era statistics)**.** *There is a set $\mathcal{Z} \subset \mathcal{M}_{+++}$, null in every regular coordinate chart and on every smooth transverse section, such that each $q \in \mathcal{M}_{+++} \setminus \mathcal{Z}$ admits an infinite sequence of successive complete eras. Their entrance times $t_m(q)$ are positive, strictly increasing, and tend to infinity. Writing $K_m(q)$ and $x_m(q)$ for their lengths and phases, at each entrance we have*

$$K_m = \lfloor \widehat{u}_m \rfloor, \qquad \widehat{u}_m > 1, \qquad x_m = 1/\widehat{u}_m \in (0,1). \tag{2.13}$$

For every such q, the following limits hold for all $f \in C([0,1])$, all finite positive words $\mathbf{b}$, and all positive integers k, R:

$$\frac{1}{n}\sum_{m=0}^{n-1} f(x_m) \longrightarrow \int_0^1 f(x)\, d\nu(x), \tag{2.14}$$

$$\frac{1}{n}\#\{m < n : (K_m, \dots, K_{m+r-1}) = \mathbf{b}\} \longrightarrow \nu(I_{\mathbf{b}}), \tag{2.15}$$

$$\frac{1}{n}\#\{m < n : K_m = k\} \longrightarrow \frac{1}{\log 2}\log\frac{(k+1)^2}{k(k+2)}, \tag{2.16}$$

$$\frac{1}{n}\#\{m < n : K_m \geq k\} \longrightarrow \frac{\log(1+1/k)}{\log 2}, \tag{2.17}$$

$$\frac{1}{n}\sum_{m=0}^{n-1} \min(K_m, R) \longrightarrow \frac{\log(R+1)}{\log 2}, \tag{2.18}$$

$$\frac{1}{n}\sum_{m=0}^{n-1} K_m \longrightarrow +\infty. \tag{2.19}$$

The set $\mathcal{Z}$ can be chosen Borel, containing $\mathcal{T}$, and invariant throughout each maximal existence interval:

$$\phi_t q \in \mathcal{Z} \quad \Longleftrightarrow \quad q \in \mathcal{Z}, \qquad q \in \mathcal{M}_{+++}, \quad t \in I(q).$$

The ordered frames $a_m(q)$ and entrance times $t_m(q)$ can be chosen as Borel functions on $\mathcal{M}_{+++} \setminus \mathcal{Z}$. Their corresponding lengths $K_m(q)$ and phases $x_m(q)$ are also Borel functions. This one exceptional set applies to all sections, densities, observation functions, words, and thresholds above.

The theorem samples eras with equal weight. The divergent mean in (2.19) counts cores per era. The invariant exceptional set also permits every finite change of the time origin within $I(q)$.

Corollary 2.5 (Two observable consequences)**.** *For each trajectory and era sequence in Theorem 2.4, the frequency of adjacent eras each containing one core is* $\log(10/9)/\log 2$*. If $p_{\geq k}$ denotes the limit in* (2.17)*, then*

$$\lim_{k\to\infty} k p_{\geq k} = \frac{1}{\log 2}. \tag{2.20}$$

Proof. The interval $I_{(1,1)}$ is $(1/2, 2/3)$, so

$$\nu(I_{(1,1)}) = \frac{\log(5/3) - \log(3/2)}{\log 2} = \frac{\log(10/9)}{\log 2}.$$

For the second assertion use

$$\frac{1}{k+1} \leq \log(1+1/k) \leq \frac{1}{k}$$

and squeeze $kp_{\geq k}$ between $k/((k+1)\log 2)$ and $1/\log 2$. □

The adjacent unit-era frequency differs from the square of the single-era frequency in (2.16); the joint law therefore records dependence between consecutive lengths. Long eras have a tail of order k^{-1}. They become rarer as the threshold increases, yet their contribution makes the arithmetic mean diverge. The frequency limit is taken at each fixed threshold before k tends to infinity.

3. Continued fractions and statistical transfer

The statistical argument needs two properties of the exact returns: their phases follow perturbed inverse Gauss branches, and the measure of each finite itinerary is controlled by its Gauss cylinder. We first prove what these properties imply. This identifies the value, derivative and measure estimates that the subsequent dynamics must provide.

3.1. **Inverse branches and phase comparison.** The statistical argument uses two descriptions of an era sequence. Its digits count the actual cores. Its phase is measured at the actual entrance. The digits determine an exact continued fraction, while the phase satisfies a perturbed inverse recurrence. The following estimates compare these objects with constants independent of the digits.

Write $\mathcal{D} = \mathbb{N}_{\geq 1}^{\mathbb{N}_0}$ for the space of positive digit sequences, with its product Borel structure, and let θ be the left shift. For a finite word $\mathbf{b} = (b_0, \dots, b_{r-1})$ put $G_{\mathbf{b}} = g_{b_0} \circ \cdots \circ g_{b_{r-1}}$, with $G_\varnothing$ the identity.

Lemma 3.1 (Contraction of inverse branches)**.** *For $a, b \geq 1$ and $x, y \geq 0$,*

$$|g_a(x) - g_a(y)| \leq |x - y|,$$

$$|g_a(g_b(x)) - g_a(g_b(y))| \leq \tfrac{1}{4}|x - y|. \tag{3.1}$$

Consequently, for a positive word of length r,

$$|G_{\mathbf{b}}(x) - G_{\mathbf{b}}(y)| \leq 4^{-\lfloor r/2 \rfloor}|x - y|. \tag{3.2}$$

Proof. The difference for a single branch is

$$g_a(x) - g_a(y) = \frac{y - x}{(a + x)(a + y)}.$$

For two branches,

$$g_a(g_b(x)) = \frac{b + x}{a(b + x) + 1},$$

and their difference at x, y is

$$\frac{x - y}{(a(b + x) + 1)(a(b + y) + 1)}.$$

The first denominator is at least 1, and each factor in the last denominator is at least 2. This proves (3.1). Pairing successive branches proves (3.2), with one unpaired nonexpanding branch when r is odd. □

Proposition 3.2 (The code of a positive digit sequence)**.** *For $\mathbf{a} = (a_0, a_1, \dots) \in \mathcal{D}$ the limit*

$$c(\mathbf{a}) = \lim_{r \to \infty} G_{(a_0, \dots, a_{r-1})}(0) \tag{3.3}$$

exists, lies in $(0, 1) \setminus \mathbb{Q}$, and is Borel as a function of $\mathbf{a}$. It satisfies

$$c(\theta^m \mathbf{a}) = g_{a_m}(c(\theta^{m+1}\mathbf{a})), \qquad T^m c(\mathbf{a}) = c(\theta^m \mathbf{a}), \tag{3.4}$$

where $T(x) = \{1/x\}$. If two sequences agree in their first r digits, their codes differ by at most $4^{-\lfloor r/2 \rfloor}$.

Proof. Each branch maps $[0,1]$ into $[0,1]$. If $s \geq r$, the approximant of length s is obtained by applying $G_{(a_0,\dots,a_{r-1})}$ to a point of $[0,1]$. Its distance from the approximant of length r is at most $4^{-\lfloor r/2 \rfloor}$ by Lemma 3.1. The approximants are therefore Cauchy and their limit belongs to $[0,1]$. Passing to the limit through the first branch gives $c(\mathbf{a}) = g_{a_0}(c(\theta\mathbf{a}))$. This first shows that every code is positive. Applying this positivity to the shifted sequence shows that every code is strictly smaller than one. Hence

$$1/c(\mathbf{a}) = a_0 + c(\theta\mathbf{a}),$$

so its integer part is a_0 and its fractional part is $c(\theta\mathbf{a})$. Iteration gives (3.4).

If $c(\mathbf{a})$ were rational, each of its iterates under T would be rational and strictly between zero and one. For a reduced rational $p/q \in (0,1)$, the reduced denominator of $T(p/q) = \{q/p\}$ is at most $p < q$. This would give an infinite strictly decreasing sequence of positive denominators, a contradiction.

Every finite approximant in (3.3) depends on finitely many discrete coordinates and is Borel. Its pointwise limit is Borel. Finally, each code with prefix $\mathbf{b}$ is $G_{\mathbf{b}}$ of its tail code; (3.2) gives the claimed bound. □

Fix $\mathbf{a} \in \mathcal{D}$ and a phase sequence $y_m \in [0,1]$. Define its inverse recurrence error by

$$e_m = |y_m - g_{a_m}(y_{m+1})|. \tag{3.5}$$

The trajectory fixes the digit sequence $\mathbf{a}$ used in this comparison.

Proposition 3.3 (Comparison with the prescribed code)**.** *Let $z_m = c(\theta^m\mathbf{a})$. For every $m \geq 0$,*

$$|y_m - z_m| \leq \sum_{j=0}^{\infty} 4^{-j}(e_{m+2j} + e_{m+2j+1}). \tag{3.6}$$

If $e_m \to 0$, then $y_m - T^m c(\mathbf{a}) \to 0$. If instead the stronger condition $\sum_m e_m < \infty$ holds, then

$$\sum_{m=0}^{\infty} |y_m - T^m c(\mathbf{a})| \leq \frac{8}{3} \sum_{m=0}^{\infty} e_m. \tag{3.7}$$

Proof. Use (3.4), insert the two intermediate inverse branches, and apply Lemma 3.1. The triangle inequality gives

$$|y_m - z_m| \leq e_m + e_{m+1} + \tfrac{1}{4}|y_{m+2} - z_{m+2}|.$$

Iterating this inequality l times yields

$$|y_m - z_m| \leq \sum_{j=0}^{l-1} 4^{-j}(e_{m+2j} + e_{m+2j+1}) + 4^{-l}|y_{m+2l} - z_{m+2l}|.$$

Both phase sequences belong to $[0,1]$, so the last absolute value is at most one. Also $e_m \leq 1$, and the series is dominated by $2\sum_j 4^{-j}$. Letting $l \to \infty$ proves (3.6).

If $e_m \to 0$, the right side is bounded by $(8/3)\sup_{k \geq m} e_k$, which tends to zero. For summable errors, nonnegativity permits interchange of the two sums. For each j, both shifted sums of errors are at most $\sum_m e_m$. Thus

$$\sum_m \sum_j 4^{-j}(e_{m+2j} + e_{m+2j+1}) \leq 2\left(\sum_j 4^{-j}\right)\sum_m e_m = \frac{8}{3}\sum_m e_m.$$

This proves (3.7). □

Corollary 3.4. *Under the hypotheses of Proposition 3.3, suppose $e_m \to 0$. Then for every $f \in C([0,1])$,*

$$\frac{1}{n}\sum_{m=0}^{n-1}\big(f(y_m) - f(T^m c(\mathbf{a}))\big) \longrightarrow 0. \tag{3.8}$$

Proof. The function f is uniformly continuous on $[0,1]$. Proposition 3.3 therefore implies $f(y_m) - f(T^m c(\mathbf{a})) \to 0$. The Cesàro averages of this sequence tend to zero. □

The role of the two comparisons is now explicit. Phase estimates transfer continuous observations to the exact code. Frequencies of digit blocks are observations of the code itself, since its digits are the prescribed a_m. To deduce almost everywhere statements for physical initial data, the next issue is the measure carried by this coding. In particular, the Birkhoff typical set must have full measure for the codes produced by the actual return branches.

3.2. **Gauss statistics on one typical set.** Gauss cylinders connect finite itineraries with distributions of infinite codes. Their distortion bounds give both a comparison of measures and a proof of ergodicity. We then place all the statistical conclusions on one Borel set.

Write $J = (0,1)$ and $J_* = J \setminus \mathbb{Q}$, and denote Lebesgue measure by λ. Throughout this subsection, a word has positive integer entries. Its length is denoted by $|\mathbf{b}|$, and the empty word is allowed. We extend the Gauss map to $\mathbb{R}$ by $T(x) = \{1/x\}$ for $x \neq 0$ and $T(0) = 0$. It is Borel, and maps J_* into itself.

3.2.1. *Cylinder geometry and comparison of measures.* Fix a sequence $\mathbf{a} \in \mathcal{D}$. For its prefix of length n write $G_n = g_{a_0} \circ \cdots \circ g_{a_{n-1}}$, and define

$$q_{-1} = 0, \qquad q_0 = 1, \qquad q_{n+1} = a_n q_n + q_{n-1}. \tag{3.9}$$

These denominators give both the length and the distortion of a cylinder.

Lemma 3.5. *For $n \geq 0$ and $x \geq 0$,*

$$G_n'(x) = \frac{(-1)^n}{(q_n + q_{n-1}x)^2}, \qquad \lambda(I_{(a_0,\dots,a_{n-1})}) = \frac{1}{q_n(q_n + q_{n-1})}. \tag{3.10}$$

Consequently, for every word $\mathbf{b}$ and $x \in [0,1]$,

$$\frac{1}{2}\lambda(I_{\mathbf{b}}) \leq |G_{\mathbf{b}}'(x)| \leq 2\lambda(I_{\mathbf{b}}). \tag{3.11}$$

Moreover, for $x, y \geq 0$,

$$\big|\log|G_{\mathbf{b}}'(x)| - \log|G_{\mathbf{b}}'(y)|\big| \leq 2|x - y|. \tag{3.12}$$

Proof. Induction in (3.9) gives $q_n \geq 1$ and $0 \leq q_{n-1} \leq q_n$. Put $D_n(x) = q_n + q_{n-1}x$. The recurrence is equivalent to

$$D_{n+1}(x) = (a_n + x)D_n(g_{a_n}(x)).$$

Since $G_{n+1} = G_n \circ g_{a_n}$, the derivative formula follows by the chain rule, starting from $G_0(x) = x$. Each branch is injective, and hence so is G_n. Integrating its absolute derivative on J gives the cylinder length. In fact,

$$\frac{d}{dx}\frac{x}{q_n(q_n + q_{n-1}x)} = \frac{1}{(q_n + q_{n-1}x)^2}.$$

For $x \in [0,1]$, the derivative divided by the cylinder length lies between $q_n/(q_n + q_{n-1})$ and $(q_n + q_{n-1})/q_n$. These bounds imply (3.11). Finally,

$$\frac{d}{dx}\log|G_n'(x)| = -\frac{2q_{n-1}}{q_n + q_{n-1}x}$$

has absolute value at most 2 on $[0,\infty)$. The mean value theorem proves (3.12). □

Every $z \in J_*$ has a code. Indeed, all its iterates remain in J_*. Taking $a_m = \lfloor 1/T^m z \rfloor$ gives $T^m z = g_{a_m}(T^{m+1}z)$. Proposition 3.3, with zero errors, yields $z = c(\mathbf{a})$.

Two intersecting cylinders have comparable words: one word is a prefix of the other. To see this, for $x \in J$ the first digit of $g_k(x)$ is $\lfloor 1/g_k(x) \rfloor = k$. Thus intersecting nonempty words have the same first digit. Cancel its inverse branch and repeat. In particular, the cylinders of a fixed length are pairwise disjoint, and their union contains J_*. They have total Lebesgue measure one. Also,

$$c(\mathbf{a}) \in I_{\mathbf{b}} \quad \Longleftrightarrow \quad (a_0, \dots, a_{|\mathbf{b}|-1}) = \mathbf{b}. \tag{3.13}$$

Lemma 3.6 (Cylinder domination)**.** *Let μ and ξ be Borel measures on $\mathbb{R}$, with ξ outer regular. Suppose that $\mu(\mathbb{R} \setminus J_*) = 0$ and*

$$\mu(I_{\mathbf{b}}) \le \xi(I_{\mathbf{b}}) \qquad \text{for every nonempty word } \mathbf{b}.$$

Then $\mu \le \xi$.

Proof. Let $O \subset \mathbb{R}$ be open. Take all nonempty words $\mathbf{b}$ such that $I_{\mathbf{b}} \subset O$, but no nonempty proper prefix has this property. Their cylinders are pairwise disjoint by comparability. They cover $O \cap J_*$: for $z = c(\mathbf{a}) \in O$, the cylinders of its successive prefixes contain z and have diameter at most $4^{-\lfloor n/2 \rfloor}$. Some cylinder is therefore contained in O, and there is a first nonempty prefix with this property. Writing $\mathcal{W}(O)$ for these words, countable subadditivity gives

$$\mu(O) \le \sum_{\mathbf{b} \in \mathcal{W}(O)} \mu(I_{\mathbf{b}}) \le \sum_{\mathbf{b} \in \mathcal{W}(O)} \xi(I_{\mathbf{b}}) = \xi\left(\bigcup_{\mathbf{b} \in \mathcal{W}(O)} I_{\mathbf{b}} \right) \le \xi(O).$$

For a Borel set A, take the infimum over open supersets of A and use outer regularity of ξ. □

The useful feature of this argument is that the covering cylinders can have different lengths. A bound at every finite depth consequently controls arbitrary Borel sets without a limiting boundary estimate.

Corollary 3.7 (Distribution of codes)**.** *Let (X, μ) be a measure space and let $\mathbf{a} : X \to \mathcal{D}$ be measurable. If $C \ge 0$ and*

$$\mu\{q : (a_0(q), \dots, a_{r-1}(q)) = \mathbf{b}\} \le C\lambda(I_{\mathbf{b}}) \tag{3.14}$$

for every nonempty word $\mathbf{b}$ of length r, then

$$(c \circ \mathbf{a})_* \mu \le C\lambda|_J. \tag{3.15}$$

In particular the distribution of codes is absolutely continuous.

Proof. The map $c \circ \mathbf{a}$ is measurable and takes its values in J_*. By (3.13), its pushforward satisfies the cylinder bounds of Lemma 3.6, with $\xi = C\lambda|_J$. □

3.2.2. *Ergodicity and a common typical set.* The density in (2.12) is strictly positive and satisfies

$$\frac{1}{2\log 2} \le \frac{1}{\log 2\,(1+x)} \le \frac{1}{\log 2} \qquad (0 \le x \le 1). \tag{3.16}$$

Its primitive is $\log(1+x)/\log 2$. Thus ν is a probability, has the same null sets as $\lambda|_J$, and gives no mass to rational points. For $0 \le a \le b \le 1$,

$$\nu((a,b)) = \frac{\log(1+b) - \log(1+a)}{\log 2}. \tag{3.17}$$

Proposition 3.8. *The Gauss map preserves ν and is ergodic with respect to ν.*

Proof. Write $w(x) = 1/(\log 2\,(1+x))$. For $k \geq 1$ and $x \in J$,

$$w(g_k(x))|g_k'(x)| = \frac{1}{\log 2}\left(\frac{1}{k+x} - \frac{1}{k+1+x}\right).$$

Summing over k telescopes to $w(x)$. For a Borel set $A \subset J$, the sets $g_k(A)$ are disjoint and their union agrees with $T^{-1}A$ on J up to rational points. Change of variables in each branch and monotone convergence therefore give

$$\nu(T^{-1}A) = \sum_{k\geq 1} \nu(g_k(A)) = \int_A \sum_{k\geq 1} w(g_k(x))|g_k'(x)|\,dx = \nu(A).$$

Since ν and its image under T are concentrated on J, this proves invariance for every Borel set in $\mathbb{R}$.

For ergodicity, let $E \subset \mathbb{R}$ be Borel with $T^{-1}E = E$, and put $B = E \cap J$. Invariance of membership, applied successively to inverse branches, gives $G_{\mathbf{b}}(B) = I_{\mathbf{b}} \cap E$. Lemma 3.5 and change of variables imply

$$\lambda(I_{\mathbf{b}} \cap E) \geq \tfrac{1}{2}\lambda(I_{\mathbf{b}})\lambda(B).$$

Apply Lemma 3.6 to $\mu = \frac{1}{2}\lambda(B)\lambda|_J$ and $\xi = \lambda|_B$. The latter is a finite Borel measure on $\mathbb{R}$, hence outer regular. It follows that

$$\tfrac{1}{2}\lambda(B)\lambda|_J \leq \lambda|_B.$$

Evaluating this inequality on E^c yields $\lambda(B)\lambda(J \setminus E) = 0$. Thus E is null or conull in J for Lebesgue measure, and hence for ν by (3.16). This is ergodicity. □

We use Birkhoff's pointwise ergodic theorem in the following form. For an ergodic measure-preserving map F of a probability space (X, μ) and a real integrable function f,

$$\frac{1}{N}\sum_{m=0}^{N-1} f(F^m q) \longrightarrow \int_X f\,d\mu \quad \text{for } \mu\text{-almost every } q. \tag{3.18}$$

See [11, Corollary 10.1.2, p. 89]. Here it is applied to the countable family of cylinder indicators.

Let $\mathcal{B}$ consist of the points $z \in \mathbb{R}$ such that, for every word $\mathbf{b}$,

$$\frac{1}{N}\sum_{m=0}^{N-1} \mathbf{1}_{I_{\mathbf{b}}}(T^m z) \longrightarrow \nu(I_{\mathbf{b}}). \tag{3.19}$$

Each convergence condition is Borel. Proposition 3.8 and (3.18) show that $\nu(\mathcal{B}) = 1$, since the indicators are bounded and there are only countably many words. Moreover, $T^{-1}\mathcal{B} = \mathcal{B}$: shifting an average of an indicator changes it by at most $1/N$.

We next pass from cylinder indicators to all continuous functions on $[0,1]$ on this same set. The following elementary observation controls the infinitely many cylinders at a fixed depth. If probability masses $p_N(i)$ on a countable set converge pointwise to probability masses $p(i)$, then

$$\sum_i |p_N(i) - p(i)| = 2 - 2\sum_i \min(p_N(i), p(i)) \longrightarrow 0. \tag{3.20}$$

Indeed, the summands in the last sum converge to $p(i)$ and are bounded by the summable sequence $p(i)$. Dominated convergence proves the limit.

Proposition 3.9. *If $\mathbf{a} \in \mathcal{D}$ and $z = c(\mathbf{a}) \in \mathcal{B}$, then all six limits* (2.14)–(2.19) *hold with $K_m = a_m$ and $x_m = T^m z$.*

Proof. The block frequencies follow from (3.13), the shift identity (3.4), and the definition of $\mathcal{B}$. For continuous observations, fix a depth r and write

$$p_N(\mathbf{b}) = \frac{1}{N}\#\{m < N : T^m z \in I_{\mathbf{b}}\}, \qquad p(\mathbf{b}) = \nu(I_{\mathbf{b}}), \qquad |\mathbf{b}| = r.$$

The depth r cylinders partition J_*, so these are probability masses. They converge pointwise by (3.19), hence in ℓ^1 by (3.20).

For each cylinder take the representative $\xi_{\mathbf{b}} = G_{\mathbf{b}}(0) \in [0,1]$. Every point of that cylinder is within $4^{-\lfloor r/2 \rfloor}$ of its representative. If $f \in C([0,1])$, choose r sufficiently large that the oscillation of f over this distance is at most ε. Then

$$\left| \frac{1}{N} \sum_{m<N} f(T^m z) - \int f \, d\nu \right| \leq 2\varepsilon + \|f\|_\infty \sum_{|\mathbf{b}|=r} |p_N(\mathbf{b}) - p(\mathbf{b})|. \tag{3.21}$$

The two approximation errors come from replacing f by its representative value in the empirical average and in the integral. Let $N \to \infty$, then $\varepsilon \to 0$. This proves the continuous observation law.

For one digit, $I_{(k)} = (1/(k+1), 1/k)$. Formula (3.17) gives

$$\nu(I_{(k)}) = \frac{1}{\log 2} \log \frac{(k+1)^2}{k(k+2)}.$$

At a fixed threshold k the tail indicator is one minus the sum of the first $k-1$ digit indicators. Its frequency is therefore

$$1 - \sum_{j=1}^{k-1} \nu(I_{(j)}) = \nu((0, 1/k)) = \frac{\log(1 + 1/k)}{\log 2}.$$

Finally, the identity $\min(a_m, R) = \sum_{k=1}^{R} \mathbf{1}_{\{a_m \geq k\}}$ gives the truncated mean. The limiting sum telescopes:

$$\sum_{k=1}^{R} \frac{\log(1 + 1/k)}{\log 2} = \frac{\log(R+1)}{\log 2}.$$

The average of a_m is bounded below by every truncated average. Given any real bound, first take R for which the last expression exceeds that bound, and then take N large. This proves divergence of the full mean to $+\infty$. □

Finite-cylinder approximation applies to every continuous observation on the same set $\mathcal{B}$. A sequence with code in $\mathcal{B}$ cannot have bounded digits: any omitted digit has a strictly positive prescribed frequency. Equivalence of ν and $\lambda|_J$ thus shows that the set of codes with bounded digits is Lebesgue null.

Corollary 3.10 (Statistical transfer)**.** *Let (X, μ) be a measure space, let $\mathbf{a} : X \to \mathcal{D}$ be measurable, and let $y_m(q) \in [0,1]$ for μ-almost every q and all m. Suppose that*

$$(c \circ \mathbf{a})_* \mu \ll \lambda|_J, \qquad |y_m(q) - g_{a_m(q)}(y_{m+1}(q))| \longrightarrow 0 \quad \text{for } \mu\text{-almost every } q.$$

Then, on one set of full μ measure, all six limits (2.14)–(2.19) *hold with $K_m = a_m(q)$ and $x_m = y_m(q)$.*

Proof. Absolute continuity puts the code in $\mathcal{B}$ for almost every q. Intersect with the full measure set on which the phase conditions hold. Proposition 3.9 gives every digit statistic; Corollary 3.4 gives the continuous observation law for the phases $y_m(q)$. □

3.3. **Measure estimates for perturbed inverse branches.** The exact return branches will be defined only on part of each phase interval. Some initial points may cease to admit a controlled return. We need to estimate both the distribution of the codes that continue indefinitely and the measure of the points that are lost. The relevant estimates depend on the errors in one branch and on the length excluded from its domain. They do not require a bound on the digits.

3.3.1. *Compositions and distortion.* Let $I_0 \subset J$ be open. For every finite word $\mathbf{b}$ and every positive integer k, let $D_{\mathbf{b},k} \subset J$ be open and let $h_{\mathbf{b},k}$ be a differentiable, injective map on this domain, with nonzero derivative. For a fixed $\mathbf{b}$, assume that the images $h_{\mathbf{b},k}(D_{\mathbf{b},k})$ are pairwise disjoint as k varies. The maps may depend on the whole preceding word.

Define the admissible terminal domains $K_{\mathbf{b}}$ and compositions $H_{\mathbf{b}}$ recursively by

$$\begin{aligned} K_{\varnothing} &= I_0, & H_{\varnothing} &= \mathrm{id}, \\ K_{\mathbf{b}k} &= D_{\mathbf{b},k} \cap h_{\mathbf{b},k}^{-1}(K_{\mathbf{b}}), & H_{\mathbf{b}k} &= H_{\mathbf{b}} \circ h_{\mathbf{b},k}. \end{aligned} \tag{3.22}$$

Here $\mathbf{b}k$ denotes the word obtained by appending k. The set of initial points carrying a finite word is $C_{\mathbf{b}} = H_{\mathbf{b}}(K_{\mathbf{b}})$. Every intermediate point of this composition lies in its prescribed domain. In particular $K_{\mathbf{b}} \subset J$.

The sets $K_{\mathbf{b}}$ are open, and $H_{\mathbf{b}}$ is continuous and injective on $K_{\mathbf{b}}$. The images $C_{\mathbf{b}}$ are Borel. Children lie in their parent and siblings are disjoint:

$$C_{\mathbf{b}k} \subset C_{\mathbf{b}}, \qquad C_{\mathbf{b}k} \cap C_{\mathbf{b}l} = \varnothing \quad (k \neq l). \tag{3.23}$$

These assertions follow inductively from (3.22). For Borelness one may use that a continuous injection on a Borel subset of $\mathbb{R}$ has Borel image. For disjointness, apply injectivity of $H_{\mathbf{b}}$ to the disjoint branch images.

Set

$$E_n = \bigcup_{|\mathbf{b}|=n} C_{\mathbf{b}}, \qquad S = \bigcap_{n\geq 0} E_n. \tag{3.24}$$

Thus $E_0 = I_0$ and $E_{n+1} \subset E_n$. Each $x \in S$ lies in a unique cell at each depth. The resulting nested words define a sequence $\mathbf{a}(x) \in \mathcal{D}$, and

$$\{x \in S : (a_0(x), \dots, a_{r-1}(x)) = \mathbf{b}\} = S \cap C_{\mathbf{b}}. \tag{3.25}$$

It follows that S, the digit sequence, and the code $\psi(x) = c(\mathbf{a}(x))$ are Borel.

For $x \in S$, let $y_n(x)$ be the unique point of $K_{(a_0(x),\dots,a_{n-1}(x))}$ mapped to x by its composition. Injectivity gives $y_0(x) = x$ and the exact recurrence

$$y_n(x) = h_{(a_0(x),\dots,a_{n-1}(x)),a_n(x)}(y_{n+1}(x)). \tag{3.26}$$

All these phases lie in J.

Suppose that $d_n, e_n \geq 0$ and that, for $|\mathbf{b}| = n$ and $t \in D_{\mathbf{b},k}$,

$$\begin{aligned} |h_{\mathbf{b},k}(t) - g_k(t)| &\leq d_n, \\ \left|\log \frac{|h'_{\mathbf{b},k}(t)|}{|g'_k(t)|}\right| &\leq e_n. \end{aligned} \tag{3.27}$$

Put

$$L_n = \sum_{j=0}^{n-1} e_j + 2\sum_{j=1}^{n-1} d_j, \qquad L_0 = 0. \tag{3.28}$$

Proposition 3.11 (Distortion on the actual domains)**.** *For $|\mathbf{b}| = n$ and $t \in K_{\mathbf{b}}$,*

$$\left|\log \frac{|H'_{\mathbf{b}}(t)|}{|G'_{\mathbf{b}}(t)|}\right| \leq L_n. \tag{3.29}$$

In particular, for every Borel set $A \subset K_{\mathbf{b}}$,

$$\lambda(H_{\mathbf{b}}(A)) \le e^{L_n}\lambda(G_{\mathbf{b}}(A)) \le 2e^{L_n}\lambda(I_{\mathbf{b}})\lambda(A). \tag{3.30}$$

Proof. Write $\ell_{\mathbf{b}} = \log|G'_{\mathbf{b}}|$ and $R_{\mathbf{b}} = \log|H'_{\mathbf{b}}| - \ell_{\mathbf{b}}$. All derivatives are nonzero on the domains under consideration, by the chain rule. For a word $\mathbf{b}$ of length n, abbreviate $h = h_{\mathbf{b},k}$ and $g = g_k$. At $t \in K_{\mathbf{b}k}$,

$$R_{\mathbf{b}k}(t) = R_{\mathbf{b}}(h(t)) + \ell_{\mathbf{b}}(h(t)) - \ell_{\mathbf{b}}(g(t)) + \log\frac{|h'(t)|}{|g'(t)|}.$$

The first term is evaluated at $h(t) \in K_{\mathbf{b}}$. Both $h(t)$ and $g(t)$ are positive, so (3.12) bounds the middle difference by $2d_n$. When $\mathbf{b}$ is empty this difference is zero, since $\ell_\varnothing = 0$. The last term is bounded by e_n. Induction proves (3.29), including the absence of a d_0 term.

Exponentiating gives $|H'_{\mathbf{b}}| \le e^{L_n}|G'_{\mathbf{b}}|$. The one-dimensional change of variables formula for injective differentiable maps, applied on A, gives the first inequality in (3.30). Integrating the upper bound in (3.11) gives the second. □

3.3.2. *The coding measure and the mass that remains.* Assume from now on that

$$L = \sum_{j=0}^{\infty} e_j + 2\sum_{j=1}^{\infty} d_j < \infty. \tag{3.31}$$

Then $L_n \le L$ and $d_n \to 0$. Define the excluded part of the terminal domain by

$$A_{\mathbf{b}} = K_{\mathbf{b}} \setminus \bigcup_{k\ge 1} h_{\mathbf{b},k}(D_{\mathbf{b},k}). \tag{3.32}$$

This is the set from which the next controlled branch cannot be taken.

Theorem 3.12. *Under* (3.27) *and* (3.31),

$$\psi_*(\lambda|_S) \le e^L\lambda|_J. \tag{3.33}$$

If $\lambda(A_{\mathbf{b}}) \le \ell_n$ whenever $|\mathbf{b}| = n$, where $\ell_n \ge 0$ and $\sum_n \ell_n < \infty$, then

$$\lambda(I_0 \setminus S) \le 2e^L \sum_{n=0}^{\infty} \ell_n. \tag{3.34}$$

For every Borel $I \subset \mathbb{R}$ the restricted coding measure has a nonnegative integrable density ρ_I such that

$$\psi_*(\lambda|_{I\cap S}) = \rho_I\lambda, \qquad 0 \le \rho_I \le e^L \quad \lambda\text{-almost everywhere}, \qquad \int_{\mathbb{R}} \rho_I\, d\lambda = \lambda(I\cap S). \tag{3.35}$$

Proof. Apply the first inequality of (3.30) to $A = K_{\mathbf{b}} \subset J$. It gives

$$\lambda(C_{\mathbf{b}}) \le e^L\lambda(I_{\mathbf{b}}).$$

Together with (3.25), these are the hypotheses of Corollary 3.7. This proves (3.33).

For the loss estimate, injectivity of $H_{\mathbf{b}}$ gives the exact identity

$$H_{\mathbf{b}}(A_{\mathbf{b}}) = C_{\mathbf{b}} \setminus \bigcup_{k\ge 1} C_{\mathbf{b}k}.$$

Every point of $I_0 \setminus S$ has a first depth at which it fails to continue. Consequently,

$$I_0 \setminus S = \bigcup_{n\ge 0} \bigcup_{|\mathbf{b}|=n} H_{\mathbf{b}}(A_{\mathbf{b}}).$$

At a fixed depth, the second inequality of (3.30) yields

$$\sum_{|\mathbf{b}|=n} \lambda(H_{\mathbf{b}}(A_{\mathbf{b}})) \le 2e^L\ell_n \sum_{|\mathbf{b}|=n} \lambda(I_{\mathbf{b}}) = 2e^L\ell_n.$$

Summing over depths proves (3.34).

Restriction of the initial measure can only decrease its pushforward. Thus the measure in (3.35) is also dominated by $e^L\lambda$ and is finite, since $S \subset J$. The Radon–Nikodym theorem gives its density. The domination implies $\rho_I \le e^L$ almost everywhere by integration on measurable sets. Its integral is the total mass of the pushforward, which equals $\lambda(I\cap S)$. □

The last equality preserves the mass of the initial data that remain. It will allow statistical estimates along different curves to be integrated without renormalizing each curve separately.

Corollary 3.13 (Statistics and restricted loss)**.** *Under the hypotheses and stopping bounds of Theorem 3.12, there is a Borel set $S_{\mathrm{stat}} \subset S$ with $\lambda(S \setminus S_{\mathrm{stat}}) = 0$ such that, for every $x \in S_{\mathrm{stat}}$, all six statistical limits in Theorem 2.4 hold for the digits $a_m(x)$ and phases $y_m(x)$. For every Borel $I \subset \mathbb{R}$,*

$$\lambda\big((I \cap I_0) \setminus S_{\mathrm{stat}}\big) \le \min\left\{\lambda(I),\ 2e^L \sum_{n\ge 0} \ell_n\right\}. \tag{3.36}$$

If $D \ge 0$ and ω is a nonnegative measurable density with $\omega \le D$ on I_0, the corresponding bound is

$$\int_{(I\cap I_0)\setminus S_{\mathrm{stat}}} \omega\, d\lambda \le \min\left\{\int_I \omega\, d\lambda,\ 2De^L \sum_{n\ge 0} \ell_n\right\}. \tag{3.37}$$

Proof. Take $S_{\mathrm{stat}} = S \cap \psi^{-1}(\mathcal{B})$, with the inverse image understood on S. It is Borel. The coding domination and the full Lebesgue measure of $\mathcal{B}$ in J show that its complement in S is null. Equation (3.26) and the value bound in (3.27) give

$$|y_m(x) - g_{a_m(x)}(y_{m+1}(x))| \le d_m \longrightarrow 0.$$

Apply Proposition 3.9 and Corollary 3.4. The loss in a restricted initial set is bounded both by its own measure and by the total loss in (3.34). Adding the null set $S \setminus S_{\mathrm{stat}}$ does not change either bound. For the weighted version, integrate $\omega \le D$ on the lost part of I_0; the other bound follows from inclusion in I. □

3.3.3. *Errors controlled by transverse amplitude.* The bounds needed from the dynamics can now be stated quantitatively. The terminal connection will decrease the transverse amplitude enough to use the sequence

$$B_0 = B, \qquad B_{m+1} = B_m^{4/3}, \qquad B_m = B^{(4/3)^m}. \tag{3.38}$$

The branch value error will be of order B_m, its logarithmic derivative error of order $\sqrt{B_m}$, and the excluded length of order B_m^η, for a fixed $\eta > 0$. The following consequence of Theorem 3.12 records how these estimates will be used.

Lemma 3.14. *Suppose $0 < B \le 1$, $\eta > 0$, and $B^\eta < 1/160$. For every $p \ge \eta$ and $k \ge 0$,*

$$\sum_{m=k}^{\infty} B_m^p \le 2B_k^p. \tag{3.39}$$

Proof. The sequence B_m is positive and at most B. Also $B^{\eta/3} < 1/2$, since its cube is less than $1/160 < 1/8$. Using $p \ge \eta$ and $B \le 1$ gives

$$B_{m+1}^p = B_m^p B_m^{p/3} \le B_m^p B^{p/3} \le \tfrac{1}{2} B_m^p.$$

Iteration from k and summation of the geometric series prove the claim. □

Corollary 3.15. *Fix $0 < \eta \le 1/2$ and $C_v, C_i \ge 0$. There is $B_* > 0$, with $B_* < 1$, such that the following holds for $0 < B \le B_*$. Suppose the branches of this subsection satisfy* (3.27) *with*

$$d_m = C_v B_m, \qquad e_m = C_i\sqrt{B_m}, \qquad \lambda(A_{\mathbf{b}}) \le 10B_{|\mathbf{b}|}^\eta.$$

Then their coding measure satisfies

$$\psi_*(\lambda|_S) \le 2\lambda|_J. \tag{3.40}$$

There is a Borel $S_{\mathrm{stat}} \subset S$ on which all six statistical limits hold and, for every Borel $I \subset \mathbb{R}$,

$$\lambda((I \cap I_0) \setminus S_{\mathrm{stat}}) \le \min\{\lambda(I), 80B^\eta\}. \tag{3.41}$$

For a nonnegative measurable density $\omega \le D$ on I_0, where $D \ge 0$, the weighted loss is at most $\min\{\int_I \omega\, d\lambda, 80DB^\eta\}$. If $I_0 = J$, then $\lambda(S_{\mathrm{stat}}) > 1/2$.

Proof. Choose $B_* < 1$ so that, for $0 < B \le B_*$,

$$B^\eta < \frac{1}{160}, \qquad 2C_i\sqrt{B} + 4C_vB^{4/3} \le \log 2.$$

Such a choice is possible because all the powers are positive. Lemma 3.14, used with $p = 1/2, 1, \eta$, gives

$$L = \sum_{m\ge 0} C_i\sqrt{B_m} + 2\sum_{m\ge 1} C_vB_m \le 2C_i\sqrt{B} + 4C_vB^{4/3} \le \log 2,$$

$$\sum_{m\ge 0} 10B_m^\eta \le 20B^\eta.$$

In the first line the value error starts at $m = 1$, as in (3.31). Thus $e^L \le 2$, and Theorem 3.12 and Corollary 3.13 give the coding and loss estimates. The errors d_m tend to zero, so the same corollary supplies all the statistical limits. When $I_0 = J$, its measure is one and the loss is at most $80B^\eta < 1/2$. □

The dependence on B in (3.41) will be retained when the initial curves are integrated into a transverse section. We now establish the estimates for the exact flow that lead to these branch bounds.

4. Passage through a curvature core

A core is the part of a curvature transition between the upward and downward crossings of the active wall. Its transverse contraction controls the error accumulated before the next core. We establish the actual passage, its ordinary initial-data derivatives, and the invariant directions needed to iterate curves of entrance states.

4.1. **Core coordinates and the first return.**

An angular coordinate turns the core passage into an interval on which the transverse decay can be integrated explicitly. We use it to prove finite arrival at the outgoing section and to compare the radius drift with the loss in the small walls. All estimates refer to the exact equations (2.8).

4.1.1. *Coordinates and estimates along the core.* For $V > 0$, set

$$R = \sqrt{(M-1)^2 + V^2}, \qquad z = \operatorname{arsinh}\frac{1-M}{V}, \qquad P = VW. \tag{4.1}$$

Then

$$M = 1 - R\tanh z, \qquad V = \frac{R}{\cosh z}, \qquad W = \frac{P}{V}. \tag{4.2}$$

The product P will decrease when $P \ge 0$ and $C = U(V + W - U) \ge 0$. Define

$$d = 1 - \frac{(U-W)^2}{R^2}.$$

Differentiation with respect to the projective time τ gives

$$\dot R = -CR - \tanh z\,(U-W)^2, \qquad \dot z = Rd, \qquad \dot P = -2(1+C)P. \tag{4.3}$$

In particular z is a time coordinate wherever $d > 0$. In that coordinate the equations for (R, U, P) become

$$\begin{aligned} \frac{dR}{dz} &= \frac{-C - \tanh z\,(U-W)^2/R}{d}, \\ \frac{dU}{dz} &= \frac{U}{d}\left(-\frac{R}{2} + \tanh z + \frac{W}{\cosh z} + \frac{U^2 - W^2}{2R}\right), \\ \frac{dP}{dz} &= -\frac{2(1+C)P}{Rd}. \end{aligned} \tag{4.4}$$

To verify (4.3), differentiate R^2 using (2.9). For z, direct differentiation yields

$$\dot{z} = \frac{(M-1)\dot{V} - V\dot{M}}{RV} = R - \frac{(U-W)^2}{R}.$$

The equation for P is already in (2.9). Dividing by Rd and substituting (4.2) gives (4.4).

On the single active wall $U = W = 0$, the radius is constant, say $R = \kappa$, and z increases at speed κ in projective time. The entrance and exit at $V = 1$ occur at $z = -\operatorname{arcosh}\kappa$ and $z = \operatorname{arcosh}\kappa$. Thus the active wall follows the explicit profile $V = \kappa/\cosh z$. The same coordinate will control the nearby nonlinear passage without making its duration a multiplier in every error estimate.

Fix $\kappa \geq 6$, and put $\varepsilon_0 = 10^{-4}$. Consider the closed region

$$|R^2 - \kappa^2| \leq \frac{1}{8}, \qquad V \geq 1, \qquad 0 \leq U \leq \frac{\varepsilon_0}{\kappa}, \qquad 0 \leq P \leq \frac{\varepsilon_0}{\kappa}. \tag{4.5}$$

Since $W = P/V$, both small walls are at most $\varepsilon_0/\kappa \leq 1/60000$. In this region,

$$\frac{99}{100}\kappa \leq R \leq \frac{101}{100}\kappa, \qquad \frac{999}{1000} \leq d \leq 1, \qquad C \geq 0. \tag{4.6}$$

The radius bounds follow by squaring, and the bounds for d follow from $|U - W| \leq \varepsilon_0/\kappa$. Also $V + W - U > 0$, which gives the sign of C.

Two further estimates keep the coordinates and their clock regular. The denominator D from (2.7) satisfies

$$2 \leq D \leq 2\kappa^2, \qquad \frac{3}{\kappa^2} \leq A = \frac{6}{D} \leq 3, \qquad \frac{dz}{dt} = ARd \geq \frac{1}{\kappa}. \tag{4.7}$$

For the lower bound, complete the square:

$$D = 3 + M^2 + (V - U - W)^2 - 4UW \geq 2.$$

For the upper bound, use

$$D = R^2 + 2(M-1) + 4 + (U-W)^2 - 2V(U+W)$$

together with $|M - 1| \leq R$ and (4.6). The bounds for A and ARd follow immediately. Furthermore $\cosh z = R/V \leq 101\kappa/100$, so

$$|z| \leq \log(4\kappa). \tag{4.8}$$

Here we used $e^{|z|} \leq 2\cosh z$.

We call U the strong transverse wall because of its rapid contraction; V is the active wall crossing the core. The strong wall U and the product P decrease throughout any passage contained in (4.5). More precisely,

$$\frac{dU}{dz} \leq -\frac{\kappa}{4}U, \qquad \frac{dP}{dz} \leq 0, \qquad \dot{U} \leq -\frac{\kappa^2}{10}U, \qquad \dot{P} \leq -2P. \tag{4.9}$$

These inequalities also hold when a wall vanishes. For the first, the logarithmic numerator in (4.4) is bounded by

$$-\frac{R}{2} + 1 + W + \frac{U^2}{2R} \leq -\frac{3\kappa}{10}.$$

It is negative, and division by $0 < d \le 1$ gives the asserted bound. For the projective-time estimate, rewrite the coefficient of U as

$$\frac{\dot U}{U} = -\frac{R^2}{2} - (M-1) + VW + \frac{U^2 - W^2}{2} \le -\frac{\kappa^2}{10}$$

when $U > 0$, and multiply by U to include $U = 0$. The two estimates for P follow from $C \ge 0$.

Lemma 4.1 (Radius drift)**.** *Suppose a solution of* (4.4) *lies in* (4.5) *for $a \le z \le b$, and $R(a) = \kappa$, $V(a) = 1$. Write $u = U(a)$ and $w = P(a) = W(a)$. Then*

$$|R(b) - \kappa| \le \frac{16}{\kappa}(u + w^2), \qquad |R(b)^2 - \kappa^2| < \frac{1}{16}. \tag{4.10}$$

Proof. The first two inequalities in (4.9) imply

$$U(z) \le u e^{-\kappa(z-a)/4}, \qquad P(z) \le w.$$

Since $(\log \operatorname{sech} z)' = -\tanh z \le 1$ and $\cosh a = \kappa$, we also have

$$V(z) \le \frac{11}{10} e^{z-a}, \qquad W(z) \le w, \qquad W(z) \le \frac{w \cosh z}{R(z)}.$$

The radius equation and $C \ge 0$ give

$$\left|\frac{dR}{dz}\right| \le \frac{1}{999/1000}\left(UV + UW + \frac{U^2}{R} + \frac{W^2}{R}\right). \tag{4.11}$$

Indeed $(U - W)^2 \le U^2 + W^2$ and $C = UV + UW - U^2 \le UV + UW$.

We estimate the four integrals separately. Using $R \ge 9\kappa/10$ and $\kappa/4 - 1 \ge \kappa/12$,

$$\begin{aligned} \int_a^b UV\,dz &\le \frac{14u}{\kappa}, & \int_a^b UW\,dz &\le \frac{4uw}{\kappa}, \\ \int_a^b \frac{U^2}{R}\,dz &\le \frac{3u^2}{\kappa^2}, & \int_a^b \frac{W^2}{R}\,dz &\le \frac{4w^2}{\kappa}. \end{aligned} \tag{4.12}$$

The first three bounds follow by integration of the exponential majorants, with decay rates $\kappa/12$, $\kappa/4$ and $\kappa/2$. For the last one, observe that

$$\int_a^b \cosh^2 z\,dz \le [\sinh z \cosh z]_a^b \le \cosh^2 a + \cosh^2 b \le 2(11\kappa/10)^2.$$

The first inequality follows by differentiating $\sinh z \cosh z$; the second uses $|\sinh z| \le \cosh z$. Combining this with $W^2/R \le w^2 \cosh^2 z/(9\kappa/10)^3$ proves the fourth estimate.

Integrate (4.11) and use (4.12). Since $u, w \le 1/60000$,

$$\frac{14u/\kappa + 4uw/\kappa + 3u^2/\kappa^2 + 4w^2/\kappa}{999/1000} \le \frac{16}{\kappa}(u + w^2).$$

This proves the radius estimate. Multiplying it by $R(b) + \kappa \le 201\kappa/100$ gives

$$|R(b)^2 - \kappa^2| \le \frac{804}{25}(u + w^2) < \frac{1}{16},$$

which strictly improves the assumed energy bound. □

The endpoint estimate for $\cosh^2 z$ is useful for every initial segment of the crossing. It retains the smallness of the weak wall even when the angle interval is long.

4.1.2. *The return and its amplitude loss.* We allow U or W to vanish when establishing local estimates. The forward-existence proof of Lemma 2.1 applies unchanged on $G = 1$ with nonnegative curvature variables: the exponential formula preserves their signs and all finite-time bounds remain valid.

Proposition 4.2 (Large-radius core passage)**.** *Let $\kappa \geq 6$ and $0 \leq u, w \leq \varepsilon_0/\kappa$. The projective initial state*

$$(M, U, V, W) = (1 + \sqrt{\kappa^2 - 1}, u, 1, w) \tag{4.13}$$

lifts by (2.7) *to a constrained solution of* (2.2)*. It has a first positive time t_* at which $V = 1$. The projective chart remains regular on $[0, t_*]$, and $V > 1$ on $(0, t_*)$. At the exit, $M < 1$ and $dV/dt < 0$. The solution lies in* (4.5) *throughout this interval, and its exit values satisfy*

$$|R_* - \kappa| \leq \frac{16}{\kappa}(u + w^2), \qquad U_* \leq e^{-\kappa/40}u, \qquad W_* \leq e^{-1/(2\kappa)}w. \tag{4.14}$$

With $\tau(0) = 0$ and $\tau_* = \int_0^{t_*} A(t)\, dt$,

$$\frac{1}{4\kappa} \leq \tau_* \leq \frac{4\log(4\kappa)}{\kappa}, \qquad t_* \leq 2\kappa \log(4\kappa). \tag{4.15}$$

Proof. The initial state has $R = \kappa$ and $D \geq 2$, so the inverse coordinate map is defined and gives $G = 1$, $A > 0$. On $V = 1$, the region (4.5) implies

$$|M - 1| \geq \frac{3\kappa}{4}, \qquad 0 \leq C = U(1 + W - U) \leq \frac{1}{1000}.$$

Hence the sign of $\dot{V} = M - 1 - C$ is positive on the incoming branch $M > 1$ and negative on the outgoing branch $M < 1$; its absolute value is at least $\kappa/2$. In particular the initial crossing is upward.

Follow the normalized solution while it lies in (4.5), also imposing $A \geq 3/(2\kappa^2)$. These closed conditions hold on an initial interval. The upper faces for U and P point strictly inward by (4.9); their zero faces are invariant. Lemma 4.1 improves the energy bound to $1/16$, and (4.7) improves the lower bound for A to $3/\kappa^2$. Consequently a finite endpoint of this interval can only occur at $V = 1$. If $V > 1$ there, continuity and the inward inequalities extend the interval. The normalized solution itself exists at every positive time, so no other finite endpoint is possible.

Such an interval cannot persist indefinitely. Indeed, (4.7) and (4.8) give, at every time in it,

$$\frac{t}{\kappa} \leq z(t) - z(0) \leq 2\log(4\kappa).$$

There is therefore a finite return to $V = 1$ with the stated bound on t_*. At any return approached from $V \geq 1$ the left derivative of V is nonpositive. The incoming sign alternative is excluded, so this return is on $M < 1$ and is strictly downward. An earlier interior touch of $V = 1$ would likewise be followed immediately by $V < 1$, contradicting the closed-region condition. Thus t_* is the first return and $V > 1$ before it.

The radius estimate is Lemma 4.1 at the exit. To estimate the projective duration, use $\dot{z} = Rd \geq \kappa/2$ and (4.8) for the upper bound. For the lower bound, the phase changes by at least κ between the two crossings. The factorization

$$-\dot{M} = (V + W - U)(V - W + MU)$$

gives $V^2/2 \leq -\dot{M} \leq 2V^2 \leq 4\kappa^2$ in the region. To check this, $|M| \leq 6\kappa/5$ and the wall bounds place each of the two factors between $9V/10$ and $11V/10$. Integrating $-\dot{M} \leq 4\kappa^2$ yields $\tau_* \geq 1/(4\kappa)$. Finally (4.9) gives

$$U_* \leq ue^{-\kappa^2\tau_*/10}, \qquad P_* \leq we^{-2\tau_*}.$$

At the exit $P_* = W_*$ because $V_* = 1$. Substitution of the lower duration bound proves both contractions in (4.14). □

Although the contraction of the weak wall in (4.14) approaches one as $\kappa \to \infty$, its loss has exactly the scale needed for the radius error. The following elementary estimate makes this relation precise.

Lemma 4.3 (Amplitude loss)**.** *Let $c, p > 0$, $\kappa \geq 2$, and $0 \leq b \leq e^{-c/\kappa}a$, where $a \geq 0$. Then*

$$\frac{a^p}{\kappa} \leq \frac{1 + pc/2}{pc}(a^p - b^p). \tag{4.16}$$

Consequently, if nonnegative amplitudes $a_0, \dots, a_n$ and intermediate values $b_0, \dots, b_{n-1}$ satisfy

$$a_{j+1} \leq b_j \leq e^{-c/\kappa_j}a_j, \qquad \kappa_j \geq 2 \quad (0 \leq j < n),$$

then

$$\sum_{j=0}^{n-1} \frac{a_j^p}{\kappa_j} \leq \frac{1 + pc/2}{pc}(a_0^p - a_n^p). \tag{4.17}$$

Proof. Put $v = pc/\kappa$. Raising the contraction inequality to the power p gives $b^p \leq e^{-v}a^p$. Since $(1 + v)e^{-v} \leq 1$,

$$va^p \leq (1 + v)(a^p - b^p) \leq (1 + pc/2)(a^p - b^p).$$

Divide by pc to obtain (4.16). For the finite sequence, use $b_j^p \geq a_{j+1}^p$ and sum. The differences telescope. □

For the large cores just constructed, both small walls, and therefore their sum, contract by at least $e^{-1/(2\kappa)}$: $\kappa/40 \geq 1/(2\kappa)$ for $\kappa \geq 6$. Thus (4.17) applies with $c = 1/2$ whenever the connections between cores do not increase the chosen amplitude. For $p = 1$ and $p = 1/2$ its constants are respectively $5/2$ and $9/2$. These are the two powers that control the accumulated phase error and the accumulated logarithmic derivative error. The next estimates concern derivatives of the actual return and the connections between successive cores; their amplitude dependence will allow the same summation.

4.2. **Variation of the core return.** The passage estimates control the position of an orbit. To control the measure carried by a family of orbits, we also need the derivative of its phase at the exit. Two features of the core equations make this possible. The strong transverse direction remains strongly contracting throughout the passage, and the coefficients that change the radius retain the small transverse amplitudes. We compare orbits at the same angle z and then account for the motion of the two section crossings.

Write the return from Proposition 4.2 as

$$\mathcal{R}(\kappa, u, w) = (R_*, U_*, W_*).$$

Since $V = 1$ at both crossings, the transverse coordinate $P = VW$ agrees with W there. Thus both the entrance and the exit are parametrized by the coordinates (R, U, P).

Proposition 4.4 (Derivative of the exit radius)**.** *The first return time and $\mathcal{R}$ are locally C^1 at every entrance with $\kappa \geq 6$ and $0 \leq u, w \leq \varepsilon_0/\kappa$. There is an absolute constant C_* such that, for every $v = (v_\kappa, v_u, v_w) \in \mathbb{R}^3$,*

$$|D\mathcal{R}_R(\kappa, u, w)v - v_\kappa| \leq C_* \left\{ \frac{|v_u|}{\kappa} + \frac{u + w}{\kappa}(|v_\kappa| + \kappa|v_u| + |v_w|) \right\}. \tag{4.18}$$

The assertion includes entrances at which either transverse wall is zero.

The dependence on the direction v is useful. Variation of the strong wall contributes at order $|v_u|/\kappa$, while all other contributions contain a transverse amplitude. We first establish this estimate along the passage, with the angle held fixed.

4.2.1. *Variation at a fixed angle.* Let $Y = (R, U, P)$ and denote the right-hand side of (4.4) by $F(z, Y)$. Along a reference orbit, let $\eta = \delta Y$ and $\zeta = \delta z$ be variations at fixed original time. Set

$$\xi = \eta - \zeta F(z, Y). \tag{4.19}$$

This is the variation after the change in the angle has been removed. It satisfies

$$\frac{d\xi}{dz} = J(z, Y)\xi, \qquad J = D_Y F. \tag{4.20}$$

Indeed, with $q = dz/dt = ARd > 0$, the original-time equations are $dY/dt = qF$ and $dz/dt = q$. Their variations give

$$\frac{d\eta}{dt} = \delta q\, F + q(F_z\zeta + J\eta), \qquad \frac{d\zeta}{dt} = \delta q, \qquad \frac{dF}{dt} = q(F_z + JF).$$

Substitution into (4.19) yields $d\xi/dt = qJ\xi$, proving (4.20). The same calculation shows that adding a multiple of the flow vector to a variation leaves ξ unchanged. This will allow us to recover the derivative at the moving exit.

For a fixed entrance radius κ, use the weighted variation

$$X = S_\kappa \xi = (\xi_R, \kappa\xi_U, \xi_P), \qquad B = S_\kappa J S_\kappa^{-1}. \tag{4.21}$$

Here κ is the radius of the reference entrance and is held constant in the linear equation $X' = BX$. This weighting balances the influence of the strong wall on the radius against its contraction. All vector norms in this subsection are supremum norms.

Lemma 4.5 (Coefficients of the variational equation)**.** *Throughout the region* (4.5)*, put* $m = \varepsilon_0/\kappa$*. The radius row satisfies*

$$|B_{RR}| \le 8m, \qquad \left|B_{RU} + \frac{V}{\kappa}\right| \le \frac{8m}{\kappa}, \qquad |B_{RP}| \le \frac{8m}{V}. \tag{4.22}$$

The transverse rows satisfy

$$\begin{aligned} &B_{UU} \le -\frac{\kappa}{4}, && |B_{UR}| \le 2\kappa U, && |B_{UP}| \le 4U, \\ &B_{PP} \le -\frac{1}{\kappa}, && |B_{PR}| \le \frac{8P}{\kappa^2}, && |B_{PU}| \le \frac{8P}{\kappa}. \end{aligned} \tag{4.23}$$

The following refinement keeps the amplitudes in the radius row:

$$\begin{aligned} &|B_{RR}| \le 8\left(U + \frac{W^2}{\kappa^2}\right), \\ &\left|B_{RU} + \frac{V}{\kappa}\right| \le \frac{8(U+W)}{\kappa}, \qquad |B_{RP}| \le \frac{8}{V}\left(U + \frac{W}{\kappa}\right). \end{aligned} \tag{4.24}$$

Proof. We give the coefficient calculations, since the negative diagonal terms and the powers of the small walls will both be used. At a fixed angle put $s = \operatorname{sech} z$, $c = -\tanh z$ and $\delta = U - W$. Then $V = Rs$, $W = P/V$, and

$$a = -\frac{R}{2} - c + sW + \frac{U^2 - W^2}{2R}, \qquad g = -C + \frac{c\delta^2}{R}, \qquad F = \left(\frac{g}{d}, \frac{Ua}{d}, -\frac{2P(1+C)}{Rd}\right).$$

In particular $W_R = -W/R$ and $W_P = 1/V$. Differentiation gives

$$\begin{aligned} &C_R = U(s - W/R), && C_U = V - 2U + W, && C_P = U/V, \\ &d_R = 2\delta(U - 2W)/R^3, && d_U = -2\delta/R^2, && d_P = 2\delta/(VR^2), \\ &a_R = -\tfrac{1}{2} - sW/R + (3W^2 - U^2)/(2R^2), && a_U = U/R, && a_P = (1 - W/V)/R. \end{aligned}$$

For the radius numerator,

$$g_R = -U(s - W/R) + c\delta(3W - U)/R^2,$$
$$g_U = -V + 2U - W + 2c\delta/R, \qquad g_P = -U/V - 2c\delta/(RV).$$

We use $R \geq 5$, $1 \leq V \leq R$, $0 \leq U, W \leq m$, $Rm \leq 1/1000$, and (4.6). These give

$$0 \leq C \leq \frac{1}{100}, \quad |a| \leq R, \quad |g| \leq 2Rm, \quad |C_R| \leq U, \quad |C_U| \leq 2R,$$
$$|d_R| \leq \frac{m}{R^2}, \quad |d_U| \leq \frac{2m}{R^2}, \quad |d_P| \leq \frac{2m}{VR^2}, \quad |a_R| \leq \frac{3}{4}, \quad |a_P| \leq \frac{1}{R}. \tag{4.25}$$

For example $|\delta| \leq m$, $|U - 2W| \leq 2m$ and $4m/R \leq 1$ give the first bound for the derivatives of d. The remaining bounds follow directly from the displayed derivatives. In applying the quotient rule we shall use

$$\left|\frac{x}{d} - \frac{y}{d^2}\right| \leq \frac{3}{2}|x| + 2|y|, \qquad d \geq \frac{999}{1000}. \tag{4.26}$$

For the rough radius bounds, the numerator derivatives obey

$$|g_R| \leq 2m, \qquad |g_U + V| \leq 3m, \qquad |Vg_P| \leq 2m.$$

Also $V(1 - d) \leq m$, and each of $|gd_R|$, $|gd_U|$, $|Vgd_P|$ is at most $m/10$. The radius row consists of

$$B_{RR} = \frac{g_R}{d} - \frac{gd_R}{d^2}, \qquad B_{RU} = \frac{1}{\kappa}\left(\frac{g_U}{d} - \frac{gd_U}{d^2}\right), \qquad B_{RP} = \frac{g_P}{d} - \frac{gd_P}{d^2}.$$

For its middle entry, combine $g_U + Vd = g_U + V - V(1 - d)$ before using (4.26). This proves (4.22).

The two diagonal entries have the exact forms

$$B_{UU} = a\,\frac{R^2 + U^2 - W^2}{R^2d^2} + \frac{U^2}{Rd},$$
$$B_{PP} = -\frac{2(1 + C)}{R}\,\frac{R^2 - U^2 + W^2}{R^2d^2} - \frac{2UW}{Rd}. \tag{4.27}$$

Both fractions with denominator R^2d^2 are at least $99/100$. Since $a \leq -3\kappa/10$, $U^2/(Rd) \leq 1/100$, and $R \leq 101\kappa/100$, the asserted negative diagonal bounds follow. For the off-diagonal entries of the strong row, the quotient rule gives

$$B_{UR} = \kappa U\left(\frac{a_R}{d} - \frac{ad_R}{d^2}\right), \qquad B_{UP} = \kappa U\left(\frac{a_P}{d} - \frac{ad_P}{d^2}\right).$$

The two parenthesized expressions have absolute values at most 2 and $2/R$, respectively, by (4.25) and (4.26). Use $\kappa/R \leq 3/2$. For the weak row, write

$$B_{PR} = \frac{P}{R^2}\left\{\frac{2(1 + C) - 2RC_R}{d} + \frac{2(1 + C)Rd_R}{d^2}\right\},$$
$$B_{PU} = \frac{P}{\kappa}\left\{-\frac{2C_U}{Rd} + \frac{2(1 + C)d_U}{Rd^2}\right\}.$$

The numerator bounds for the two terms in braces are respectively $(3, 1/10)$ and $(4, 1/10)$. Applying (4.26) and $\kappa^2/R^2 \leq 3/2$ proves (4.23).

To retain the amplitudes in the radius row, use

$$|\delta|\,|U - 2W| \leq 4mU + 2W^2, \qquad |\delta|\,|3W - U| \leq 5mU + 3W^2.$$

These inequalities follow by replacing the absolute differences by $U + W$, $U + 2W$ and $U + 3W$, and then using $U^2, UW \leq mU$. They give the sharper bounds

$$|g_R| \leq 2U + 3W^2/R^2, \qquad |g_U + Vd| \leq 4(U + W), \qquad |Vg_P| \leq 2(U + W/R),$$
$$|gd_R| \leq (U + W^2/R^2)/10, \qquad |gd_U| \leq (U + W/R)/10, \quad |Vgd_P| \leq (U + W/R)/10.$$

For the middle numerator, $\delta^2 \le m(U+W)$ controls $V(1-d)$. For the products with derivatives of d, multiply the two preceding quadratic bounds by $|g| \le 2Rm$; in particular

$$|gd_R| \le \frac{16m^2}{R^2}U + 8m\frac{W^2}{R^2}, \qquad |gd_U| \le \frac{4m}{R}U + 4m\frac{W}{R}.$$

Their coefficients are at most $1/10$, and $Vd_P = -d_U$. One last application of (4.26), with $\kappa/R, \kappa^2/R^2 \le 3/2$, proves (4.24). □

The negative entries in (4.27) prevent a large growth factor. More precisely, the sum of the diagonal entry and the absolute values of the off-diagonal entries in each row obeys

$$\begin{aligned} &B_{RR} + |B_{RU}| + |B_{RP}| \le 2\operatorname{sech} z + 24m, \\ &|B_{UR}| + B_{UU} + |B_{UP}| \le 0, \\ &|B_{PR}| + |B_{PU}| + B_{PP} \le 0. \end{aligned} \tag{4.28}$$

For the first row use $V/\kappa \le 2\operatorname{sech} z$ and $V \ge 1$. The other two follow from $U, P \le m$ and (4.23).

Lemma 4.6 (Uniform growth of the weighted variation)**.** *On every interval $[\alpha, \beta]$ in the core region, every solution of* (4.20) *satisfies*

$$\|S_\kappa \xi(\beta)\|_\infty \le K \|S_\kappa \xi(\alpha)\|_\infty, \qquad K = \exp(2\pi + 48\varepsilon_0). \tag{4.29}$$

Proof. If the signed row sums of a matrix B are at most ρ, then for sufficiently small $h \ge 0$,

$$\|X + hBX\|_\infty \le (1 + h\rho)\|X\|_\infty.$$

Indeed $1 + hB_{ii} \ge 0$ for every i, so the diagonal term can be estimated with its sign intact. Applied to $X' = BX$, this inequality and the first-order remainder give the upper right slope bound $D^+\|X\|_\infty \le \rho\|X\|_\infty$. Multiplication by $\exp(-\int \rho)$ and scalar comparison imply

$$\|X(\beta)\|_\infty \le \|X(\alpha)\|_\infty \exp\left(\int_\alpha^\beta \rho(z)\, dz\right).$$

Take $\rho(z) = 2\operatorname{sech} z + 24m$ in (4.28). A primitive is

$$H(z) = 2\arctan(\sinh z) + 24mz.$$

By (4.8),

$$H(\beta) - H(\alpha) \le 2\pi + 48m\log(4\kappa) \le 2\pi + 48\varepsilon_0.$$

The last inequality uses $\log(4\kappa) \le \kappa$ for $\kappa \ge 6$, which follows from $e^\kappa \ge 1 + \kappa + \kappa^2/2 \ge 4\kappa$. □

4.2.2. *Integrated decay and the radius component.* The uniform bound controls the entire variation. The radius estimate requires more information about the strong component, whose forcing decays together with U. Let α be the entrance angle, so that $\cosh\alpha = \kappa$, and suppose $\|X(z)\|_\infty \le Q$ on an initial segment $[\alpha, \beta]$. Write $x = X_U$ and $x_0 = x(\alpha)$. By (4.23),

$$x' = B_{UU}x + h, \qquad B_{UU} \le -\kappa/4, \qquad |h| \le 3\kappa UQ.$$

Multiply this equation by $e^{\kappa(z-\alpha)/4}$. The new damping coefficient is $B_{UU} + \kappa/4 \le 0$, and the absolute value of its forcing is at most $3\kappa uQ$, since $U(z) \le ue^{-\kappa(z-\alpha)/4}$. The scalar comparison argument used above therefore gives

$$|x(z)| \le e^{-\kappa(z-\alpha)/4}(|x_0| + 3\kappa uQ(z-\alpha)). \tag{4.30}$$

There is no division by U in this estimate.

Two integrals of (4.30) will be used:

$$\int_\alpha^\beta \frac{|x(z)|}{\cosh z}\,dz \le \frac{12|x_0|+432uQ}{\kappa^2},$$
$$\int_\alpha^\beta |x(z)|\,dz \le \frac{4|x_0|+48uQ}{\kappa}. \tag{4.31}$$

For the first, use $\operatorname{sech}(\alpha+s)\le e^s/\kappa$ and $\kappa/4-1\ge\kappa/12$. Both estimates then follow from

$$\int_0^L e^{-cs}(A+Bs)\,ds \le \frac{A}{c}+\frac{B}{c^2} \qquad (A,B,L\ge 0,\ c>0).$$

The same endpoint estimate for $\cosh^2$ used in Lemma 4.1 gives, with $\gamma=(101/99)^2$,

$$\int_\alpha^\beta U\,dz \le \frac{4u}{\kappa}, \qquad \int_\alpha^\beta \frac{1}{V^2}\,dz \le 2\gamma,$$
$$\int_\alpha^\beta \frac{W}{\kappa V}\,dz \le \frac{2\gamma w}{\kappa}, \qquad \int_\alpha^\beta W^2\,dz \le 2\gamma w^2. \tag{4.32}$$

Here $P(z)\le w$, $R\ge 99\kappa/100$, and the two endpoint values of $\cosh z$ are at most $101\kappa/100$. These facts prove the inverse-wall integral; the last two bounds follow from $W=P/V$.

Lemma 4.7 (Drift of a radial variation)**.** *Under the preceding assumptions,*

$$|\xi_R(\beta)-\xi_R(\alpha)| \le 256\left\{\frac{|x_0|}{\kappa^2}+\frac{(u+w)Q}{\kappa}\right\}. \tag{4.33}$$

Proof. The refined radius row (4.24) yields

$$|\xi_R'| \le \left(16U+\frac{8W^2}{\kappa^2}+\frac{8W}{\kappa V}\right)Q+\frac{2|x|}{\cosh z}+\frac{|x|}{\kappa}. \tag{4.34}$$

Indeed $V/\kappa\le 2\operatorname{sech} z$, $1/V\le 1$, and $8(U+W)\le 1$. Integrating with (4.31) and (4.32) bounds the result by

$$\frac{28|x_0|}{\kappa^2}+\left(\frac{64}{\kappa}+\frac{912}{\kappa^2}\right)uQ+\frac{16\gamma w^2Q}{\kappa^2}+\frac{16\gamma wQ}{\kappa}.$$

Use $\kappa\ge 6$, $w\le\kappa$ and $\gamma<2$ to obtain (4.33). □

4.2.3. *The two moving crossings.* We now identify the derivatives just estimated with derivatives of the actual first return. The vector field and coordinate changes are smooth on a neighborhood of the compact crossing segment. At the exit $dV/dt<0$, so the implicit function theorem gives a locally C^1 crossing time near t_*. This crossing remains the first positive return. To see this, choose small intervals about 0 and t_* on which the corresponding crossings are unique by transversality. On the intervening compact interval the reference solution has $V>1$, uniformly separated from 1. Continuous dependence preserves this separation for nearby entrance data. At the entrance the crossing time stays exactly zero because the incoming section is $V=1$. Thus no earlier positive crossing can appear. The same compactness argument preserves the flow domain and $A>0$ up to the moving return time. It also supplies a local C^1 extension when an entrance wall is zero.

For the entrance parametrization (4.13), $Y=(\kappa,u,w)$ and $z=\alpha=-\operatorname{arcosh}\kappa$. Consequently the initial value of the variation at fixed angle is

$$\xi(\alpha)=v+\frac{v_\kappa}{\sqrt{\kappa^2-1}}F(\alpha,\kappa,u,w). \tag{4.35}$$

At the entrance, the equations imply

$$|F_R|\le 2(u+w^2/\kappa), \qquad |F_U|\le 2\kappa u, \qquad |F_P|\le 4w/\kappa.$$

The first bound follows from (4.11) with $V = 1$; the other two follow from $|a| \le \kappa$, $C \le 1/100$ and $d \ge 999/1000$. Since $\sqrt{\kappa^2 - 1} \ge \kappa/2$,

$$|\xi_R(\alpha) - v_\kappa| \le 4(u/\kappa + w^2/\kappa^2)|v_\kappa|,$$
$$|\kappa\xi_U(\alpha)| \le \kappa|v_u| + 4\kappa u|v_\kappa|,$$
$$|\xi_P(\alpha) - v_w| \le 8w|v_\kappa|/\kappa^2. \tag{4.36}$$

In particular, if

$$Q_0 = |v_\kappa| + \kappa|v_u| + |v_w|,$$

then $\|S_\kappa\xi(\alpha)\|_\infty \le 2Q_0$. Lemma 4.6 therefore gives the uniform bound $Q = 2KQ_0$ for use in (4.33).

At the exit put $r = R_*$, $c_* = -\tanh z_*$ and $f_* = F_R(z_*, Y_*)$. A variation tangent to $V = 1$ satisfies

$$\delta R + rc_*\delta z = 0, \qquad \xi_R = \delta R - f_*\delta z.$$

The first identity is the differential of $V = R/\cosh z$ on that section. Eliminating δz gives

$$D\mathcal{R}_R v = \sigma\,\xi_R(z_*), \qquad \sigma = \frac{c_*}{c_* + f_*/r}. \tag{4.37}$$

The change in the exit time adds a multiple of the flow vector and hence does not change the projected variation in (4.19).

At this crossing $c_* \le -1/2$, $C \ge 0$ and $\tanh z_* > 0$. Thus $f_* \le 0$, and

$$|f_*| \le 2(U_* + W_*^2/r), \qquad 0 < \sigma \le 1, \qquad |\sigma - 1| \le \frac{2|f_*|}{r} \le 8(u/\kappa + w^2/\kappa^2).$$

The last estimate uses $U_* \le u$, $W_* \le w$ and $r \ge 99\kappa/100$. The exit projection therefore preserves the amplitude dependence of the interior estimate.

Proof of Proposition 4.4. Local C^1 regularity and first-return persistence were proved above. Since $0 < \sigma \le 1$, equation (4.37) gives

$$\begin{aligned}|D\mathcal{R}_R v - v_\kappa| &\le |\xi_R(z_*) - \xi_R(\alpha)| + |\xi_R(\alpha) - v_\kappa| + |\sigma - 1|\,|v_\kappa| \\ &\le 256\left\{\frac{|\kappa\xi_U(\alpha)|}{\kappa^2} + \frac{(u+w)2KQ_0}{\kappa}\right\} + 12(u/\kappa + w^2/\kappa^2)|v_\kappa|.\end{aligned}$$

Use (4.36), $|v_\kappa| \le Q_0$ and $w^2/\kappa^2 \le w/\kappa$. The right-hand side is at most

$$\frac{256|v_u|}{\kappa} + (1040 + 512K)\frac{(u+w)Q_0}{\kappa}.$$

Taking $C_* = 1040 + 512K$ proves the proposition. □

The preceding estimate becomes small along families whose tangent directions satisfy

$$v_\kappa = 1, \qquad |v_u| \le \sqrt{u}, \qquad |v_w| \le \sqrt{w}. \tag{4.38}$$

This choice permits unequal wall sizes and allows either wall to vanish.

Corollary 4.8. *For every entrance in Proposition 4.4 and every direction satisfying* (4.38)*,*

$$|D\mathcal{R}_R v - 1| \le C_1\frac{\sqrt{u+w}}{\kappa}, \qquad C_1 = 5C_*. \tag{4.39}$$

There is an absolute $\varepsilon_1 \in (0, \varepsilon_0]$ such that if $0 \le u, w \le \varepsilon_1/\kappa$, then

$$\frac{3}{4} \le D\mathcal{R}_R v \le \frac{5}{4}, \qquad |\log(D\mathcal{R}_R v)| \le 2C_1\frac{\sqrt{u+w}}{\kappa}. \tag{4.40}$$

Proof. Put $b = u + w$. Then $b \le 1$ and $\kappa b \le 2\varepsilon_0 \le 2$. Hence

$$b(1 + \kappa|v_u| + |v_w|) \le 4\sqrt{b}, \qquad |v_u| \le \sqrt{b}.$$

Substitution into (4.18) proves (4.39). Choose $\varepsilon_1 \le \min\{\varepsilon_0, (4C_1)^{-2}\}$. Since $b \le \varepsilon_1$, the error in (4.39) is at most $1/4$. Finally, if $|t - 1| \le e \le 1/4$, then $t \ge 1/2$ and $1/t \le 1 + 2e$. The elementary inequalities $1 - 1/t \le \log t \le t - 1$ give $|\log t| \le 2e$. Apply this with $t = D\mathcal{R}_R v$ and $e = C_1\sqrt{b}/\kappa$. □

The two scales now agree with the losses from Lemma 4.3: phase displacement is bounded by a constant times $(u + w)/\kappa$, and logarithmic phase distortion by a constant times $\sqrt{u + w}/\kappa$. Their sums over many cores can therefore be controlled by the decrease in the corresponding powers of the transverse amplitude. To apply this observation along successive returns, the transverse derivative must also preserve the directions (4.38) after normalization by the new radial component. This requires the derivative estimates for the two wall multipliers.

4.3. **Transverse contraction and invariant directions.** The weak transverse wall contracts by a factor approaching one as $\kappa \to \infty$. Its derivative must therefore be controlled on the same scale as this small loss. The variation of the strong wall U enters the weak-wall equation. Integrating the decay of this variation supplies an additional inverse power of the radius in the weak multiplier estimate.

4.3.1. *Positive multipliers and their derivatives.* For the angle system (4.4), define

$$L_U = \frac{a}{d}, \qquad L_P = -\frac{2(1 + C)}{Rd}, \qquad a = -\frac{R}{2} + \tanh z + \frac{W}{\cosh z} + \frac{U^2 - W^2}{2R}. \tag{4.41}$$

Thus $U' = L_U U$ and $P' = L_P P$, where primes denote angle derivatives. For an entrance $Y_0 = (\kappa, u, w)$, let $z_-(Y_0)$ and $z_+(Y_0)$ be its entrance and first-exit angles. Write $Y_* = \mathcal{R}(Y_0)$ and set

$$\lambda_j(Y_0) = \exp\left(\int_{z_-(Y_0)}^{z_+(Y_0)} L_j(z, Y(z; Y_0))\, dz\right), \qquad j \in \{U, P\}. \tag{4.42}$$

The scalar equations give

$$U_* = u\lambda_U, \qquad W_* = w\lambda_P. \tag{4.43}$$

The second identity uses $P = W$ at both crossings. The factors are strictly positive, including at entrances with a zero wall.

These factors and their logarithms are locally C^1. To see this directly in the original time, write the integral in (4.42) as

$$\int_0^{t_*(Y_0)} q(t; Y_0) L_j(z(t; Y_0), Y(t; Y_0))\, dt, \qquad q = \frac{dz}{dt} = ARd.$$

The integrand is jointly C^1 on a neighborhood of the compact crossing segment, and t_* is locally C^1 by Proposition 4.4. Differentiation of this moving-endpoint integral proves the claim. Only R, V and d occur as denominators in the rates, and all three stay positive.

The proof of (4.9) bounds the coefficients themselves, even when the corresponding wall is zero. In projective time they are at most $-\kappa^2/10$ and -2. Integrating these inequalities and using $\tau_* \ge 1/(4\kappa)$ gives

$$0 < \lambda_U \le e^{-\kappa/40}, \qquad 0 < \lambda_P \le e^{-1/(2\kappa)}. \tag{4.44}$$

Proposition 4.9 (Logarithmic derivatives of the wall multipliers)**.** *For $\kappa \geq 6$ and $0 \leq u, w \leq \varepsilon_0/\kappa$, put $Q_0 = |v_\kappa| + \kappa|v_u| + |v_w|$. Then*

$$|D \log \lambda_U(Y_0)v| \leq A_U \log(4\kappa) Q_0,$$

$$|D \log \lambda_P(Y_0)v| \leq A_P \frac{\log(4\kappa)}{\kappa^2} Q_0, \tag{4.45}$$

where $A_U = 20K + 4$, $A_P = 118K + 8$, and K is the constant in (4.29)*.*

Proof. At fixed angle, the two rates satisfy the following bounds on the core region. We use them first to estimate the multiplier derivatives and verify their coefficients at the end of the proof:

$$|L_U| \leq 2\kappa, \qquad |L_P| \leq 4/\kappa,$$

	$\lvert\partial_R\rvert$	$\lvert\partial_U\rvert$	$\lvert\partial_P\rvert$
L_U	2	$4\varepsilon_0/\kappa^2$	$4/\kappa$
L_P	$8/\kappa^2$	8	$8\varepsilon_0/\kappa^2$

(4.46)

The entries in the array are upper bounds for the indicated derivatives.

Let ξ be the variation at fixed angle corresponding to v, and put $X = S_\kappa \xi$, $x = X_U$ and $Q = 2KQ_0$. The preceding subsection gives $\|X\|_\infty \leq Q$ throughout the passage. Consequently

$$|D_Y L_U \xi| \leq 3Q, \qquad |D_Y L_P \xi| \leq \frac{9Q}{\kappa^2} + \frac{8}{\kappa}|x|. \tag{4.47}$$

The strong component is kept separately in the second estimate. By (4.31),

$$\frac{8}{\kappa} \int_{z_-}^{z_+} |x|\, dz \leq \frac{32|x(z_-)| + 384uQ}{\kappa^2} \leq \frac{33Q}{\kappa^2}.$$

The last step uses $|x(z_-)| \leq Q$ and $u \leq 1/60000$. Since $z_+ - z_- \leq 2\log(4\kappa)$, integration of (4.47) gives

$$|I_U| \leq 6 \log(4\kappa) Q, \qquad |I_P| \leq 51 \frac{\log(4\kappa)}{\kappa^2} Q, \quad I_j = \int_{z_-}^{z_+} D_Y L_j \xi \, dz. \tag{4.48}$$

Here $1 \leq \log(4\kappa)$ absorbs the constant term in the weak estimate.

It remains to include the moving angles. The exact derivative formula is

$$D \log \lambda_j(Y_0)v = I_j + L_j(z_+, Y_*)\, Dz_+(Y_0)v - L_j(z_-, Y_0)\, Dz_-(Y_0)v. \tag{4.49}$$

One can verify this formula before changing time. Let $\zeta = \delta z$ and $\eta = \delta Y$ be variations at fixed original time, so that $\xi = \eta - \zeta F$. The chain rule yields

$$\delta(qL_j) = qD_Y L_j \xi + \frac{d}{dt}(L_j \zeta).$$

Integrate this identity from 0 to t_*. The derivative of the upper time endpoint contributes $q_* L_j(z_+, Y_*) Dt_* v$; together with $L_j(z_+, Y_*)\zeta(t_*)$ it is exactly $L_j(z_+, Y_*) Dz_+ v$. The lower boundary term is $-L_j(z_-, Y_0) Dz_- v$. This proves (4.49).

The entrance angle derivative and the exit tangency calculation give

$$|Dz_- v| \leq \frac{2Q_0}{\kappa}, \qquad |Dz_+ v| \leq \frac{2Q}{\kappa}. \tag{4.50}$$

For the second bound, the notation in (4.37) gives $Dz_+ v = -\xi_R(z_+)/(rc_* + f_*)$. Since $rc_* = M_* - 1 \leq -3\kappa/4$ and $f_* \leq 0$, its denominator has absolute value at least $3\kappa/4$. Thus the two boundary contributions have total absolute value at most $4Q + 4Q_0$ for L_U, and $(8Q + 8Q_0)/\kappa^2$ for L_P. Combine these bounds with (4.48), use $1 \leq \log(4\kappa)$, and substitute $Q = 2KQ_0$. The resulting constants are $20K + 4$ and $118K + 8$.

Verification of the coefficient bounds. Put $m = \varepsilon_0/\kappa$. In terms of the current radius R, the bounds underlying (4.46) are

	$\lvert\partial_R\rvert$	$\lvert\partial_U\rvert$	$\lvert\partial_P\rvert$
L_U	2	$7m/(2R)$	$2/R$
L_P	$5/R^2$	6	$7m/(2R)$.

The quotient rule gives $(L_U)_i = a_i/d - ad_i/d^2$. For the weak rate, set $n = -2(1+C)/R$. Then $(L_P)_i = n_i/d - nd_i/d^2$, with

$$n_R = \frac{2(1+C)}{R^2} - \frac{2C_R}{R}, \qquad n_U = -\frac{2C_U}{R}, \qquad n_P = -\frac{2C_P}{R}.$$

The auxiliary bounds (4.25) for a, C, d and their derivatives give

$$\begin{aligned}(|a_R|, |a_U|, |a_P|) &\le (3/4, m/R, 1/R),\\ (|ad_R|, |ad_U|, |ad_P|) &\le (m/R, 2m/R, 2m/R),\\ (|n_R|, |n_U|, |n_P|) &\le (3/R^2, 4, 2m/R),\\ (|nd_R|, |nd_U|, |nd_P|) &\le (1/R^2, 1, m/R),\end{aligned}$$

where vector inequalities are componentwise. Applying

$$\left|\frac{x}{d} - \frac{y}{d^2}\right| \le \frac{11}{10}(|x| + |y|) \qquad (d \ge 999/1000)$$

proves the radius-dependent bounds. Converting them using $\kappa/R \le 10/9$ and $\kappa^2/R^2 \le 3/2$ proves (4.46). The rate bounds themselves follow from $|a| \le R$ and $0 \le C \le 1/100$. □

4.3.2. *An invariant cone.* For $Y_0 = (\kappa, u, w)$ define the forward radial cone

$$\mathcal{C}(Y_0) = \left\{v : \ v_\kappa \ge 0, \ |v_u| \le \sqrt{u}\, v_\kappa, \ |v_w| \le \sqrt{w}\, v_\kappa\right\}. \tag{4.51}$$

Every nonzero vector in this cone has $v_\kappa > 0$. We may therefore normalize it by $v_\kappa = 1$, as in (4.38). At the exit the cone is defined in the same way using (R_*, U_*, W_*).

Proposition 4.10 (Preservation with a quantitative margin)**.** *There is an absolute $\varepsilon_c \in (0, \varepsilon_0]$ such that for $\kappa \ge 6$ and $0 \le u, w \le \varepsilon_c/\kappa$, the derivative of the core return maps $\mathcal{C}(Y_0)$ into $\mathcal{C}(\mathcal{R}(Y_0))$. More precisely, if $v \in \mathcal{C}(Y_0)$ and $v_\kappa = 1$, write $d = D\mathcal{R}(Y_0)v = (d_R, d_U, d_W)$. Then*

$$\begin{aligned} &d_R > 0, && |d_R - 1| \le \frac{1}{32\kappa},\\ &|d_U| \le \left(1 - \frac{1}{16\kappa}\right)\sqrt{U_*}\, d_R, && |d_W| \le \left(1 - \frac{1}{16\kappa}\right)\sqrt{W_*}\, d_R. \end{aligned} \tag{4.52}$$

Proof. Write $v = (1, a, b)$, $Q_0 = 1 + \kappa|a| + |b|$ and $\ell_j = D \log \lambda_j(Y_0)v$. Differentiating (4.43) gives

$$d_U = \lambda_U(a + u\ell_U), \qquad d_W = \lambda_P(b + w\ell_P). \tag{4.53}$$

If $0 \le u, w \le \varepsilon/\kappa$ and $\varepsilon \le 1$, the bounds on a, b imply

$$\sqrt{u}\, Q_0 \le 3\sqrt{\varepsilon}, \qquad \sqrt{w}\, Q_0 \le 3\sqrt{\varepsilon}, \qquad \sqrt{u} + (u+w)Q_0 \le 7\sqrt{\varepsilon}. \tag{4.54}$$

For example, $\sqrt{u}\, Q_0 \le \sqrt{u} + \kappa u + \sqrt{uw}$, where each term is at most $\sqrt{\varepsilon}$. For the weak component use $\kappa\sqrt{uw} \le \varepsilon$ and $w \le \varepsilon$. Multiplying these first two estimates by $\sqrt{u}$ and $\sqrt{w}$ proves the last one, since $\varepsilon \le \sqrt{\varepsilon}$.

Choose $\varepsilon_c > 0$, at most ε_0, so small that

$$7C_*\sqrt{\varepsilon_c} \le \frac{1}{32}, \qquad 38400A_U\sqrt{\varepsilon_c} \le \frac{1}{32}, \qquad 3A_P\sqrt{\varepsilon_c} \le \frac{1}{32}.$$

All constants here are independent of κ. Set $e = 1/(32\kappa)$ and $L = \log(4\kappa)$. Proposition 4.4 and (4.54) imply $|d_R - 1| \le e$, so $d_R \ge 1 - e > 0$.

The square roots of both multipliers are bounded by $1-4e$. Indeed,

$$\sqrt{\lambda_U} \le e^{-\kappa/80} \le e^{-1/(4\kappa)}, \qquad \sqrt{\lambda_P} \le e^{-1/(4\kappa)} \le 1-4e.$$

The last inequality follows from $e^{-x} \le (1+x)^{-1} \le 1 - x/2$ for $0 \le x \le 1$. The strong multiplier also satisfies

$$\kappa^2 e^{-\kappa/80} \le 12800, \qquad e^{-\kappa/80} L \le \frac{12800}{\kappa}. \tag{4.55}$$

The first estimate follows from $e^x \ge x^2/2$ with $x = \kappa/80$; the second uses $L \le \kappa$.

Proposition 4.9 and (4.54) now give

$$\sqrt{\lambda_U}\sqrt{u}\,|\ell_U| \le 3A_U\sqrt{\varepsilon_c}\,e^{-\kappa/80}L \le \frac{38400A_U\sqrt{\varepsilon_c}}{\kappa} \le e,$$
$$\sqrt{\lambda_P}\sqrt{w}\,|\ell_P| \le \frac{3A_P L\sqrt{\varepsilon_c}}{\kappa^2} \le \frac{3A_P\sqrt{\varepsilon_c}}{\kappa} \le e.$$

For either wall, the product derivative in (4.53) has the bound

$$|\lambda(a_0 + j\ell)| \le \sqrt{j\lambda}\left(\sqrt{\lambda} + \sqrt{\lambda}\sqrt{j}\,|\ell|\right) \quad \text{if } |a_0| \le \sqrt{j}, j \ge 0. \tag{4.56}$$

This follows by the triangle inequality and $j = (\sqrt{j})^2$; it includes $j = 0$. The parenthesis is at most $1 - 3e$. Finally,

$$1 - 3e \le (1-2e)(1-e) \le (1-2e)d_R.$$

Use $1 - 2e = 1 - 1/(16\kappa)$ and (4.43) to obtain (4.52). Homogeneity gives the assertion for every vector of the cone. □

The margin in (4.52) tends to zero with $1/\kappa$, just as the weak-wall loss does. It is nevertheless uniform relative to that loss: the same entrance threshold works for arbitrarily large radii.

For a family almost tangent to the radial direction, the following refinement retains its actual slope and measures the slope generated from an initially radial family.

Corollary 4.11. *Under the entrance assumptions of Proposition 4.10, let* $0 \le \theta \le 1$ *and*

$$v = (1, a, b), \qquad |a| \le \theta\sqrt{u}, \qquad |b| \le \theta\sqrt{w}.$$

Put

$$S = \frac{\sqrt{u+w}\log(4\kappa)}{\kappa^2}, \qquad C_2 = 2(1 + 12800A_U + 2A_P), \qquad d = D\mathcal{R}(Y_0)v.$$

Then $d_R \ge 3/4$, *the radial estimate* (4.39) *holds, and*

$$|d_U| \le C_2(\theta + S)\sqrt{U_*}\,d_R, \qquad |d_W| \le C_2(\theta + S)\sqrt{W_*}\,d_R. \tag{4.57}$$

Proof. The direction belongs to the cone, so the radial conclusions follow from Proposition 4.10 and Corollary 4.8. Write $\varepsilon = \varepsilon_c$, $L = \log(4\kappa)$, $E = e^{-\kappa/80}$ and $Q_0 = 1 + \kappa|a| + |b|$. Keeping θ in the calculation of (4.54) gives

$$\sqrt{u}\,Q_0 \le \sqrt{u} + 2\varepsilon\theta, \qquad \sqrt{w}\,Q_0 \le \sqrt{w} + 2\varepsilon\theta.$$

The logarithmic derivative estimates therefore imply

$$\sqrt{u}\,|\ell_U| \le A_U L(\sqrt{u} + 2\varepsilon\theta), \qquad \sqrt{w}\,|\ell_P| \le \frac{A_P L}{\kappa^2}(\sqrt{w} + 2\varepsilon\theta).$$

By (4.55),

$$EL\sqrt{u} \le 12800S, \qquad 2\varepsilon EL \le 12800,$$
$$L\sqrt{w}/\kappa^2 \le S, \qquad 2\varepsilon L/\kappa^2 \le 2.$$

For the second estimate in the first row, use $2\varepsilon L \le 2\kappa \le \kappa^2$. Together with $\sqrt{\lambda_U} \le E \le 1$ and $\sqrt{\lambda_P} \le 1$, these bounds yield

$$\sqrt{\lambda_U}\theta + \sqrt{\lambda_U}\sqrt{u}\,|\ell_U| \le (1 + 12800A_U)\theta + 12800A_U S,$$
$$\sqrt{\lambda_P}\theta + \sqrt{\lambda_P}\sqrt{w}\,|\ell_P| \le (1 + 2A_P)\theta + A_P S.$$

Both right-hand sides are at most $(1+12800A_U+2A_P)(\theta+S)$. The product estimate (4.56), with its first $\sqrt{\lambda}$ term replaced by $\theta\sqrt{\lambda}$, then bounds $|d_U|$ and $|d_W|$ by this factor times $\sqrt{U_*}$ and $\sqrt{W_*}$, respectively. Since $2d_R \ge 1$, this is (4.57). □

For positive walls, division by $\sqrt{U_*}d_R$ and $\sqrt{W_*}d_R$ expresses (4.57) as a bound on the normalized slopes of the image family. In particular a radial tangent, for which $\theta = 0$, acquires a slope of order $\sqrt{u+w}\log(4\kappa)/\kappa^2$.

4.4. **Uniform core returns.** The large-radius estimates control cores arbitrarily close to a Taub point. The remaining radii form a compact interval, where the exact single-wall orbits provide a family of transverse crossings. Their wall multipliers are strictly less than one. We first preserve these properties in a common neighborhood, and then combine the two ranges in one return estimate.

4.4.1. *The single-wall family and its neighborhood.* For $\kappa > 1$, put $h = \operatorname{arcosh}(\kappa)/\kappa$. With projective time centered at the maximum of V, the single-wall orbit from Section 4.1 is

$$(M, U, V, W)(\tau) = \left(1 - \kappa\tanh(\kappa\tau),\, 0,\, \frac{\kappa}{\cosh(\kappa\tau)},\, 0\right). \tag{4.58}$$

Its two crossings of $V = 1$ are at $\tau = -h$ and $\tau = h$, and $V > 1$ between them. The projective derivatives of V there are $\sqrt{\kappa^2-1}$ and $-\sqrt{\kappa^2-1}$. Along this orbit,

$$D = \kappa^2 + 4 - 2\kappa\tanh(\kappa\tau) > 0, \qquad \frac{dt}{d\tau} = \frac{D}{6}.$$

Thus these are also transverse crossings for the original flow. The original time elapsed between them is

$$T_0(\kappa) = \frac{(\kappa^2+4)\operatorname{arcosh}\kappa}{3\kappa}. \tag{4.59}$$

Indeed, a primitive of $D/6$ is $(\kappa^2+4)\tau/6 - \log\cosh(\kappa\tau)/3$, and the logarithmic terms cancel at $\pm h$.

For an entrance $Y = (\kappa, u, w)$, use the same embedding

$$(M, U, V, W) = (1 + \sqrt{\kappa^2-1}, u, 1, w)$$

as before. Denote its first positive return time to $V = 1$ by $T(Y)$, and its returned radius and walls by $\mathcal{R}(Y) = (R_*, U_*, W_*)$. Equivalently, $T(Y)$ is the infimum of the positive times in the maximal flow domain at which $V = 1$; in all domains considered below this infimum is attained.

Lemma 4.12 (A common neighborhood of the bounded cores)**.** *There are $\delta > 0$ and $B > 0$ such that, for*

$$2 \le \kappa \le 6, \qquad |u|, |w| \le \delta, \tag{4.60}$$

the first return exists, the projective chart remains regular on $[0, T(Y)]$, and

$$V(t) > 1 \quad (0 < t < T(Y)), \qquad \frac{dV}{dt}(T(Y)) < 0.$$

Both T and $\mathcal{R}$ are C^2 on a neighborhood of this box, and their first and second derivatives are bounded there by B after possibly decreasing δ. All operator norms in this section use the maximum norm on $\mathbb{R}^3$.

Proof. Fix first $\kappa > 1$ and $Z = (\kappa, 0, 0)$. The explicit orbit (4.58) lies in the projective chart throughout its finite crossing segment. Smooth dependence of the original flow and the implicit function theorem at its downward crossing give a local C^2 exit time near Z.

This exit is the first return. At the entrance, transversality makes $t = 0$ the only nearby crossing for all nearby initial points of the section. At the exit, transversality makes the constructed event the only nearby crossing. On a closed interval between these two event neighborhoods, the reference orbit has $V > 1$, with a positive minimum gap from the section. Continuity preserves this gap for nearby entrances. These three intervals exclude an earlier return. The sign of $V - 1$ is positive throughout the intervening open interval by continuity, and the exit derivative remains negative. Compactness of the reference segment also preserves the flow domain and $A > 0$ up to the moving exit.

Consequently T agrees locally with the implicit exit time, and $\mathcal{R}$ is C^2 by composition with the flow and the radius coordinates. These local functions agree wherever their neighborhoods overlap, since each is the first positive return. The segment

$$\{(\kappa, 0, 0) : 2 \le \kappa \le 6\}$$

is compact. A closed neighborhood of sufficiently small positive width lies in the union of the regular neighborhoods just constructed. The first and second derivatives of T and $\mathcal{R}$ are continuous there and hence bounded. Choose a smaller box (4.60) inside this neighborhood. This proves both the uniform bounds and the asserted regularity around the box. □

The transverse derivatives on the reference segment can be computed without solving the full variational equation. The equations for the two transverse components of a variation are

$$\dot{\xi}_U = \left(-\frac{\kappa^2}{2} + \kappa \tanh(\kappa\tau)\right)\xi_U, \qquad \dot{\xi}_W = (-2 + \kappa \tanh(\kappa\tau))\xi_W.$$

For a coefficient $-r + \kappa \tanh(\kappa\tau)$, a positive fundamental solution is $e^{-r\tau}\cosh(\kappa\tau)$. Its ratio between h and $-h$ is e^{-2rh}. A change of time in a variation adds a multiple of the vector field; on the single-wall orbit that field has zero U and W components. The exit-time correction has the same property. Therefore the actual return derivative satisfies

$$\begin{aligned} D\mathcal{R}_U(Z)v &= m_U(\kappa)v_u, & m_U(\kappa) &= e^{-\kappa \operatorname{arcosh}\kappa}, \\ D\mathcal{R}_W(Z)v &= m_W(\kappa)v_w, & m_W(\kappa) &= e^{-4\operatorname{arcosh}(\kappa)/\kappa}. \end{aligned} \tag{4.61}$$

Both multipliers belong to $(0, 1)$. Moreover,

$$\mathcal{R}(\kappa, 0, 0) = (\kappa, 0, 0), \qquad D\mathcal{R}(\kappa, 0, 0)e_R = e_R, \quad e_R = (1, 0, 0). \tag{4.62}$$

The second identity follows by differentiating the first along the reference segment.

The invariant hyperplanes $u = 0$ and $w = 0$ give $\mathcal{R}_U(\kappa, 0, w) = 0$ and $\mathcal{R}_W(\kappa, u, 0) = 0$. On the box of Lemma 4.12, define

$$\begin{aligned} \mu_U(\kappa, u, w) &= \int_0^1 \partial_u \mathcal{R}_U(\kappa, su, w)\, ds, \\ \mu_W(\kappa, u, w) &= \int_0^1 \partial_w \mathcal{R}_W(\kappa, u, sw)\, ds. \end{aligned} \tag{4.63}$$

The coordinate segments in these integrals remain in the box. The fundamental theorem of calculus and differentiation under the integral give C^1 factors satisfying

$$U_* = u\mu_U(Y), \qquad W_* = w\mu_W(Y). \tag{4.64}$$

This representation includes zero and signed wall coordinates. The identities hold locally around each point of the box, since its coordinate segments remain in a neighborhood of regular returns. At $Z = (\kappa, 0, 0)$ the factors equal the multipliers in (4.61).

Let q_0 be the maximum of m_U and m_W over $[2,6]$. It satisfies $0 < q_0 < 1$. Taking $q = (1+q_0)/2$ and reducing the box, compactness and continuity give constants $L > 0$ such that

$$0 < \mu_U, \mu_W \le q < 1, \qquad \|D\log\mu_U\|, \|D\log\mu_W\| \le L. \tag{4.65}$$

For the derivative bounds, first choose a closed neighborhood on which the factors remain positive and C^1, and then bound the continuous derivatives of their logarithms there. Thus the same neighborhood supports both the factorization and the bounds.

We also need the phase errors to vanish with the walls. Set $Z = (\kappa, 0, 0)$. Apply the mean-value bound on the convex box to $\mathcal{R}$ and $D\mathcal{R}$, using the first and second derivative bounds of Lemma 4.12. After enlarging B,

$$\begin{aligned} \|\mathcal{R}(Y) - Z\| &\le B(|u| + |w|), \\ \|D\mathcal{R}(Y) - D\mathcal{R}(Z)\| &\le B(|u| + |w|), \qquad \|D\mathcal{R}(Y)\| \le B. \end{aligned} \tag{4.66}$$

Now suppose $u, w \ge 0$, $u + w \le 1$, and $v = (1, a, b)$ belongs to the cone (4.51). Since $\|v - e_R\| \le \sqrt{u+w}$, the radial identity (4.62) implies

$$\begin{aligned} |D\mathcal{R}_R(Y)v - 1| &\le |D\mathcal{R}_R(Y)e_R - 1| + |D\mathcal{R}_R(Y)(v - e_R)| \\ &\le B(u+w) + B\sqrt{u+w} \le 2B\sqrt{u+w}. \end{aligned}$$

Using $\kappa \le 6$ in this inequality and (4.66), we obtain

$$|R_* - \kappa| \le C_b \frac{u+w}{\kappa}, \qquad |D\mathcal{R}_R(Y)v - 1| \le C_b \frac{\sqrt{u+w}}{\kappa}, \quad C_b = 12B. \tag{4.67}$$

Lemma 4.13 (Strict cone preservation for bounded radii)**.** *There are $\delta_c > 0$ and $\sigma \in (0,1)$ such that, if $2 \le \kappa \le 6$, $0 \le u, w \le \delta_c$, and $v = (1, a, b) \in \mathcal{C}(Y)$, then $d = D\mathcal{R}(Y)v$ satisfies*

$$d_R > 0, \qquad |d_U| \le \sigma\sqrt{U_*}\, d_R, \qquad |d_W| \le \sigma\sqrt{W_*}\, d_R. \tag{4.68}$$

Proof. Put $s = \sqrt{q}$ and $e = (1-s)/4$, so $0 < e < 1/4$. By continuity of $D\mathcal{R}$, its radial column satisfies $|D\mathcal{R}_R(Y)e_R - 1| \le e$ throughout a sufficiently small common box. Keep the bounds B and L above, and choose $\delta_c \le 1$ small enough that

$$(1 + B + L)\sqrt{\delta_c} \le e.$$

Then $\|v\| \le 1$, $\|v - e_R\| \le \sqrt{\delta_c}$, and

$$|d_R - 1| \le 2e, \qquad d_R \ge 1 - 2e > 0.$$

For either wall write j for its amplitude, a_0 for the corresponding component of v, and μ for its factor. Put $\ell = D\log\mu(Y)v$. The factorization gives the returned component as $\mu(a_0 + j\ell)$, with $|a_0| \le \sqrt{j}$. Moreover,

$$|\ell| \le L, \qquad \sqrt{j}\,|\ell| \le e, \qquad \sqrt{\mu} \le s < 1.$$

The product estimate (4.56) bounds this component by $\sqrt{j\mu}(s + e)$. Finally,

$$s + e = 1 - 3e \le (1-e)(1-2e) \le (1-e)d_R.$$

Taking $\sigma = 1 - e$ proves the lemma, including at a zero wall. □

4.4.2. *One estimate for all radii.* The compact and large-radius constructions identify the same return. Indeed, both give the first positive crossing of $V = 1$ along the original flow. Their local exit functions therefore agree on a neighborhood of every entrance where both apply. Their derivatives agree there as well, including at the common endpoint $\kappa = 6$.

Proposition 4.14 (Uniform core estimates)**.** *There are constants $\varepsilon \in (0, 10^{-4}]$, $C_0 > 0$ and $c_0 \in (0, 1/2]$ such that, for*

$$\kappa \geq 2, \qquad 0 \leq u, w \leq \frac{\varepsilon}{\kappa}, \tag{4.69}$$

the actual first return T and $\mathcal{R}$ are locally C^1. The flow and the projective chart remain regular throughout the crossing, with $V > 1$ in its interior and $dV/dt < 0$ at its exit. Writing $b = u + w$, one has

$$|R_* - \kappa| \leq C_0 \frac{b}{\kappa}, \qquad 0 \leq U_* \leq e^{-c_0/\kappa} u, \qquad 0 \leq W_* \leq e^{-c_0/\kappa} w. \tag{4.70}$$

An output wall is positive exactly when its input wall is positive. For every $v \in \mathcal{C}(Y)$ with $v_\kappa = 1$, put $d = D\mathcal{R}(Y)v$. Then

$$\frac{3}{4} \leq d_R \leq \frac{5}{4}, \qquad |d_R - 1| \leq C_0 \frac{\sqrt{b}}{\kappa}, \qquad |\log d_R| \leq C_0 \frac{\sqrt{b}}{\kappa}, \tag{4.71}$$

$$|d_U| \leq \sqrt{U_*}\, d_R, \qquad |d_W| \leq \sqrt{W_*}\, d_R. \tag{4.72}$$

Proof. Take ε smaller than every wall threshold used in the compact construction and in Proposition 4.10. For $2 \leq \kappa \leq 6$, the bounds $u, w \leq \varepsilon/\kappa$ then place the entrance in each compact box. For $\kappa \geq 6$, the large-radius results apply directly. The preceding uniqueness argument supplies one locally C^1 map and one locally C^1 return time on the resulting domain.

For the wall contraction, let

$$c_0 = \min\{1/2, -2\log q\} > 0.$$

Since $\log q < 0$ and $\kappa \geq 2$, $\kappa \log q \leq 2 \log q \leq -c_0$, whence $q \leq e^{-c_0/\kappa}$. This proves the compact part of the wall estimate by (4.64). In the large-radius range the weak multiplier is at most $e^{-1/(2\kappa)}$, while $e^{-\kappa/40} \leq e^{-1/(2\kappa)}$ for $\kappa \geq 6$. Thus (4.44) gives the same bound. The positive factors in both constructions also give the assertion about positivity.

In the large-radius value estimate, $w^2 \leq w$ reduces Proposition 4.2 to $|R_* - \kappa| \leq 16b/\kappa$. Combine this with (4.67) and Corollary 4.8. There is one $C > 0$ such that both the value error and the ordinary radial derivative error have the bounds Cb/κ and $C\sqrt{b}/\kappa$, respectively, in the full range. The two cone results give (4.72) on this same domain.

Finally decrease ε so that $\varepsilon \leq (4C)^{-2}$. Since $b \leq 2\varepsilon/\kappa \leq \varepsilon$, the radial derivative error is at most $1/4$. Hence $3/4 \leq d_R \leq 5/4$, and the logarithmic inequality used in Corollary 4.8 gives $|\log d_R| \leq 2|d_R - 1|$. Taking $C_0 = 2C$ proves all the estimates. □

Both the phase error and its logarithmic derivative can now be compared with the wall loss. For any finite chain whose connecting passages do not increase the wall sum, Lemma 4.3 applies with $c = c_0$, and with $p = 1$ or $p = 1/2$ for the two errors. To use this comparison for a complete era, we next determine how the exact connecting passages change the radius, the two walls and the transverse directions.

5. Reflections at large radius

After a nonterminal core the two active axes exchange roles. At large radius the resulting connection may take a long time. The estimates below retain the small wall factors throughout that interval, so that both the phase error and its derivatives remain controlled by transverse amplitude.

5.1. **Passage to the growing wall.**

We use the growing wall as a clock throughout the connection. A phase function relating the radii of the adjacent cores on their single-wall boundaries then makes the accumulated error proportional to the small wall amplitudes, uniformly in the waiting time.

5.1.1. *The growing wall as a clock.* Keep the frame of the preceding core during the connection and set $m = -M$. Consider the initial state

$$(M,U,V,W)(0) = (-\mu, u, 1, w), \qquad \mu \geq 4, \quad 0 \leq u \leq \frac{\varepsilon_0}{\mu}, \quad 0 < w \leq \varepsilon_0, \tag{5.1}$$

where $\varepsilon_0 = 10^{-4}$ as before. The next entrance occurs when W first reaches one. After the reflection $(M,U,V,W) \mapsto (-M,U,W,V)$, this becomes an upward crossing of the new active wall. Equivariance under this fixed reflection makes the result identical to changing the frame at the preceding core exit.

Write

$$C = U(V+W-U), \qquad q = m-1-C, \qquad a = \frac{1-m^2-(V-W)^2+U^2}{2}. \tag{5.2}$$

On an interval where $W > 0$ and $q > 0$, the variable $\ell = \log(W/w)$ has derivatives

$$\frac{d\ell}{d\tau} = q, \qquad \frac{d\ell}{dt} = Aq. \tag{5.3}$$

The projective equations become

$$\begin{aligned} m' &= \frac{V^2 - W^2 - U(V-W) - mC}{q}, & U' &= \frac{a}{q}U, \\ V' &= \frac{-m-1-C}{q}V, & W' &= W, \end{aligned} \tag{5.4}$$

where primes now mean $d/d\ell$. In particular $W = we^{\ell}$, so the target $W = 1$ is reached at the prescribed clock value $L = \log(1/w)$ whenever the passage exists.

We control every initial segment in the closed region

$$|m-\mu| \leq \frac{1}{2}, \qquad 0 \leq U \leq \varepsilon_0, \qquad 0 \leq V, W \leq 1. \tag{5.5}$$

The elementary bounds needed below are

$$-U^2 \leq C \leq 2U, \qquad \frac{\mu}{2} \leq q \leq \mu, \qquad a \leq -\frac{\mu^2}{4}, \qquad a+m+1+C < 0, \tag{5.6}$$

$$0 < 3 + m^2 - 4U \leq D \leq 2\mu^2, \qquad A \geq \frac{3}{\mu^2}, \qquad Aq \geq \frac{1}{\mu}. \tag{5.7}$$

For the first line, use $\mu - 1/2 \leq m \leq \mu + 1/2$ and $U \leq 10^{-4}$. In particular $m \geq 7\mu/8$ gives the stated bound on a, and $m \geq 7/2$ gives the negative ratio rate. For the second line, expand D in (2.7): the term $(V-W)^2$ is nonnegative, while $-2U(V+W) \geq -4U$. The upper bound follows from $m^2 \leq (\mu + 1/2)^2$ and $|V - W| \leq 1$. The two clock bounds then follow from $A = 6/D$ and $q \geq \mu/2$.

Lemma 5.1 (Decay and finite integrals before arrival)**.** *Suppose an initial segment from* (5.1) *remains in* (5.5)*, with $V, W > 0$. At every clock value $s \geq 0$ in this segment,*

$$\begin{aligned} U &\leq V, & C &\geq 0, \\ U(s) &\leq ue^{-\mu s/2}, & V(s) &\leq e^{-(1+2/\mu)s}, \\ V(s)W(s) &\leq we^{-2s/\mu}, & W(s) &= we^{s}. \end{aligned} \tag{5.8}$$

Moreover,

$$\begin{aligned} \int_0^s \frac{V^2}{q}\, d\ell &\leq \frac{1}{2\mu}, & \int_0^s \frac{W^2}{q}\, d\ell &\leq \frac{1}{\mu}, \\ \int_0^s \frac{U}{q}\, d\ell &\leq \frac{4u}{\mu^2}, & \int_0^s \frac{(VW)^2}{q}\, d\ell &\leq \frac{w^2}{4}, \end{aligned} \tag{5.9}$$

and the phase band improves to

$$|m(s)-\mu| \le \frac{3/2+12u}{\mu} < \frac{1}{2}. \tag{5.10}$$

Proof. The ratio equation is

$$\left(\frac{U}{V}\right)' = \frac{a+m+1+C}{q}\frac{U}{V}.$$

Its coefficient is negative by (5.6). Since $U(0)/V(0) = u \le 1$, it follows that $U \le V$, and hence $C \ge 0$. With this ordering the rates satisfy

$$\frac{a}{q} \le -\frac{\mu}{2}, \qquad \frac{-m-1-C}{q} = -1 - \frac{2(1+C)}{q} \le -1 - \frac{2}{\mu}.$$

To check the first inequality, multiply by $2q > 0$. The required numerator is

$$1 - m^2 - (V-W)^2 + U^2 + \mu(m-1-C) \le 1 - \mu + U^2 + \frac{m}{2} < 0.$$

Here $\mu - m \le 1/2$, $C \ge 0$, and $m \le \mu + 1/2$ were used. The scalar differential inequalities give the two wall bounds in (5.8). The product equation is

$$(VW)' = -\frac{2(1+C)}{q}VW \le -\frac{2}{\mu}VW,$$

which gives the product bound as well.

For the integrals, retain the exact clock denominator in the dissipation inequalities:

$$(V^2)' \le -\frac{2\mu V^2}{q}, \qquad (1-W^2)' \le -\frac{\mu W^2}{q},$$
$$U' \le -\frac{\mu^2 U}{4q}, \qquad ((VW)^2)' \le -\frac{4(VW)^2}{q}.$$

The first uses $-m-1-C \le -\mu$, the second uses $2q \ge \mu$, and the last uses $C \ge 0$. Integrate each inequality and discard its nonnegative terminal value. The initial values are 1, $1-w^2 \le 1$, u and w^2, respectively. This proves (5.9).

Finally, the numerator of m' has absolute value at most $V^2 + W^2 + 3\mu U$. Indeed, $|V-W| \le 1$, $0 \le C \le 2U$, and $1 + 2m \le 3\mu$. Integrating and using the first three bounds in (5.9) gives (5.10). Its strict inequality follows from $\mu \ge 4$ and $u \le 10^{-4}$. □

The integrals in (5.9) are projective time integrals, written using $d\tau = d\ell/q$. Their bounds have no factor L. This is essential when w is much smaller than the other initial quantities.

Proposition 5.2 (First arrival at the growing wall)**.** *For every input* (5.1)*, the original flow has a first positive time T_c at which $W = 1$. Up to that time it remains in* (5.5)*, with $A \ge 3/\mu^2$. The wall V is strictly decreasing and W strictly increasing; there is exactly one time in $(0, T_c)$ at which $V = W$. The exit is transverse, with $dW/dt > 0$.*

Put $L = \log(1/w)$, and let τ_c be the elapsed projective time. Then

$$\frac{L}{\mu} \le \tau_c \le \frac{2L}{\mu}, \qquad 0 < T_c \le \mu L. \tag{5.11}$$

After reflection, the two transverse walls are $u_+ = U(T_c)$ and $w_+ = V(T_c)$, and they satisfy

$$0 \le u_+ \le u e^{-\mu L/2}, \qquad 0 < w_+ \le w e^{-2L/\mu}. \tag{5.12}$$

Here $u_+ > 0$ if and only if $u > 0$. The time T_c and the returned state are locally C^2 functions of (μ, u, w), also at $u = 0$.

Proof. The denominator is positive at the entrance by (5.7). Lift the initial state to the normalized constraint surface and use its forward flow. At any unfinished segment in the proposed region, positivity of V, W persists by their multiplicative equations. The strong wall cannot cross its upper boundary because $a < 0$ there; its zero boundary is invariant. The old active wall points strictly into $V < 1$, since $-m - 1 - C < 0$ throughout the region. Initially $W < 1$, so continuity gives a first short segment with the required bounds.

For continuation in the original coordinates, include the closed condition $A \geq 3/(2\mu^2)$. Within the region it improves to $A \geq 3/\mu^2$ by (5.7). The phase condition improves strictly by Lemma 5.1. Thus, as long as $W < 1$, none of these boundaries can terminate the passage. The original constrained flow exists at every finite forward time by Lemma 2.1; the positive lower bound on A keeps the projective coordinates defined. Applying the same local continuation argument at a finite endpoint extends the segment unless $W = 1$ there.

There must be such a finite endpoint. Otherwise (5.7) and (5.3) give

$$\frac{t}{\mu} \leq \log \frac{W(t)}{w} \leq L$$

for arbitrarily large t, a contradiction. This also proves $T_c \leq \mu L$. Since q lies between $\mu/2$ and μ, integrating $d\tau = d\ell/q$ from 0 to L proves the other duration bounds.

The rates of V and W have strict opposite signs. Hence the hit is the first one, and $V - W$ decreases from $1 - w > 0$ to $V(T_c) - 1 < 0$, giving one and only one equality $V = W$. At the hit, $dW/dt = Aq > 0$. Substitution of $s = L$ in (5.8) gives (5.12); the sign statements follow from wall sign preservation.

To obtain local regularity, apply the implicit function theorem to the actual equation $W(t; \mu, u, w) = 1$ at its transverse hit. The flow and the event are C^2 on a neighborhood of the whole finite segment. The derivative dW/dt is positive on this segment and remains positive on a neighborhood of it, while the initial value remains below one. The nearby implicit hit is therefore still the first hit, with the intervening flow in the same chart. Composition with the flow gives the C^2 returned state. This argument also provides a local extension through $u = 0$. □

5.1.2. *Phase change across the connection.* The band estimate (5.10) bounds m throughout the passage, but it does not yet give an error that vanishes with both incoming walls. The appropriate quantity combines the radii of the two adjacent cores. Set

$$H(a, b) = \sqrt{a^2 + b^2}, \qquad J(m, V, W) = H(m + 1, V) + H(m - 1, W) - m. \tag{5.13}$$

For $m > 1$,

$$J(m, V, 0) = H(m + 1, V) - 1, \qquad J(m, 0, W) = H(m - 1, W) + 1. \tag{5.14}$$

The first expression is the preceding core radius minus one; the second is the reflected core radius plus one. Thus a small change of J gives a decrement close to two between the physical core radii.

Write $\rho_+ = H(m + 1, V)$, $\rho_- = H(m - 1, W)$, and define

$$S = \frac{m + 1}{\rho_+} + \frac{m - 1}{\rho_-} - 1,$$
$$B = mS + \frac{V^2}{\rho_+} + \frac{W^2}{\rho_-}, \qquad K_\pm = \frac{1}{\rho_\pm(\rho_\pm + m \pm 1)}. \tag{5.15}$$

Direct differentiation in projective time gives the exact identity

$$\dot{J} = (VW)^2(K_+ - K_-) - US(V - W) - CB. \tag{5.16}$$

To verify its cancellation, use

$$H(a, b) - a = \frac{b^2}{H(a, b) + a}, \qquad 1 - \frac{a}{H(a, b)} = \frac{b^2}{H(a, b)(H(a, b) + a)} \quad (a > 0).$$

Before this substitution, the terms in $\dot{J}$ independent of U are

$$V^2\left(\frac{m-1}{\rho_-}-1\right)+W^2\left(1-\frac{m+1}{\rho_+}\right).$$

The two identities turn their sum into $(VW)^2(K_+-K_-)$. The remaining terms are precisely $-US(V-W)-CB$.

In (5.5), these coefficients satisfy

$$\frac{3}{4}\le S\le 1,\qquad 0\le B\le 2\mu,\qquad 0\le K_\pm\le\frac{2}{\mu^2},\quad |K_+-K_-|\le\frac{4}{\mu^2}. \tag{5.17}$$

Indeed, if $a\ge 2$ and $0\le b\le 1$, then $7/8\le a/H(a,b)\le 1$, which gives the bounds on S. Also $V^2/\rho_+\le 1$ and $W^2/\rho_-\le 1$, so $B\le m+2\le 2\mu$. For the kernels use $m\pm 1\ge\mu/2$ and $H(a,b)\ge a$; their denominators are at least $\mu^2/2$.

Since $|C|\le 2U$ and $|V-W|\le 1$, the derivative identity yields

$$|\dot{J}|\le 5\mu U+\frac{4}{\mu^2}(VW)^2. \tag{5.18}$$

Consequently the last two integrals in (5.9) give, for every completed initial segment,

$$|J(m(s),V(s),W(s))-J(\mu,1,w)|\le\frac{20u}{\mu}+\frac{w^2}{\mu^2}. \tag{5.19}$$

The interaction of the two walls is controlled by the integral of $(VW)^2$, whose initial value is w^2. Its contribution stays small even when the clock interval becomes long.

Proposition 5.3 (The radius decrement)**.** *For the passage in Proposition 5.2, put $m_*=m(T_c)$ and*

$$r_-=H(\mu+1,1),\qquad r_+=H(m_*-1,1).$$

These are the preceding core exit radius and the next core entrance radius after reflection. They satisfy

$$|r_+-(r_--2)|\le\frac{24(u+w^2)}{\mu}. \tag{5.20}$$

Proof. Let $v_*=V(T_c)$. By (5.12), $0<v_*\le w$. The two endpoint defects

$$e_-=H(\mu-1,w)-(\mu-1),\qquad e_+=H(m_*+1,v_*)-(m_*+1)$$

belong to $[0,w^2/\mu]$. This follows from the identity for $H(a,b)-a$, since $\mu-1$ and m_*+1 are at least $\mu/2$. Expanding (5.13) at the two ends gives

$$r_+-r_-+2=J(m_*,v_*,1)-J(\mu,1,w)+e_--e_+.$$

Apply (5.19) and the defect bounds. The absolute value is at most $20u/\mu+w^2/\mu^2+2w^2/\mu$, which is bounded by the right-hand side of (5.20). □

To express the result directly in core coordinates, take a downward exit with radius $R\ge 11/2$ and walls u,w. Its state is

$$(M,U,V,W)=(1-\sqrt{R^2-1},u,1,w),\qquad \mu=\sqrt{R^2-1}-1.$$

Then $\mu\ge 4$, $R/2\le\mu<R$, and $r_-=R$. If $0\le u\le\varepsilon_0/R$ and $0<w\le\varepsilon_0/R$, all the entrance hypotheses above hold. Define the reflection map by

$$\mathcal{F}(R,u,w)=(r_+,u_+,w_+). \tag{5.21}$$

It is locally C^2, and its core entrance embedding is exactly $(m_*,U(T_c),1,V(T_c))$. Here $m_*>1$, so the positive entrance branch of the radius coordinates is the correct one. The estimates give

$$|\mathcal{F}_R(R,u,w)-(R-2)|\le 48\frac{u+w}{R},\qquad 0\le u_+\le u,\qquad 0<w_+\le w. \tag{5.22}$$

The value error is now measured by the same wall amplitudes as in the core estimate. To control repeated reflections, we also need its derivative along the transverse cone. The next estimates keep the cancellation in (5.16) when the entrance and the logarithmic clock interval vary.

5.2. **Variations across long connections.** The matching phase also controls nearby orbits. Its usefulness at the level of derivatives comes from two facts: its rate retains a small wall factor, and the transverse linear equation has bounded growth throughout a connection, however long the waiting time. We first establish these facts in coordinates adapted to the phase. We then treat variations of the entrance phase and of the two incoming walls. The growing wall requires separate attention because it changes the length of the logarithmic clock interval.

Throughout this subsection the entrance satisfies (5.1), with $u > 0$ as well as $w > 0$. All three walls are therefore positive on the finite passage. We use s for the clock ℓ, write $L = \log(1/w)$, and retain $P = VW$. The quantities $S, B, \rho_\pm$ are those in (5.15).

5.2.1. *The phase coordinates and the actual variation.* Since $\partial_m J = S \geq 3/4$, the variables $(J, \log U, \log V, \log W)$ form local coordinates. In these coordinates the equation is

$$J' = G, \qquad (\log U)' = f, \qquad (\log V)' = g, \qquad (\log W)' = 1, \tag{5.23}$$

where

$$f = \frac{a}{q}, \qquad g = -1 - \frac{2(1+C)}{q}, \qquad G = U\alpha + P^2\beta, \tag{5.24}$$

and the cancellation in (5.16) gives

$$\alpha = -\frac{S(V-W) + (V+W-U)B}{q}, \qquad \beta = \frac{K_+ - K_-}{q}. \tag{5.25}$$

We identify the variations in this equation with derivatives of the original flow. Let $Y = (m, U, V, W)$, let $F(Y)$ denote the right-hand side of (5.4), and let η be a variation of Y at fixed original time. For an entrance direction $v = (v_\mu, v_u, v_w)$, put $\zeta = v_w/w$ and

$$\xi = \eta - \left(\frac{\eta_W}{W} - \zeta\right) F(Y). \tag{5.26}$$

This compares orbits at the same value of $\log(W/w)$. In particular, $\xi_W = \zeta W$, and its initial value is $(v_\mu, v_u, 0, v_w)$.

For completeness, set $h = Aq = d\ell/dt$. The original equation is $dY/dt = hF$. Its variation gives $d\eta/dt = \delta h\, F + hDF\eta$. Also $d(\eta_W/W)/dt = \delta h$, by varying $d\log W/dt = h$. Differentiating (5.26) therefore cancels the δh term and yields $d\xi/dt = hDF\xi$. The strictly increasing clock gives $\xi' = DF\xi$ on the entire passage.

Write the induced variation in the phase coordinates as

$$Z = \delta J = S\xi_m + \frac{V}{\rho_+}\xi_V + \frac{W}{\rho_-}\xi_W, \qquad x = \frac{\xi_U}{U}, \qquad y = \frac{\xi_V}{V}. \tag{5.27}$$

Its initial values are

$$(Z, x, y, \zeta)(0) = \left(S(\mu, 1, w)v_\mu + \frac{w}{H(\mu-1, w)}v_w, \frac{v_u}{u}, 0, \frac{v_w}{w}\right). \tag{5.28}$$

Thus the logarithmic wall variations may be large at the entrance. Their coefficients in the phase equation will retain the corresponding small wall amplitudes.

To specify these coefficients, introduce differentiation at fixed values of the other phase coordinates:

$$\begin{aligned} &\mathcal{D}_J = S^{-1}\partial_m, && \mathcal{D}_U = U\partial_U, && \\ &\mathcal{D}_V = V\partial_V + e_V\partial_m, && \mathcal{D}_W = W\partial_W + e_W\partial_m, && e_V = -\frac{V^2}{\rho_+ S}, \quad e_W = -\frac{W^2}{\rho_- S}. \end{aligned} \tag{5.29}$$

For any function h, write $h_J = \mathcal{D}_J h$ and similarly for the other three subscripts. The exact variational equation is

$$\begin{pmatrix} Z \\ x \\ y \end{pmatrix}' = \begin{pmatrix} G_J & G_U & G_V & G_W \\ f_J & f_U & f_V & f_W \\ g_J & g_U & g_V & g_W \end{pmatrix} \begin{pmatrix} Z \\ x \\ y \\ \zeta \end{pmatrix}, \qquad \zeta' = 0. \tag{5.30}$$

Lemma 5.4 (Coefficients retaining the wall amplitudes)**.** *In the region* (5.5)*, the phase row satisfies*

$$\begin{aligned} &|G_J| \le 64\left(\frac{U}{\mu^2} + \frac{P^2}{\mu^4}\right), && |G_U| \le 32U, \\ &|G_V| \le 32\left(UV + \frac{P^2}{\mu^3}\right), && |G_W| \le 32\left(UW + \frac{P^2}{\mu^3}\right). \end{aligned} \tag{5.31}$$

The two amplitude rows satisfy

$$\begin{aligned} &|f_J| \le 32, && |f_U| \le 32U, \\ &|f_V| \le 32\left(\frac{V^2+P}{\mu} + UV\right), && |f_W| \le 32\left(\frac{W^2+P}{\mu} + UW\right), \end{aligned} \tag{5.32}$$

$$\begin{aligned} &|g_J| \le \frac{64}{\mu^2}, && |g_U| \le \frac{64U}{\mu}, \\ &|g_V| \le 64\left(\frac{V^2}{\mu^3} + \frac{UV}{\mu}\right), && |g_W| \le 64\left(\frac{W^2}{\mu^3} + \frac{UW}{\mu}\right). \end{aligned} \tag{5.33}$$

Proof. The coordinate derivatives have

$$S^{-1} \le \frac{4}{3}, \qquad |e_V| \le \frac{3V^2}{\mu}, \qquad |e_W| \le \frac{3W^2}{\mu}. \tag{5.34}$$

Here $\rho_+ \ge \mu$ and $\rho_- \ge 5\mu/8$. We also have $q \ge \mu/2$, $|1+C| \le 2$, $|f| \le 2\mu$, $|m+f| \le 4\mu$, and $|V+W-2U| \le 3$. For a variation (Z, x, y, ζ), put

$$\delta m = S^{-1}Z + e_V y + e_W \zeta, \qquad \delta C = U(V+W-2U)x + UVy + UW\zeta.$$

The quotient rule gives

$$\begin{aligned} \delta f &= \frac{-(m+f)\delta m - (V-W)(Vy - W\zeta) + U^2 x + f\delta C}{q}, \\ \delta g &= \frac{2(1+C)\delta m - 2m\delta C}{q^2}. \end{aligned}$$

Substitution of (5.34) gives (5.32)–(5.33). For example, the numerator of f_V has absolute value at most $12V^2 + V(V+W) + 2\mu UV$; division by q gives $26V^2/\mu + 2P/\mu + 4UV$, within the stated bound. The weak wall columns are bounded in the same calculation by $48V^2/\mu^3 + 16UV/\mu$ and its W counterpart.

For the phase row, differentiating the factored expression (5.25) preserves the cancellation. Let $\mathcal{K} = K_+ - K_-$. The elementary kernel $k(a,b) = [H(a,b)(H(a,b)+a)]^{-1}$ has derivatives

$$k_a = -H^{-3}, \qquad k_b = -\frac{b(2H+a)}{H^3(H+a)^2}.$$

For $a \ge d > 0$, these imply $0 \le k \le 1/(2d^2)$, $|k_a| \le d^{-3}$ and $|bk_b| \le 3b^2/(4d^4)$. Applying these bounds at the two arguments, and then using (5.29), gives

$$|\mathcal{K}| \le \frac{3}{\mu^2}, \quad |\partial_m \mathcal{K}| \le \frac{9}{\mu^3}, \quad |\mathcal{K}_J| \le \frac{12}{\mu^3}, \quad |\mathcal{K}_V| \le \frac{32V^2}{\mu^4}, \quad |\mathcal{K}_W| \le \frac{32W^2}{\mu^4}.$$

Indeed, each wall derivative consists of $e_V\partial_m\mathcal{K}$ or $e_W\partial_m\mathcal{K}$ and one kernel wall derivative, bounded respectively by $27r^2/\mu^4$ and $5r^2/\mu^4$. Since

$$(q_J, q_U, q_V, q_W) = (S^{-1}, -U(V+W-2U), e_V - UV, e_W - UW),$$

the quotient $\beta = \mathcal{K}/q$ satisfies

$$|\beta| \le \frac{6}{\mu^3}, \qquad |\beta_J| \le \frac{40}{\mu^4}, \qquad |\beta_U| \le \frac{36U}{\mu^4},$$
$$|\beta_V| \le \frac{100V^2}{\mu^5} + \frac{12UV}{\mu^4}, \qquad |\beta_W| \le \frac{100W^2}{\mu^5} + \frac{12UW}{\mu^4}.$$

The phase derivative of α needs one further cancellation. The identity

$$B = J - \frac{m+1}{\rho_+} + \frac{m-1}{\rho_-}$$

gives

$$S_m = \frac{V^2}{\rho_+^3} + \frac{W^2}{\rho_-^3}, \qquad B_m = S - \frac{V^2}{\rho_+^3} + \frac{W^2}{\rho_-^3},$$
$$0 \le S_m \le \frac{9(V^2+W^2)}{2\mu^3}, \qquad |B_m - 1| \le \frac{3(V^2+W^2)}{\mu^2}.$$

Moreover,

$$B - m = \frac{V^2(\rho_+ + 1)}{\rho_+(\rho_+ + m + 1)} + \frac{W^2(\rho_- - 1)}{\rho_-(\rho_- + m - 1)}.$$

Both terms are nonnegative, and their sum is at most $8(V^2+W^2)/(5\mu) \le 4/5$. It follows that $0 \le B - q \le 2$. Set

$$N_0 = S(V-W) + (V+W-U)(B-q).$$

Then $|N_0| \le 6$, $|(N_0)_m/S| \le 20/\mu^2$, and

$$\alpha = -(V+W-U) - \frac{N_0}{q}, \qquad \alpha_J = -\frac{(N_0)_m}{qS} + \frac{N_0}{q^2 S}.$$

Consequently $|\alpha_J| \le 40/\mu^3 + 32/\mu^2 \le 48/\mu^2$.

For the other columns, let $N = S(V-W)+(V+W-U)B$, so $\alpha = -N/q$. Differentiation of the displayed formulas for S, B gives, for $r = V, W$,

$$|S_r|, |B_r| \le \frac{5r^2}{\mu^2}, \qquad |N| \le 4\mu, \quad |N_U| \le \frac{3}{2}\mu U, \quad |N_r| \le 3\mu r.$$

For clarity, the wall derivatives of B are

$$B_V = \frac{(m+1)V^2}{\rho_+^3} + e_V(B_m - S), \qquad B_W = -\frac{(m-1)W^2}{\rho_-^3} + e_W(B_m - S).$$

The quotient rule now yields

$$|\alpha| \le 8, \qquad |\alpha_U| \le 16U, \qquad |\alpha_V| \le 16V, \qquad |\alpha_W| \le 16W. \tag{5.35}$$

Finally,

$$G_J = U\alpha_J + P^2\beta_J, \qquad G_U = U\alpha_U + U\alpha + P^2\beta_U,$$

and $G_r = U\alpha_r + P^2\beta_r + 2P^2\beta$ for $r = V, W$. The phase column is bounded by $48U/\mu^2 + 40P^2/\mu^4$. For a wall column, the last two terms have sum at most $32P^2/\mu^3$, using $\mu \ge 4$ and $U, V, W \le 1$. Together with (5.35), these prove (5.31). □

5.2.2. *Bounded transverse growth.* The phase column of the weak amplitude equation is of order μ^{-2}, whereas that of the strong equation is bounded independently of μ. Set $X = (x, \mu^2 y)$, with the maximum norm on $\mathbb{R}^2$, and write

$$X' = TX + Zc + \zeta d, \qquad T = \begin{pmatrix} f_U & f_V/\mu^2 \\ \mu^2 g_U & g_V \end{pmatrix}, \quad c = \begin{pmatrix} f_J \\ \mu^2 g_J \end{pmatrix}, \quad d = \begin{pmatrix} f_W \\ \mu^2 g_W \end{pmatrix}. \tag{5.36}$$

The coefficient bounds give $\|c\| \le 64$ and bound each absolute row sum of T by

$$r(s) = 64\left(2\mu U + \frac{V^2}{\mu^3} + \frac{P}{\mu^3}\right).$$

Its integral is controlled by the decreasing quantity

$$\mathcal{E}(s) = 64\left(4U + \frac{V^2}{2\mu^3} + \frac{P}{2\mu^2}\right). \tag{5.37}$$

In fact $U' \le -\mu U/2$, $V' \le -V$, and $P' \le -2P/\mu$ imply $\mathcal{E}' \le -r$. At the entrance,

$$0 \le \mathcal{E}(0) \le \frac{2533}{5000} < \log 2.$$

The last comparison follows, for example, from $2533/5000 < 2/3$ and $e^{2/3} < 2$.

Lemma 5.5 (Growth on every subinterval)**.** *If $X' = TX + b$ on $[s,t] \subset [0,L]$, with continuous forcing b, then*

$$\|X(t)\| \le 2\left(\|X(s)\| + \int_s^t \|b(r)\|\,dr\right). \tag{5.38}$$

Proof. The right upper derivative of the maximum norm is at most $r\|X\| + \|b\|$. This follows by an Euler step in each component, as in the core variation estimate; the diagonal term can retain its sign, and its signed row sum is bounded by the absolute row sum. Multiplication by $e^{\mathcal{E}}$, using $\mathcal{E}' \le -r$, gives

$$\|X(t)\| \le e^{\mathcal{E}(s)-\mathcal{E}(t)}\|X(s)\| + \int_s^t e^{\mathcal{E}(r)-\mathcal{E}(t)}\|b(r)\|\,dr.$$

The nonnegative function $\mathcal{E}$ is decreasing and $e^{\mathcal{E}(0)} < 2$, proving the result. □

The bound is independent of L. The phase equation couples back to X, but its total effect is small. To quantify it, the zero and first clock moments of the amplitudes suffice:

$$\begin{aligned} &\int_0^s U\,dr \le \frac{2u}{\mu}, \qquad \int_0^s rU\,dr \le \frac{4u}{\mu^2}, \\ &\int_0^s P^2\,dr \le \frac{\mu w^2}{4}, \quad \int_0^s rP^2\,dr \le \frac{\mu^2 w^2}{16}. \end{aligned} \tag{5.39}$$

Indeed, if $h' \le -ch$ and $h \ge 0$, integration gives $\int_0^s h \le h(0)/c$. Differentiating $(1+cr)h(r)$ gives an upper bound $-c^2 r h(r)$, so $\int_0^s rh(r)\,dr \le h(0)/c^2$. Apply this to U with $c = \mu/2$, and to P^2 with $c = 4/\mu$.

For $\zeta = 0$, the phase row in Lemma 5.4 gives

$$|Z'| \le A_0|Z| + B_0\|X\|, \quad A_0 = 64\left(\frac{U}{\mu^2} + \frac{P^2}{\mu^4}\right), \quad B_0 = 64\left(U + \frac{P^2}{\mu^5}\right). \tag{5.40}$$

Here the transverse row bound is $|G_U| + |G_V|/\mu^2 \le B_0$; keeping the factor 32 in the two wall columns proves this bound directly. Define

$$k(s) = A_0(s) + 128 s B_0(s), \qquad \Lambda = \frac{128u}{\mu^3} + \frac{32768u}{\mu^2} + \frac{528w^2}{\mu^3}. \tag{5.41}$$

The four moments give $\int_0^s k \le \Lambda$ for every $s \le L$. Moreover $\Lambda \le 1/16$, by $\mu \ge 4$, $u \le 1/40000$ and $w^2 \le 10^{-8}$.

We will repeatedly use the following elementary absorption argument. Suppose that, on a compact interval, every bound $|Z| \le Q$ implies $|Z'| \le e + kQ$, where $e, k \ge 0$ and $\int k \le \Lambda < 1$. Choose $Q = \max|Z|$ on this interval and integrate up to a point where the maximum is attained. Then

$$\max|Z| \le \frac{|Z(0)| + \int e}{1-\Lambda}, \qquad |Z(s) - Z(0)| \le \int e + \Lambda \frac{|Z(0)| + \int e}{1-\Lambda}. \tag{5.42}$$

In our application $(1-\Lambda)^{-1} \le 16/15$.

5.2.3. *Entrance derivatives and the moving arrival.*

Proposition 5.6 (Two parameter responses)**.** *Let (Z_μ, x_μ, y_μ) and (Z_u, x_u, y_u) denote the fixed-clock variations induced respectively by the ordinary entrance directions $(1,0,0)$ and $(0,1,0)$. Put $S_0 = S(\mu, 1, w)$. For every $0 \le s \le L$,*

$$\begin{gathered} |Z_\mu(s)| \le \frac{16}{15}, \qquad |Z_\mu(s) - S_0| \le \frac{16}{15}\Lambda, \\ \|(x_\mu, \mu^2 y_\mu)(s)\| \le \frac{2048}{15} s, \end{gathered} \tag{5.43}$$

$$|Z_u(s)| \le \frac{70}{\mu}, \qquad \left\|(x_u, \mu^2 y_u)(s) - (u^{-1}, 0)\right\| \le 256 + \frac{8960 s}{\mu}. \tag{5.44}$$

Proof. In the phase direction $X(0) = 0$ and $\zeta = 0$. Lemma 5.5 gives

$$\|X(s)\| \le 128 \int_0^s |Z(r)|\, dr.$$

If $|Z| \le Q$ on the interval, this is at most $128sQ$. Equation (5.40) now gives $|Z'| \le kQ$. Apply (5.42) with $e = 0$, and use $0 < S_0 \le 1$. This proves (5.43).

For the strong direction $Z(0) = 0$, $\zeta = 0$ and $X(0) = (u^{-1}, 0)$. Subtract this constant vector before estimating the transverse equation. Its extra forcing satisfies

$$\|T(u^{-1}, 0)\| \le 64\mu U/u.$$

By (5.39) and the factor two in (5.38),

$$\|X(s) - (u^{-1}, 0)\| \le 256 + 128 \int_0^s |Z(r)|\, dr. \tag{5.45}$$

The constant entry vector contributes only G_U/u to Z', so

$$|Z'| \le 32U/u + A_0|Z| + B_0\|X - (u^{-1}, 0)\|.$$

For $|Z| \le Q$, the right-hand side is bounded by $e_u + kQ$, where

$$e_u = 32U/u + 16384\left(U + P^2/\mu^5\right).$$

Its integral is at most

$$\frac{64}{\mu} + \frac{32768u}{\mu} + \frac{4096w^2}{\mu^4} \le \frac{65}{\mu}.$$

Absorption gives $|Z| \le (16/15)(65/\mu) \le 70/\mu$. Substitution into (5.45) proves the second assertion. □

The singular entry derivative u^{-1} has thus remained in the strong logarithmic component. Its influence on the phase was integrated against the decaying wall U.

The direction $(0,0,1)$ has $\zeta = 1/w$ and $Z(0) = w/H(\mu - 1, w) \le 2w/\mu$. In this direction a large direct forcing appears in x. We integrate that forcing explicitly before estimating how it influences y and Z.

We first record the needed mixed integrals, all in the logarithmic clock:

$$\begin{aligned} &\int_0^s W^2\,dr \le \frac{W(s)^2}{2}, \qquad \int_0^s UW\,dr \le \frac{4uw}{\mu}, \\ &\int_0^s UW^2\,dr \le uw, \qquad \int_0^s P\,dr \le \frac{\mu w}{2}, \\ &\int_0^s P^2W^2\,dr \le \frac{w^2W(s)^2}{2}. \end{aligned} \tag{5.46}$$

The first follows from $(W^2)' = 2W^2$. The second follows from $(UW)' \le -\mu UW/4$, and the fourth from $P' \le -2P/\mu$. For the third, $(UW^2)' \le 0$, so its integral is at most $uw^2 s$. Since W is increasing and $W' = W$, $ws \le \int_0^s W\,dr = W(s) - w \le 1$. For the last inequality, use $P \le w$ and the first bound. These arguments include the endpoint $\mu = 4$, where UW^2 need not have a strictly negative decay rate.

Define the primitive

$$F_w(s) = \frac{1}{w}\int_0^s f_W(r)\,dr. \tag{5.47}$$

The strong row and (5.46) give

$$|F_w(s)| \le 32\left(1 + \frac{W(s)^2}{\mu w}\right), \qquad \int_0^s 64\mu U|F_w|\,dr \le 6144u. \tag{5.48}$$

For the first inequality, the integral of $|f_W|/w$ is at most $16W(s)^2/(\mu w) + 16 + 128u/\mu$. For the second, insert the first bound and use $\int U \le 2u/\mu$ and $\int UW^2 \le uw$.

The residual $\widetilde{X} = X - (F_w, 0)$ starts at zero and solves

$$\widetilde{X}' = T\widetilde{X} + T(F_w, 0) + \left(0, \frac{\mu^2 g_W}{w}\right) + Zc.$$

Its three forcing terms have norm at most $64\mu U|F_w|$, $64(W^2/\mu + \mu UW)/w$, and $64|Z|$. Their integrals, together with Lemma 5.5, give

$$\|\widetilde{X}(s)\| \le 12800u + \frac{64W(s)^2}{\mu w} + 128\int_0^s |Z(r)|\,dr. \tag{5.49}$$

Indeed, the middle forcing has integral at most $256u + 32W(s)^2/(\mu w)$, while the first contributes at most $6144u$ before the factor two.

Proposition 5.7 (Response to the growing wall)**.** *For the fixed-clock variation (Z_w, x_w, y_w) in the ordinary direction $(0,0,1)$, and every $0 \le s \le L$,*

$$\begin{aligned} |Z_w(s)| &\le \frac{8000(u+w)}{\mu}, \\ |x_w(s)| &\le 32 + 12800u + \frac{96W(s)^2}{\mu w} + \frac{1024000s(u+w)}{\mu}, \\ |y_w(s)| &\le \frac{1}{\mu^2}\left(12800u + \frac{64W(s)^2}{\mu w} + \frac{1024000s(u+w)}{\mu}\right). \end{aligned} \tag{5.50}$$

Proof. For a bound $|Z| \le Q$, put $q_w = W^2/(\mu w)$. Equations (5.48) and (5.49) give

$$|x| \le 32(1+q_w) + (12800u + 64q_w + 128sQ), \qquad |y| \le \frac{12800u + 64q_w + 128sQ}{\mu^2}.$$

Apply the phase row separately to these two contributions to x. The result is $|Z'| \le e_w + kQ$, with the same k as before and

$$e_w = 1024U(1+q_w) + 64\left(U + \frac{P^2}{\mu^5}\right)(12800u + 64q_w) + \frac{32}{w}\left(UW + \frac{P^2}{\mu^3}\right).$$

The last term is the direct contribution G_W/w. Expansion and the moments in (5.39) and (5.46) yield

$$\int_0^s e_w\,dr \le \frac{u(7296 + 1638400u + 204800w^2/\mu^3) + w(2048/\mu^5 + 8/\mu)}{\mu}.$$

In detail, the four expanded terms before the direct contribution are bounded by

$$\frac{2048u + 1638400u^2}{\mu},\quad \frac{5120u}{\mu},\quad \frac{204800uw^2}{\mu^4},\quad \frac{2048w}{\mu^6}.$$

The direct contribution has integral at most $128u/\mu + 8w/\mu^2$. Adding $|Z(0)| \le 2w/\mu$ gives a total at most $7500(u+w)/\mu$: here $u \le 1/40000$, $w \le 10^{-4}$ and $\mu \ge 4$ suffice. Equation (5.42) proves the phase bound $|Z| \le 8000(u+w)/\mu$. Substituting it into the primitive and residual bounds proves the two amplitude assertions. □

The arrival is at $W = 1$, or $s = L(p)$ for $p = (\mu, u, w)$. Since $DL(p)v = -v_w/w = -\zeta$, differentiation at this moving endpoint gives

$$D(J_*, \log u_+, \log w_+)(p)v = (Z, x, y)(L) - \zeta(G, f, g)_*. \tag{5.51}$$

Here $J_* = J(m_*, V_*, 1)$, $u_+ = U_*$, and $w_+ = V_*$. This identity also follows directly from the original-time event: varying $W(T_c; p) = 1$ removes $(\eta_W/W)F$ from the fixed-time variation. Comparing with (5.26) leaves exactly $-\zeta F$. Thus (5.51) is the derivative of the actual first arrival established in Proposition 5.2.

Corollary 5.8 (Derivatives of the returned amplitudes)**.** *The ordinary parameter derivatives of the two returned logarithmic amplitudes satisfy*

$$\begin{aligned}
&|\partial_\mu \log u_+| \le 137L, && |\partial_\mu \log w_+| \le \frac{137L}{\mu^2},\\
&|\partial_u \log u_+| \le 9000\left(\frac{1}{u} + \frac{L}{\mu}\right), && |\partial_u \log w_+| \le \frac{9000(1+L)}{\mu^2},\\
&|\partial_w \log u_+| \le \frac{32\mu}{w}, && |\partial_w \log w_+| \le \frac{1 + 32/\mu}{w}.
\end{aligned} \tag{5.52}$$

Proof. For the first two parameter directions $\zeta = 0$, so Proposition 5.6 applies directly at $s = L$. Use $2048/15 < 137$, $u \le 1$ and $\mu \ge 1$ to obtain the first four bounds.

For the growing direction, $W(L) = 1$ and $wL \le 1$. The amplitude bounds in (5.50) then imply

$$|x_w(L)| \le \frac{16\mu}{w}, \qquad |y_w(L)| \le \frac{200}{\mu^3 w}.$$

To check the constants, multiply the first proposed bound by μw and the second by $\mu^3 w$. The resulting numerators are

$$32\mu w + 12800\mu uw + 96 + 1024000wL(u+w),$$
$$12800\mu uw + 64 + 1024000wL(u+w).$$

Use $\mu uw \le 10^{-8}$, $u+w \le 1/8000$ and $wL \le 1$; these are at most $16\mu^2$ and 200, respectively. Finally $|f_*| \le 2\mu$ and $|g_*| \le 1 + 8/\mu$. The endpoint correction in (5.51) therefore gives

$$|\partial_w \log u_+| \le \frac{18\mu}{w} \le \frac{32\mu}{w}, \qquad |\partial_w \log w_+| \le \frac{1 + 8/\mu + 200/\mu^3}{w} \le \frac{1 + 32/\mu}{w}.$$

□

The large logarithmic derivatives are paired with the exponential decay of the returned walls in (5.12). For the radial derivative, we next convert the matching-phase variation back to the radius of the next core. This conversion must retain the small endpoint defects, just as in the value estimate.

5.3. **The differential of a long reflection.** The matching phase differs from the radial phase by a constant on each single-wall boundary. At the actual entrance and exit, the remaining wall produces a small norm defect. The same defects enter the derivative, where their wall factors are essential: they absorb the large logarithmic derivatives from the preceding subsection. We first recover the radial derivative, then combine it with transverse contraction to show that a long connection preserves the square-root cone.

Retain the connection parameters $p = (\mu, u, w)$, with $\mu \ge 4$, $0 < u \le 10^{-4}/\mu$ and $0 < w \le 10^{-4}$. Write $L = \log(1/w)$ and

$$r_- = H(\mu + 1, 1), \qquad r_+ = H(m_* - 1, 1), \qquad \Delta(p) = r_+ - r_- + 2. \tag{5.53}$$

The output in core coordinates is $\mathcal{C}(p) = (r_+, u_+, w_+)$.

5.3.1. *The endpoint factors.* The exponential bounds (5.12) give three useful endpoint estimates:

$$\frac{u_+}{w} \le \frac{8u}{\mu}, \qquad w_+^2 \le w^2, \qquad (1 + L)w_+^2 \le \mu w^2. \tag{5.54}$$

Here is a direct verification that also explains the loss of one radial power in the last estimate. The inequality $(1 + x)e^{-x} \le 1$, valid for $x \ge 0$, implies $xe^{-x} \le 1$. Since $L \ge 1/2$ and $\mu \ge 4$,

$$\frac{u_+}{w} \le ue^{-\mu L/2 + L} \le ue^{-\mu L/4} \le \frac{8u}{\mu}.$$

Also $w_+^2 \le w^2 e^{-4L/\mu}$. Multiplying by $1 + L$ and using $1 + L \le \mu(1 + 4L/\mu)$ proves the last assertion in (5.54).

The moving-clock term in the phase derivative is small for the same reason. At the exit $W = 1$, the phase-rate bound (5.18), divided by $q \ge \mu/2$, yields

$$|G_*| \le 10u_+ + \frac{8w_+^2}{\mu^3}.$$

Consequently

$$\frac{|G_*|}{w} \le \frac{80u + 8w}{\mu} \le \frac{100(u + w)}{\mu}. \tag{5.55}$$

To convert to the radius, put $v_* = w_+$ and define

$$\chi = \frac{\mu + 1}{r_-}, \qquad \sigma = \frac{(m_* - 1)/r_+}{S(m_*, v_*, 1)}, \qquad \omega = \frac{v_*^2}{H(m_* + 1, v_*)}. \tag{5.56}$$

The input radius has derivative $Dr_- v = \chi v_\mu$. At the output, differentiating $J_* = J(m_*, v_*, 1)$ gives

$$DJ_* v = S_* Dm_* v + \frac{v_*^2}{H(m_* + 1, v_*)} D \log w_+ v.$$

Eliminating $Dm_* v$ from $Dr_+ v = ((m_* - 1)/r_+)Dm_* v$ thus gives the exact identity

$$D\Delta(p)v = \sigma(\widehat{Z}_v - \omega \widehat{y}_v) - \chi v_\mu, \quad \widehat{Z}_v = DJ_* v, \quad \widehat{y}_v = D \log w_+ v. \tag{5.57}$$

The hats distinguish actual arrival derivatives from the fixed-clock variations of Section 5.2.

The factors in (5.56) satisfy

$$0 \le \sigma \le \frac{4}{3}, \qquad |\sigma - 1| \le \frac{4v_*^2}{\mu^2}, \qquad 0 \le \omega \le \frac{v_*^2}{\mu}, \qquad |S_0 - \chi| \le \frac{2w^2}{\mu^2}, \tag{5.58}$$

where $S_0 = S(\mu, 1, w)$. Indeed, the ratio-defect identity used in the phase cancellation gives

$$\sigma - 1 = \frac{v_*^2 K_+(m_*, v_*, 1)}{S(m_*, v_*, 1)}, \qquad S_0 - \chi = -w^2 K_-(\mu, 1, w). \tag{5.59}$$

Now use $S \ge 3/4$, $K_\pm \le 2/\mu^2$ and $H(m_* + 1, v_*) \ge \mu$. These formulas retain the square of the small endpoint wall even when the waiting time is arbitrarily large.

Proposition 5.9 (Derivative of the radial error)**.** *For $\mu \ge 4$, $0 \le u \le 10^{-4}/\mu$ and $0 < w \le 10^{-4}$, the actual first-arrival error in* (5.53) *satisfies*

$$\begin{aligned} |\partial_\mu \Delta| &\le \frac{50000(u + w^2)}{\mu^2}, \quad |\partial_u \Delta| \le \frac{50000}{\mu}, \\ |\partial_w \Delta| &\le \frac{50000(u + w)}{\mu}. \end{aligned} \tag{5.60}$$

Proof. First assume $u > 0$. For $i = \mu, u, w$, write $\widehat{Z}_i = \partial_i J_*$ and $\widehat{y}_i = \partial_i \log w_+$. The phase and strong-wall responses in Proposition 5.6 have no clock correction. They give

$$|\widehat{Z}_\mu| \le \frac{16}{15}, \qquad |\widehat{Z}_\mu - S_0| \le \frac{35000(u + w^2)}{\mu^2}, \qquad |\widehat{Z}_u| \le \frac{70}{\mu}. \tag{5.61}$$

For the middle estimate, multiply the expression for Λ in (5.41) by $16/15$ and use $\mu^{-3} \le \mu^{-2}/4$. In the growing-wall direction, Proposition 5.7 and the actual clock correction (5.55) give

$$|\widehat{Z}_w| \le \frac{8100(u + w)}{\mu}.$$

The weak logarithmic derivatives enter multiplied by v_*^2/μ. Equations (5.52) and (5.54) imply

$$\frac{v_*^2}{\mu}|\widehat{y}_\mu| \le \frac{137w^2}{\mu^2}, \qquad \frac{v_*^2}{\mu}|\widehat{y}_u| \le \frac{9000w^2}{\mu^2}, \qquad \frac{v_*^2}{\mu}|\widehat{y}_w| \le \frac{9w}{\mu}. \tag{5.62}$$

For the first two bounds, use

$$|Y| \le \frac{C(1 + L)}{\mu^2} \quad \Longrightarrow \quad \frac{v_*^2}{\mu}|Y| \le \frac{Cw^2}{\mu^2}.$$

For the third, $1 + 32/\mu \le 9$ and $v_*^2 \le w^2$ suffice.

In the radial direction, expand the exact formula as

$$\partial_\mu \Delta = (\widehat{Z}_\mu - S_0) + (\sigma - 1)\widehat{Z}_\mu - \sigma\omega\widehat{y}_\mu + (S_0 - \chi).$$

Its absolute value is at most

$$|\widehat{Z}_\mu - S_0| + \frac{4v_*^2}{\mu^2}|\widehat{Z}_\mu| + \frac{4}{3}\frac{v_*^2}{\mu}|\widehat{y}_\mu| + \frac{2w^2}{\mu^2}.$$

The second term is at most $5w^2/\mu^2$. Substitution of (5.61) and (5.62) proves the first bound in (5.60). For the other two directions the last term of (5.57) is zero, so

$$|\partial_i \Delta| \le \frac{4}{3}\left(|\widehat{Z}_i| + \frac{v_*^2}{\mu}|\widehat{y}_i|\right), \qquad i = u, w.$$

The remaining estimates follow from the same bounds, using $w^2/\mu^2 \le 1/\mu$ in the strong direction.

At $u = 0$, the actual return is locally C^2 by Proposition 5.2; the two norms defining Δ are smooth there. Apply the positive-wall estimates to

$$p_n = \left(\mu, \frac{10^{-4}}{\mu(n+1)}, w\right), \qquad n \geq 0.$$

They tend to $(\mu, 0, w)$ within the admissible region. Continuity of $D\Delta$ and of the right-hand sides proves all three bounds on this boundary. □

The derivative estimates also retain the actual amplitude on a square-root direction. If

$$\chi v_\mu = 1, \qquad |v_u| \leq \sqrt{u}, \qquad |v_w| \leq \sqrt{w}, \tag{5.63}$$

then $|v_\mu| \leq 2$, since $\chi \geq 7/8$. Linearity and (5.60) give

$$|D\Delta(p)v| \leq \frac{50000}{\mu}\left(\frac{2(u+w^2)}{\mu} + \sqrt{u} + (u+w)\sqrt{w}\right) \leq \frac{150000\sqrt{u+w}}{\mu}. \tag{5.64}$$

For the last step put $b = u + w \leq 1$. Each of the three terms in parentheses is at most $\sqrt{b}$, using $b \leq \sqrt{b}$, $w^2 \leq w$ and $\mu \geq 4$. If instead the size is expressed through $u \leq \varepsilon/\mu$, $w \leq \varepsilon \leq 1$, the same parenthesis is at most $4\sqrt{\varepsilon}$. In this form,

$$|D\Delta(p)v| \leq \frac{200000\sqrt{\varepsilon}}{\mu}. \tag{5.65}$$

Indeed, its first term is at most $\sqrt{\varepsilon}$, its second at most $\sqrt{\varepsilon}$, and its last at most $2\sqrt{\varepsilon}$.

5.3.2. *An invariant cone through the connection.* Choose $\varepsilon_\ell > 0$ so that

$$\varepsilon_\ell \leq 10^{-4}, \qquad \sqrt{\varepsilon_\ell} \leq 10^{-6}, \qquad \varepsilon_\ell \leq e^{-256}. \tag{5.66}$$

For example, the minimum of 10^{-12} and e^{-256} has these properties. The small threshold supplies a lower bound on the waiting length, while the estimates remain uniform as both μ and L become unbounded.

Proposition 5.10 (The connection cone)**.** *Suppose that*

$$\mu \geq 4, \qquad 0 < u \leq \varepsilon_\ell/\mu, \qquad 0 < w \leq \varepsilon_\ell.$$

For a direction (5.63)*, put* $d = D\mathcal{C}(p)v = (d_r, d_u, d_w)$ *and* $\gamma = 1 - e^{-L/(2\mu)}$*. Then*

$$d_r \geq \frac{3}{4}, \qquad |d_r - 1| \leq \frac{1}{4\mu}, \tag{5.67}$$

$$|d_u| \leq \frac{1}{3}\sqrt{u_+}\, d_r, \qquad |d_w| \leq (1 - \gamma/2)\sqrt{w_+}\, d_r. \tag{5.68}$$

In particular the cone $|v_u| \leq \sqrt{u}\,|\chi v_\mu|$, $|v_w| \leq \sqrt{w}\,|\chi v_\mu|$ *is mapped into the corresponding square-root cone at the output.*

Proof. The physical differential has components

$$d_r = 1 + D\Delta(p)v, \qquad d_u = u_+ D\log u_+(p)v, \qquad d_w = w_+ D\log w_+(p)v. \tag{5.69}$$

The first follows by differentiating $r_+ = r_- - 2 + \Delta$; the others use positivity of the returned walls. Equation (5.65) gives $|d_r - 1| \leq 1/(4\mu)$, hence $d_r \geq 3/4$.

Let $\ell_u = D\log u_+(p)v$ and $\ell_w = D\log w_+(p)v$. The six amplitude derivatives and $|v_\mu| \leq 2$ imply

$$\begin{aligned} |\ell_u| &\leq 274L + 9000\left(\frac{1}{u} + \frac{L}{\mu}\right)\sqrt{u} + \frac{32\mu}{w}\sqrt{w}, \\ |\ell_w| &\leq \frac{274L}{\mu^2} + \frac{9000(1+L)}{\mu^2}\sqrt{u} + \frac{1 + 32/\mu}{w}\sqrt{w}. \end{aligned} \tag{5.70}$$

The wall decay supplies

$$\sqrt{u_+} \le \sqrt{u}\, e^{-\mu L/4}, \qquad \sqrt{w_+} \le \sqrt{w}\, e^{-L/\mu}. \tag{5.71}$$

We estimate these products before dividing by the radial derivative.

For the strong wall, set $E = e^{-\mu L/4}$. Multiplying its row in (5.70) by $\sqrt{u}E$ gives

$$\sqrt{u_+}|\ell_u| \le 274\sqrt{u}\, LE + 9000E + 9000\frac{u}{\mu}LE + 32\sqrt{u}\,\mu e^{(-\mu/4+1/2)L}.$$

Here $w = e^{-L}$ was used in the last term. Since $L \ge 256$, we have $E \le w$, $LE \le 1$, and

$$\mu e^{(-\mu/4+1/2)L} \le \mu e^{-\mu/8} \le 8.$$

The first inequality uses $\mu \ge 4$ and $L \ge 1$; the second again follows from $xe^{-x} \le 1$. As $\sqrt{u} \le \sqrt{\varepsilon_\ell}$, $E \le \varepsilon_\ell \le \sqrt{\varepsilon_\ell}$ and $u/\mu \le \sqrt{\varepsilon_\ell}$, the four terms sum to at most $20000\sqrt{\varepsilon_\ell}$. Thus

$$\sqrt{u_+}|\ell_u| \le \frac{1}{4} \le \frac{1}{3}d_r. \tag{5.72}$$

For the weak wall, $u \le 1$ bounds the first two terms of its logarithmic row by $10000(1 + L)/\mu^2$. Consequently

$$\sqrt{w_+}|\ell_w| \le e^{-L/\mu}\left(1 + \frac{32}{\mu} + \frac{10000(1+L)\sqrt{w}}{\mu^2}\right).$$

The threshold gives $40000\sqrt{w} \le 1$. Together with $\mu \ge 4$ and $L \ge 256$, this implies

$$\frac{32}{\mu} + \frac{10000(1+L)\sqrt{w}}{\mu^2} \le \frac{32}{\mu} + \frac{1+L}{4\mu^2} \le \frac{L}{4\mu}.$$

The last inequality is $128\mu + 1 + L \le \mu L$. Using $1 + x \le e^x$ now gives

$$\sqrt{w_+}|\ell_w| \le e^{-L/\mu}(1 + L/(4\mu)) \le e^{-3L/(4\mu)} \le e^{-L/(2\mu)} = 1 - \gamma. \tag{5.73}$$

It remains to compare this last bound with d_r. For $0 \le x \le 1$, $e^{-x} \le (1+x)^{-1} \le 1 - x/2$. Apply this with $x = 2/\mu$ and use $L \ge 4$ to obtain

$$\frac{1}{\mu} \le \gamma \le 1, \qquad d_r \ge 1 - \frac{\gamma}{4}. \tag{5.74}$$

It follows that

$$(1 - \gamma/2)d_r \ge (1 - \gamma/2)(1 - \gamma/4) \ge 1 - \gamma.$$

Equations (5.69), (5.72) and (5.73) prove both margins.

For a general cone direction with $\chi v_\mu \ne 0$, divide by χv_μ to reduce to the unit direction just treated, then use linearity and absolute values. If $\chi v_\mu = 0$, the cone inequalities force $v = 0$. This proves the full cone assertion, including both orientations. □

The strong component has a fixed margin. The weak margin retains its dependence on L/μ and can become small at large radius. Its comparison with the radial error in (5.74) is what allows the same cone to pass through the whole unbounded range.

For later control of families, it is useful to know how much slope a radial direction acquires. This follows from the same amplitude derivatives without repeating the variational argument.

Corollary 5.11 (Slope of a narrower direction)**.** *Under the hypotheses of Proposition 5.10, let* $0 \le \theta \le 1$, $\chi v_\mu = 1$ *and*

$$|v_u| \le \theta\sqrt{u}, \qquad |v_w| \le \theta\sqrt{w}.$$

Put $d = D\mathcal{C}(p)v$ *and*

$$S_\ell = \frac{\sqrt{u} + (1+L)\sqrt{w}}{\mu^2}.$$

Then $d_r \ge 3/4$ and

$$\max\left\{\frac{|d_u|}{\sqrt{u_+}\,d_r}, \frac{|d_w|}{\sqrt{w_+}\,d_r}\right\} \le 30000(\theta + S_\ell). \tag{5.75}$$

Proof. First consider the radial direction $v^0 = (\chi^{-1}, 0, 0)$. Its logarithmic responses are bounded by $274L$ and $274L/\mu^2$. For $L \ge 1$,

$$Le^{-\mu L/4} \le \frac{64}{\mu^2}. \tag{5.76}$$

To see this, let $E_0 = e^{-\mu L/8}$. The elementary exponential inequality gives $\mu L E_0 \le 8$; since $L \ge 1$, also $\mu E_0 \le 8$. Multiplication proves (5.76). Combining the logarithmic derivatives with (5.71) therefore gives

$$\begin{aligned} |D\mathcal{C}_u(p)v^0| &\le \frac{17536\sqrt{u}}{\mu^2}\sqrt{u_+}, \\ |D\mathcal{C}_w(p)v^0| &\le \frac{274L\sqrt{w}}{\mu^2}\sqrt{w_+}. \end{aligned} \tag{5.77}$$

Both coefficients are at most $20000S_\ell$.

For any unit-phase direction in the full cone, Proposition 5.10 gives $d_r \le 2$ and hence $|d_u| \le 2\sqrt{u_+}$, $|d_w| \le 2\sqrt{w_+}$. If $\theta > 0$, write

$$v = \theta z + (1-\theta)v^0, \qquad z = v^0 + \theta^{-1}(v - v^0).$$

The vector z is in that full cone and has unit input phase. Linearity and (5.77) imply, for either output wall $j = u, w$,

$$|d_j| \le (2\theta + 20000S_\ell)\sqrt{j_+}.$$

For $\theta = 0$ the same assertion follows directly from $v = v^0$. Finally $d_r \ge 3/4$ and $2\theta + 20000S_\ell \le 30000(\theta + S_\ell)d_r$. The output walls are positive, so division proves (5.75). □

We can now express the estimates in the radius used at successive cores. The reflection map $\mathcal{F}$ was defined in (5.21); its input is the downward core exit (R, u, w), and its output is the next upward core entrance.

Corollary 5.12 (Long reflection in core coordinates)**.** *There exists $\varepsilon_\ell > 0$ such that, for*

$$R \ge \frac{11}{2}, \qquad 0 < u \le \frac{\varepsilon_\ell}{R}, \qquad 0 < w \le \frac{\varepsilon_\ell}{R},$$

the actual reflection time and map $\mathcal{F} = (R_+, u_+, w_+)$ are locally C^2. The value estimates are

$$|R_+ - (R-2)| \le \frac{48(u+w)}{R}, \qquad 0 < u_+ \le u, \qquad 0 < w_+ \le w. \tag{5.78}$$

For every $v = (1, a, b)$ with $|a| \le \sqrt{u}$ and $|b| \le \sqrt{w}$, the differential $d = D\mathcal{F}(R,u,w)v$ satisfies

$$\begin{aligned} d_R &\ge \frac{1}{2}, & |d_R - 1| &\le \frac{300000\sqrt{u+w}}{R}, \\ |d_u| &\le \sqrt{u_+}\,d_R, & |d_w| &\le \sqrt{w_+}\,d_R. \end{aligned} \tag{5.79}$$

Proof. Choose the threshold in (5.66). The coordinate change

$$(R, u, w) \longmapsto p = (\sqrt{R^2 - 1} - 1, u, w)$$

has $\mu \ge 4$ and $R/2 \le \mu < R$ on this range. Thus $u \le \varepsilon_\ell/\mu$ and $w \le \varepsilon_\ell$, as required for the connection. The first arrival and its local C^2 dependence were proved in Proposition 5.2.

By reflection equivariance, its output is the next positive entrance, as in (5.21). The value estimates follow from (5.22) and positivity of the walls.

The coordinate differential leaves the two wall components unchanged and sends a unit radial input to a vector with $\chi v_\mu = 1$, because $H(\mu+1,1) = R$. The chain rule therefore identifies d with the differential in Proposition 5.10. Its radial lower bound is at least $3/4$, and both transverse margins are at most one. The error bound follows from (5.64) and

$$\frac{150000\sqrt{u+w}}{\mu} \le \frac{300000\sqrt{u+w}}{R}.$$

□

The connection now has the same form of phase control as the core: a value error proportional to the incoming amplitude, a derivative error proportional to its square root, and preservation of the same cone. To use this control throughout an era, the reflection estimates must also cover the remaining bounded range of exit radii. There the waiting time can still be long, so the small growing wall must remain part of the estimates.

6. Reflections at bounded radius

The bounded range completes the reflection estimates down to the terminal threshold. Here a finite outgoing passage and a possibly long incoming passage are joined through the single-wall geometry. The stable boundary phase organizes their derivatives; the final result combines this range with the large-radius estimate.

6.1. **Outgoing and incoming passages.** At bounded exit radii, the outgoing part of a connection belongs to a compact family of finite orbit segments. Its endpoint can be chosen where the departing wall has a fixed small height η. After reflection, the next wall is still small and must grow to height one. In the projective clock, this second passage takes a time of order $\log(1/v)$, where v is the initial height of the growing wall. We first estimate that passage, retaining the decay accumulated during the wait, and then join it to the finite outgoing segment.

6.1.1. *Arrival from a small active wall.* Use the projective coordinates in the incoming frame, so that V is the growing wall. Write the initial point as $y = (m,u,v,w)$ and put

$$k = H(m-1,v), \qquad \varepsilon_0 = 10^{-4}.$$

The radius k labels this incoming passage. A dot denotes projective time and t denotes the original normalized time.

Proposition 6.1 (Incoming passage)**.** *Suppose*

$$k \ge \frac{7}{4}, \qquad m \ge 1, \qquad 0 \le u,w \le \varepsilon_0, \qquad 0 < v < 1. \tag{6.1}$$

There is a first positive time $T_{\mathrm{in}}(y)$ at which $V = 1$. The projective chart remains regular up to that time, V is strictly increasing, and the arrival is transverse with $M > 1$. The arrival time and state $\mathcal{P}(y)$ are locally C^2 in all four ordinary initial coordinates, including at $u = 0$ or $w = 0$.

Set $L = \log(1/v)$ and use $s = \log(V/v)$ along the passage. For $0 \le s \le L$,

$$0 \le U(s) \le ue^{-(1+k/3)s}, \qquad 0 \le W(s) \le we^{-(1+1/k)s}, \tag{6.2}$$

$$|\rho(s) - k| \le \frac{12}{k}\left[u(v+w) + u^2 + w^2\right] < \frac{k}{200}, \qquad \rho = H(M-1,V). \tag{6.3}$$

The durations in the two time variables satisfy

$$T_{\mathrm{in}}(y) \le \frac{2(k+1)^2}{k}L, \qquad \frac{L}{2k} \le \tau_{\mathrm{in}} \le \frac{2L}{k}. \tag{6.4}$$

At arrival the wall bounds become

$$\mathcal{P}_U(y) \le u v^{1+k/3}, \qquad \mathcal{P}_W(y) \le w v^{1+1/k}. \tag{6.5}$$

Each output wall is positive exactly when its input wall is positive. If in addition $k \le 4$, then

$$\mathcal{P}_U(y) \le u v^{19/12}, \qquad \mathcal{P}_W(y) \le w v^{13/9}. \tag{6.6}$$

Proof. We first work on any initial segment in the closed region

$$M \ge 1, \qquad |\rho - k| \le \frac{k}{100}, \qquad 0 \le U, W \le \varepsilon_0, \qquad 0 \le V \le 1. \tag{6.7}$$

Along the actual orbit, V remains positive by its multiplicative equation and the assumption $v > 0$. The identity $\rho^2 = (M-1)^2 + V^2$, together with $k \ge 7/4$, gives

$$\frac{4}{5}k \le M - 1 \le \frac{101}{100}k. \tag{6.8}$$

For the lower bound, square both sides and use $(99k/100)^2 - 1 \ge (4k/5)^2$; the upper bound follows from $M - 1 \le \rho$.

Recall $C = U(V + W - U)$ and put

$$q = M - 1 - C, \qquad a = \tfrac{1}{2}(1 - M^2 - (V - W)^2 + U^2).$$

Since $-U^2 \le C \le 2U$, the numerical margins in (6.8) imply

$$\frac{3}{4}k \le q \le \frac{21}{20}k. \tag{6.9}$$

In particular, $ds/d\tau = q > 0$. The two decreasing walls obey

$$\frac{a}{q} \le -\left(1 + \frac{k}{3}\right), \qquad \frac{-M - 1 - C}{q} \le -\left(1 + \frac{1}{k}\right). \tag{6.10}$$

Here the identities that retain the needed signs are

$$\frac{a}{q} = -\frac{M+1}{2} + \frac{-(V-W)^2 + U^2 - (M+1)C}{2q},$$
$$\frac{-M-1-C}{q} = -1 - \frac{2(1+C)}{q}.$$

For the first remainder, $C \ge -U^2$ and $M + 2 \le 4q$ give an upper bound $2U^2$. Thus the first rate is at most $-1 - 2k/5 + 2\varepsilon_0^2 \le -1 - k/3$. For the second rate, $2(1 + C)/q \ge 2(1 - \varepsilon_0^2)/((21/20)k) \ge 1/k$. Integrating the scalar inequalities proves (6.2) on every such segment. A zero wall remains zero by its multiplicative equation, so these estimates also hold on the boundary.

The radial equation, obtained from (2.9), is

$$\dot\rho = -C\rho + \frac{M-1}{\rho}(U - W)^2.$$

In the present region, $\rho/q \le 2$, $0 \le (M-1)/\rho \le 1$, and $1/q \le 2/k$. Also $|C| \le U(V + W) + U^2$ and $(U - W)^2 \le U^2 + W^2$. Consequently

$$\left|\frac{d\rho}{ds}\right| \le 4\left[U(V+W) + U^2 + \frac{W^2}{k}\right].$$

The time-dependent terms in this bound are integrable uniformly in the length of the passage. Indeed, $V = ve^s$ and (6.2) give

$$U(V+W) + U^2 + \frac{W^2}{k} \le \big[u(v+w) + u^2\big]e^{-ks/3} + \frac{w^2}{k}e^{-2s}.$$

Integration from zero to any current clock value yields

$$|\rho(s)-k| \le \frac{12}{k}\big[u(v+w)+u^2\big] + \frac{2w^2}{k} \le \frac{12}{k}\big[u(v+w)+u^2+w^2\big]. \tag{6.11}$$

The bracket is at most $\varepsilon_0 + 3\varepsilon_0^2$. Since $12(\varepsilon_0 + 3\varepsilon_0^2) < (7/4)^2/200$, this improves the radius band strictly to (6.3).

We next show that the actual solution reaches the section while these estimates hold. On (6.7),

$$0 < D \le 6(k+1)^2, \qquad A = \frac{6}{D} \ge \frac{1}{(k+1)^2}. \tag{6.12}$$

For positivity, use $D \ge 3 + M^2 - 2U(V+W) > 0$. For the upper bound, drop the negative cross terms in D, use $M \le 2(k+1)$ and $U, V, W \le 1$, and obtain $D \le 4(k+1)^2+6 \le 6(k+1)^2$. Thus the projective chart is regular initially. The normalized solution exists for all positive time by the proof of Lemma 2.1, which also applies when a wall is zero.

Continue the initial segment in the closed region with the additional condition $A \ge 1/[2(k+1)^2]$. At an endpoint before $V = 1$, (6.11) puts the radius strictly inside its allowed band, and (6.8) puts M strictly above one. The upper faces $U = \varepsilon_0$ and $W = \varepsilon_0$ point inward by (6.10); zero walls are invariant. Finally (6.12) improves the additional chart condition. Continuity and these strict margins extend the segment whenever $V < 1$.

An indefinitely long segment with $V < 1$ is impossible, because

$$\frac{ds}{dt} = Aq \ge \frac{k}{2(k+1)^2}.$$

The clock starts at zero and is less than L before arrival. This proves existence of a first arrival and the original-time bound in (6.4). Strict positivity of AqV proves monotonicity and transversality. Integrating $d\tau = ds/q$ and using (6.9) gives the projective-time bounds. Substituting $s = L$ into (6.2) proves (6.5). The multiplicative equations give positivity of every wall that was initially positive.

For local regularity, fix one such finite passage. Its flow domain, positive chart and positive derivative dV/dt persist on the compact segment for nearby initial points and nearby endpoint times. The implicit function theorem gives a local C^2 solution of the arrival equation. Since initially $V < 1$ and its derivative remains positive up to that event, this solution is still the first arrival. Composition with the flow gives the C^2 arrival state. This argument takes place in an ordinary open neighborhood in $\mathbb{R}^4$, so it also covers zero initial U or W.

Finally suppose $k \le 4$. Then $q \le 21/5$, and

$$\frac{2(1+C)}{q} \ge \frac{2(1-\varepsilon_0^2)}{21/5} \ge \frac{4}{9}.$$

The weak-wall rate in (6.10) improves to $-13/9$, while the strong-wall rate is at most $-19/12$ because $k \ge 7/4$. Integrating to $s = L$ proves (6.6). □

6.1.2. *A finite outgoing segment.* Return now to the frame of the preceding core. Its downward exit is

$$y_-(R,u,w) = (1-\sqrt{R^2-1}, u, 1, w), \qquad \frac{31}{8} \le R \le 6. \tag{6.13}$$

The interval extends below the reflection threshold $R = 4$, providing a neighborhood in which the returning maps can be differentiated. Fix a height $0 < \eta \le 1/8$. Let S_η be the first positive time at which the old wall V reaches η, and let $y_\eta = (M_\eta, U_\eta, \eta, W_\eta)$ denote the state there. The axis reflection acts in these coordinates by

$$\mathsf{S}(M,U,V,W) = (-M,U,W,V).$$

Thus $\mathsf{S}y_\eta$ is an entrance for Proposition 6.1 once its walls and radius lie in the stated range.

Lemma 6.2 (The outgoing segment and its reflected entrance)**.** *For each fixed $\eta \in (0, 1/8]$ there are $\delta > 0$ and $B > 0$ such that, on*

$$\frac{31}{8} \le R \le 6, \qquad |u|, |w| \le \delta, \tag{6.14}$$

the first arrival at $V = \eta$ exists, its whole segment lies in the projective chart, and

$$\frac{dV}{dt} < 0, \qquad |W| < \frac{1}{2} \quad \text{on } [0, S_\eta].$$

The time and state are C^2 on a neighborhood of this box. There are positive C^2 factors B_U, B_W such that

$$\mathsf{S} y_\eta = (-M_\eta, uB_U, wB_W, \eta). \tag{6.15}$$

For $j = U, W$ they satisfy

$$B_j + B_j^{-1} + \|D \log B_j\| + \|D^2 \log B_j\| \le B. \tag{6.16}$$

With $k_\eta = H(-M_\eta - 1, W_\eta)$, one has

$$-M_\eta - 1 \ge \frac{7}{4}, \qquad \frac{7}{4} \le k_\eta \le 4 - \frac{\eta^2}{24} < 4, \tag{6.17}$$

$$\left| k_\eta - \left(\sqrt{R^2 - \eta^2} - 2 \right) \right| \le B(|u| + |w|) \le \frac{\eta^2}{24}. \tag{6.18}$$

The norms in (6.16) *are induced by the maximum norm on the three entrance parameters.*

Proof. On the single-wall orbit (4.58), with $\kappa = R$, this segment runs from $\tau = \operatorname{arcosh}(R)/R$ to $\tau = \operatorname{arcosh}(R/\eta)/R$. Its endpoint is

$$y_\eta(R, 0, 0) = (1 - \sqrt{R^2 - \eta^2}, 0, \eta, 0).$$

The reference V is strictly decreasing throughout, its derivative is negative at the endpoint, and $D > 0$. The segment is finite in original time as well. Smooth dependence and the implicit function theorem therefore give a C^2 first arrival near each $(R, 0, 0)$. Negativity of dV/dt on the compact segment ensures that the nearby arrival is still the first one. It also persists up to the moving endpoint, as do the chart condition and $|W| < 1/2$.

The reference parameter interval is compact. As in Lemma 4.12, take a closed neighborhood inside the open set of these regular first arrivals. The time, state and their first two derivatives are bounded there. Restricting to a smaller box (6.14) and applying the mean-value bound along its wall directions gives

$$\|y_\eta(R, u, w) - y_\eta(R, 0, 0)\| \le B_0(|u| + |w|) \tag{6.19}$$

for some $B_0 > 0$.

The positive factors can be defined directly from the normalized wall equations. If x is the normalized initial point associated with (6.13), set

$$B_j = \exp\left(\log A(0) - \log A(S_\eta) + \int_0^{S_\eta} \lambda_{i(j)}(\phi_t x)\, dt \right), \qquad i(U) = 1, \quad i(W) = 3, \tag{6.20}$$

where the rates λ_i are those in the proof of Lemma 2.1. Multiplying the exact normalized wall formula by $\sqrt{3}/A(S_\eta)$ gives (6.15), including for zero or signed walls. The logarithms of the factors are C^2: the finite moving integral may be written as $S_\eta \int_0^1 \lambda_{i(j)}(\phi_{aS_\eta} x)\, da$ and differentiated twice on a neighborhood of the compact segment. The factors are positive, so compactness bounds their inverses and the derivatives of their logarithms. This proves (6.16) after enlarging B.

Put $r_0 = \sqrt{R^2 - \eta^2} - 2$. The reference radius has the bounds

$$\frac{15}{8} - \frac{8\eta^2}{31} \le r_0 \le 4 - \frac{\eta^2}{12}. \tag{6.21}$$

Indeed, with $b = \sqrt{R^2 - \eta^2}$,

$$R - b = \frac{\eta^2}{R + b}, \qquad \frac{\eta^2}{2R} \le R - b \le \frac{\eta^2}{R},$$

and $31/8 \le R \le 6$. The state estimate (6.19) bounds both $|(-M_\eta - 1) - r_0|$ and $|W_\eta|$ by $B_0(|u| + |w|)$. For $r_0 \ge 0$ the norm inequality

$$|H(a, b) - r_0| \le |a - r_0| + |b|$$

therefore gives $|k_\eta - r_0| \le 2B_0(|u| + |w|)$. Take $B \ge 2B_0$ and then $\delta \le \eta^2/(48B)$. Both k_η and $-M_\eta - 1$ differ from r_0 by at most $\eta^2/24$. Equation (6.21) and $\eta \le 1/8$ now give (6.17) and (6.18). The lower bound on $-M_\eta - 1$ also makes k_η a C^2 function on a neighborhood of the box. □

6.1.3. *The actual reflection and its wall loss.* The two segments now fit on one exact orbit. The lower radius bound in (6.17) permits the incoming passage; the upper bound gives the fixed powers in (6.6).

Proposition 6.3 (Existence and wall decay at bounded radii)**.** *Fix $0 < \eta \le \varepsilon_0$. There are $\varepsilon > 0$ and $C_\eta > 0$ such that, for*

$$\frac{31}{8} \le R \le 6, \qquad 0 < u, w \le \varepsilon, \tag{6.22}$$

the exact orbit from (6.13) *has a first positive arrival at the old wall $W = 1$. Reflecting at that time gives a state $(M_+, u_+, 1, w_+)$ with $M_+ > 1$. Its time and state, and hence its core coordinates $\mathcal{F} = (H(M_+ - 1, 1), u_+, w_+)$, are locally C^2. Moreover,*

$$0 < u_+ \le C_\eta u w^{19/12} \le u, \qquad 0 < w_+ \le C_\eta w^{13/9} \le w. \tag{6.23}$$

The map and time agree locally with the composition of the two passages just constructed.

Proof. Let B and δ be given by Lemma 6.2. Decrease the input threshold so that

$$\varepsilon \le \delta, \qquad B\varepsilon \le \varepsilon_0, \qquad B\varepsilon \le \frac{1}{2}.$$

Then the reflected entrance

$$y_e = \mathsf{S} y_\eta = (-M_\eta, uB_U, wB_W, \eta)$$

has radius $k_\eta \in [7/4, 4)$, first coordinate greater than one, strong wall in $(0, \varepsilon_0]$, growing wall in $(0, 1/2]$, and weak wall equal to $\eta \le \varepsilon_0$. Proposition 6.1 applies. The total time and the reflected final state are

$$T = S_\eta + T_{\mathrm{in}}(y_e), \qquad y_+ = \mathcal{P}(y_e). \tag{6.24}$$

Flow uniqueness and reflection equivariance identify this state with the reflection of the original orbit at time T.

This is the first arrival at the old wall $W = 1$. During the first segment $|W| < 1/2$. During the second segment, the same coordinate is the new V and remains below one until its first arrival. The final derivative of this coordinate is positive, and (6.8) gives $M_+ > 1$. The two constituent first arrivals remain valid in full neighborhoods of the given input and of y_e. Shrink the input neighborhood so that its reflected endpoints lie in the latter neighborhood. The same first-arrival argument applies throughout this neighborhood, identifying the actual map with (6.24). The C^2 assertion follows by composition. Taking the core radius is C^2 as well, since its final active wall equals one.

The compact incoming powers and $B_U, B_W \le B$ give

$$u_+ \le (uB_U)(wB_W)^{19/12} \le B^{1+19/12} u w^{19/12}, \qquad w_+ \le \eta (wB_W)^{13/9} \le \eta B^{13/9} w^{13/9}.$$

Choose $C_\eta = B^{1+19/12} + \eta B^{13/9}$ and reduce ε once more so that

$$C_\eta \varepsilon^{19/12} \le 1, \qquad C_\eta \varepsilon^{4/9} \le 1.$$

These inequalities prove the two nonincrease statements in (6.23). Positivity follows from the positive finite factors and the multiplicative equations during the incoming passage. □

The powers in (6.23) quantify the wall loss accumulated during a possibly long wait at bounded radius. They will also control the transverse derivatives after multiplication by the output walls. The radial estimates require finer information: the incoming drift bound (6.3) still contains the fixed height η. We therefore next retain the cancellation between the outgoing segment and the incoming passage in their common radial phase.

6.2. **Phase along the stable boundary.** During the incoming passage, the two decaying walls move the shear coordinate even before the growing wall becomes appreciable. The endpoint of this boundary motion provides a useful phase. Its level sets collect boundary orbits approaching the same Kasner state. We construct this phase directly from the flow, prove its smoothness, and use it to match the two parts of a bounded-radius connection. It is an auxiliary coordinate for the connection estimates. Its comparison with the radius at the next entrance will recover the observable era phase.

6.2.1. *The limiting Kasner coordinate.* On the invariant boundary $V = 0$, write $y = (m, u, w)$. The vector field in projective time is

(6.25)
$$f(y) = \left(mu^2 - (m+1)uw + w^2, \quad \frac{1 - m^2 - w^2 + u^2}{2} u, \quad -(m + 1 + uw - u^2)w \right).$$

Its expression in the original normalized time is

$$g(y) = \frac{6}{D(y)} f(y), \qquad D(y) = 3 + m^2 + (u - w)^2. \tag{6.26}$$

In particular, the projective denominator is positive throughout this boundary. Signed walls will be allowed in the construction, so that the resulting differentiability statements hold in open sets.

Proposition 6.4 (The stable phase)**.** *Fix $c \geq 2$, put $K = c + 1$, and suppose that*

$$|m - c| \leq \frac{1}{40}, \qquad |u|, |w| \leq \frac{1}{200}. \tag{6.27}$$

The solution $(M(t), U(t), W(t))$ of (6.26) exists for all $t \geq 0$ and remains in

$$|M - c| \leq \frac{1}{10}, \qquad |U|, |W| \leq \frac{1}{100}. \tag{6.28}$$

There is a limit

$$P(m, u, w) = \lim_{t \to \infty} M(t), \qquad U(t), W(t) \longrightarrow 0.$$

For every $t \geq 0$,

$$|P(m, u, w) - M(t)| \leq \frac{8}{K} \big(|U(t)| + |W(t)| \big)^2. \tag{6.29}$$

The value of P is unchanged by restarting this boundary orbit. In particular, $P(m, 0, 0) = m$.

Proof. Let $a = (1 - M^2 - W^2 + U^2)/2$ and $b = -(M + 1 + UW - U^2)$ be the two wall rates in projective time. Throughout (6.28),

$$\frac{5K}{8} \leq M \leq K, \qquad \frac{a}{K} \leq -\frac{K}{8}, \qquad \frac{b}{K} \leq -\frac{1}{4}, \qquad D \leq 2K^2.$$

These follow by substitution, using $K \geq 3$ and the two wall bounds. Set

$$\xi(t) = K \int_0^t \frac{6}{D(M(r), U(r), W(r))} \, dr.$$

The multiplicative wall equations give, on any initial interval in the band,

$$|U(t)| \le |u|e^{-K\xi(t)/8}, \qquad |W(t)| \le |w|e^{-\xi(t)/4}. \tag{6.30}$$

The first equation of (6.25) gives

$$\left|\frac{dM}{d\xi}\right| \le U^2 + 2|U||W| + \frac{W^2}{K} \le (u^2 + 2|u||w|)e^{-K\xi/8} + \frac{w^2}{K}e^{-\xi/2}.$$

Integrating proves

$$|M(t) - m| \le \frac{8}{K}(|u| + |w|)^2 < \frac{1}{40}.$$

Thus $|M - c| < 1/20$ and the wall bounds improve to $1/200$. Also $A = 6/D \ge 3/K^2$. To apply continuation in the original normalized variables, include $A \ge 3/(2K^2)$ in the closed conditions. All conditions improve strictly. The proof of Lemma 2.1 also covers signed curvature variables: $|N_i(t)| \le |N_i(0)|e^t$, and the constraint bounds the shear using the absolute values of the products N_iN_j. Hence the normalized solution continues for all positive time; the lower bound for A keeps its projective coordinates regular. The band therefore holds globally.

We now have $\xi(t) \ge 3t/K$, so both walls tend to zero. The same integral estimate, begun at any time t_0, yields

$$|M(t) - M(t_0)| \le \frac{8}{K}\big(|U(t_0)| + |W(t_0)|\big)^2, \qquad t \ge t_0.$$

It makes $M(t)$ Cauchy at infinity and proves (6.29). Flow uniqueness identifies a restarted orbit with the time translate of the original one. Their first coordinates have the same limit. The zero-wall orbit is constant, giving the last assertion. □

6.2.2. *Smooth dependence of the limiting phase.* The limit must be differentiated with respect to its initial state. For this purpose we keep c, and therefore K, fixed and use

$$Y(s; y) = (M(Ks; y), U(Ks; y), W(Ks; y)), \qquad \frac{dY}{ds} = Kg(Y). \tag{6.31}$$

This is a fixed rescaling of the original time. In particular, varying y leaves the elapsed original time Ks unchanged. The preceding decay estimates become

$$\begin{gathered} |Y_u(s)| \le |u|e^{-3Ks/8}, \qquad |Y_w(s)| \le |w|e^{-3s/4}, \\ B(s) := |Y_u(s)| + |Y_w(s)| \le b_0 e^{-s/2}, \end{gathered} \tag{6.32}$$

where $b_0 = |u| + |w| \le 1/100$. All norms below are supremum norms and their induced operator norms. For a direction $h = (h_m, h_u, h_w)$, write $h_s = (h_u, h_w)$.

Lemma 6.5 (Differential estimates on the boundary)**.** *On* (6.28)*, put* $\beta = |u| + |w|$ *and* $\mathcal{A} = KDg(y)$*. Then*

$$\begin{aligned} |\mathcal{A}_{mm}| &\le 64\beta^2, & |\mathcal{A}_{mu}|, |\mathcal{A}_{mw}| &\le 64\beta, \\ |\mathcal{A}_{um}| &\le 64|u|, & |\mathcal{A}_{wm}| &\le 64|w|, \end{aligned} \tag{6.33}$$

$$\mathcal{A}_{uu} + |\mathcal{A}_{uw}| \le -3K/16, \qquad \mathcal{A}_{ww} + |\mathcal{A}_{wu}| \le -3/8. \tag{6.34}$$

Every full signed row sum of $\mathcal{A}$ *is at most* 256β*. Moreover,*

$$\|KD^2g(y)[h, z]\| \le 1024\left(\beta\|h\|\|z\| + \|h_s\|\|z\| + \|h\|\|z_s\|\right). \tag{6.35}$$

Proof. The polynomial Jacobian is

$$Df = \begin{pmatrix} u^2 - uw & 2mu - (m+1)w & -(m+1)u + 2w \\ -mu & (1 - m^2 - w^2 + 3u^2)/2 & -uw \\ -w & (2u - w)w & -(m + 1 + 2uw - u^2) \end{pmatrix}.$$

The denominator satisfies $K^2 \le 3D$ and $K \le D$, while

$$|f_m/D| \le \beta^2, \qquad |f_u/D| \le |u|, \qquad |f_w/D| \le |w|, \qquad dD = (2m, 2(u-w), 2(w-u)).$$

Insert these bounds into

$$\mathcal{A}_{ij} = \frac{6K}{D}\left((Df)_{ij} - \frac{f_i}{D}\partial_j D\right). \tag{6.36}$$

The center row of Df is bounded by $(\beta^2, 2K\beta, 2K\beta)$, and its center column below the diagonal by $(K|u|, |w|)$. This gives (6.33). For the stable rows, direct substitution in the displayed matrix gives

$$(Df)_{uu} + |(Df)_{uw}| + 4\beta^2 \le -K^2/16, \qquad (Df)_{ww} + |(Df)_{wu}| + 4\beta^2 \le -K/8.$$

The two rational corrections in each stable row of (6.36) contribute at most $4\beta^2$ inside the parentheses. Since $D \le 2K^2$, this proves (6.34). Both stable row sums are at most $-1/4$. Adding the center column, and using $\beta \le 1$, proves the bound for the full rows.

To estimate the Hessian, retain the distinction between center and stable directions. Differentiating the matrix above gives

$$\|D^2 f[h,z]\| \le 12K\|h\|\|z\|, \qquad D^2 f[(h_m,0,0),(z_m,0,0)] = (0, -uh_m z_m, 0).$$

For the first bound, the sums of the absolute coefficient bounds in the three rows are at most $12K$, $9K$, and $10K$, respectively. Splitting each direction into its center and stable parts therefore gives, with

$$R_0 = \beta\|h\|\|z\|, \qquad S_0 = \|h_s\|\|z\| + \|h\|\|z_s\|,$$

the sharper bound $\|D^2 f[h,z]\| \le R_0 + 12KS_0$. We also have

$$\|g\| \le 6\beta, \qquad |dD[h]| \le 6K\|h\|, \qquad |D^2 D[h,z]| \le 10\|h\|\|z\|,$$
$$\|\mathcal{A}h\| \le 64\beta\|h\| + 36K\|h_s\|.$$

Here $D^2 D[h,z] = 2h_m z_m + 2(h_u - h_w)(z_u - z_w)$. For the last bound, the center column is controlled by (6.33); each wall column of Dg is bounded by 18, using $|(Df)_{ij}| \le 2D$ for $j = u, w$ in (6.36).

Twice differentiating the product identity $D(y)g(y) = 6f(y)$ gives

$$KD^2 g[h,z] = \frac{6K}{D}D^2 f[h,z] - \frac{dD[h]}{D}\mathcal{A}z - \frac{dD[z]}{D}\mathcal{A}h - \frac{D^2 D[h,z]}{D}Kg. \tag{6.37}$$

Using $K/D \le 1$ and $K^2/D \le 3$, the four terms are bounded by

$$6R_0 + 216S_0, \qquad 384R_0 + 648\|h\|\|z_s\|,$$
$$384R_0 + 648\|h_s\|\|z\|, \qquad 60R_0.$$

Their sum is at most $834R_0 + 864S_0$, which proves (6.35). □

Proposition 6.6 (Regularity of the stable phase)**.** *The function P is C^2 on each open box*

$$\Omega_c = \{(m,u,w) : |m-c| < 1/40,\ |u|, |w| < 1/200\}, \qquad c \ge 2.$$

Set $b_0 = |u| + |w|$ and $E = e^{512 b_0}$. On this box,

$$|DP(y)h - h_m| \le 256 b_0\|h_s\| + 66000 b_0^2 E\|h\|, \tag{6.38}$$

$$\|D^2 P(y)\| \le 40000 E^3. \tag{6.39}$$

For $m \ge 2$, we also have $DP(y)f(y) = 0$ and $DP(m,0,0)h = h_m$.

Proof. Every finite-time map in (6.31) is C^2. Write $J_h(s) = D_y Y(s;y)h$. Its variational equation is

$$J_h' = \mathcal{A}(Y)J_h, \qquad J_h(0) = h.$$

The signed row comparison used in Lemma 4.6, together with (6.32), yields

$$\|J_h(s)\| \le \exp\left(\int_0^s 256 b_0 e^{-r/2}\,dr\right)\|h\| \le E\|h\|. \tag{6.40}$$

The stable block has row sums at most $-1/4$. Its forcing by the center component has norm at most $64b_0E\|h\|e^{-s/2}$. Multiplying by $e^{s/4}$ and integrating gives

$$\|(J_h(s))_s\| \le (\|h_s\| + 256b_0E\|h\|)e^{-s/4} \le 4E\|h\|e^{-s/4}. \tag{6.41}$$

In the center row, keep the amplitudes in (6.33). They give

$$|(J_h)_m'| \le 64B^2|(J_h)_m| + 128B\|(J_h)_s\| \le Q_1(y,h)e^{-s/2},$$

where

$$Q_1(y,h) = 128b_0\|h_s\| + 33000b_0^2E\|h\|.$$

Consequently, for $t \ge a \ge 0$,

$$|(J_h(t))_m - (J_h(a))_m| \le 2Q_1(y,h)e^{-a/2}. \tag{6.42}$$

Since $b_0 \le 1/100$ and $E < e^6$, the linear maps $D_yY_m(s;y)$ converge uniformly on Ω_c in operator norm.

We next control their derivatives. Put $H_{h,z}(s) = D_y^2Y(s;y)[h,z]$. Differentiating the flow equation once more gives

$$H_{h,z}' = \mathcal{A}(Y)H_{h,z} + KD^2g(Y)[J_h,J_z], \qquad H_{h,z}(0) = 0.$$

By (6.35), (6.40), and (6.41), its forcing obeys

$$\|KD^2g(Y)[J_h,J_z]\| \le 10000E^2\|h\|\|z\|e^{-s/4}. \tag{6.43}$$

The three terms in (6.35) contribute at most E^2, $4E^2$, and $4E^2$, respectively, after removal of the common factor $\|h\|\|z\|e^{-s/4}$. Since $9 \cdot 1024 < 10000$, this proves the forcing bound. The homogeneous growth on every subinterval is at most E. The forced signed row estimate therefore gives

$$\|H_{h,z}(s)\| \le 40000E^3\|h\|\|z\|. \tag{6.44}$$

The center row has absolute row sum at most $192B$. Combining this with (6.43) yields

$$|(H_{h,z})_m'| \le Q_2(y)\|h\|\|z\|e^{-s/4}, \qquad Q_2(y) = 192b_0(40000E^3) + 10000E^2.$$

Thus

$$|(H_{h,z}(t))_m - (H_{h,z}(a))_m| \le 4Q_2(y)\|h\|\|z\|e^{-a/4}.$$

The second derivatives of $Y_m(s;\cdot)$ also converge uniformly on Ω_c.

These uniform limits identify the derivatives of the limiting phase. Take the finite-time functions $F_n(y) = Y_m(n;y)$ and let L_1 be the uniform limit of DF_n. For a line segment in the convex box Ω_c,

$$F_n(z) - F_n(y) = \int_0^1 DF_n(y + \theta(z-y))(z-y)\,d\theta.$$

Pass to the limit using $F_n \to P$ and uniform convergence of DF_n. The resulting identity and continuity of L_1 imply $DP = L_1$. Apply the same argument to the operator-valued functions DF_n and their uniformly convergent derivatives D^2F_n. It follows that DP is C^1 and that its derivative is the second limit. Taking $a = 0$ in (6.42) proves (6.38); (6.44) proves (6.39).

At an interior point the boundary orbit remains in Ω_c for sufficiently small positive time. Differentiate the invariance from Proposition 6.4 at time zero. This gives $DP(y)g(y) = 0$, and hence $DP(y)f(y) = 0$. The zero-wall differential follows from (6.38) with $b_0 = 0$. □

6.2.3. *The phase formula and its wall dependence.* The stable phase has a useful approximation in which the weak wall is kept exactly. Define

$$P_0(m,u,w) = H(m+1,w) - 1 + \frac{m}{m^2-1}u^2 - \frac{2}{m+1}uw. \tag{6.45}$$

The first term is the endpoint phase on the plane $u=0$. The two quadratic corrections cancel the leading motion produced by the strong wall.

Proposition 6.7 (Boundary phase expansion)**.** *Under* (6.27)*,*

$$|P(m,u,w) - P_0(m,u,w)| \le \frac{1024}{K^3}|u|(|u|+|w|)^3. \tag{6.46}$$

In particular,

$$P(m,0,w) = H(m+1,w) - 1. \tag{6.47}$$

Proof. Put $\ell = m+1$, $d = m^2-1$, and $\sigma = H(m+1,w)$. Differentiation along (6.25) gives

$$DP_0 f = r_4u^4 + r_3u^3w + r_2u^2w^2 + r_1uw^3, \tag{6.48}$$

where

$$r_4 = -\frac{2m}{d^2}, \qquad r_3 = \frac{2(3m-1)}{d^2},$$
$$r_2 = -\frac{m^2+1}{d^2} - \frac{m}{d} + \frac{\sigma+1}{\sigma(\sigma+\ell)}, \qquad r_1 = \frac{2}{\ell^2} + \frac{1}{\ell} - \frac{1}{\sigma+\ell}.$$

To see the cancellation, first differentiate σ:

$$D\sigma f = mu^2 - (m+1)uw + \frac{\sigma+1}{\sigma(\sigma+\ell)}u^2w^2 - \frac{1}{\sigma+\ell}uw^3.$$

This uses $\sigma - \ell = w^2/(\sigma+\ell)$ and $1 - \ell/\sigma = w^2/[\sigma(\sigma+\ell)]$. The derivatives of the two rational quadratic terms in (6.45) cancel the first two terms and give (6.48).

For $m \ge 3/2$, the inequalities

$$\frac{1}{d} \le \frac{2}{\ell}, \qquad \frac{m}{d} \le \frac{3}{\ell}, \qquad \frac{m^2+1}{d} \le 3, \qquad \sigma \ge \ell$$

bound $|r_4|, |r_3|, |r_2|, |r_1|$ by $12/\ell, 36/\ell, 11/\ell, 4/\ell$, respectively. All four are therefore at most $64/\ell$. On (6.28), $\ell \ge K/2$, and hence

$$|DP_0 f| \le \frac{128}{K}|u|(|u|+|w|)^3.$$

Along the boundary orbit, the sum of the wall magnitudes does not increase, while (6.30) retains the faster factor $|U| \le |u|e^{-K\xi/8}$. Since $d\xi = KA\,dt$, integration gives

$$|P_0(M(t),U(t),W(t)) - P_0(m,u,w)|$$
$$\le \frac{128}{K^2}|u|(|u|+|w|)^3 \int_0^{\xi(t)} e^{-Kx/8}\,dx \le \frac{1024}{K^3}|u|(|u|+|w|)^3.$$

As $t \to \infty$, the left-hand phase converges to $P_0(P(m,u,w),0,0) = P(m,u,w)$. This proves (6.46); setting $u=0$ proves (6.47). □

Extend P to four coordinates by ignoring V, and define

$$\widehat{\rho}(M,U,V,W) = H(P(M,U,W) - 1, V). \tag{6.49}$$

For $M \ge 2$ and $|U|, |W| < 1/200$, the stable phase is C^2. Its boundary invariance removes all terms without an active wall from its derivative along the full projective field:

$$\dot{P} = UV\big((1-M)P_M + W(P_U - P_W)\big) - V^2\big(P_M + (U/2)P_U\big). \tag{6.50}$$

Indeed, subtract the boundary field (6.25) from the M, U, W components of (2.8). The differences are

$$(-V^2 + (1-M)UV, \quad UVW - UV^2/2, \quad -UVW).$$

Applying DP and using $DPf = 0$ proves (6.50). For $V > 0$ the radial differential is

$$D\widehat{\rho}[h] = \frac{(P-1)DP[h_M, h_U, h_W] + Vh_V}{\widehat{\rho}}. \tag{6.51}$$

The same formula applies whenever the defining radicand is positive.

Recall the finite outgoing endpoint $y_\eta = (M_\eta, U_\eta, \eta, W_\eta)$ from Lemma 6.2. Its reflected entrance is $(-M_\eta, U_\eta, W_\eta, \eta)$. We associate to this endpoint the phase

$$F_\eta(R, u, w) = H(P(-M_\eta, U_\eta, \eta) - 1, W_\eta). \tag{6.52}$$

Proposition 6.8 (Cancellation of the fixed section height)**.** *Fix $0 < \eta \le 10^{-4}$. On a signed neighborhood of $[31/8, 6] \times \{(0,0)\}$, the function F_η is C^2. There are $\delta > 0$ and $C_\eta < \infty$ such that, for $R \in [31/8, 6]$ and $|u|, |w| \le \delta$,*

$$|F_\eta| + \|DF_\eta\| + \|D^2 F_\eta\| \le C_\eta. \tag{6.53}$$

On the reference line,

$$F_\eta(R, 0, 0) = R - 2, \qquad \partial_R F_\eta(R, 0, 0) = 1. \tag{6.54}$$

Proof. For $u = w = 0$, the reflected finite endpoint is

$$(\sqrt{R^2 - \eta^2} - 1, 0, 0, \eta).$$

Its first coordinate is greater than two, even for $R \ge 7/2$. By (6.47),

$$P(\sqrt{R^2 - \eta^2} - 1, 0, \eta) = H(\sqrt{R^2 - \eta^2}, \eta) - 1 = R - 1.$$

Substitution in (6.52) gives $F_\eta = R - 2$. The finite endpoint is C^2 in a full neighborhood, as established in Lemma 6.2. At each reference point the composition defining P lies in its C^2 domain, and the radicand defining F_η equals $(R-2)^2 > 0$. The composition is therefore C^2 locally. Compactness of the reference segment provides a uniform signed neighborhood and a bound for its first two derivatives, proving (6.53). The reference identity holds on an open interval around every $R \in [31/8, 6]$, so it may be differentiated there. □

The ordinary reflected radius at the finite endpoint is $\sqrt{R^2 - \eta^2} - 2$ on the reference line. Following the stable boundary to its Kasner endpoint replaces this quantity by exactly $R - 2$. Thus the fixed section height has been absorbed into the phase before the incoming passage is estimated. The factors UV and V^2 in (6.50) will allow the remaining phase change to be integrated as the active wall grows.

We first compute the differential of P on the weak-wall plane. The transverse derivative on this plane contributes to the linear response of the outgoing phase.

Lemma 6.9 (The weak-wall differential)**.** *If $m \ge 2$ and $|w| < 1/200$, then, for every direction $h = (h_m, h_u, h_w)$,*

$$DP(m, 0, w)h = \frac{(m+1)h_m + w(h_w - 2h_u)}{H(m+1, w)}. \tag{6.55}$$

Proof. Use the fixed rescaling (6.31), with $c = m$, and let $Y(s) = (M(s), 0, W(s))$ be the orbit from $(m, 0, w)$. For its initial variation $J(s) = D_y Y(s; y)h$, consider

$$L(s) = (M(s) + 1)J_m(s) + W(s)\big(J_w(s) - 2J_u(s)\big).$$

This quantity is constant. To verify it, put $\ell = (m+1, -2w, w)$ on $u = 0$. The boundary field and its Jacobian there are

$$f = (w^2, 0, -(m+1)w), \qquad Df = \begin{pmatrix} 0 & -(m+1)w & 2w \\ 0 & (1-m^2-w^2)/2 & 0 \\ -w & -w^2 & -(m+1) \end{pmatrix}.$$

Along this field,

$$\dot{\ell} = (w^2, 2(m+1)w, -(m+1)w), \qquad \dot{\ell} + \ell Df = 0, \qquad \ell f = 0.$$

The derivative of $g = Af$ is $Dg = ADf + f\,dA$. The last identity removes the dA term, so the same conservation holds for the original flow and for its fixed rescaling. Thus

$$L(s) = L(0) = (m+1)h_m + w(h_w - 2h_u).$$

As $s \to \infty$, $M(s) \to P(m, 0, w)$, $W(s) \to 0$, and $J_m(s) \to DP(m, 0, w)h$. The whole variation is bounded by (6.40), so both products with $W(s)$ vanish. Consequently

$$(P(m, 0, w) + 1)DP(m, 0, w)h = (m+1)h_m + w(h_w - 2h_u).$$

Now use (6.47), whose right-hand norm is strictly positive. □

The term $-2wh_u$ in (6.55) measures the response to a strong wall introduced transversely to the weak-wall plane. A related linear invariant carries this response through the finite outgoing segment.

Proposition 6.10 (Strong-wall response of the finite phase)**.** *For $R \in [31/8, 6]$ and $0 < \eta \le 10^{-4}$,*

$$\partial_u F_\eta(R, 0, 0) = -\frac{2}{R}. \tag{6.56}$$

Proof. Write X for the full projective vector field in (2.8), and let $Z(\tau) = (M(\tau), 0, V(\tau), 0)$ be the reference pulse (4.58) with radius R. An initial-state variation of the original flow, evaluated at the original times corresponding to this pulse, satisfies

$$\dot{\xi} = DX(Z)\xi + a(\tau)X(Z), \qquad a(\tau) = -\frac{dD(Z)\xi}{D(Z)}. \tag{6.57}$$

Indeed, the original field is AX, and multiplication of its variational equation by $dt/d\tau = 1/A$ gives $DX\xi + (dA\xi/A)X$. Since $A = 6/D$, this is (6.57).

For every such variation,

$$I(\tau) = (M-1)\xi_M + V\xi_V - 2V\xi_U$$

is constant. The coefficient row is $\lambda = (M-1, -2V, V, 0)$. On the reference pulse,

$$\dot{M} = -V^2, \qquad \dot{V} = (M-1)V, \qquad \dot{\lambda} + \lambda DX = 0, \qquad \lambda X = 0.$$

The two row identities follow by differentiating (2.8) at $U = W = 0$. They imply $\dot{I} = 0$ even with the tangent forcing in (6.57).

Vary u in the downward entry $y_-(R, u, 0) = (1 - \sqrt{R^2 - 1}, u, 1, 0)$, keeping R fixed. The initial variation is $e_U = (0, 1, 0, 0)$, so $I = -2$. Let

$$z = y_\eta(R, 0, 0), \qquad v = \partial_u y_\eta(R, 0, 0)$$

be the actual endpoint and its derivative. The latter is the fixed-time variation plus $\partial_u S_\eta$ times the original vector field at z. Since $\lambda X = 0$, this endpoint correction also disappears from the invariant. We obtain

$$(z_M - 1)v_M + \eta(v_V - 2v_U) = -2.$$

The stable point after reflection is $(-z_M, 0, \eta)$. Its variation is $(-v_M, v_U, v_V)$. Let q be the derivative of P in this direction. Lemma 6.9 gives

$$\big(P(-z_M, 0, \eta) + 1\big)q = (z_M - 1)v_M + \eta(v_V - 2v_U) = -2.$$

The reference value is $P = R - 1$, so $q = -2/R$. Finally $F_\eta = H(P - 1, W_\eta)$, with $W_\eta = 0$ and $P - 1 = R - 2 > 0$ at this endpoint. Differentiating this norm gives $\partial_u F_\eta = q$, as asserted. □

On the plane $u = 0$, the energy of the outgoing segment changes quadratically in the initial weak wall. Retaining this factor through the two radial norms will also control the radial derivative.

For $Y = (R, u, w)$ in a neighborhood of the reference segment, let $b_W(Y, t)$ be the positive wall factor from (6.20), with S_η replaced by t. Thus

$$W(t; Y) = w b_W(Y, t), \qquad b_W(Y, S_\eta(Y)) = B_W(Y).$$

The factor is jointly C^2 in Y and t around the finite orbit segment. Define

$$\begin{aligned} r_\eta(Y) &= H(M_\eta(Y) - 1, \eta), \\ I_\eta(Y) &= \int_0^{S_\eta(Y)} 2A(t; Y)(M(t; Y) - 1) b_W(Y, t)^2 \, dt, \\ Q_\eta(Y) &= \frac{I_\eta(Y)\left(1 - \dfrac{4}{r_\eta(Y) + R}\right) + B_W(Y)^2}{F_\eta(Y) + R - 2}. \end{aligned} \tag{6.58}$$

All three functions are C^2 on a sufficiently small signed neighborhood. For r_η the radicand is positive because $\eta > 0$. For I_η, write the moving integral over $[0, 1]$ and differentiate the finite flow and wall factor twice. The two denominators in Q_η are positive near the reference line, where $r_\eta = R$ and $F_\eta = R - 2$.

Lemma 6.11 (Quadratic weak-wall factorization)**.** *After reducing the signed neighborhood if necessary,*

$$F_\eta(R, 0, w) = R - 2 + w^2 Q_\eta(R, 0, w). \tag{6.59}$$

In particular, $\partial_w F_\eta(R, 0, 0) = 0$. *There are* $\delta > 0$ *and* $B_\eta > 0$ *such that*

$$|\partial_R F_\eta(R, 0, w) - 1| \le B_\eta w^2 \quad \text{for } 31/8 \le R \le 6, \quad |w| \le \delta. \tag{6.60}$$

Proof. The plane $U = 0$ is invariant. On it, (2.9) in original time gives

$$\frac{d}{dt}\big((M - 1)^2 + V^2\big) = 2A(M - 1)W^2.$$

Substitute $W = w b_W$ and integrate from the downward core exit to $V = \eta$. The initial energy is R^2, so

$$r_\eta^2 = R^2 + w^2 I_\eta. \tag{6.61}$$

At the reflected endpoint, the exact weak-wall formula (6.47) gives

$$P(-M_\eta, 0, \eta) = r_\eta - 1, \qquad F_\eta^2 = (r_\eta - 2)^2 + w^2 B_W^2.$$

Here the neighborhood has been chosen so that $-M_\eta > 2$. Rationalizing (6.61) first yields $r_\eta - R = w^2 I_\eta/(r_\eta + R)$. Hence

$$\begin{aligned} \big(F_\eta - (R - 2)\big)\big(F_\eta + R - 2\big) &= r_\eta^2 - R^2 - 4(r_\eta - R) + w^2 B_W^2 \\ &= w^2 \left[I_\eta \left(1 - \frac{4}{r_\eta + R}\right) + B_W^2 \right]. \end{aligned}$$

Division by $F_\eta + R - 2$ proves (6.59).

This identity holds on an open neighborhood in the two variables (R, w), so both of its partial derivatives can be taken there. The w derivative vanishes at $w = 0$, and

$$\partial_R F_\eta(R, 0, w) = 1 + w^2 \partial_R Q_\eta(R, 0, w).$$

The derivative of Q_η is continuous. Compactness of the reference segment provides a common closed strip on which it is bounded, giving (6.60). □

We can now gather the finite phase estimates in the form needed for the incoming passage. In particular, the weak-wall square survives radial differentiation.

Proposition 6.12 (Expansion of the finite outgoing phase)**.** *Fix $0 < \eta \le 10^{-4}$. There are $\delta > 0$ and $C_\eta > 0$ such that, for*

$$31/8 \le R \le 6, \qquad |u|, |w| \le \delta,$$

the same finite outgoing map is defined and F_η is C^2. With $b = |u| + |w|$,

$$\left| F_\eta(R,u,w) - \left(R - 2 - \frac{2u}{R} \right) \right| \le C_\eta b^2, \tag{6.62}$$

$$|\partial_R F_\eta(R,u,w) - 1| \le C_\eta(|u| + w^2), \tag{6.63}$$

$$\left| \partial_u F_\eta(R,u,w) + \frac{2}{R} \right| + |\partial_w F_\eta(R,u,w)| \le C_\eta b. \tag{6.64}$$

Proof. Choose a closed convex box contained in the common signed neighborhood of the preceding results. Let A_η bound $\|D^2 F_\eta\|$ on this box. The derivative is then Lipschitz:

$$\|DF_\eta(Y) - DF_\eta(Z)\| \le A_\eta \|Y - Z\|. \tag{6.65}$$

The reference value and all three partial derivatives have now been computed:

$$F_\eta(R,0,0) = R - 2, \qquad DF_\eta(R,0,0) = (1, -2/R, 0).$$

Set $Z = (R,0,0)$ and $Y = (R,u,w)$. Applying (6.65) to the unit wall directions proves (6.64) with constant $2A_\eta$. To obtain the value estimate, subtract the linear functional $DF_\eta(Z)$ from F_η. Its derivative at any point of the segment from Z to Y has norm at most $A_\eta\|Y - Z\|$. The mean-value bound therefore gives

$$|F_\eta(Y) - F_\eta(Z) - DF_\eta(Z)(Y - Z)| \le A_\eta \|Y - Z\|^2 \le A_\eta b^2,$$

which is (6.62).

For the radial derivative, compare instead with $Z_w = (R,0,w)$. The distance to this point is $|u|$, so (6.65) and (6.60) give

$$|\partial_R F_\eta(Y) - 1| \le A_\eta |u| + B_\eta w^2.$$

Taking $C_\eta = 2A_\eta + B_\eta + 1$ proves all three estimates. □

The coefficient $-2/R$ is independent of the section height. On $u = 0$, the weak wall contributes only at quadratic order to the phase and its radial derivative. These are properties of the finite outgoing phase. Its comparison with the radius at the next core will use (6.50) during the intervening growth of V.

6.3. **Differential bounds for the stable phase.** The finite outgoing segment has now been matched to the stable phase. To follow that phase through the incoming segment, we need its response to both shear and wall variations. These responses have different orders in the wall amplitudes. They also become smaller as the Kasner parameter increases. The following estimates retain both effects. Here D_s denotes differentiation in the two wall coordinates (u,w), with their supremum norm.

Proposition 6.13 (Refined phase derivatives)**.** *There is a universal constant $C > 0$ such that, for $m \ge 2$, $|u|, |w| < 1/200$, $\ell = m + 1$, and $b = |u| + |w|$,*

$$|P_m - 1| \le C\frac{b^2}{\ell^2}, \qquad \|D_s P\| \le C\frac{b}{\ell}, \tag{6.66}$$

$$|P_{mm}| \le C\frac{b^2}{\ell^3}, \qquad \|D_s P_m\| \le C\frac{b}{\ell^2}, \qquad \|D_s^2 P\| \le \frac{C}{\ell}. \tag{6.67}$$

All derivatives are evaluated at (m, u, w).

We prove the proposition by differentiating the residual integral for $P - P_0$. The strong wall in that residual decays on the scale K^{-1} of the rescaled time (6.31). Its first two variations retain this decay, which supplies an additional factor K^{-1} after integration. Throughout this subsection, C denotes a positive universal constant whose value may increase between estimates.

6.3.1. *Variations retaining the wall amplitudes.* Fix $c \geq 2$, put $K = c + 1$, and take $y = (m, u, w) \in \Omega_c$. As in the preceding subsection, K is fixed when the initial state varies. Write $Y(s) = Y(s; y)$, $J_h(s) = D_y Y(s; y)h$, and $H_{h,z}(s) = D_y^2 Y(s; y)[h, z]$. For $b = |u| + |w|$ define

$$\omega_b(h) = b\|h\| + \|h_s\|,$$
$$\nu_b(h, z) = b\|h\|\|z\| + \|h_s\|\|z\| + \|h\|\|z_s\|.$$

The first expression records that a pure shear variation forces the walls only through their existing amplitudes. The elementary relation

$$b\nu_b(h, z) \leq \omega_b(h)\omega_b(z) \tag{6.68}$$

will preserve that distinction in the second derivative of the residual.

Lemma 6.14 (Weighted boundary variations)**.** *For every* $s \geq 0$ *and all directions* h, z,

$$\|J_h(s)\| \leq C\|h\|, \qquad \|(J_h(s))_s\| \leq C\omega_b(h)e^{-s/4}, \tag{6.69}$$

$$|(J_h(s))_u| \leq C\omega_b(h)e^{-3Ks/16}, \tag{6.70}$$

$$\|H_{h,z}(s)\| \leq C\nu_b(h, z), \qquad |(H_{h,z}(s))_u| \leq \frac{C}{K}\nu_b(h, z)e^{-3Ks/32}. \tag{6.71}$$

Proof. Set $E = e^{512b} \leq e^6$. The two bounds in (6.69) follow directly from (6.40) and (6.41). To retain the faster decay of the strong component, use its scalar variational equation. On (6.28), the Jacobian formula (6.36) gives

$$\mathcal{A}_{uu} \leq -\frac{3K}{16}, \qquad |\mathcal{A}_{uw}| \leq 18|u|(|u| + |w|), \qquad |\mathcal{A}_{um}v_m + \mathcal{A}_{uw}v_w| \leq 96|u|\|v\|. \tag{6.72}$$

The second inequality uses the factor u in both $(Df)_{uw} = -uw$ and f_u; the last also uses (6.33). Thus the forcing of $(J_h)_u$ has magnitude at most $96|u|E\|h\|e^{-3Ks/8}$. Scalar comparison yields

$$|(J_h(s))_u| \leq \left(|h_u| + \frac{512}{K}|u|E\|h\|\right)e^{-3Ks/16}. \tag{6.73}$$

Since $|u| \leq b$ and $K \geq 3$, this proves (6.70).

For the full second variation, first suppose that $h_s = z_s = 0$. The stable first variations in (6.41) then carry the factor b. Substitution in (6.35), using $|Y_u| + |Y_w| \leq be^{-s/2}$, bounds the second-order forcing by

$$\|KD^2g(Y)[J_h, J_z]\| \leq CbE^2\|h\|\|z\|e^{-s/4}.$$

The homogeneous growth on every subinterval is at most E, and $H_{h,z}(0) = 0$. Integration therefore gives $\|H_{h,z}\| \leq CbE^3\|h\|\|z\|$ for these two center inputs. For general inputs, decompose $h = h_c + h_s^0$ and $z = z_c + z_s^0$, where the superscript denotes inclusion of the stable pair in $\mathbb{R}^3$. Bilinearity gives

$$H_{h,z} = H_{h_c,z_c} + H_{h_s^0,z} + H_{h_c,z_s^0}.$$

Apply the improved bound to the first term and (6.44) to the other two. This proves the first bound in (6.71).

The strong component of the Hessian has more structure than its full norm. Put $\beta = |u| + |w|$. On the same band,

$$\begin{aligned}|(KD^2g(y)[h,z])_u| \le C(&|u|\|h\|\|z\| + |h_m||z_u| + |h_u||z_m| \\ &+ \beta(|h_u|\|z\| + \|h\||z_u|)).\end{aligned} \tag{6.74}$$

For completeness, its polynomial part is

$$\begin{aligned}(D^2f[h,z])_u = &-uh_mz_m - m(h_mz_u + h_uz_m) + 3uh_uz_u \\ &- w(h_uz_w + h_wz_u) - uh_wz_w.\end{aligned}$$

In the quotient terms of (6.37), use

$$|dD[h]| \le 2K|h_m| + 4\beta\|h\|, \qquad |(\mathcal{A}z)_u| \le 96|u|\|z\| + 18K|z_u|, \qquad |g_u| \le 6|u|.$$

Together with $K/D \le 1$, $K^2/D \le 3$, and the bound for D^2D, these inequalities give (6.74). In particular, each term contains either the strong wall or a strong component of a direction.

Insert Y, J_h, J_z in this estimate. Equations (6.32) and (6.73) give

$$|(KD^2g(Y)[J_h,J_z])_u| \le C\nu_b(h,z)e^{-3Ks/16}.$$

For example, $\|h\|\omega_b(z)$ and $\omega_b(h)\|z\|$ are each bounded by $\nu_b(h,z)$. The off-diagonal linear forcing of $(H_{h,z})_u$ is bounded by $96|Y_u|\|H_{h,z}\|$, so it obeys the same estimate. Use the diagonal bound in (6.72) with the weaker decay rate $3K/32$. The total forcing decays at twice that rate and the initial value is zero. Its convolution with the scalar decay is at most

$$C\nu_b(h,z)e^{-3Ks/32}\int_0^s e^{-3Kr/32}\,dr \le \frac{C}{K}\nu_b(h,z)e^{-3Ks/32}.$$

This proves the remaining bound. □

6.3.2. *Differentiating the residual integral.* Let r_4, r_3, r_2, r_1 be the four coefficients in (6.48). In the fixed time scale used here, the residual is

$$\begin{aligned}S_K(y) &= \frac{6K}{D(y)}DP_0(y)f(y) = uG_K(y), \\ G_K &= \frac{6K}{D}(r_4u^3 + r_3u^2w + r_2uw^2 + r_1w^3).\end{aligned} \tag{6.75}$$

The coefficient bounds must hold after differentiation as well. A reciprocal coordinate makes their uniformity in K transparent.

Lemma 6.15 (Residual coefficients)**.** *On* (6.28)*, every coefficient* $(6K/D)r_i$ *is* C^2*, and its value, first derivative norm, and second derivative norm are bounded by* C/K^2*.*

Proof. Use the reciprocal coordinate $x = (m+1)^{-1}$ and put $\varrho = H(1, xw)$. The coefficients have the form $r_i(m,w) = x\phi_i(x,w)$, where

$$\phi_4 = -\frac{2(1-x)x^2}{(1-2x)^2}, \qquad \phi_3 = \frac{2(3-4x)x^2}{(1-2x)^2},$$

$$\phi_2 = -\frac{x((1-x)^2+x^2)}{(1-2x)^2} - \frac{1-x}{1-2x} + \frac{\varrho + x}{\varrho(\varrho+1)}, \qquad \phi_1 = 2x + 1 - \frac{1}{\varrho+1}.$$

These identities follow by substituting $m = x^{-1} - 1$ and $H(m+1,w) = (m+1)H(1,xw)$ in the coefficients. All four profiles are smooth on a neighborhood of $[0, 2/5] \times [-1, 1]$: their denominators stay positive there. Their first two derivatives are therefore uniformly bounded. Since $m \ge 3/2$ on the band and

$$\partial_m x = -x^2, \qquad \partial_m^2 x = 2x^3,$$

the product and chain rules give

$$|r_i| + \|Dr_i\| + \|D^2r_i\| \le \frac{C}{m+1} \le \frac{2C}{K}.$$

The derivatives here may also be taken in (m, u, w), as r_i does not depend on u.

For $q_K = 6K/D$, direct differentiation gives

$$Dq_K[h] = -q_K \frac{dD[h]}{D}, \qquad D^2 q_K[h,z] = q_K \left(2\frac{dD[h]dD[z]}{D^2} - \frac{D^2 D[h,z]}{D} \right).$$

The band estimates give $q_K \le 18/K$, $|dD[h]|/D \le 6\|h\|$, and $|D^2 D[h,z]|/D \le 10\|h\|\|z\|$. Hence q_K and its first two derivatives are bounded by C/K. The product rule proves the assertion. □

Define the residual along a boundary orbit by

$$q(s,y) = S_K(Y(s;y)) = \frac{d}{ds} P_0(Y(s;y)).$$

Here q is a scalar function; the time scale K remains fixed. Its decay and the decay of its derivatives can now be estimated in the same integrable expression.

Lemma 6.16 (Differentiable residual tails)**.** *For every $a \ge 0$, the function*

$$T_a(y) = P(y) - P_0(Y(a;y))$$

is C^2 on Ω_c. Its derivatives are obtained by differentiating

$$T_a(y) = \int_a^\infty q(s,y)\, ds \tag{6.76}$$

under the integral sign. Moreover,

$$\begin{aligned} |T_a(y)| &\le \frac{C}{K^3} |u| b^3 e^{-Ka/16}, \\ |DT_a(y)h| &\le \frac{C}{K^3} b^3 \omega_b(h) e^{-Ka/16}, \\ |D^2 T_a(y)[h,z]| &\le \frac{C}{K^3} b^2 \omega_b(h)\omega_b(z) e^{-Ka/16}. \end{aligned} \tag{6.77}$$

Proof. Write $\beta = |Y_u| + |Y_w| \le b$. Product differentiation of the cubic expression G_K in (6.75) gives, for arbitrary directions v, z,

$$\begin{aligned} |G_K(Y)| &\le \frac{C}{K^2}\beta^3, \\ |DG_K(Y)v| &\le \frac{C}{K^2}(\beta^3\|v\| + 3\beta^2\|v_s\|), \\ |D^2 G_K(Y)[v,z]| &\le \frac{C}{K^2}(\beta^3\|v\|\|z\| + 3\beta^2(\|v\|\|z_s\| + \|v_s\|\|z\|) + 6\beta\|v_s\|\|z_s\|). \end{aligned}$$

The three contributions correspond to differentiating a coefficient twice, once, or not at all. Lemma 6.15 bounds each coefficient and its derivatives by the same C/K^2. With the weighted variation bounds, these imply

$$\begin{aligned} |DG_K(Y)J_h| &\le \frac{C}{K^2} b^2 \omega_b(h), \\ |D^2 G_K(Y)[J_h, J_z]| &\le \frac{C}{K^2} b\omega_b(h)\omega_b(z), \\ |DG_K(Y)H_{h,z}| &\le \frac{C}{K^2} b^2 \nu_b(h,z). \end{aligned}$$

In the last line we used $b \le 1$ and $\|(H_{h,z})_s\| \le \|H_{h,z}\|$.

Put $Q(s) = e^{-Ks/16}$. The stronger decay rates already proved give $|Y_u| \le |u|Q$, $|(J_h)_u| \le C\omega_b(h)Q$, and $|(H_{h,z})_u| \le C\nu_b(h,z)Q$. The exact product and chain rules for $q = Y_u G_K(Y)$

are

$$\begin{aligned} D_y q[h] &= (J_h)_u G_K(Y) + Y_u DG_K(Y) J_h, \\ D_y^2 q[h,z] &= (H_{h,z})_u G_K(Y) + (J_h)_u DG_K(Y) J_z + (J_z)_u DG_K(Y) J_h \\ &\quad + Y_u (D^2 G_K(Y)[J_h, J_z] + DG_K(Y) H_{h,z}). \end{aligned}$$

Thus

$$\begin{aligned} |q(s,y)| &\le \frac{C}{K^2} |u| b^3 Q(s), \\ |D_y q(s,y)h| &\le \frac{C}{K^2} b^3 \omega_b(h) Q(s), \\ |D_y^2 q(s,y)[h,z]| &\le \frac{C}{K^2} b^2 \omega_b(h) \omega_b(z) Q(s). \end{aligned} \tag{6.78}$$

For the terms containing $H_{h,z}$, the required conversion is $b^3 \nu_b(h,z) \le b^2 \omega_b(h) \omega_b(z)$, which is (6.68). This step also covers $b = 0$ without division by an amplitude.

Integrating the finite-time identity for $P_0(Y)$ and using $P_0(Y(s;y)) \to P(y)$ proves (6.76). For differentiation, note that q and its first two initial-state derivatives are continuous in (s,y) on every finite time interval. Since $b \le 1$ and $\omega_b(h) \le 2\|h\|$, all three operator bounds in (6.78) have the common integrable majorant $CK^{-2} e^{-Ks/16}$, uniformly for $y \in \Omega_c$. Dominated differentiation, applied twice on this open box, therefore identifies the first two derivatives of the improper integral. Finally,

$$\int_a^\infty e^{-Ks/16}\, ds = \frac{16}{K} e^{-Ka/16}.$$

Integrating the three bounds gives (6.77). □

Proof of Proposition 6.13. Let $R = P - P_0$. Take $a = 0$ in Lemma 6.16. At a given point with $m \ge 2$, choose $c = m$, so $K = \ell = m + 1$. The differentiation has already been performed with K fixed on Ω_c; we now evaluate those estimates at the chosen initial point. We obtain

$$\begin{aligned} |DR(y)h| &\le \frac{C}{\ell^3} b^3 (b\|h\| + \|h_s\|), \\ |D^2 R(y)[h,z]| &\le \frac{C}{\ell^3} b^2 (b\|h\| + \|h_s\|)(b\|z\| + \|z_s\|). \end{aligned} \tag{6.79}$$

It remains to differentiate the explicit approximation. Put $\sigma = H(\ell, w)$, $\alpha = m/(m^2-1)$, and $\gamma = -2/\ell$. Then

$$DP_0 = \left(\frac{\ell}{\sigma} + \alpha' u^2 + \gamma' u w, \quad 2\alpha u + \gamma w, \quad \frac{w}{\sigma} + \gamma u \right),$$

and the distinct Hessian entries are

$$\begin{aligned} (P_0)_{mm} &= \frac{w^2}{\sigma^3} + \alpha'' u^2 + \gamma'' u w, & (P_0)_{mu} &= 2\alpha' u + \gamma' w, \\ (P_0)_{mw} &= -\frac{\ell w}{\sigma^3} + \gamma' u, & (P_0)_{uu} &= 2\alpha, \\ (P_0)_{uw} &= \gamma, & (P_0)_{ww} &= \frac{\ell^2}{\sigma^3}. \end{aligned}$$

The partial-fraction identity

$$\alpha = \frac{1}{2} \left(\frac{1}{m-1} + \frac{1}{m+1} \right)$$

and $m + 1 \le 3(m-1)$ give

$$|\alpha| \le \frac{2}{\ell}, \quad |\alpha'| \le \frac{5}{\ell^2}, \quad |\alpha''| \le \frac{28}{\ell^3}, \qquad |\gamma^{(j)}| = \frac{2j!}{\ell^{j+1}} \quad (0 \le j \le 2).$$

Using $\sigma \geq \ell$ and $1 - \ell/\sigma = w^2/[\sigma(\sigma+\ell)]$ now yields

$$|(P_0)_m - 1| \leq 8b^2/\ell^2, \qquad \|D_s P_0\| \leq 9b/\ell,$$
$$|(P_0)_{mm}| \leq 33b^2/\ell^3, \qquad \|D_s(P_0)_m\| \leq 15b/\ell^2, \qquad \|D_s^2 P_0\| \leq 9/\ell.$$

For a center direction, $\omega_b(h) = b\|h\|$; for a stable direction, $\omega_b(h) \leq 2\|h\|$. Thus (6.79) contributes at most Cb^4/ℓ^3 and Cb^3/ℓ^3 to the two first-derivative bounds, and Cb^4/ℓ^3, Cb^3/ℓ^3, and Cb^2/ℓ^3 to the three second-derivative bounds. Since $b \leq 1$ and $\ell \geq 1$, these are absorbed by the stated bounds for P_0. □

The phase is therefore close to the shear coordinate in a way that also controls its differential. In the incoming passage, these amplitude factors accompany the UV and V^2 terms in (6.50). They will allow the phase variation to be estimated through the long interval before the active wall reaches the next core.

6.4. **Phase transport into the next core.** The stable phase absorbs the shear motion caused by the two decaying walls. During the incoming passage, its remaining motion is coupled to the active wall. We now estimate this motion up to the actual arrival at $V = 1$. The useful quantities are the products UV and VW: although V grows, both products decay, and their integrals control the accumulated phase error.

Retain the incoming map $\mathcal{P}$ and the notation of Proposition 6.1. Thus $y = (m, u, v, w)$ satisfies (6.1), $k = H(m-1, v)$, and $L = \log(1/v)$. Write

$$r(y) = \widehat{\rho}(y) = H(P(m, u, w) - 1, v), \qquad \rho_+(y) = H(\mathcal{P}_M(y) - 1, 1).$$

The first quantity is the stable radial phase at entry, and the second is the radius at the next core entrance.

Proposition 6.17 (Incoming phase error)**.** *There is a universal constant $C > 0$ such that*

$$|\rho_+(y) - r(y)| \leq \frac{C}{k}\left[uv + (vw)^2(1 + e^{-2L/k})\right] \leq \frac{2C}{k}\left[uv + v^2(u+w)^2\right]. \tag{6.80}$$

The estimate includes $u = 0$ and $w = 0$ and holds for the same locally C^2 first-arrival map $\mathcal{P}$.

We first describe the radial coordinate in which this phase can be varied. The coefficient estimates below also prepare the differential analysis of the arrival map. The constant C will be universal and may increase from one estimate to the next.

6.4.1. *A radial coordinate adapted to the boundary.* For the local estimates, fix $k \geq 7/4$ and let $y = (M, U, V, W)$ belong to (6.7), with $V > 0$. Set

(6.81)
$$\chi = P(M, U, W) - 1, \qquad r = H(\chi, V), \qquad b = U + W, \qquad q = M - 1 - U(V + W - U).$$

Subscripts on P always denote its ordinary derivatives in (M, U, W). The reference parameter k labels the region and is held fixed in the estimates; the new radial variable is r.

Lemma 6.18 (Radial coordinates)**.** *At every such point the map*

$$(M, U, V, W) \longmapsto (r, U, V, W)$$

has a C^2 local inverse, which preserves the last three coordinates. In the inverse coordinates, the first coordinate has derivatives

$$M_r = \frac{r}{\chi P_M}, \qquad M_U = -\frac{P_U}{P_M}, \qquad M_V = -\frac{V}{\chi P_M}, \qquad M_W = -\frac{P_W}{P_M}. \tag{6.82}$$

They satisfy

$$\frac{2}{3} \leq M_r \leq 8, \qquad |M_U| + |M_W| \leq \frac{Cb}{k}, \qquad |M_V| \leq \frac{4V}{k}. \tag{6.83}$$

Proof. The phase band (6.8) gives $M > 2$. Since $b \le 1/5000$, the explicit estimates (6.29) and (6.38) give

$$|P - M| \le \frac{8b^2}{M+1} \le \frac{1}{100}, \qquad |P_M - 1| \le 66000 b^2 e^{512b} \le \frac{1}{2}.$$

For the last inequality one may use $e^{512b} \le 2$. Combining these bounds with the phase band and $0 < V \le 1$ yields

$$\frac{k}{2} \le \chi \le \frac{3k}{2}, \qquad \frac{k}{2} \le r \le 2k, \qquad \frac{1}{2} \le P_M \le \frac{3}{2}. \tag{6.84}$$

Indeed, $\chi \le r$ and $r^2 = \chi^2 + V^2$ give the radius bounds. The ordinary M derivative is

$$\partial_M r = \frac{\chi P_M}{r} \in [1/8, 3/2].$$

The coordinate change is C^2 by Proposition 6.6. Its triangular Jacobian is invertible, so the inverse function theorem gives the claimed local inverse. Solving the differential of $r^2 = (P-1)^2 + V^2$ proves (6.82). The radial and active-wall bounds follow from (6.84). For the two other columns, use Proposition 6.13 and $M + 1 \ge k/2$. Two local inverses agree on a neighborhood of their common base point, so these derivatives do not depend on the chosen inverse. □

In what follows, $\partial_r, \partial_U, \partial_V, \partial_W$ denote derivatives in this radial chart. We use d for a differential in the ordinary coordinates and D_z for a differential in $z = (r, U, V, W)$. This distinction matters for the cancellations below. The boundary estimates, transferred using $M + 1 \ge k/2$, read

$$\begin{aligned} &|P - M| \le Cb^2/k, && |P_M - 1| \le Cb^2/k^2, && |P_U| + |P_W| \le Cb/k, \\ &|P_{MM}| \le Cb^2/k^3, && |P_{MU}| + |P_{MW}| \le Cb/k^2. \end{aligned} \tag{6.85}$$

The full norms of dP and d^2P are uniformly bounded.

6.4.2. *The exact radial drift.* The cancellation of the boundary motion is expressed by two scalar coefficients. Define

$$\begin{aligned} F &= \chi - (M-1)P_M, \\ N &= M - 1 - U(W-U) - \chi P_M, \\ E &= \chi\left[F + W(P_U - P_W) - \frac{V}{2}P_U\right], \end{aligned} \tag{6.86}$$

$$\mathcal{A} = -\frac{r}{q} + \frac{E}{rq}, \qquad \mathcal{B} = \frac{N}{rq}. \tag{6.87}$$

The denominators are positive by (6.9) and (6.84). Along an actual incoming orbit, with $s = \log(V/v)$,

$$\frac{dr}{ds} = G, \qquad G = UV\mathcal{A} + V^2\mathcal{B}. \tag{6.88}$$

To verify this identity, insert (6.50) in the following expression, where a dot denotes projective time:

$$\frac{dr}{ds} = \frac{\chi\dot{P} + V^2 q}{rq}.$$

The coefficient of V^2 is $N/(rq)$ after removing the term $-UV^3$. The numerator of the remaining UV coefficient is

$$\chi\big((1-M)P_M + W(P_U - P_W)\big) - V^2 - \frac{\chi V}{2}P_U.$$

Using $r^2 = \chi^2 + V^2$, this equals $-r^2 + E$, which gives (6.87).

Proposition 6.19 (Radial coefficient bounds)**.** *The coefficients are C^1 in every local radial chart, and*

$$|\mathcal{A}| + \|D_z\mathcal{A}\| \le C, \qquad |\partial_r\mathcal{A}| \le C/k^2, \tag{6.89}$$

$$|\mathcal{B}| \le Cb^2/k^2, \qquad |\partial_r\mathcal{B}| \le Cb^2/k^3, \qquad |\partial_U\mathcal{B}| + |\partial_W\mathcal{B}| \le Cb/k^2. \tag{6.90}$$

These bounds hold on the full region under consideration, including zero stable walls.

Proof. All expressions involve at most first derivatives of the C^2 function P and positive denominators, so they are C^1 on an ordinary open neighborhood of each point.

Start with the weak numerator. The identity

$$N = -(P - M) - U(W - U) - \chi(P_M - 1)$$

and (6.85) imply $|N| \le Cb^2$. Its ordinary partial derivatives are

$$\begin{aligned} N_M &= 1 - P_M^2 - \chi P_{MM}, \\ N_U &= 2U - W - P_M P_U - \chi P_{MU}, \\ N_W &= -U - P_M P_W - \chi P_{MW}, \qquad N_V = 0. \end{aligned}$$

Since $1 - P_M^2 = -(P_M - 1)(P_M + 1)$, they satisfy

$$|N_M| \le Cb^2/k^2, \qquad |N_U| + |N_W| \le Cb.$$

The inverse columns (6.82) therefore give

$$|\partial_r N| \le Cb^2/k^2, \qquad |\partial_U N| + |\partial_W N| \le Cb.$$

Here and below an ordinary numerator is regarded as a function of the radial chart when a ∂ is used. Likewise, $\partial_r q = M_r$, and the other clock columns are

$$\partial_U q = M_U - (V + W - 2U), \qquad \partial_V q = M_V - U, \qquad \partial_W q = M_W - U.$$

In particular, $\|D_z q\| \le C$. For a direction h in the radial chart, quotient differentiation gives

$$D_z\mathcal{B}[h] = \frac{D_z N[h]}{rq} - \mathcal{B}\left(\frac{h_r}{r} + \frac{D_z q[h]}{q}\right).$$

Use $r, q \ge k/2$ in this formula. The value bound is Cb^2/k^2. For the radial column, the first term is at most Cb^2/k^4 and the second at most Cb^2/k^3. For a stable-wall column they are at most Cb/k^2 and Cb^2/k^3. Since $k \ge 1$ and $b \le 1$, these prove (6.90).

For the strong coefficient, the useful ordinary-coordinate estimates are

$$|F| \le Cb^2/k, \qquad |E| \le C(Vb + b^2), \qquad |E_M| \le C(Vb + b^2)/k. \tag{6.91}$$

The first follows from $F = (P - M) - (M - 1)(P_M - 1)$. For the third, the derivative $F_M = -(M - 1)P_{MM}$ retains the cancellation of the two first-order terms. In full,

$$\begin{aligned} E_M = P_M &\left[F + W(P_U - P_W) - \frac{V}{2}P_U\right] \\ &+ \chi\left[-(M-1)P_{MM} + W(P_{MU} - P_{MW}) - \frac{V}{2}P_{MU}\right]. \end{aligned}$$

The bracket in the definition of E is bounded by $C(Vb + b^2)/k$, and the second bracket here by $C(Vb + b^2)/k^2$. With $\chi \le 2k$ and $P_M \le 3/2$, this proves the other two estimates in (6.91). Consequently $|E| \le C$ and $|\partial_r E| = |E_M M_r| \le C/k$.

The leading quotient $-r/q$ has the exact radial identity

$$r\partial_r q - q = \frac{F}{P_M} + \frac{V^2}{\chi P_M} + U(V + W - U).$$

Indeed, substitute $\partial_r q = r/(\chi P_M)$ and $r^2 = \chi^2 + V^2$. The right side is bounded by $C(V^2/k + UV + b^2) \le C$. Differentiating (6.87) gives

$$\partial_r \mathcal{A} = \frac{r\partial_r q - q}{q^2} + \frac{\partial_r E}{rq} - \frac{E}{rq}\left(\frac{1}{r} + \frac{\partial_r q}{q}\right).$$

The three terms are bounded by C/k^2, C/k^3, and C/k^3, respectively. The value bound $|\mathcal{A}| \le C$ follows from $r/q \le 4$ and $|E|/(rq) \le C/k^2$.

To control all remaining columns, a coarse bound suffices. The uniform bounds on dP, d^2P imply $\|dF\| \le Ck$. The bracket defining E is bounded by C, and its ordinary differential by Ck. Hence $\|dE\| \le Ck^2$. The inverse chart has bounded differential by (6.83), so $\|D_z E\| \le Ck^2$ as well. For an arbitrary radial-chart direction h,

$$D_z\mathcal{A}[h] = \frac{rD_z q[h] - qh_r}{q^2} + \frac{D_z E[h]}{rq} - \frac{E}{rq}\left(\frac{h_r}{r} + \frac{D_z q[h]}{q}\right).$$

The bounds $r, q \asymp k$, $\|D_z q\| \le C$, and $|E| \le C$ give $|D_z\mathcal{A}[h]| \le C\|h\|$. This completes the proof. □

The distinction between the two coefficients now gives useful bounds for the drift and its derivatives.

Corollary 6.20 (Drift estimates)**.** *In the radial chart,*

$$|G| \le C\left(UV + \frac{(VW)^2}{k^2}\right), \qquad |\partial_r G| \le C\left(\frac{UV}{k^2} + \frac{(VW)^2}{k^3}\right), \tag{6.92}$$

$$|U\partial_U G| \le CUV, \qquad |W\partial_W G| \le C\left(UVW + \frac{(VW)^2}{k^2}\right). \tag{6.93}$$

For a positive wall the weighted derivative is its logarithmic derivative; the displayed expressions also make sense when a wall vanishes.

Proof. First use $G = UV\mathcal{A} + V^2\mathcal{B}$ and Proposition 6.19, which give

$$|G| \le C(UV + V^2b^2/k^2), \qquad |\partial_r G| \le C(UV/k^2 + V^2b^2/k^3).$$

Since $U, W, V \le 1$,

$$b^2 \le 3U + W^2, \qquad V^2U \le UV.$$

Substitution proves (6.92). For the wall derivatives, use the exact formulas

$$\partial_U G = V\mathcal{A} + UV\partial_U\mathcal{A} + V^2\partial_U\mathcal{B},$$
$$\partial_W G = UV\partial_W\mathcal{A} + V^2\partial_W\mathcal{B}.$$

Multiply by U or W, respectively. The first line is bounded by $C(UV + U^2V + UV^2b/k^2) \le CUV$. In the second, split $Wb = UW + W^2$ and use $UV^2W/k^2 \le UVW$. This proves (6.93) without dividing by either wall. □

We return to the actual orbit from $y = (m, u, v, w)$. The products in (6.92) have simple dissipation inequalities in $s = \log(V/v)$:

$$(UV)' \le -\frac{k}{3}UV, \qquad ((VW)^2)' \le -\frac{2}{k}(VW)^2. \tag{6.94}$$

These follow by adding the active logarithmic rate 1 to each of the two rates in (6.10). Their zeroth and first moments obey, for every $0 \le S \le L$,

$$\int_0^S UV\,ds \le \frac{3uv}{k}, \qquad \int_0^S sUV\,ds \le \frac{9uv}{k^2},$$

$$\int_0^S (VW)^2\,ds \le \frac{k}{2}(vw)^2, \qquad \int_0^S s(VW)^2\,ds \le \frac{k^2}{4}(vw)^2. \tag{6.95}$$

For example, the first product is bounded by $uve^{-ks/3}$; integrating that exponential and s times it proves the first row. The second row follows with decay rate $2/k$. These are also the corresponding integrals along the normalized flow with $ds = Aq\,dt$, since that clock is strictly increasing throughout the passage.

Integrating (6.88) with the first bound in (6.92) gives

$$|r(Y(S)) - r(y)| \le \frac{C}{k}\left[uv + (vw)^2\right], \qquad 0 \le S \le L. \tag{6.96}$$

Here $Y(S)$ is the state at logarithmic clock value S. To convert the endpoint to the core radius, we need the difference between the two radial coordinates.

Lemma 6.21 (Radius correction)**.** *On* (6.7)*, with $V > 0$, put $J = r - \rho$, where $\rho = H(M-1, V)$. Then J is C^2 locally and*

$$|J| \le Cb^2/k, \qquad |\partial_r J| \le Cb^2/k^2, \qquad |\partial_U J| + |\partial_W J| \le Cb/k. \tag{6.97}$$

Proof. Set $a = M - 1 > 0$. Rationalization gives

$$r - \rho = \frac{(\chi - a)(\chi + a)}{r + \rho}, \qquad |r - \rho| \le |\chi - a| = |P - M|.$$

This proves the value bound, while positivity of r, ρ gives local C^2 regularity. By (6.82),

$$\partial_r J = 1 - \frac{ar}{\rho\chi P_M}, \qquad \partial_U J = \frac{aP_U}{\rho P_M}, \qquad \partial_W J = \frac{aP_W}{\rho P_M}.$$

Let $\delta = Cb^2/k$ bound both $|r - \rho|$ and $|a - \chi|$. Since $a \le \rho$ and $\chi \ge k/2$,

$$|ar - \rho\chi| = |a(r - \rho) + \rho(a - \chi)| \le 2\rho\delta, \qquad \left|\frac{ar}{\rho\chi} - 1\right| \le \frac{4\delta}{k}.$$

Finally,

$$1 - \frac{ar}{\rho\chi P_M} = \frac{(P_M - 1) - (ar/(\rho\chi) - 1)}{P_M}.$$

Use $P_M \ge 1/2$ and (6.85) to obtain the radial derivative bound. The wall bounds follow from $a/\rho \le 1$, $1/P_M \le 2$, and the same projection estimates. □

Proof of Proposition 6.17. At the first arrival $S = L$, write U_+, W_+ for the two stable walls and put $\omega = e^{-2L/k}$. The decay estimates (6.5) give

$$U_+^2 \le u^2 v^{2+2k/3} \le uv, \qquad W_+^2 \le w^2 v^{2+2/k} = (vw)^2\omega.$$

Thus

$$(U_+ + W_+)^2 \le 2\left[uv + (vw)^2\omega\right].$$

The endpoint remains in the same region. Apply Lemma 6.21 there and combine it with (6.96) at $S = L$. The identity

$$\rho_+(y) - r(y) = r(\mathcal{P}(y)) - r(y) - J(\mathcal{P}(y))$$

proves the first inequality in (6.80). Since $0 < \omega \le 1$ and $w^2 \le (u + w)^2$, the second follows. All estimates were obtained on the actual passage of Proposition 6.1; its first-arrival property and local C^2 regularity therefore apply to this same map. □

At fixed incoming shear and stable walls, the phase error tends to zero with the height of the entering active wall, even though the passage time diverges. The first moments in (6.95) will also control the additional time factors produced by variations of this passage.

6.5. **Variations of the incoming passage.** The phase error is small because its rate contains decaying wall products. Its derivative also depends on how the walls respond to a change of the incoming phase. We estimate that response at equal values of $s = \log(V/v)$. The strong and weak logarithmic wall variations have different radial scales; weighting the latter by k^2 makes their coupling uniformly integrable.

Fix an incoming point $y = (m, u, v, w)$ as in Proposition 6.1, and choose the local inverse Ψ of Lemma 6.18. In the input coordinates $z = (r, u, v, w)$, define

$$Q(z) = r(\mathcal{P}(\Psi(z))), \qquad \ell_U(z) = \log \mathcal{P}_U(\Psi(z)), \qquad \ell_W(z) = \log \mathcal{P}_W(\Psi(z)). \tag{6.98}$$

Thus Q is the stable radial phase evaluated at the actual arrival. The parameter $k = H(m-1, v)$ and the weight k^2 are fixed at the reference input throughout differentiation. All vector norms below are maximum norms, with the induced operator norms on differentials.

Proposition 6.22 (Three arrival columns)**.** *There are universal constants $C > 0$ and $0 < \varepsilon \le 10^{-4}$ such that, for every input satisfying*

$$k \ge 7/4, \qquad m \ge 1, \qquad 0 < u, w \le \varepsilon, \qquad 0 < v < 1,$$

the functions in (6.98) *are locally C^2. At $z = (r(y), u, v, w)$, with $L = \log(1/v)$, they satisfy*

$$|\partial_r Q - 1| \le \frac{C}{k^2}[uv + (vw)^2], \qquad |\partial_u Q| \le \frac{Cv}{k},$$
$$|\partial_w Q| \le \frac{C}{k}(uv + v^2 w), \tag{6.99}$$

and

$$|\partial_r \ell_U| + k^2 |\partial_r \ell_W| \le CL,$$
$$|\partial_u \ell_U - 1/u| + k^2 |\partial_u \ell_W| \le C,$$
$$|\partial_w \ell_U| + k^2 |\partial_w \ell_W - 1/w| \le C. \tag{6.100}$$

The wall derivatives retain the logarithmic singularity of their initial amplitudes. After that term is removed, the two wall columns are uniformly bounded, even as v tends to zero. The radial column grows at most with the logarithmic waiting length.

6.5.1. *Comparison at equal clock values.* Write f_{p} for the projective vector field in (2.8), and put

$$\mathcal{V} = \frac{f_{\mathrm{p}}}{q}, \qquad \mathrm{D}_t \gamma = A f_{\mathrm{p}}(\gamma), \qquad A = \frac{6}{D}, \qquad q = M - 1 - U(V + W - U).$$

The vector field $\mathcal{V}$ describes the same orbit in the clock s, and $\mathcal{V}_V = V$. Let $\gamma(t; y)$ be the actual flow starting at y, and let $J_h(t) = D_y\gamma(t; y)h$ be its variation at fixed original time. Set

$$\zeta = \frac{h_v}{v}, \qquad \xi_h(t) = J_h(t) - \left(\frac{(J_h)_V(t)}{V(t)} - \zeta\right)\mathcal{V}(\gamma(t)). \tag{6.101}$$

Then

$$(\xi_h)_V = \zeta V, \qquad \xi_h(0) = h, \qquad \frac{d\xi_h}{ds} = D\mathcal{V}(\gamma)\xi_h. \tag{6.102}$$

For the differential identity, let $\lambda = (J_h)_V/V - \zeta$ and $a = Aq$. Since the original vector field is $a\mathcal{V}$ and its V component is aV, differentiation gives $\mathrm{D}_t \lambda = da[J_h]$. Consequently

$$\mathrm{D}_t \xi_h = D(a\mathcal{V})J_h - \mathrm{D}_t \lambda\, \mathcal{V} - \lambda D\mathcal{V}(a\mathcal{V}) = aD\mathcal{V}\, \xi_h.$$

Division by the positive clock speed a proves (6.102).

This projection is the derivative of nearby actual trajectories. At any point of the passage, the equation

$$\log V(\gamma(t; \widetilde{y})) - \log \widetilde{v} = s$$

has a locally C^2 solution for t, because its time derivative is $Aq > 0$. Implicit differentiation and the chain rule give exactly (6.101). The resulting local branches agree wherever they overlap.

For a direction $d = (d_r, d_u, d_v, d_w)$ in the input radial chart, take $h = D\Psi(z)d$. The inverse chart preserves the three walls, so $\zeta = d_v/v$. The arrival clock $L = \log(1/v)$ has derivative $-d_v/v$, and hence

$$D(\mathcal{P} \circ \Psi)(z)d = \xi_h(L) - \frac{d_v}{v}\mathcal{V}(\mathcal{P}(y)). \tag{6.103}$$

In this formula and below, the variation is evaluated using the clock s. In particular, the correction vanishes for the three directions considered in Proposition 6.22.

6.5.2. *The weighted wall equation.* Assume $d_v = 0$ and $u, w > 0$. Positivity of the walls allows us to set

$$X = \frac{(\xi_h)_U}{U}, \qquad Y = \frac{(\xi_h)_W}{W}, \qquad Z = dr(\gamma)[\xi_h], \qquad \mathbf{x} = (X, k^2 Y). \tag{6.104}$$

Their initial values are

$$X_0 = d_u/u, \qquad Y_0 = d_w/w, \qquad Z_0 = d_r.$$

The stable logarithmic rates in the clock s are

$$\alpha = \frac{1 - M^2 - (V - W)^2 + U^2}{2q}, \qquad \beta = -1 - \frac{2(1 + \mathcal{C})}{q}, \qquad \mathcal{C} = U(V + W - U). \tag{6.105}$$

Thus $U' = \alpha U$ and $W' = \beta W$. All partial derivatives of these rates in the next formulas are in the local coordinates (r, U, V, W). Differentiating the two logarithmic equations gives

$$\mathbf{x}' = B\mathbf{x} + \mathbf{b}Z, \qquad B = \begin{pmatrix} U\partial_U \alpha & (W/k^2)\partial_W \alpha \\ k^2 U \partial_U \beta & W \partial_W \beta \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \partial_r \alpha \\ k^2 \partial_r \beta \end{pmatrix}. \tag{6.106}$$

The derivatives defining these coefficients are independent of the choice of local inverse, so the coefficients are continuous along the full passage.

Lemma 6.23 (Wall response)**.** *There is a universal constant $T > 0$ such that, for $0 \le a \le b \le L$,*

$$|X(b) - X(a)| + k^2 |Y(b) - Y(a)| \le T \left[u|X(a)| + w|Y(a)| + \int_a^b |Z(s)|\, ds \right]. \tag{6.107}$$

Proof. We first record the coefficient bounds that make the estimate independent of L. In ordinary coordinates,

$$\begin{aligned} d\alpha &= \frac{-M\, dM - (V - W)(dV - dW) + U\, dU - \alpha\, dq}{q}, \\ d\beta &= \frac{2(1 + \mathcal{C})\, dM - 2M\, d\mathcal{C}}{q^2}, \\ d\mathcal{C} &= (V + W - 2U)\, dU + U(dV + dW), \qquad dq = dM - d\mathcal{C}. \end{aligned}$$

On the incoming region, $M \le 3k$, $q \ge k/2$, $|1 + \mathcal{C}| \le 2$, and $|\alpha| \le 12k$. The first formula therefore gives $\|d\alpha\| \le 180$. Composing with (6.82) bounds all radial chart derivatives of α by a universal constant. For β, the same inverse columns give

$$\begin{aligned} |\partial_r \beta| &\le 128/k^2, \\ |\partial_U \beta| &\le C(U + W)/k^3 + 48(V + U + W)/k, \\ |\partial_W \beta| &\le C(U + W)/k^3 + 24U/k. \end{aligned}$$

Indeed, the three numerators are respectively $2(1+\mathcal{C})M_r$, $2(1+\mathcal{C})M_U - 2M(V+W-2U)$, and $2(1+\mathcal{C})M_W - 2MU$. The estimates for M_r, M_U, M_W and $q^2 \geq k^2/4$ give the displayed constants and scales.

It follows that $\|\mathbf{b}\| \leq C$ and

$$\begin{aligned} &|B_{11}| \leq CU, && |B_{12}| \leq CW/k^2, \\ &|B_{21}| \leq CkU(V+U+W), && |B_{22}| \leq C\left(\frac{UW}{k} + \frac{W(U+W)}{k^3}\right). \end{aligned} \tag{6.108}$$

Let c_1, c_2 be the sums of the absolute values in the two columns of B. Their integrals satisfy

$$\int_a^b c_1\, ds \leq C_0 u, \qquad \int_a^b c_2\, ds \leq C_0 w/k^2 \quad (0 \leq a \leq b \leq L). \tag{6.109}$$

To see this, the decay in (6.2) gives $\int_0^L U\, ds \leq 3u/k$ and $\int_0^L W\, ds \leq w$, while $W \leq w$. Using $U, V, W \leq 1$, the two envelopes in (6.108) obey

$$\int_0^L [U + kU(V+U+W)]\, ds \leq (1+3k)\frac{3u}{k} \leq 12u,$$

$$\int_0^L \left[\frac{W}{k^2} + \frac{UW}{k} + \frac{W(U+W)}{k^3}\right] ds \leq \frac{w}{k}\frac{3u}{k} + \left(\frac{1}{k^2} + \frac{2}{k^3}\right) w \leq \frac{6w}{k^2}.$$

Nonnegativity gives the same bounds on every subinterval.

For completeness, apply a scalar growth estimate to the change $\mathbf{h}(s) = \mathbf{x}(s) - \mathbf{x}(a)$. It satisfies

$$\mathbf{h}' = B\mathbf{h} + B\mathbf{x}(a) + \mathbf{b}Z, \qquad \mathbf{h}(a) = 0.$$

In the maximum norm, every row sum of absolute values of B is at most $c_1 + c_2$, and

$$\|B\mathbf{x}(a)\| \leq c_1|X(a)| + c_2k^2|Y(a)|.$$

Multiplication by the scalar integrating factor therefore gives

$$\|\mathbf{h}(b)\| \leq e^{\int_a^b (c_1+c_2)\, ds} \int_a^b \left[c_1|X(a)| + c_2k^2|Y(a)| + C|Z|\right] ds.$$

This comparison also follows at zeros of a component by taking upper one-sided derivatives of the absolute values. The exponent is at most $2C_0$, since $u, w \leq 1$ and $k \geq 1$. Substitute (6.109) and use $|h_1| + |h_2| \leq 2\|\mathbf{h}\|$ to obtain (6.107). □

6.5.3. *Closing the phase estimate.* The radial equation $r' = G$ gives a scalar companion to (6.106):

$$Z' = a_0 Z + a_1 X + a_2 Y, \qquad a_0 = \partial_r G, \quad a_1 = U\partial_U G, \quad a_2 = W\partial_W G. \tag{6.110}$$

This identity follows by differentiating $dr[\mathcal{V}] = G$. The C^2 regularity of r justifies the two derivatives, and $(\xi_h)_V = 0$ removes the active column. Corollary 6.20 gives

$$\begin{aligned} &|a_0| \leq C\left(\frac{UV}{k^2} + \frac{(VW)^2}{k^3}\right), && |a_1| \leq CUV, \\ &|a_2| \leq C\left(UVW + \frac{(VW)^2}{k^2}\right). \end{aligned} \tag{6.111}$$

Lemma 6.24 (Phase response at fixed clock)**.** *There are universal $K, F > 0$ such that, after reducing the allowed stable amplitudes to a universal $\varepsilon > 0$, the following holds. Put*

$$\begin{aligned} A_0 &= u|X_0| + w|Y_0|, \\ I_0 &= uv|X_0| + \left[uvw + (vw)^2\right]|Y_0|, \\ B_0 &= KI_0/k, \qquad \Lambda = F\left[uv + (vw)^2\right]/k^2. \end{aligned} \tag{6.112}$$

Then $0 \le \Lambda \le 1/2$, *and for* $0 \le s \le L$,

$$|Z(s) - Z_0| \le 2B_0 + 2\Lambda|Z_0|,$$
$$|Z(s)| \le (1+2\Lambda)|Z_0| + 2B_0,$$
$$|X(s) - X_0| + k^2|Y(s) - Y_0| \le T\big[A_0 + 2s(|Z_0| + B_0)\big]. \tag{6.113}$$

Here T *is the constant of Lemma 6.23.*

Proof. The moments (6.95) and $W \le w$ give the three integral bounds

$$\int_0^L |a_1|\,ds \le Cuv/k,$$
$$\int_0^L |a_2|\,ds \le C\big[uvw + (vw)^2\big]/k,$$
$$\int_0^L \left[|a_0| + s\left(|a_1| + \frac{|a_2|}{k^2}\right)\right] ds \le C\big[uv + (vw)^2\big]/k^2. \tag{6.114}$$

The last integral is at most a universal constant times

$$\frac{3uv}{k^3} + \frac{(vw)^2}{2k^2} + \left(1 + \frac{w}{k^2}\right)\frac{9uv}{k^2} + \frac{(vw)^2}{4k^2}.$$

Thus the linear growth in s allowed by the wall response is still integrable in the phase equation.

The inhomogeneous terms from the initial wall data also preserve their amplitudes. More precisely,

$$A_0\left[uv + \frac{uvw + (vw)^2}{k^2}\right] \le 3I_0. \tag{6.115}$$

To verify this, write $b_0 = uv$ and $j = uvw + (vw)^2$. Then $b_0 \le v$, $j \le 2v$, $u(b_0 + j/k^2) \le 3b_0$, and $w(b_0 + j/k^2) \le wb_0 + wj \le 2j$. Multiplication by $|X_0|$ and $|Y_0|$ proves (6.115). It follows from (6.114) that

$$\int_0^L \left[|a_1|(|X_0| + TA_0) + |a_2|\left(|Y_0| + \frac{TA_0}{k^2}\right)\right] ds \le B_0,$$
$$\int_0^L \left[|a_0| + Ts\left(|a_1| + \frac{|a_2|}{k^2}\right)\right] ds \le \Lambda \tag{6.116}$$

for universal choices of K and F. Choose $\varepsilon \le \min(10^{-4}, 1/(4F))$. Since $uv \le \varepsilon$ and $(vw)^2 \le \varepsilon$, this makes $\Lambda \le 1/2$.

Fix $S \le L$ and let $M_S = \max_{0\le s\le S} |Z(s)|$. Lemma 6.23 gives, on $[0, S]$,

$$|X(s)| \le |X_0| + T(A_0 + sM_S), \qquad |Y(s)| \le |Y_0| + \frac{T}{k^2}(A_0 + sM_S).$$

Insert these two inequalities in (6.110) and integrate. By (6.116),

$$|Z(s) - Z_0| \le B_0 + \Lambda M_S, \qquad M_S \le |Z_0| + B_0 + \Lambda M_S.$$

Therefore $M_S \le (|Z_0| + B_0)/(1-\Lambda) \le 2(|Z_0| + B_0)$. Substituting this bound into the first inequality, and using $2\Lambda B_0 \le B_0$, proves the first two lines of (6.113). The wall response with $a = 0$ gives the last line. □

Proof of Proposition 6.22. The arrival map is locally C^2 by Proposition 6.1. The radial coordinate is C^2, and the positive output walls remain positive on a neighborhood of the given input. This proves the asserted regularity.

For any direction d with $d_v = 0$, set

$$J_0 = v|d_u| + (uv + v^2w)|d_w|, \qquad H_0 = uv + (vw)^2.$$

The initial values in (6.104) give $A_0 = |d_u| + |d_w|$ and $I_0 = J_0$. At $s = L$, equation (6.103) identifies $Z(L)$ with $DQ[d]$, and $X(L), Y(L)$ with $D\ell_U[d], D\ell_W[d]$. Hence

$$|DQ[d] - d_r| \le \frac{2K}{k} J_0 + \frac{2F}{k^2} H_0 |d_r|, \tag{6.117}$$

$$|D\ell_U[d] - d_u/u| + k^2 |D\ell_W[d] - d_w/w| \le T\left[|d_u| + |d_w| + 2L\left(|d_r| + \frac{K}{k} J_0\right)\right]. \tag{6.118}$$

Apply these inequalities to the three coordinate directions. The phase bounds follow immediately. For the radial wall column, $J_0 = 0$ and $d_r = 1$, giving $2TL$. For the strong wall column, $J_0 = v$ and $Lv \le 1$, so its remainder is at most $T(1 + 2K)$. For the weak wall column, $J_0 = uv + v^2 w$ and $LJ_0 \le 2$, giving a remainder at most $T(1 + 4K)$. Here $Lv \le 1$ follows by writing $v = e^{-L}$ and using $L \le e^L$; the second bound follows from $u, w, v \le 1$. A common larger constant proves (6.99) and (6.100). □

Flow transport determines the fourth column from the three already estimated. Moving the input along its orbit preserves the arrival state. In differential form,

$$D\mathcal{P}(y)\mathcal{V}(y) = 0. \tag{6.119}$$

Indeed, the differential of a flow transports its vector field to the vector field at the endpoint. In the derivative of a transverse event map, the change of arrival time subtracts precisely this endpoint vector. The positive change of clock preserves the identity.

For any differentiable scalar observable p at the arrival, let $F_p = p \circ \mathcal{P} \circ \Psi$. The radial coordinates of $\mathcal{V}(y)$ are $(G_0, u\alpha_0, v, w\beta_0)$, where $G_0 = G(y)$, $\alpha_0 = \alpha(y)$ and $\beta_0 = \beta(y)$. Equation (6.119) gives

$$v\,\partial_v F_p = -DF_p[d_*], \qquad d_* = (G_0, u\alpha_0, 0, w\beta_0). \tag{6.120}$$

The direction on the right has zero active component, so the mixed response estimates (6.117) and (6.118) apply to it.

Proposition 6.25 (The active arrival column)**.** *On the domain of Proposition 6.22, after enlarging its universal constant if necessary,*

$$|\partial_v Q| \le C(u + vw^2). \tag{6.121}$$

With $H_0 = uv + (vw)^2$, the logarithmic wall derivatives satisfy

$$|v\partial_v \ell_U + \alpha_0| + k^2 |v\partial_v \ell_W + \beta_0| \le C(ku + w + LH_0). \tag{6.122}$$

In particular,

$$|\partial_v \ell_U| \le Ck/v, \qquad |\partial_v \ell_W| \le C/v. \tag{6.123}$$

Proof. The initial drift bound in (6.92) gives $|G_0| \le C_1 H_0$. Also $|\alpha_0| \le 12k$ and $|\beta_0| \le 9$ by (6.105) and the incoming bounds. For the direction d_*, the quantities entering the mixed response obey

$$|(d_*)_u| + |(d_*)_w| \le 12ku + 9w,$$

$$J_* := v|(d_*)_u| + (uv + v^2 w)|(d_*)_w| \le 12kuv + 9uvw + 9v^2 w^2 \le 21kH_0. \tag{6.124}$$

Since $H_0 \le 2$ and $k \ge 1$, (6.117) yields

$$|DQ[d_*] - G_0| \le 42KH_0 + 4F|G_0|, \qquad |DQ[d_*]| \le C_2 H_0.$$

Apply (6.120) with $p = r$ and divide by $v > 0$. The identity $H_0/v = u + vw^2$ proves (6.121).

For the walls, $(d_*)_u/u = \alpha_0$ and $(d_*)_w/w = \beta_0$. Equations (6.118) and (6.124) give

$$|D\ell_U[d_*] - \alpha_0| + k^2 |D\ell_W[d_*] - \beta_0| \le T\big[12ku + 9w + 2L(C_1 + 21K)H_0\big].$$

The transport identity for each logarithmic observable proves (6.122), including its signs. Finally $Lv \le 1$ implies $LH_0 \le u + w \le 2$, and hence $ku + w + LH_0 \le 4k$. The first term

of (6.122) is therefore at most $4Ck$; the second, after division by k^2, is at most $4C$. Adding $|\alpha_0| \le 12k$ or $|\beta_0| \le 9$ and dividing by v proves (6.123). □

The two propositions estimate every input direction of the stable radial phase and both wall amplitudes at the same arrival. The physical core radius is obtained from this phase by subtracting the endpoint radius correction of Lemma 6.21.

6.6. **The radius at the next core.** The stable radial coordinate controls the passage through a long incoming segment. At the next core, the return map is expressed in the core radius $\rho = H(M-1, V)$. The difference between these two radii is quadratic in the output walls. Retaining those walls in the chain rule makes the conversion uniform in the waiting time.

Use the input coordinates and arrival map of Section 6.5, and define

$$F(z) = \rho(\mathcal{P}(\Psi(z))), \qquad z = (r, u, v, w). \tag{6.125}$$

All derivatives below are in these input coordinates, with $k = H(m-1, v)$ fixed at the reference point.

Proposition 6.26 (Physical arrival estimates)**.** *There are universal constants $C > 0$ and $0 < \varepsilon \le 10^{-4}$ such that, if*

$$k \ge 7/4, \qquad m \ge 1, \qquad 0 < u, w \le \varepsilon, \qquad 0 < v < 1,$$

then F is locally C^2. At $z = (r(y), u, v, w)$, put $L = \log(1/v)$ and $\omega = e^{-2L/k}$. Then

$$|F - r| \le \frac{C}{k}\big[uv + v^2w^2(1+\omega)\big], \tag{6.126}$$

$$|\partial_r F - 1| \le \frac{C}{k^2}\big[uv + v^2w^2\{1 + (1+L/k)\omega\}\big], \tag{6.127}$$

$$|\partial_u F| \le Cv/k, \qquad |\partial_v F| \le C(u + vw^2),$$

$$|\partial_w F| \le \frac{C}{k}\big[uv + v^2w(1+\omega)\big]. \tag{6.128}$$

Write $U_+ = \mathcal{P}_U(y)$, $W_+ = \mathcal{P}_W(y)$ and $b_+ = U_+ + W_+$. The decay estimates give

$$U_+ \le uv\,a, \qquad W_+ \le wv\,c, \qquad a = e^{-kL/3}, \quad c = e^{-L/k}, \quad c^2 = \omega. \tag{6.129}$$

Both a and c belong to $(0, 1]$. The elementary inequalities $xe^{-x} \le 1$ and $(1+x)e^{-x} \le 1$, for $x \ge 0$, imply

$$La \le 3/k, \qquad (1 + L/k)\omega \le 1. \tag{6.130}$$

These bounds absorb the waiting length in radial derivatives. The other columns require the wall amplitudes to remain visible.

Lemma 6.27 (Endpoint products)**.** *Set $S = v^2w^2\omega$. For the positive inputs of Proposition 6.26,*

$$\begin{aligned} b_+^2 &\le 2(uv + S), & kLb_+U_+ &\le 6uv, \\ (L/k)b_+W_+ &\le 3uv + (L/k)S, & b_+U_+ &\le 2uv, \\ b_+W_+ &\le w(uv + v^2w\omega), & b_+ &\le 2v. \end{aligned} \tag{6.131}$$

Proof. All initial amplitudes are at most one. Equation (6.129) gives $U_+^2 \le uv$, $W_+^2 \le S$, and $U_+, W_+ \le v$. This proves the first and last estimates. Since $U_+, W_+ \le 1$, it also gives

$$b_+U_+ \le 2uv\,a, \qquad b_+W_+ \le uv\,a + S.$$

Multiplication by kL or L/k, followed by (6.130) and $k \ge 1$, proves the two estimates containing L. Dropping $a \le 1$ gives $b_+U_+ \le 2uv$. Finally $W_+ \le w$, so $U_+W_+ \le uvw$ and $b_+W_+ \le uvw + S = w(uv + v^2w\omega)$. □

Let $J = r - \rho$ be the correction from Lemma 6.21, and set

$$E = J \circ \mathcal{P} \circ \Psi.$$

The identity $F = Q - E$ holds on a neighborhood of the input. For a direction d, let $\xi = D(\mathcal{P} \circ \Psi)[d]$ be the actual endpoint variation. The output active wall is identically one, hence $\xi_V = 0$. In endpoint radial coordinates the remaining components of ξ are

$$DQ[d], \qquad U_+ D\ell_U[d], \qquad W_+ D\ell_W[d].$$

The correction bounds therefore yield

$$|DE[d]| \le T\left[\frac{b_+^2}{k^2}|DQ[d]| + \frac{b_+}{k}(U_+|D\ell_U[d]| + W_+|D\ell_W[d]|)\right] \tag{6.132}$$

for a universal $T > 0$. The output remains in the incoming region with reference radius k, so this is the same k as in the input estimates.

Proof of Proposition 6.26. Choose a common constant $A \ge 1$ for the phase and logarithmic wall estimates in Propositions 6.22 and 6.25. Decrease ε to their common threshold. The regularity of the actual arrival and the positivity of its radius give the stated C^2 regularity. The value estimate is Proposition 6.17. It remains to convert the four derivative columns.

For the radial column, put $H = uv + (vw)^2 \le 2$. Then

$$|Q_r| \le 1 + 2A, \qquad |(\ell_U)_r| \le AL, \qquad |(\ell_W)_r| \le AL/k^2.$$

Using Lemma 6.27 in (6.132), we obtain

$$\begin{aligned} |E_r| &\le \frac{T}{k^2}\left[2(1+2A)(uv+S) + A\{9uv + (L/k)S\}\right] \\ &\le \frac{T(2+13A)}{k^2}\big[uv + (1+L/k)S\big]. \end{aligned}$$

Adding $|Q_r - 1| \le AH/k^2$ proves (6.127).

For the wall columns, define

$$J_0 = uv + v^2 w(1+\omega), \qquad K_0 = u + vw^2.$$

The endpoint products imply

$$\frac{b_+^2}{k^2} \le 4, \qquad b_+^2 \le 4v, \qquad b_+^2 \le 2J_0,$$
$$\frac{b_+U_+}{u} \le 2v, \qquad \frac{b_+W_+}{w} \le J_0, \qquad \frac{b_+U_+}{v} \le 2u, \qquad \frac{b_+W_+}{v} \le K_0.$$

For example, $b_+^2 \le 4v^2 \le 4v$ follows from $b_+ \le 2v$, and $b_+^2 \le 2J_0$ follows from $w^2 \le w$. The final inequality follows from $b_+W_+ \le uvw + v^2w^2\omega$ using $w, \omega \le 1$. The divisions are legitimate because $u, v, w > 0$.

The logarithmic estimates, with their singular initial terms retained, read

$$\begin{aligned} &|(\ell_U)_u| \le 1/u + A, && |(\ell_W)_u| \le A/k^2, \\ &|(\ell_U)_w| \le A, && |(\ell_W)_w| \le 1/w + A/k^2, \\ &|(\ell_U)_v| \le Ak/v, && |(\ell_W)_v| \le A/v. \end{aligned}$$

For the strong column, the second term inside the brackets of (6.132) is at most

$$\frac{1}{k}\left(\frac{b_+U_+}{u} + Ab_+^2\right) \le (2+4A)\frac{v}{k}.$$

For the weak column it is at most

$$\frac{1}{k}\left(\frac{b_+W_+}{w} + Ab_+^2\right) \le (1+2A)\frac{J_0}{k}.$$

For the active column it is at most

$$A\left(\frac{b_+U_+}{v}+\frac{b_+W_+}{kv}\right)\le A(2u+K_0)\le 3AK_0.$$

The first term in the correction has the same respective scales: $|Q_u|\le Av/k$, $|Q_w|\le AJ_0/k$ and $|Q_v|\le AK_0$, while $b_+^2/k^2\le 4$. Thus, with $B=T(2+8A)$,

$$|E_u|\le Bv/k,\qquad |E_w|\le BJ_0/k,\qquad |E_v|\le BK_0.$$

Since $DF=DQ-DE$, addition of the corresponding phase bounds proves (6.128). A common enlargement of C includes the value and radial bounds. □

One useful consequence expresses the estimates without a waiting factor. Since $(1+L/k)\omega\le 1$ and $u+w\le 1$, a universal constant B gives

$$\begin{aligned}|F-r|+|F_r-1|&\le Bv(u+v),\\ |F_u|\le Bv,\qquad |F_v|&\le B(u+v).\end{aligned}\tag{6.133}$$

Indeed, $uv+v^2w^2\le v(u+v)$, $k\ge 1$ and $u+vw^2\le u+v$. These bounds are adapted to composition with the finite outgoing segment: its new growing wall is proportional to the old weak wall.

6.7. **The connection phase at bounded radii.** The finite outgoing phase has the first-order expansion $R-2-2u/R$. The incoming estimates now show that this expansion also describes the radius at the next core entrance. The remaining error contains the old weak wall as a factor, because that wall becomes the growing wall of the incoming passage.

For the actual reflection of Proposition 6.3, denote the next core radius by $R_+(R,u,w)$. Choose the intermediate height η sufficiently small and then hold it fixed. Its reflected entrance, in radial coordinates, is

$$z_\eta(R,u,w)=\big(F_\eta(R,u,w),\,uB_U(R,u,w),\,wB_W(R,u,w),\,\eta\big).\tag{6.134}$$

Here F_η is the finite outgoing phase and B_U, B_W are the positive factors from Lemma 6.2.

Lemma 6.28 (Error through the incoming segment)**.** *There exists $\delta>0$ such that, for every fixed $0<\eta\le\delta$, there are $\varepsilon>0$ and $C_\eta>0$ with the following property. For*

$$31/8\le R\le 6,\qquad 0<u,w\le\varepsilon,$$

the actual reflection is defined and the function $E_\eta=R_+-F_\eta$ is locally C^2. It satisfies

$$\begin{aligned}|E_\eta|+|\partial_R E_\eta|&\le C_\eta w(u+w),\\ |\partial_u E_\eta|\le C_\eta w,\qquad |\partial_w E_\eta|&\le C_\eta(u+w).\end{aligned}\tag{6.135}$$

Proof. Take δ at most the universal stable-wall threshold in Proposition 6.26. For fixed η, compactness of the finite outgoing family gives a common signed neighborhood on which F_η, B_U, B_W are C^2, the two factors are positive, and

$$\|DF_\eta\|,\quad B_U,\quad B_W,\quad \|DB_U\|,\quad \|DB_W\|\le A_\eta$$

for some $A_\eta\ge 1$. The bounds follow from the finite phase regularity and (6.16). Reduce the input threshold so that $u+w\le 1$, $uB_U\le\delta$, and $wB_W\le 1/2$. The reflected entrance has $k_\eta\ge 7/4$ and first coordinate greater than one. The physical incoming estimates therefore apply.

At any given input choose its local incoming radial chart, and write F for the physical arrival radius in that chart. The composition of actual first-arrival maps holds on a neighborhood, so

$$R_+=F\circ z_\eta,\qquad E_\eta=F\circ z_\eta-F_\eta$$

there. This also proves the C^2 assertion. In (6.133), substitute the incoming amplitudes $\widetilde{u} = uB_U$, $\widetilde{v} = wB_W$ and $\widetilde{w} = \eta$. With $b = u + w$, the result is

$$|E_\eta| + |F_r - 1| \le BA_\eta^2 wb,$$
$$|F_u| \le BA_\eta w, \qquad |F_v| \le BA_\eta b.$$

The derivatives of F here are evaluated at z_η.

For $h = (h_R, h_u, h_w)$, the chain rule gives the exact identity

$$\begin{aligned} DE_\eta[h] = {} & (F_r - 1)DF_\eta[h] \\ & + F_u(B_U h_u + u\, DB_U[h]) \\ & + F_v(B_W h_w + w\, DB_W[h]). \end{aligned}$$

The fourth coordinate of z_η is constant. Each of the three terms containing a derivative of F_η, B_U, B_W is bounded, for any unit coordinate direction, by a constant depending on η times wb. For example, the latter two are at most $BA_\eta^2 wu$ and $BA_\eta^2 bw$. In the radial direction these are all the terms. In the u direction there is in addition $F_u B_U$, bounded by $BA_\eta^2 w$; in the w direction there is $F_v B_W$, bounded by $BA_\eta^2 b$. Use $b, w \le 1$ and enlarge the common constant to obtain (6.135). □

Proposition 6.29 (Expansion of the actual connection phase)**.** *There are $\varepsilon > 0$ and $C > 0$ such that, for $31/8 \le R \le 6$ and $0 < u, w \le \varepsilon$, the same actual reflection is locally C^2 and its next core radius satisfies*

$$\left| R_+(R, u, w) - \left(R - 2 - \frac{2u}{R} \right) \right| \le C(u + w)^2, \tag{6.136}$$

$$|\partial_R R_+ - 1| \le C(u + w^2), \tag{6.137}$$

$$\left| \partial_u R_+ + \frac{2}{R} \right| + |\partial_w R_+| \le C(u + w). \tag{6.138}$$

Proof. Fix one height η allowed by Lemma 6.28, and take a common input threshold with Proposition 6.12. The identity $R_+ = F_\eta + E_\eta$ and $w(u + w) \le (u + w)^2$ give (6.136). For the radial derivative, use the stronger finite estimate $|\partial_R F_\eta - 1| \le C_\eta(u + w^2)$ and observe that

$$w(u + w) = uw + w^2 \le u + w^2.$$

This proves (6.137). The two wall derivatives follow by adding (6.64) to the corresponding bounds in (6.135). The constant is fixed once η has been chosen. □

Thus the first-order strong-wall response survives the entire connection. The expansion has no linear weak-wall term, and the radial derivative retains the more precise scale $u + w^2$. These distinctions will matter when the phase estimate is combined with the contraction of the two output walls.

6.8. **Cone control and uniform reflections.** The phase estimate controls how a connection changes the radial parameter. To carry a family of orbits through the connection, we also need its transverse slope. The powers of the small growing wall in (6.23) provide enough decay to absorb both the logarithmic waiting length and the singular derivatives of the initial amplitudes. We first prove this at bounded radii, then combine the result with the long reflection.

For now let $31/8 \le R \le 6$ and fix an intermediate height η small enough for the incoming estimates. Write

$$\mathcal{F}(R, u, w) = (R_+, u_+, w_+), \qquad b = u + w.$$

All wall amplitudes are positive. The directions considered below are the normalized vectors in the radial cone (4.51):

$$h = (1, \xi, \zeta), \qquad |\xi| \le \sqrt{u}, \qquad |\zeta| \le \sqrt{w}. \tag{6.139}$$

We decrease the wall threshold as needed, always uniformly in R. Constants may depend on the fixed height η.

Lemma 6.30 (Logarithmic wall response)**.** *For sufficiently small positive u, w, the actual output walls satisfy, in every direction* (6.139)*,*

$$|D\log u_+[h]| \le C_\eta\left(1+|\log w|+\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{w}}\right),$$
$$|D\log w_+[h]| \le C_\eta\left(1+|\log w|+\frac{1}{\sqrt{w}}\right). \tag{6.140}$$

Proof. Recall the reflected entrance coordinates

$$z_\eta = (F_\eta, \widetilde{u}, \widetilde{v}, \eta), \qquad \widetilde{u} = uB_U, \quad \widetilde{v} = wB_W,$$

from (6.134). The finite factors, their inverses, and the differentials of their logarithms are uniformly bounded on a fixed neighborhood of the reference segment. The same is true of DF_η and DB_U. Since $u, w \le 1$, every direction in (6.139) has maximum norm one. It follows that, for $d = Dz_\eta[h]$, there is $A_\eta \ge 1$ such that

$$|d_r|,\ |d_u| \le A_\eta, \qquad d_w = 0,$$
$$|d_u| \le \widetilde{u}\left(A_\eta+\frac{1}{\sqrt{u}}\right), \qquad |d_v| \le \widetilde{v}\left(A_\eta+\frac{1}{\sqrt{w}}\right). \tag{6.141}$$

For the relative bounds, differentiate the factored coordinates:

$$d_u = B_U(\xi + u\,D\log B_U[h]), \qquad d_v = B_W(\zeta + w\,D\log B_W[h]).$$

Then use $\sqrt{u} = u/\sqrt{u}$ and $\sqrt{w} = w/\sqrt{w}$. The separate absolute bound on d_u follows directly from $d_u = B_U\xi + u\,DB_U[h]$.

The incoming waiting length satisfies

$$0 \le L_\eta = \log(1/\widetilde{v}) \le A_\eta + |\log w|. \tag{6.142}$$

Indeed, $-\log B_W \le B_W^{-1}$, and the inverse factor is bounded. The entrance has $7/4 \le k_\eta < 4$. Apply Propositions 6.22 and 6.25 in its radial chart. For a universal $B > 0$, the columns used in this composition obey

$$|(\ell_U)_r| \le BL_\eta, \qquad |(\ell_U)_u - 1/\widetilde{u}| \le B, \qquad |(\ell_U)_v| \le 4B/\widetilde{v},$$
$$|(\ell_W)_r| \le BL_\eta, \qquad |(\ell_W)_u| \le B, \qquad |(\ell_W)_v| \le B/\widetilde{v}.$$

The coordinate w in this chart equals the fixed height η, so its column makes no contribution.

The actual composition holds on a neighborhood of the input. Its chain rule identifies $D\log u_+[h] = D\ell_U[d]$ and $D\log w_+[h] = D\ell_W[d]$. Thus

$$|D\log u_+[h]| \le BA_\eta L_\eta + \frac{|d_u|}{\widetilde{u}} + B|d_u| + 4B\frac{|d_v|}{\widetilde{v}},$$
$$|D\log w_+[h]| \le BA_\eta L_\eta + B|d_u| + B\frac{|d_v|}{\widetilde{v}}.$$

Substitution of (6.141) and (6.142) proves the result. In particular, the weak output uses the absolute bound on d_u; its estimate has no inverse power of u. □

Lemma 6.31 (Transverse response)**.** *After reducing the input threshold if necessary, there is $C_\eta > 0$ such that*

$$|Du_+[h]| \le C_\eta b^{19/24}\sqrt{u_+}, \qquad |Dw_+[h]| \le C_\eta b^{2/9}\sqrt{w_+} \tag{6.143}$$

for every direction (6.139)*.*

Proof. For $0 < w \le 1$,

$$\sqrt{w}(1 + |\log w|) \le 2.$$

To prove this, set $s = \sqrt{w}$. The inequality $\log s \ge 1 - 1/s$ gives $s(1 - 2\log s) \le 2 - s \le 2$. The wall decay of Proposition 6.3 gives, with a fixed $A_\eta > 0$,

$$\sqrt{u_+} \le \sqrt{A_\eta}\sqrt{u}\, w^{19/24}, \qquad \sqrt{w_+} \le \sqrt{A_\eta}\, w^{13/18}. \tag{6.144}$$

For the strong wall, the factor in (6.140) has the decomposition

$$\begin{aligned}
&\sqrt{u}\, w^{19/24}\left(1 + |\log w| + \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{w}}\right) \\
&\quad = \sqrt{u}\, w^{7/24}[\sqrt{w}(1 + |\log w|) + 1] + w^{19/24} \\
&\quad \le 3\sqrt{u}\, w^{7/24} + w^{19/24} \le 4b^{19/24}.
\end{aligned}$$

The last step uses $\sqrt{u}\, w^{7/24} \le b^{1/2} b^{7/24}$ and $w \le b$. For the weak wall the corresponding calculation is

$$\begin{aligned}
w^{13/18}\left(1 + |\log w| + \frac{1}{\sqrt{w}}\right) &= w^{2/9}[\sqrt{w}(1 + |\log w|) + 1] \\
&\le 3w^{2/9} \le 3b^{2/9}.
\end{aligned}$$

Multiply the logarithmic response bounds by the appropriate square root in (6.144). Since $Du_+ = u_+ D\log u_+$ and $Dw_+ = w_+ D\log w_+$, one further multiplication by the output square root proves (6.143). □

These estimates are uniform over the ratio of the two incoming walls. In particular, the exponent $2/9$ in the weak output comes from $13/18 - 1/2$: the available wall decay exceeds the square-root singularity of its input derivative.

The phase derivative has a positive lower bound on the same directions. This supplies the normalization needed to compare input and output slopes.

Proposition 6.32 (Strict cone preservation at bounded radii)**.** *There are $\varepsilon > 0$ and $C > 0$ such that, for $31/8 \le R \le 6$ and $0 < u, w \le \varepsilon$, the actual reflection is locally C^2, and for every h in* (6.139)*, with $d = D\mathcal{F}[h]$,*

$$\begin{aligned}
&d_R \ge \tfrac{1}{2}, && |d_R - 1| \le C(\sqrt{u} + w\sqrt{w}), \\
&|d_u| \le Cb^{2/9}\sqrt{u_+}\, d_R, && |d_w| \le Cb^{2/9}\sqrt{w_+}\, d_R, \\
&Cb^{2/9} < \tfrac{1}{2}.
\end{aligned} \tag{6.145}$$

Consequently the radial cone is strictly preserved.

Proof. Let P be a common constant for the phase bounds of Proposition 6.29. For $u, w \le 1$, those bounds give

$$|(R_+)_R - 1| \le P(u + w^2), \qquad |(R_+)_u| \le 1 + 2P, \qquad |(R_+)_w| \le P(u + w),$$

where $2/R \le 1$ was used in the strong column. The differential on $h = (1, \xi, \zeta)$ therefore satisfies

$$\begin{aligned}
|d_R - 1| &\le P(u + w^2) + (1 + 2P)\sqrt{u} + P(u + w)\sqrt{w} \\
&\le (4P + 1)(\sqrt{u} + w\sqrt{w}).
\end{aligned} \tag{6.146}$$

Here $u \le \sqrt{u}$, $w^2 \le w\sqrt{w}$, and $u\sqrt{w} \le \sqrt{u}$. Put $K = 4P + 1$ and choose the threshold so that

$$\varepsilon \le \tfrac{1}{2}, \qquad \sqrt{\varepsilon} \le \frac{1}{4K}.$$

Then $\sqrt{u} + w\sqrt{w} \le 2\sqrt{\varepsilon}$, and (6.146) implies $d_R \ge 1/2$.

Let W be the constant in Lemma 6.31. Since $b \le 1$ and $19/24 > 2/9$, both transverse estimates are bounded by $Wb^{2/9}$ times the respective output square root. Choose $C \ge \max(K, 2W)$. The lower bound on d_R gives $W \le Cd_R$, which proves the two slope estimates in (6.145). Finally reduce ε so that $C(2\varepsilon)^{2/9} < 1/2$. For a nonzero vector of the full radial cone, division by its positive radial component reduces to (6.139); homogeneity gives the asserted strict preservation. □

The connection is now controlled across the whole reflection range. We state the result in the same coordinates as the uniform core return. The arrival time is measured in the original normalized time t.

Theorem 6.33 (Uniform reflection estimates)**.** *There are constants $C > 0$ and $0 < \varepsilon \le 10^{-4}$ such that, for*

$$R \ge 31/8, \qquad 0 < u, w \le \varepsilon/R, \tag{6.147}$$

the exact orbit from $(1 - \sqrt{R^2 - 1}, u, 1, w)$ has a first positive arrival at the old wall $W = 1$. Its time T and the reflected return $\mathcal{F}(R, u, w) = (R_+, u_+, w_+)$ are locally C^2. The reflected arrival state is $(1 + \sqrt{R_+^2 - 1}, u_+, 1, w_+)$, and

$$|R_+ - (R - 2)| \le \frac{C(u + w)}{R}, \qquad 0 < u_+ \le u, \qquad 0 < w_+ \le w. \tag{6.148}$$

For every h in (6.139)*, write $d = D\mathcal{F}[h]$. Then*

$$d_R \ge \tfrac{1}{2}, \qquad |d_R - 1| \le \frac{C\sqrt{u + w}}{R},$$
$$|\log d_R| \le \frac{C\sqrt{u + w}}{R}, \qquad |d_u| \le \sqrt{u_+}\, d_R, \qquad |d_w| \le \sqrt{w_+}\, d_R. \tag{6.149}$$

Proof. First put the estimates for bounded radii in the stated form. Fix a height η allowed by the preceding results, and take a common threshold for their existence, phase, wall and cone estimates. Restrict it further so that $b = u + w \le 1$. With P denoting the constant in Proposition 6.29, its expansion gives

$$|R_+ - (R - 2)| \le Pb^2 + 2u/R \le (P + 1)b.$$

As $R \le 6$, this is at most $6(P + 1)b/R$. Likewise

$$\sqrt{u} + w\sqrt{w} \le 2\sqrt{b} \quad \Longrightarrow \quad |d_R - 1| \le 12K\sqrt{b}/R$$

for a suitable constant K from Proposition 6.32. Its transverse factor is less than $1/2$, hence at most one. The wall nonincrease is (6.23). The value, additive derivative and cone bounds therefore hold on $[31/8, 6]$, with some constants $\varepsilon_c, C_c > 0$.

On $R \ge 11/2$, Corollary 5.12 gives the same conclusions with threshold ε_ℓ/R and error constant at most 300000. Both constructions describe the first arrival at the old $W = 1$ section. The arrival remains valid on neighborhoods, so uniqueness identifies their times and maps there, including on the overlapping interval. Their differentials thus belong to this single reflection map.

Take

$$\varepsilon = \min(\varepsilon_c, \varepsilon_\ell, 10^{-4}), \qquad C_0 = \max(C_c, 300000).$$

For $R \ge 11/2$ use the long estimate. For $31/8 \le R < 11/2$, the hypothesis $u, w \le \varepsilon/R$ implies $u, w \le \varepsilon_c$, and the estimate for bounded radii applies. This proves the value and additive derivative estimates with constant C_0, together with the regularity and cone assertions. At the actual arrival the reflected shear satisfies $M_+ > 1$ and the active wall equals one. Solving $R_+^2 = (M_+ - 1)^2 + 1$ yields the stated reconstruction.

Finally, for every $a \ge 1/2$,

$$|\log a| \le 2|a - 1|. \tag{6.150}$$

Indeed, $1/a - 1 \le 2|a-1|$, while $1 - 1/a \le \log a \le a - 1$. Apply (6.150) to $a = d_R$ and replace C_0 by $C = 2C_0$. □

Core passages and reflections now preserve the same radial cone. Their phase errors are controlled by the amplitudes, and their derivative errors by the square roots of those amplitudes. The logarithmic estimate is suited to successive passages, because logarithms turn the product of radial derivatives into a sum. To close an era, it remains to estimate the final connection, where the change of frame differs from a reflection.

7. The terminal connection

At the last core of an era, a cyclic change of axes replaces the reflection. The new reciprocal radius is the bounded phase that enters the next inverse branch. We determine this phase and its derivatives, together with the transverse contraction needed at the next entrance.

7.1. **Passage across the cyclic change of axes.**

Write the terminal exit radius as

$$R_0 = 2 + 2\alpha, \qquad 0 < \alpha \le 1.$$

The new projective denominator is of order α^2. To enter the next core with small transverse walls, we follow the departing wall down to a height

$$h = \eta\alpha^J, \qquad L = \log(1/h) = \log(1/\eta) + J\log(1/\alpha). \tag{7.1}$$

Here η and the integer J will be chosen once. The outgoing segment can become long as α tends to zero, but its wall rates have size α apart from integrable terms. This permits uniform control before changing frames.

7.1.1. *Descent to the auxiliary section.* Start at the downward core exit

$$y_-(R_0, u, w) = (1 - \sqrt{R_0^2 - 1}, u, 1, w). \tag{7.2}$$

We allow signed u, w in this first passage. This will give ordinary smooth dependence at points where a wall vanishes. Denote the first arrival at $V = h$ by S_h, and the state there by y_h.

Lemma 7.1 (Uniform outgoing passage)**.** *Fix $0 < \eta < 1$ and an integer $J \ge 1$. There are constants $B \ge 2$ and $\varepsilon > 0$ such that, if*

$$0 < \alpha \le 1, \qquad 0 < b \le \varepsilon\alpha^J, \qquad |u| + |w| \le b, \tag{7.3}$$

the actual orbit from (7.2) reaches $V = h$ for the first time at $S_h > 0$. Throughout this segment,

$$A \ge \frac{1}{8}, \qquad M < 0, \qquad \frac{7}{4} < \rho < \frac{17}{4}, \qquad h \le V \le 1, \quad \rho = H(M-1, V), \tag{7.4}$$

$$|U| + |W| \le \frac{Bb}{2}, \qquad |\rho - R_0| \le 40Bb, \qquad \rho > 2 + \alpha, \qquad |U| \le \frac{V}{2}. \tag{7.5}$$

There are positive wall multipliers $B_U(t), B_W(t)$ with

$$U(t) = uB_U(t), \qquad W(t) = wB_W(t), \qquad \frac{2}{B} \le B_U(t), B_W(t) \le \frac{B}{2}. \tag{7.6}$$

The departing wall is strictly decreasing and

$$0 < S_h \le 16L. \tag{7.7}$$

For each such passage, holding its numerical height h fixed, the arrival time and state are C^2 on a neighborhood of (R_0, u, w).

Proof. Use the decreasing wall as a clock, $\sigma = -\log V$. On the downward branch set

$$g = \sqrt{\rho^2 - V^2} = 1 - M, \qquad q_- = g + C, \qquad C = U(V + W - U).$$

Then $d\sigma/d\tau = q_-$ and $V = e^{-\sigma}$. The energy identity and the two transverse equations give

$$\begin{aligned} \frac{d\rho}{d\sigma} &= \frac{-C\rho - g(U-W)^2/\rho}{q_-}, \\ \frac{dU}{d\sigma} &= a_U U, \qquad a_U = \frac{2g - \rho^2 + 2VW - W^2 + U^2}{2q_-}, \\ \frac{dW}{d\sigma} &= a_W W, \qquad a_W = \frac{g - 2 - C}{q_-}. \end{aligned} \tag{7.8}$$

On the region

$$\frac{7}{4} \le \rho \le \frac{17}{4}, \qquad |U|, |W| \le \frac{1}{100}, \qquad 0 \le V \le 1, \tag{7.9}$$

one has $1 < g \le \rho$ and $q_- \ge 1/2$. The latter follows from $|C| \le |U|V + (|U| + |W|)^2$. At zero transverse walls the rates are

$$\begin{aligned} a_U^0 &= -\frac{\rho - 2}{2} - \frac{\rho V^2}{2g(\rho + g)}, \\ a_W^0 &= \frac{\rho - 2}{\rho} - \frac{2V^2}{\rho g(\rho + g)}. \end{aligned} \tag{7.10}$$

These identities use $\rho - g = V^2/(\rho + g)$. In particular,

$$|a_U^0| + |a_W^0| \le 10(\alpha + |\rho - R_0| + V^2). \tag{7.11}$$

For example, the two fractions involving V^2 in (7.10) are at most $5V^2$ and $2V^2$, respectively, and $|\rho - 2| \le |\rho - R_0| + 2\alpha$.

Put $\beta = |U| + |W|$. The differences from the boundary rates are

$$a_U - a_U^0 = \frac{2VW - W^2 + U^2 - 2Ca_U^0}{2q_-}, \qquad a_W - a_W^0 = -\frac{C(1 + a_W^0)}{q_-}.$$

On (7.9), $|a_U^0| \le 20$, $|a_W^0| \le 7$, and hence

$$\begin{aligned} \left|\frac{d\rho}{d\sigma}\right| &\le 20\big(|U|V + \beta^2\big), \\ |a_j - a_j^0| &\le 100(\beta V + \beta^2), \qquad j = U, W. \end{aligned} \tag{7.12}$$

For the first inequality, use $g/\rho \le 1$, $(U - W)^2 \le \beta^2$, $\rho \le 5$ and $q_- \ge 1/2$. The other two follow directly from the displayed difference formulas.

We now close these estimates uniformly over the full clock interval. The elementary inequality $xe^{-x} \le e^{-1}$ for $x \ge 0$ gives

$$\begin{aligned} \alpha L &\le K := \log(1/\eta) + Je^{-1}, \\ \alpha^J(1 + L) &\le Q := 1 + \log(1/\eta) + e^{-1}. \end{aligned} \tag{7.13}$$

Set $B = 2\exp(10K + 6)$. On any existing prefix $0 \le \sigma \le T \le L$ assume, in addition to (7.9),

$$\beta \le Bb, \qquad |\rho - R_0| \le 80Bb. \tag{7.14}$$

Integrating (7.11) and (7.12), using $\int_0^T e^{-\sigma}\, d\sigma \le 1$ and $\int_0^T e^{-2\sigma}\, d\sigma \le 1/2$, yields

$$\begin{aligned} \left|\int_0^T a_j\, d\sigma\right| &\le 10\alpha T + 800BbT + 100(Bb)^2T + 100Bb + 5, \\ |\rho(T) - R_0| &\le 20\big[Bb + (Bb)^2T\big]. \end{aligned}$$

Choose a constant $\delta > 0$, depending on B, so small that $b(1+T) \le \delta$ makes the first right-hand side at most $10K+6$ and the second at most $40Bb$. For instance,

$$\delta \le \frac{1}{900B + 100B^2 + 1}$$

suffices. Define $\Lambda_j(T) = \exp(\int_0^T a_j \, d\sigma)$. The scalar equations give $U(T) = u\Lambda_U(T)$ and $W(T) = w\Lambda_W(T)$ in this clock, including for signed or zero inputs. Thus (7.14) improves to

$$\frac{2}{B} \le \Lambda_j(T) \le \frac{B}{2}, \qquad \beta(T) \le \frac{Bb}{2}, \qquad |\rho(T) - R_0| \le 40Bb.$$

Take $\varepsilon Q \le \delta$ and $B\varepsilon \le \min(1/1000, \eta)$. Equations (7.3) and (7.13) ensure all the preceding smallness conditions. They also give

$$Bb \le \frac{\alpha}{1000}, \qquad Bb \le h.$$

The improved bounds put ρ strictly between $7/4$ and $17/4$ and both walls strictly below $1/100$. Continuity therefore extends the improved bounds over every existing prefix. Their last two consequences in (7.5) follow from $\rho \ge 2 + 2\alpha - 40Bb$ and $V \ge h \ge Bb$.

To prove arrival at the lower section, return to the normalized flow. It exists for all positive time also for signed initial walls on $G = 1$: the proof of Lemma 2.1 bounds $|N_i(t)|$ by $|N_i(0)|e^t$ and then bounds the shear using absolute values of the curvature products. Continue the orbit in the closed region given by $M \le 0$, (7.9), $h \le V \le 1$ and $A \ge 1/16$. In this region,

$$0 < D \le 48, \qquad A = \frac{6}{D} \ge \frac{1}{8}.$$

Indeed, $-5 \le M < 0$ follows from the radius and active-wall bounds. In (2.7) the cross term has absolute value at most one, while $U^2 \le 1$ and $(V-W)^2 \le 4$. These bounds give both positivity of D and its stated upper bound.

On every prefix of this region, the clock is strictly increasing and its inverse converts the actual orbit into (7.8). Thus all the improved estimates already proved apply. Before $V = h$, each boundary condition has a strict margin, except $V = 1$, where $dV/dt = -Aq_- V < 0$ points inward. The region can therefore be continued until the lower section is reached. Moreover,

$$\frac{d\sigma}{dt} = Aq_- \ge \frac{1}{16}.$$

As long as $V > h$ one has $\sigma < L$, so the segment cannot last longer than $16L$. This proves existence, strict decrease, first arrival and (7.7). The multipliers in original time are $B_j(t) = \Lambda_j(-\log V(t))$, giving (7.6).

Finally fix h at its value for the given input. On this finite segment the chart is positive and $dV/dt < 0$. These properties persist on a neighborhood of the segment and its moving endpoint. The implicit function theorem gives a C^2 arrival time, and strict decrease identifies it with the first arrival. Composing with the flow gives the C^2 arrival state, including for zero or signed transverse walls. □

7.1.2. *The cyclic change of frame.* The normalized permutation C in (2.4) has the following projective expression:

$$\mathcal{H} = (M+1)^2 + U^2 + V^2 + W^2 - 2UV - 2UW - 2VW,$$

$$\mathsf{C}_p(M,U,V,W) = \left(1 - \frac{4(M+1)}{\mathcal{H}}, \frac{4V}{\mathcal{H}}, \frac{4W}{\mathcal{H}}, \frac{4U}{\mathcal{H}}\right). \tag{7.15}$$

Substitution in (2.6) gives

$$A_{\mathrm{new}} = \frac{A\mathcal{H}}{4}. \tag{7.16}$$

Thus $\mathcal{H} > 0$ ensures that both frames describe the same orbit in valid charts. The permutation commutes with the flow in original time; the two projective clocks have the ratio displayed in (7.16).

The small denominator can be estimated directly along the whole outgoing segment. For $M = 1 - g$ and $g = \sqrt{\rho^2 - V^2}$,

$$\mathcal{H} = (\rho - 2)^2 + \frac{4V^2}{\rho + g} + (U - W)^2 - 2V(U + W). \tag{7.17}$$

This identity separates the radial scale from the wall error.

Lemma 7.2 (Cyclic entrance estimates)**.** *Fix $0 < \eta \le 1/100$ and an integer $J \ge 2$. There are $C_* > 0$ and $\varepsilon > 0$ such that, if*

$$0 < \alpha \le 1, \qquad 0 < b \le \varepsilon\alpha^{2J}, \qquad |u| + |w| \le b, \tag{7.18}$$

the outgoing segment of Lemma 7.1 satisfies

$$\mathcal{H} > \frac{39}{10}\alpha^2 + \frac{2}{5}V^2, \qquad |V_{\mathrm{new}}| \le \frac{2C_*b}{\alpha^2} \le \frac{1}{5000}. \tag{7.19}$$

At its endpoint write $z = \mathsf{C}_p y_h = (m_e, u_e, v_e, w_e)$ and $k = H(m_e - 1, v_e)$. Then

$$\frac{399}{100}\alpha^2 < \mathcal{H}(y_h) < \frac{401}{100}\alpha^2, \tag{7.20}$$

$$\frac{19}{10\alpha} < m_e - 1 < \frac{21}{10\alpha}, \qquad \frac{19}{10\alpha} < k < \frac{11}{5\alpha}, \tag{7.21}$$

$$\frac{\eta}{2}\alpha^{J-2} \le u_e \le 2\eta\alpha^{J-2}, \qquad |v_e|, |w_e| \le \frac{2C_*b}{\alpha^2}. \tag{7.22}$$

For positive input walls, v_e and w_e are positive. The cyclic chart remains positive throughout the outgoing segment.

Proof. Use Lemma 7.1 and take $C_* = 40B$. After decreasing ε, its estimates apply and, with $E = C_*b$, give

$$|\rho - R_0| \le E, \qquad |U| + |W| \le E, \qquad E \le \frac{\alpha^2}{10000} \le \frac{\alpha}{10000}.$$

Here $\alpha^{2J} \le \alpha^J$ ensures the previous smallness condition, and $\alpha^{2J} \le \alpha^2$ gives the new one. In (7.17), $\rho + g \le 9$ and $\rho - 2 \ge (199/100)\alpha$. The inequality $2VE \le V^2/100 + 100E^2$ therefore yields

$$\mathcal{H} \ge \left(\frac{199}{100}\right)^2 \alpha^2 - 100\left(\frac{\alpha}{10000}\right)^2 + \left(\frac{4}{9} - \frac{1}{100}\right)V^2 > \frac{39}{10}\alpha^2 + \frac{2}{5}V^2.$$

The wall formula in (7.15) implies $|4W/\mathcal{H}| \le 2E/\alpha^2 \le 1/5000$. Together with (7.16), this proves the assertions along the segment.

At $V = h$ we have $V \le \alpha/100$. Now $\rho + g \ge 3$ and

$$\frac{19999}{10000}\alpha \le \rho - 2 \le \frac{20001}{10000}\alpha.$$

The part of $\mathcal{H}$ involving the two transverse walls has absolute value at most

$$E^2 + 2VE \le \left(\frac{\alpha}{10000}\right)^2 + 2\frac{\alpha}{100}\frac{\alpha}{10000}.$$

The remaining correction $4V^2/(\rho + g)$ lies between zero and $4\alpha^2/(3 \cdot 10000)$. Substitution in (7.17) proves (7.20).

At the same endpoint,

$$\frac{199}{100}\alpha < g - 2 < \frac{201}{100}\alpha. \tag{7.23}$$

The upper bound follows from $g \le \rho$. For the lower bound, the preceding lower bound for ρ and $V \le \alpha/100$ give $\rho^2 - V^2 > (2 + (199/100)\alpha)^2$ by expansion. Since $g \ge 0$, taking square roots proves it. As $m_e - 1 = 4(g-2)/\mathcal{H}$, equations (7.20) and (7.23) give the first pair of bounds in (7.21). The second follows from

$$m_e - 1 \le H(m_e - 1, v_e) \le m_e - 1 + |v_e|, \qquad |v_e| \le 1/5000, \quad \alpha \le 1.$$

Finally $u_e = 4h/\mathcal{H}$ gives its two bounds, and $v_e = 4W_h/\mathcal{H}$, $w_e = 4U_h/\mathcal{H}$ give the others. Their positivity follows from (7.6). □

The purpose of the exponent J is now explicit. Suppose the incoming estimates require its two decreasing walls to be small even after multiplication by k^q, where $q \in \mathbb{N}$. Choose

$$J \ge q + 2, \qquad 0 < \varepsilon_{\mathrm{in}} \le 10^{-4}, \qquad 0 < \eta \le \min\left(\frac{1}{100}, \frac{\varepsilon_{\mathrm{in}}}{4(11/5)^q}\right). \tag{7.24}$$

The next result records both this smallness and the actual event that it makes possible.

7.1.3. *Arrival at the next core.*

Proposition 7.3 (The actual terminal passage)**.** *For the choices in* (7.24)*, there is* $\varepsilon > 0$ *such that every positive pair* u, w *satisfying* (7.18) *has the following properties. The cyclic endpoint* $z = (m_e, u_e, v_e, w_e)$ *is an incoming point with*

$$m_e > 2, \qquad k > \frac{7}{4}, \qquad 0 < v_e < \frac{1}{2},$$
$$0 < u_e, w_e \le \varepsilon_{\mathrm{in}}, \qquad u_e k^q,\ w_e k^q \le \frac{\varepsilon_{\mathrm{in}}}{2}. \tag{7.25}$$

For the exact flow starting at the cyclic permutation of the normalized state associated with (7.2)*, the time and state*

$$T_h = S_h + T_{\mathrm{in}}(z), \qquad y_+ = \mathcal{P}(z) \tag{7.26}$$

give its first positive arrival at the new $V = 1$*. The new chart is positive up to that time, the new* V *is less than one before arrival, and the final crossing is upward with* $M_+ > 1$*. For each input, the two expressions in* (7.26)*, with* h *held fixed, are locally* C^2 *functions of* (R, u, w) *at* (R_0, u, w)*.*

If η *is replaced by another value satisfying* (7.24)*, then on a common input domain the resulting total time and final state are unchanged.*

Proof. Apply Lemma 7.2 and put

$$d = \frac{\varepsilon_{\mathrm{in}}}{4(11/5)^q}.$$

Decrease the input threshold so that $C_*\varepsilon \le d$. The endpoint bounds give

$$u_e k^q \le 2\eta(11/5)^q \alpha^{J-q-2} \le \frac{\varepsilon_{\mathrm{in}}}{2},$$
$$w_e k^q \le 2d(11/5)^q \alpha^{2J-q-2} \le \frac{\varepsilon_{\mathrm{in}}}{2}.$$

Both exponents are nonnegative. Since $k > 1$, these inequalities also imply the unweighted wall bounds. The remaining entrance conditions follow from (7.19) and (7.21). Thus Proposition 6.1 applies to z.

The original-time flow commutes with C. Consequently its cyclic state at time S_h is exactly z, and flow uniqueness identifies the continuation from z with the same normalized orbit. It reaches y_+ at time T_h. During $[0, S_h]$, the new chart is positive and the new active wall is at most $1/5000$ by (7.19). During the incoming segment, that wall is strictly increasing and stays below one until its first arrival. This proves that T_h is the first positive event for the whole orbit. Its transversality and $M_+ > 1$ follow from the incoming passage.

For fixed h, Lemma 7.1 gives C^2 dependence of S_h and y_h. The rational map C_p is smooth wherever $\mathcal{H} \neq 0$, and $\mathcal{H}(y_h) > 0$ here. The incoming arrival time and state are C^2 on an ordinary neighborhood of z. Composition proves the stated local regularity.

Finally, for two admissible values η_1, η_2, choose the smaller of their input thresholds. Both constructions give the first positive arrival of the same cyclic orbit at the same section. Uniqueness of this first time gives equality of the total times, and flow uniqueness gives equality of the final states. □

The terminal passage therefore admits a decomposition adapted to any admissible auxiliary height. The radius bounds show how an exit with $R_0 - 2 = 2\alpha$ feeds an incoming radius of order α^{-1}. To determine the era map quantitatively, we next estimate the phase and derivatives accumulated in this passage, retaining the wall factors through the cyclic denominator.

7.2. **Differentiation of the terminal passage.** The auxiliary height separates two parts of a single orbit. Its choice can therefore be held fixed when differentiating the full terminal return. We first establish this local independence, including ordinary neighborhoods of the initial data. We then estimate the outgoing derivatives. Although the outgoing clock interval grows like $\log(1/\alpha)$, its integrated Jacobian is bounded. First derivatives consequently remain bounded, while second derivatives grow at most linearly in the clock length.

Write $Y = (R, u, w)$ for the downward-exit parameters. Make the height dependence in Section 7.1 explicit by setting $z(h, Y) = \mathsf{C}_p y_h(Y)$ and

$$\begin{aligned} \mathcal{T}(h, Y) &= S_h(Y) + T_{\mathrm{in}}(z(h, Y)), \\ \mathcal{E}(h, Y) &= \mathcal{P}(z(h, Y)). \end{aligned} \tag{7.27}$$

All the time variables in this definition are original normalized times.

7.2.1. *Local independence of the section height.*

Lemma 7.4 (The height can be held fixed)**.** *At every positive-wall input in Proposition 7.3, put $Y_0 = (2 + 2\alpha, u, w)$ and $h_0 = \eta\alpha^J$. The functions $\mathcal{T}$ and $\mathcal{E}$ are jointly C^2 near (h_0, Y_0). On a neighborhood of that point,*

$$\mathcal{T}(h, Y) = \mathcal{T}(h_0, Y), \qquad \mathcal{E}(h, Y) = \mathcal{E}(h_0, Y). \tag{7.28}$$

In particular, their height derivatives vanish. For any function a continuous at Y_0 with $a(Y_0) = h_0$, the maps $\mathcal{T}(a(Y), Y)$ and $\mathcal{E}(a(Y), Y)$ agree near Y_0 with their fixed-height versions and are C^2 there.

Proof. Consider first the outgoing event equation $V(\phi_t x(Y)) - h = 0$, where $x(Y)$ is the normalized state corresponding to $y_-(Y)$. Its time derivative is strictly negative at $t = S_{h_0}(Y_0)$. The implicit function theorem, with (h, Y) as parameters, gives a jointly C^2 event time. The chart and strict negativity of dV/dt persist on the whole compact segment, including a small extension beyond its endpoints. For nearby (h, Y), they therefore persist up to the moving event time. The event is still the first arrival at $V = h$. Hence this local implicit time equals $S_h(Y)$, and its state is jointly C^2.

The cyclic denominator is positive at the endpoint. Moreover, all incoming conditions hold there with strict margins. In particular, (7.25) and $k^q \geq 1$ give

$$0 < u_e, w_e \leq \varepsilon_{\mathrm{in}}/2 < 10^{-4}, \qquad 0 < v_e < 1, \qquad k > 7/4, \qquad m_e > 1.$$

They persist near (h_0, Y_0). The rational cyclic map and the incoming arrival are thus C^2 throughout a smaller neighborhood, which proves joint regularity in (7.27).

We also need to identify these nearby compositions as physical first arrivals. Along the original outgoing segment, the cyclic chart is positive and its active wall is less than one. These are open conditions on a compact segment. By continuity of the flow and of $S_h(Y)$,

they continue to hold up to the nearby moving endpoint. The incoming segment then stays below its own arrival section until its first crossing. The proof of Proposition 7.3 applies to every point of this neighborhood. Thus $\mathcal{T}(h, Y)$ and $\mathcal{T}(h_0, Y)$ are both the first positive arrival time of the same cyclic orbit at $V = 1$. They are equal, and their states are equal as well. This proves (7.28).

For fixed Y_0 the two functions are locally constant in h, so their height derivatives vanish. Finally, $Y \mapsto (a(Y), Y)$ is continuous at Y_0 and takes that point to (h_0, Y_0). Equation (7.28) therefore holds along this graph near Y_0. The fixed-height functions are C^2, giving the last assertion without any differentiability requirement on a. □

For the choice used in the terminal passage, $a(R, u, w) = \eta((R-2)/2)^J$. The lemma identifies its derivatives with those computed at fixed numerical height. The same local identity also gives zero derivative with respect to the logarithmic height ℓ when $h = e^{-\ell}$.

Uniform estimates over a range of heights follow by stopping one longer outgoing segment earlier. This avoids choosing a different initial neighborhood for each height.

Lemma 7.5 (Uniformity above a lower height)**.** *Fix $0 < \eta_0 \le 1/100$ and $J \ge 2$. The constants C_* and ε in Lemma 7.2 can be chosen to work simultaneously for every $\eta \in [\eta_0, 1/100]$. If $q + 2 \le J$ and $0 < \varepsilon_{\mathrm{in}} \le 10^{-4}$ are also fixed, then one input threshold gives all the conclusions of Proposition 7.3 for every*

$$\eta_0 \le \eta \le \min\left(\frac{1}{100}, \frac{\varepsilon_{\mathrm{in}}}{4(11/5)^q}\right). \tag{7.29}$$

The local height independence of Lemma 7.4 holds at each such input and height.

Proof. Construct the outgoing segment down to $h_0 = \eta_0\alpha^J$. Its strictly decreasing active wall crosses each $h = \eta\alpha^J$ with $\eta \ge \eta_0$ and $\eta < 1$ exactly once no later than that endpoint. The intermediate value theorem gives the crossing, and strict decrease proves its first-arrival property. Thus

$$S_{\eta\alpha^J}(Y) \le S_{\eta_0\alpha^J}(Y).$$

All estimates along the longer segment apply up to the earlier endpoint. Reduce ε so that $C_*\varepsilon \le 1/10000$. Then, with the fixed constant C_* from that segment,

$$|\rho - R_0| \le C_* b, \qquad |U| + |W| \le C_* b, \qquad C_* b \le \alpha^{2J}/10000.$$

At the earlier endpoint, $V = \eta\alpha^J \le \alpha/100$. The algebra in Lemma 7.2 therefore gives exactly the same denominator and radius bounds, and the same wall bounds with this value of η.

For the weighted incoming walls, reduce ε so that $C_*\varepsilon \le \varepsilon_{\mathrm{in}}/[4(11/5)^q]$. The proof of (7.25) is then uniform over (7.29). The entire cyclic outgoing segment still has active wall at most $1/5000$, since it is a subsegment of the longer one. Composition with the incoming passage gives the actual first arrival. The strict margins used in Lemma 7.4 hold at each input, proving its local conclusion as well. □

7.2.2. *The Jacobian along the outgoing segment.* Let $Y = (\rho, U, W)$ and regard V as a prescribed clock value. Write the vector field in (7.8) as

$$f_V(Y) = (f, a_U U, a_W W), \qquad f = \frac{-C\rho - g(U-W)^2/\rho}{q_-}.$$

All derivatives in these estimates hold V fixed. Norms on $\mathbb{R}^3$ are maximum norms, with their induced operator norms.

Lemma 7.6 (An integrable Jacobian)**.** *On the signed region* (7.9)*, with $\beta = |U| + |W|$,*

$$\|D_Y f_V\| \le |a_U| + |a_W| + 2048(V + \beta). \tag{7.30}$$

The field and its first two state derivatives are uniformly bounded on this region, including at $V = 0$.

Proof. The small factors in the radial row and in the off-diagonal wall entries are visible before taking norms. Put $c_U = V + W - 2U$. Quotient differentiation gives

$$\partial_\rho f = \frac{-C - (1/g - g/\rho^2)(U-W)^2 - f\rho/g}{q_-},$$
$$\partial_U f = \frac{-c_U\rho - 2g(U-W)/\rho - fc_U}{q_-},$$
$$\partial_W f = \frac{-U\rho + 2g(U-W)/\rho - fU}{q_-}.$$

The corresponding rate derivatives are

$$\partial_\rho a_U = \frac{\rho/g - \rho - a_U\rho/g}{q_-}, \qquad \partial_\rho a_W = \frac{(1-a_W)\rho/g}{q_-},$$
$$\partial_U a_U = \frac{U - a_U c_U}{q_-}, \qquad \partial_U a_W = -\frac{(1+a_W)c_U}{q_-},$$
$$\partial_W a_U = \frac{V - W - a_U U}{q_-}, \qquad \partial_W a_W = -\frac{(1+a_W)U}{q_-}.$$

To bound them, use $q_- \geq 1/2$, $g \geq 1$, $1 \leq \rho \leq 5$, $g/\rho \leq 1$, $\rho/g \leq 5$, and

$$|c_U| \leq V + 2\beta, \qquad |1/g - g/\rho^2| \leq 2.$$

Equation (7.12) and $\beta \leq 1/50$ give $|f| \leq 1$ and $|a_U|, |a_W| \leq 32$. Keeping the sharper bound $|f| \leq 20(|U|V + \beta^2)$ in the radial derivative gives

$$|\partial_\rho f| \leq 256(|U|V + \beta^2), \qquad |\partial_U f| \leq 32(V+\beta), \qquad |\partial_W f| \leq 16\beta,$$
$$|\partial_\rho a_j| \leq 512, \qquad |\partial_U a_j|,\ |\partial_W a_j| \leq 256(V+\beta), \qquad j = U, W. \tag{7.31}$$

For example, the three numerator terms in $\partial_\rho f$ are bounded by H, $2\beta^2$ and $100H$, respectively, where $H = |U|V + \beta^2$. Division by $q_- \geq 1/2$ gives the stated $256H$. The other bounds follow from the same displayed formulas: the radial U numerator is at most $6V + 14\beta$, and the radial W numerator at most 8β; the rate numerators are bounded by 256 in the radial direction and $128(V+\beta)$ in either wall direction.

The Jacobian entries in a wall row retain the factor of that wall:

$$D(a_U U)[\xi] = U\, Da_U[\xi] + a_U\xi_U, \qquad D(a_W W)[\xi] = W\, Da_W[\xi] + a_W\xi_W.$$

Since $H \leq V + \beta$ and $V + \beta \leq 2$, the sum of absolute values in the radial row is at most $512(V+\beta)$, while each rate gradient has row sum at most 1536. Each wall row is consequently bounded by $|a_j| + 1536\beta$. Taking the maximum of the three row sums proves (7.30).

For the second derivatives, consider the compact box

$$7/4 \leq \rho \leq 17/4, \qquad |U|, |W| \leq 1/100, \qquad 0 \leq V \leq 1.$$

On it, $\rho^2 - V^2$ is bounded away from zero, $\rho > 0$, and $q_- \geq 1/2$. The formulas defining f_V are therefore smooth on a neighborhood of the box. Their derivatives through order two are continuous there and have a finite common bound. □

Fix $\eta_0 \in (0,1)$ and $J \geq 1$, and set $L_0 = \log(1/\eta_0) + J\log(1/\alpha)$. For the outgoing orbit constructed at this lowest height, denote its clock state by $Y(\sigma)$ and its state Jacobian by $L_Y(\sigma) = D_Y f_{e^{-\sigma}}(Y(\sigma))$. The pointwise bounds (7.11) and (7.12) give, for $T \leq L_0$,

$$\int_0^T |a_j|\, d\sigma \leq \mathcal{R}(T),$$
$$\mathcal{R}(T) = 10\alpha T + 800BbT + 100(Bb)^2 T + 100Bb + 5. \tag{7.32}$$

Here B is chosen using η_0 as in Lemma 7.1. Integrating (7.30) now yields

$$\int_0^T \|L_Y(\sigma)\|\, d\sigma \le 2\mathcal{R}(T) + 2048(1 + BbT). \tag{7.33}$$

With the same smallness bounds as before, $\mathcal{R}(T) \le 10K + 6$, where $K = \log(1/\eta_0) + J/e$. Reducing the input threshold if necessary also gives $bT \le 1$. Thus the right-hand side is at most

$$B_* = 2(10K + 6) + 2048(1 + B), \tag{7.34}$$

uniformly in α, the signed wall inputs, and the duration $T \le L_0$.

For fixed h, express the outgoing state in its radius and two small walls:

$$E_h(Y) = \big(H(M_h - 1, h), U_h, W_h\big). \tag{7.35}$$

The active wall has the fixed value h at this event.

Proposition 7.7 (First and second outgoing derivatives)**.** *Fix $0 < \eta_0 < 1$ and $J \ge 1$. There are $C > 0$ and $\varepsilon > 0$ such that, for*

$$0 < \alpha \le 1, \qquad 0 < b \le \varepsilon\alpha^J, \qquad |u| + |w| \le b, \tag{7.36}$$

and every $\eta \in [\eta_0, 1)$, the map E_h, with $h = \eta\alpha^J$ held fixed, is C^2 near $Y_0 = (2 + 2\alpha, u, w)$. At this input,

$$\|DE_h(Y_0)\| \le C, \qquad \|D^2E_h(Y_0)\| \le C\log(1/h). \tag{7.37}$$

More generally, the outgoing state at any fixed clock value $0 \le \sigma \le L_0$ has these bounds with $\log(1/h)$ replaced by σ.

Proof. We first construct the clock family from the actual flow. Along the segment down to $\eta_0\alpha^J$, the functions A, V, q_- are positive and $M < 1$. Extend the compact segment slightly in original time. Smooth dependence gives a common open interval and a neighborhood of Y_0 on which these strict conditions persist. The map

$$(Y, t) \longmapsto (Y, -\log V(\phi_t x(Y)))$$

has positive time derivative Aq_-. The inverse function theorem gives a joint C^2 inverse time $t = t(Y, \sigma)$. For each nearby Y, the clock is strictly increasing on the common interval, so the local inverses agree wherever they overlap. At Y_0, the extended interval has clock image containing $[0, L_0]$ in its interior. This remains true after restricting the initial neighborhood, so the inverses cover that entire clock interval for every nearby input.

In radius and wall coordinates the resulting actual family, denoted by $\Phi(Y, \sigma)$, satisfies

$$\Phi(Y, 0) = Y, \qquad \partial_\sigma\Phi(Y, \sigma) = f_{e^{-\sigma}}(\Phi(Y, \sigma)). \tag{7.38}$$

For every fixed $h \in [\eta_0\alpha^J, 1)$, strict monotonicity and persistence of the first arrival identify

$$E_h(Y) = \Phi(Y, \log(1/h)) \tag{7.39}$$

on a neighborhood of Y_0. This is an equality of local functions, so it identifies their first and second derivatives.

For an input direction ξ, put $X(\sigma) = D_Y\Phi(Y_0, \sigma)[\xi]$. Differentiating (7.38) gives

$$X' = L_Y X, \qquad X(0) = \xi.$$

The estimate needed also applies to a forced equation $X' = L_Y X + P$. A signed row sum of a three-dimensional matrix is bounded by $3\|L_Y\|$. As in Lemma 4.6, the upper right derivative of $\|X\|$ is therefore at most $3\|L_Y\|\|X\| + \|P\|$. Multiply by the integrating factor

$$\exp\left(-3\int_a^t \|L_Y(s)\|\, ds\right).$$

Integration, together with $\int_a^t \|L_Y\| \le B_*$, gives

$$\|X(t)\| \le e^{3B_*}\left(\|X(a)\| + \int_a^t \|P(s)\|\,ds\right), \qquad 0 \le a \le t \le L_0. \tag{7.40}$$

In particular,

$$\|D_Y\Phi(Y_0,\sigma)[\xi]\| \le e^{3B_*}\|\xi\|.$$

For two input directions ξ, ζ, let $X = D_Y\Phi[\xi]$, $Z = D_Y\Phi[\zeta]$, and $Q = D_Y^2\Phi[\xi,\zeta]$, all evaluated along the reference clock orbit. The second variation satisfies

$$Q' = L_Y Q + D_Y^2 f_{e^{-\sigma}}(Y(\sigma))[X,Z], \qquad Q(0) = 0. \tag{7.41}$$

This identity follows by differentiating the integral equation for the first variation with respect to its input. On each compact clock interval the first-variation integrand is C^1 in that input and has continuous derivatives. Differentiation under the finite integral is therefore valid and gives (7.41).

Let C_2 bound the state Hessian in Lemma 7.6. The forcing in (7.41) is bounded by $C_2 e^{6B_*}\|\xi\|\|\zeta\|$. Applying (7.40) once more gives

$$\|Q(\sigma)\| \le C_2 e^{9B_*}\sigma\|\xi\|\|\zeta\|.$$

Taking operator norms and using (7.39) proves the result, with $C \ge \max(e^{3B_*}, C_2 e^{9B_*})$. The argument takes place in ordinary initial coordinates and allows signed or zero walls. □

7.2.3. *The incoming phase at terminal scales.* The incoming portion of a terminal connection has a large radius $k \gtrsim \alpha^{-1}$. Its weak wall is smaller than its strong wall once the old input is sufficiently small. These two facts simplify the estimates of Proposition 6.26.

Lemma 7.8 (Terminal incoming phase)**.** *There are constants $A_1 \ge 1$ and $0 < \delta \le 10^{-4}$ with the following property. For an incoming point $y = (m,u,v,w)$ satisfying*

$$0 < \alpha \le 1, \qquad k = H(m-1,v) \ge \max(7/4, 1/\alpha),$$
$$m \ge 1, \qquad 0 < w \le u \le \delta, \qquad 0 < v < 1,$$

write its stable radial coordinates as $z = (r,u,v,w)$ and use the physical arrival function F of (6.125)*. At z,*

$$|F - r| \le A_1\alpha uv, \qquad |\partial_r F - 1| \le A_1 uv,$$
$$|\partial_u F| \le A_1\alpha v, \qquad |\partial_v F| \le A_1 u,$$
$$|\partial_w F| \le A_1\alpha uv. \tag{7.42}$$

The function F is locally C^2.

Proof. Let $s = u + w$. By (6.130), both ω and $(1 + L/k)\omega$ are at most one. Enlarging the constant in Proposition 6.26 gives its following common bounds:

$$|F - r| \le C(uv + v^2 s^2)/k, \qquad |\partial_r F - 1| \le C(uv + v^2 s^2)/k^2,$$
$$|\partial_u F| \le C(v + v^2 s)/k, \qquad |\partial_v F| \le C(u + v s^2),$$
$$|\partial_w F| \le C(uv + v^2 s)/k.$$

Here $w \le u \le 1$ implies $s \le 2u$ and $s^2 \le 4u$, while $v^2 \le v$. Consequently the four numerators in these bounds are at most $5uv$, $3v$, $5u$ and $3uv$, respectively. Use $1/k \le \alpha$ in the value and two decreasing-wall columns, and $1/k^2 \le 1$ in the radial column. Taking $A_1 = 5C + 1$ proves (7.42). The C^2 conclusion is the one for the same physical arrival function in Proposition 6.26. □

The ordering $w_e \le u_e$ required by this lemma holds uniformly on a terminal height interval. Indeed, fix $0 < \eta_0 \le \delta_1 := \min(\delta/4, 1/1000)$ and $J \ge 2$. Lemma 7.5, with $q = 0$ and $\varepsilon_{\mathrm{in}} = \delta$, puts every cyclic endpoint with $\eta \in [\eta_0, \delta_1]$ in the same incoming domain. Decrease the old input threshold further so that $C_*\varepsilon \le \eta_0/4$. Then

$$w_e \le \frac{\eta_0}{2}\alpha^{2J-2} \le \frac{\eta}{2}\alpha^{J-2} \le u_e. \tag{7.43}$$

Also (7.21) gives $k > 1/\alpha$. Thus every one of these actual cyclic endpoints satisfies (7.42), with one constant A_1. These estimates retain the growing-wall factor v_e. To obtain derivatives in the original exit parameters, the remaining step is to control the cyclic entrance coordinates and their stable radial phase.

7.3. **Phase after the terminal connection.** A terminal exit of radius $R_0 = 2 + 2\alpha$ leads to a new core whose radius is close to $2/\alpha$. The cyclic entrance already has this scale, but its ordinary radius still depends on the auxiliary height. The stable phase removes that dependence exactly on the reference orbit. Comparing at the same height then gives the phase error for a perturbed orbit.

We first record the resulting value and wall estimates. Retain the actual terminal arrival $\mathcal{E}(h, Y)$ from (7.27), and write

$$Y = (2 + 2\alpha, u, w), \qquad h = \eta\alpha^J, \qquad y_+ = \mathcal{E}(h, Y), \qquad \kappa = H(M_+ - 1, 1).$$

Proposition 7.9 (The next core and its phase)**.** *Fix $0 < \varepsilon_{\mathrm{in}} \le 10^{-4}$ and integers $q \ge 16$, $J \ge q + 2$. Choose*

$$0 < \eta \le \min\left(\frac{1}{100}, \frac{\varepsilon_{\mathrm{in}}}{4(11/5)^q}\right). \tag{7.44}$$

There are $C \ge 1$ and $\varepsilon > 0$ such that, whenever

$$0 < \alpha \le 1, \qquad 0 < u, w, \qquad u + w \le b \le \varepsilon\alpha^{2J}, \tag{7.45}$$

the actual first arrival of Proposition 7.3 satisfies

$$\kappa \ge \frac{1}{\alpha}, \qquad \left|\kappa - \frac{2}{\alpha}\right| \le \frac{Cb}{\alpha^2}, \qquad \left|\frac{2}{\kappa} - \alpha\right| \le Cb, \tag{7.46}$$

$$0 < U_+ \le \frac{\varepsilon_{\mathrm{in}}}{\kappa^{16}}, \qquad 0 < W_+ \le \frac{\varepsilon_{\mathrm{in}}}{\kappa^{16}}. \tag{7.47}$$

If $Cb < 1 - \alpha$, then $2/\kappa < 1$ and $\kappa > 2$.

The last condition ensures that the arrival lies in the core domain for the next era. The wall estimates will carry a radius-dependent smallness condition through successive eras.

7.3.1. *The reference phase.* On the boundary with only the strong wall present, the stable system (6.25) reduces to

$$\dot m = mu^2, \qquad \dot u = \frac{1 - m^2 + u^2}{2}u, \qquad w = 0.$$

Its conserved quantity is

$$I(m, u) = \frac{1 + m^2 + u^2}{m}, \qquad \lambda_+(I) = \frac{I + \sqrt{I^2 - 4}}{2}. \tag{7.48}$$

Lemma 7.10 (The strong-wall boundary limit)**.** *For $m \ge 2$ and $|u| \le 1/200$,*

$$P(m, u, 0) = \lambda_+(I(m, u)). \tag{7.49}$$

For $0 < \alpha \le 1$ and $0 \le v \le \alpha^2/400$, set

$$R_0 = 2 + 2\alpha, \qquad y_0 = (1 - \sqrt{R_0^2 - v^2}, 0, v, 0), \qquad z_0 = \mathsf{C}_p y_0.$$

Then

$$P((z_0)_M,(z_0)_U,0)=1+\frac{2}{\alpha},\qquad \widehat{\rho}(z_0)=\frac{2}{\alpha}. \tag{7.50}$$

Proof. The derivative of I along the displayed two-dimensional system vanishes:

$$\left(1-\frac{1+u^2}{m^2}\right)mu^2+\frac{2u}{m}\frac{1-m^2+u^2}{2}u=0.$$

The same identity holds in original time, since the time change multiplies both components by the same positive factor. Proposition 6.4 keeps $m(t)>1$ and gives $(m(t),u(t))\to(P,0)$ with $P>1$. Passing to the limit in I yields

$$P^2-I(m,u)P+1=0.$$

Moreover, $I(m,u)-2=((m-1)^2+u^2)/m>0$. The two roots are positive, their product is one, and exactly one exceeds one. This proves (7.49).

For the cyclic reference point, put $g_0=\sqrt{R_0^2-v^2}$ and

$$\mathcal{H}_0=(2-g_0)^2+v^2=R_0^2+4-4g_0.$$

The cyclic formula gives

$$m_0=\frac{R_0^2-4}{\mathcal{H}_0},\qquad u_0=\frac{4v}{\mathcal{H}_0},\qquad v_0=w_0=0.$$

Since $v\le\alpha/100$, the endpoint algebra in the proof of Lemma 7.2 gives

$$1.99\alpha<g_0-2<2.01\alpha,\qquad 3.99\alpha^2<\mathcal{H}_0<4.01\alpha^2.$$

It follows that $m_0-1>1.9/\alpha$ and $0\le u_0\le 2v/\alpha^2\le 1/200$. Thus (7.49) applies.

The identity $g_0^2+v^2=R_0^2$ gives

$$\mathcal{H}_0^2+(R_0^2-4)^2+16v^2=(2(R_0^2-4)+16)\mathcal{H}_0.$$

Substitution into (7.48) therefore gives

$$I(m_0,u_0)=2+\frac{16}{R_0^2-4},\qquad \lambda_+(I(m_0,u_0))=\frac{R_0+2}{R_0-2}=1+\frac{2}{\alpha}. \tag{7.51}$$

For the second equality, the displayed fraction exceeds one and satisfies the defining quadratic, so it is the upper root. Since $v_0=0$, the stable radial phase is $P(m_0,u_0,0)-1$, which proves (7.50). □

On the actual outgoing orbit with $u=w=0$, the radius is constant: the radial equation in (7.8) has zero right-hand side. Thus its state at $V=h$ is exactly the point y_0 above with $v=h$. For $J\ge 2$ and $\eta\le 1/400$, the reference stable radius is consequently $2/\alpha$ at every chosen height $h=\eta\alpha^J$.

We shall use the stable phase on the signed strip

$$\mathcal{S}=\{(m,u,w):m\ge 2,\ |u|,|w|\le 1/400\}. \tag{7.52}$$

There is a constant $L\ge 1$ such that

$$\|DP\|\le L,\qquad |P(X)-P(X_0)|\le L\|X-X_0\|,\qquad X,X_0\in\mathcal{S}. \tag{7.53}$$

Here and below the finite-dimensional norms are maximum norms. Indeed, (6.38) bounds DP independently of m whenever $|u|+|w|<1/100$. Integrate this bound along the line segment joining X_0 to X, which stays in $\mathcal{S}$. This argument also includes its zero-wall faces.

Lemma 7.11 (Stable radius at the cyclic entrance)**.** *Suppose*

$$0<\alpha\le 1,\qquad 0\le E\le \alpha^2/10000,\qquad |\rho-(2+2\alpha)|\le E,\\ |U|+|W|\le E,\qquad 0\le v\le \alpha^2/1000. \tag{7.54}$$

For $y=(1-\sqrt{\rho^2-v^2},U,v,W)$,

$$\left|\widehat{\rho}(\mathsf{C}_p y)-\frac{2}{\alpha}\right|\le \frac{42LE}{\alpha^2}. \tag{7.55}$$

In particular, for fixed $0<\eta\le 1/1000$ *and* $J\ge 2$, *there are* $C\ge 1$ *and* $\varepsilon>0$ *such that*

$$\left|\widehat{\rho}(\mathsf{C}_p y_h(Y))-\frac{2}{\alpha}\right|\le \frac{Cb}{\alpha^2} \tag{7.56}$$

whenever $0<\alpha\le 1$, $0<b\le \varepsilon\alpha^{2J}$, $|u|+|w|\le b$, $Y=(2+2\alpha,u,w)$ *and* $h=\eta\alpha^J$.

Proof. Set $R_0=2+2\alpha$, $g=\sqrt{\rho^2-v^2}$ and $g_0=\sqrt{R_0^2-v^2}$. Denote the two cyclic denominators by $\mathcal{H}$ and $\mathcal{H}_0$, comparing y with the zero-wall point y_0 at this same v. The elementary endpoint bounds from the proof of Lemma 7.2 give

$$\mathcal{H},\mathcal{H}_0\ge 3.9\alpha^2,\qquad 0<g-2<3\alpha,\qquad 0<g_0-2<3\alpha.$$

Rationalizing the square roots gives

$$(g-g_0)(g+g_0)=(\rho-R_0)(\rho+R_0).$$

Here $g+g_0\ge 3$ and $\rho+R_0\le 9$, so $|g-g_0|\le 3E$. Also,

$$\mathcal{H}-\mathcal{H}_0=(g-g_0)(g+g_0-4)+(U-W)^2-2v(U+W).$$

The first term is at most $18\alpha E$ in absolute value. The remaining terms are at most $E^2+2vE\le \alpha E$. Consequently

$$|g-g_0|\le 3E,\qquad |\mathcal{H}-\mathcal{H}_0|\le 20\alpha E.$$

To compare a cyclic quotient, use

$$\frac{n}{\mathcal{H}}-\frac{n_0}{\mathcal{H}_0}=\frac{n-n_0}{\mathcal{H}}+\frac{n_0(\mathcal{H}_0-\mathcal{H})}{\mathcal{H}\mathcal{H}_0}.$$

If $|n-n_0|\le 12E$ and $|n_0|\le 12\alpha$, the two terms are at most $4E/\alpha^2$ and $30E/\alpha^2$, respectively. Thus their sum is at most $40E/\alpha^2$. This applies to $n=4(g-2)$, $n_0=4(g_0-2)$ and to $n=n_0=4v$. Writing $z=\mathsf{C}_p y=(m_e,u_e,v_e,w_e)$, we obtain

$$|m_e-m_0|,\ |u_e-u_0|\le 40E/\alpha^2,\qquad |v_e|,\ |w_e|\le 2E/\alpha^2.$$

Both (m_e,u_e,w_e) and $(m_0,u_0,0)$ lie in $\mathcal{S}$: the shear coordinates exceed two, while

$$|u_e|,\ |u_0|\le 2v/\alpha^2\le 1/400,\qquad |w_e|\le 2E/\alpha^2\le 1/400.$$

Equations (7.53) and (7.50) therefore give

$$|P(m_e,u_e,w_e)-(1+2/\alpha)|\le 40LE/\alpha^2.$$

The norm inequality $|H(a,c)-d|\le |a-d|+|c|$ for $d\ge 0$, applied with $a=P(m_e,u_e,w_e)-1$ and $d=2/\alpha$, proves (7.55), since $L\ge 1$.

For the actual endpoint, the outgoing estimates give $|\rho-R_0|\le C_*b$ and $|U|+|W|\le C_*b$. Choose ε so that $C_*\varepsilon\le 1/10000$. Then $E=C_*b\le \alpha^2/10000$, while $v=h=\eta\alpha^J\le \alpha^2/1000$. Equation (7.55) applies, proving the last assertion with $C=42LC_*$. □

7.3.2. *The actual next core.*

Proof of Proposition 7.9. Proposition 7.3 supplies an admissible incoming endpoint

$$z = \mathsf{C}_p y_h(Y) = (m_e, u_e, v_e, w_e).$$

Since $\eta \le \varepsilon_{\mathrm{in}}/4 \le 1/40000$, Lemma 7.11 applies there. Put $k = H(m_e - 1, v_e)$ and $r = \widehat{\rho}(z)$.

The incoming phase bound (6.80) implies

$$|\kappa - r| \le D v_e \tag{7.57}$$

with a universal $D > 0$. Indeed, $u_e + w_e \le 1$, $u_e \le 1$, $0 < v_e < 1$, and $k \ge 1$, so

$$u_e v_e + v_e^2 (u_e + w_e)^2 \le 2 v_e.$$

Since $v_e \le 2C_* b/\alpha^2$, adding (7.57) to (7.56) gives $|\kappa - 2/\alpha| \le Cb/\alpha^2$ after increasing C.

Reduce ε so that $C\varepsilon \le 1$. Then $Cb \le \alpha$, and the lower bound in the last estimate gives $\kappa \ge 1/\alpha > 0$. The identity

$$\frac{2}{\kappa} - \alpha = -\frac{\alpha}{\kappa}\left(\kappa - \frac{2}{\alpha}\right)$$

and $\alpha/\kappa \le \alpha^2$ prove the bounded phase estimate in (7.46). If $Cb < 1 - \alpha$, its upper bound gives $2/\kappa < 1$, hence $\kappa > 2$.

For the walls, the incoming radius control and decay give

$$\frac{99}{100} k \le \kappa \le \frac{101}{100} k, \qquad 0 < U_+ \le u_e, \qquad 0 < W_+ \le w_e. \tag{7.58}$$

The weighted entrance estimates (7.25) state $u_e k^q, w_e k^q \le \varepsilon_{\mathrm{in}}/2$. As $k \ge 1$ and $q \ge 16$,

$$\kappa^{16} \le (101/100)^{16} k^{16} \le 2k^{16} \le 2k^q.$$

Multiplying the two wall bounds in (7.58) by this inequality proves (7.47). Positivity follows from the positive wall multipliers along the same finite orbit. □

The phase comparison above uses finite differences at a common height. For the differential analysis we also need the ordinary cyclic coordinates and their stable radial correction. Define

$$\begin{aligned} Z_h(Y) &= \mathsf{C}_p y_h(Y) = (m_e, u_e, v_e, w_e), \\ \widehat{Z}_h(Y) &= \big(\widehat{\rho}(Z_h(Y)), u_e, v_e, w_e\big). \end{aligned} \tag{7.59}$$

In all derivatives of these maps, h is held fixed.

Proposition 7.12 (Cyclic entrance derivatives)**.** *Fix $0 < \eta_0 \le 1/1000$ and $J \ge 2$. There are $C \ge 1$ and $\varepsilon > 0$ such that, for*

$$\begin{aligned} &\eta_0 \le \eta \le 1/1000, \qquad 0 < \alpha \le 1, \\ &0 < b \le \varepsilon \alpha^{2J}, \qquad |u| + |w| \le b, \end{aligned} \tag{7.60}$$

the maps Z_h and $\widehat{Z}_h$ are C^2 near $Y = (2 + 2\alpha, u, w)$, where $h = \eta\alpha^J$. At this point,

$$\|DZ_h\|,\ \|D\widehat{Z}_h\| \le C/\alpha^2, \tag{7.61}$$

$$\|D \log(\mathcal{H} \circ y_h)\| \le C/\alpha. \tag{7.62}$$

The estimates include signed and zero old walls.

Proof. At fixed height, reconstruct the full outgoing state from the radius map E_h of (7.35):

$$(\rho, U, W) \longmapsto (1 - \sqrt{\rho^2 - h^2}, U, h, W).$$

For $\rho \ge 7/4$ and $|h| \le 1$, the norm of this derivative is at most two, since $\rho/\sqrt{\rho^2 - h^2} \le 2$. The reconstruction agrees with y_h on an initial neighborhood: $M_h < 1$ and the first-arrival identity persist there. Proposition 7.7 therefore gives

$$\|Dy_h(Y)\| \le 2C_0 \tag{7.63}$$

with one C_0 for the whole height interval.

At the endpoint, the uniform geometry gives

$$\mathcal{H} \geq 3.9\alpha^2, \qquad |M+1| \leq 3\alpha, \qquad |U|+|V|+|W| \leq 3\alpha.$$

Write the four cyclic components as $c_i + N_i/\mathcal{H}$, where $c = (1,0,0,0)$ and

$$N = \big(-4(M+1), 4V, 4W, 4U\big).$$

For each ordinary coordinate j,

$$|N_i| \leq 12\alpha, \qquad |\partial_j N_i| \leq 4, \qquad |\partial_j \mathcal{H}| \leq 6\alpha.$$

The last bound follows directly from

$$d\mathcal{H} = 2(M+1)dM + 2(U-V-W)dU + 2(V-U-W)dV + 2(W-U-V)dW.$$

Quotient differentiation yields

$$|\partial_j(\mathsf{C}_p)_i| \leq \frac{4}{3.9\alpha^2} + \frac{72}{3.9^2\alpha^2} \leq \frac{6}{\alpha^2}.$$

Summing the four entries in each row, and differentiating the logarithm of the positive denominator, gives

$$\|D\mathsf{C}_p\| \leq 24/\alpha^2, \qquad \|D\log\mathcal{H}\| \leq 24\alpha/\mathcal{H} \leq 7/\alpha.$$

Together with (7.63), these prove the bound for Z_h and (7.62).

It remains to apply the stable radial coordinate change. On $\mathcal{S}$, (6.29) gives $|P-m| \leq 1/100$, hence $\chi = P - 1 > 0$. For arbitrary signed v, put $r = H(\chi, v)$. Then

$$dr = \frac{\chi\, dP + v\, dv}{r}, \qquad \chi/r \leq 1, \qquad |v|/r \leq 1.$$

Equation (7.53) bounds the derivative of $(m,u,v,w) \mapsto (r,u,v,w)$ by $L+1$. This map is locally C^2 because P is C^2 and $r > 0$.

At the actual cyclic endpoint, the estimates of Lemma 7.5 give $m_e > 2$, $|u_e| \leq 2\eta \leq 1/400$ and $|w_e| \leq 2C_*b/\alpha^2 \leq 1/400$, after decreasing ε. Thus the signed radial bound applies. Composition proves (7.61) for $\widehat{Z}_h$ and its C^2 regularity. □

One consequence is useful when varying the phase while keeping a numerical height fixed. For $J \geq 2$, choose $\eta > 0$ with $2\eta \leq 1/1000$ and set $h = \eta\alpha_0^J$, where $\alpha_0 > 0$. If

$$0 < \alpha \leq 1, \qquad \left|\log\frac{\alpha}{\alpha_0}\right| < \frac{\log 2}{J}, \qquad 0 < b \leq \varepsilon\alpha_0^{2J+1}, \qquad |u|+|w| \leq b, \tag{7.64}$$

then the conclusions of Proposition 7.12 hold at $Y = (2+2\alpha, u, w)$ with the same fixed h, after choosing C, ε depending only on η, J. Indeed, exponentiating the logarithmic inequality gives

$$\frac{\alpha_0^J}{2} < \alpha^J < 2\alpha_0^J, \qquad \frac{\eta}{2} < \eta_{\mathrm{eff}} := \frac{\eta\alpha_0^J}{\alpha^J} < 2\eta.$$

Also $\alpha_0 \leq 2\alpha$, so $\alpha_0^{2J+1} \leq 8\alpha^{2J+1} \leq 8\alpha^{2J}$. Apply the proposition with lower coefficient $\eta/2$ and effective coefficient η_{eff}, reducing its input threshold by a factor of eight. The identity $h = \eta_{\mathrm{eff}}\alpha^J$ leaves the numerical height unchanged.

These coordinate estimates account for the singular change of scale at the terminal cycle. To estimate the bounded phase derivative close to its reference value $1/2$, we next retain the cancellation in (7.51) before taking norms.

7.4. **Radial variation of the terminal boundary phase.** The cyclic coordinates have derivatives of order α^{-2}, but their contribution to the bounded phase is much smaller. The single-wall invariant makes this cancellation explicit. To preserve it along the outgoing passage, we first refine the radial derivative of that passage: its radius changes almost by translation, while each wall response retains its own amplitude.

Using I and λ_+ from (7.48), define

$$\vartheta(m,u) = \frac{2}{\lambda_+(I(m,u)) - 1}, \qquad m > 1. \tag{7.65}$$

In the domain of Lemma 7.10, this is $2/(P(m,u,0)-1)$. At the actual cyclic entrance $Z_h(Y) = (m_e, u_e, v_e, w_e)$, put

$$\psi_h(Y) = \vartheta(m_e, u_e), \qquad Y = (R,u,w). \tag{7.66}$$

Thus ψ_h retains the actual entrance shear and strong wall, and evaluates their single-wall phase. The weak wall and the subsequent incoming passage will be included in the phase correction below.

Proposition 7.13. *Fix $0 < \eta_0 \le 1/100$ and an integer $J \ge 2$. There are $C \ge 1$ and $\varepsilon > 0$, depending only on η_0, J, with the following property. If*

$$\eta_0 \le \eta \le 1/100, \quad 0 < \alpha \le 1, \quad 0 < b \le \varepsilon\alpha^{2J}, \quad |u| + |w| \le b, \tag{7.67}$$

then, at $Y_0 = (2 + 2\alpha, u, w)$ and the fixed numerical height $h = \eta\alpha^J$, the function ψ_h is C^2 near Y_0 and

$$|\psi_h(Y_0) - \alpha| + |\partial_R\psi_h(Y_0) - 1/2| \le Cb, \qquad \|D\psi_h(Y_0)\| \le C. \tag{7.68}$$

The estimates allow signed and zero old walls.

7.4.1. *The outgoing radial response.* Recall the radius map $E_h(Y) = (\rho_h, U_h, W_h)$ from (7.35) and set $L = \log(1/h)$. All norms below are maximum norms and their induced operator norms.

Lemma 7.14. *Fix $0 < \eta_0 < 1$ and an integer $J \ge 1$. There are $C \ge 1$ and $\varepsilon > 0$ such that, whenever*

$$\eta_0 \le \eta < 1, \quad 0 < \alpha \le 1, \quad 0 < b \le \varepsilon\alpha^J, \quad |u| + |w| \le b,$$

the fixed-height map E_h at $Y_0 = (2 + 2\alpha, u, w)$ is C^2, and

$$\|DE_h\| \le C, \qquad |\partial_R\rho_h - 1| \le C(|u| + b^2(1+L)^2), \tag{7.69}$$

$$|U_h| \le C|u|, \qquad |W_h| \le C|w|, \tag{7.70}$$

$$|\partial_R U_h| \le C|u|/\alpha, \qquad |\partial_R W_h| \le C|w|/\alpha. \tag{7.71}$$

All derivatives are evaluated at Y_0, with $h = \eta\alpha^J$ held fixed.

Proof. Use the actual clock family $\Phi(Y,s)$ of (7.38), constructed first up to the lowest height $\eta_0\alpha^J$. Every height in the statement corresponds to a shorter interval $0 \le s \le L$. For the two wall rates in (7.8), define

$$\ell_j(Y,s) = \int_0^s a_j(e^{-r}, \Phi(Y,r))\, dr, \qquad j \in \{U, W\}. \tag{7.72}$$

On a common initial neighborhood, the scalar wall equations give

$$\Phi_j(Y,s) = Y_j e^{\ell_j(Y,s)}, \qquad D_Y\Phi_j[d] = e^{\ell_j} d_j + \Phi_j D_Y\ell_j[d].$$

Here $Y_U = u$ and $Y_W = w$. The integral defining ℓ_j is C^1 in Y: the joint clock family is C^2, and differentiation is under a finite integral with continuous integrand and derivative. Consequently these identities also hold at zero walls.

The absolute rate integrals (7.32), the first-variation bound in Proposition 7.7, and the rate gradients (7.31) give one $K \geq 1$ such that, along the whole reference segment,

$$\|D_Y\Phi\| \leq K, \qquad K^{-1} \leq e^{\ell_j} \leq K, \qquad \|D_Y\ell_U\| + \|D_Y\ell_W\| \leq Ks. \tag{7.73}$$

For the last inequality, each differentiated integrand has norm at most $1536\|D_Y\Phi\|$; enlarge K to absorb the two resulting constants.

Let $\xi(s) = D_Y\Phi(Y_0, s)[e_R]$, where $e_R = (1, 0, 0)$. The factor identities and (7.73) imply

$$\begin{gathered} |U(s)| \leq K|u|, \qquad |U(s)| + |W(s)| \leq Kb, \qquad |\xi_\rho(s)| \leq K, \\ |\xi_U(s)| \leq K^2|u|s, \qquad |\xi_W(s)| \leq K^2|w|s. \end{gathered} \tag{7.74}$$

The radial row of the variational equation has no constant term in the old walls. In fact, (7.31), with $v = e^{-s}$ and $\beta = |U| + |W|$, yields

$$|\xi_\rho'| \leq 256(|U|v + \beta^2)|\xi_\rho| + 32(v + \beta)|\xi_U| + 16\beta|\xi_W|.$$

Substitution of (7.74) gives

$$|\xi_\rho'(s)| \leq 512K^3\big(|u|(1 + s)e^{-s} + b^2(1 + s)\big).$$

Since $\xi_\rho(0) = 1$,

$$|\xi_\rho(s) - 1| \leq 512K^3\big(2|u| + b^2s(1 + s)\big).$$

We used $\int_0^s (1 + r)e^{-r}\, dr \leq 2$ and $\int_0^s (1 + r)\, dr \leq s(1 + s)$.

At the endpoint, $E_h(Y) = \Phi(Y, L)$ on an ordinary neighborhood of Y_0, by (7.39). Thus the last bound proves (7.69); the factor identity proves (7.70). Finally,

$$\alpha L \leq \log(1/\eta_0) + J/e$$

turns the last two bounds in (7.74) into (7.71). Increasing C gives all conclusions with the same constants. □

7.4.2. *The invariant in outgoing coordinates.* Consider an outgoing point $(1 - g, U, v, W)$, where $g = \sqrt{\rho^2 - v^2}$. Introduce

$$\begin{aligned} H_0 &= (2 - g)^2 + v^2, \quad \Delta = (U - W)^2 - 2v(U + W), \\ \mathcal{H} &= H_0 + \Delta, \qquad\qquad N = \rho^2 - 4 + \Delta, \\ e &= \Delta\left(1 + \frac{N}{H_0}\right), \quad X = \sqrt{\rho^2 + e}. \end{aligned} \tag{7.75}$$

The cyclic denominator is $\mathcal{H}$, and its shear and strong-wall coordinates are

$$m_e = N/\mathcal{H}, \qquad u_e = 4v/\mathcal{H}. \tag{7.76}$$

The following identities express their invariant without separately differentiating these two singular quotients:

$$\mathcal{H}^2 + N^2 + 16v^2 = 2\mathcal{H}N + 16H_0, \tag{7.77}$$

$$\rho^2 + e = 4 + \frac{\mathcal{H}N}{H_0}. \tag{7.78}$$

To check the first, use $N = \mathcal{H} - 4(2 - g)$; the second follows by expanding $(H_0 + \Delta)(\rho^2 - 4 + \Delta)$. If $g > 2$ and $H_0, \mathcal{H}$ are positive, then $N = \mathcal{H} + 4(g - 2) > 0$, $m_e > 1$, and $X > 2$. Moreover,

$$I(m_e, u_e) = 2 + \frac{16H_0}{\mathcal{H}N} = 2 + \frac{16}{X^2 - 4}.$$

The upper root is $(X + 2)/(X - 2)$. Therefore

$$\vartheta(m_e, u_e) = s_v(\rho, U, W), \qquad s_v(\rho, U, W) := \frac{\sqrt{\rho^2 + e} - 2}{2}. \tag{7.79}$$

At $U = W = 0$ the defect Δ vanishes and $s_v = (\rho - 2)/2$, independently of v.

Lemma 7.15. *Suppose*

$$\begin{aligned} &0<\alpha\le 1, \qquad |\rho-(2+2\alpha)|\le \alpha/10000,\\ &|U|+|W|\le \beta\le \alpha/10000, \qquad 0\le v\le \alpha/100. \end{aligned} \tag{7.80}$$

Then s_v is C^2 near (ρ,U,W), with v held fixed, and

$$|s_v-(\rho-2)/2|\le 23\beta(v+\beta)/\alpha, \tag{7.81}$$

$$|\partial_\rho s_v-1/2|\le 1024\beta(v+\beta)/\alpha^2, \tag{7.82}$$

$$|\partial_U s_v|+|\partial_W s_v|\le 1024(v+\beta)/\alpha. \tag{7.83}$$

Proof. The algebraic gap and denominator estimates used in the proof of Lemma 7.2, with the bounds in (7.80), give

$$\begin{aligned} &2<\rho\le 5, \qquad 2<g\le 5, \qquad H_0\ge\alpha^2,\\ &|\Delta|\le 2\beta(v+\beta)\le \alpha^2/1000, \qquad |N|\le 22\alpha, \qquad \mathcal{H}>0. \end{aligned} \tag{7.84}$$

In particular, $1.99\alpha<g-2<2.01\alpha$ and $H_0=(g-2)^2+v^2\ge\alpha^2$. The estimate for Δ follows from $|U-W|,|U+W|\le\beta$, and $\rho^2-4\le(3\alpha)7$ gives the bound on N. Also $N=\mathcal{H}+4(g-2)>0$, so (7.78) gives $X>2$. All square roots and denominators are consequently smooth on a neighborhood.

For the derivative in ρ, let

$$H_{0,\rho}=2(g-2)\rho/g, \qquad |H_{0,\rho}|\le 20\alpha.$$

Differentiating (7.75), while keeping v fixed, gives

$$\begin{aligned} e_\rho &= \Delta\left(\frac{2\rho}{H_0}-\frac{NH_{0,\rho}}{H_0^2}\right),\\ e_\Delta &= 1+\frac{\rho^2-4+2\Delta}{H_0},\\ d\Delta &= (2(U-W)-2v)dU+(-2(U-W)-2v)dW,\\ ds_v &= \frac{(2\rho+e_\rho)d\rho+e_\Delta d\Delta}{4X}. \end{aligned} \tag{7.85}$$

The scalar estimates in (7.84) imply

$$|e|\le\frac{46\beta(v+\beta)}{\alpha}, \qquad |e_\rho|\le\frac{900\beta(v+\beta)}{\alpha^2}, \qquad |e_\Delta|\le\frac{24}{\alpha}. \tag{7.86}$$

Indeed, the factors multiplying Δ in e and e_ρ are at most $23/\alpha$ and $450/\alpha^2$:

$$1+\frac{22\alpha}{\alpha^2}\le\frac{23}{\alpha}, \qquad \frac{10}{\alpha^2}+\frac{22\alpha\,20\alpha}{\alpha^4}=\frac{450}{\alpha^2}.$$

Adding one more Δ gives $|\rho^2-4+2\Delta|\le 23\alpha$, which proves the last bound in (7.86).

Since $(X-\rho)(X+\rho)=e$ and $X,\rho>2$, we have $|X-\rho|\le|e|$. This proves (7.81). For the radial derivative, retain the subtraction before estimating:

$$\partial_\rho s_v-\frac12=\frac{\rho-X}{2X}+\frac{e_\rho}{4X}.$$

Equation (7.86) and $\alpha\le1$ bound this by $1024\beta(v+\beta)/\alpha^2$. Finally $|\Delta_U|+|\Delta_W|\le4(v+\beta)$. The last line of (7.85) then proves (7.83). □

Proof of Proposition 7.13. Choose the constants in Lemma 7.14 and in the uniform terminal geometry. Reduce ε so that both apply and $\varepsilon\le1$. Write

$$z=E_h(Y_0)=(\rho,U,W), \qquad \beta=|U|+|W|.$$

For a constant $B\ge1$ depending only on η_0,J,

$$|\rho-(2+2\alpha)|\le Bb, \qquad \beta\le Bb, \qquad Bb\le\alpha/10000.$$

Also $h \le \alpha/100$, so Lemma 7.15 applies at z. The strict inequalities used in (7.79) persist on an initial neighborhood. There the actual outgoing state satisfies $M_h < 1$ and is reconstructed from E_h with $V = h$. Hence

$$\psi_h = s_h \circ E_h \quad \text{on a neighborhood of } Y_0. \tag{7.87}$$

This identity proves C^2 regularity and identifies the ordinary derivatives of the two sides.

Because $J \ge 2$, we have $b \le \alpha^2$ and $h \le \alpha^2$. Thus $h + \beta \le (1 + B)\alpha^2$. The radical estimates give constants C_1, C_2 with

$$|s_{h,\rho}(z) - 1/2| \le C_1 b, \qquad |s_{h,U}(z)| + |s_{h,W}(z)| \le C_2 \alpha. \tag{7.88}$$

They also give

$$|\psi_h(Y_0) - \alpha| \le \frac{23\beta(h+\beta)}{\alpha} + \frac{1}{2}|\rho - (2 + 2\alpha)| \le C_3 b.$$

For the radial derivative put $d = DE_h(Y_0)[e_R]$ and $\overline{L} = 1 + \log(1/\eta_0) + J/e$. The fine outgoing bounds give

$$|d_\rho| \le A, \qquad |d_U| + |d_W| \le Ab/\alpha,$$

with a uniform $A \ge 1$. Moreover, $\alpha(1 + L) \le \overline{L}$, and therefore

$$b^2(1 + L)^2 \le \overline{L}^2 b, \qquad |d_\rho - 1| \le A(1 + \overline{L}^2)b.$$

The first inequality uses $b \le \alpha^2$. Apply the chain rule in (7.87) and subtract the reference derivative:

$$\begin{aligned} |\partial_R \psi_h - 1/2| \le{}& |s_{h,\rho} - 1/2|\,|d_\rho| + \tfrac{1}{2}|d_\rho - 1| \\ &+ (|s_{h,U}| + |s_{h,W}|)(|d_U| + |d_W|) \le C_4 b. \end{aligned}$$

In particular, the factor α in the wall derivative of the radical compensates for the factor $1/\alpha$ in the outgoing wall response.

Finally (7.88) bounds $\|Ds_h(z)\|$ by a uniform constant, since $b, \alpha \le 1$. Combining this with $\|DE_h\| \le A$ proves the operator bound in (7.68). Increase C to cover the sum of the value and radial errors. □

The cancellation has now been retained through the actual outgoing segment. To obtain the bounded phase of the next core, the remaining comparison is between this single-wall phase, the full stable projection at Z_h, and the subsequent incoming arrival.

7.5. **The physical phase at the terminal return.** We now compare the single-wall phase with the phase of the actual next core. There are two corrections. The first restores the weak stable wall and the active wall at the cyclic entrance. The second follows that entrance through its incoming passage. Both retain wall factors that compensate for the singular derivatives of the cyclic coordinates.

Write the ordinary and stable radial entrance coordinates as

$$Z_h = (m_e, u_e, v_e, w_e), \qquad \widehat{Z}_h = (r_h, u_e, v_e, w_e),$$

and let κ_h be the radius at the actual first arrival of Proposition 7.3. Define

$$Q_h = \frac{2}{\kappa_h}, \qquad \Xi_h = \frac{2}{r_h} - \psi_h, \qquad \Psi_h = \frac{2}{\kappa_h} - \frac{2}{r_h}. \tag{7.89}$$

Thus

$$Q_h = \psi_h + \Xi_h + \Psi_h = \frac{2}{r_h} + \Psi_h. \tag{7.90}$$

The first decomposition preserves the reference radial derivative $1/2$. The second will give a uniform bound on the two wall derivatives.

Proposition 7.16 (The terminal phase)**.** *There is $0 < \delta \le 1/40000$ such that, for each $0 < \eta_0 \le \delta$ and integer $J \ge 4$, there are $C \ge 1$ and $\varepsilon > 0$ with the following property. Suppose*

$$\eta_0 \le \eta \le \delta, \quad 0 < \alpha \le 1, \quad 0 < b \le \varepsilon\alpha^{2J}, \quad u, w > 0, \quad u + w \le b. \tag{7.91}$$

At $Y_0 = (2 + 2\alpha, u, w)$ and fixed $h = \eta\alpha^J$, the physical phase Q_h is locally C^2 and satisfies

$$|Q_h(Y_0) - \alpha| + |\partial_R Q_h(Y_0) - 1/2| \le Cb, \tag{7.92}$$

$$|\partial_u Q_h(Y_0)| + |\partial_w Q_h(Y_0)| \le C. \tag{7.93}$$

The constants are uniform over the indicated height interval.

7.5.1. *Wall factors at the cyclic entrance.* The ordinary derivative bound for Z_h loses the amplitude of an individual wall. To keep that amplitude, use the outgoing multipliers before applying the cyclic map.

Lemma 7.17. *Fix $0 < \eta_0 \le 1/1000$ and $J \ge 2$. There are $C \ge 1$ and $\varepsilon > 0$ such that, for $\eta_0 \le \eta \le 1/1000$, $0 < \alpha \le 1$ and $|u| + |w| \le b \le \varepsilon\alpha^{2J}$ with $b > 0$, the entrance at fixed $h = \eta\alpha^J$ has the form*

$$u_e = \frac{4h}{\mathcal{H}(y_h)}, \qquad v_e = w\mathcal{B}_W, \qquad w_e = u\mathcal{B}_U. \tag{7.94}$$

The factors $\mathcal{B}_U, \mathcal{B}_W$ are positive and locally C^2, including at signed or zero old walls, and

$$\mathcal{B}_j \le C/\alpha^2, \qquad \mathcal{B}_j^{-1} \le C\alpha^2, \tag{7.95}$$

$$\|D \log \mathcal{B}_j\| \le C/\alpha, \qquad \|D \log u_e\| \le C/\alpha. \tag{7.96}$$

In particular,

$$|\partial_R z_i| \le C|z_i|/\alpha, \qquad z = Z_h(Y_0), \quad i \in \{u, v, w\}. \tag{7.97}$$

Proof. Let B_j denote the outgoing wall multiplier evaluated at the arrival at $V = h$. The exact cyclic formulas give $\mathcal{B}_j = 4B_j/\mathcal{H}(y_h)$ and (7.94). These factors are C^2 in the initial data. Indeed, if $N_j' = \gamma_j N_j$ in original normalized time, the projective multiplier has the expression

$$B_j = \exp\left(\log A(0) - \log A(S_h) + \int_0^{S_h} \gamma_j(\phi_t x(Y))\, dt\right).$$

The event time and flow are C^2 on a neighborhood of this finite segment, the endpoint values of A are positive, and the moving integral is C^2. One may see the last assertion by writing it as $S_h \int_0^1 \gamma_j(\phi_{sS_h} x(Y))\, ds$.

On the same neighborhood, $B_j = \exp \ell_j(Y, \log(1/h))$, where ℓ_j is defined in (7.72). Equation (7.73) and $\alpha \log(1/h) \le \log(1/\eta_0) + J/e$ imply

$$C_0^{-1} \le B_j \le C_0, \qquad \|D \log B_j\| \le C_0/\alpha.$$

Combine these with $3.99\alpha^2 < \mathcal{H} < 4.01\alpha^2$ and (7.62). The identities

$$D \log \mathcal{B}_j = D \log B_j - D \log \mathcal{H}, \qquad D \log u_e = -D \log \mathcal{H}$$

prove (7.95)–(7.96). In the second identity h is fixed. Finally, differentiating (7.94) in R gives (7.97); the old wall coordinates are constant in this direction. □

7.5.2. *Restoring the weak wall and the active wall.* First consider an arbitrary entrance $y = (m, a, v, c)$ with $m \geq 2$ and $|a|, |c| \leq 10^{-4}$. Put

$$\begin{aligned} &\chi = P(m,a,c) - 1, && q = P(m,a,0) - 1, && r = H(\chi, v), \\ &E_0 = P(m,a,c) - P(m,a,0), && E = |c|(|a| + |c|) + v^2. \end{aligned} \tag{7.98}$$

Here q is the single-wall radius. The decomposition

$$r - q = E_0 + \frac{v^2}{r + \chi} \tag{7.99}$$

separates the removed weak wall from the active-wall square.

Lemma 7.18. *Suppose $\alpha > 0$, $m + 1 \geq 1/\alpha$, and a direction $d = (d_m, d_a, d_v, d_c)$ satisfies, for some $D \geq 0$,*

$$\|d\| \leq D/\alpha^2, \qquad |d_a| \leq D|a|/\alpha, \quad |d_v| \leq D|v|/\alpha, \quad |d_c| \leq D|c|/\alpha. \tag{7.100}$$

There is a universal constant C such that

$$|r - q| \leq C\alpha E, \qquad |D(r-q)[d]| \leq CDE, \tag{7.101}$$

$$\left|\frac{2}{r} - \frac{2}{q}\right| \leq C\alpha^3 E, \qquad \left|D\left(\frac{2}{r} - \frac{2}{q}\right)[d]\right| \leq CD\alpha^2 E. \tag{7.102}$$

These are ordinary derivatives of locally C^2 functions.

Proof. Write $\ell = m + 1$ and $s = |a| + |c|$. Integrate the derivatives of P along (m, a, tc), $0 \leq t \leq 1$. Proposition 6.13 gives

$$\begin{aligned} &|E_0| \leq C|c|s/\ell, && |(E_0)_m| \leq C|c|s/\ell^2, \\ &|(E_0)_a| \leq C|c|/\ell, && |(E_0)_c| \leq Cs/\ell. \end{aligned} \tag{7.103}$$

For the middle two bounds, apply the fundamental theorem of calculus to P_m and P_a along this segment, using the mixed and stable Hessian estimates. The last bound is the first derivative of P itself. All points of the segment have total wall amplitude at most s.

The bound (6.29) gives $|P - m| \leq 1/100$ at these inputs, and hence

$$\chi, q \geq \ell/4 \geq 1/(4\alpha), \qquad r \geq \chi > 0, \tag{7.104}$$

since $m \geq 2$. Equation (7.99) now proves the first value estimate. To differentiate it, use $r^2 = \chi^2 + v^2$. After subtracting $D\chi[d]$ from $Dr[d]$, the exact formula is

$$D(r-q)[d] = DE_0[d] + \frac{vd_v}{r} - \frac{v^2 DP[d]}{r(r+\chi)}. \tag{7.105}$$

The four bounds in (7.103), together with (7.100) and $\ell^{-1} \leq \alpha$, give $|DE_0[d]| \leq 3CD|c|s$. The uniform first derivative of P gives $|DP[d]| \leq CD/\alpha^2$. Since $r^{-1}, (r+\chi)^{-1} \leq 4\alpha$, the last two terms in (7.105) are bounded by $4Dv^2$ and $16CDv^2$. This proves the derivative estimate in (7.101).

For completeness, the reciprocal difference of two positive functions f, g satisfies, with $e = f - g$,

$$\frac{2}{f} - \frac{2}{g} = -\frac{2e}{gf}, \qquad D\left(\frac{2}{f} - \frac{2}{g}\right)[d] = -\frac{2De[d]}{f^2} + \frac{2e(f+g)Dg[d]}{g^2 f^2}.$$

If $\lambda > 0$, $f, g \geq 1/\lambda$ and $|Dg[d]| \leq D_0/\lambda^2$, then

$$\left|\frac{2}{f} - \frac{2}{g}\right| \leq 2\lambda^2|e|, \qquad \left|D\left(\frac{2}{f} - \frac{2}{g}\right)[d]\right| \leq 2\lambda^2|De[d]| + 4D_0\lambda|e|. \tag{7.106}$$

Indeed, $(f+g)/(g^2f^2) = 1/(g^2 f) + 1/(gf^2) \leq 2\lambda^3$. Apply this with $f = r$, $g = q$, $\lambda = 4\alpha$ and a constant D_0 bounded by a universal multiple of D. The latter bound follows from the uniform derivative of $P(m, a, 0)$ and $\|d\| \leq D/\alpha^2$. Equation (7.101) gives (7.102). Positivity in (7.104) and the regularity of P also prove the stated C^2 property. □

Apply the lemma to the cyclic entrance. Fix $0 < \eta_0 \le 1/40000$, $J \ge 2$, and $\eta \in [\eta_0, 1/40000]$. For sufficiently small $b \le \varepsilon\alpha^{2J}$, including signed old walls, the geometry gives

$$m_e > 2, \qquad m_e + 1 \ge 1/\alpha, \qquad |u_e|, |w_e| \le 10^{-4}.$$

Proposition 7.12 and Lemma 7.17 show that $d = \partial_R Z_h$ satisfies (7.100) with a constant D depending only on η_0, J. The identity $2/q = \vartheta(m_e, u_e)$ holds on an ordinary neighborhood by Lemma 7.10. The chain rule therefore identifies the composed reciprocal correction with Ξ_h, including its derivative.

Put $N_0 = |u|(h + b) + w^2$. The wall factors give

$$|u_e| \le 2h/\alpha^2, \qquad |v_e| \le C|w|/\alpha^2, \qquad |w_e| \le C|u|/\alpha^2.$$

Consequently

$$|w_e|(|u_e| + |w_e|) + v_e^2 \le CN_0/\alpha^4.$$

Equation (7.102) now gives the useful amplitude estimates

$$|\Xi_h| \le CN_0/\alpha, \qquad |\partial_R \Xi_h| \le CN_0/\alpha^2. \tag{7.107}$$

After reducing ε to at most one, $b, h \le \alpha^2$. Thus $N_0 \le 3b\alpha^2$, and

$$|\Xi_h| + |\partial_R \Xi_h| \le Cb. \tag{7.108}$$

7.5.3. *The actual incoming correction.* Return now to positive old walls. Let δ_{in} be the entrance threshold in Lemma 7.8 and choose $\delta = \min(\delta_{\mathrm{in}}/4, 1/40000)$. By (7.43), that lemma applies uniformly to all cyclic entrances with $\eta \le \delta$, after reducing the old input threshold. In particular, $0 < w_e \le u_e \le 1$ and $0 < v_e < 1$. At fixed height write

$$\Theta_h = \kappa_h - r_h = F(\widehat{Z}_h) - r_h. \tag{7.109}$$

The equality holds on an initial neighborhood: the local inverse of the stable radial coordinate takes $\widehat{Z}_h$ back to the actual cyclic entrance, and F is its physical incoming arrival radius.

Lemma 7.19. *Fix $0 < \eta_0 \le \delta$ and $J \ge 2$. For the positive inputs above, after choosing a sufficiently small uniform ε, the functions Θ_h and Ψ_h are locally C^2 and*

$$\begin{aligned} |\Theta_h| &\le C\alpha u_e v_e, & |\partial_R \Theta_h| + |\partial_u \Theta_h| &\le C u_e v_e/\alpha^2, \\ |\partial_w \Theta_h| &\le C u_e/\alpha^2, & & \\ r_h, \kappa_h &\ge 1/\alpha, & |\Psi_h| &\le C\alpha^3 u_e v_e, \\ |\partial_R \Psi_h| + |\partial_u \Psi_h| &\le C u_e v_e, & |\partial_w \Psi_h| &\le C u_e. \end{aligned} \tag{7.110, 7.111}$$

If $J \ge 4$, these imply

$$\begin{aligned} |\Psi_h| &\le Cw\alpha^{J-1}, & & \\ |\partial_R \Psi_h| + |\partial_u \Psi_h| &\le Cw\alpha^{J-4}, & |\partial_w \Psi_h| &\le C\alpha^{J-2}. \end{aligned} \tag{7.112}$$

Proof. For an input direction $d = (d_R, d_u, d_w)$, differentiate the three wall factors:

$$\begin{aligned} Du_e[d] &= u_e D \log u_e[d], \\ Dv_e[d] &= \mathcal{B}_W d_w + v_e D \log \mathcal{B}_W[d], \\ Dw_e[d] &= \mathcal{B}_U d_u + w_e D \log \mathcal{B}_U[d]. \end{aligned}$$

The chain rule in (7.109) therefore gives

$$\begin{aligned} D\Theta_h[d] = (F_r - 1) Dr_h[d] &+ F_u u_e D \log u_e[d] \\ &+ F_v v_e D \log \mathcal{B}_W[d] + F_w w_e D \log \mathcal{B}_U[d] \\ &+ F_v \mathcal{B}_W d_w + F_w \mathcal{B}_U d_u. \end{aligned}$$

All derivatives of F are evaluated at $\widehat{Z}_h$. For a coordinate unit vector d, the first four terms are bounded in absolute value by

$$Cu_e v_e(\alpha^{-2} + 1 + \alpha^{-1} + w_e) \le 4Cu_e v_e/\alpha^2.$$

Here we used (7.42), $\|D\widehat{Z}_h\| \le C/\alpha^2$, and (7.96). The last two terms retain different amplitudes:

$$|F_v \mathcal{B}_W| \le Cu_e/\alpha^2, \qquad |F_w \mathcal{B}_U| \le Cu_e v_e/\alpha.$$

For $d = e_R$ neither occurs; for e_u only the second occurs; for e_w only the first occurs. This proves the derivative bounds for Θ_h. Its value bound is the first estimate of Lemma 7.8. Local C^2 regularity follows from the same composition.

The lower bound for r_h follows from (7.21) and $|P(m_e, u_e, w_e) - m_e| \le 1/100$:

$$r_h \ge P(m_e, u_e, w_e) - 1 > \frac{1.9}{\alpha} - \frac{1}{100} \ge \frac{3}{2\alpha}.$$

Since $u_e \le 1$ and $v_e \le Cb/\alpha^2$, the value bound for Θ_h is at most $C'b/\alpha$. Decreasing ε so that $C'b \le 1/2$ gives $\kappa_h = r_h + \Theta_h \ge 1/\alpha$. In particular, all reciprocal functions are locally C^2.

For each input coordinate j, the derivative of r_h is at most D_0/α^2. Apply (7.106) with $f = \kappa_h$, $g = r_h$ and $\lambda = \alpha$:

$$|\Psi_h| \le 2\alpha^2 |\Theta_h|, \qquad |\partial_j \Psi_h| \le 2\alpha^2 |\partial_j \Theta_h| + 4D_0 \alpha |\Theta_h|.$$

Equation (7.110), $\alpha \le 1$ and $v_e \le 1$ prove (7.111). Finally

$$u_e \le 2\eta\alpha^{J-2} \le \alpha^{J-2}, \qquad v_e \le Cw/\alpha^2$$

give $u_e v_e \le Cw\alpha^{J-4}$. Substitution yields (7.112). □

Proof of Proposition 7.16. Choose a common input threshold for Proposition 7.13, (7.108), and Lemma 7.19. The choice of δ also places every endpoint in the incoming domain of Proposition 7.3 with $q = 0$, so all three terms in (7.90) refer to the same physical first arrival. Since $J \ge 4$, $w \le b$ and $0 < \alpha \le 1$, (7.112) gives

$$|\Psi_h| + |\partial_R \Psi_h| \le Cb.$$

Adding this to (7.108) and (7.68) proves (7.92).

For the wall derivatives, use the second decomposition in (7.90). As $r_h \ge 1/\alpha$ and $\|Dr_h\| \le C_0/\alpha^2$,

$$\left\| D\left(\frac{2}{r_h}\right) \right\| \le 2C_0.$$

The two wall columns of Ψ_h are bounded by (7.112), reducing ε to at most one if necessary. Their sum proves (7.93). The local C^2 assertions proved above give the regularity of Q_h. □

The auxiliary height may now follow the phase parameter. For fixed η, J, set

$$a(Y) = \eta\left(\frac{R-2}{2}\right)^J, \qquad Q(Y) = Q_{a(Y)}(Y). \tag{7.113}$$

Define the corresponding time and state by $T(Y) = \mathcal{T}(a(Y), Y)$ and $\mathcal{Y}(Y) = \mathcal{E}(a(Y), Y)$, using (7.27).

Corollary 7.20. *For $0 < \eta \le \delta$ and $J \ge 4$, there are $C \ge 1$ and $\varepsilon > 0$ such that all positive inputs with $0 < \alpha \le 1$ and $u + w \le b \le \varepsilon\alpha^{2J}$, $b > 0$, have the following properties at $Y_0 = (2 + 2\alpha, u, w)$. The functions $T, \mathcal{Y}, Q$ are locally C^2; T is the actual first positive arrival at the new $V = 1$, its crossing is upward, and $\mathcal{Y}$ is the state there. Moreover,*

$$|Q(Y_0) - \alpha| \le Cb. \tag{7.114}$$

For every direction $d = (1, d_u, d_w)$ *with* $|d_u| \le \sqrt{u}$, $|d_w| \le \sqrt{w}$,

$$DQ(Y_0)[d] \ge \frac{1}{4}, \qquad |2DQ(Y_0)[d] - 1| \le C\sqrt{b}, \qquad |\log(2DQ(Y_0)[d])| \le C\sqrt{b}. \tag{7.115}$$

Proof. The height function a is continuous and $a(Y_0) = \eta\alpha^J$. Lemma 7.4 identifies the time and state with their fixed-height versions on an ordinary neighborhood of Y_0. The same identity holds for their radius and its reciprocal, which are positive there. It proves the first-arrival statements, regularity, and equality of the derivatives of Q and Q_h at this input.

Let A be a common constant in (7.92)–(7.93). For $b \le 1$, the directional derivative obeys

$$|2DQ[d] - 1| \le 2Ab + 2A\max(|d_u|, |d_w|) \le 4A\sqrt{b}.$$

Reduce ε further so that $4A\sqrt{b} \le 1/4$. Then $2DQ[d] \ge 1/2$, hence $DQ[d] \ge 1/4$. The elementary bound $|\log t| \le 2|t-1|$ for $t \ge 1/2$ gives the logarithmic estimate. Increasing C also includes the value estimate inherited from Proposition 7.16. □

The bounded phase therefore has a positive derivative on the same square-root directions used in the core and reflection estimates. To carry these directions through the whole terminal return, we next estimate the two wall components of its derivative.

7.6. **The terminal cone.** The terminal passage sends a radius close to two to a large radius in the next core. Its radial derivative has size α^{-2}. To compare the transverse derivatives with this scale, we retain the two logarithmic wall responses separately and multiply each by the square root of its own output wall. The decay during the incoming passage then controls both slopes, even when the two input walls have very different sizes.

For the fixed-height passage, write

$$\mathcal{R}_h(Y) = (\kappa_h(Y), U_+(Y), W_+(Y)), \qquad Y = (R, u, w),$$

where U_+, W_+ are the wall components of $\mathcal{E}(h, Y)$. All derivatives with a subscript h below hold this height fixed. The radial cone at Y is

$$\mathcal{K}_Y = \{d : \ |d_u| \le \sqrt{u}\,|d_R|, \ |d_w| \le \sqrt{w}\,|d_R|\}. \tag{7.116}$$

Every nonzero vector in it can be divided by d_R to give a direction $d = (1, d_u, d_w)$ with $|d_u| \le \sqrt{u}$, $|d_w| \le \sqrt{w}$.

Theorem 7.21 (Terminal cone preservation)**.** *There is* $0 < \delta \le 1/40000$ *such that, for every* $0 < \eta_0 \le \delta$ *and integer* $J \ge 5$*, there are* $C \ge 1$ *and* $\varepsilon > 0$ *with the following property. Suppose*

$$\eta_0 \le \eta \le \delta, \qquad 0 < \alpha \le 1, \qquad u, w > 0, \qquad u + w \le b \le \varepsilon\alpha^{2J}, \quad b > 0. \tag{7.117}$$

At $Y = (2 + 2\alpha, u, w)$ *and* $h = \eta\alpha^J$*, the actual return* $\mathcal{R}_h$ *is locally* C^2 *and has positive output walls. For every normalized direction in* (7.116)*, put* $p = D\kappa_h(Y)[d]$*. Then*

$$\begin{aligned} &p < 0, \qquad |p| \ge \frac{3}{4\alpha^2}, \qquad |\alpha^2 p + 1| \le C\sqrt{b}, \\ &\max\left\{\frac{|DU_+[d]|}{\sqrt{U_+}\,|p|}, \frac{|DW_+[d]|}{\sqrt{W_+}\,|p|}\right\} \le Cb^{7/24} < \frac{1}{2}. \end{aligned} \tag{7.118}$$

Thus $D\mathcal{R}_h$ *maps every nonzero vector of* $\mathcal{K}_Y$ *strictly inside the radial cone at its output, with the radial orientation reversed.*

We prove the radial bound first, then the two wall bounds. Throughout the proof, constants may depend on η_0, J but are independent of α, η, u, w. Each smallness condition is imposed by reducing the same positive ε.

7.6.1. *Radius and wall responses.* The inversion $\kappa_h = 2/Q_h$ converts the bounded phase estimates into ordinary radius derivatives.

Lemma 7.22. *On the domain* (7.117) *with* $J \geq 4$*, after choosing* δ *and* ε *small enough,*

$$\left|\kappa_h - \frac{2}{\alpha}\right| \leq \frac{Cb}{\alpha^2}, \qquad \left|(\kappa_h)_R + \frac{1}{\alpha^2}\right| \leq \frac{Cb}{\alpha^3},$$

$$|(\kappa_h)_u| + |(\kappa_h)_w| \leq \frac{C}{\alpha^2}. \tag{7.119}$$

The radius is positive and locally C^2*.*

Proof. Let $A \geq 1$ be a common constant in Proposition 7.16. Since $\alpha^{2J} \leq \alpha$, we can require $Ab \leq \alpha/4$. Then

$$\alpha/2 \leq Q_h \leq 2\alpha, \qquad D\kappa_h = -\frac{2DQ_h}{Q_h^2}.$$

The value and wall estimates follow from $|Q_h - \alpha| \leq Ab$ and $Q_h^{-2} \leq 4/\alpha^2$. For the radial column, subtract its reference value before estimating:

$$-\frac{2(Q_h)_R}{Q_h^2} + \frac{1}{\alpha^2} = -\frac{2((Q_h)_R - 1/2)}{Q_h^2} + \frac{(Q_h - \alpha)(Q_h + \alpha)}{\alpha^2 Q_h^2}.$$

The two terms are bounded by $8Ab/\alpha^2$ and $12Ab/\alpha^3$, respectively. Since $\alpha \leq 1$, their sum is at most $20Ab/\alpha^3$. The value constant is $4A$ and the sum of the wall constants is $8A$, so a single enlarged C gives all three bounds. Positivity and local regularity follow from those of Q_h. □

For a normalized cone direction, these columns give

$$\begin{aligned} |\alpha^2 D\kappa_h[d] + 1| &\leq \alpha^2|(\kappa_h)_R + \alpha^{-2}| + \alpha^2\big(|(\kappa_h)_u| + |(\kappa_h)_w|\big)\max(|d_u|, |d_w|) \\ &\leq Cb/\alpha + C\sqrt{b} \leq 2C\sqrt{b}. \end{aligned} \tag{7.120}$$

Here $b \leq \alpha^2$, after taking $\varepsilon \leq 1$, implies $b/\alpha \leq \sqrt{b}$. Requiring $2C\sqrt{b} \leq 1/4$ proves the negative sign and the lower bound in (7.118).

At the cyclic entrance use the notation

$$Z_h = (m_e, u_e, v_e, w_e), \qquad \widehat{Z}_h = (r_h, u_e, v_e, w_e), \qquad L = \log(1/v_e).$$

The geometric bounds and the wall factors give

$$\frac{1}{\alpha} \leq k = H(m_e - 1, v_e) \leq \frac{3}{\alpha}, \qquad 0 < v_e < 1, \tag{7.121}$$

with u_e, w_e in the domain of Propositions 6.22 and 6.25. This follows by taking η below one quarter of their wall threshold and then using Proposition 7.3 with $q = 0$.

Lemma 7.23. *For* $J \geq 2$*, uniformly on a domain of the form* (7.117)*, the output walls are positive and locally* C^2*. In every normalized cone direction,*

$$\begin{aligned} |D\log U_+[d]| &\leq C\left(\frac{1+L}{\alpha^2} + \frac{1}{\alpha\sqrt{w}}\right), \\ |D\log W_+[d]| &\leq C\left(L + \frac{1}{\alpha} + \frac{1}{\sqrt{w}} + \frac{1}{\sqrt{u}}\right). \end{aligned} \tag{7.122}$$

Proof. The two passive coordinates at the entrance retain the input wall as a factor:

$$v_e = w\mathcal{B}_W, \qquad w_e = u\mathcal{B}_U.$$

By Lemma 7.17, the factors are positive and locally C^2, and their logarithmic derivatives are bounded by C/α. The same bound holds for $D\log u_e$. Since $b \leq 1$, a normalized cone

direction has maximum norm at most one. Proposition 7.12 and logarithmic differentiation therefore give

$$|Dr_h[d]| \le C/\alpha^2, \qquad |D\log u_e[d]| \le C/\alpha,$$
$$|D\log v_e[d]| \le C/\alpha + 1/\sqrt{w}, \qquad |D\log w_e[d]| \le C/\alpha + 1/\sqrt{u}.$$

For example, $D\log v_e[d] = d_w/w + D\log \mathcal{B}_W[d]$.

Put $a = u_e$, $v = v_e$, $c = w_e$, and $e = D\widehat{Z}_h[d]$. With a common $B \ge 1$, the bounds just proved read

$$|e_r| \le B/\alpha^2, \qquad |e_a| \le aB/\alpha,$$
$$|e_v| \le v(B/\alpha + s), \qquad |e_c| \le c(B/\alpha + t), \qquad s = 1/\sqrt{w}, \quad t = 1/\sqrt{u}.$$

One further product is useful in the strong output row:

$$ct = \mathcal{B}_U\sqrt{u} \le B/\alpha^2. \tag{7.123}$$

This uses $u \le 1$ and the upper factor bound. It is uniform in the ratio u/w.

The local inverse of the entrance radial coordinates identifies the terminal output with the incoming return on a neighborhood of Y. Thus its logarithmic derivative is the incoming logarithmic differential applied to e. Using (6.100) and (6.123), with a common constant $A \ge 1$, its two rows satisfy

$$|D\log U_+[d]| \le AL|e_r| + (1/a + A)|e_a| + (Ak/v)|e_v| + A|e_c|,$$
$$|D\log W_+[d]| \le (AL/k^2)|e_r| + (A/k^2)|e_a| + (A/v)|e_v| + (1/c + A/k^2)|e_c|.$$

In the first row, the four terms are bounded by

$$\frac{ABL}{\alpha^2}, \qquad \frac{2AB}{\alpha^2}, \qquad \frac{3AB}{\alpha^2} + \frac{3As}{\alpha}, \qquad \frac{2AB}{\alpha^2},$$

respectively. Here $a, c \le 1$, $k \le 3/\alpha$, and (7.123) were used. In the second row, $k^{-2} \le \alpha^2$ gives the bounds

$$ABL, \qquad AB/\alpha, \qquad AB/\alpha + As, \qquad 2AB/\alpha + 2At.$$

Adding each row proves (7.122). The regularity and positivity are those of the actual passage and its multiplicative wall equations. □

The strong output row has no $u^{-1/2}$ term. Its dependence on that input is multiplied by $w_e = u\mathcal{B}_U$ before the estimate is taken. Keeping this factor is what allows the strong and weak walls to be treated independently.

7.6.2. *Decay absorbs the logarithmic singularities.* The incoming wall estimates in Proposition 6.1 and $k \ge 7/4$ imply

$$\sqrt{U_+} \le \sqrt{u_e}\, v_e^{19/24}, \qquad \sqrt{W_+} \le \sqrt{w_e}\sqrt{v_e}. \tag{7.124}$$

Indeed $1 + k/3 \ge 19/12$ and $1 + 1/k \ge 1$. The entrance estimates also give, with $A \ge 1$,

$$\sqrt{u_e} \le \alpha^{(J-2)/2}, \qquad v_e \le Aw/\alpha^2,$$
$$\sqrt{v_e} \le A\sqrt{w}/\alpha, \qquad \sqrt{w_e} \le A\sqrt{u}/\alpha. \tag{7.125}$$

For the first inequality, use $u_e \le 2\eta\alpha^{J-2}$ and $2\eta \le 1$. The others follow directly from the cyclic wall factors, enlarging A to cover their square roots.

We first record the powers left by the strong output. For $0 < v \le 1$,

$$v^{19/24}(1 + |\log v|) \le 24v^{3/4}. \tag{7.126}$$

To see this, put $t = v^{1/24}$. The elementary inequality $1 - 1/t \le \log t$ implies $t(1 - \log t) \le 1$; hence $t(1 - 24\log t) \le 24$. Also, if $v \le Aw/\alpha^2$ and $0 \le q \le 1$, then

$$v^q \le A\alpha^{-2q}w^q, \tag{7.127}$$

because $A^q \le A$. With $\beta = (J-2)/2$, these two inequalities yield

$$\alpha^\beta \left[v^{19/24}(1 + |\log v|) + \frac{\alpha v^{19/24}}{\sqrt{w}} \right] \le 24A \left[\alpha^{(J-5)/2} w^{3/4} + \alpha^{J/2-19/12} w^{7/24} \right]. \tag{7.128}$$

The second exponent of w is $19/24 - 1/2 = 7/24$.

For the weak output, the four terms can be bounded without comparing u and w. From (7.125), $v_e \le 1$, and $\sqrt{v_e}|\log v_e| \le 2$, we obtain

$$\begin{aligned} &\alpha^2 \sqrt{w_e}\sqrt{v_e}\, L \le 2A\sqrt{u}, && \alpha \sqrt{w_e}\sqrt{v_e} \le A\sqrt{u}, \\ &\frac{\alpha^2 \sqrt{w_e}\sqrt{v_e}}{\sqrt{w}} \le A^2\sqrt{u}, && \frac{\alpha^2 \sqrt{w_e}\sqrt{v_e}}{\sqrt{u}} \le A^2\sqrt{w}. \end{aligned} \tag{7.129}$$

Their sum is bounded by $4A^2(\sqrt{u} + \sqrt{w})$.

We now combine these powers with the logarithmic derivatives. For $J \ge 4$, the radial lower bound already proved gives $\alpha^2|p| \ge 3/4$. For either positive output wall x,

$$\frac{|Dx[d]|}{\sqrt{x}\,|p|} = \frac{\alpha^2\sqrt{x}\,|D\log x[d]|}{\alpha^2|p|} \le 2\alpha^2\sqrt{x}\,|D\log x[d]|. \tag{7.130}$$

The first row of (7.122), multiplied by $\alpha^2\sqrt{U_+}$ and estimated with (7.124) and (7.128), therefore gives

$$\frac{|DU_+[d]|}{\sqrt{U_+}\,|p|} \le C\left[\alpha^{(J-5)/2} w^{3/4} + \alpha^{J/2-19/12} w^{7/24}\right]. \tag{7.131}$$

The second row, together with (7.124) and (7.129), gives

$$\frac{|DW_+[d]|}{\sqrt{W_+}\,|p|} \le C(\sqrt{u} + \sqrt{w}). \tag{7.132}$$

Completion of the proof of Theorem 7.21. Choose δ to satisfy the phase, entrance-column, and wall-factor thresholds above. For $J \ge 5$, the powers of α in (7.131) are nonnegative. As $u, w \le b \le 1$,

$$w^{3/4} + w^{7/24} \le 2b^{7/24}, \qquad \sqrt{u} + \sqrt{w} \le 2b^{7/24}.$$

After enlarging C, both slopes are at most $Cb^{7/24}$. Reduce ε once more so that $C\varepsilon^{7/24} < 1/2$ and the radial error in (7.120) is at most $1/4$. All thresholds are independent of the individual walls. The actual-return regularity and positivity were established in the preceding lemmas. Finally, the inequalities are homogeneous in the direction, so the normalized conclusion extends to every nonzero vector of $\mathcal{K}_Y$. □

The same cone estimate holds for the actual return when the auxiliary height follows the input phase. With $a(Y)$ and $\mathcal{Y}(Y)$ as in (7.113), set

$$\kappa(Y) = H(\mathcal{Y}_M(Y) - 1, \mathcal{Y}_V(Y)), \qquad \mathcal{R}(Y) = (\kappa(Y), \mathcal{Y}_U(Y), \mathcal{Y}_W(Y)).$$

Corollary 7.24. *For fixed $0 < \eta \le \delta$ and $J \ge 5$, there are $C \ge 1$ and $\varepsilon > 0$ such that, whenever $0 < \alpha \le 1$ and $u, w > 0$ satisfy $u + w \le b \le \varepsilon\alpha^{2J}$, $b > 0$, the map $\mathcal{R}$ is locally C^2 at $Y = (2 + 2\alpha, u, w)$. Its differential satisfies all the cone estimates of Theorem 7.21 at that point.*

Proof. Apply Theorem 7.21 with $\eta_0 = \eta$. After intersecting its amplitude threshold with that of Lemma 7.4, the latter identifies $\mathcal{E}(a(Z), Z)$ and $\mathcal{E}(h, Z)$ on a neighborhood of Y, where $h = a(Y) = \eta\alpha^J$. Consequently $\mathcal{R}$ and $\mathcal{R}_h$ agree there. Their values and ordinary differentials agree, giving all the stated estimates. □

The terminal passage therefore preserves the radial cones used through the preceding core passages and reflections. Its reversal of the radius is accompanied by a positive bounded-phase derivative, as in Corollary 7.20. These two descriptions provide the radial orientation and the transverse control needed to compose a complete era.

8. Complete era returns and their statistics

The core and connection estimates now form an entire era return. Their errors are summed against transverse losses, giving constants independent of the number of cores. At each entrance we retain the branches whose radii and wall amplitudes satisfy the estimates for another return. Iterating this selection gives a set of entrances with an infinite itinerary. The coding measure estimate then yields the statistics of these itineraries in Theorem 8.20. We recover their complete events in the original time and estimate the measure left outside this construction along entrance families.

8.1. **Accumulated error and the terminal core.** A core followed by a reflection lowers the radius by approximately two. Its radius error has the scale of the lost transverse amplitude, and its logarithmic derivative error has the scale of the lost square root of that amplitude. Applying Lemma 4.3 to these two powers controls the entire finite sequence and locates its terminal core.

Write F for the core return of Proposition 4.14, and retain $\mathcal{F}$ for the reflection of Theorem 6.33. Their composition

$$G = \mathcal{F} \circ F \tag{8.1}$$

takes an upward core entrance to the next upward entrance, in the reflected frame. Its elapsed original time is $T_G(Y) = T_c(Y) + T_r(F(Y))$, where T_c and T_r are the core and reflection times. We use the embeddings

$$\iota_\pm(r, u, w) = (1 \pm \sqrt{r^2 - 1}, u, 1, w)$$

for upward and downward core states. Denote by Φ_t the original normalized flow expressed in projective coordinates, on its regular coordinate domain.

Lemma 8.1. *There are constants $0 < \varepsilon \le 10^{-4}$, $C > 0$, and $0 < c \le 1/2$ with the following properties. Let $Y = (r, u, w)$ satisfy*

$$r \ge 2, \qquad 0 < u, w \le \varepsilon/r, \qquad b = u + w, \tag{8.2}$$

and put $Z = F(Y) = (R, U, W)$. The core and its time are locally C^1, and

$$|R - r| \le \min(1/8, Cb/r), \qquad 0 < U \le e^{-c/r}u, \qquad 0 < W \le e^{-c/r}w. \tag{8.3}$$

If $R \ge 31/8$, then $G(Y) = (r', u', w')$ and T_G are locally C^1, $T_G > 0$, and

$$\iota_+(G(Y)) = \mathsf{S}\Phi_{T_G(Y)}(\iota_+(Y)). \tag{8.4}$$

Moreover,

$$\begin{aligned} |R - r| + |r' - (R - 2)| &\le Cb/r, & |r' - (r - 2)| &\le Cb/r, \\ 3/2 \le r - r' &\le 5/2, & 0 < u' &\le U, \qquad 0 < w' \le W. \end{aligned} \tag{8.5}$$

If $r' \ge 2$, then $u', w' \le \varepsilon/r'$. For either $H = F$ or, when defined, $H = G$, every direction $v = (1, v_u, v_w)$ with $|v_u| \le \sqrt{u}$ and $|v_w| \le \sqrt{w}$ has image $d = DH(Y)[v]$ satisfying

$$d_r > 0, \qquad |\log d_r| \le C\sqrt{b}/r, \qquad |d_u| \le \sqrt{H_u(Y)}\, d_r, \quad |d_w| \le \sqrt{H_w(Y)}\, d_r. \tag{8.6}$$

Proof. Let K, c, ε_c be the constants for the uniform core return, and let L, ε_r be those for the uniform reflection. Set $C = K + 2L$. Choose

$$\varepsilon \le \min\{\varepsilon_c, \varepsilon_r/2, 10^{-4}, 1/(4K), 1/C\}.$$

Then $b \le 2\varepsilon/r \le \varepsilon$. The core estimate gives $|R - r| \le K\varepsilon/r \le 1/8$, hence

$$r/2 \le R \le 2r.$$

Its walls do not increase, so

$$U, W \le \varepsilon/r \le \varepsilon_r/R.$$

For $R \ge 31/8$, the reflection is therefore defined. Its radius error is at most $L(U+W)/R \le 2Lb/r$. Adding the core error proves the first two estimates in (8.5). Their right-hand side is at most $C\varepsilon/2 \le 1/2$, which gives the stated decrease in radius. The walls remain positive and decrease through the reflection. When $r' \ge 2$, the inequality $r' < r$ gives $u' \le u \le \varepsilon/r \le \varepsilon/r'$, and similarly for w'.

To identify the actual core endpoint, consider a downward $V = 1$ crossing with radius $R \ge 2$ and $0 \le U, W \le 1/4$. The crossing condition gives $M - 1 < U(1+W-U) \le 5/16$. If $M \ge 1$, this would give $0 \le M-1 < 5/16$, contrary to $(M-1)^2+1 = R^2 \ge 4$. Hence $M < 1$, so the state is $\iota_-(R, U, W)$. Our exit satisfies these bounds. Thus $\iota_-(Z) = \Phi_{T_c(Y)}(\iota_+(Y))$, and the reflected arrival is $\iota_+(G(Y)) = \mathsf{S}\Phi_{T_r(Z)}(\iota_-(Z))$. The flow restart identity gives (8.4), with positive elapsed time. Composition gives the local C^1 assertions.

It remains to combine the derivatives. Put $z = DF(Y)[v]$. The core estimates give $z_r > 0$ and place $\widehat{z} = z/z_r$ in the normalized radial cone at Z. Thus, with $q = D\mathcal{F}(Z)[\widehat{z}]$,

$$DG(Y)[v] = z_r q, \qquad q_r > 0, \qquad |\log q_r| \le L\sqrt{U+W}/R.$$

The transverse inequalities are homogeneous. For the radial component, positivity allows the logarithms to add:

$$\begin{aligned} |\log(z_r q_r)| &\le |\log z_r| + |\log q_r| \\ &\le K\sqrt{b}/r + 2L\sqrt{b}/r = C\sqrt{b}/r. \end{aligned}$$

This proves (8.6) for G; the assertion for F follows directly from the core estimate. □

Consider a finite sequence $Y_{j+1} = G(Y_j)$ and write

$$Y_j = (r_j, u_j, w_j), \qquad F(Y_j) = (R_j, U_j, W_j), \qquad b_j = u_j + w_j, \quad B_j = U_j + W_j. \tag{8.7}$$

We shall use the constants

$$A_p = \frac{1 + pc/2}{pc} \quad (p > 0), \qquad A = CA_1, \qquad D_0 = CA_{1/2}, \tag{8.8}$$

where C, c are from Lemma 8.1.

Proposition 8.2 (Uniform control of a finite sequence)**.** *Suppose Y_0 satisfies* (8.2) *and the first n steps satisfy*

$$r_j \ge 2, \qquad R_j \ge 31/8 \quad (0 \le j < n).$$

All the walls at $Y_0, \dots, Y_n$ are positive, and

$$\sum_{j=0}^{n-1} \frac{b_j^p}{r_j} \le A_p(b_0^p - b_n^p) \qquad (p > 0). \tag{8.9}$$

The accumulated radius error obeys

$$\begin{aligned} \sum_{j=0}^{n-1} \big(|R_j - r_j| + |r_{j+1} - (R_j - 2)|\big) &\le A(b_0 - b_n), \\ |r_n - (r_0 - 2n)| &\le A(b_0 - b_n). \end{aligned} \tag{8.10}$$

The iterate G^n is locally C^1 at Y_0. For every normalized radial cone direction v there, its differential $d = D(G^n)(Y_0)[v]$ satisfies

$$d_r > 0, \qquad |\log d_r| \le D_0(\sqrt{b_0} - \sqrt{b_n}), \qquad |d_u| \le \sqrt{u_n}\, d_r, \quad |d_w| \le \sqrt{w_n}\, d_r. \tag{8.11}$$

If also $r_n \geq 2$, the last core is defined and $\mathcal{H}_n = F \circ G^n$ and $T_c \circ G^n$ are locally C^1 at Y_0. Its output walls are positive, $B_n \leq b_0$, and

$$|R_n - (r_0 - 2n)| \leq A(b_0 - B_n). \tag{8.12}$$

For $e = D\mathcal{H}_n(Y_0)[v]$,

$$e_r > 0, \qquad |\log e_r| \leq D_0(\sqrt{b_0} - \sqrt{B_n}), \qquad |e_u| \leq \sqrt{U_n}\, e_r, \quad |e_w| \leq \sqrt{W_n}\, e_r. \tag{8.13}$$

Proof. The amplitude bound in Lemma 8.1 propagates the input bound $u_j, w_j \leq \varepsilon/r_j$ whenever the next radius is at least two. Induction therefore applies the lemma at every indicated step. In particular,

$$0 < b_{j+1} \leq B_j \leq e^{-c/r_j} b_j.$$

Lemma 4.3 now gives (8.9). Sum the first estimate in (8.5) and use $p = 1$ to obtain the first line of (8.10). For the second line, sum the identity

$$r_{j+1} - (r_j - 2) = (R_j - r_j) + (r_{j+1} - (R_j - 2)).$$

The left side telescopes to $r_n - (r_0 - 2n)$.

Each map G is C^1 on a neighborhood of the point where it is used. A finite sequence of compositions is consequently C^1 near Y_0. To estimate its derivative, put $d_j = D(G^j)(Y_0)[v]$, so $d_0 = v$. Inductively $d_{j,r} > 0$, and $d_j/d_{j,r}$ lies in the normalized cone at Y_j. If

$$q_j = DG(Y_j)[d_j/d_{j,r}],$$

then $d_{j+1} = d_{j,r} q_j$. The cone inequalities propagate, and

$$|\log d_{n,r}| \leq \sum_{j=0}^{n-1} |\log q_{j,r}| \leq C \sum_{j=0}^{n-1} \frac{\sqrt{b_j}}{r_j} \leq D_0(\sqrt{b_0} - \sqrt{b_n}).$$

This proves (8.11), including $n = 0$, where the sum is empty and the derivative is the identity.

When $r_n \geq 2$, the propagated amplitude bound also permits the last core. Its loss satisfies $B_n \leq e^{-c/r_n} b_n$. Applying (4.16) to this last passage gives

$$\frac{b_n}{r_n} \leq A_1(b_n - B_n), \qquad \frac{\sqrt{b_n}}{r_n} \leq A_{1/2}(\sqrt{b_n} - \sqrt{B_n}).$$

The first inequality bounds $|R_n - r_n|$ and extends the radius telescope to B_n. The second extends the logarithmic derivative bound: normalize the derivative of G^n by its positive radial component, apply the core estimate, and add the two logarithms. This gives (8.12) and (8.13). Local regularity of the last core and its time gives the remaining composition statements. □

The finite-sequence estimate also constructs the sequence. A fixed margin from the endpoints of a radius interval prevents a perturbed orbit from changing its number of reflections.

Proposition 8.3 (The terminal index)**.** *There are $0 < \varepsilon \leq 10^{-4}$ and $K > 0$ such that the following holds for every integer $m \geq 0$ and $0 < \beta < 1/2$. Let $X = (r, u, w)$ satisfy*

$$0 < u, w \leq \varepsilon/r, \qquad m + 1 + \beta \leq r/2 \leq m + 2 - \beta, \qquad K(u + w) < \beta/2. \tag{8.14}$$

Then all $m + 1$ cores at $Y_j = G^j(X)$, $0 \leq j \leq m$, exist. In the notation (8.7)*,*

$$\begin{aligned} r_j &> 2 + 3\beta/2 && (0 \leq j \leq m), \\ R_j &> 4 + 3\beta/2 && (0 \leq j < m), \\ 2 + 3\beta/2 < R_m &< 4 - 3\beta/2. \end{aligned} \tag{8.15}$$

The m intervening reflections are the actual positive-time passages in (8.4). *For every $j \le m$, G^j, $\mathcal{H}_j = F \circ G^j$, and $T_c \circ G^j$ are locally C^1 at X, their core output walls are positive, and*

$$B_j \le b_0 = u + w, \qquad |r_j - (r - 2j)| + |R_j - (r - 2j)| \le K b_0. \tag{8.16}$$

For every normalized cone direction v at X, put $d = D\mathcal{H}_j(X)[v]$. Then

$$d_r > 0, \qquad |\log d_r| \le K\sqrt{b_0}, \qquad |d_u| \le \sqrt{U_j}\, d_r, \quad |d_w| \le \sqrt{W_j}\, d_r. \tag{8.17}$$

Finally,

$$\lfloor r/2 \rfloor = m + 1, \qquad \frac{3\beta}{4} < \alpha := \frac{R_m - 2}{2} < 1 - \frac{3\beta}{4}. \tag{8.18}$$

Proof. Use the amplitude threshold of Proposition 8.2 and put $K = 2A + D_0$, with (8.8). Suppose the first j reflections have been constructed, where $j \le m$. The prefix estimate gives

$$|r_j - (r - 2j)| \le A b_0 < \beta/2.$$

Since $r - 2j \ge 2 + 2\beta$, this shows $r_j > 2 + 3\beta/2$ and permits the next core. The last-core estimate then gives

$$|R_j - (r - 2j)| \le A b_0 < \beta/2.$$

For $j < m$, the reference radius is at least $4 + 2\beta$. Thus $R_j > 4 + 3\beta/2 > 31/8$, so the next reflection exists. This closes the induction, starting from $j = 0$ with an empty reflection sequence.

At $j = m$, the reference radius lies in $[2 + 2\beta, 4 - 2\beta]$. The same error bound places R_m in the interval in (8.15). All the regularity, positivity, and cone statements follow from Proposition 8.2. The two radius errors sum to at most $2Ab_0 \le Kb_0$, and the logarithmic error is at most $D_0\sqrt{b_0} \le K\sqrt{b_0}$. The input interval gives $m + 1 < r/2 < m + 2$, proving the floor identity; dividing the final radius interval by two after subtracting two gives the bounds on α. □

The constants are independent of m. The entrance condition scales the walls with r, while the accumulated error is controlled by the amplitude lost along the actual sequence.

8.2. **Phase through an era.** The terminal connection closes the sequence constructed in Proposition 8.3. Its bounded phase can be compared directly with the fractional part of the initial half-radius. The comparison remains uniform in the number of intervening reflections.

Fix η, J in the terminal construction. Let Q, T, and $\mathcal{Y}$ denote its bounded phase, elapsed time, and four-coordinate output state at the natural auxiliary height, as in (7.113). For a prescribed number m of reflections, define

$$\mathcal{H}_m = F \circ G^m, \qquad P_m = Q \circ \mathcal{H}_m, \qquad \mathcal{Y}_m = \mathcal{Y} \circ \mathcal{H}_m. \tag{8.19}$$

Thus $\mathcal{H}_m$ is the last downward core state, and P_m is twice the reciprocal radius at the following upward entrance. The time $T \circ \mathcal{H}_m$ measures only the terminal connection; each preceding core and reflection has its own positive elapsed time.

Theorem 8.4 (The bounded era phase)**.** *There is $0 < \delta \le 1/40000$ such that, for each $0 < \eta \le \delta$ and integer $J \ge 4$, there are $0 < \varepsilon \le 10^{-4}$ and $C \ge 1$ with the following property. Suppose $m \in \mathbb{N}$, $0 < \beta < 1/2$, and $X = (r, u, w)$ satisfy*

$$\begin{aligned} 0 < u, w \le \varepsilon/r, \qquad b_0 = u + w \le \varepsilon(\beta/2)^{2J}, \\ m + 1 + \beta \le r/2 \le m + 2 - \beta, \qquad C b_0 < \beta/4. \end{aligned} \tag{8.20}$$

The $m + 1$ cores and m reflections exist, and their last core has radius $2 < R_m < 4$. The subsequent terminal connection has a first positive upward arrival at the new $V = 1$ section.

The maps $\mathcal{H}_m, P_m, \mathcal{Y}_m$ and $T \circ \mathcal{H}_m$ are locally C^1 at X. Writing $f = r/2 - (m+1)$, one has

$$\beta/2 < P_m(X) < 1 - \beta/2, \qquad |P_m(X) - f| \le Cb_0. \tag{8.21}$$

The next core radius is $\kappa_+ = 2/P_m(X) > 2$, and $\lfloor r/2 \rfloor = m+1$. For every normalized radial cone direction v at X,

$$DP_m(X)[v] > 0, \qquad |\log(2DP_m(X)[v])| \le C\sqrt{b_0}. \tag{8.22}$$

Proof. Let ε_k, K be the constants of Proposition 8.3, and let ε_t, A be those of Corollary 7.20, with $A \ge 1$. Choose

$$\varepsilon = \min(\varepsilon_k, \varepsilon_t), \qquad C = K + A + 1.$$

The hypotheses imply $Kb_0 < \beta/2$, so the core itinerary is defined. Set

$$Z = \mathcal{H}_m(X) = (R, U, W), \qquad b = U + W, \qquad \alpha = (R-2)/2.$$

Its terminal estimates give

$$0 < b \le b_0, \qquad 3\beta/4 < \alpha < 1 - 3\beta/4. \tag{8.23}$$

In particular,

$$b \le \varepsilon(\beta/2)^{2J} \le \varepsilon_t \alpha^{2J}.$$

At the last downward crossing, $R > 2$ and $0 < U, W \le b_0 \le 10^{-4}$. The branch identification in the proof of Lemma 8.1 shows that the actual state there is $\iota_-(Z)$. The terminal passage therefore applies at this endpoint. It gives the stated first arrival, its upward crossing, and locally C^2 time and state as functions of Z. Composing with the C^1 map $\mathcal{H}_m$ gives all the regularity assertions at X.

The terminal phase estimate reads

$$|P_m(X) - \alpha| \le Ab \le Ab_0. \tag{8.24}$$

As $Ab_0 \le Cb_0 < \beta/4$, combining this with (8.23) gives the interval in (8.21). Also,

$$|\alpha - f| = \tfrac{1}{2}|R - (r - 2m)| \le \tfrac{K}{2} b_0.$$

Adding this estimate to (8.24) gives $|P_m(X) - f| \le (A + K/2)b_0 \le Cb_0$. The output radius is positive and satisfies $P_m = 2/\kappa_+$. Since $0 < P_m < 1$, it follows that $\kappa_+ > 2$. The floor identity comes from Proposition 8.3.

For the derivative, put $d = D\mathcal{H}_m(X)[v]$ and $s = d_r$. By (8.17), $s > 0$, $|\log s| \le K\sqrt{b_0}$, and $\widehat{d} = d/s$ lies in the normalized cone at Z. The terminal bounded phase satisfies

$$DQ(Z)[\widehat{d}] > 0, \qquad |\log(2DQ(Z)[\widehat{d}])| \le A\sqrt{b}.$$

The chain rule and linearity give

$$2DP_m(X)[v] = s\, 2DQ(Z)[\widehat{d}] > 0.$$

Both factors are positive. Taking logarithms and using $b \le b_0$ yields

$$|\log(2DP_m(X)[v])| \le K\sqrt{b_0} + A\sqrt{b} \le C\sqrt{b_0},$$

as required. □

The constructed segment contains $m+1$ successive cores and ends at the entrance after its first terminal connection. When its initial entrance also follows a terminal connection, it is a complete era in Definition 2.3, of length $\lfloor r/2 \rfloor$. The theorem identifies its output phase and its derivative from the actual crossings.

The differential estimate is adapted to curves transverse to the small wall directions. Let U, W be C^1 near a radius r and consider the graph

$$\gamma(s) = (s, U(s), W(s)), \qquad |U'(r)| \le \sqrt{U(r)}, \quad |W'(r)| \le \sqrt{W(r)}.$$

Its tangent at r is a normalized radial cone direction.

Corollary 8.5. *Suppose $X = \gamma(r)$ satisfies (8.20), with $u = U(r)$, $w = W(r)$. Then $\varphi = P_m \circ \gamma$ is C^1 near r, has $\varphi'(r) > 0$, and admits a C^1 local inverse g at $\varphi(r)$. At the corresponding points,*

$$g(\varphi(r)) = r, \qquad g'(\varphi(r)) = \frac{1}{\varphi'(r)} > 0, \tag{8.25}$$

and

$$|\log(2\varphi'(r))| \le C\sqrt{U(r) + W(r)}, \qquad |\log(g'(\varphi(r))/2)| \le C\sqrt{U(r) + W(r)}. \tag{8.26}$$

The identities $g \circ \varphi = \mathrm{id}$ and $\varphi \circ g = \mathrm{id}$ hold on suitable open neighborhoods of r and $\varphi(r)$, respectively.

Proof. Differentiating the graph gives

$$\varphi'(r) = DP_m(X)[(1, U'(r), W'(r))].$$

Theorem 8.4 supplies its positivity and the first logarithmic bound. The one-dimensional inverse function theorem gives the local inverse and (8.25). Finally,

$$g'(\varphi(r))/2 = (2\varphi'(r))^{-1},$$

so the two logarithmic errors have equal absolute value. □

The local inverse is therefore controlled by the same initial amplitude as the forward phase, independently of the era length. To obtain the interval branches used in the measure argument, we next need the output wall estimates and a common phase interval on which these local inverses join.

8.3. **Contraction of the era amplitude.** At their endpoints, the core passages and reflections do not increase either small wall. The terminal connection supplies a stronger gain: their sum at the next entrance is bounded by a power greater than one of their previous sum. This gain will allow the phase and derivative errors to be summed over successive eras. To obtain it uniformly near a terminal phase of zero, we retain the radius in the strong-wall equation during the incoming passage.

Lemma 8.6. *Under the hypotheses of Proposition 6.1, the same first arrival satisfies*

$$\mathcal{P}_U(y) \le u \exp\left(\frac{2w + 3u^2}{k}\right) v^{1+99k/200}. \tag{8.27}$$

In particular, if $k \ge 19/10$, then

$$\mathcal{P}_U(y) \le e^{1/1000} u v^{19/10}, \qquad \mathcal{P}_W(y) \le wv. \tag{8.28}$$

Proof. Use the increasing clock $s = \log(V/v)$ on the passage, and write

$$\rho = H(M-1, V), \qquad c = U(V + W - U), \qquad q = M - 1 - c,$$
$$a = \tfrac{1}{2}(1 - M^2 - (V - W)^2 + U^2).$$

Thus $dU/ds = (a/q)U$. The region (6.7) and its clock bounds hold up to the actual first arrival. The identity

$$\frac{a}{q} = -1 - \frac{\rho^2}{2(M-1)} + \frac{2VW - W^2 + U^2 - \big(2 + \rho^2/(M-1)\big)c}{2q} \tag{8.29}$$

keeps the full radius in the negative term. Since $0 < M - 1 \le \rho$ and $\rho \ge 99k/100$,

$$\frac{\rho^2}{2(M-1)} \ge \frac{\rho}{2} \ge \frac{99k}{200}.$$

For the remainder, the bounds $M-1 \ge 4k/5$ and $\rho \le 101k/100$ give

$$\frac{\rho^2}{M-1} \le \frac{32}{25}k, \qquad 3+\frac{\rho^2}{M-1} \le 4q.$$

The second inequality uses $q \ge 3k/4$ and $k \ge 7/4$. Together with $c \ge -U^2$, these estimates bound the last term of (8.29) by

$$\frac{VW}{q} + 2U^2 \le \frac{2W}{k} + 2U^2.$$

Consequently

$$\frac{a}{q} \le -\left(1+\frac{99k}{200}\right) + \frac{2W}{k} + 2U^2.$$

The previously established decay (6.2) controls the two error integrals without a factor of the waiting length:

$$\int_0^s W(\ell)\,d\ell \le w, \qquad \int_0^s U(\ell)^2\,d\ell \le \frac{3u^2}{2k}.$$

Indeed, it is enough to use $W(\ell) \le we^{-\ell}$ and $U(\ell)^2 \le u^2 e^{-2k\ell/3}$. Integrating the scalar equation for U now gives, for every $0 \le s \le \log(1/v)$,

$$U(s) \le u \exp\left(\frac{2w+3u^2}{k}\right) e^{-(1+99k/200)s}.$$

This also holds for $u=0$, by invariance of that wall. At the first arrival $s=\log(1/v)$, this is (8.27).

On the incoming domain, $(2w+3u^2)/k \le 1/1000$. If $k \ge 19/10$, then $1+99k/200 \ge 19/10$. Since $0<v<1$, the first bound in (8.28) follows. The second follows from $\mathcal{P}_W(y) \le wv^{1+1/k} \le wv$. □

Let $\mathcal{Y}$ be the actual terminal output at the natural height, as in (7.113). The cyclic change of axes amplifies two incoming walls by at most α^{-2}. The following estimate shows how the incoming decay compensates for this amplification under the same smallness conditions.

Proposition 8.7. *Fix $0<\eta\le 1/40000$ and an integer $J \ge 20$. There are $C \ge 1$ and $0<\varepsilon\le 1$ such that, for $0<\alpha\le 1$, $u,w>0$, and*

$$u+w \le b \le \varepsilon\alpha^{2J}, \qquad Y=(2+2\alpha,u,w),$$

the two small walls at the next entrance satisfy

$$\mathcal{Y}_U(Y)>0, \qquad \mathcal{Y}_W(Y)>0, \qquad \mathcal{Y}_U(Y)+\mathcal{Y}_W(Y) \le Cb^{19/10}. \tag{8.30}$$

Proof. For this fixed η, choose a common amplitude threshold for the cyclic geometry, the wall factors, and the first incoming arrival. At $h=\eta\alpha^J$, let $y_e=(m_e,u_e,v_e,w_e)$ be the cyclic entrance. Lemma 7.2, Proposition 7.3, and Lemma 7.17 give, with $q=0$ and $\varepsilon_{\mathrm{in}}=10^{-4}$ in the first-arrival proposition,

$$k=H(m_e-1,v_e) > \frac{19}{10\alpha} \ge \frac{19}{10}, \qquad m_e>2, \qquad 0<v_e<\tfrac12,$$
$$0<u_e,w_e\le 10^{-4}, \qquad u_e \le 2\eta\alpha^{J-2}, \qquad v_e \le \frac{Aw}{\alpha^2}, \quad w_e \le \frac{Au}{\alpha^2}$$

for a constant $A \ge 1$. Lemma 8.6 therefore applies to this actual entrance. At its first arrival,

$$\mathcal{Y}_U(Y) \le e^{1/1000}u_e v_e^{19/10}, \qquad \mathcal{Y}_W(Y) \le w_e v_e. \tag{8.31}$$

Both walls are positive by their multiplicative equations. The state in (8.31) is $\mathcal{Y}(Y)$ because the natural height at this Y is exactly $\eta\alpha^J$.

Shrink the threshold so that $\varepsilon \le 1$. Since $J \ge 20$, $0 < \alpha \le 1$, and $2\eta \le 1$,

$$u_e \le \alpha^4, \qquad b \le \alpha^{40}, \qquad b^{1/10} \le \alpha^4.$$

It follows that

$$u_e v_e^{19/10} \le A^{19/10} w^{19/10} \alpha^{1/5} \le A^2 b^{19/10},$$
$$w_e v_e \le \frac{A^2 uw}{\alpha^4} \le \frac{A^2 b^2}{\alpha^4} \le A^2 b^{19/10}.$$

Insert these bounds into (8.31) and take $C = (e^{1/1000} + 1)A^2$. □

Return to the era maps (8.19), and denote their two small output walls and radius by

$$U_m^+ = (\mathcal{Y}_m)_U, \qquad W_m^+ = (\mathcal{Y}_m)_W, \qquad \kappa_m^+ = H((\mathcal{Y}_m)_M - 1, (\mathcal{Y}_m)_V). \tag{8.32}$$

All are functions of the initial core coordinates X. On the era domain, $\kappa_m^+ = 2/P_m$.

Theorem 8.8 (Amplitude and cone contraction through an era)**.** *The constants in Theorem 8.4 can be chosen so that, for each $J \ge 20$, the same domain* (8.20) *also has the following properties. The maps $\mathcal{Y}_m$ and κ_m^+ are locally C^1, their small output walls are positive, and*

$$U_m^+(X) + W_m^+(X) \le C b_0^{19/10}, \qquad C b_0^{7/24} < \tfrac{1}{2}. \tag{8.33}$$

For every normalized radial cone direction v at X, put $p = D\kappa_m^+(X)[v]$. Then

$$p < 0,$$
$$\max\left\{ \frac{|DU_m^+(X)[v]|}{\sqrt{U_m^+(X)}\,|p|}, \frac{|DW_m^+(X)[v]|}{\sqrt{W_m^+(X)}\,|p|} \right\} \le C b_0^{7/24}. \tag{8.34}$$

The constants are independent of m and of the ratio of the two input walls.

Proof. Intersect the allowed ranges of η in the bounded phase and terminal cone estimates. For fixed η, J, let K be the core itinerary constant, A the constant of Corollary 7.24, and B the constant of Proposition 8.7. Choose $C \ge K + A + B$ also large enough for Theorem 8.4. Choose ε below all the corresponding amplitude thresholds and so small that

$$C\varepsilon^{7/24} < \tfrac{1}{2}. \tag{8.35}$$

For $Z = \mathcal{H}_m(X) = (R, U, W)$, set $b = U + W$ and $\alpha = (R-2)/2$. The core itinerary and the domain give

$$0 < b \le b_0, \qquad 3\beta/4 < \alpha < 1 - 3\beta/4, \qquad b \le \varepsilon(\beta/2)^{2J} \le \varepsilon\alpha^{2J}.$$

Thus both terminal estimates apply at Z. The terminal amplitude is at most $Bb^{19/10}$, which is at most $Cb_0^{19/10}$. The actual terminal state and radius are locally C^2 at Z, so their compositions with $\mathcal{H}_m$ are locally C^1 at X. Positivity is inherited from the terminal return. Also $b_0 \le \varepsilon$, and hence (8.35) gives the second inequality in (8.33).

To compare the cones, put $d = D\mathcal{H}_m(X)[v]$. The core itinerary gives $s = d_r > 0$ and a normalized cone vector $\widehat{d} = d/s$ at Z. If $p_t = D\kappa(Z)[\widehat{d}]$, then the terminal cone estimate gives $p_t < 0$ and

$$\frac{|D\mathcal{Y}_U(Z)[\widehat{d}]|}{\sqrt{\mathcal{Y}_U(Z)}\,|p_t|} \le Ab^{7/24}, \qquad \frac{|D\mathcal{Y}_W(Z)[\widehat{d}]|}{\sqrt{\mathcal{Y}_W(Z)}\,|p_t|} \le Ab^{7/24}.$$

By the chain rule, $p = sp_t < 0$ and each output wall derivative is s times its terminal derivative in direction $\widehat{d}$. This positive factor cancels from each ratio. Finally $Ab^{7/24} \le Cb_0^{7/24}$, which proves (8.34). □

In particular, along a radial graph satisfying the input cone bound, the output radius has negative derivative. When the image is parametrized locally by that radius, its two wall slopes satisfy the same square-root bounds with a factor at most $Cb_0^{7/24} < 1/2$. Thus the class of graphs used to control the phase is preserved with a strict margin at the next entrance.

8.4. **A common interval for the era branches.** The local inverse in Corollary 8.5 becomes an interval branch once the initial radius runs across a prescribed era. The phase error controls its two endpoints, while the positive derivative ensures uniqueness throughout. These facts give a common output interval for every radial graph in the same amplitude range.

Theorem 8.9 (The interval inverse)**.** *Choose η, J, ε, C as in Theorem 8.4. Let $m \in \mathbb{N}$, $0 < \beta < 1/4$, and set*

$$a_- = 2(m+1+\beta), \qquad a_+ = 2(m+2-\beta), \qquad I_m = [a_-, a_+]. \tag{8.36}$$

Suppose $B > 0$ satisfies

$$Ba_+ \le \varepsilon, \qquad B \le \varepsilon(\beta/2)^{2J}, \qquad CB < \beta/4. \tag{8.37}$$

Let U, W be C^1 on a neighborhood of I_m and assume, for every $r \in I_m$,

$$U(r), W(r) > 0, \qquad U(r)+W(r) \le B, \qquad |U'(r)| \le \sqrt{U(r)}, \quad |W'(r)| \le \sqrt{W(r)}. \tag{8.38}$$

Write $\gamma(r) = (r, U(r), W(r))$ and $\varphi = P_m \circ \gamma$. Then φ is strictly increasing on I_m, with

$$\varphi(a_-) < 2\beta, \qquad 1 - 2\beta < \varphi(a_+). \tag{8.39}$$

Its inverse g on $\varphi(I_m)$ satisfies $g(\varphi(r)) = r$ for every $r \in I_m$. For every $y \in [2\beta, 1-2\beta]$, one has

$$a_- < g(y) < a_+, \qquad \varphi(g(y)) = y, \tag{8.40}$$

and $g(y)$ is the unique radius in I_m with this phase. The inverse is C^1 near each such y, including the endpoints of the retained interval, and

$$g'(y) = \frac{1}{\varphi'(g(y))} > 0,$$

$$\left| \frac{g(y)}{2} - (m+1+y) \right| \le CB, \qquad \left| \log \frac{g'(y)}{2} \right| \le C\sqrt{B}. \tag{8.41}$$

Proof. For $r \in I_m$, one has $0 < r \le a_+$ and hence $Br \le Ba_+ \le \varepsilon$. In particular,

$$0 < U(r), W(r) \le B \le \varepsilon/r.$$

The other conditions of (8.20) follow from (8.37) and (8.38). Thus the actual era phase is defined along the whole graph. By Theorem 8.4 and the chain rule, it is locally C^1 at every $r \in I_m$ and satisfies

$$\left| \varphi(r) - \left(\frac{r}{2} - (m+1) \right) \right| \le CB,$$

$$\varphi'(r) > 0, \qquad |\log(2\varphi'(r))| \le C\sqrt{B}. \tag{8.42}$$

The mean value theorem makes φ strictly increasing on I_m. Continuity gives

$$\varphi(I_m) = [\varphi(a_-), \varphi(a_+)],$$

so there is a unique inverse on this image. At the two endpoints, the first bound in (8.42) gives

$$\varphi(a_-) \le \beta + CB < \tfrac{5}{4}\beta < 2\beta, \qquad \varphi(a_+) \ge 1 - \beta - CB > 1 - \tfrac{5}{4}\beta > 1 - 2\beta.$$

This proves (8.39) and places the entire closed interval $[2\beta, 1-2\beta]$ strictly inside the open image $(\varphi(a_-), \varphi(a_+))$. Strict monotonicity then gives (8.40) and uniqueness.

For the ordinary differentiability assertion, fix such a y and put $r = g(y)$. Since $r \in (a_-, a_+)$ and $\varphi'(r) > 0$, the inverse function theorem supplies a C^1 local inverse j near y. By continuity, shrink its neighborhood so that its values remain in (a_-, a_+) and its arguments remain in $(\varphi(a_-), \varphi(a_+))$. Both j and g then invert φ inside I_m, where it is injective. Hence $j = g$ on this ordinary open neighborhood of y, and $g'(y) = 1/\varphi'(g(y)) > 0$. This argument also applies when $y = 2\beta$ or $y = 1 - 2\beta$ because both endpoints lie in the open image.

Finally apply (8.42) at $r = g(y)$. The value estimate follows from $\varphi(g(y)) = y$ and

$$\frac{g(y)}{2} - (m + 1 + y) = -\left[y - \left(\frac{g(y)}{2} - (m + 1)\right)\right].$$

For the derivative, the reciprocal identity gives

$$\log \frac{g'(y)}{2} = -\log(2\varphi'(g(y))),$$

which proves the last bound. □

The retained interval depends only on β. The input radius interval depends on the era length, but the inverse error and logarithmic derivative bounds depend only on the common amplitude B. Together with Theorem 8.8, this prepares the choice of a phase margin and an amplitude scale that can be propagated from one era to the next.

8.5. **Retained phases and inverse branches.** The amplitude now determines how much of the phase interval we retain. Smaller walls permit both a smaller margin at the ends of an era and a larger number of possible era lengths. We choose these two restrictions together, so that the length omitted from the initial phase interval is controlled by the same margin.

Fix $0 < \eta \le \delta$ and $J \ge 20$ in the common parameter range of Theorems 8.8 and 8.9. Choose

$$0 < \sigma < \frac{1}{2J}, \qquad \beta = B^\sigma, \qquad N = \lfloor 1/\beta \rfloor. \tag{8.43}$$

The retained digits will be $1, \dots, N$, corresponding to $m = 0, \dots, N - 1$ reflections.

Lemma 8.10. *For every $\varepsilon > 0$ and $C \ge 0$, there is $B_* \in (0, 1)$ such that $0 < B \le B_*$ implies*

$$0 < \beta < \frac{1}{160}, \qquad B \le \varepsilon(\beta/2)^{2J}, \qquad CB < \beta/4,$$
$$B\,2(m + 2 - \beta) \le \varepsilon \quad \text{whenever } m + 1 \le 1/\beta. \tag{8.44}$$

In particular, the last inequality holds for every $0 \le m < N$.

Proof. All the powers of B in

$$4B^{1-\sigma} < \varepsilon, \qquad 2^{2J} B^{1-2J\sigma} < \varepsilon, \qquad 4CB^{1-\sigma} < 1, \qquad B^\sigma < 1/160$$

have positive exponents. They therefore hold below one positive threshold, which we may take smaller than one. The identities

$$B = B^{1-\sigma}\beta = B^{1-2J\sigma}\beta^{2J}$$

give the terminal smallness and phase margin in (8.44). If $(m + 1)\beta \le 1$ and $0 < \beta < 1$, then

$$\beta\,2(m + 2 - \beta) = 2\big((m + 1)\beta + \beta - \beta^2\big) \le 4.$$

Multiplication by $B^{1-\sigma}$ proves the radius condition. Finally $m < N$ implies $m + 1 \le N \le 1/\beta$. □

We shall use graphs defined on every radius $r > 2$. Call

$$\Gamma(r) = (r, U(r), W(r))$$

admissible at amplitude B if U, W are C^1 on $(2, \infty)$ and, throughout that interval,

$$U, W > 0, \qquad U + W \le B, \qquad |U'| \le \sqrt{U}, \quad |W'| \le \sqrt{W}. \tag{8.45}$$

Lemma 8.10 places each restriction of such a graph to I_m in the domain of Theorem 8.9, for $m < N$.

For the moment, let m, β, B and the graph satisfy the hypotheses of Theorem 8.9. Write ζ_m for the radius inverse constructed there, and put

$$K_\beta = [2\beta, 1 - 2\beta], \qquad h_m(y) = \frac{2}{\zeta_m(y)}, \qquad \varphi_m = P_m \circ \Gamma. \tag{8.46}$$

Thus h_m maps the next entrance phase y to the initial phase $x = 2/r$. Recall $g_n(y) = 1/(n + y)$.

Proposition 8.11. *The map h_m is strictly decreasing on K_β. For every $y \in K_\beta$, it is C^1 near y and*

$$\frac{1}{m+2} < h_m(y) < \frac{1}{m+1}, \qquad \varphi_m(2/h_m(y)) = y, \qquad h_m'(y) < 0,$$

$$|h_m(y) - g_{m+1}(y)| \le \frac{CB}{(m+1)^2}, \qquad \left| \log \frac{|h_m'(y)|}{|g_{m+1}'(y)|} \right| \le 5C\sqrt{B}. \tag{8.47}$$

Here C is the constant in Theorem 8.9.

Proof. Set $n = m + 1$ and $q(y) = \zeta_m(y)/2$. The interval inverse gives

$$n < q(y) < n + 1, \qquad q'(y) > 0, \qquad |q(y) - (n + y)| \le E, \qquad |\log q'(y)| \le D, \tag{8.48}$$

where $E = CB$ and $D = C\sqrt{B}$. In particular, $h_m = 1/q$ is locally C^1 and

$$h_m'(y) = -\frac{q'(y)}{q(y)^2} < 0.$$

Its range and its inverse identity follow from (8.48) and $2/h_m(y) = \zeta_m(y)$. The mean value theorem proves strict decrease on the whole interval.

Since $q(y) \ge n \ge 1$ and $y \ge 0$,

$$|h_m(y) - g_n(y)| = \frac{|q(y) - (n + y)|}{q(y)(n + y)} \le \frac{E}{n^2}.$$

For the derivatives, the exact ratio is

$$\frac{|h_m'(y)|}{|g_n'(y)|} = q'(y) \left(\frac{n + y}{q(y)} \right)^2. \tag{8.49}$$

The positive factor $t = (n + y)/q(y)$ satisfies $|t - 1| \le E/q(y) \le E$. The phase margin gives $E < \beta/4 < 1/4$; hence $1 - 1/t \le \log t \le t - 1$ implies $|\log t| \le 2E$. Taking logarithms in (8.49) therefore yields an error at most $D + 4E$. The input conditions imply $B \le \varepsilon \le 10^{-4}$, so $B \le \sqrt{B}$ and $D + 4E \le 5C\sqrt{B}$. □

For an admissible graph and the choices (8.43), define its retained phases by

$$\mathcal{V}_\Gamma(B) = \bigcup_{m=0}^{N-1} (h_m(1 - 2\beta), h_m(2\beta)). \tag{8.50}$$

These are exactly the images of the open interval $(2\beta, 1 - 2\beta)$ under the retained inverse branches.

Theorem 8.12. *There are $C \geq 1$ and $B_* \in (0,1)$, depending only on η, J, σ, such that every graph admissible at $0 < B \leq B_*$ has the following properties. One has $0 < \beta < 1/160$, and each $m < N$ admits the inverse branch and all the estimates of Proposition 8.11. Its closed image is*

$$h_m(K_\beta) = [h_m(1-2\beta), h_m(2\beta)]. \tag{8.51}$$

The open branch images are pairwise disjoint, and

$$\lambda\big((0,1) \setminus \mathcal{V}_\Gamma(B)\big) \leq 10B^\sigma. \tag{8.52}$$

The omitted set is Borel.

Proof. Choose ε, C for the interval inverse and use Lemma 8.10 to fix B_*. Every I_m, $m < N$, lies in $(2,\infty)$, and the admissible graph obeys all the required bounds there. Proposition 8.11 therefore applies with the same constants. Continuity and strict decrease give both the closed image (8.51) and its open-interval version.

Put $L_m = h_m(2\beta)$, $R_m = h_m(1-2\beta)$. Each interval (R_m, L_m) lies inside $(1/(m+2), 1/(m+1))$, so these intervals are disjoint subsets of $(0,1)$. Their union is open. It remains to estimate its total length. For $n = m+1$, the elementary difference formula gives

$$|g_n(s) - g_n(t)| \leq \frac{|s-t|}{n^2}, \qquad s,t \geq 0.$$

Apply this at $(s,t) = (2\beta, 0)$ and $(1-2\beta, 1)$, and add the two endpoint errors from (8.47). With $E = CB$,

$$L_m - R_m \geq \frac{1}{m+1} - \frac{1}{m+2} - \frac{4\beta + 2E}{(m+1)^2}. \tag{8.53}$$

The sums in this inequality satisfy

$$\sum_{m=0}^{N-1} \left(\frac{1}{m+1} - \frac{1}{m+2}\right) = 1 - \frac{1}{N+1}, \qquad \sum_{m=0}^{N-1} \frac{1}{(m+1)^2} \leq 2.$$

For the second bound, sum $n^{-2} \leq 2(n^{-1} - (n+1)^{-1})$ over $1 \leq n \leq N$. Disjointness and (8.53) now give

$$\lambda\big((0,1) \setminus \mathcal{V}_\Gamma(B)\big) \leq \frac{1}{N+1} + 8\beta + 4E.$$

Since $N = \lfloor 1/\beta \rfloor$, one has $1/(N+1) < \beta$. Also $E < \beta/4$. The right-hand side is therefore less than 10β, which proves (8.52). □

The estimate includes the tail of digits larger than N and the gaps at both ends of every retained digit interval. It uses only the branch endpoints; logarithmic derivative control is reserved for the distortion of successive compositions.

8.6. **Successor graphs.** The image of a retained era is a radial graph over a closed interval of next radii. To repeat the same construction, we extend its two small walls to all radii while preserving their positivity and slope bounds. The superlinear amplitude estimate leaves enough room for this extension.

The cone inequality becomes especially simple after taking square roots: $|U'| \leq \theta\sqrt{U}$ is equivalent to $|(\sqrt{U})'| \leq \theta/2$ when $U > 0$. This gives an explicit extension with a controlled amplitude increase.

Lemma 8.13. *Let $l < r$ and $0 \leq \theta \leq 1$. Suppose u, w are positive C^1 functions on $[l, r]$, with continuous one-sided derivatives at its endpoints, and*

$$u + w \leq A, \qquad |u'| \leq \theta\sqrt{u}, \qquad |w'| \leq \theta\sqrt{w}. \tag{8.54}$$

They admit positive C^1 extensions $\widetilde{u}, \widetilde{w}$ to $\mathbb{R}$, agreeing with both values and derivatives on $[l, r]$, such that

$$\widetilde{u} + \widetilde{w} \le \frac{25}{16} A, \qquad |\widetilde{u}'| \le \theta\sqrt{\widetilde{u}}, \qquad |\widetilde{w}'| \le \theta\sqrt{\widetilde{w}}. \tag{8.55}$$

Proof. Define

$$c(t) = \max(1 - t, 0), \qquad H(t) = \int_0^t c(s)\, ds.$$

The function H is C^1, with $H' = c$ and $H(0) = 0$. For $t \ge 0$,

$$H(t) = \begin{cases} t - t^2/2, & 0 \le t \le 1, \\ 1/2, & t \ge 1, \end{cases} \qquad 0 \le H(t) \le \tfrac{1}{2}, \quad 0 \le c(t) \le 1.$$

Given an endpoint value $s > 0$, a derivative d, and an outward orientation $e \in \{-1, 1\}$, put

$$E_{s,d,e}(x) = s + eds\, H(ex/s). \tag{8.56}$$

Here x is the signed displacement from the endpoint, so $ex \ge 0$ on the exterior half-line. Direct differentiation gives

$$E_{s,d,e}(0) = s, \qquad E'_{s,d,e}(x) = d\, c(ex/s), \qquad E'_{s,d,e}(0) = d. \tag{8.57}$$

If $|d| \le \theta/2$, then on this half-line

$$(1 - \theta/4)s \le E_{s,d,e}(x) \le (1 + \theta/4)s, \qquad |E'_{s,d,e}(x)| \le \theta/2. \tag{8.58}$$

Indeed, $|edsH(ex/s)| \le \theta s/4$. The lower bound is positive because $\theta \le 1$.

Apply this construction to $f = \sqrt{u}$. On $[l, r]$ its derivative is the continuous function $f' = u'/(2\sqrt{u})$, whose absolute value is at most $\theta/2$. Set

$$\widetilde{f}(t) = \begin{cases} E_{f(l),f'(l),-1}(t - l), & t < l, \\ f(t), & l \le t \le r, \\ E_{f(r),f'(r),1}(t - r), & r < t. \end{cases} \tag{8.59}$$

The values and one-sided derivatives agree at both joining points by (8.57). The derivative is continuous there, since c is continuous and $c(0) = 1$. Thus $\widetilde{f}$ is positive and C^1 on $\mathbb{R}$, with $|\widetilde{f}'| \le \theta/2$. Use the same formula for $g = \sqrt{w}$, obtaining $\widetilde{g}$.

Define $\widetilde{u} = \widetilde{f}^2$ and $\widetilde{w} = \widetilde{g}^2$. Squaring preserves their values on $[l, r]$, and $2ff' = u'$, $2gg' = w'$ there, including at the endpoints. Everywhere on $\mathbb{R}$,

$$|\widetilde{u}'| = 2\widetilde{f}|\widetilde{f}'| \le \theta\widetilde{f} = \theta\sqrt{\widetilde{u}},$$

and the same holds for $\widetilde{w}$. On each exterior half-line, the two upper bounds in (8.58) refer to the same endpoint. Therefore

$$\widetilde{u} + \widetilde{w} \le (1 + \theta/4)^2 A \le \frac{25}{16} A.$$

On $[l, r]$ the original bound is stronger. □

Use the input interval $I_m = [a_-, a_+]$ of (8.36). Along its graph, write

$$k(r) = \kappa_m^+(\Gamma(r)), \qquad u_0(r) = U_m^+(\Gamma(r)), \qquad w_0(r) = W_m^+(\Gamma(r)). \tag{8.60}$$

The next bounded phase is $\varphi_m(r) = 2/k(r)$. The retained phase interval corresponds to

$$L_\beta = [l_\beta, r_\beta], \qquad l_\beta = \frac{2}{1 - 2\beta}, \qquad r_\beta = \frac{1}{\beta} \tag{8.61}$$

in the next radius. Let ξ_m be the inverse of k on its full interval image, and reparametrize the two wall functions there by

$$u(t) = u_0(\xi_m(t)), \qquad w(t) = w_0(\xi_m(t)). \tag{8.62}$$

The next proposition verifies these definitions on the whole interval needed for the retained phases.

Proposition 8.14. *For $J \geq 20$, the constants ε, C in the interval and output estimates can be chosen so that every graph satisfying (8.37)–(8.38) has positive C^1 extensions U^+, W^+ of the functions (8.62) on L_β to all of $\mathbb{R}$. These extensions are given by the construction in Lemma 8.13. With $\theta = CB^{7/24} < 1/2$, they satisfy*

$$U^+ + W^+ \leq B^{4/3}, \qquad |(U^+)'| \leq \theta\sqrt{U^+}, \qquad |(W^+)'| \leq \theta\sqrt{W^+}. \tag{8.63}$$

For every $y \in K_\beta$,

$$k(\zeta_m(y)) = 2/y, \qquad U^+(2/y) = u_0(\zeta_m(y)), \quad W^+(2/y) = w_0(\zeta_m(y)),$$

$$(U^+)'(2/y) = \frac{u_0'(\zeta_m(y))}{k'(\zeta_m(y))}, \qquad (W^+)'(2/y) = \frac{w_0'(\zeta_m(y))}{k'(\zeta_m(y))}. \tag{8.64}$$

Proof. Take common constants for Theorems 8.8 and 8.9, enlarging C if necessary. Shrink ε so that

$$C\varepsilon^{7/24} < \tfrac{1}{2}, \qquad \frac{25}{16}C\varepsilon^{17/30} \leq 1.$$

The hypotheses imply $B \leq \varepsilon$. At every $r \in I_m$, the graph tangent is a normalized radial cone vector. The chain rule and Theorem 8.8 therefore give

$$k'(r) < 0, \qquad u_0(r), w_0(r) > 0, \qquad u_0(r) + w_0(r) \leq CB^{19/10},$$
$$|u_0'(r)| \leq \theta\sqrt{u_0(r)}\,|k'(r)|, \qquad |w_0'(r)| \leq \theta\sqrt{w_0(r)}\,|k'(r)|.$$

All three functions are locally C^1 on I_m. The function k is positive there, and is strictly decreasing by the mean value theorem. Consequently it has an inverse ξ_m on $[k(a_+), k(a_-)]$.

The strict phase endpoint bounds (8.39), together with $\varphi_m = 2/k$, give

$$k(a_+) < l_\beta < r_\beta < k(a_-).$$

The middle inequality follows from $\beta < 1/4$. In particular, L_β lies strictly inside the image of k. At every point of this window, its interval inverse agrees near that point with the ordinary local inverse, by injectivity of k. Thus ξ_m is C^1 near L_β and

$$\xi_m'(t) = \frac{1}{k'(\xi_m(t))}.$$

The functions in (8.62) are positive and C^1 there, with

$$u'(t) = \frac{u_0'(\xi_m(t))}{k'(\xi_m(t))}, \qquad w'(t) = \frac{w_0'(\xi_m(t))}{k'(\xi_m(t))},$$

$$u(t) + w(t) \leq CB^{19/10}, \qquad |u'(t)| \leq \theta\sqrt{u(t)}, \quad |w'(t)| \leq \theta\sqrt{w(t)}. \tag{8.65}$$

Their derivatives are continuous through the endpoints of the window.

Apply Lemma 8.13 on L_β, with $A = CB^{19/10}$, to define U^+, W^+. It preserves the cone constant θ and gives

$$U^+ + W^+ \leq \frac{25}{16}CB^{19/10} = \left(\frac{25}{16}CB^{17/30}\right)B^{4/3} \leq B^{4/3}.$$

This proves (8.63).

Finally, $\varphi_m(\zeta_m(y)) = y > 0$ gives $k(\zeta_m(y)) = 2/y$, and $y \in K_\beta$ implies $2/y \in L_\beta$. Uniqueness of the radius inverse then gives $\xi_m(2/y) = \zeta_m(y)$. The extension preserves both values and derivatives on the closed window, so (8.65) yields all the identities in (8.64). □

We can now combine the retained phase intervals with the successor graphs. The parameters η, J, σ remain fixed as in (8.43).

Theorem 8.15. *There are $C \geq 1$ and $B_* \in (0,1)$ such that every graph Γ admissible at amplitude $0 < B \leq B_*$ has the following properties. Set $\beta = B^\sigma$ and $N = \lfloor 1/\beta \rfloor$. The retained inverse branches h_m, $0 \leq m < N$, satisfy* (8.47)*; their open images are disjoint and omit length at most $10B^\sigma$ from $(0,1)$.*

For each such m there is a successor graph

$$\Gamma_m^+(t) = (t, \widehat{U}_m(t), \widehat{W}_m(t))$$

admissible at amplitude $B^{4/3}$, defined by the extension in Proposition 8.14. For $y \in K_\beta$, the actual era segment starting at $\Gamma(2/h_m(y))$ has next core entrance $\Gamma_m^+(2/y)$ in core coordinates. It consists of $m+1$ successive cores, m reflections, and the following terminal connection. The two output wall derivatives also satisfy (8.64)*.*

Proof. Take the constants of Theorem 8.12 and Proposition 8.14, and enlarge C to dominate both. Apply Lemma 8.10 to the latter proposition's input constants. Intersect the resulting amplitude threshold with that of the retained-phase theorem. All choices are independent of the graph and of $m < N$.

The retained-phase theorem gives the branches, disjointness, and omitted-length estimate. For each branch, the parameter lemma verifies the interval conditions for the extension proposition. Its positive global extensions satisfy $\widehat{U}_m + \widehat{W}_m \leq B^{4/3}$ and the square-root slope bounds with a coefficient smaller than $1/2$. They therefore define an admissible successor graph. Since $2/h_m(y) = \zeta_m(y)$, the value and derivative identities (8.64) identify this graph with the actual output for every retained y. The count and order of the intervening events are those of Theorem 8.4. □

The successor graph agrees with the actual era image over L_β. Its extension supplies a common radius domain for choosing the following branches. The intersections in (3.22) retain only the parts that are images of the preceding actual segments.

8.7. **The selected return on the space of entrance states.** The estimates along a single graph will ultimately be integrated over families of initial data. For this purpose the statistical set must be defined on the space of entrance states, so that its intersections with different graphs describe the same condition. We first express the branch selection in these coordinates. The only additional coordinate is an upper bound for the transverse amplitude.

Fix η, J, σ in the range of Theorem 8.15. Let $B_{\mathrm{cap}} \in (0,1)$ be a sufficiently small constant, to be chosen below. Write $p = (b, X) = (b, r, u, w) \in \mathbb{R} \times \mathbb{R}^3$. For a positive integer a, define the entrance set

$$\begin{aligned} \mathcal{U}_a = \{(b,r,u,w) : 0 < b \leq B_{\mathrm{cap}}, \quad & a - 1 < \lfloor b^{-\sigma} \rfloor, \\ & u, w > 0, \quad u + w \leq b, \\ & 2(a + b^\sigma) \leq r \leq 2(a + 1 - b^\sigma)\}. \end{aligned} \tag{8.66}$$

The index a counts the cores in the return. In the notation of (8.32), put

$$\begin{aligned} \mathcal{R}_a(b, X) &= (b^{4/3}, (\kappa_{a-1}^+(X), U_{a-1}^+(X), W_{a-1}^+(X))), \\ \mathcal{V}_a &= \{(b, X) \in \mathcal{U}_a : 2b^\sigma < 2/\kappa_{a-1}^+(X) < 1 - 2b^\sigma\}. \end{aligned} \tag{8.67}$$

Only the restriction of $\mathcal{R}_a$ to $\mathcal{V}_a$ will be iterated. The amplitude bound is updated by $b \mapsto b^{4/3}$, while the other three coordinates are those of the actual next core entrance.

Proposition 8.16. *The constant B_{cap} can be chosen so that all $\mathcal{U}_a$ and $\mathcal{V}_a$ are Borel, the sets $\mathcal{V}_a$ are pairwise disjoint, and $\mathcal{R}_a : \mathcal{V}_a \to \mathbb{R} \times \mathbb{R}^3$ is Borel. At every $(b, X) \in \mathcal{U}_a$, the three output coordinates in* (8.67) *are C^1 functions of X in an ordinary neighborhood of X.*

Proof. Choose the common constants ε, C of the phase and output estimates, and apply Lemma 8.10. For $(b, r, u, w) \in \mathcal{U}_a$, with $\beta = b^\sigma$, it gives

$$br \le 2b(a + 1 - \beta) \le \varepsilon, \qquad u + w \le b \le \varepsilon(\beta/2)^{2J}, \qquad C(u + w) \le Cb < \beta/4.$$

Here $a \le \lfloor 1/\beta \rfloor \le 1/\beta$. Since $r > 2$, each wall is at most ε/r. Thus Theorems 8.4 and 8.8 apply at every point of $\mathcal{U}_a$, and give the asserted local regularity and the actual return.

The inequalities in (8.66) are Borel conditions: real powers are continuous for $b > 0$, and the integer part is a Borel function. On the Borel subspace $\mathcal{U}_a$, the output state is continuous, by the local regularity just proved. Its positive radius therefore gives a continuous output phase there. The two strict phase inequalities define a Borel subset $\mathcal{V}_a$ of $\mathcal{U}_a$, hence a Borel subset of the ambient space. The same local regularity and continuity of $b^{4/3}$ make the restricted return Borel.

Finally, two different indices cannot satisfy the radial inequalities in (8.66) at the same point. If $a < c$, then $c \ge a + 1$ and

$$r/2 \le a + 1 - b^\sigma < c + b^\sigma \le r/2,$$

a contradiction. The entrance sets, and consequently their selected subsets, are disjoint. □

The selection can be read equally from the actual return or from an inverse branch. This equality will identify the histories in state space with the cells used in the measure estimates.

Lemma 8.17. *After decreasing B_{cap} if necessary, let $0 < B \le B_{\mathrm{cap}}$ and let Γ be admissible at amplitude B. Set $\beta = B^\sigma$ and suppose $a - 1 < \lfloor 1/\beta \rfloor$. For $x \in (0, 1)$,*

$$(B, \Gamma(2/x)) \in \mathcal{V}_a \quad \Longleftrightarrow \quad x \in h_{a-1}\big((2\beta, 1 - 2\beta)\big), \tag{8.68}$$

where h_{a-1} is the branch of (8.46) *for this graph.*

Proof. Write $\phi = P_{a-1} \circ \Gamma$ and let ζ be its inverse from Theorem 8.9. If the state on the left is selected, put $y = \phi(2/x)$. Then $y \in (2\beta, 1 - 2\beta)$ and $2/x$ belongs to the prescribed closed radial interval. Uniqueness of the inverse on that interval gives $\zeta(y) = 2/x$, whence $h_{a-1}(y) = x$.

Conversely, if $x = h_{a-1}(y)$ with $2\beta < y < 1 - 2\beta$, then $2/x = \zeta(y)$ lies inside the radial interval and $\phi(2/x) = y$. Admissibility of Γ supplies the two positive walls and their sum bound. These are precisely the entrance and phase conditions in (8.67). □

For a word $\mathbf{a} = (a_0, \dots, a_{n-1})$, let $\mathcal{D}_{\mathbf{a}}$ be the set of states on which these selected returns can be made, and let $F_{\mathbf{a}}$ be their composition. More precisely, set

$$\begin{aligned} \mathcal{D}_\varnothing &= \mathbb{R} \times \mathbb{R}^3, & F_\varnothing &= \mathrm{id}, \\ \mathcal{D}_{\mathbf{a}a} &= \{p \in \mathcal{D}_{\mathbf{a}} : F_{\mathbf{a}}(p) \in \mathcal{V}_a\}, & F_{\mathbf{a}a} &= \mathcal{R}_a \circ F_{\mathbf{a}} \quad \text{on } \mathcal{D}_{\mathbf{a}a}. \end{aligned} \tag{8.69}$$

Thus every composition is evaluated only where the successive returns are selected. For any two words $\mathbf{a}, \mathbf{b}$,

$$\begin{aligned} \mathcal{D}_{\mathbf{ab}} &= \{p \in \mathcal{D}_{\mathbf{a}} : F_{\mathbf{a}}(p) \in \mathcal{D}_{\mathbf{b}}\}, \\ F_{\mathbf{ab}}(p) &= F_{\mathbf{b}}(F_{\mathbf{a}}(p)). \end{aligned} \tag{8.70}$$

Both identities follow by induction on the second word. The first coordinate of a composition of length n is

$$\big(F_{\mathbf{a}}(b, X)\big)_b = b^{(4/3)^n} \qquad ((b, X) \in \mathcal{D}_{\mathbf{a}},\ b \ge 0). \tag{8.71}$$

Define the set of indefinitely continuing states by

$$\mathcal{S} = \bigcap_{n \ge 0} \bigcup_{|\mathbf{a}| = n} \mathcal{D}_{\mathbf{a}}. \tag{8.72}$$

The amplitude coordinate is part of this definition. In particular, it specifies the increasingly small phase margins used in all later returns.

Proposition 8.18. *Every $\mathcal{D}_{\mathbf{a}}$ is Borel and $F_{\mathbf{a}} : \mathcal{D}_{\mathbf{a}} \to \mathbb{R} \times \mathbb{R}^3$ is Borel. The set $\mathcal{S}$ is Borel. Each $p \in \mathcal{S}$ has a unique sequence of positive integers $\mathbf{a}(p) = (a_n(p))_{n\geq 0}$ whose prefix of length n is an allowed history of p. The following maps on $\mathcal{S}$ are Borel:*

$$p \longmapsto \mathbf{a}(p), \qquad \chi(p) = c(\mathbf{a}(p)), \qquad p_n = F_{(a_0(p),\dots,a_{n-1}(p))}(p), \qquad x_n(p) = \frac{2}{(p_n)_r}.$$

They satisfy

$$p_0 = p, \qquad p_n \in \mathcal{V}_{a_n(p)}, \qquad p_{n+1} = \mathcal{R}_{a_n(p)}(p_n). \tag{8.73}$$

If $p = (b, X)$, the first coordinate of p_n is $b^{(4/3)^n}$ and $0 < x_n(p) < 1$.

Proof. Suppose the Borel assertions hold for $\mathbf{a}$. The inverse image of $\mathcal{V}_a$ under the restricted Borel map $F_{\mathbf{a}}$ is Borel in $\mathcal{D}_{\mathbf{a}}$, hence in the ambient space. On this inverse image $F_{\mathbf{a}}$ takes its values in $\mathcal{V}_a$, so composition with the Borel restriction of $\mathcal{R}_a$ proves the assertions for $\mathbf{a}a$. Starting with the empty word proves the induction. There are countably many finite words, so (8.72) is Borel.

Two allowed words of the same length must be equal. For nonempty words, disjointness of the $\mathcal{V}_a$ gives equality of their first digits; apply the same argument to the returned state and the remaining words. A state in $\mathcal{S}$ has a word at every length, and uniqueness makes these words compatible under truncation. This defines $\mathbf{a}(p)$ and proves (8.73).

For a word $\mathbf{b}$ of length n, its prefix fiber is

$$\{p \in \mathcal{S} : (a_0(p), \dots, a_{n-1}(p)) = \mathbf{b}\} = \mathcal{S} \cap \mathcal{D}_{\mathbf{b}}. \tag{8.74}$$

In particular the fiber of the nth digit at a is the countable union of $\mathcal{S} \cap \mathcal{D}_{\mathbf{b}a}$ over $|\mathbf{b}| = n$. This proves measurability of every digit and hence of the sequence in the product of the discrete digit spaces. The continued fraction coding map c is Borel, so χ is Borel.

To prove measurability of p_n, extend each restricted map $F_{\mathbf{b}}$ measurably to the ambient space, for example by a constant off its Borel domain. The word of length n is a Borel map into a countable discrete space by (8.74). Evaluating the corresponding extension gives a Borel map, and on $\mathcal{S}$ its value is p_n, independently of the extension. The amplitude identity follows from (8.71). Since $p_n \in \mathcal{V}_{a_n(p)}$, its radius is greater than 2. Taking its reciprocal proves the last assertion and the Borelness of x_n. □

Let $\mathcal{B} \subset \mathbb{R}$ be the common Gauss typical set in (3.19). We shall use the single Borel set

$$\mathcal{S}_{\mathrm{stat}} = \{p \in \mathcal{S} : \chi(p) \in \mathcal{B}\}. \tag{8.75}$$

It depends on $\eta, J, \sigma, B_{\mathrm{cap}}$, but not on a choice of initial graph. We now estimate its intersections with all admissible graphs by identifying their finite histories.

8.8. **Iteration and statistics on entrance graphs.** The rapid decrease of the transverse amplitude makes the errors of all later returns summable. The graphs constructed in Section 8.6 let us apply the same return estimate after every finite history. Their inverse branches then fit the measure argument of Section 3.3.

Fix an admissible graph Γ at amplitude B, where B is smaller than the thresholds of Theorem 8.15 and Section 8.7. Use the amplitude sequence (3.38), and put

$$\beta_n = B_n^{\sigma}, \qquad N_n = \lfloor 1/\beta_n \rfloor. \tag{8.76}$$

Define $\Gamma_{\varnothing} = \Gamma$. If $|\mathbf{a}| = n$ and $1 \leq a \leq N_n$, let $\Gamma_{\mathbf{a}a}$ be the successor graph for the branch with a cores, as in Theorem 8.15. For the other digits take

$$\Gamma_{\mathbf{a}a}(r) = (r, B_{n+1}/2, B_{n+1}/2), \qquad r > 2.$$

The latter graphs are admissible at amplitude B_{n+1}; the corresponding branches will have empty domains. Induction proves that every $\Gamma_{\mathbf{a}}$ is admissible at amplitude $B_{|\mathbf{a}|}$. Indeed $0 < B_n \le B < 1$, so the same smallness threshold applies at every depth.

For $|\mathbf{a}| = n$ define

$$D_{\mathbf{a},a} = \begin{cases} (2\beta_n, 1-2\beta_n), & 1 \le a \le N_n, \\ \varnothing, & a > N_n. \end{cases} \tag{8.77}$$

On a nonempty domain, let $h_{\mathbf{a},a}$ be the inverse phase branch of $\Gamma_{\mathbf{a}}$ with a cores. Empty domains contribute no image or constraint. Theorem 8.15 gives, for $y \in D_{\mathbf{a},a}$,

$$\begin{aligned} |h_{\mathbf{a},a}(y) - g_a(y)| &\le CB_n/a^2 \le CB_n, \\ \left|\log \frac{|h'_{\mathbf{a},a}(y)|}{|g'_a(y)|}\right| &\le 5C\sqrt{B_n}. \end{aligned} \tag{8.78}$$

Each branch is C^1 with negative derivative, its image lies in $(1/(a+1), 1/a)$, and images with different digits are disjoint. Thus these are branches of the kind considered in Section 3.3, with root interval $(0,1)$. Use $K_{\mathbf{a}}, H_{\mathbf{a}}, C_{\mathbf{a}}$ and S for their terminal domains, compositions, cells and surviving set, as defined there.

The excluded terminal set $A_{\mathbf{a}}$ in (3.32) is a subset of $(0,1) \setminus \mathcal{V}_{\Gamma_{\mathbf{a}}}(B_n)$. Indeed the union of the images of the nonempty domains in (8.77) is precisely the retained set (8.50) for $\Gamma_{\mathbf{a}}$. Theorem 8.12 therefore gives

$$\lambda(A_{\mathbf{a}}) \le 10 B_{|\mathbf{a}|}^{\sigma}. \tag{8.79}$$

The intersections in (3.22) remain part of this construction: a successor graph contributes only those points whose entire preceding history lies in the actual branch domains.

For $x \in (0,1)$, set

$$Z_{\mathbf{a}}(x) = (B_{|\mathbf{a}|}, \Gamma_{\mathbf{a}}(2/x)). \tag{8.80}$$

The successor matching identities give the following two relations:

$$\begin{aligned} \mathcal{R}_a(Z_{\mathbf{a}}(h_{\mathbf{a},a}(y))) &= Z_{\mathbf{a}a}(y) && (y \in D_{\mathbf{a},a}), \\ Z_{\mathbf{a}}(x) \in \mathcal{V}_a &\iff x \in h_{\mathbf{a},a}(D_{\mathbf{a},a}) && (x \in (0,1)). \end{aligned} \tag{8.81}$$

The first is Theorem 8.15, including the identity $B_{n+1} = B_n^{4/3}$ in the amplitude coordinate. For a retained digit, the second is Lemma 8.17. For an unretained digit both sides are false, by the digit condition in (8.66) and the empty branch domain.

Proposition 8.19. *For every finite word* $\mathbf{a}$ *and* $y \in K_{\mathbf{a}}$,

$$F_{\mathbf{a}}(Z_{\varnothing}(H_{\mathbf{a}}(y))) = Z_{\mathbf{a}}(y). \tag{8.82}$$

For $x \in (0,1)$,

$$\begin{aligned} Z_{\varnothing}(x) \in \mathcal{D}_{\mathbf{a}} &\iff x \in C_{\mathbf{a}}, \\ Z_{\varnothing}(x) \in \mathcal{S} &\iff x \in S. \end{aligned} \tag{8.83}$$

On these corresponding surviving sets, the digits, continued fraction codes and phases agree.

Proof. The prefix identity is immediate for the empty word. Suppose it holds for $\mathbf{a}$. If $y \in K_{\mathbf{a}a}$, then $y \in D_{\mathbf{a},a}$ and $h_{\mathbf{a},a}(y) \in K_{\mathbf{a}}$. Since $H_{\mathbf{a}a} = H_{\mathbf{a}} \circ h_{\mathbf{a},a}$, the induction hypothesis identifies the state after $\mathbf{a}$ with $Z_{\mathbf{a}}(h_{\mathbf{a},a}(y))$. The second identity in (8.81) says that this state belongs to $\mathcal{V}_a$, so the next composition is defined. The first identity then gives

$$\begin{aligned} F_{\mathbf{a}a}(Z_{\varnothing}(H_{\mathbf{a}a}(y))) &= \mathcal{R}_a(Z_{\mathbf{a}}(h_{\mathbf{a},a}(y))) \\ &= Z_{\mathbf{a}a}(y). \end{aligned}$$

We prove the first equivalence in (8.83) by the same induction. It holds for the empty word. If $Z_{\varnothing}(x) \in \mathcal{D}_{\mathbf{a}}$, the induction hypothesis gives a unique $z \in K_{\mathbf{a}}$ with $H_{\mathbf{a}}(z) = x$. The

prefix identity then says $F_{\mathbf{a}}(Z_\varnothing(x)) = Z_{\mathbf{a}}(z)$. The next digit a is selected exactly when $z \in h_{\mathbf{a},a}(D_{\mathbf{a},a})$. Writing $z = h_{\mathbf{a},a}(y)$, this is exactly the condition $y \in K_{\mathbf{a}a}$ and $x = H_{\mathbf{a}a}(y)$. This proves the equivalence for $\mathbf{a}a$ in both directions. Taking the union at each depth and then the intersection over depths gives the second equivalence.

Let $x \in S$ and put $p = Z_\varnothing(x)$. The unique allowed word of length n for p is the word of the unique cell of depth n containing x. Their digits and codes are therefore equal. For that word $\mathbf{a}$, let $y_n(x) \in K_{\mathbf{a}}$ satisfy $H_{\mathbf{a}}(y_n(x)) = x$. Equation (8.82) gives

$$p_n = \big(B_n, \Gamma_{\mathbf{a}}(2/y_n(x))\big).$$

Its radial coordinate is $2/y_n(x) > 2$, so $x_n(p) = 2/(2/y_n(x)) = y_n(x)$, as required. □

For fixed B, Γ, define

$$G_{\Gamma,B} = \{x \in (0,1) : (B, \Gamma(2/x)) \in \mathcal{S}_{\mathrm{stat}}\}. \tag{8.84}$$

The equality of codes and surviving sets in Proposition 8.19 gives the exact identity

$$G_{\Gamma,B} = \{x \in S : \psi(x) \in \mathcal{B}\}. \tag{8.85}$$

Here ψ is the code associated with the inverse branches, as in Section 3.3. Thus the set on which the branch measure argument gives statistics is the restriction of the same Borel set of entrance states for every graph.

Theorem 8.20 (Statistics on entrance graphs)**.** *For the fixed parameters η, J, σ, there are constants $0 < B_* \le B_{\mathrm{cap}} < 1$ with the following properties. The set $\mathcal{S}_{\mathrm{stat}}$ is Borel. For every $0 < B \le B_*$ and every graph Γ admissible at amplitude B, $G_{\Gamma,B}$ is Borel,*

$$\lambda(G_{\Gamma,B}) > \frac{1}{2}, \qquad \lambda\big((I \cap (0,1)) \setminus G_{\Gamma,B}\big) \le \min\{\lambda(I), 80B^\sigma\} \tag{8.86}$$

for every Borel set $I \subset \mathbb{R}$. For every $x \in G_{\Gamma,B}$, the selected returns starting at $p = (B, \Gamma(2/x))$ continue indefinitely, and all six limits of Theorem 2.4 hold for their digits $a_n(p)$ and phases $x_n(p)$.

Proof. First choose B_{cap} for Proposition 8.16 and Lemma 8.17. This fixes the Borel set $\mathcal{S}_{\mathrm{stat}}$. Choose $B_* > 0$ no greater than B_{cap} and the graph iteration threshold, and small enough that

$$B^\sigma < \frac{1}{160}, \qquad 10C\sqrt{B} + 4CB^{4/3} \le \log 2 \quad (0 < B \le B_*).$$

This is possible since all exponents are positive. The geometric assumption $2J\sigma < 1$, with $J \ge 20$, implies $0 < \sigma < 1/2$.

Equations (8.78) and (8.79) are the hypotheses of Corollary 3.15, with exponent σ, $C_v = C$ and $C_i = 5C$. In particular,

$$\sum_{n\ge 0} 5C\sqrt{B_n} + 2\sum_{n\ge 1} CB_n \le 10C\sqrt{B} + 4CB^{4/3} \le \log 2.$$

The coding measure is bounded by $2\lambda|_{(0,1)}$, and the total statistical loss is at most $80B^\sigma < 1/2$. Together with (8.85), these facts give the Borel assertion and (8.86).

For $x \in G_{\Gamma,B}$, the same corollary gives all six limits for the digits and phases of the inverse tree. Proposition 8.19 identifies them with the actual selected digits and phases. It also supplies each finite prefix of the sequence of returns. □

In particular the loss tends to zero with the initial amplitude, uniformly over the admissible graph. The definition (8.75) retains the dependence on the entrance state and its amplitude bound; this will allow the graph estimates to be integrated over transverse families. Before doing so, we relate the indefinitely continued returns to the original normalized time and verify the ordering of their constituent events.

8.9. **Successive complete eras in the original time.** The selected histories describe successive exact return states. To interpret their digits as the lengths of complete eras, we now follow the intervening events in the original normalized time t. The finite return times add, with the axis frame changed at each connection. For an infinite history these sums tend to infinity: the shrinking transverse amplitude forces a positive, continuous curvature product to tend to zero along the arrivals, which cannot happen at a finite time.

Let $\mathcal{N}$ denote the inverse projective map in (2.7), and set

$$\mathcal{E}_{\pm}(X) = \mathcal{N}(\iota_{\pm}(X)).$$

These are normalized states on the upward and downward core sections. We retain F, G, T_c, T_r, T_G from Section 8.1, and $T, \mathcal{Y}$ for the terminal time and state from Section 8.2. All times in this subsection are measured in t.

For an input $X = (r, u, w)$ whose return has $m+1$ cores, define, for $0 \le j \le m$,

$$\begin{aligned} X_j &= G^j(X), & Z_j &= F(X_j), \\ s_j(X) &= \sum_{i=0}^{j-1} T_G(X_i), & e_j(X) &= s_j(X) + T_c(X_j), \\ \Theta_m(X) &= e_m(X) + T(Z_m). \end{aligned} \tag{8.87}$$

Thus s_j and e_j are the entrance and exit times of the jth core, measured from the first entrance. The total time Θ_m includes the terminal connection. Define also the next core coordinates

$$X^+ = (\kappa_m^+(X), U_m^+(X), W_m^+(X)). \tag{8.88}$$

Proposition 8.21. *The common thresholds can be chosen so that, whenever $(b, X) \in \mathcal{U}_{m+1}$, the following statements hold. All times in* (8.87) *are defined, $\Theta_m(X) > 0$, and Θ_m is C^1 near X. For $0 \le j \le m$,*

$$\begin{aligned} \mathcal{E}_+(X_j) &= \mathsf{S}^j \phi_{s_j(X)} \mathcal{E}_+(X), \\ \mathcal{E}_-(Z_j) &= \mathsf{S}^j \phi_{e_j(X)} \mathcal{E}_+(X), \\ \mathcal{E}_+(X^+) &= \mathsf{CS}^m \phi_{\Theta_m(X)} \mathcal{E}_+(X). \end{aligned} \tag{8.89}$$

In addition,

$$\begin{gathered} s_j < e_j \quad (0 \le j \le m), \qquad e_j < s_{j+1} \quad (0 \le j < m), \qquad e_m < \Theta_m, \\ \#\{s_0, \dots, s_m\} = m + 1 = \lfloor r/2 \rfloor. \end{gathered} \tag{8.90}$$

At s_j the active wall in frame S^j crosses $V = 1$ upward; e_j is its first subsequent $V = 1$ event and the crossing there is downward. If $j < m$, the exit radius is greater than 4, and s_{j+1} is the first subsequent $V = 1$ event in frame S^{j+1}, with upward crossing. At $j = m$ the exit radius lies in $(2, 4)$, and Θ_m is the first subsequent $V = 1$ event in frame CS^m, also with upward crossing. Finally,

$$\begin{gathered} \iota_+(X^+) = \mathcal{Y}(Z_m), \qquad \kappa_m^+(X) > 2, \\ U_m^+(X), W_m^+(X) > 0, \qquad U_m^+(X) + W_m^+(X) \le b^{4/3}. \end{gathered} \tag{8.91}$$

Proof. The parameter choice used in Proposition 8.16 places every point of $\mathcal{U}_{m+1}$ in the common domain of the core itinerary, phase and output estimates. We make that choice with the common constants of the core and reflection estimates as well. In particular the positive wall sums at every entrance and exit are at most b, all entrances have radius greater than 2, all nonterminal exits have radius greater than 4, and the last exit has radius in $(2, 4)$.

Write $Z_j = (R_j, U_j, W_j)$. To obtain regularity of the reflection times, let ε_r be the smallness constant in Theorem 6.33. Shrink the common entrance constant to $\varepsilon \le \varepsilon_r/4$.

The itinerary estimates give $R_j \le 2r$ and $U_j, W_j \le u + w$, whereas $(u + w)r \le 2\varepsilon$. Consequently

$$U_j R_j,\ W_j R_j \le 4\varepsilon \le \varepsilon_r.$$

Thus T_r is locally C^1 at every nonterminal exit. The core times, the maps $F \circ G^j$, and the composed core times are locally C^1 by Proposition 8.3. Theorem 8.4 gives the same regularity for $T \circ \mathcal{H}_m$. Finite sums and compositions now prove the assertion for Θ_m.

Each entrance is an upward crossing. Indeed, at $r \ge 2$ and $0 \le u, w \le 1/4$, the projective rate is

$$\dot{V} = \sqrt{r^2 - 1} - u(1 + w - u) > 0,$$

since $\sqrt{r^2 - 1} \ge 1$ and $u(1 + w) \le 5/16$. The projective clock is positive, so this is also an upward crossing in t. The first downward core return and its identification with $\iota_-(Z_j)$ were proved in Lemma 8.1. The reflection theorem gives the first arrival at the old wall $W = 1$; the swap S makes this the new active wall V. Its endpoint is the next upward core entrance. The terminal theorem gives the corresponding first upward event after the cycle C.

For the terminal endpoint, Theorem 8.8 and a further decrease of B_{cap} give

$$U_m^+ + W_m^+ \le C b^{19/10} \le b^{4/3} \le \tfrac{1}{4}.$$

The exponent difference is $19/10 - 4/3 = 17/30 > 0$. At a unit active wall with two nonnegative walls at most $1/4$,

$$D = 3 + M^2 + u^2 + (1 - w)^2 - 2u(1 + w) > 0,$$

because $u(1 + w) \le 5/16$. The normalized constraint therefore reconstructs the physical state by (2.7) with positive clock. Upward crossing implies $M - 1 > u(1 + w - u) \ge 0$. Hence $M = 1 + \sqrt{\kappa^2 - 1}$, which proves the first identity in (8.91). The positive walls and $\kappa > 2$ follow from the output and phase estimates.

It remains to place these local events on one normalized orbit. The flow law and its equivariance under axis permutations give

$$\phi_t(\mathsf{A}\phi_s q) = \mathsf{A}\phi_{s+t} q \qquad (s, t \ge 0) \tag{8.92}$$

for any ordered frame A. The initial state is in $\mathcal{M}_{+++}$, so Lemma 2.1 makes all the times in this identity available. Induction using the core-reflection endpoint identity (8.4) gives the entrance formula in (8.89). The core return gives its exit formula. Applying the terminal event at the last exit, and then (8.91), gives the final formula.

Restarting also preserves first arrivals. If T_0 is the first positive time at which the orbit through $y = \phi_s q$ meets a specified section, then $s + T_0$ is its first meeting after s on the orbit through q: an earlier meeting would, after subtracting s, contradict the definition of T_0. The event derivatives agree by (8.92). Apply this observation at each entrance and exit in the appropriate fixed frame. It proves all the stated first-event assertions.

Each core and each connection has positive duration. Thus $e_j - s_j = T_c(X_j) > 0$, $s_{j+1} - e_j = T_r(Z_j) > 0$, and $\Theta_m - e_m = T(Z_m) > 0$. The entrance times are distinct, so their number is $m + 1$. The identity with $\lfloor r/2 \rfloor$ follows from the entrance margin in (8.66). Since every earlier exit has radius greater than 4, the terminal core is the first one with exit radius in $(2, 4)$. □

The identities (8.89) use the normalized flow through each change of frame. The projective coordinates are needed only on the individual segments where their clock is positive.

Let $p \in \mathcal{S}$, and write its successive states from (8.73) as $p_n = (B_n, X_n)$, with $p_0 = (B, X_0)$ and digits $a_n = a_n(p)$. Define

$$d_n(p) = \Theta_{a_n - 1}(X_n), \qquad t_n(p) = \sum_{j=0}^{n-1} d_j(p),$$

$$\mathsf{A}_0(p) = \mathrm{id}, \qquad \mathsf{A}_{n+1}(p) = \mathsf{CS}^{a_n - 1}\mathsf{A}_n(p). \tag{8.93}$$

The frame maps take values in the six permutations generated by S and C.

Proposition 8.22. *For sufficiently small common B_{cap}, every $p \in \mathcal{S}$ satisfies $d_n(p) > 0$, $t_0(p) = 0$, and $t_{n+1}(p) > t_n(p)$. Its successive entrances lie on the same orbit:*

$$\mathcal{E}_+(X_n) = \mathsf{A}_n(p)\phi_{t_n(p)}\mathcal{E}_+(X_0). \tag{8.94}$$

Each segment from t_n to t_{n+1} has the events of Proposition 8.21, translated by t_n and with the initial frame A_n. Its last terminal arrival is exactly the entrance at t_{n+1}.

Proof. The state p_n belongs to $\mathcal{V}_{a_n} \subset \mathcal{U}_{a_n}$, so Proposition 8.21 applies at every n. It gives the positive duration and

$$\mathcal{E}_+(X_{n+1}) = \mathsf{CS}^{a_n - 1}\phi_{d_n(p)}\mathcal{E}_+(X_n).$$

Starting with $t_0 = 0$ and $\mathsf{A}_0 = \mathrm{id}$, induction with (8.92) proves (8.94). Positivity of d_n gives strict increase of t_n. The reconstruction identity (8.91) identifies the entire terminal state with the next entrance. The first-event assertions are transported by the same restart argument as in the preceding proposition. □

Lemma 8.23. *Let $X = (r, u, w)$ with $r \geq 2$, $u, w \geq 0$ and $u + w \leq b \leq 1/4$. For the normalized curvature product $P(q) = N_1(q)N_2(q)N_3(q)$ one has*

$$0 \leq P(\mathcal{E}_+(X)) \leq b^2. \tag{8.95}$$

The function P is unchanged by every axis permutation.

Proof. At $\iota_+(X)$, let $M = 1 + \sqrt{r^2 - 1}$. The denominator in (2.7) becomes

$$D = 4 + M^2 + (u - w)^2 - 2(u + w) \geq 4.$$

Thus $0 < \lambda = 6/(\sqrt{3}D) \leq 1$, using $\sqrt{3} \geq 3/2$. The three normalized walls are $\lambda u, \lambda, \lambda w$, and therefore $P = \lambda^3 uw \leq b^2$. They are nonnegative. The invariance follows because axis permutations merely permute these three walls. □

Theorem 8.24. *For B_{cap} as above, with $B_{\mathrm{cap}} \leq 1/4$, every $p \in \mathcal{S}$ satisfies*

$$t_n(p) \longrightarrow +\infty. \tag{8.96}$$

Proof. The amplitude coordinates are $B_n = B^{(4/3)^n}$ with $0 < B \leq B_{\mathrm{cap}} < 1$, so $B_n \to 0$. At each entrance the two positive small walls have sum at most B_n. Lemma 8.23 and the frame invariance of P, applied to (8.94), give

$$0 < P\big(\phi_{t_n(p)}\mathcal{E}_+(X_0)\big) \leq B_n^2 \longrightarrow 0.$$

The increasing nonnegative sequence $t_n(p)$ either tends to infinity or has a finite limit $T \geq 0$. In the latter case, Lemma 2.1 gives a well-defined state $\phi_T\mathcal{E}_+(X_0)$ with three strictly positive curvature variables. Continuity of the flow and of P makes the displayed sequence converge to its positive product at time T, contradicting its limit zero. □

This argument is uniform in the entrance radii and the era lengths. It uses the decrease of the transverse amplitude and positivity on finite intervals of the normalized time.

Proposition 8.25. *For every n, the maps $p \mapsto d_n(p)$, $p \mapsto t_n(p)$ and $p \mapsto \mathsf{A}_n(p)$ are Borel on $\mathcal{S}$.*

Proof. For a fixed digit a, Proposition 8.21 makes $\Theta_{a-1}(X)$ locally C^1 at every $(b, X) \in \mathcal{V}_a$. Its restriction to the Borel set $\mathcal{V}_a$ is therefore Borel. Extend this restriction measurably to the whole entrance state space, for example by zero off $\mathcal{V}_a$. The family of extensions is jointly Borel in the positive integer a and the state, since the digit space is countable and discrete. The maps $p \mapsto a_n(p)$ and $p \mapsto p_n$ are Borel by Proposition 8.18, and $p_n \in \mathcal{V}_{a_n(p)}$. Evaluating the extension at this pair therefore gives the actual Borel duration d_n. The finite sum defining t_n is Borel.

The frame assertion follows by induction in (8.93). Multiplication of the finite frame group and the map $a \mapsto \mathsf{CS}^{a-1}$ are Borel, so the Borel digit and the preceding Borel frame give the next one. □

An initial core entrance need not have a terminal connection immediately before it. Its first selected segment ends with such a connection, so every later segment begins a complete era. Discarding this initial segment preserves the statistical limits.

Lemma 8.26. *For any fixed $k \in \mathbb{N}$, the six statistical limits of Theorem 2.4 hold for sequences (a_n, x_n) if and only if they hold for (a_{k+n}, x_{k+n}).*

Proof. For a real sequence u_j, write $A_n(u) = n^{-1} \sum_{j<n} u_j$. For $n > 0$,

$$A_n\big((u_{k+j})_{j\ge 0}\big) = \left(1 + \frac{k}{n}\right) A_{k+n}(u) - \frac{1}{n} \sum_{j<k} u_j.$$

This identity and its rearrangement show that a finite Cesaro limit is preserved in both directions, without any boundedness assumption on u. Apply it to continuous phase observables, the indicators of fixed digit blocks, single digits and tails, and the truncated digits $\min(a_j, R)$. For a block, the indicator in the shifted sequence is exactly the original block indicator at the shifted starting index. Finally the means of a_j tend to infinity in both sequences by their truncated lower bounds: the limits $\log(R+1)/\log 2$ tend to infinity as $R \to \infty$. □

Corollary 8.27. *Suppose $p = (B, X_0) \in \mathcal{S}$ has the six statistical limits for its digits and phases. Let $q \in \mathcal{M}_{+++}$, $s > 0$ and an ordered frame A satisfy*

$$\mathcal{E}_+(X_0) = \mathsf{A}\phi_s q. \tag{8.97}$$

Then the orbit through q has successive complete eras with entrance times, states and frames

$$\tau_n = s + t_{n+1}(p), \qquad Y_n = X_{n+1}, \qquad \widehat{\mathsf{A}}_n = \mathsf{A}_{n+1}(p)\mathsf{A}. \tag{8.98}$$

Their lengths are $K_n = a_{n+1}(p)$, their phases are $2/(Y_n)_r = x_{n+1}(p)$, and all six statistical limits hold. The times are positive, strictly increasing and tend to infinity. Each length counts distinct actual core entrances and satisfies $K_n = \lfloor (Y_n)_r/2 \rfloor$.

Proof. Equations (8.92) and (8.94) give

$$\mathcal{E}_+(X_j) = \mathsf{A}_j(p)\mathsf{A}\phi_{s+t_j(p)} q \qquad (j \ge 0).$$

The times $s + t_j(p)$ are positive, strictly increasing and unbounded, by Proposition 8.22 and Theorem 8.24. All core and connection events translate to these absolute times. The segment from s to $s + t_1(p)$ supplies a terminal connection immediately preceding the entrance $Y_0 = X_1$. Every later segment ends at the next entrance. By Proposition 8.21, all earlier connections within each segment are reflections. These are therefore successive complete eras in the sense of Definition 2.3.

Within the nth era, the entrance times are $\tau_n + s_j(Y_n)$, $0 \le j < a_{n+1}(p)$. They are distinct by (8.90), and their number equals the integer part of the entrance half-radius. Lemma 8.26, with $k = 1$, gives the statistical conclusions. □

Theorem 8.20 supplies the statistical histories required in this corollary on every admissible entrance graph. Their common Borel definition, together with the Borel times and frames, will also preserve measurability when the initial entrance is chosen from a transverse family.

8.10. **Statistics in radial families.** To use the statistical set on a transverse section, we integrate along curves arriving at core entrances. A lower bound for the derivative of the entrance phase transfers the loss estimate to the original curve coordinate. The common set $\mathcal{S}_{\mathrm{stat}}$ then makes these estimates compatible across a measurable family of curves.

Throughout this subsection, fix η, J, σ and the constants B_*, B_{cap} of Theorem 8.20. We write λ for Lebesgue measure on $\mathbb{R}$. All weights are nonnegative, and their integrals may be infinite.

The following elementary estimate records the dependence on the phase derivative and on a density.

Lemma 8.28. *Let $I = [a, b]$, let $c > 0$, and suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at every point of I, with*

$$f'(\rho) \geq c, \qquad f(\rho) \in (0, 1) \quad (\rho \in I).$$

Let $G \subset (0, 1)$ be Borel and suppose $\lambda((0, 1) \setminus G) \leq L$. For every Borel set $A \subset I$ and every nonnegative Borel weight ω satisfying $\omega \leq M$ on A, where $M \geq 0$, the set $A \setminus f^{-1}(G)$ is Borel and

$$\int_{A \setminus f^{-1}(G)} \omega(\rho)\, d\rho \leq \min\left\{ \int_A \omega(\rho)\, d\rho, \frac{M}{c} L \right\}. \tag{8.99}$$

Proof. Continuity and the mean value theorem imply that f is strictly increasing on I. In particular it is injective there. For a Borel set $E \subset I$, the one-dimensional change of variables formula gives

$$\lambda(f(E)) = \int_E |f'(\rho)|\, d\rho \geq c\, \lambda(E).$$

Consequently, whenever $f(E) \subset H$,

$$\lambda(E) \leq \min\left\{ (b - a)_+, \frac{1}{c} \lambda(H) \right\}, \qquad (b - a)_+ = \max\{b - a, 0\}. \tag{8.100}$$

The continuity of $f|_I$ shows that $E = A \setminus f^{-1}(G)$ is Borel. Its image lies in $(0, 1) \setminus G$, so $\lambda(E) \leq L/c$. Thus

$$\int_E \omega\, d\rho \leq M \lambda(E) \leq \frac{M}{c} L.$$

The other bound in (8.99) follows from $E \subset A$. □

Here differentiability at an endpoint of I means ordinary differentiability, as in the return estimates used below.

Suppose now that Γ is admissible at amplitude $0 < B \leq B_*$ and that an entrance state $Y(\rho)$ has the representation

$$Y(\rho) = \Gamma(2/f(\rho)), \qquad \rho \in I, \tag{8.101}$$

where f satisfies the hypotheses of Lemma 8.28. Define the excluded set

$$F_I = \{\rho \in I : (B, Y(\rho)) \notin \mathcal{S}_{\mathrm{stat}}\}. \tag{8.102}$$

This set records the points left outside the chosen statistical entrance construction. Such a point may enter the statistical set at a later return; the capture argument will use precisely this possibility. By (8.84) and (8.101),

$$F_I = I \setminus f^{-1}(G_{\Gamma,B}). \tag{8.103}$$

This identity proves measurability even if Y is only specified on I.

Corollary 8.29. *The set F_I is Borel. For every Borel $A \subset I$ and every nonnegative Borel weight $\omega \le M$ on A,*

$$\int_{A\cap F_I} \omega(\rho)\,d\rho \le \min\left\{\int_A \omega(\rho)\,d\rho, \frac{80M}{c}B^\sigma\right\}. \tag{8.104}$$

For every $\rho \in I \setminus F_I$, the selected digits and phases starting at $(B, Y(\rho))$ satisfy all six limits of Theorem 2.4.

Proof. Theorem 8.20 gives

$$\lambda((0,1)\setminus G_{\Gamma,B}) \le 80B^\sigma.$$

Apply Lemma 8.28 with $G = G_{\Gamma,B}$ and use (8.103). For $\rho \notin F_I$, $f(\rho) \in G_{\Gamma,B}$, and the same theorem gives the statistical conclusion at $(B, \Gamma(2/f(\rho))) = (B, Y(\rho))$. □

Let Λ be a measurable space of labels, and equip $\mathbb{R}$ with its Borel σ-algebra. For each $\ell \in \Lambda$, choose an interval $I_\ell = [a_\ell, b_\ell]$, an amplitude $B_\ell \in (0, B_*]$, and a graph Γ_ℓ admissible at amplitude B_ℓ. Assume that $\ell \mapsto B_\ell$ is measurable. Let $D \subset \Lambda \times \mathbb{R}$ be measurable, with $(\ell,\rho) \in D$ only if $\rho \in I_\ell$. An entrance family $Y(\ell,\rho)$ is required to satisfy

$$\begin{aligned}(\ell,\rho) &\longmapsto (B_\ell, Y(\ell,\rho)) \quad \text{is measurable on } D,\\ Y(\ell,\rho) &= \Gamma_\ell(2/f_\ell(\rho)) \quad (\rho \in I_\ell),\end{aligned} \tag{8.105}$$

where each $f_\ell : \mathbb{R} \to \mathbb{R}$ is differentiable at every point of I_ℓ and, for one constant $c > 0$,

$$f_\ell'(\rho) \ge c, \qquad f_\ell(\rho) \in (0,1) \quad (\ell \in \Lambda,\ \rho \in I_\ell). \tag{8.106}$$

The graphs and phase functions are used separately on each fiber; joint measurability is imposed on the entrance state in (8.105).

Define one excluded set in the full family by

$$F = \{(\ell,\rho) \in D : (B_\ell, Y(\ell,\rho)) \notin \mathcal{S}_{\mathrm{stat}}\}. \tag{8.107}$$

For any $A \subset \Lambda \times \mathbb{R}$, write $A_\ell = \{\rho : (\ell,\rho) \in A\}$.

Theorem 8.30 (Integration over entrance families)**.** *Under* (8.105) *and* (8.106)*, the set F is measurable. For every $(\ell,\rho) \in D \setminus F$, the selected returns starting at $(B_\ell, Y(\ell,\rho))$ satisfy all six statistical limits of Theorem 2.4.*

Let μ be a σ-finite measure on Λ, let $A \subset D$ be measurable, and let $\omega : \Lambda \times \mathbb{R} \to [0,\infty]$ be measurable. Suppose that $M : \Lambda \to [0,\infty)$ is measurable and

$$\omega(\ell,\rho) \le M_\ell \quad ((\ell,\rho) \in A).$$

Put

$$m_A(\ell) = \int_{A_\ell} \omega(\ell,\rho)\,d\rho, \qquad L_\ell = \frac{80M_\ell}{c}B_\ell^\sigma. \tag{8.108}$$

Then

$$\begin{aligned}\int_{A\cap F} \omega\,d(\mu\times\lambda) &\le \int_\Lambda \min\{m_A(\ell), L_\ell\}\,d\mu(\ell)\\ &\le \min\left\{\int_A \omega\,d(\mu\times\lambda), \int_\Lambda L_\ell\,d\mu(\ell)\right\}.\end{aligned} \tag{8.109}$$

Proof. Since $\mathcal{S}_{\mathrm{stat}}$ is Borel, its complement has a measurable inverse image under the restricted state map in (8.105). The domain D is measurable, so this inverse image is also measurable in $\Lambda \times \mathbb{R}$. This is precisely F.

Fix ℓ. The representation of Y gives

$$F_\ell = D_\ell \setminus f_\ell^{-1}(G_{\Gamma_\ell,B_\ell}).$$

If $(\ell,\rho) \in D \setminus F$, then $f_\ell(\rho) \in G_{\Gamma_\ell,B_\ell}$. Theorem 8.20 therefore supplies the statistical conclusion for the stated entrance.

For the integral estimate, the section A_ℓ is Borel and lies in I_ℓ. Since $A \subset D$,

$$(A \cap F)_\ell = A_\ell \setminus f_\ell^{-1}(G_{\Gamma_\ell, B_\ell}).$$

Corollary 8.29, with weight $\omega(\ell, \cdot)$ and bound M_ℓ, gives

$$\int_{(A\cap F)_\ell} \omega(\ell, \rho)\, d\rho \le \min\{m_A(\ell), L_\ell\}. \tag{8.110}$$

For a measurable set $E \subset \Lambda \times \mathbb{R}$, the function $\mathbf{1}_E \omega$ is measurable and nonnegative. Both μ and Lebesgue measure are σ-finite. Tonelli's theorem shows that the fiber integral is measurable and that

$$\int_E \omega\, d(\mu \times \lambda) = \int_\Lambda \left(\int_{E_\ell} \omega(\ell, \rho)\, d\rho \right) d\mu(\ell). \tag{8.111}$$

Apply this identity to $E = A \cap F$ and integrate (8.110). This proves the first inequality in (8.109). For the second, the integrand is bounded both by $m_A(\ell)$ and by L_ℓ. Another application of (8.111), now with $E = A$, completes the proof. □

The first bound in (8.109) retains the mass of every restricted fiber. In particular, integrability of $M_\ell B_\ell^\sigma$ gives finite weighted mass of the excluded set. This is the form needed for the section flux, whose density grows near the boundary. We next construct the entrance family from initial data on a fixed small section and prove the amplitude and phase derivative estimates required above.

9. Transport from a small physical section

The next three sections construct a return from a small transverse section near Taub to the entrance graphs of Theorem 8.20. Where shear dominates the two weak walls, varying the shear supplies a radial parameter. The return must preserve a nonzero phase derivative, control the transverse slopes, and leave only a small measure of parameters outside the statistical set. The resulting statement is Theorem 11.30, extended to both shear signs in Proposition 11.31.

The two wall orderings require different waiting times. We first let the dominant wall decay and follow the orbit through the remaining core to the next entrance. For the other ordering, an additional slow interval reduces the wall that would otherwise be too large. Both constructions use the same transverse event on their overlap. Their measure bounds will be integrated against section flux in Section 12. The purpose of that integration is capture: a set of finite flux cannot retain a positive measure of recurrent points under a strictly expanding return.

Initial-data derivatives are taken in the ambient coordinate charts. On closed parameter intervals they are the restrictions of derivatives on the open neighborhoods supplied by the flow and transverse event maps. Prescribed waiting times are held fixed when differentiating at a fixed scale; derivatives of arrival times are included when an endpoint is defined by an event.

9.1. **The reference flow and its radial phase.** Fix the section $N_1 = h$ near $T_1 = (-1, 0)$. The orbit with vanishing transverse coordinates provides an explicit linear transport for the shear and the two weak walls. Its transported radial phase has a positive derivative on the initial families used below.

9.1.1. *Section coordinates and the reference orbit.* Write $Z = (s, n_2, n_3) \in \mathbb{R}^3$ and define

$$\begin{aligned} \mathcal{D}_h(Z) &= 1 - s^2 - Q(h, n_2, n_3), \\ \Xi_h(Z) &= (-\sqrt{\mathcal{D}_h(Z)}, s, h, n_2, n_3), \end{aligned} \tag{9.1}$$

where Q is the curvature polynomial in (2.1). Wherever $\mathcal{D}_h > 0$, this is a smooth chart on the negative shear branch of the constraint with $N_1 = h$. The projection

$$\Pi(p, s, N_1, N_2, N_3) = (s, N_2, N_3)$$

satisfies $\Pi\Xi_h(Z) = Z$. We use the maximum norm on these three coordinates and its induced operator norm.

For $|h| \le 1/16$ and $\|Z\| \le 1/16$, direct estimates of the three squares and three products in Q give

$$|Q(h, n_2, n_3)| \le \frac{9}{256}, \qquad \frac{246}{256} \le \mathcal{D}_h(Z) \le \frac{265}{256}, \qquad \frac{15}{16} \le \sqrt{\mathcal{D}_h(Z)} \le \frac{17}{16}. \tag{9.2}$$

On the constraint the logarithmic rate of N_1 is $Q - 1 + 2p$. Thus, at $\Xi_h(Z)$,

$$-4 < Q(h, n_2, n_3) - 1 - 2\sqrt{\mathcal{D}_h(Z)} < -2. \tag{9.3}$$

For $h > 0$, the flow crosses the section towards decreasing N_1. The signed coordinate neighborhood in (9.2) also provides ordinary derivatives at points where either of the two small walls vanishes.

Set

$$\begin{aligned} z(n) &= \sqrt{1 - n^2}, \\ F(n) &= n^2 - 1 - 2z(n) = -z(n)(2 + z(n)), \\ b(n) &= n^2 - 1 + z(n) = z(n)(1 - z(n)). \end{aligned} \tag{9.4}$$

The identities on the right hold for $n^2 < 1$. The reference solution is $q_h(t) = \phi_t \Xi_h(0)$. Denote its strong wall by $n_h(t)$. Here this is also the dominant wall in amplitude. The word "strong" refers to decay, while "dominant" refers to size.

Lemma 9.1. *For $0 < h \le 1/16$, the reference solution exists for every $t \ge 0$ and has the form*

$$q_h(t) = (-z(n_h(t)), 0, n_h(t), 0, 0), \qquad \mathrm{D}_t\, n_h = n_h F(n_h), \qquad 0 < n_h(t) \le h e^{-2t}. \tag{9.5}$$

Proof. The wall equations preserve $N_2 = N_3 = 0$. On this invariant set the shear equation is $\mathrm{D}_t\, s = N_1^2 s$, so $s = 0$ is preserved as well. The constraint reduces to $p^2 + N_1^2 = 1$, which bounds the solution and gives forward continuation. Moreover,

$$\mathrm{D}_t\, p = N_1^2 (p - 2) \le 0, \qquad p(t) \le p(0) = -z(h) < 0.$$

The constraint therefore selects $p(t) = -z(N_1(t))$ throughout. The multiplicative wall equation preserves $N_1 > 0$ and gives $\mathrm{D}_t\, N_1 = N_1 F(N_1)$.

Since $z(h) \ge 15/16$, we have $p(t) \le -15/16$. The original wall rate $-p^2 + 2p$ is consequently at most -2. Multiplication by e^{2t} shows that $N_1(t) \le h e^{-2t}$, proving the assertion. □

Define three smooth coefficients for $n^2 < 1$ by

$$A(n) = \frac{3}{2 + z(n)}, \qquad C(n) = \frac{9(1 + z(n))}{2(2 + z(n))^2}, \qquad K(n) = \frac{3\sqrt{3}\, n}{(2 + z(n))^2}. \tag{9.6}$$

The linear map

$$\mathcal{T}(n)(s, n_2, n_3) = (A(n)s + K(n)(n_2 - n_3), C(n)n_2, C(n)n_3) \tag{9.7}$$

will express the transverse variation in coordinates conserved along the reference orbit. In particular, $\mathcal{T}(0)$ is the identity.

To see why these coefficients arise, differentiate the exact vector field at $(-z(n), 0, n, 0, 0)$. Its three transverse components form the closed system

$$\mathrm{D}_t\, \xi = n^2 \xi + \sqrt{3}\, n(\alpha - \beta), \qquad \mathrm{D}_t\, \alpha = b(n)\alpha, \qquad \mathrm{D}_t\, \beta = b(n)\beta. \tag{9.8}$$

The variations of p and N_1 do not enter these equations. Thus (9.8) is the transverse part of the actual flow differential.

Lemma 9.2. *Along any solution of* $\mathrm{D}_t\, n = nF(n)$ *and* (9.8) *with* $n^2 < 1$*, the three components of* $\mathcal{T}(n)(\xi, \alpha, \beta)$ *are constant.*

Proof. Since $z' = -n/z$, differentiation of (9.6) gives

$$A' = \frac{3n}{z(2+z)^2}, \qquad C' = \frac{9n}{2(2+z)^3}, \qquad K' = \frac{3\sqrt{3}\{z(2+z)+2n^2\}}{z(2+z)^3}.$$

Using $n^2 = 1 - z^2$, these formulas yield

$$nFA' + n^2 A = 0, \qquad nFC' + bC = 0, \qquad nFK' + \sqrt{3}\, nA + bK = 0. \tag{9.9}$$

For example, the derivative of $A\xi + K(\alpha - \beta)$ is

$$(nFA' + n^2 A)\xi + (nFK' + \sqrt{3}\, nA + bK)(\alpha - \beta) = 0.$$

The other two conserved quantities follow from the middle identity in (9.9). □

For a fixed time, let

$$Y_h(t) = D_Z\big(\Pi \phi_t \Xi_h\big)(0). \tag{9.10}$$

The chart is smooth in an ordinary neighborhood of zero, and the flow is differentiable in its initial state. Differentiating $\Pi\Xi_h = \mathrm{id}$ gives $Y_h(0) = \mathrm{id}$. Equation (9.8) and Lemma 9.2 therefore give the finite identity

$$\mathcal{T}(n_h(t))Y_h(t) = \mathcal{T}(h), \qquad t \geq 0. \tag{9.11}$$

Proposition 9.3. *For* $0 < h \leq 1/16$ *and* $t \geq 0$*,*

$$\|Y_h(t)\| \leq 2, \qquad \|Y_h(t) - \mathcal{T}(h)\| \leq 6he^{-2t}. \tag{9.12}$$

In particular, the transverse differential converges to $\mathcal{T}(h)$ *as* $t \to \infty$*. For* $0 < a \leq 1$*,* $q \geq 0$ *and* $T = (q/2)\log(1/a)$*,*

$$0 < n_h(T) \leq ha^q, \qquad \|Y_h(T) - \mathcal{T}(h)\| \leq 6ha^q. \tag{9.13}$$

Proof. For $0 \leq n \leq 1/16$, one has $1/2 \leq z \leq 1$ and $1 - z \leq n$. The identities

$$A - 1 = \frac{1-z}{2+z}, \qquad C - 1 = \frac{(1-z)(2z+1)}{2(2+z)^2}$$

and $3\sqrt{3} \leq (2+z)^2$ imply

$$0 \leq A(n) - 1 \leq n, \qquad 0 \leq C(n) - 1 \leq n, \qquad 0 \leq K(n) \leq n. \tag{9.14}$$

For the maximum norm, the first coordinate of $(\mathcal{T}(n) - \mathrm{id})Z$ is bounded by $n|s| + n|n_2 - n_3| \leq 3n\|Z\|$, and each remaining coordinate is bounded by $n\|Z\|$. Hence

$$\|\mathcal{T}(n) - \mathrm{id}\| \leq 3n, \qquad \|\mathcal{T}(n)\| \leq 1 + 3n.$$

Put $n = n_h(t)$. Equation (9.11) can be rewritten as

$$Y_h(t) - \mathcal{T}(h) = (\mathrm{id} - \mathcal{T}(n))Y_h(t).$$

It follows that $\|Y_h(t)\| \leq 1 + 3h + 3n\|Y_h(t)\|$. Since $h, n \leq 1/16$, this implies $\|Y_h(t)\| \leq 2$. The same identity now gives $\|Y_h(t) - \mathcal{T}(h)\| \leq 6n$. Apply (9.5) to obtain (9.12). Finally $e^{-2T} = a^q$ proves (9.13). □

9.1.2. *The transported radius along the nonlinear flow.* The limiting transverse transfer suggests the quadratic form

$$\mathcal{Q}(n;s,v) = (A(n)s + K(n)v)^2 + C(n)^2 v^2. \tag{9.15}$$

For $n^2 < 1$, both $A(n)$ and $C(n)$ are positive, so $\mathcal{Q}(n;s,v) > 0$ unless $s = v = 0$. At $n = 0$ it is $s^2 + v^2$. For $0 \le n \le 1/16$, the coefficient bounds also imply

$$s^2 + v^2 \le 2\mathcal{Q}(n;s,v). \tag{9.16}$$

Indeed, $A, C \ge 1$ and $2K^2 \le 1$, and

$$s^2 + v^2 \le 2(As + Kv)^2 + (1 + 2K^2)v^2 \le 2(As + Kv)^2 + 2C^2 v^2.$$

We now compute its drift on the exact flow. At a constrained state $x = (p, s, n, N_2, N_3)$, set

$$\begin{gathered} v = N_2 - N_3, \qquad w = N_2 + N_3, \qquad \Delta_Q = v^2 - 2nw, \\ \Delta_z = -p - z(n), \qquad \Delta_F = \Delta_Q - 2\Delta_z. \end{gathered} \tag{9.17}$$

Here $Q = n^2 + \Delta_Q$. If $p < 0$ and $n^2 < 1$, the constraint gives the useful rationalized expression

$$\Delta_z = -\frac{s^2 + \Delta_Q}{-p + z(n)}. \tag{9.18}$$

The denominator is positive. The factor n in the term $-2nw$ remains explicit.

Substitution into (2.2) yields

$$\begin{aligned} &\mathrm{D}_t\, n = n(F + \Delta_F), \\ &\mathrm{D}_t\, s = n^2 s + \sqrt{3}\, nv + e_s, \quad e_s = \Delta_Q s - \sqrt{3}\, wv, \\ &\mathrm{D}_t\, v = bv + e_v, \qquad\qquad\quad e_v = (\Delta_Q + \Delta_z)v + \sqrt{3}\, ws. \end{aligned} \tag{9.19}$$

All rates are in the original normalized time.

Proposition 9.4. *Along an exact constrained solution with $n^2 < 1$, define*

$$\mathcal{R}(n;s,v) = AK(s^2 - v^2) + (K^2 + C^2 - A^2)sv.$$

Then

$$\begin{aligned} \frac{d}{dt}\mathcal{Q}(n;s,v) &= n\Delta_F\, \partial_n \mathcal{Q}(n;s,v) + 2\Delta_Q\, \mathcal{Q}(n;s,v) \\ &\quad + 2\Delta_z v\{AKs + (K^2 + C^2)v\} + 2\sqrt{3}\, w\mathcal{R}(n;s,v). \end{aligned} \tag{9.20}$$

Moreover, $\mathcal{R}(0;s,v) = 0$.

Proof. Write $U = As + Kv$ and $V = Cv$. The cancellations (9.9), applied to (9.19), give

$$\mathrm{D}_t\, U = n\Delta_F(A's + K'v) + Ae_s + Ke_v, \qquad \mathrm{D}_t\, V = n\Delta_F C'v + Ce_v.$$

Differentiate $\mathcal{Q} = U^2 + V^2$ using these two identities. The terms involving Δ_F give $n\Delta_F \partial_n \mathcal{Q}$. Those involving Δ_Q give $2\Delta_Q(U^2 + C^2v^2)$, and the terms involving Δ_z give $2\Delta_z v(KU + C^2 v)$. The remaining terms are

$$2\sqrt{3}\, w\{U(-Av + Ks) + C^2 vs\} = 2\sqrt{3}\, w\mathcal{R}(n;s,v).$$

This proves (9.20). At zero, $A = C = 1$ and $K = 0$, so $\mathcal{R} = 0$. □

In particular, the terms $-\sqrt{3}\, wv$ and $\sqrt{3}\, ws$ in (9.19) act as a rotation of (s, v). Their contribution to the transported radius vanishes when $n = 0$. Keeping this cancellation before estimating the drift will preserve the small factor supplied by the decaying wall.

Define the reference radial phase on the section by

$$\Phi_h(Z) = \frac{1}{\sqrt{3}} \sqrt{\mathcal{Q}(h;s,n_2 - n_3)}. \tag{9.21}$$

This is the Euclidean radius formed from the shear and wall difference after the limiting linear transfer, divided by $\sqrt{3}$. We will compare it with the phase of the actual entrance.

Fix $\varsigma \in \{-1, 1\}$ and vary the shear while keeping both small walls fixed:

$$Z_\varsigma(\rho) = (\varsigma\rho, n_2, n_3).$$

For $a > 0$, $0 \le \varepsilon \le 1/64$ and $0 \le n_2, n_3 \le 2\varepsilon a$, one has, for $a \le \rho \le 2a$,

$$\|Z_\varsigma(\rho)\| \le 2a, \qquad \sqrt{\rho^2 + (n_2 - n_3)^2} \ge a, \qquad |n_2 - n_3| \le 2\varepsilon a. \tag{9.22}$$

Proposition 9.5. *If $0 \le h \le 1/16$ and the preceding conditions hold, then $\rho \mapsto \Phi_h(Z_\varsigma(\rho))$ is ordinarily differentiable at every $\rho \in [a, 2a]$, and*

$$\frac{1}{2\sqrt{3}} \le \frac{d}{d\rho}\Phi_h(Z_\varsigma(\rho)) \le \frac{2}{\sqrt{3}}. \tag{9.23}$$

Proof. Let $d = n_2 - n_3$, and use the Euclidean norm for vectors in $\mathbb{R}^2$. Put

$$P = (A(h)\varsigma, 0), \qquad W = (K(h)d, C(h)d).$$

Then $\sqrt{3}\,\Phi_h(Z_\varsigma(\rho)) = \|\rho P + W\|_2$. By (9.14), $\|P\|_2 = A(h) \in [1, 2]$ and $\|W\|_2 \le 3|d|$. Consequently,

$$3\|W\|_2 \le 9|d| \le 18\varepsilon a \le a \le \rho\|P\|_2.$$

The reverse triangle inequality gives $\|\rho P + W\|_2 \ge \rho\|P\|_2 - \|W\|_2 > 0$, so the norm is differentiable at the point in question. Cauchy–Schwarz and the triangle inequality show that

$$\frac{d}{d\rho}\|\rho P + W\|_2 = \frac{P \cdot (\rho P + W)}{\|\rho P + W\|_2} \le \|P\|_2,$$

whereas

$$\frac{P \cdot (\rho P + W)}{\|\rho P + W\|_2} \ge \frac{\rho\|P\|_2^2 - \|P\|_2\|W\|_2}{\rho\|P\|_2 + \|W\|_2} \ge \frac{1}{2}\|P\|_2.$$

Division by $\sqrt{3}$ proves (9.23). □

The reference derivative is thus controlled for both signs of the shear and for every relative size of the two small walls in (9.22). The next estimates control the nonlinear drift over the initial interval of length $(q/2)\log(1/a)$ and its ordinary derivatives. They will allow this reference phase to be compared with an actual core entrance.

9.2. **Nonlinear transport from the small section.** The strong wall decays exponentially along the reference orbit. In logarithmic strong-wall coordinates, the derivative of the vector field near this orbit is controlled by the decaying wall and by the size of the three transverse coordinates. Its time integral can therefore remain small on an interval whose length grows logarithmically as the initial transverse size tends to zero. We use this observation to compare the nonlinear flow with the transfer map $\mathcal{T}(h)$. All finite-dimensional spaces in this subsection carry the maximum norm and its induced operator norms.

9.2.1. *Coordinates and integrated growth.* For $X = (n, s, n_2, n_3)$, put

$$\zeta(X) = \sqrt{1 - s^2 - Q(n, n_2, n_3)}.$$

The following map records the logarithmic rate of the strong wall and the three transverse velocities:

$$\mathcal{V}(X) = \begin{pmatrix} Q - 1 - 2\zeta \\ Qs + \sqrt{3}(n - n_2 - n_3)(n_2 - n_3) \\ n_2(Q - 1 + \zeta + \sqrt{3}s) \\ n_3(Q - 1 + \zeta - \sqrt{3}s) \end{pmatrix}. \tag{9.24}$$

Here $Q = Q(n, n_2, n_3)$ and $\zeta = \zeta(X)$. The map is smooth on a neighborhood of the signed box $\|X\| \le 1/16$, by (9.2), and

$$\mathcal{V}(0) = (-3, 0, 0, 0), \qquad D\mathcal{V}(0) = 0.$$

The second identity follows directly from (9.24): Q and $1-\zeta$ have vanishing first derivatives at zero, the second row has no linear term, and each of the last two rows is a coordinate times a function that vanishes at zero. Compactness and the mean value theorem now give a constant $B \ge 1$ such that

$$\|D\mathcal{V}(X)\| \le B\|X\|, \qquad \|D^2\mathcal{V}(X)\| \le B \quad (\|X\| \le 1/16). \tag{9.25}$$

Use the logarithmic coordinate $\ell = \log n$ when $n > 0$, and set

$$E(\ell, s, n_2, n_3) = (e^\ell, s, n_2, n_3), \qquad \mathcal{W} = \mathcal{V} \circ E.$$

The corresponding domain

$$\mathcal{D} = \{(\ell, s, n_2, n_3) : \ell \le \log(1/16),\ |s|, |n_2|, |n_3| \le 1/16\}$$

is convex. On this domain $\|DE\| \le 1$ and $\|D^2E\| \le 1$. The chain rule and (9.25) therefore give a constant $C \ge 32$ such that

$$\|D\mathcal{W}(y)\| \le C\|E(y)\|, \qquad \|D^2\mathcal{W}(y)\| \le C, \tag{9.26}$$

$$\|D\mathcal{W}(y) - D\mathcal{W}(\widetilde{y})\| \le C\|y - \widetilde{y}\| \qquad (y, \widetilde{y} \in \mathcal{D}). \tag{9.27}$$

Choose C also to dominate B, and fix it throughout the section. The second line follows by applying the mean value theorem to $D\mathcal{W}$ along the line segment in $\mathcal{D}$. The unbounded range of ℓ causes no loss in these estimates.

We will repeatedly use the following norm comparison. If $\Gamma(0) = 0$, $\Gamma(t) \ge 0$, $\varepsilon \ge 0$, and differentiable functions satisfy

$$\|f'(t)\| \le \Gamma'(t)\|f(t)\| + \varepsilon \quad (0 \le t \le T),$$

then

$$\|f(t)\| \le (\|f(0)\| + \varepsilon t)e^{\Gamma(t)}. \tag{9.28}$$

Indeed, for $\delta > 0$ compare $\|f(t)\|$ with

$$B_\delta(t) = (\|f(0)\| + \delta + \varepsilon t)e^{\Gamma(t)+\delta t}.$$

Its initial value is strictly larger, and at any contact with $\|f(t)\|$ it satisfies $B_\delta'(t) > \Gamma'(t)B_\delta(t) + \varepsilon$. The norm comparison principle gives $\|f(t)\| \le B_\delta(t)$; letting δ decrease to zero proves (9.28). This argument uses the bound on $\|f'\|$ and remains valid at zeros of f.

The signed coordinates used for ordinary differentiation also have forward solutions. To see this, the multiplicative wall equations in the proof of Lemma 2.1 give

$$|N_i(t)| \le |N_i(0)|e^t$$

for any solution on $G = 1$, with no sign restriction on the walls. If $|N_i| \le M$, the identity $Q = 2\sum_i N_i^2 - (\sum_i N_i)^2$ and the constraint give $p^2 + s^2 \le 1 + 9M^2$. Thus the full state has norm at most $1 + 3M$. The same continuation argument proves forward existence, and every initially positive wall remains positive.

For $h, b > 0$, define

$$\Gamma(t) = C\left(\frac{h}{2}(1 - e^{-2t}) + bt\right), \qquad \Gamma'(t) = C(he^{-2t} + b). \tag{9.29}$$

The conditions used below are

$$Ch \le \frac{1}{4}, \qquad 0 < b \le \frac{1}{16}, \qquad T \ge 0, \qquad CbT \le \frac{1}{8}. \tag{9.30}$$

They imply $h \le 1/128$ and $0 \le \Gamma(t) \le 1/4$ for $0 \le t \le T$.

Proposition 9.6. *Suppose $h > 0$ and (9.30) holds. Let $x = (p, s, N_1, N_2, N_3)$ satisfy $G(x) = 1$, $p < 0$, $0 < N_1 \le h$, and $\|\Pi x\| \le b/2$. For $0 \le t \le T$, the actual orbit satisfies*

$$0 < N_1(\phi_t x) \le h e^{-2t}, \qquad p(\phi_t x) \le -\frac{15}{16}, \tag{9.31}$$
$$\|\Pi \phi_t x\| \le \|\Pi x\| e^{\Gamma(t)} < b.$$

In particular these conclusions hold for $x = \Xi_h(Z)$ with $\|Z\| \le b/2$.

Proof. First consider a time interval on which

$$p \le -\frac{1}{2}, \qquad N_1 \le 2h, \qquad \|\Pi \phi_t x\| \le b. \tag{9.32}$$

Positivity of N_1 and $2h \le 1/16$ place the four physical coordinates in the signed box. The constraint selects its negative root, so $p \le -15/16$, and (9.3) gives $N_1'/N_1 \le -2$. Consequently $N_1(t) \le h e^{-2t}$.

Write $Z(t) = \Pi \phi_t x$ and $n = N_1(t)$. The last three rows of $\mathcal{V}(n, 0)$ vanish. Along the segment from $(n, 0)$ to $(n, Z(t))$, (9.25) gives

$$\|Z'(t)\| \le C \max(n, \|Z(t)\|)\|Z(t)\| \le C(n + \|Z(t)\|)\|Z(t)\|.$$

Thus $\|Z'(t)\| \le \Gamma'(t)\|Z(t)\|$ on the interval under consideration. Equation (9.28) with $\varepsilon = 0$ yields the last bound in (9.31), including its strict inequality because

$$\|Z(0)\| e^{\Gamma(t)} \le \frac{b}{2} e^{1/4} < b.$$

Initially all three inequalities in (9.32) are strict: the constraint and the box estimate give $p \le -15/16$, while $N_1 \le h < 2h$ and $\|Z(0)\| \le b/2 < b$. On every closed interval where they hold, the preceding estimates improve them strictly. Continuity therefore extends such an interval past its right endpoint whenever that endpoint is smaller than T. Taking the supremum of these intervals proves (9.32) throughout $[0, T]$, and hence (9.31). The chart $\Xi_h(Z)$ satisfies the required initial conditions by (9.2). □

9.2.2. *Ordinary initial derivatives.* For $y = (\ell, s, n_2, n_3)$, let

$$L(p, s, N_1, N_2, N_3) = (\log N_1, s, N_2, N_3), \qquad \widehat{\Xi}(y) = (-\zeta(E(y)), s, e^{\ell}, n_2, n_3).$$

On the open domain where the initial radicand is positive, the actual flow exists, and its endpoint has $p < 0$ and $N_1 > 0$, define

$$\Psi_t(y) = L\big(\phi_t \widehat{\Xi}(y)\big).$$

The flow, the chart, and L are jointly C^2 there. The constraint and (2.2) give the local identity

$$\partial_t \Psi_t(y) = \mathcal{W}(\Psi_t(y)), \qquad \Psi_0(y) = y. \tag{9.33}$$

Because this identity holds on an open neighborhood, differentiation in y gives the ordinary variational equation

$$J_v'(t) = D\mathcal{W}(\Psi_t(y)) J_v(t), \qquad J_v(t) = D_y \Psi_t(y) v, \qquad J_v(0) = v. \tag{9.34}$$

For initial data on the fixed section, write

$$U_{h,t}(Z) = \Psi_t(\log h, Z), \qquad H_{h,t}(Z) = \Pi \phi_t \Xi_h(Z). \tag{9.35}$$

The last three components of $U_{h,t}$ are $H_{h,t}$. Their initial values are $(\log h, Z)$ and Z respectively.

Proposition 9.7. *Under* (9.30)*, with $h > 0$, the maps in* (9.35) *are differentiable at every $\|Z\| \le b/2$ for $0 \le t \le T$. For $\|Z\|, \|W\| \le b/2$,*

$$\|DU_{h,t}(Z)\| \le e^{\Gamma(t)} \le e^{1/4} < 2, \tag{9.36}$$

$$\|U_{h,t}(Z) - U_{h,t}(W)\| \le e^{1/4}\|Z - W\|, \tag{9.37}$$

$$\|DU_{h,t}(Z) - DU_{h,t}(W)\| \le 8Ct\|Z - W\|. \tag{9.38}$$

The derivative comparison (9.38) *also holds with $H_{h,t}$ in place of $U_{h,t}$.*

Proof. Proposition 9.6 places each orbit in the open regularity domain of Ψ and gives

$$\|E(U_{h,t}(Z))\| \le he^{-2t} + b, \qquad U_{h,t}(Z) \in \mathcal{D}.$$

It follows from (9.26) that the coefficient in (9.34) has norm at most $\Gamma'(t)$. The inclusion $v \mapsto (0, v)$ has norm one, so (9.28) proves (9.36). The mean value theorem on the convex ball $\|Z\| \le b/2$ gives (9.37). Ordinary differentiability at its boundary follows from the same open regularity domain.

For the final estimate, fix $v \in \mathbb{R}^3$ and put

$$J_Z(t) = DU_{h,t}(Z)v, \quad J_W(t) = DU_{h,t}(W)v, \quad A_Z(t) = D\mathcal{W}(U_{h,t}(Z)).$$

Define A_W in the same way. The difference of the two variational equations is

$$(J_Z - J_W)' = A_Z(J_Z - J_W) + (A_Z - A_W)J_W.$$

Equations (9.27), (9.37), and (9.36) imply

$$\|A_Z - A_W\| \le 2C\|Z - W\|, \qquad \|J_W(t)\| \le 2\|v\|.$$

The difference starts at zero. Applying (9.28) with $\varepsilon = 4C\|Z - W\|\|v\|$ therefore gives

$$\|J_Z(t) - J_W(t)\| \le 4Ct\, e^{\Gamma(t)}\|Z - W\|\|v\| \le 8Ct\|Z - W\|\|v\|.$$

Taking the operator norm proves (9.38). Projection onto the last three coordinates has norm one, which gives the assertion for $H_{h,t}$. □

The reference orbit has zero transverse coordinates, so $H_{h,t}(0) = 0$. Its initial derivative is precisely the map $Y_h(t)$ of (9.10). Define the remainder relative to the limiting transfer by

$$R_{h,t}(Z) = H_{h,t}(Z) - \mathcal{T}(h)Z. \tag{9.39}$$

Proposition 9.8. *For $h > 0$ satisfying* (9.30)*, $0 \le t \le T$, and $\|Z\| \le b/2$, the remainder is differentiable at Z and satisfies*

$$\|R_{h,t}(Z)\| \le 4Ct\|Z\|^2 + 6he^{-2t}\|Z\|, \tag{9.40}$$

$$\|DR_{h,t}(Z)\| \le 8Ct\|Z\| + 6he^{-2t}. \tag{9.41}$$

Proof. The derivative splits as

$$DR_{h,t}(Z) = DH_{h,t}(Z) - DH_{h,t}(0) + Y_h(t) - \mathcal{T}(h).$$

Proposition 9.7 and (9.12) prove (9.41). To recover the value, set $g(\theta) = R_{h,t}(\theta Z)$ for $0 \le \theta \le 1$. Then $g(0) = 0$ and

$$\|g'(\theta)\| \le 8Ct\theta\|Z\|^2 + 6he^{-2t}\|Z\|.$$

Norm comparison with the scalar polynomial $4Ct\theta^2\|Z\|^2 + 6he^{-2t}\theta\|Z\|$ at $\theta = 1$ proves (9.40). □

Theorem 9.9 (Transport at logarithmic time)**.** *There exists $h_* > 0$ such that, for every $0 < h \le h_*$, $C_0 > 0$, and $q > 2$, there are $K > 0$ and $a_* > 0$ with the following property. If $0 < a \le a_*$ and $\|Z\| \le C_0 a$, put*

$$T = \frac{q}{2}\log(1/a).$$

The exact flow through $\Xi_h(Z)$ exists at time T, and

$$0 < N_1(\phi_T \Xi_h(Z)) \le h a^q, \qquad H_{h,T}(Z) = \mathcal{T}(h)Z + R_{h,T}(Z). \tag{9.42}$$

The remainder has an ordinary derivative in Z and

$$\begin{aligned} \|R_{h,T}(Z)\| &\le K a^2(1 + |\log a|), \\ \|DR_{h,T}(Z)\| &\le K a(1 + |\log a|). \end{aligned} \tag{9.43}$$

Here the derivative is taken with h and T fixed.

Proof. Take $h_* = 1/(4C)$ and set $b = 2C_0 a$. Since $a|\log a| \to 0$ as $a \downarrow 0$, there is an $a_* \le 1$ such that

$$2C_0 a \le \frac{1}{16}, \qquad C(2C_0 a)\frac{q}{2}\log(1/a) \le \frac{1}{8} \quad (0 < a \le a_*).$$

Thus (9.30) holds. Proposition 9.6 gives the wall bound because $e^{-2T} = a^q$. Proposition 9.8 gives the more explicit estimates

$$\begin{aligned} \|R_{h,T}(Z)\| &\le 4CC_0^2 a^2 T + 6hC_0 a^{q+1}, \\ \|DR_{h,T}(Z)\| &\le 8CC_0 a T + 6h a^q. \end{aligned}$$

For $a \le 1$ and $q > 2$, one has $a^{q+1} \le a^2$ and $a^q \le a$. Hence (9.43) holds with

$$K = \max\{2CC_0^2 q + 6hC_0,\ 4CC_0 q + 6h\} > 0.$$

□

The theorem controls the full transverse displacement and its ordinary derivative while the strong wall decreases from h to at most ha^q. For the radial phase, a finer estimate comes from the cancellation in the transported quadratic form (9.20). We turn next to that scalar estimate before following the orbit to a core entrance.

9.3. **Radial phase during the initial decay.** The radial phase changes less than the full transverse vector. The rotation visible in (9.19) makes no contribution to the Euclidean radius when the strong wall vanishes. The transported quadratic form retains this cancellation at positive strong wall. Its drift has an additional small factor, which removes the logarithmic loss from the phase estimate and its initial derivative.

For $Z = (s, n_2, n_3)$, write

$$v = n_2 - n_3, \qquad w = n_2 + n_3, \qquad \varrho(Z) = \sqrt{s^2 + v^2}, \qquad \mathcal{J}(n, Z) = \mathcal{Q}(n; s, v).$$

The function ϱ is the Euclidean radius in the shear and wall-difference coordinates. We continue to use maximum norms for Z, for (n, Z), and for their derivatives.

9.3.1. *The vanishing drift coefficients.* Use the coefficients A, C, K of (9.6) to define

$$\begin{aligned} g(n) &= (A^2,\ 2AK,\ K^2 + C^2), \\ k(n) &= (0,\ AK,\ K^2 + C^2), \\ o(n) &= (AK,\ K^2 + C^2 - A^2,\ -AK). \end{aligned} \tag{9.44}$$

The coefficients on the right are evaluated at n. Thus $\mathcal{J}(n, Z) = g(n) \cdot (s^2, sv, v^2)$. On the negative constraint branch put

$$\Delta_Q = v^2 - 2nw, \qquad \Delta_z = \sqrt{\mathcal{D}_n(Z)} - z(n).$$

These are the differences in (9.17). Define

$$\begin{aligned} m(n,Z) &= n(\Delta_Q - 2\Delta_z)g'(n) + 2\Delta_Q g(n) + 2\Delta_z k(n) + 2\sqrt{3}\, w\, o(n), \\ \mathcal{L}(n,Z) &= m(n,Z)\cdot(s^2, sv, v^2). \end{aligned} \tag{9.45}$$

Proposition 9.4 gives the exact identity

$$\frac{d}{dt}\mathcal{J}(N_1(t), \Pi x(t)) = \mathcal{L}(N_1(t), \Pi x(t)) \tag{9.46}$$

along the constrained flow wherever $p < 0$ and $N_1^2 < 1$.

All coefficient functions in (9.45) are smooth on a neighborhood of the signed box $|n|, \|Z\| \le 1/16$. Two identities determine their size:

$$m(n,0) = 0, \qquad Dm(0,0) = 0. \tag{9.47}$$

The first follows from $\Delta_Q = \Delta_z = w = 0$ at $Z = 0$. For the second, both differences have zero value and zero first derivative at the origin, and $o(0) = 0$. In particular, the term linear in w also has zero first derivative there.

There is a constant $B \ge 1$ such that on this box

$$\begin{aligned} \|m(n,Z)\| &\le B(|n| + \|Z\|)\|Z\|, \\ \|Dm(n,Z)\| &\le B(|n| + \|Z\|), \\ \|\partial_n m(n,Z)\| &\le B\|Z\|. \end{aligned} \tag{9.48}$$

Indeed, a compact bound on the Hessian makes Dm Lipschitz on the box. Comparing $Dm(n,Z)$ with $Dm(0,0)$ gives the second inequality. Integrating this bound on the segment from $(n,0)$ to (n,Z) gives the first, by (9.47). Finally, $\partial_n m(n,0) = 0$; comparing the derivatives at $(n,0)$ and (n,Z) gives the third.

We may choose the same B so that

$$\begin{aligned} \|D\mathcal{J}(n,Z)\| &\le B\|Z\|, \\ \|g(n) - g(0)\| &\le B|n|, \qquad \|g'(n)\| \le B. \end{aligned} \tag{9.49}$$

For the first inequality, $D\mathcal{J}(n,0) = 0$ because $\mathcal{J}$ is quadratic in s, v. A compact Hessian bound and the same segment comparison apply. The other two inequalities follow from a compact derivative bound for g, with $g(0) = (1,0,1)$.

For a coefficient vector a, the quadratic polynomial $a_1 s^2 + a_2 sv + a_3 v^2$ has absolute value at most $2\|a\|(s^2+v^2)$. Its differential, when $|s| \le r$, $|v| \le 2r$, $|\delta s| \le d$, and $|\delta v| \le 2d$, has absolute value at most

$$12\|\delta a\| r^2 + 24\|a\| r d.$$

These bounds follow by expanding the three monomials and using $|sv| \le s^2 + v^2$. Apply them to $a = m(n,Z)$, and use (9.16). For $0 \le n \le 1/16$ and $\|Z\| \le 1/16$ we obtain

$$\begin{aligned} |\mathcal{L}(n,Z)| &\le C(n\|Z\| + \|Z\|^2)\mathcal{J}(n,Z), \\ \|D_Z\mathcal{L}(n,Z)\| &\le C(n\|Z\|^2 + \|Z\|^3), \\ |\partial_n \mathcal{L}(n,Z)| &\le C\|Z\|^3, \end{aligned} \tag{9.50}$$

with $C = 36B$. The last line uses the third bound in (9.48), keeping the strong-wall direction separate from the transverse directions.

We also need to compare the transported and Euclidean radii. The bounds (9.14) give $\|g(n) - g(0)\| \le 4n$ for $0 \le n \le 1/16$. Consequently

$$\begin{aligned} |\mathcal{J}(n,Z) - \varrho(Z)^2| &\le 8n\varrho(Z)^2, \\ \tfrac{1}{2}\varrho(Z)^2 \le \mathcal{J}(n,Z) &\le 2\varrho(Z)^2. \end{aligned} \tag{9.51}$$

The lower bound is (9.16). For $\varrho(Z) > 0$, rationalizing the two square roots gives

$$|\sqrt{\mathcal{J}(n,Z)} - \varrho(Z)| \le 8n\varrho(Z).$$

When $\varrho(Z) = 0$, both sides vanish. If $\|Z\| \le b$, then $|s| \le b$, $|v| \le 2b$, and $\varrho(Z) \le 3b$. Hence

$$\sqrt{\mathcal{J}(n,Z)} \le 4b, \qquad |\sqrt{\mathcal{J}(n,Z)} - \varrho(Z)| \le 24nb. \tag{9.52}$$

9.3.2. *The accumulated radial displacement.* Choose $C \ge 32$ large enough for (9.50), (9.49), and the estimates of Section 9.2, and fix this constant. We impose

$$h > 0, \qquad Ch \le \frac{1}{4}, \qquad 0 < b \le \frac{1}{16}, \qquad T \ge 0, \qquad CbT \le \frac{1}{8}. \tag{9.53}$$

For $\|Z\| \le b/2$, write

$$\begin{aligned} \nu_t(Z) &= N_1(\phi_t \Xi_h(Z)), \qquad Z_t = H_{h,t}(Z), \\ J_t(Z) &= \mathcal{J}(\nu_t(Z), Z_t), \qquad r_t(Z) = \sqrt{J_t(Z)}. \end{aligned} \tag{9.54}$$

The height h is fixed in this notation. In particular, $J_0(Z) = \mathcal{J}(h,Z)$ and $r_0(Z)/\sqrt{3} = \Phi_h(Z)$.

Proposition 9.10. *Under* (9.53)*, for* $\|Z\| \le b/2$ *and* $0 \le t \le T$*,*

$$\tfrac{1}{2} J_0(Z) \le J_t(Z) \le 2J_0(Z), \tag{9.55}$$

$$|r_t(Z) - r_0(Z)| \le Chb^2 + 2Cb^3 t, \tag{9.56}$$

$$|\varrho(Z_t) - r_0(Z)| \le Chb^2 + 2Cb^3 t + 24hbe^{-2t}. \tag{9.57}$$

The value estimates include $\varrho(Z) = 0$.

Proof. Proposition 9.6 gives $0 < \nu_t \le he^{-2t}$ and $\|Z_t\| \le b$. By (9.46) and (9.50),

$$|J_t'| \le C(\nu_t \|Z_t\| + \|Z_t\|^2) J_t \le C(he^{-2t} + b) J_t. \tag{9.58}$$

In the last step we used $\|Z_t\| \le b \le 1$. Multiplying the two differential inequalities by $e^{-\Gamma(t)}$ and $e^{\Gamma(t)}$, respectively, where Γ is (9.29), gives

$$J_0 e^{-\Gamma(t)} \le J_t \le J_0 e^{\Gamma(t)}.$$

Since $\Gamma(t) \le 1/4$ and $e^{1/4} < 2$, this proves (9.55) without dividing by J_t.

If $J_0 = 0$, the upper bound forces $J_t = 0$ throughout, and (9.56) follows. Otherwise the lower bound keeps $r_t > 0$, and (9.58), before its last simplification, yields

$$|r_t'| = \frac{|J_t'|}{2r_t} \le \frac{C}{2}(\nu_t b + b^2) r_t \le 2C(\nu_t b^2 + b^3).$$

Here $r_t \le 4b$ follows from (9.52). Comparing with the primitive $Chb^2(1 - e^{-2t}) + 2Cb^3 t$ proves (9.56). Finally, (9.52) at the endpoint gives $|r_t - \varrho(Z_t)| \le 24hbe^{-2t}$. The triangle inequality proves (9.57). □

9.3.3. *Initial derivatives on a radial annulus.* For differentiation of the radius, assume in addition that

$$\varrho(Z) \ge cb, \qquad c > 0. \tag{9.59}$$

Equations (9.51) and (9.55) give

$$r_t(Z) \ge \frac{cb}{2}, \qquad \varrho(Z_t) \ge \frac{cb}{4}. \tag{9.60}$$

These positive lower bounds provide ordinary neighborhoods on which both square roots are smooth.

Proposition 9.11. *For every $c > 0$ there is $K_c > 0$ such that, under* (9.53) *and* (9.59), *with $\|Z\| \le b/2$ and $0 \le t \le T$,*

$$\|D(r_t - r_0)(Z)\| \le K_c(hb + b^2 t), \tag{9.61}$$

$$\|D(r_t - \varrho \circ H_{h,t})(Z)\| \le K_c \nu_t(Z). \tag{9.62}$$

All derivatives are ordinary initial derivatives at fixed h, t. The constant K_c is independent of h, b, T.

Proof. First recover the strong-wall derivative from its logarithm. Proposition 9.7 gives $\|DU_{h,t}(Z)\| \le 2$, and $\log \nu_t$ is its first component. Since $\nu_t > 0$,

$$\|D\nu_t(Z)\| \le 2\nu_t(Z), \qquad \|DH_{h,t}(Z)\| \le 2, \qquad \|D(\nu_t, H_{h,t})(Z)\| \le 2. \tag{9.63}$$

The last bound uses $\nu_t \le 1$. In the rest of the proof D denotes differentiation in the initial variable. Put

$$F_t(Z) = \mathcal{L}(\nu_t(Z), H_{h,t}(Z)), \qquad V_t(Z) = \frac{F_t(Z)}{2r_t(Z)}.$$

The chain rule, (9.49), and the three estimates in (9.50) give

$$\begin{aligned} \|DJ_t\| &\le 2Cb, \\ |F_t| &\le C(\nu_t b + b^2) J_t, \\ \|DF_t\| &\le 4C(\nu_t b^2 + b^3). \end{aligned} \tag{9.64}$$

For the last inequality the strong-wall contribution is at most $2C\nu_t b^3$, whereas the transverse contribution is at most $2C(\nu_t b^2 + b^3)$. Since $b \le 1$, their sum has the stated bound.

Using (9.60), differentiation of the square root and of its time rate gives

$$Dr_t = \frac{DJ_t}{2r_t}, \qquad DV_t = \frac{DF_t}{2r_t} - \frac{F_t}{2J_t} Dr_t.$$

It follows that, for a constant depending only on c,

$$\|Dr_t\| \le K_c, \qquad \|DV_t\| \le K_c(\nu_t b + b^2).$$

The actual flow and the positive square root defining r_t are jointly C^2 on an open neighborhood of each point under consideration. The local time identity $\partial_t r_t = V_t$ therefore implies, for every fixed initial direction u,

$$\frac{d}{dt}(Dr_t(Z)u) = DV_t(Z)u.$$

Compare its absolute derivative with $K_c\|u\|(hbe^{-2t} + b^2)$. The primitive $K_c\|u\|(hb(1 - e^{-2t})/2 + b^2 t)$ gives (9.61) after taking the operator norm.

It remains to compare the two endpoint radii at the derivative level. Define the quadratic defect

$$\mathcal{E}(n, Z) = \mathcal{J}(n, Z) - \varrho(Z)^2 = (g(n) - g(0)) \cdot (s^2, sv, v^2).$$

The quadratic differential estimate used above and (9.49) give, for a direction (η, W),

$$|D\mathcal{E}(n, Z)(\eta, W)| \le 12C|\eta|\|Z\|^2 + 24C|n|\|Z\|\|W\|.$$

Let $S_t(Z) = \varrho(H_{h,t}(Z))^2$. Composing with the actual state and using (9.63) yields

$$\|D(J_t - S_t)\| \le 24C\nu_t b^2 + 48C\nu_t b \le 72C\nu_t b. \tag{9.65}$$

Both J_t and S_t are positive by (9.60). The exact differential identity

$$D(r_t - \sqrt{S_t}) = \frac{D(J_t - S_t)}{2\sqrt{S_t}} + \frac{\sqrt{S_t} - r_t}{2r_t\sqrt{S_t}} DJ_t \tag{9.66}$$

preserves the small difference of the two radii. Use (9.65), $|r_t - \sqrt{S_t}| \le 24\nu_t b$ from (9.52), and $\|DJ_t\| \le 2Cb$ from (9.64). Each term on the right of (9.66) is bounded by a

constant depending on c times ν_t. This proves (9.62), after enlarging K_c to cover both conclusions. □

Define the phase error

$$E_{h,t}(Z) = \frac{\varrho(H_{h,t}(Z))}{\sqrt{3}} - \Phi_h(Z) = \frac{\varrho(Z_t) - r_0(Z)}{\sqrt{3}}. \tag{9.67}$$

Combining the two parts of Proposition 9.11, and using $\nu_t \le he^{-2t}$, gives

$$\|DE_{h,t}(Z)\| \le K_c(hb + b^2 t + he^{-2t}) \tag{9.68}$$

on the annulus (9.59). The value estimate comes from (9.57); division by $\sqrt{3} > 1$ only decreases its right-hand side.

Theorem 9.12 (Radial phase approximation)**.** *There exists $h_* > 0$ such that for every $0 < h \le h_*$ and $c_0, C_0 > 0$ there is a constant $K > 0$ with the following property. For each $q > 2$ there is $a_* > 0$ such that, if*

$$0 < a \le a_*, \qquad \|Z\| \le C_0 a, \qquad \varrho(Z) \ge c_0 a, \qquad T = \frac{q}{2}\log(1/a),$$

then the actual flow through $\Xi_h(Z)$ exists at T, $E_{h,T}$ has an ordinary derivative at Z, and

$$|E_{h,T}(Z)| \le Ka^2, \qquad \|DE_{h,T}(Z)\| \le Ka. \tag{9.69}$$

The derivative is taken with h, T fixed. The constant K can be chosen before q.

Proof. Take $h_* = 1/(4C)$, $b = 2C_0 a$, and $c = c_0/(2C_0)$. For each fixed $q > 2$, the limit $a|\log a| \to 0$ allows us to choose $a_* \le 1$ so that

$$2C_0 a \le \frac{1}{16}, \qquad C(2C_0 a)T \le \frac{1}{8}, \qquad aT \le 1 \quad (0 < a \le a_*). \tag{9.70}$$

The orbit exists and satisfies all previous estimates. Since $e^{-2T} = a^q$, its phase value error obeys

$$|E_{h,T}(Z)| \le 4CC_0^2 h a^2 + 16CC_0^3 a^3 T + 48C_0 h a^{q+1}.$$

The middle term is at most $C_0^2 a^2$ by the second condition in (9.70), and $a^{q+1} \le a^2$. Thus the value constant may be taken to be

$$K_{\mathrm{v}} = 4CC_0^2 h + C_0^2 + 48C_0 h.$$

Equation (9.68) gives a constant K_{d} depending only on c_0, C_0 such that

$$\|DE_{h,T}(Z)\| \le K_{\mathrm{d}}(ha + a^2 T + ha^q).$$

For example, $K_{\mathrm{d}} = K_c(2C_0 + 4C_0^2 + 1)$ suffices. Now $aT \le 1$ and $a^q \le a$, so the last bound is at most $K_{\mathrm{d}}(2h+1)a$. Taking $K = \max\{K_{\mathrm{v}}, K_{\mathrm{d}}(2h+1)\}$ proves (9.69). This choice is independent of q. □

Return to the curves $Z_\varsigma(\rho) = (\varsigma\rho, n_2, n_3)$ of Section 9.1. For

$$\varsigma \in \{-1, 1\}, \qquad 0 \le \varepsilon \le \frac{1}{64}, \qquad 0 \le n_2, n_3 \le 2\varepsilon a, \tag{9.71}$$

define the phase after the initial decay by

$$f(\rho) = \frac{\varrho(H_{h,T}(Z_\varsigma(\rho)))}{\sqrt{3}}, \qquad T = \frac{q}{2}\log(1/a).$$

The time T is fixed along the entire interval $[a, 2a]$.

Corollary 9.13. *For each $0 < h \le h_*$ there is $K > 0$ such that, for every $q > 2$, the following holds for all sufficiently small $a > 0$, uniformly over* (9.71)*. The function f has an ordinary derivative at every $\rho \in [a, 2a]$, and*

$$\begin{aligned} |f(\rho) - \Phi_h(Z_\varsigma(\rho))| &\le Ka^2, \\ \left| f'(\rho) - \frac{d}{d\rho}\Phi_h(Z_\varsigma(\rho)) \right| &\le Ka, \\ \frac{1}{4\sqrt{3}} \le f'(\rho) &\le \frac{2}{\sqrt{3}} + 1. \end{aligned} \tag{9.72}$$

In particular, f is strictly increasing. For $a \le x \le y \le 2a$,

$$\frac{y-x}{4\sqrt{3}} \le f(y) - f(x) \le \left(\frac{2}{\sqrt{3}} + 1\right)(y-x). \tag{9.73}$$

Proof. Equation (9.22) gives $\|Z_\varsigma(\rho)\| \le 2a$ and $\varrho(Z_\varsigma(\rho)) \ge a$. Apply Theorem 9.12 with $c_0 = 1$ and $C_0 = 2$. Since $Z_\varsigma' = (\varsigma, 0, 0)$ has maximum norm one, the chain rule gives the first two inequalities in (9.72). Proposition 9.5 bounds the reference derivative between $1/(2\sqrt{3})$ and $2/\sqrt{3}$. Reduce the scale threshold so that $Ka \le 1/(4\sqrt{3}) \le 1$. The third line follows, and the mean value theorem gives strict monotonicity and (9.73). □

The actual phase after the initial decay thus has a uniform positive derivative on both shear families. We next carry its comparison with Φ_h through the remaining passage to a core entrance, keeping track of the arrival time and the entrance coordinates.

9.4. **From the small section to core coordinates.** After the initial decay, the shear still has size a, whereas the denominator $1 + p$ of the core coordinates has size a^2. The core radius is consequently large. Its reciprocal is the quantity that remains comparable to the radial phase of the preceding subsection. To make this comparison also at the level of derivatives, we retain the small factors in the weak walls and then rescale the coordinate change by a.

Throughout this subsection, finite-dimensional norms are maximum norms. We enlarge the constant $C \ge 32$ from Sections 9.2 and 9.3 when necessary, and thereafter keep it fixed.

9.4.1. *Weak-wall factors.* For $X = (n, s, w_2, w_3)$, let

$$g_\pm(X) = Q(n, w_2, w_3) - 1 + \zeta(X) \pm \sqrt{3}s, \qquad \zeta(X) = \sqrt{1 - s^2 - Q(n, w_2, w_3)}.$$

These are the two weak-wall rates. They are smooth on a neighborhood of $\|X\| \le 1/16$ and vanish at zero. Compact derivative bounds, the mean value theorem, and $\|DE\| \le 1$ in logarithmic coordinates give

$$\max\{|g_+(X)|, |g_-(X)|\} \le C\|X\|, \qquad \|D(g_\pm \circ E)(y)\| \le C \quad (\|E(y)\| \le 1/16). \tag{9.74}$$

Lemma 9.14. *Suppose $h > 0$ and* (9.30) *holds. For $Z = (s, n_2, n_3)$ with $\|Z\| < b/2$ and $i = 2, 3$, there are positive C^2 functions $\mu_i(Z)$, defined on ordinary neighborhoods of these points, such that*

$$\begin{aligned} N_i(\phi_T \Xi_h(Z)) &= n_i \mu_i(Z), \\ |\log \mu_i(Z)| \le \frac{1}{4}, \qquad & \frac{1}{2} < \mu_i(Z) < 2, \\ \|D \log \mu_i(Z)\| \le 2CT, \qquad & \|D\mu_i(Z)\| \le 4CT. \end{aligned} \tag{9.75}$$

For a direction $v = (v_s, v_2, v_3)$ with $v_i = 0$,

$$\left|D_Z[N_i(\phi_T \Xi_h(Z))]v\right| \le 8CT\,|N_i(\phi_T \Xi_h(Z))|\,\|v\|. \tag{9.76}$$

Proof. Use the actual logarithmic-coordinate flow $U_{h,t}$ and put

$$L_i(Z) = \int_0^T g_i(E(U_{h,t}(Z)))\,dt, \qquad \mu_i(Z) = e^{L_i(Z)}, \qquad g_2 = g_+, \quad g_3 = g_-.$$

The scalar wall equation gives the first identity in (9.75), including at zero or signed initial walls. The orbit estimate and (9.74) imply

$$|L_i(Z)| \le C\int_0^T (he^{-2t} + b)\,dt \le \frac{Ch}{2} + CbT \le \frac{1}{4}.$$

Since $e^{1/4} < 2$, this gives both multiplier bounds.

The actual flow is jointly C^2 on a neighborhood of the compact time segment. Differentiating its rate integral at fixed T, and using $\|D_Z U_{h,t}\| \le 2$, yields

$$\|DL_i(Z)\| \le \int_0^T 2C\,dt = 2CT.$$

The same regularity permits a second differentiation under the finite integral, so L_i and μ_i are C^2. The chain rule gives $\|D\mu_i\| \le 4CT$. Finally, when $v_i = 0$, differentiation of the wall identity leaves only $n_i D\mu_i\, v$. Its absolute value is at most $4CT|n_i|\|v\|$. The lower bound $\mu_i > 1/2$ gives $|n_i| \le 2|N_i(\phi_T \Xi_h(Z))|$ and proves (9.76). □

Fix the shear families (9.71), and write

$$\begin{aligned} T &= \frac{q}{2}\log(1/a), \qquad x(\rho) = \phi_T \Xi_h(Z_\varsigma(\rho)),\\ x(\rho) &= (p(\rho), s(\rho), n(\rho), w_2(\rho), w_3(\rho)),\\ X(\rho) &= (n(\rho), s(\rho), w_2(\rho), w_3(\rho)), \qquad a \le \rho \le 2a. \end{aligned} \tag{9.77}$$

All derivatives in ρ below keep a, h, q, n_2, n_3 fixed; in particular they keep T fixed. For an arbitrary four-coordinate state $X = (n, s, w_2, w_3)$, define

$$\begin{aligned} d(X) &= s^2 + Q(n, w_2, w_3) = s^2 + n^2 + (w_2 - w_3)^2 - 2n(w_2 + w_3),\\ \mathcal{H}(X) &= \frac{d(X)}{1 + \sqrt{1 - d(X)}}. \end{aligned} \tag{9.78}$$

When $d(X) < 1$, rationalization gives $\mathcal{H}(X) = 1 - \sqrt{1 - d(X)}$. Consequently, on the negative constraint branch,

$$\mathcal{H}(X) = 1 + p.$$

Proposition 9.15. *There is $h_* > 0$ such that, for every $0 < h \le h_*$ and $q > 2$, the following holds for all sufficiently small $a > 0$, uniformly over* (9.71)*. The state* (9.77) *exists, has ordinary derivatives in ρ, and satisfies*

$$\begin{aligned} &p \le -\frac{15}{16}, \qquad 0 < n \le ha^q, \qquad \|X\| \le 8a, \qquad \|X'\| \le 2,\\ &\frac{3a}{4} \le \varsigma s \le \frac{9a}{4}, \qquad |n'| \le 2n,\\ &\frac{n_i}{2} \le w_i \le 2n_i, \qquad |w_i'| \le 8CTw_i \le \frac{8Cw_i}{a} \quad (i = 2, 3). \end{aligned} \tag{9.79}$$

Its height $A = 1 + p$ is positive, and

$$\frac{a^2}{4} \le A \le 6a^2, \qquad |A'| \le 512a. \tag{9.80}$$

The actual core coordinates

$$Y(\rho) = (M, U, V, W) = \frac{\sqrt{3}}{A}(s, n, w_2, w_3)$$

have ordinary derivatives and obey

$$\begin{aligned} \frac{1}{8a} \le \varsigma M \le \frac{18}{a}, \qquad & 0 < U \le 8ha^{q-2}, \\ \frac{n_2}{12a^2} \le V \le \frac{16n_2}{a^2}, \quad & \frac{n_3}{12a^2} \le W \le \frac{16n_3}{a^2}, \\ 0 \le V + W \le \frac{64\varepsilon}{a}, \qquad & 0 \le VW \le \frac{256n_2n_3}{a^4}. \end{aligned} \tag{9.81}$$

Proof. Choose h_* below the threshold of Theorem 9.9 and so that $Ch_* \le 1/4$ and $h_* \le 1/64$. Set $b = 8a$. By decreasing the threshold for a, arrange

$$0 < a \le 1, \qquad 8a \le 1/16, \qquad C(8a)T \le 1/8, \qquad aT \le 1.$$

The initial family has norm at most $2a < b/2$. Proposition 9.6 gives the claims about p, n and the center coordinates, since $e^{-2T} = a^q$. The physical-state derivative bound (9.63) gives $\|X'\| \le 2$ and $|n'| \le 2n$. Lemma 9.14 gives the two weak-wall bounds, including when an initial wall is zero; the last inequality uses $aT \le 1$.

To retain the sign and size of the shear, use the full transverse approximation of Theorem 9.9 with $C_0 = 2$. Decrease the threshold again so that its remainder is at most $a/8$. Also,

$$\|(\mathcal{T}(h) - I)Z_\varsigma(\rho)\| \le 3h\|Z_\varsigma(\rho)\| \le 6ha \le a/8.$$

Thus the center coordinates at time T differ from $Z_\varsigma(\rho)$ by at most $a/4$. This proves $3a/4 \le \varsigma s \le 9a/4$.

The remaining assertions concern the coordinate change. The preceding bounds, $h \le 1$, and $q > 2$ give

$$0 < n \le a^2, \qquad 0 \le w_2, w_3 \le 4\varepsilon a \le a/16.$$

Since $a \le 1/128$, the expression for d in (9.78) satisfies

$$\begin{aligned} d(X) &\ge \frac{9a^2}{16} - \frac{a^3}{4} \ge \frac{a^2}{2}, \\ d(X) &\le \frac{81a^2}{16} + a^4 + \frac{a^2}{256} \le 6a^2 \le \frac{1}{2}. \end{aligned}$$

The denominator of $\mathcal{H}$ lies between one and two. This proves the two height bounds. For a tangent vector $v = (v_n, v_s, v_2, v_3)$, an equivalent expression for the defect gives

$$Dd(X)v = 2sv_s + 4(nv_n + w_2v_2 + w_3v_3) - 2(n + w_2 + w_3)(v_n + v_2 + v_3).$$

Hence $|Dd(X)v| \le 32\|X\|\|v\|$. Along our curve,

$$A' = \frac{Dd(X)X'}{2\sqrt{1 - d(X)}}, \qquad |A'| \le 32(8a)2 = 512a.$$

This is an ordinary derivative: the negative constraint branch persists in a neighborhood of each initial point, so the identity $A = \mathcal{H}(X)$ holds there as well. The positivity of A also makes Y differentiable in such a neighborhood.

Finally, divide the shear and wall estimates by the two height bounds, using $1 \le \sqrt{3} \le 2$. For example,

$$\frac{1}{8a} = \frac{3a/4}{6a^2} \le \frac{\sqrt{3}\,\varsigma s}{A} \le \frac{2(9a/4)}{a^2/4} = \frac{18}{a}.$$

The same calculation gives the separate bounds on U, V, W. Adding the bounds on V, W and multiplying them proves the last line of (9.81). □

9.4.2. *Rescaling the reciprocal radius.* In core coordinates the radius is

$$r(Y) = \sqrt{(M-1)^2 + V^2}, \qquad Y = (M, U, V, W).$$

Its reciprocal phase in physical coordinates is

$$\mathcal{P}(X) = \frac{2\mathcal{H}(X)}{\sqrt{(\sqrt{3}s - \mathcal{H}(X))^2 + 3w_2^2}}. \tag{9.82}$$

When X belongs to the negative constraint branch and $\mathcal{H}(X) > 0$, direct substitution in the core coordinates gives

$$\frac{2}{r(Y)} = \mathcal{P}(X). \tag{9.83}$$

We compare this expression with $\sqrt{s^2 + (w_2 - w_3)^2}/\sqrt{3}$. Both quantities have size a. The numerator of $\mathcal{P}$ has size a^2 and its denominator has size a; rescaling makes their relative orders explicit.

To remove the scale, let $u = (\lambda, \alpha, \beta) \in \mathbb{R}^3$ and $z = (z_1, z_2) \in \mathbb{R}^2$. On the open set where the square roots below have positive arguments, define

$$\begin{aligned} X(u,z) &= (\lambda\alpha, \lambda z_1, \lambda z_2, \lambda\beta), \\ d_*(u,z) &= z_1^2 + Q(\alpha, z_2, \beta), \\ H_*(u,z) &= \frac{d_*(u,z)}{1 + \sqrt{1 - \lambda^2 d_*(u,z)}}, \\ D_*(u,z) &= (\sqrt{3}z_1 - \lambda H_*(u,z))^2 + 3z_2^2, \\ F(u,z) &= \frac{2H_*(u,z)}{\sqrt{D_*(u,z)}} - \frac{\sqrt{z_1^2 + (z_2 - \beta)^2}}{\sqrt{3}}. \end{aligned} \tag{9.84}$$

Homogeneity gives $d(X(u,z)) = \lambda^2 d_*(u,z)$ and $\mathcal{H}(X(u,z)) = \lambda^2 H_*(u,z)$. Thus, on this domain with $\lambda > 0$,

$$\mathcal{P}(X(u,z)) - \frac{\sqrt{(\lambda z_1)^2 + (\lambda z_2 - \lambda\beta)^2}}{\sqrt{3}} = \lambda F(u,z). \tag{9.85}$$

The decisive feature of this expression is its vanishing on the whole slice $u = 0$. For $z \neq 0$,

$$H_*(0,z) = \frac{z_1^2 + z_2^2}{2}, \qquad D_*(0,z) = 3(z_1^2 + z_2^2), \qquad F(0,z) = 0.$$

Fix $c > 0$ and $M_0 \geq c$, and restrict z to the compact set

$$\mathcal{K} = \{z : \|z\| \leq M_0,\ |z_1| \geq c\}.$$

All radicands and denominators in (9.84) are positive at $(0,z)$ for $z \in \mathcal{K}$. They remain positive on a neighborhood of this compact slice. Choose $\delta > 0$ so that a closed neighborhood of radius δ lies in this smooth domain, and let $B \geq 1$ bound both $\|DF\|$ and $\|D^2F\|$ there. The mean value theorem along the segment from $(0,z)$ to (u,z) gives

$$\begin{aligned} |F(u,z)| &\leq B\|u\|, \qquad \|DF(u,z)\| \leq B, \\ |DF(u,z)(0,\delta z)| &\leq B\|u\|\|\delta z\| \quad (z \in \mathcal{K},\ \|u\| \leq \delta). \end{aligned} \tag{9.86}$$

For the last inequality, differentiate $F(0,z) = 0$ to obtain $DF(0,z)(0,\delta z) = 0$, and apply the same mean value estimate to DF. In particular,

$$|DF(u,z)(\delta u, \delta z)| \leq B\|\delta u\| + B\|u\|\|\delta z\|. \tag{9.87}$$

This retains the small factor in the directions tangent to the limiting slice.

We now impose the additional smallness condition

$$n_3 \leq a^P, \qquad P > 2. \tag{9.88}$$

The scaled normal coordinates then tend to zero, while the shear remains in a fixed compact annulus.

Theorem 9.16. *There is $h_* > 0$ such that for each $0 < h \le h_*$ one can choose $K > 0$ with the following property. For every $q > 2$ and $P > 2$, there is $a_* > 0$ such that, whenever $0 < a \le a_*$, (9.71) and (9.88) hold, the actual phase*

$$f_{\mathrm{p}}(\rho) = \frac{2}{r(Y(\rho))} \quad (a \le \rho \le 2a)$$

has an ordinary derivative at every point of the interval and

$$\begin{aligned} |f_{\mathrm{p}}(\rho) - \Phi_h(Z_\varsigma(\rho))| &\le Ka^2, \\ \left| f_{\mathrm{p}}'(\rho) - \frac{d}{d\rho}\Phi_h(Z_\varsigma(\rho)) \right| &\le Ka, \\ \frac{1}{4\sqrt{3}} \le f_{\mathrm{p}}'(\rho) &\le \frac{2}{\sqrt{3}} + 1. \end{aligned} \tag{9.89}$$

In particular the reciprocal core radius is strictly increasing along both shear families.

Proof. Take h_* below the thresholds of Proposition 9.15 and Corollary 9.13, and also below $1/16$. For the scaling estimates fix $c = 3/4$ and $M_0 = 8$. The resulting constants δ, B are independent of h, q, P.

At the actual state $X(\rho)$, use

$$u(\rho) = \left(a, \frac{n(\rho)}{a}, \frac{w_3(\rho)}{a}\right), \qquad z(\rho) = \left(\frac{s(\rho)}{a}, \frac{w_2(\rho)}{a}\right).$$

Then $X(u(\rho), z(\rho)) = X(\rho)$ and $z(\rho) \in \mathcal{K}$ by (9.79). For $a \le 1$, the assumptions $q, P > 2$ give

$$n \le a^2, \qquad 0 \le w_3 \le 2n_3 \le 2a^2, \qquad \|u\| \le 2a.$$

Choose the scale threshold so that $2a \le \delta$. Since a is fixed when differentiating in ρ,

$$u' = \left(0, \frac{n'}{a}, \frac{w_3'}{a}\right), \qquad z' = \left(\frac{s'}{a}, \frac{w_2'}{a}\right).$$

The cutoff estimates imply

$$\|u'\| \le 16C, \qquad \|z'\| \le \frac{2}{a}.$$

Indeed, $|n'| \le 2n \le 2n/a$ for $a \le 1$, while $|w_3'| \le 8Cw_3/a$; division by a gives the stated uniform bound on u'.

Let

$$e(\rho) = \mathcal{P}(X(\rho)) - \frac{\sqrt{s(\rho)^2 + (w_2(\rho) - w_3(\rho))^2}}{\sqrt{3}}.$$

Equation (9.85) reads $e(\rho) = aF(u(\rho), z(\rho))$. The value and the separated derivative estimates yield

$$|e(\rho)| \le 2Ba^2, \qquad |e'(\rho)| \le aB(16C + 4).$$

Thus both errors are bounded by $K_{\mathrm{e}}a^2$ and $K_{\mathrm{e}}a$, respectively, where $K_{\mathrm{e}} = B(16C + 6)$ is independent of h, q, P.

At each initial point the negative constraint branch and positive height persist on an ordinary neighborhood. The scaled formula is smooth there as well. Consequently (9.83) is a local identity of differentiable functions. It gives

$$f_{\mathrm{p}}(\rho) = e(\rho) + \frac{\varrho(H_{h,T}(Z_\varsigma(\rho)))}{\sqrt{3}},$$

together with the corresponding derivative identity. Corollary 9.13 compares the second term with Φ_h with constants $K_{\mathrm{r}}a^2$ and $K_{\mathrm{r}}a$. Choose $K = K_{\mathrm{e}} + K_{\mathrm{r}}$ before choosing q, P. The first two bounds of (9.89) follow.

Finally, Proposition 9.5 gives the reference derivative between $1/(2\sqrt{3})$ and $2/\sqrt{3}$. Decrease a_* so that $Ka \le 1/(4\sqrt{3}) \le 1$. This proves the final line, and strict monotonicity follows from the mean value theorem. □

The reciprocal radius has therefore retained the reference phase through both the initial evolution and the change to core coordinates. We next connect the resulting state to a complete core entrance, using its wall sizes and the ordinary derivatives established above.

10. From a remaining core to the next entrance

The section transport can end inside a core, or just before its upward entrance. In both cases we follow the orbit to the same downward exit and then to the next entrance. The estimates retain the moving initial angle and the small wall product. They give the monotone phase and transverse slope bounds used in Theorem 10.35, which establishes statistical entrance families for positive shear with the second weak wall sufficiently small.

10.1. **Initial data and the remaining core passage.** We first control the core coordinates and their variation along the shear family, then follow the state to its outgoing section. Selecting the exit by its downward crossing makes the return smooth across initial states on the incoming boundary.

10.1.1. *Entrance data and the incoming passage.* For the cutoff state in (9.77), write $Y(\rho) = (M, U, V, W)$ and $P(\rho) = V(\rho)W(\rho)$. The following bounds hold for both shear signs. As in the preceding subsection, ρ varies in $[a, 2a]$ while the time $T = (q/2)\log(1/a)$ is fixed.

Proposition 10.1. *There are constants $h_* > 0$ and $K_0 > 0$ such that, for every $0 < h \le h_*$ and $q > 2$, all sufficiently small $a > 0$ satisfy the following estimates, uniformly over* (9.71)*:*

$$\begin{aligned} &|M'| \le \frac{K_0}{a^2}, \\ &|U'| \le \frac{K_0 U}{a}, \qquad |V'| \le \frac{K_0 V}{a}, \qquad |W'| \le \frac{K_0 W}{a}, \\ &|P'| \le \frac{K_0 P}{a}. \end{aligned} \tag{10.1}$$

Each positive wall has logarithmic derivative bounded in absolute value by K_0/a. If $n_2, n_3 > 0$, then

$$|(\log U)'| + |(\log V)'| + |(\log W)'| \le \frac{K_0}{a}. \tag{10.2}$$

If $n_2 = 0$, then $V = V' = 0$; if $n_3 = 0$, then $W = W' = 0$. In either case $P = P' = 0$.

Proof. The height estimates give

$$|A'| \le 512a \le \frac{2048A}{a}.$$

For nonnegative f, positive g, and a constant $c \ge 0$, the quotient rule shows that

$$|f'| \le K_f f/a, \quad |g'| \le K_g g/a \quad \Longrightarrow \quad \left|\left(\frac{cf}{g}\right)'\right| \le (K_f + K_g)\frac{cf/g}{a}.$$

This calculation remains valid when $f = 0$. By (9.79), each physical wall n, w_2, w_3 has relative derivative bounded by $8C/a$. Applying the quotient estimate with $g = A$, $c = \sqrt{3}$, and

$$B = 8C + 2048$$

gives the three wall inequalities with B in place of K_0. The product rule then gives $|P'| \le 2BP/a$. Division by a positive wall gives its logarithmic derivative, so the sum of the three logarithmic bounds is at most $3B/a$.

For the shear, (9.79) gives $|s'| \le 2$ and $\varsigma s \ge 3a/4$. Consequently

$$|(\varsigma s)'| \le \frac{4\varsigma s}{a}.$$

The same quotient estimate, now followed by $\varsigma M \le 18/a$, yields

$$|M'| \le (4 + 2048)\frac{\varsigma M}{a} \le \frac{18(4 + 2048)}{a^2}.$$

Thus $K_0 = 4B + 40000$ suffices for all the estimates. It is chosen before h and q. The vanishing assertions follow first from the value bounds in (9.81), and then from the corresponding relative derivative bounds. □

From now on take the positive shear sign $\varsigma = 1$, and define

$$k(\rho) = \sqrt{(M(\rho) - 1)^2 + V(\rho)^2}.$$

When $V(\rho) > 0$, also define

$$z(\rho) = \operatorname{arsinh} \frac{1 - M(\rho)}{V(\rho)}.$$

The symbol z here is the actual core angle (4.1).

Proposition 10.2. *There are constants $h_* > 0$ and $K_1 > 0$ such that for every $0 < h \le h_*$ and $q, \ell > 2$, all sufficiently small $a > 0$ have the following properties. For the positive-shear family* (9.71) *with $n_3 \le a^\ell$,*

$$\begin{aligned} &M > 1, \qquad k \ge 6, \qquad \frac{1}{16a} \le k \le \frac{32}{a}, \\ &\frac{1}{4096a^2} \le -k' \le \frac{1536}{a^2}. \end{aligned} \tag{10.3}$$

The radius has an ordinary derivative. Wherever $V > 0$, the angle also has an ordinary derivative and

$$|z'| \le \frac{K_1}{a}. \tag{10.4}$$

If $V \ge 1$, then

$$-\operatorname{arcosh} k \le z \le 0. \tag{10.5}$$

For $q > 3$, $\ell > 4$, and any $0 < \eta \le 10^{-4}$, one can further decrease the scale threshold so that

$$U + P \le \frac{\eta}{k}. \tag{10.6}$$

In particular, when $V \ge 1$, the actual state belongs to (4.5) *with $\kappa = k$.*

Proof. Take the common threshold of the coordinate and phase estimates from the preceding subsection, and require $a \le 1/128$. Equation (9.81) gives

$$\frac{1}{8a} \le M \le \frac{18}{a}, \qquad 0 \le V \le V + W \le \frac{1}{a}.$$

Thus $M > 1$, and

$$k \ge M - 1 \ge \frac{1}{16a} \ge 6, \qquad k \le M + V \le \frac{32}{a}.$$

If $V \ge 1$, the identity $\cosh z = k/V$ implies $\cosh |z| \le k$. Since $M > 1$, we have $z \le 0$, which proves (10.5).

For the radial derivative use the actual phase $f = 2/k$. The radius bounds give $a/16 \le f \le 32a$, and Theorem 9.16 gives $1/8 \le f' \le 3$. Hence $k = 2/f$ is ordinarily differentiable and

$$-k' = \frac{2f'}{f^2}, \qquad \frac{1}{4096a^2} \le \frac{2f'}{f^2} \le \frac{1536}{a^2}.$$

These numerical bounds are independent of the phase comparison constant; its dependence on h only changes the permitted threshold for a.

For $V > 0$, direct differentiation gives

$$z' = \frac{(M-1)V' - VM'}{kV}.$$

Use $|M-1| \le k$, $|V'| \le K_0 V/a$, and $|M'| \le K_0/a^2 \le 16K_0 k/a$. The factor V cancels in both terms, proving (10.4) with $K_1 = 17K_0$. No lower bound on a positive V is needed in this estimate.

It remains to obtain (10.6). The coordinate bounds and $n_2 \le a$ imply

$$0 < U \le 32ha^{q-2}, \qquad 0 \le P \le \frac{2048 n_3}{a^3}.$$

Together with $k \le 32/a$ and $n_3 \le a^\ell$, these give

$$k(U+P) \le 1024h\, a^{q-3} + 65536 a^{\ell-4}.$$

Both powers tend to zero for $q > 3$ and $\ell > 4$. Choose the scale threshold so that the right-hand side is at most η. At $V \ge 1$ the initial radius equals k, while $U, P \le \eta/k \le 10^{-4}/k$. These are precisely the remaining inequalities of (4.5). □

10.1.2. *Radius drift from an arbitrary initial angle.* The initial angle need not be the incoming endpoint $-\operatorname{arcosh} k$. In particular the initial active wall can be comparable to k. The following version of the radius estimate keeps this possibility.

Lemma 10.3. *Let a solution of* (4.4) *remain in* (4.5), *with* $\kappa = k \ge 6$, *for* $z_0 \le z \le z_1$. *Suppose* $R(z_0) = k$, *and put* $u_0 = U(z_0)$ *and* $P_0 = P(z_0)$. *Then*

$$|R(z_1) - k| \le 16\left(u_0 + \frac{P_0^2}{k}\right), \qquad |R(z_1)^2 - k^2| < \frac{1}{16}. \tag{10.7}$$

Proof. The contraction estimates (4.9) give

$$U(z) \le u_0 e^{-k(z-z_0)/4}, \qquad P(z) \le P_0.$$

Thus

$$W(z) \le P_0, \qquad W(z) \le \frac{P_0 \cosh z}{R(z)}, \qquad V(z) \le \frac{11k}{10}.$$

The four integral estimates in the proof of Lemma 4.1 now apply with V/k in place of V. Indeed, $V/k \le 11/10 \le (11/10)e^{z-z_0}$. Multiplying the resulting first integral by k gives

$$\int_{z_0}^{z_1} UV\,dz \le 14u_0, \quad \int_{z_0}^{z_1} UW\,dz \le \frac{4u_0P_0}{k},$$
$$\int_{z_0}^{z_1} \frac{U^2}{R}\,dz \le \frac{3u_0^2}{k^2}, \quad \int_{z_0}^{z_1} \frac{W^2}{R}\,dz \le \frac{4P_0^2}{k}.$$

The last estimate uses only the endpoint bounds $\cosh z_0, \cosh z_1 \le 11k/10$, which still follow from $V \ge 1$.

Integrating (4.11), we obtain

$$|R(z_1) - k| \le \frac{14u_0 + 4u_0P_0/k + 3u_0^2/k^2 + 4P_0^2/k}{999/1000} \le 16(u_0 + P_0^2/k).$$

For the last inequality it suffices to use $4u_0P_0/k \le u_0$ and $3u_0^2/k^2 \le u_0/2$. Multiplication by $R(z_1) + k \le 201k/100$ then gives

$$|R(z_1)^2 - k^2| \le \frac{804}{25}(ku_0 + P_0^2) < \frac{1}{16},$$

because $ku_0 \le 10^{-4}$ and $P_0 \le 1/60000$. □

10.1.3. *The first outgoing event.* We formulate the passage for a general initial core state. This also identifies the event reached by the small-section family.

Theorem 10.4. *Let* $y_0 = (M_0, U_0, V_0, W_0)$ *satisfy*

$$\begin{aligned} k &= \sqrt{(M_0-1)^2 + V_0^2} \geq 6, \\ V_0 &> 1 \quad \textit{or} \quad (V_0 = 1 \textit{ and } M_0 \geq 1), \\ 0 \leq U_0 &\leq 10^{-4}/k, \qquad 0 \leq P_0 := V_0 W_0 \leq 10^{-4}/k. \end{aligned} \tag{10.8}$$

Lift y_0 *by* (2.7) *and follow the normalized flow. Write its core coordinates as*

$$\beta(t) = (M(t), U(t), V(t), W(t)).$$

There is a first positive time t_* *with* $V(t_*) = 1$. *For* $0 < t < t_*$ *we have* $V(t) > 1$, *and*

$$M(t_*) < 1, \qquad z(t_*) > 0, \qquad \frac{dV}{dt}(t_*) \leq -\frac{A(t_*)k}{2} < 0, \qquad t_* \leq 2k\log(4k). \tag{10.9}$$

Throughout $0 \leq t \leq t_*$, *the projective chart is regular,* (4.5) *holds, and*

$$\begin{aligned} A(t) &\geq \frac{3}{2k^2}, \\ |R(t) - k| &\leq 16\left(U_0 + \frac{P_0^2}{k}\right), \qquad |R(t)^2 - k^2| < \frac{1}{16}. \end{aligned} \tag{10.10}$$

Proof. The initial state lies in (4.5), so (4.7) makes its inverse chart regular. The lift has $G = 1$ and nonnegative walls, and hence exists for every forward normalized time. If $V_0 > 1$, the active-wall inequality is initially strict. If $V_0 = 1$, the radius condition and $M_0 \geq 1$ place the state on the incoming branch, where $dV/dt \geq A(0)k/2 > 0$.

As in the proof of Proposition 4.2, continue the actual orbit under the closed conditions (4.5) and $A \geq 3/(2k^2)$. The wall equations preserve their zero faces and point inward at the positive upper faces. On each closed time segment the quantities A, V, d are positive. They remain so on an open interval containing that segment, where the strictly increasing angle gives a legitimate change of time. Lemma 10.3, applied on its angle image, improves the energy bound to $1/16$. Equation (4.7) improves the height bound to $3/k^2$. Thus the closed region continues past every finite endpoint at which $V > 1$.

It cannot contain the forward orbit indefinitely. On any such segment,

$$\frac{t}{k} \leq z(t) - z(0) \leq 2\log(4k),$$

by (4.7) and (4.8). The first boundary encountered is consequently $V = 1$, at some $t_* \leq 2k\log(4k)$. At any return approached from $V \geq 1$, the left derivative of V is nonpositive. The two crossing alternatives on $V = 1$, proved in Proposition 4.2, then give $M < 1$ and $dV/dt \leq -Ak/2$. In particular an interior touch would immediately leave $V \geq 1$, so no earlier return occurs. The definition of z gives $z(t_*) > 0$. Applying the radius lemma to every initial segment proves (10.10). □

For a projective state y near an initial state in (10.8), let $\beta_y(t)$ denote the core coordinates of the normalized flow starting at its inverse image. Define $\tau_\downarrow(y)$ to be the first positive time such that

$$V_y(\tau_\downarrow(y)) = 1, \qquad \partial_t V_y(\tau_\downarrow(y)) < 0,$$

whenever that time exists. Write its output in radius and wall coordinates as

$$\mathcal{R}_\downarrow(y) = (R, U, P)(\beta_y(\tau_\downarrow(y))). \tag{10.11}$$

Since $V = 1$ at this event, its third component is the outgoing weak wall W.

Proposition 10.5. *At every state satisfying* (10.8), $\tau_\downarrow$ *and* $\mathcal{R}_\downarrow$ *are defined and* C^2 *on an ordinary open neighborhood in* $\mathbb{R}^4$. *At the stated initial state,* $\tau_\downarrow = t_*$ *from Theorem 10.4. The nearby flow segments up to their moving exit times remain in the projective chart. If* $V_0 > 1$, *the same event is the first positive crossing of* $V = 1$ *throughout a sufficiently small neighborhood.*

Proof. The normalized flow and coordinate changes are jointly C^2 on a neighborhood of the compact orbit segment. At t_* the event function has a strictly negative time derivative. The implicit function theorem gives a C^2 time $\tau(y)$ and a unique nearby crossing. Its orientation remains downward by continuity.

To prove that this branch is the first downward crossing, consider the whole interval $[0, t_*]$. Near t_* local uniqueness excludes a second crossing. At each earlier positive time, $V > 1$, so continuity excludes a nearby hit of $V = 1$. At time zero there are two possibilities. If $V_0 > 1$, the same avoidance applies. If $V_0 = 1$, the initial crossing speed is strictly positive and remains positive on a neighborhood of that point in state and time. Any nearby crossing there is therefore upward. These neighborhoods cover the compact segment. Taking a finite subcover and using the continuity of τ gives one initial neighborhood on which there is no downward crossing in $0 < t < \tau(y)$. The same argument preserves the flow domain and $A > 0$ on the entire moving interval $[0, \tau(y)]$. Thus $\tau = \tau_\downarrow$ on this neighborhood. When $V_0 > 1$, the argument excludes every earlier crossing, proving the last assertion.

The endpoint depends C^2 on y. At the endpoint $V = 1 > 0$, so the radius and product coordinates in (10.11) are smooth. Their composition proves the asserted regularity of $\mathcal{R}_\downarrow$. □

We can now apply these results to the positive-shear family. Fix $q > 3$, $\ell > 4$, $0 < \eta \le 10^{-4}$, and $n_3 \le a^\ell$, and choose the common scale threshold in Proposition 10.2. Whenever its actual state at time T has $V \ge 1$, it satisfies (10.8). Its first outgoing time, measured from the original small section, is

$$t_{\mathrm{out}}(\rho) = T + \tau_\downarrow(Y(\rho)) > T.$$

The group property of the normalized flow identifies this concatenation with the same original orbit. The time and endpoint are C^2 functions of the initial coordinates near each such point, with T fixed. The bounds (10.9) and (10.10) apply after translating time by T. Together with the relative wall, radius, and angle derivatives, this supplies the initial data for estimating the variation of the remaining core passage.

10.2. **Variation of the remaining core passage.** The initial state now moves in both radius and angle. To follow its phase to the outgoing section, we keep these two variations separate until the last crossing. The weighted variational equation from Section 4.2 still has uniformly bounded growth. Its initial condition acquires an angular correction, and the integral of its strong component determines the resulting change in the radial derivative.

For a projective state $y = (M, U, V, W)$ with $V > 0$, write

$$\mathcal{C}(y) = (r(y), U, VW), \qquad r(y) = \sqrt{(M-1)^2 + V^2}, \qquad z(y) = \operatorname{arsinh} \frac{1-M}{V}.$$

Let $\mathcal{R}_\downarrow$ be the actual downward return defined in (10.11). Denote its radius component by $\mathcal{R}_{\downarrow,R}$, and put

$$K_v = \exp(2\pi + 48\varepsilon_0), \qquad C_p = 4096 K_v.$$

Thus K_v is the growth constant in (4.29).

Theorem 10.6. *Let* y_0 *satisfy* (10.8), *and set* $k = r(y_0)$, $u_0 = U_0$, $P_0 = V_0 W_0$. *For any ordinary initial direction* $w \in \mathbb{R}^4$, *define*

$$v = (v_R, v_U, v_P) = D\mathcal{C}(y_0)w, \qquad \delta z = Dz(y_0)w.$$

Then

$$|D\mathcal{R}_{\downarrow,R}(y_0)w - v_R| \le C_p\Big\{|v_U| + \frac{u_0+P_0}{k}(|v_R|+|v_P|) + \left(ku_0 + \frac{P_0^2}{k}\right)|\delta z|\Big\}. \tag{10.12}$$

The estimate holds at interior entrances and at the incoming boundary $V_0 = 1$, including states with a zero transverse wall.

The first two terms measure changes in the initial radius and wall amplitudes. The last term accounts for moving the starting point along the core. Its strong-wall coefficient is ku_0, whereas its weak-wall coefficient is quadratic in P_0. We retain this distinction when passing to the reciprocal radius.

10.2.1. *The moving initial angle.* Let $\mathcal{Y} = (R, U, P)$ along the reference passage, and let $F(z, \mathcal{Y})$ be the angular vector field in (4.4). Write $\alpha = z(y_0)$ and $\beta = z(t_*)$ for its initial and final angles. For the initial direction w, the variation at fixed angle is $\xi = \delta\mathcal{Y} - \delta z\, F$, as in (4.19). At the initial point,

$$\xi(\alpha) = v - \delta z\, F(\alpha, k, u_0, P_0). \tag{10.13}$$

Here v and δz are derivatives of the actual initial coordinates. The identity follows by differentiating the smooth coordinate changes and the actual flow. In particular the variational equation (4.20) applies to this ξ. In the original time it reads $d\xi/\,dt = ARd\, J\xi$; the angular integrals below use the positive weight $dz = ARd\, dt$ along the actual orbit.

At a core point with $R = k$, the field satisfies

$$|F_R| \le 2\left(ku_0 + \frac{P_0^2}{k}\right), \qquad |F_U| \le 2ku_0, \qquad |F_P| \le \frac{4P_0}{k}. \tag{10.14}$$

For the first bound, use the radial integrand estimate (4.11) at this point. Since $1 \le V \le k$ and $W = P_0/V \le P_0$,

$$UV \le ku_0, \quad UW \le u_0/60000, \quad U^2/k \le u_0/60000, \quad W^2/k \le P_0^2/k.$$

Division by $d \ge 999/1000$ gives the stated bound. The other two follow from the wall factors $F_U = UL_U$, $F_P = PL_P$ and (4.46).

Recall $S_k\xi = (\xi_R, k\xi_U, \xi_P)$. As $P_0 \le 1$ and $k \ge 6$, (10.14) implies

$$\|S_k F(\alpha, k, u_0, P_0)\|_\infty \le 4(k^2u_0 + P_0/k).$$

Introduce the quantity

$$Q_0 = |v_R| + k|v_U| + |v_P| + (k^2u_0 + P_0/k)|\delta z|. \tag{10.15}$$

The triangle inequality in (10.13) gives $\|S_k\xi(\alpha)\|_\infty \le 4Q_0$. Lemma 4.6, applied on every initial part of the passage, therefore gives

$$\|S_k\xi(z)\|_\infty \le Q, \qquad Q = 4K_vQ_0, \qquad \alpha \le z \le \beta. \tag{10.16}$$

The radius k in this weighting is fixed at the reference initial state; it is not differentiated inside the linear equation.

We first work with any bound $\|S_k\xi\|_\infty \le Q$ on a part of the core. Set

$$x = k\xi_U, \qquad x_0 = x(\alpha).$$

The strong row of (4.23) and the decay of U give

$$|x(z)| \le e^{-k(z-\alpha)/4}(|x_0| + 3ku_0Q(z-\alpha)). \tag{10.17}$$

Indeed $x' = B_{UU}x + h$, with $B_{UU} \le -k/4$ and $|h| \le 3kUQ$. Multiplication by $e^{k(z-\alpha)/4}$ leaves a nonpositive damping coefficient and forcing at most $3ku_0Q$. This is the scalar

comparison used in (4.30); it uses only the angle difference and applies to the present initial point. Its integral is

$$\int_\alpha^\beta |x(z)|\,dz \le \frac{4|x_0| + 48u_0Q}{k}. \tag{10.18}$$

The constants follow by integrating $e^{-ks/4}$ and $se^{-ks/4}$ over $s \ge 0$.

Put $\gamma = (101/99)^2$. The amplitude integrals needed for the radius row are

$$\int_\alpha^\beta U\,dz \le \frac{4u_0}{k}, \qquad \int_\alpha^\beta W^2\,dz \le 2\gamma P_0^2, \qquad \int_\alpha^\beta \frac{W}{kV}\,dz \le \frac{2\gamma P_0}{k}. \tag{10.19}$$

These are the estimates in (4.32), with the initial product P_0 in place of the entrance weak wall. For clarity, $P \le P_0$ and $R \ge 99k/100$, while $V \ge 1$ gives $\cosh\alpha, \cosh\beta \le 101k/100$. Thus the same endpoint estimate for $\int \cosh^2 z\,dz$ yields $\int V^{-2}\,dz \le 2\gamma$. The last two integrals follow from $W = P/V$.

Lemma 10.7. *On every part of the passage on which $\|S_k\xi\|_\infty \le Q$,*

$$|\xi_R(\beta) - \xi_R(\alpha)| \le 256\left\{\frac{|x_0|}{k} + \frac{(u_0 + P_0)Q}{k}\right\}. \tag{10.20}$$

Proof. The radius row estimate (4.34) gives

$$\left|\frac{d\xi_R}{dz}\right| \le \left(16U + \frac{8W^2}{k^2} + \frac{8W}{kV}\right)Q + 3|x|.$$

Here $2/\cosh z + 1/k \le 3$. Integrating with (10.18) and (10.19) gives

$$\frac{12|x_0|}{k} + \frac{208u_0Q}{k} + \frac{16\gamma P_0^2 Q}{k^2} + \frac{16\gamma P_0 Q}{k}.$$

Since $P_0 \le k$ and $\gamma < 2$, this is bounded by the right-hand side of (10.20). □

The strong contribution here is $|x_0|/k$. For a complete core passage the incoming condition $\cosh\alpha = k$ supplies the further factor $1/k$ in (4.33). Keeping the present estimate allows the starting angle to vary through the core.

10.2.2. *Projection onto the moving exit.* At the exit set

$$r_* = R(\beta), \qquad c_* = -\tanh\beta, \qquad f_* = F_R(\beta, \mathcal{Y}(\beta)).$$

The endpoint variation is tangent to $V = 1$. The same elimination as in (4.37) gives

$$D\mathcal{R}_{\downarrow,R}(y_0)w = s_*\xi_R(\beta), \qquad s_* = \frac{c_*}{c_* + f_*/r_*}. \tag{10.21}$$

This identity includes the derivative of the true exit time. Its contribution is a multiple of the flow vector, which is removed by the fixed-angle projection.

At the downward exit, $c_* \le -1/2$ and $f_* \le 0$. The exit factor bounds proved in Section 4.2, together with $U(\beta) \le u_0$ and $W(\beta) = P(\beta) \le P_0$, give

$$0 < s_* \le 1, \qquad |s_* - 1| \le 8\left(\frac{u_0}{k} + \frac{P_0^2}{k^2}\right). \tag{10.22}$$

The use of the initial product follows from $V = 1$ at the exit.

Proof of Theorem 10.6. The actual return is C^2 by Proposition 10.5. Equations (10.13) and (10.14) imply

$$\begin{aligned} |\xi_R(\alpha) - v_R| &\le 2(ku_0 + P_0^2/k)|\delta z|, \\ |x_0| &\le k|v_U| + 2k^2u_0|\delta z|. \end{aligned} \tag{10.23}$$

Write

$$N = |v_U| + \frac{u_0 + P_0}{k}(|v_R| + |v_P|) + (ku_0 + P_0^2/k)|\delta z|.$$

The core bounds imply $u_0 + P_0 \le 1$ and $P_0 \le k$. They yield

$$\frac{(u_0 + P_0)Q_0}{k} \le 2N, \qquad \frac{|x_0|}{k} \le 2N, \qquad |\xi_R(\alpha) - v_R| \le 2N. \tag{10.24}$$

To verify the first inequality, the $k|v_U|$ term contributes at most $|v_U|$, and

$$\frac{u_0 + P_0}{k}(k^2 u_0 + P_0/k) \le 2(ku_0 + P_0^2/k).$$

For this last bound, use $u_0 + P_0 \le 1$ in the first product, then $u_0 P_0/k^2 \le ku_0$ and $P_0^2/k^2 \le P_0^2/k$. The other two inequalities follow from (10.23). Also (10.22) gives

$$|s_* - 1|\,|v_R| \le 8\frac{u_0 + P_0}{k}|v_R| \le 8N.$$

Use $Q = 4K_v Q_0$ in (10.20). The first two inequalities in (10.24) give

$$|\xi_R(\beta) - \xi_R(\alpha)| \le (512 + 2048K_v)N.$$

Because $0 < s_* \le 1$, the exact exit identity implies

$$\begin{aligned} |s_*\xi_R(\beta) - v_R| &\le |\xi_R(\beta) - \xi_R(\alpha)| + |\xi_R(\alpha) - v_R| + |s_* - 1|\,|v_R| \\ &\le (522 + 2048K_v)N \le 4096K_v N. \end{aligned}$$

This proves the theorem. □

Let $Y(\rho)$ be an ordinarily differentiable curve of projective initial states, and set

$$k(\rho) = r(Y(\rho)), \quad U(\rho) = Y_U(\rho), \quad P_{\mathrm{in}}(\rho) = Y_V(\rho)Y_W(\rho), \quad z(\rho) = z(Y(\rho)).$$

Its outgoing radius is

$$f(\rho) = \mathcal{R}_{\downarrow,R}(Y(\rho)).$$

The following statement is local at the parameter in question; nearby points need only lie in the ordinary domain of the downward return.

Proposition 10.8. *Suppose $Y(\rho)$ satisfies* (10.8). *Let $0 < a \le 1$ and $K \ge 1$, and assume at this parameter that*

$$\begin{gathered} \frac{2}{Ka} \le k \le \frac{K}{a}, \qquad |k'| \le \frac{K}{a^2}, \\ |U'| \le \frac{KU}{a}, \qquad |P_{\mathrm{in}}'| \le \frac{KP_{\mathrm{in}}}{a}, \qquad |z'| \le \frac{K}{a}. \end{gathered} \tag{10.25}$$

Then f is ordinarily differentiable, $k/2 \le f \le 2k$, and

$$\begin{aligned} |f - k| &\le 16K(U + aP_{\mathrm{in}}^2), \\ |(f - k)'| &\le 4C_p K^2\,\frac{U + aP_{\mathrm{in}}}{a^2}. \end{aligned} \tag{10.26}$$

For the reciprocal phase,

$$\begin{aligned} \left|\frac{2}{f} - \frac{2}{k}\right| &\le 32K^3 a^2(U + aP_{\mathrm{in}}^2), \\ \left|\left(\frac{2}{f} - \frac{2}{k}\right)'\right| &\le (8C_p K^4 + 64K^5)(U + aP_{\mathrm{in}}). \end{aligned} \tag{10.27}$$

Proof. The chain rule and Proposition 10.5 give ordinary differentiability. The radius bounds and drift in (10.10) give $k/2 \le f \le 2k$ and

$$|f-k| \le 16(U + P_{\mathrm{in}}^2/k) \le 16K(U + aP_{\mathrm{in}}^2),$$

where $1/k \le Ka$. Apply Theorem 10.6 with $w = Y'(\rho)$. Its four input derivatives become k', U', P_{in}' and z' by the chain rule. Put $B = (U + aP_{\mathrm{in}})/a^2$. Since $a, P_{\mathrm{in}} \le 1$,

$$\frac{U}{a} \le B, \qquad \frac{U + P_{\mathrm{in}}}{a} \le B, \qquad (U + P_{\mathrm{in}})P_{\mathrm{in}} \le B, \qquad \frac{U}{a^2} + P_{\mathrm{in}}^2 \le B.$$

These inequalities and (10.25) bound each of the four terms

$$|U'|, \quad \frac{U + P_{\mathrm{in}}}{k}|k'|, \quad \frac{U + P_{\mathrm{in}}}{k}|P_{\mathrm{in}}'|, \quad (kU + P_{\mathrm{in}}^2/k)|z'|$$

by K^2B. Their sum proves the derivative estimate in (10.26).

For the reciprocal error set $e = f - k$ and $\psi = 2/f - 2/k$. The exact identities are

$$\psi = -\frac{2e}{kf}, \qquad \psi' = -\frac{2e'}{f^2} + \frac{2e(f+k)k'}{k^2f^2}.$$

Both f and k are at least $1/(Ka)$. Consequently

$$|\psi| \le 2K^2a^2|e|, \qquad \frac{f+k}{k^2f^2} = \frac{1}{k^2f} + \frac{1}{kf^2} \le 2(Ka)^3.$$

The value estimate follows at once. For the derivative,

$$|\psi'| \le 2K^2a^2|e'| + 4K^4a|e|.$$

Insert (10.26) and use $a(U + aP_{\mathrm{in}}^2) \le U + aP_{\mathrm{in}}$, which follows from $a \le 1$ and $aP_{\mathrm{in}} \le 1$. This proves (10.27). □

10.2.3. *Application to the small section.*

Corollary 10.9. *There are absolute constants $h_* > 0$ and $K \ge 1$ such that for every $0 < h \le h_*$, $q > 3$ and $\ell > 4$, all sufficiently small scales $a > 0$ have the following property. Consider the positive-shear family* (9.71) *with $n_3 \le a^\ell$, at the fixed time $T = (q/2)\log(1/a)$. At every $\rho \in [a, 2a]$ for which its actual core state $Y(\rho)$ has $V \ge 1$, the outgoing radius $f = \mathcal{R}_{\downarrow,R} \circ Y$ satisfies* (10.26) *and* (10.27). *These are ordinary derivatives along the original shear curve, with h, a, q, n_2, n_3 fixed.*

Proof. Take the common small-h threshold of Propositions 10.1 and 10.2. Choose

$$K = 2000 + K_0 + K_1$$

before h, q, ℓ, and take the small-a thresholds from those propositions, also requiring $a \le 1$. The radius bounds give

$$\frac{2}{Ka} \le \frac{1}{16a} \le k \le \frac{32}{a} \le \frac{K}{a}, \qquad |k'| \le \frac{1536}{a^2} \le \frac{K}{a^2}.$$

The wall and angle bounds give the other three inequalities in (10.25). Choose the entry bound in (10.6) with $\eta = \varepsilon_0$. When $V \ge 1$, the state satisfies (10.8). Proposition 10.8 now applies, including at $V = 1$. The flow identity at the end of Section 10.1 identifies this endpoint with the outgoing event on the original orbit, at time $T + \tau_\downarrow(Y(\rho))$. □

Thus the radial and angular derivatives supplied by the initial decay control the phase at the actual outgoing section. The remaining derivative estimates concern the two wall amplitudes. Their multiplier form will retain the small factors needed to pass from this exit to the next incoming section.

10.3. **Transverse variation of the remaining core.** The radius estimate controls the direction in which an initial curve reaches the outgoing section. We now compare its two transverse derivatives with this radial speed. The strong wall decays exponentially during the remaining passage. The weak wall needs no comparable contraction: its relative derivative grows more slowly than the radial speed. Together these estimates place the outgoing curve in a narrow radial cone.

For an initial state y_0 satisfying (10.8), keep the notation $k, u_0, P_0, \alpha, \beta$ of the preceding subsection. Write

$$(r_*, u_*, w_*) = \mathcal{R}_\downarrow(y_0).$$

The last coordinate is the outgoing wall W, since $V = 1$ at the return. Let L_U, L_P be the angular rates in (4.41). Along the actual orbit, put $q = dz/dt = ARd$ and define

$$\mu_j(y) = \exp\left(\int_0^{\tau_\downarrow(y)} q(t;y) L_j(z(t;y), \mathcal{Y}(t;y))\, dt\right), \qquad j \in \{U, P\}. \tag{10.28}$$

Here $\mathcal{Y} = (R, U, P)$ and the downward return time is the one defined in (10.11). The scalar wall equations give

$$u_*(y) = U(y)\mu_U(y), \qquad w_*(y) = V(y)W(y)\mu_P(y). \tag{10.29}$$

These identities hold on an ordinary neighborhood of y_0. Indeed, the compact base segment has positive V, projective denominator and angular denominator d. Continuous dependence and the smooth return time preserve these inequalities along every nearby segment. The original flow and the integrands in (10.28) are jointly C^1 there. Thus μ_U, μ_P and their logarithms are C^1 on that neighborhood, and both multipliers are strictly positive. This construction includes the incoming boundary $V_0 = 1$ and zero transverse walls.

Proposition 10.10. *For $w \in \mathbb{R}^4$, let $v = D\mathcal{C}(y_0)w$, $\delta z = Dz(y_0)w$, and*

$$Q_0 = |v_R| + k|v_U| + |v_P| + (k^2 u_0 + P_0/k)|\delta z|.$$

Then

$$\begin{aligned} |D\log\mu_U(y_0)w| &\le 40K_v \log(4k) Q_0 + 2k|\delta z|, \\ |D\log\mu_P(y_0)w| &\le 236K_v \frac{\log(4k)}{k^2} Q_0 + \frac{4}{k}|\delta z|. \end{aligned} \tag{10.30}$$

Proof. Let ξ be the variation at fixed angle. By (10.16), $\|S_k\xi\|_\infty \le Q = 4K_v Q_0$ throughout the passage. With $x = k\xi_U$, the rate bounds (4.47) give

$$|D_{\mathcal{Y}} L_U \xi| \le 3Q, \qquad |D_{\mathcal{Y}} L_P \xi| \le \frac{9Q}{k^2} + \frac{8}{k}|x|.$$

The angle interval has length at most $2\log(4k)$. Moreover, (10.18) implies

$$\frac{8}{k}\int_\alpha^\beta |x|\, dz \le \frac{32|x(\alpha)| + 384u_0 Q}{k^2} \le \frac{33Q}{k^2},$$

because $|x(\alpha)| \le Q$ and $u_0 \le 1/60000$. Consequently, for $I_j = \int_\alpha^\beta D_{\mathcal{Y}} L_j \xi\, dz$,

$$|I_U| \le 6\log(4k)Q, \qquad |I_P| \le 51\frac{\log(4k)}{k^2}Q. \tag{10.31}$$

Differentiating the original-time integral in (10.28) yields

$$D\log\mu_j(y_0)w = I_j + L_j(\beta, \mathcal{Y}_*)D\beta(y_0)w - L_j(\alpha, \mathcal{Y}_0)\delta z.$$

This is the moving-endpoint identity (4.49) with the actual initial angle. More explicitly, if ζ is the angle variation at fixed original time, the identity

$$\delta(qL_j) = qD_{\mathcal{Y}} L_j \xi + \frac{d}{dt}(L_j\zeta)$$

leaves the lower term $-L_j(\alpha, \mathcal{Y}_0)\delta z$. At the upper endpoint, $\zeta(t_*) + q(t_*)D\tau_\downarrow(y_0)w = D\beta(y_0)w$.

The exit tangency calculation used in (4.50) applies to this same downward branch. It gives

$$|D\beta(y_0)w| \le \frac{2|\xi_R(\beta)|}{k} \le \frac{2Q}{k}.$$

Together with $|L_U| \le 2k$ and $|L_P| \le 4/k$, this bounds the upper endpoint terms by $4Q$ and $8Q/k^2$, respectively. The lower terms are bounded by $2k|\delta z|$ and $4|\delta z|/k$. Combining these estimates with (10.31) and $\log(4k) \ge 1$ gives

$$|D\log\mu_U w| \le 10\log(4k)Q + 2k|\delta z|, \qquad |D\log\mu_P w| \le 59\frac{\log(4k)}{k^2}Q + \frac{4}{k}|\delta z|.$$

Substitute $Q = 4K_vQ_0$. □

Let $Y(\rho)$ be an ordinarily differentiable family of projective states. As in Proposition 10.8, write

$$k = r(Y(\rho)), \quad U = Y_1(\rho), \quad P_{\mathrm{in}} = Y_2(\rho)Y_3(\rho), \quad z = z(Y(\rho)), \quad (f, u_+, w_+) = \mathcal{R}_\downarrow(Y(\rho)).$$

Primes in the rest of this subsection denote ordinary derivatives with respect to ρ at the point under consideration. The scale a is held fixed.

Proposition 10.11. *Suppose that $Y(\rho)$ satisfies* (10.8), $0 < a \le 1$, $K \ge 1$, *and*

$$\begin{aligned} &\frac{1}{k} \le Ka, \qquad k \le \frac{K}{a}, \qquad |k'| \le \frac{K}{a^2}, \\ &|U'| \le \frac{KU}{a}, \qquad |P_{\mathrm{in}}'| \le \frac{KP_{\mathrm{in}}}{a}, \qquad |z'| \le \frac{K}{a}. \end{aligned} \tag{10.32}$$

Set

$$A_m = (200K_v + 2)K^2, \qquad B_m = 1180K_vK^4 + 4K^2, \qquad L = \log(4k). \tag{10.33}$$

The two outgoing walls are nonnegative and ordinarily differentiable, and their multipliers satisfy

$$|(\log\mu_U \circ Y)'| \le \frac{A_mL}{a^2}, \qquad |(\log\mu_P \circ Y)'| \le B_mL. \tag{10.34}$$

Moreover,

$$|u_+'| \le u_+\left(\frac{K}{a} + \frac{A_mL}{a^2}\right), \qquad |w_+'| \le w_+\left(\frac{K}{a} + B_mL\right). \tag{10.35}$$

If $U = 0$, then $u_+ = u_+' = 0$. If $P_{\mathrm{in}} = 0$, then $w_+ = w_+' = 0$.

Proof. Apply Proposition 10.10 with $w = Y'(\rho)$. The input derivatives are $k', U', P_{\mathrm{in}}', z'$. Since $kU \le 1$ and $P_{\mathrm{in}} \le 1$, the five terms in

$$Q_0 = |k'| + k|U'| + |P_{\mathrm{in}}'| + k^2U|z'| + \frac{P_{\mathrm{in}}}{k}|z'|$$

are each at most K^2/a^2. Hence $Q_0 \le 5K^2/a^2$. For the strong multiplier,

$$40K_vLQ_0 + 2k|z'| \le (200K_v + 2)\frac{K^2L}{a^2}.$$

For the weak multiplier, use $1/k^2 \le K^2a^2$:

$$\frac{236K_vLQ_0}{k^2} + \frac{4}{k}|z'| \le 1180K_vK^4L + 4K^2 \le B_mL.$$

These prove (10.34).

The local identities (10.29) give

$$u_+' = \mu_U(U' + U(\log\mu_U \circ Y)'), \qquad w_+' = \mu_P(P_{\mathrm{in}}' + P_{\mathrm{in}}(\log\mu_P \circ Y)').$$

Their factors are positive, so the relative input bounds in (10.32) prove (10.35). If an input wall is zero, its assumed derivative bound forces its derivative to vanish as well. The displayed product identities then give both zero-wall conclusions. □

To obtain a small transverse slope, the initial point must leave a definite angular interval before the exit. The positive-shear families from the small section have $z \le 0$, so they satisfy the following condition with $Z = 0$.

Lemma 10.12. *Suppose that y_0 satisfies* (10.8). *If $Z \ge 0$, $2\cosh Z \le k$ and $\alpha \le Z$, then*

$$0 \le u_* \le u_0 e^{-k/12}, \qquad 0 \le w_* \le P_0. \tag{10.36}$$

Proof. At the downward exit, $\beta > 0$ and $\cosh\beta = r_* \ge 99k/100$. Thus $Z \le \beta$ and

$$\frac{99}{50} \le \frac{\cosh\beta}{\cosh Z} \le e^{\beta - Z}.$$

The second inequality follows from the addition formula for cosh and $0 \le \tanh Z \le 1$. Since $\log x \ge 1 - 1/x$ for $x > 0$,

$$\beta - \alpha \ge \log(99/50) \ge 49/99 > 1/3.$$

Integrating the coefficient bounds $L_U \le -k/4$ and $L_P \le 0$ proves (10.36). □

Theorem 10.13. *Assume the hypotheses of Proposition 10.8, and let A_m, B_m be as in* (10.33). *Suppose in addition that, for some $D > 0$ and $Z \ge 0$,*

$$-k' \ge \frac{1}{Da^2}, \qquad 2\cosh Z \le k, \qquad z \le Z, \qquad 4C_p K^2 (U + aP_{\mathrm{in}}) \le \frac{1}{2D}. \tag{10.37}$$

Then

$$-f' \ge \frac{1}{2Da^2}, \qquad |f'| \le \frac{K + 1/(2D)}{a^2}, \tag{10.38}$$

and $0 \le u_+ \le U e^{-k/12}$, $0 \le w_+ \le P_{\mathrm{in}}$. The transverse derivatives satisfy

$$\begin{aligned} |u_+'| &\le 2D(K + 24A_m)\sqrt{U}\,\sqrt{u_+}\,|f'|, \\ |w_+'| &\le 2DK(1 + B_m)a\sqrt{P}_{\mathrm{in}}\,\sqrt{w_+}\,|f'|. \end{aligned} \tag{10.39}$$

The zero-wall conclusions of Proposition 10.11 hold as well.

Proof. The derivative error (10.26) and (10.37) give

$$|f' - k'| \le \frac{1}{2Da^2}.$$

Combining this with the lower bound for $-k'$ and the upper bound for $|k'|$ proves (10.38). Lemma 10.12 supplies the two value bounds. The stronger lower radius bound $2/(Ka) \le k$ implies $1/k \le Ka$, so Proposition 10.11 applies.

For the strong wall, take a square root in its value bound: $\sqrt{u_+} \le \sqrt{U}\, e^{-k/24}$. For $k \ge 6$, we have $\log(4k) \le k$ and $e^{-k/24}\log(4k) \le 24$. The latter follows from $xe^{-x} \le 1$ at $x = k/24$. Since $a \le 1$,

$$e^{-k/24}\left(\frac{K}{a} + \frac{A_m L}{a^2}\right) \le \frac{K + 24A_m}{a^2}.$$

Factor one square root from u_+ in (10.35) to obtain

$$|u_+'| \le \sqrt{u_+}\sqrt{U}\,\frac{K + 24A_m}{a^2}.$$

For the weak wall, $\sqrt{w_+} \le \sqrt{P}_{\mathrm{in}}$ and $L \le k \le K/a$ give

$$|w_+'| \le \sqrt{w_+}\sqrt{P}_{\mathrm{in}}\,\frac{K(1 + B_m)}{a}.$$

Finally, $1 \le 2Da^2|f'|$ converts these two inequalities into (10.39). All factors are nonnegative, and no division by a wall amplitude has been used. □

Corollary 10.14. *The constants h_*, K in Corollary 10.9 can also be used for the following conclusion. For $0 < h \le h_*$, $q > 3$, $\ell > 4$ and sufficiently small $a > 0$, every positive-shear input in that corollary with $V \ge 1$ satisfies* (10.38) *and* (10.39) *with $D = 4096$. Its outgoing walls satisfy $0 \le u_+ \le Ue^{-k/12}$ and $0 \le w_+ \le P_{\mathrm{in}}$. These are the walls on the original orbit at its first $V = 1$ event strictly after $T = (q/2)\log(1/a)$.*

Proof. Proposition 10.2 gives $-k' \ge 1/(4096a^2)$, $z \le 0$ and $U + P_{\mathrm{in}} \le 1/k$. Use $Z = 0$, for which $2\cosh Z \le k$ follows from $k \ge 6$. The radius bound gives $1/k \le Ka$, so $U + aP_{\mathrm{in}} \le U + P_{\mathrm{in}} \le Ka$. Restrict the scale further to

$$a \le \frac{1}{8C_pK^3 \cdot 4096}.$$

Then $4C_pK^2(U + aP_{\mathrm{in}}) \le 1/(2 \cdot 4096)$. All hypotheses of Theorem 10.13 hold. The orbit and first-event identities follow from the same time shift as in Corollary 10.9. □

The two cone coefficients depend on $\sqrt{U}$ and $a\sqrt{P}_{\mathrm{in}}$, respectively. The exponential decay absorbs the logarithmic derivative of the strong multiplier. For the weak wall, the extra factor a comes from comparing a derivative of order a^{-1} with a radial speed of order a^{-2}. This is the quantitative transverse control needed to continue the outgoing curve through the following connection.

10.4. **From the remaining core to the next entrance.** After the downward core crossing, the growing wall determines the next entrance. The estimates for a long connection apply once its parameters are expressed in terms of the actual core exit. We first make this identification, then follow the radius and its derivative through the connection. The resulting phase differs from the fractional-linear expression associated with a radius decrement of two by an error that retains the initial wall amplitudes.

Let y_0 satisfy (10.8), and write

$$k = r(y_0), \quad U = U_0, \quad P_{\mathrm{in}} = V_0W_0, \qquad (f, u, w) = \mathcal{R}_{\downarrow}(y_0).$$

The connection parameters are

$$\pi = (\mu, u, w), \qquad \mu = \sqrt{f^2 - 1} - 1, \qquad H(\mu + 1, 1) = f, \tag{10.40}$$

where $H(a, b) = \sqrt{a^2 + b^2}$. Denote by $T_c(\pi)$ the first $W = 1$ time of the connection in Section 5.1. At that event, write the state in the original frame as $(-m_*, U_*, V_*, 1)$. Its core coordinates in the reflected frame are

$$\Psi(y_0) = (g, \widehat{u}, \widehat{w}) = (H(m_* - 1, 1), U_*, V_*). \tag{10.41}$$

Proposition 10.15. *Suppose that $Z \ge 0$, $2\cosh Z \le k$, $z(y_0) \le Z$, and*

$$0 < \eta \le \varepsilon_0, \qquad 0 < U, P_{\mathrm{in}} \le \eta/k.$$

Then the connection is defined, and

$$4 \le \mu, \qquad k/2 \le \mu < k, \qquad 0 < u \le \frac{\eta}{\mu}, \qquad 0 < w \le \eta. \tag{10.42}$$

More precisely, $u \le Ue^{-k/12}$ and $w \le P_{\mathrm{in}}$. The parameter map π, the total time $\tau_{\downarrow} + T_c \circ \pi$, and Ψ are locally C^2. With the embeddings and reflection of Section 8.1,

$$\iota_+(\Psi(y_0)) = \mathsf{S}\Phi_{\tau_{\downarrow}(y_0)+T_c(\pi(y_0))}(y_0). \tag{10.43}$$

The connection ends at the first $W = 1$ event after the downward core exit, and $g \ge k/4$.

Proof. At the downward crossing, $V = 1$ and $M < 1$, so the actual state is

$$(1 - \sqrt{f^2 - 1}, u, 1, w) = (-\mu, u, 1, w).$$

The core energy bound gives $|(\mu + 1)^2 + 1 - k^2| \le 1/8$. Together with $k \ge 6$ and $\mu + 1 > 0$, this implies $4 \le \mu$, $k/2 \le \mu < k$. For the first two inequalities, an assumed upper bound $\mu < 4$ or $\mu < k/2$ contradicts the lower energy bound. For the third, $\mu \ge k$ contradicts the upper bound.

Positivity of the scalar wall multipliers and Lemma 10.12 give

$$0 < u \le Ue^{-k/12} \le \eta/k \le \eta/\mu, \qquad 0 < w \le P_{\mathrm{in}} \le \eta/k \le \eta.$$

Thus Proposition 5.2 applies to the actual exit state. The flow composition identity gives (10.43). The connection band has $|m_* - \mu| \le 1/2$; hence $m_* - 1 \ge \mu - 3/2 \ge \mu/2 \ge k/4$. It follows that $g = H(m_* - 1, 1) \ge k/4$ and that the reflected state belongs to the positive entrance branch ι_+.

For regularity, $f > 1$ makes the change (10.40) smooth. Proposition 10.5 and Proposition 5.2 give the C^2 time and state maps. Their composition proves the claim, including at an incoming boundary point with $V_0 = 1$. □

Consider a differentiable family $Y(\rho)$ satisfying the hypotheses of Proposition 10.8 at the parameter under consideration. Use U, P_{in}, k, z for its initial core quantities and $(f, u, w) = \mathcal{R}_\downarrow \circ Y$ for the outgoing quantities. Suppose also that $U, P_{\mathrm{in}} > 0$ and $z \le Z$, $2\cosh Z \le k$ for some $Z \ge 0$. Write $(g, \widehat{u}, \widehat{w}) = \Psi \circ Y$. All primes below are ordinary derivatives in ρ, with the scale a fixed.

The connection error from (5.53), evaluated on this curve, is

$$\Delta = g - f + 2. \tag{10.44}$$

This equality holds on a neighborhood of the base parameter. Indeed, (10.40) there identifies the connection's entrance radius with f, so it can also be differentiated.

Proposition 10.16. *For each $K \ge 1$ there is $C_K > 0$, depending only on K, such that*

$$|\Delta| \le C_K a(U + P_{\mathrm{in}}^2), \qquad |\Delta'| \le C_K(U + P_{\mathrm{in}}^2). \tag{10.45}$$

Put $E_K = 16K + 4C_pK^2 + C_K$. Then $g \ge k/4$ and

$$|g - (k-2)| \le E_K(U + aP_{\mathrm{in}}^2), \qquad |g' - k'| \le E_K \frac{U + aP_{\mathrm{in}}}{a^2}. \tag{10.46}$$

The reciprocal phase error

$$\mathcal{E} = \frac{2}{g} - \frac{2/k}{1 - 2/k} \tag{10.47}$$

is differentiable and satisfies

$$\begin{aligned} |\mathcal{E}| &\le 8E_K K^2 a^2 (U + aP_{\mathrm{in}}^2), \\ |\mathcal{E}'| &\le (8E_K K^2 + 32E_K K^4)(U + aP_{\mathrm{in}}). \end{aligned} \tag{10.48}$$

Proof. Proposition 10.15, with $\eta = \varepsilon_0$, supplies the connection and its regularity. In particular $g \ge k/4$. The preceding radius and multiplier estimates give

$$\begin{gathered} |f'| \le F_K/a^2, \qquad F_K = K + 8C_pK^2, \\ 0 < u \le Ue^{-k/12}, \qquad 0 < w \le P_{\mathrm{in}}, \\ |u'| \le u\left(\frac{K}{a} + \frac{A_m \log(4k)}{a^2}\right), \qquad |w'| \le w\left(\frac{K}{a} + B_m \log(4k)\right). \end{gathered}$$

The first estimate follows from (10.26) and $U + aP_{\mathrm{in}} \le 2$. The constants A_m, B_m are those in (10.33).

The exponential decay also removes the extra inverse scale in the strong derivative. Since $1/a \le Kk$,

$$\frac{e^{-k/12}\log(4k)}{a} \le K\big(e^{-k/24}\log(4k)\big)\big(ke^{-k/24}\big) \le 576K.$$

The two factors in parentheses are each at most 24. Using $\log(4k) \le k \le K/a$ for the weak wall, we obtain

$$a|u'| \le A_c U, \qquad a|w'| \le B_c P_{\mathrm{in}}, \qquad A_c = K(1+576A_m), \quad B_c = K(1+B_m).$$

Differentiate $H(\mu+1,1) = f$:

$$\chi\mu' = f', \qquad \chi = \frac{\mu+1}{H(\mu+1,1)} \ge \frac{1}{2}.$$

Thus $|\mu'| \le 2F_K/a^2$. Also $\mu \ge k/2$ and $k \ge 2/(Ka)$ imply $1/\mu \le Ka$. The value bound (5.20) therefore yields

$$|\Delta| \le 24Ka(U + P_{\mathrm{in}}^2).$$

Apply the three ordinary derivative bounds in (5.60) to (μ', u', w'). They give

$$\begin{aligned}|\Delta'| &\le 50000\left(\frac{u+w^2}{\mu^2}|\mu'| + \frac{|u'|}{\mu} + \frac{u+w}{\mu}|w'|\right)\\ &\le 50000\big(2K^2F_K + KA_c + KB_c\big)(U + P_{\mathrm{in}}^2).\end{aligned}$$

For the last term, use $(U + P_{\mathrm{in}})P_{\mathrm{in}} \le U + P_{\mathrm{in}}^2$, since $P_{\mathrm{in}} \le 1$. We may consequently choose

$$C_K = 1 + 24K + 50000(2K^2F_K + KA_c + KB_c).$$

This proves (10.45).

Now use the local identity $g = f - 2 + \Delta$. Adding the radius errors, with $a(U + P_{\mathrm{in}}^2) \le U + aP_{\mathrm{in}}^2$, gives the value estimate in (10.46). For the derivative, use $a^2(U + P_{\mathrm{in}}^2) \le U + aP_{\mathrm{in}}$ and (10.26). The single constant E_K bounds both sums.

Finally set $b = k - 2$ and $e = g - b$. Since $k \ge 6$, the fractional expression in (10.47) equals $2/b$ on a neighborhood of the base point. Moreover,

$$g, b \ge \frac{k}{4} \ge \frac{1}{2Ka}.$$

The reciprocal identities give

$$\mathcal{E} = -\frac{2e}{gb}, \qquad \mathcal{E}' = -\frac{2e'}{g^2} + \frac{2e(g+b)k'}{g^2b^2}.$$

They imply

$$|\mathcal{E}| \le 8K^2a^2|e|, \qquad |\mathcal{E}'| \le 8K^2a^2|e'| + 32K^4a|e|.$$

Insert (10.46) and use $a(U + aP_{\mathrm{in}}^2) \le U + aP_{\mathrm{in}}$. This proves (10.48). □

We next apply the cone estimate to the same family. For the positive-shear inputs from the small section, the outgoing cone can be made arbitrarily narrow before the connection is taken.

Lemma 10.17. *After decreasing h_* if necessary, for each $0 < h \le h_*$ there is $C_h > 0$, chosen before $q > 3$ and $\ell > 4$, with the following property. For every $\delta > 0$ and sufficiently small $a > 0$, the inputs of Corollary 10.14 satisfy*

$$E_a = a^{(q-2)/2} + \sqrt{n_3/a}, \qquad C_hE_a \le \delta, \tag{10.49}$$

and

$$|u'| \le C_hE_a\sqrt{u}\,|f'|, \qquad |w'| \le C_hE_a\sqrt{w}\,|f'|. \tag{10.50}$$

Here $(f,u,w) = \mathcal{R}_\downarrow \circ Y$ and $f' < 0$.

Proof. The initial core amplitudes obey $U \le 32ha^{q-2}$ and $P_{\mathrm{in}} \le 2048n_3/a^3$. Thus

$$\sqrt{U} \le \sqrt{32h}\, a^{(q-2)/2}, \qquad a\sqrt{P_{\mathrm{in}}} \le \sqrt{2048}\sqrt{n_3/a}.$$

Use (10.39) with $D = 4096$. Its two coefficients are bounded by C_hE_a if

$$C_h = 1 + 8192(K + 24A_m)\sqrt{32h} + 8192K(1 + B_m)\sqrt{2048}.$$

Finally $n_3 \le a^\ell$ implies $\sqrt{n_3/a} \le a^{(\ell-1)/2}$. Both exponents are positive, so the further scale restriction $C_hE_a \le \delta$ is uniform in the permitted n_3. The sign of f' follows from (10.38). □

Theorem 10.18. *There are $h_* > 0$ and constants $K \ge 1$, $C_K > 0$ such that the following holds for $0 < h \le h_*$, $q > 3$, $\ell > 4$ and sufficiently small $a > 0$. Take the positive-shear family* (9.71)*, now with $0 < n_3 \le a^\ell$, and its actual state $Y(\rho)$ at $T = (q/2)\log(1/a)$. At every $\rho \in [a, 2a]$ with $V(Y(\rho)) \ge 1$, the next entrance $G(\rho) = \Psi(Y(\rho)) = (g, \widehat{u}, \widehat{w})$ is on the original orbit, at time*

$$T + \tau_\downarrow(Y(\rho)) + T_c(\pi(\rho)), \tag{10.51}$$

with the reflected frame. It satisfies the phase estimates (10.45)–(10.48)*.*

The maps f and G are differentiable, $g > 0$, $f', g' < 0$, and

$$\left|\frac{g'}{f'} - 1\right| \le \frac{1}{4\mu}, \qquad \frac{3}{4} \le \frac{g'}{f'} \le \frac{5}{4}. \tag{10.52}$$

The two walls are positive, and

$$|\widehat{u}'| \le \sqrt{\widehat{u}}\,|g'|, \qquad |\widehat{w}'| \le \sqrt{\widehat{w}}\,|g'|. \tag{10.53}$$

Both reciprocal phase derivatives are positive, and

$$\frac{(2/g)'}{(2/f)'} = \left(\frac{f}{g}\right)^2 \frac{g'}{f'} > 0. \tag{10.54}$$

Proof. Choose the common small-h threshold for the preceding small-section estimates. Their K is chosen before h, q, ℓ, and Proposition 10.16 chooses C_K from K alone. The actual strong wall is positive because $h > 0$. The sign preservation of N_3 and the positive projective height give $W(Y(\rho)) > 0$; hence $P_{\mathrm{in}} > 0$. The other hypotheses of the phase proposition follow from Corollary 10.9 and $z \le 0$. Take $Z = 0$.

For the cone, use the threshold ε_ℓ in (5.66). Proposition 10.2, with entry bound $\eta = \varepsilon_\ell$, and Proposition 10.15 give

$$0 < u \le \varepsilon_\ell/\mu, \qquad 0 < w \le \varepsilon_\ell.$$

Apply Lemma 10.17 with $\delta = 1$. The input curve then satisfies $|u'| \le \sqrt{u}\,|f'|$ and $|w'| \le \sqrt{w}\,|f'|$. Differentiating (10.40) shows that its connection tangent is $v = (\mu', u', w')$ with $\chi\mu' = f'$. Since $f' < 0$, the vector v/f' has unit input radial derivative and lies in the cone of Proposition 5.10.

Let d be its image under the connection differential. Linearity gives $G' = f'd$. That proposition yields $d_r \ge 3/4$ and $|d_r - 1| \le 1/(4\mu)$, which prove (10.52) and $g' < 0$. Its transverse bounds imply (10.53). Positive wall multipliers along both passages give $\widehat{u}, \widehat{w} > 0$. The chain rule for reciprocal functions proves (10.54).

Finally, the downward event is the first $V = 1$ crossing strictly after T by Corollary 10.14. The following connection ends at its first $W = 1$ crossing. Applying (10.43) after the fixed initial time T gives (10.51). All derivatives refer to these actual event maps, including when $V(Y(\rho)) = 1$. □

The same original curve has now reached an upward core entrance with a controlled phase and a tangent in the radial cone. The next step is to obtain such an entrance also

when the fixed initial decay ends outside the core, giving a common section on which to apply the estimates.

10.5. **A common return from the positive shear family.** If the initial decay ends with $0 < V < 1$, the active wall must first grow to the incoming core section. This additional passage preserves the small wall amplitudes needed for the core estimates. Its endpoint then leads to the same downward event used in the preceding subsections. Selecting this event by its direction gives a smooth return as the intermediate state passes through $V = 1$.

We first work with a general projective state $y = (m, u, v, w)$. Write $r(y) = H(m-1, v)$ and retain T_{in} and $\mathcal{P}$ from Proposition 6.1.

Lemma 10.19. *Suppose that*

$$\begin{aligned} &k = r(y) \geq 8, \qquad m \geq 1, \qquad 0 < v < 1, \\ &0 \leq u, w \leq \varepsilon_0, \qquad 0 < \eta \leq \varepsilon_0, \qquad u + vw \leq \frac{\eta}{2k}. \end{aligned} \tag{10.55}$$

Let $y_+ = \mathcal{P}(y)$ and $\kappa = r(y_+)$. Then

$$\begin{aligned} &\kappa \geq 6, \qquad \frac{99}{100}k \leq \kappa \leq \frac{101}{100}k, \\ &(y_+)_V = 1, \qquad (y_+)_M > 1, \\ &0 \leq (y_+)_U \leq u, \qquad 0 \leq (y_+)_W \leq vw, \\ &(y_+)_U + (y_+)_W \leq \frac{\eta}{\kappa}. \end{aligned} \tag{10.56}$$

In particular, y_+ belongs to the closed core region (4.5)*, with reference radius κ, and its angle is $-\operatorname{arcosh}\kappa$.*

Proof. Proposition 6.1 applies to y. Its radius band gives the first line of (10.56), and the arrival is upward with $(y_+)_M > 1$. Set $L = \log(1/v) > 0$. The wall estimates at arrival give

$$(y_+)_U \leq u, \qquad (y_+)_W \leq vwe^{-L/k} \leq vw.$$

Since $\kappa \leq 2k$,

$$(y_+)_U + (y_+)_W \leq u + vw \leq \frac{\eta}{2k} \leq \frac{\eta}{\kappa}.$$

At this point the energy difference from its own reference radius is zero, and $V = 1$. These facts give all the defining core inequalities. Finally $\cosh z(y_+) = \kappa$ and $z(y_+) < 0$, which identify the incoming angle. □

Recall that $\tau_\downarrow$ selects the first positive $V = 1$ event with negative normalized-time derivative, and $\mathcal{R}_\downarrow$ records its radius and walls as in (10.11).

Proposition 10.20. *Under* (10.55)*, these maps are defined and C^2 on an ordinary open neighborhood of y. On a possibly smaller neighborhood they satisfy*

$$\begin{aligned} \tau_\downarrow &= T_{\mathrm{in}} + \tau_\downarrow \circ \mathcal{P}, \\ \mathcal{R}_\downarrow &= \mathcal{R}_\downarrow \circ \mathcal{P}. \end{aligned} \tag{10.57}$$

The corresponding orbit segments remain in the projective chart.

Proof. Put $S = T_{\mathrm{in}}(y) > 0$ and $y_+ = \mathcal{P}(y)$. Lemma 10.19 and Proposition 10.5 give a first downward time $t_+ = \tau_\downarrow(y_+) > 0$. The inverse projective chart at y_+ reconstructs the normalized state reached from y at time S. The group property of the normalized flow therefore gives

$$\Phi_t(y_+) = \Phi_{S+t}(y), \qquad t \geq 0, \tag{10.58}$$

whenever these projective expressions are used along the passages. The underlying normalized states agree, so their active-wall rates agree as well. Thus $S + t_+$ is a downward event for the orbit from y.

There is no downward event in $[0, S]$, because the active-wall rate is strictly positive throughout the incoming passage. Any downward event at a time $t > S$ corresponds, by (10.58), to one at time $t - S$ on the orbit from y_+. The minimality of t_+ gives $t - S \geq t_+$. This proves both identities at y.

For nearby initial points, the incoming passage still has positive height and positive active-wall rate on its whole moving time interval. This persistence is part of the ordinary-neighborhood construction in Proposition 6.1. Its endpoint varies continuously and hence lies in the ordinary neighborhood on which the first downward event from y_+ is smooth. Shrink the initial neighborhood also to preserve the positive denominator of the inverse chart. The same time-translation and minimality argument now proves (10.57) throughout this neighborhood. Both right-hand sides are C^2. The positive charts on the two compact orbit segments also persist there, completing the proof. □

Fix the positive shear sign and write the initial normalized state and its projective image at time T as

$$\begin{aligned} x_0(n_2, n_3, \rho) &= \Xi_h(\rho, n_2, n_3), \\ T &= \frac{q}{2}\log(1/a), \\ Y(n_2, n_3, \rho) &= \mathrm{proj}(\phi_T x_0(n_2, n_3, \rho)). \end{aligned} \tag{10.59}$$

Here proj is the coordinate map (2.6), and T is held fixed when the three initial parameters vary.

Lemma 10.21. *There is $h_* > 0$ such that, for each $0 < h \leq h_*$, $q > 3$, $\ell > 4$, and $0 < \eta \leq \varepsilon_0$, all sufficiently small $a > 0$ have the following property. Suppose*

$$\begin{aligned} &0 \leq \varepsilon \leq 1/64, \qquad 0 < n_2 \leq 2\varepsilon a, \\ &0 \leq n_3 \leq 2\varepsilon a, \qquad n_3 \leq a^\ell, \qquad a \leq \rho \leq 2a. \end{aligned} \tag{10.60}$$

If $Y_V < 1$, then $y = Y$ satisfies (10.55), with $y_U > 0$. Its upward arrival has radius

$$\frac{1}{32a} \leq \kappa \leq \frac{64}{a}. \tag{10.61}$$

The choice of the scale threshold is uniform in $\varepsilon, n_2, n_3, \rho$.

Proof. Apply Proposition 10.2 with $\eta/2$ in place of η and require $a \leq 1/512$. At the intermediate state,

$$M > 1, \qquad \frac{1}{16a} \leq k \leq \frac{32}{a}, \qquad U + VW \leq \frac{\eta}{2k}.$$

In particular $k \geq 8$ and $U \leq \varepsilon_0$. The value bounds (9.81) give $U > 0$ and $V > 0$ because $h, n_2 > 0$. They also give

$$0 \leq W \leq \frac{16 n_3}{a^2} \leq 16a^2 \leq \varepsilon_0,$$

where we used $\ell > 4$ and $a \leq 1$. Thus all the incoming conditions hold whenever $V < 1$. Lemma 10.19 gives $99k/100 \leq \kappa \leq 101k/100$, which implies (10.61). All thresholds used above precede the choice of the four parameters in (10.60). □

Theorem 10.22 (A common outgoing core crossing)**.** *There is $h_* > 0$ such that, for every $0 < h \leq h_*$, $q > 3$ and $\ell > 4$, all sufficiently small $a > 0$ have the following property on* (10.60)*. The time*

$$t_{\mathrm{out}} = T + \tau_\downarrow(Y) > T \tag{10.62}$$

is the first time strictly after T at which the original orbit has $V = 1$ with negative normalized-time derivative. Its core output is

$$\mathcal{R}_\downarrow(Y) = \mathcal{C}\big(\mathrm{proj}(\phi_{t_{\mathrm{out}}} x_0)\big), \qquad \mathcal{C}(M, U, V, W) = (H(M-1, V), U, VW). \tag{10.63}$$

The time and output are C^2 functions of (n_2, n_3, ρ) near each such point. The conclusion includes $n_3 = 0$ and intermediate states with $V = 1$.

Proof. Choose a common threshold for Proposition 10.2 and Lemma 10.21, taking $\eta = \varepsilon_0$. If $Y_V \geq 1$, then $Y_M > 1$ and (10.8) holds. Proposition 10.5 gives the first downward event and its C^2 maps. If $Y_V < 1$, use Proposition 10.20. Both cases select the event defining $\tau_\downarrow$.

The inverse chart at Y equals $\phi_T x_0$, so the normalized flow composition gives

$$\Phi_t(Y) = \operatorname{proj}(\phi_{T+t} x_0).$$

Equality of the normalized states also identifies the crossing rates. A candidate downward event after T therefore corresponds to one after time zero from Y, with elapsed time reduced by T. This proves the first-event claim and (10.63).

The radicand of Ξ_h is strictly positive at the initial state, the normalized flow is smooth near (x_0, T), and its height at T is positive. Consequently Y is jointly C^2 in all three initial parameters on an ordinary open neighborhood. Composition with the ordinary maps $\tau_\downarrow$ and $\mathcal{R}_\downarrow$ proves the stated regularity, also at a zero weak wall. □

For positive n_3, the weak wall grows after the core exit and reaches the next incoming section. Define $(f, u, w) = \mathcal{R}_\downarrow(Y)$ and use the same connection parameters and reflected output as in (10.40) and (10.41):

$$\begin{aligned} \pi &= (\sqrt{f^2 - 1} - 1, u, w), \qquad G = \Psi(Y) = (g, \widehat{u}, \widehat{w}), \\ t_{\text{next}} &= T + \tau_\downarrow(Y) + T_c(\pi). \end{aligned} \tag{10.64}$$

Corollary 10.23. *After decreasing the common thresholds in Theorem 10.22, assume also $n_3 > 0$. The connection in* (10.64) *is defined, ends at the first $W = 1$ event after t_{out}, and satisfies*

$$\iota_+(G) = \mathsf{S} \operatorname{proj}(\phi_{t_{\text{next}}} x_0). \tag{10.65}$$

The maps $\pi, G, 2/g, t_{\text{next}}$ are jointly C^2 in (n_2, n_3, ρ) near every point of the stated domain. All three coordinates of G are positive.

Proof. For $Y_V \geq 1$, apply Proposition 10.15 with $Z = 0$ and $\eta = \varepsilon_0$; the angle and wall conditions follow from Proposition 10.2.

For $Y_V < 1$, put $y_+ = \mathcal{P}(Y)$ and $\kappa = r(y_+)$. Positivity of the wall multipliers gives $(y_+)_U, (y_+)_W > 0$. Lemma 10.19 puts this state on the incoming boundary with total wall amplitude at most ε_0/κ. Apply Proposition 10.15 at y_+, again with $Z = 0$. Its parameters obey

$$\mu \geq 4, \quad \kappa/2 \leq \mu < \kappa, \quad 0 < u \leq \varepsilon_0/\mu, \quad 0 < w \leq \varepsilon_0,$$

and more precisely $u \leq (y_+)_U e^{-\kappa/12}$ and $w \leq (y_+)_W$. By (10.57), these are exactly the parameters computed from $\mathcal{R}_\downarrow(Y)$ in (10.64). The time identity in the same proposition and the group property give (10.65). In both cases the first-arrival statement follows from the actual connection, and its positive walls and positive radius give the positivity of G.

Finally $f > 1$, so the parameter change defining π is smooth. Theorem 10.22 gives its joint C^2 dependence. The connection time and state are C^2 functions of π by Proposition 5.2. Composing these maps proves the regularity of G and t_{next}, and $g > 0$ permits division to obtain $2/g$. □

The same event formulas now apply throughout each positive-wall shear interval. To control its image as a radial graph, it remains to estimate the differential during the additional incoming passage. The decay in (6.5) will retain the wall amplitudes in that comparison.

10.6. **Radial directions through the incoming passage.** An arbitrarily small active wall can produce a long wait before the core. The wall derivatives grow with this waiting time, but the walls themselves decay. Keeping half of that decay gives uniform estimates for the square-root slopes used by the core and connection maps. The core radius has a simpler response: its derivative changes by one power of the small scale less than its leading radial derivative.

For $y = (m, u, v, w)$ write

$$k = r(y) = H(m-1, v), \qquad b = u + w, \qquad J = uv + v^2 b^2, \qquad L = \log(1/v).$$

Let $\mathcal{P}(y)$ be the actual first upward arrival, and denote its core radius and two walls by

$$\kappa(y) = r(\mathcal{P}(y)), \qquad U_+(y) = \mathcal{P}_U(y), \qquad W_+(y) = \mathcal{P}_W(y).$$

The stable radius used in Sections 6.4–6.6 will be denoted here by

$$r_s(y) = H(P(m, u, w) - 1, v).$$

This notation distinguishes it from the input core radius r.

Proposition 10.24. *There are universal constants $C > 0$ and $0 < \varepsilon_1 \le \varepsilon_0$ with the following property. Suppose*

$$k \ge 7/4, \qquad m \ge 1, \qquad 0 < u, w \le \varepsilon_1, \qquad 0 < v < 1.$$

For an ordinary initial direction $d = (d_m, d_u, d_v, d_w)$ and a number $D_0 \ge 0$, assume

$$|d_m| \le D_0 k^2, \qquad |d_u| \le D_0 k u, \qquad |d_v| \le D_0 k v, \qquad |d_w| \le D_0 k w. \tag{10.66}$$

Then

$$|\kappa(y) - k| \le C\frac{b^2 + J}{k}, \tag{10.67}$$

$$|D\kappa(y)d - Dr(y)d| \le CD_0(b^2 + kJ), \tag{10.68}$$

$$|DU_+(y)d| \le CD_0 k^2 \sqrt{uv}\sqrt{U_+(y)},$$

$$|DW_+(y)d| \le CD_0 k \sqrt{wv}\sqrt{W_+(y)}. \tag{10.69}$$

The derivatives are those of the actual arrival maps on their ordinary open domains.

Proof. We first compare the two input radii, then use the arrival columns already proved in the stable radial coordinates. Put

$$c = m - 1, \qquad \chi = P(m, u, w) - 1, \qquad s = r_s(y).$$

The incoming radius band gives $m > 2$ and $c, \chi > 0$. Choose $A > 0$ for the projection estimates (6.85). They imply

$$|\chi - c| \le Ab^2/k, \qquad |s - k| \le Ab^2/k.$$

Since $s, k \ge k/2$ and $c/k \le 1$, the identities

$$\frac{\chi}{s} - \frac{c}{k} = \frac{\chi - c}{s} - \frac{c}{k}\frac{s-k}{s}, \qquad \frac{1}{s} - \frac{1}{k} = -\frac{s-k}{sk}$$

give

$$\left|\frac{\chi}{s} - \frac{c}{k}\right| \le \frac{4Ab^2}{k^2}, \qquad \left|\frac{1}{s} - \frac{1}{k}\right| \le \frac{4Ab^2}{k^3}.$$

The same projection bounds, applied to (10.66), give

$$|DP(y)d - d_m| \le \frac{Ab^2}{k^2}|d_m| + \frac{Ab}{k}(|d_u| + |d_w|) \le 2AD_0 b^2.$$

The exact differential comparison is

$$Dr_s(y)d - Dr(y)d = \frac{\chi}{s}(DP(y)d - d_m) + \left(\frac{\chi}{s} - \frac{c}{k}\right) d_m + v\left(\frac{1}{s} - \frac{1}{k}\right) d_v.$$

Here $\chi/s \le 1$ and $v \le k$. Substitution yields

$$|Dr_s(y)d - Dr(y)d| \le 10AD_0b^2, \qquad |Dr_s(y)d| \le (10A+2)D_0k^2. \tag{10.70}$$

For the second inequality, use $Dr(y)d = (cd_m + vd_v)/k$, so that $|Dr(y)d| \le 2D_0k^2$, and $b^2 \le 1 \le k^2$.

In the local input coordinates (r_s, u, v, w), let F be the physical arrival radius of (6.125). Its derivative columns, with the waiting factors bounded by (6.130), imply

$$\begin{aligned} |F - s| &\le CJ/k, \qquad |F_s - 1| \le CJ/k^2, \\ |F_u| &\le C(v + v^2b)/k, \qquad |F_v| \le C(u + vb^2), \\ |F_w| &\le C(uv + v^2b)/k. \end{aligned}$$

These are direct weakenings of Proposition 6.26, after a universal enlargement of C. The transformed direction is $e = (Dr_s(y)d, d_u, d_v, d_w)$. Put $D_1 = (10A+2)D_0$. All its components satisfy (10.66) with D_1 and with the first component interpreted as radial. Expanding $DF\,e - e_s$ by its four columns gives

$$|DF\,e - e_s| \le CD_1\{J + u(v + v^2b) + kJ + w(uv + v^2b)\} \le 4CD_1kJ.$$

The last inequality uses $u(v + v^2b) \le J$ and $w(uv + v^2b) \le J$, together with $k \ge 1$. The local inverse chart identifies $DF\,e$ with $D\kappa(y)d$. Combining this estimate with (10.70) proves (10.68). The value estimate follows in the same way from $|\kappa - r_s| \le CJ/k$ and $|r_s - r| \le Ab^2/k$.

For the walls, use the logarithmic arrival columns (6.100) and (6.123). In the same radial coordinates they give

$$\begin{aligned} |D\ell_U\,e| &\le CL|e_s| + (1/u + C)|e_u| + (Ck/v)|e_v| + C|e_w|, \\ |D\ell_W\,e| &\le (CL/k^2)|e_s| + (C/k^2)|e_u| + (C/v)|e_v| + (1/w + C/k^2)|e_w|. \end{aligned}$$

The relative wall bounds cancel the inverse input amplitudes. Using $u, w \le 1$ and $k \ge 1$ gives, after another universal enlargement of C,

$$\begin{aligned} |DU_+(y)d| &\le CD_0k^2(1+L)U_+(y), \\ |DW_+(y)d| &\le CD_0(k+L)W_+(y). \end{aligned} \tag{10.71}$$

The actual decay at arrival is

$$U_+ \le uve^{-kL/3}, \qquad W_+ \le wve^{-L/k}.$$

Taking square roots, while leaving one square-root factor of each output wall unchanged in (10.71), reduces the remaining time factors to

$$(1+L)e^{-kL/6} \le 6, \qquad (k+L)e^{-L/(2k)} \le 2k.$$

Indeed, both follow from $(1+x)e^{-x} \le 1$ for $x \ge 0$, with $x = kL/6$ and $x = L/(2k)$ respectively. This proves (10.69). □

Return to the positive shear family (10.59). In this subsection we assume (10.60) with $n_3 > 0$. Write

$$Y(\rho) = (M, U, V, W), \qquad k(\rho) = r(Y(\rho)), \qquad Z(\rho) = \mathcal{P}(Y(\rho)),$$

and

$$\kappa(\rho) = r(Z(\rho)), \qquad U_+(\rho) = Z_U(\rho), \qquad W_+(\rho) = Z_W(\rho).$$

The incoming maps in these expressions are used at parameters with $V(\rho) < 1$. Derivatives keep a, h, q, n_2, n_3 fixed.

Lemma 10.25. *There are $h_* > 0$ and $D_* > 0$, chosen before h, q, ℓ, such that the following holds. For every $0 < h \le h_*$, $q > 3$, $\ell > 4$ and $0 < \delta \le \varepsilon_0$, all sufficiently small $a > 0$ satisfy, at every parameter with $V < 1$,*

$$\begin{aligned} &0 < U, W \le \delta, \qquad U + VW \le \frac{\delta}{2k}, \\ &|M'| \le D_* k^2, \qquad |U'| \le D_* k U, \qquad |V'| \le D_* k V, \qquad |W'| \le D_* k W. \end{aligned} \tag{10.72}$$

The radius bounds $1/(16a) \le k \le 32/a$ and $k \ge 8$ also hold, and Y is ordinarily differentiable.

Proof. Use Lemma 10.21 with $\eta = \delta$, and Proposition 10.1. Their common small-h threshold is independent of q, ℓ, δ. The first lemma gives $U + VW \le \delta/(2k)$ and hence $U \le \delta$. Require also $a \le \min\{1, \delta/16\}$. Since $n_3 \le a^\ell$ and $\ell > 4$,

$$W \le 16 n_3/a^2 \le 16 a^2 \le 16 a \le \delta.$$

Both walls are positive by their multiplicative equations and $h, n_3 > 0$. The lower radius bound gives $1/a \le 16k$. Thus the relative derivative estimates (10.1) imply (10.72) with $D_* = 256 K_0$. □

Proposition 10.26. *There are constants $h_* > 0$ and $B > 0$, chosen before h, q, ℓ, such that for every $0 < h \le h_*$, $q > 3$, $\ell > 4$, and all sufficiently small $a > 0$, the preceding family satisfies, wherever $V < 1$,*

$$|\kappa - k| \le Ba, \qquad |\kappa' - k'| \le B/a, \qquad \frac{1}{8192 a^2} \le -\kappa' \le \frac{1537}{a^2}. \tag{10.73}$$

Moreover, for any prescribed $\theta > 0$ the scale threshold can be decreased so that

$$|U_+'| \le \theta \sqrt{U_+}\, |\kappa'|, \qquad |W_+'| \le \theta \sqrt{W_+}\, |\kappa'|. \tag{10.74}$$

All these are ordinary derivatives, and $U_+, W_+ > 0$.

Proof. Take $\delta \le \varepsilon_1$ in Lemma 10.25. Proposition 10.24 applies to $d = Y'(\rho)$ with $D_0 = D_*$. Here $b = U + W \le 1$ and $J = UV + V^2 b^2 \le 2$. Using the two radius bounds gives

$$|\kappa - k| \le 3C/k \le 48 C a, \qquad |\kappa' - k'| \le C D_* (1 + 2k) \le 96 C D_*/a.$$

Choose $B = 48C + 96 C D_* + 1$. The original radial speed is bounded by (10.3). If $a \le 1/(8192 B)$, then $B/a \le 1/(8192 a^2)$. Subtracting this error from the lower bound for $-k'$, and adding it to the upper bound, proves (10.73).

For the wall slopes, the radius upper bound and the new lower speed give

$$k^2 \le \frac{1024}{a^2} \le 2^{23} |\kappa'|.$$

Let $E = 2^{23} C D_*$ and choose

$$\delta = \min\{\varepsilon_1, (\theta/E)^2\}.$$

After decreasing a as required by the preceding lemma, both $\sqrt{UV}$ and $\sqrt{WV}$ are at most $\sqrt{\delta}$, because $V < 1$. Use (10.69) and $k \le k^2$ to obtain

$$|U_+'| \le E\sqrt{\delta}\, \sqrt{U_+}\, |\kappa'|, \qquad |W_+'| \le E\sqrt{\delta}\, \sqrt{W_+}\, |\kappa'|.$$

Since $E\sqrt{\delta} \le \theta$, these are (10.74). Differentiability and positivity follow from the ordinary arrival map and its positive wall multipliers. □

The first downward event from Y follows a full core passage from Z. To transfer derivatives as well as states, this identification is needed on a neighborhood of the parameter.

Put $X = (\kappa, U_+, W_+)$ and let $\mathcal{R}$ denote the full core return in radius and wall coordinates. Near a parameter with $V < 1$, the upward arrival satisfies $Z_V = 1$ and $Z_M > 1$, so $Z = \iota_+(X)$. The full core time and the downward time from this state agree at the reference point. Both are continuous nearby and solve the same transverse arrival equation near that time. The local uniqueness from the implicit function theorem makes them equal on a neighborhood. Combining this fact with (10.57) gives

$$\mathcal{R}_\downarrow(Y(\rho)) = \mathcal{R}(X(\rho)) \quad \text{on a neighborhood of the reference parameter.} \tag{10.75}$$

Consequently their ordinary derivatives agree there.

Proposition 10.27. *For the positive-wall family above, after a common decrease of the scale thresholds, write*

$$(f, u, w) = \mathcal{R}_\downarrow(Y), \qquad G = \Psi(Y) = (g, \widehat{u}, \widehat{w}), \qquad \mu = \sqrt{f^2 - 1} - 1.$$

At every parameter with $V < 1$, $\kappa', f', g' < 0$, and

$$\left|\frac{f'}{\kappa'} - 1\right| \le \frac{1}{32\kappa}, \qquad \frac{3}{4} \le \frac{f'}{\kappa'} \le \frac{5}{4}, \tag{10.76}$$

$$|u'| \le \left(1 - \frac{1}{16\kappa}\right)\sqrt{u}\,|f'|, \qquad |w'| \le \left(1 - \frac{1}{16\kappa}\right)\sqrt{w}\,|f'|, \tag{10.77}$$

$$\left|\frac{g'}{f'} - 1\right| \le \frac{1}{4\mu}, \qquad \frac{3}{4} \le \frac{g'}{f'} \le \frac{5}{4}, \tag{10.78}$$

$$|\widehat{u}'| \le \sqrt{\widehat{u}}\,|g'|, \qquad |\widehat{w}'| \le \sqrt{\widehat{w}}\,|g'|. \tag{10.79}$$

Both reciprocal phase derivatives are positive, and

$$\frac{(2/g)'}{(2/f)'} = \left(\frac{f}{g}\right)^2 \frac{g'}{f'}. \tag{10.80}$$

Proof. Choose the core threshold ε_c from Proposition 4.10. Use Lemma 10.25 with $\delta = \varepsilon_c$, and Proposition 10.26 with $\theta = 1$. By Lemma 10.19, the upward arrival has

$$\kappa \ge 6, \qquad 0 < U_+, W_+ \le \varepsilon_c/\kappa.$$

Since $\kappa' < 0$, the tangent X'/κ' has radial component one and wall components bounded by $\sqrt{U_+}$ and $\sqrt{W_+}$. Apply the core cone estimate to this vector. By linearity and (10.75), the radial component of its image is f'/κ'. The estimate (4.52) gives (10.76); in particular $f' < 0$. Multiplying the two wall inequalities by $|\kappa'|$ gives (10.77).

Next choose the long-connection threshold ε_ℓ of (5.66). Apply Lemma 10.21 with $\eta = \varepsilon_\ell$. At the ensuing core exit, Proposition 10.15 gives

$$\mu \ge 4, \qquad 0 < u \le \varepsilon_\ell/\mu, \qquad 0 < w \le \varepsilon_\ell.$$

The factor in (10.77) is at most one, so the input tangent to this connection satisfies $|u'| \le \sqrt{u}\,|f'|$ and $|w'| \le \sqrt{w}\,|f'|$. In connection coordinates, set $\chi = (\mu+1)/H(\mu+1, 1)$; differentiation gives $\chi\mu' = f'$. Thus $(\mu', u', w')/f'$ has unit physical radial derivative. Proposition 5.10, followed by multiplication by $f' < 0$, proves (10.78) and (10.79), with $g' < 0$. Finally $f, g > 0$. Differentiating their reciprocals gives the positivity and (10.80). □

Corollary 10.28 (The whole positive shear interval)**.** *There is $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 3$, $\ell > 4$, and all sufficiently small $a > 0$, the common return $G = (g, \widehat{u}, \widehat{w})$ of* (10.64) *satisfies*

$$g' < 0, \qquad |\widehat{u}'| \le \sqrt{\widehat{u}}\,|g'|, \qquad |\widehat{w}'| \le \sqrt{\widehat{w}}\,|g'|, \qquad (2/g)' > 0 \tag{10.81}$$

at every $\rho \in [a, 2a]$ under (10.60) *with $n_3 > 0$. The return and its phase are C^2 functions throughout this interval.*

Proof. Take the common thresholds of Theorem 10.18 and Proposition 10.27. At a point with $V \geq 1$, the former applies; at a point with $V < 1$, the latter applies. Corollary 10.23 identifies their return with the same C^2 map G, including at $V = 1$. Thus the derivative bounds hold at every parameter. The phase is C^2 because $g > 0$, and its derivative is $-2g'/g^2 > 0$. □

The incoming wait has therefore preserved the radial orientation and supplied the transverse slope control needed at the next core. The subsequent phase comparison can now be made along the whole positive shear interval.

10.7. **Phase and amplitude at the next entrance.** The return constructed above reaches a complete core entrance from every point of the positive shear family. We now compare its phase with the reference phase on the original small section. A bounded change of a radius of order a^{-1} produces a phase change of order a^2. For derivatives, the corresponding estimate follows by combining the radial responses of the core and the connection before taking reciprocals. The wall estimates retain the product of the two initial weak walls through the same passage.

Throughout this subsection we use the family (10.59) under (10.60), with $n_3 > 0$, $q > 3$ and $\ell > 4$. Write

$$Y = (M, U, V, W), \qquad k = r(Y), \qquad G = \Psi(Y) = (g, \widehat{u}, \widehat{w}),$$

and set

$$x = \frac{2}{k}, \qquad \xi = \frac{2}{g}, \qquad F(\rho) = \Phi_h(Z_+(\rho)). \tag{10.82}$$

Here Z_+ is the positive shear curve on the original section. All derivatives are ordinary derivatives in ρ, with a, h, q, n_2, n_3 fixed. Thresholds for a are uniform in ε, n_2, n_3 and $\rho \in [a, 2a]$ subject to the stated conditions.

At a parameter with $V < 1$, let (κ, U_+, W_+) be the coordinates of the preceding upward arrival, and let (f, u, w) be those of the following downward arrival. The connection parameter is $\mu = \sqrt{f^2 - 1} - 1$.

Lemma 10.29. *There are absolute constants $h_* > 0$ and $D > 0$ such that for every $0 < h \leq h_*$, $q > 3$, $\ell > 4$, and all sufficiently small $a > 0$, at every parameter with $V < 1$ one has*

$$k, g \geq \frac{1}{64a}, \qquad |g - k| \leq D, \qquad |k'| \leq \frac{D}{a^2}, \qquad |g' - k'| \leq \frac{D}{a}. \tag{10.83}$$

Both radii are ordinarily differentiable there.

Proof. Choose the common thresholds of the incoming radius and cone estimates, with the input bound $\eta = \varepsilon_0$. By Lemma 10.19 and (10.75), the downward arrival is the full core return from (κ, U_+, W_+). Applying Proposition 10.15 at this upward state gives $\mu \geq \kappa/2$. Put $b = U_+ + W_+^2$. The full core radius estimate (4.14) and the connection decrement (5.20) give

$$|f - \kappa| \leq \frac{16b}{\kappa}, \qquad |g - f + 2| \leq \frac{24(u + w^2)}{\mu} \leq \frac{48b}{\kappa}.$$

We used $u \leq U_+$ and $w \leq W_+$ in the second inequality. Consequently

$$|g - (\kappa - 2)| \leq \frac{64b}{\kappa}, \qquad 1 \leq \kappa - g \leq 3, \qquad \frac{\kappa}{2} \leq g \leq \kappa. \tag{10.84}$$

Indeed, $\kappa \geq 6$ and $U_+, W_+ \leq \varepsilon_0/\kappa$ make the first right-hand side at most one; the last pair then follows from $\kappa - 3 \geq \kappa/2$.

For the derivative, use the two ratios in Proposition 10.27. Since $\kappa', f' < 0$,

$$\begin{aligned} |g' - \kappa'| &\le |g' - f'| + |f' - \kappa'| \\ &\le \frac{|f'|}{4\mu} + \frac{|\kappa'|}{32\kappa} \\ &\le \left(\frac{5}{8} + \frac{1}{32}\right)\frac{|\kappa'|}{\kappa} \le \frac{|\kappa'|}{\kappa}. \end{aligned}$$

The middle step uses $|f'| \le (5/4)|\kappa'|$ and $\mu \ge \kappa/2$. Together with (10.61) and (10.73), this yields

$$|g' - \kappa'| \le \frac{49184}{a}, \qquad |g' - k'| \le \frac{49184 + B}{a}.$$

For $a \le 1$, the same estimates imply

$$|g - k| \le 3 + Ba \le 3 + B, \qquad |k'| \le \frac{1537}{a^2} + \frac{B}{a} \le \frac{1537 + B}{a^2}.$$

The lower bounds for k and κ, followed by $g \ge \kappa/2$, give those in (10.83). Thus $D = B + 50000$ suffices. Differentiability follows from the ordinary arrival maps already identified. □

Proposition 10.30. *There is $h_* > 0$ such that for each $0 < h \le h_*$ one can choose $C_h > 0$ before q and ℓ with the following property. For every $q > 3$ and $\ell > 4$, all sufficiently small $a > 0$ satisfy*

$$|\xi - F| \le C_h a^2, \qquad |\xi' - F'| \le C_h a, \qquad \frac{1}{8\sqrt{3}} \le \xi' \le C_h \tag{10.85}$$

at every $\rho \in [a, 2a]$ in the positive-wall family. The phase ξ is a C^2 function near each such parameter, including those with $V = 1$.

Proof. The common return is C^2 and has $g > 0$ by Corollary 10.23. Theorem 9.16 supplies a constant $K_h > 0$, independent of q, ℓ, such that

$$|x - F| \le K_h a^2, \qquad |x' - F'| \le K_h a, \qquad \frac{1}{4\sqrt{3}} \le x' \le D_0 := \frac{2}{\sqrt{3}} + 1. \tag{10.86}$$

We compare ξ with x in the two possible locations of Y.

If $V < 1$, Lemma 10.29 gives positive radii $g, k \ge 1/(64a)$. The exact reciprocal identities are

$$\xi - x = -\frac{2(g-k)}{gk}, \qquad \xi' - x' = -\frac{2(g'-k')}{g^2} + \frac{2k'(g-k)(g+k)}{g^2k^2}. \tag{10.87}$$

In particular,

$$\frac{1}{gk} \le 64^2 a^2, \qquad \frac{g+k}{g^2k^2} = \frac{1}{gk^2} + \frac{1}{g^2k} \le 2 \cdot 64^3 a^3.$$

Substitution of (10.83) therefore gives

$$|\xi - x| \le A_- a^2, \qquad |\xi' - x'| \le A_- a, \qquad A_- = 2^{13}D + 2^{20}D^2.$$

This constant is independent of h, q, ℓ.

Suppose now that $V \ge 1$. Use the constant K from Corollary 10.9 and the associated E_K in Proposition 10.16. Write $P_{\mathrm{in}} = VW$ and

$$\mathcal{E} = \xi - \frac{x}{1-x}, \qquad Q = \frac{x}{1-x} - x.$$

The input estimates, with $a \le 1$, give

$$U, P_{\mathrm{in}} \le 1, \qquad U + P_{\mathrm{in}} \le Ka, \qquad 0 \le x \le \frac{1}{2}, \qquad x \le Ka.$$

Here the amplitude bound follows from (10.6) and $k \geq 2/(Ka)$. Since $aP_{\rm in}^2 \leq P_{\rm in}$ and $aP_{\rm in} \leq P_{\rm in}$, (10.48) implies

$$|\mathcal{E}| \leq V_+ a^2, \qquad |\mathcal{E}'| \leq A_+ a, \quad V_+ = 8E_K K^3, \quad A_+ = (8E_K K^2 + 32E_K K^4)K.$$

The correction Q has the explicit value and derivative

$$Q = \frac{x^2}{1-x}, \qquad Q' = \frac{x(2-x)x'}{(1-x)^2}.$$

Thus (10.86) gives

$$|Q| \leq 2K^2 a^2, \qquad |Q'| \leq 8KD_0 a.$$

Since $\xi - x = \mathcal{E} + Q$ as ordinary functions,

$$|\xi - x| \leq (V_+ + 2K^2)a^2, \qquad |\xi' - x'| \leq H_+ a, \qquad H_+ = A_+ + 8KD_0.$$

The same functions are used at $V = 1$: the downward event and its connection agree on an ordinary neighborhood with the common return of Corollary 10.23.

Choose

$$C_- = A_- + K_h + D_0 + 1, \qquad C_+ = 1 + K_h + V_+ + 2K^2 + H_+ + D_0, \qquad C_h = C_- + C_+.$$

All these constants are chosen before q, ℓ. Take the common scale threshold from the preceding estimates and decrease it so that

$$A_- a \leq \frac{1}{8\sqrt{3}}, \qquad H_+ a \leq \frac{1}{8\sqrt{3}}.$$

In either case, the derivative differs from x' by at most $1/(8\sqrt{3})$; hence it is at least $1/(8\sqrt{3})$ and at most $D_0 + 1$. Finally, adding the value and derivative errors in (10.86) proves (10.85). □

The connection converts a bound on its incoming weak wall into a bound on both outgoing walls. This permits the product VW to control the entire output, even when the incoming strong wall is much larger.

For connection parameters (μ, u, w) with

$$\mu \geq 4, \qquad 0 < u \leq \varepsilon_0/\mu, \qquad 0 < w \leq \varepsilon_0,$$

put $L = \log(1/w) \geq 0$. The decay estimates (5.12) imply

$$0 < \widehat{u} \leq u e^{-\mu L/2} \leq w, \qquad 0 < \widehat{w} \leq w e^{-2L/\mu} \leq w, \qquad \widehat{u} + \widehat{w} \leq 2w. \tag{10.88}$$

Indeed $u \leq 1$ and $\mu/2 \geq 1$, so the first exponential is at most $e^{-L} = w$. The connection band also gives

$$\frac{\mu}{2} \leq g \leq 2\mu. \tag{10.89}$$

To see this, write $g = H(m_* - 1, 1)$. From $|m_* - \mu| \leq 1/2$ and $\mu \geq 4$ one has $\mu/2 \leq m_* - 1 \leq \mu$, and hence $g^2 \leq \mu^2 + 1 \leq 4\mu^2$.

Proposition 10.31. *There is $h_* > 0$ such that for every $0 < h \leq h_*$, $q > 3$, $\ell > 4$, and all sufficiently small $a > 0$, the common return satisfies*

$$\frac{1}{64a} \leq g \leq \frac{64}{a}, \qquad \widehat{u}, \widehat{w} > 0, \qquad \widehat{u} + \widehat{w} \leq \frac{512 n_2 n_3}{a^4} \tag{10.90}$$

throughout the positive-wall family. This is the actual reflected arrival at time $t_{\rm next} > T$ in (10.65).

Proof. Use the common existence thresholds with the input bound $\eta = \varepsilon_0$. If $V \geq 1$, Proposition 10.15 gives $k/2 \leq \mu < k$ and $w \leq VW$ at the downward arrival. The radius bounds $1/(16a) \leq k \leq 32/a$, combined with (10.89), yield

$$\frac{1}{64a} \leq \frac{k}{4} \leq g \leq 2k \leq \frac{64}{a}.$$

If $V < 1$, the incoming passage has $1/(32a) \leq \kappa \leq 64/a$ by (10.61). Equation (10.84) then gives the same output radius bounds. For the walls in this case, Lemma 10.19 gives $W_+ \leq VW$, and the full core has $w \leq W_+$. Thus $w \leq VW$ in both cases.

Apply (10.88), followed by the initial coordinate bound (9.81), to obtain

$$\widehat{u} + \widehat{w} \leq 2w \leq 2VW \leq \frac{512 n_2 n_3}{a^4}.$$

All walls remain positive by their scalar multiplier equations. The common return has already been identified with the original orbit in Corollary 10.23; the positive downward and connection times give $t_{\mathrm{next}} > T$. □

The passage to the next entrance has thus retained both the reference phase and a bound in terms of the initial weak-wall product. Together with the cone estimate (10.81), these bounds describe the size, orientation and transverse slopes of the returned shear family.

10.8. **Statistical entrance families from the small section.** The returned shear curves have a phase with a uniform positive derivative and two small transverse walls. To apply the statistical estimates of Section 8.10, we express each curve as a graph over its output radius. The square-root cone bounds allow this graph to be extended to all radii with a fixed amplitude cost. We then verify that the construction is measurable in the two weak-wall labels. This makes the excluded set a single Borel set on the original family of initial conditions.

Fix $h, q, \ell, a, \varepsilon$ as in Section 10.7, and put

$$\begin{aligned} I &= [a, 2a], \\ \Lambda &= \{\nu = (n_2, n_3) : 0 < n_2 \leq 2\varepsilon a, \quad 0 < n_3 \leq 2\varepsilon a, \quad n_3 \leq a^{\ell}\}. \end{aligned} \tag{10.91}$$

We display the label by writing

$$G_\nu(\rho) = (g_\nu(\rho), \widehat{u}_\nu(\rho), \widehat{w}_\nu(\rho)), \qquad \xi_\nu(\rho) = 2/g_\nu(\rho).$$

All scale thresholds below are uniform over $\Lambda \times I$. The time $T = (q/2)\log(1/a)$ remains fixed when differentiating in the labels or in ρ.

Proposition 10.32. *Under the thresholds of Proposition 10.30, for every $\nu \in \Lambda$ and $a \leq b \leq c \leq 2a$, the phase ξ_ν is strictly increasing on $[b, c]$ and*

$$\xi_\nu([b, c]) = [\xi_\nu(b), \xi_\nu(c)], \qquad \frac{c-b}{8\sqrt{3}} \leq \xi_\nu(c) - \xi_\nu(b) \leq C_h(c-b). \tag{10.92}$$

For every Borel set $A \subset [b, c]$ and every set $E \subset \mathbb{R}$ with $\xi_\nu(A) \subset E$,

$$\lambda(A) \leq \min\{c - b,\, 8\sqrt{3}\,\lambda(E)\}. \tag{10.93}$$

Here λ is Lebesgue measure, interpreted as outer measure for a set E that is not measurable.

Proof. The mean value theorem applied on each subinterval gives

$$\frac{t-s}{8\sqrt{3}} \leq \xi_\nu(t) - \xi_\nu(s) \leq C_h(t-s) \quad (b \leq s \leq t \leq c).$$

It proves strict monotonicity; continuity and the intermediate value theorem give the exact interval image. For the measure estimate, use the one-dimensional change of variables

argument in (8.100):

$$\frac{1}{8\sqrt{3}}\lambda(A) \le \int_A |\xi_\nu'(\rho)|\, d\rho = \lambda(\xi_\nu(A)) \le \lambda(E).$$

Combining this with $A \subset [b,c]$ proves (10.93). □

Since $g_\nu' < 0$, its image of I is the interval

$$J_\nu = [l_\nu, r_\nu], \qquad l_\nu = g_\nu(2a) < g_\nu(a) = r_\nu. \tag{10.94}$$

Let $j_\nu : J_\nu \to I$ be its inverse, and define

$$u_\nu(r) = \widehat{u}_\nu(j_\nu(r)), \qquad w_\nu(r) = \widehat{w}_\nu(j_\nu(r)). \tag{10.95}$$

Their derivatives on the closed interval, including one-sided derivatives at its endpoints, are

$$u_\nu'(r) = \frac{\widehat{u}_\nu'(j_\nu(r))}{g_\nu'(j_\nu(r))}, \qquad w_\nu'(r) = \frac{\widehat{w}_\nu'(j_\nu(r))}{g_\nu'(j_\nu(r))}. \tag{10.96}$$

The radius extends to a C^1 function with negative derivative on a neighborhood of I. The inverse function theorem gives $j_\nu' = 1/g_\nu'(j_\nu(r))$ up to both endpoints of J_ν. Thus the expressions in (10.96) are continuous on the closed interval.

Define $\widetilde{u}_\nu, \widetilde{w}_\nu$ by the explicit square-root extension of Lemma 8.13, applied on J_ν to (10.95) and (10.96). Put

$$B_\nu = \frac{800 n_2 n_3}{a^4}, \qquad \Gamma_\nu(r) = (r, \widetilde{u}_\nu(r), \widetilde{w}_\nu(r)). \tag{10.97}$$

Proposition 10.33. *After a common decrease of the preceding scale thresholds, both extensions are positive C^1 functions on $\mathbb{R}$, and*

$$\widetilde{u}_\nu + \widetilde{w}_\nu \le B_\nu, \qquad |\widetilde{u}_\nu'| \le \sqrt{\widetilde{u}_\nu}, \qquad |\widetilde{w}_\nu'| \le \sqrt{\widetilde{w}_\nu}. \tag{10.98}$$

For every $\rho \in I$ they agree with the actual output in both value and radial derivative:

$$\Gamma_\nu(g_\nu(\rho)) = G_\nu(\rho),$$
$$\widetilde{u}_\nu'(g_\nu(\rho)) = \frac{\widehat{u}_\nu'(\rho)}{g_\nu'(\rho)}, \qquad \widetilde{w}_\nu'(g_\nu(\rho)) = \frac{\widehat{w}_\nu'(\rho)}{g_\nu'(\rho)}. \tag{10.99}$$

In particular Γ_ν is admissible at amplitude B_ν.

Proof. Divide the two cone inequalities in (10.81) by $|g_\nu'|$. Equations (10.95)–(10.96) give

$$|u_\nu'| \le \sqrt{u_\nu}, \qquad |w_\nu'| \le \sqrt{w_\nu} \quad \text{on } J_\nu.$$

The functions are positive, with continuous one-sided derivatives, and their sum is at most $512 n_2 n_3/a^4$ by (10.90). Lemma 8.13, with $\theta = 1$, gives global positivity and the two slope bounds. Its amplitude cost is exactly

$$\frac{25}{16}\frac{512 n_2 n_3}{a^4} = \frac{800 n_2 n_3}{a^4} = B_\nu.$$

The same lemma preserves values and derivatives on the whole closed interval. Using $j_\nu(g_\nu(\rho)) = \rho$ proves (10.99). These global estimates imply the required admissibility conditions on $r > 2$. □

The intervals J_ν depend on the labels. We verify measurability for the explicit extensions just constructed, so the same graphs can be used for all labels in the statistical estimate.

Proposition 10.34. *The map $(\nu, \rho) \mapsto G_\nu(\rho)$ is Borel on $\Lambda \times I$. The four functions*

$$(\nu, r) \longmapsto \widetilde{u}_\nu(r), \quad \widetilde{w}_\nu(r), \quad \widetilde{u}_\nu'(r), \quad \widetilde{w}_\nu'(r) \tag{10.100}$$

are Borel on $\Lambda \times \mathbb{R}$. Extending any of them by zero outside $\Lambda \times \mathbb{R}$ gives a Borel function on $\mathbb{R}^2 \times \mathbb{R}$.

Proof. The set Λ is Borel. The joint C^2 regularity from Corollary 10.23 implies continuity of the restricted map $(\nu,\rho)\mapsto G_\nu(\rho)$, including at parameters where $V=1$. In particular it is Borel, as are the two endpoint functions l_ν, r_ν.

Consider the Borel map

$$H:\Lambda\times I\longrightarrow\Lambda\times\mathbb{R},\qquad H(\nu,\rho)=(\nu,g_\nu(\rho)).$$

It is injective: equality of its values first fixes the label, and strict decrease of g_ν then fixes ρ. Both spaces are standard Borel spaces. The injective Borel mapping theorem therefore gives a Borel inverse on its image

$$\mathcal{J}=\{(\nu,r):r\in J_\nu\}.$$

Transporting $\widehat{u}_\nu,\widehat{w}_\nu$ by this inverse proves that the radius graphs are jointly Borel on $\mathcal{J}$. In particular their values restricted to the interval interiors, extended by zero elsewhere, are jointly Borel.

We next recover the endpoint values and slopes from interior samples. This also gives a direct measurable description of the derivative used in the extension. Let v_ν be either radius graph, or its positive square root, and let v_ν' be its continuous one-sided derivative. For $t\in J_\nu$ and $n\geq 0$ set

$$h_n(\nu,t)=\begin{cases}(r_\nu-t)/(3(n+1)), & t<r_\nu,\\ (l_\nu-t)/(3(n+1)), & t=r_\nu.\end{cases}\tag{10.101}$$

This nonzero step tends to zero, and both $t+h_n$ and $t+2h_n$ lie in (l_ν,r_ν). Continuity and the one-sided derivative imply

$$\begin{aligned}v_\nu(t)&=\lim_{n\to\infty}v_\nu(t+h_n),\\ v_\nu'(t)&=\lim_{n\to\infty}\frac{v_\nu(t+2h_n)-v_\nu(t+h_n)}{h_n}.\end{aligned}\tag{10.102}$$

For the second identity, write the quotient as twice the secant from t to $t+2h_n$, minus the secant from t to $t+h_n$. Each secant tends to $v_\nu'(t)$.

If t is any Borel selector of J_ν, all the samples and quotients in (10.102) are Borel. Their pointwise limits are therefore Borel. Apply this first to $t=l_\nu,r_\nu$. For an arbitrary (ν,r) apply it also, on the product label space, to the selector

$$t=\max\{l_\nu,\min\{r,r_\nu\}\}.$$

Thus the endpoint data and the interior value and slope are measurable wherever the piecewise extension uses them. The formulas (8.56)–(8.59) now show that the extended square roots and their derivatives are jointly Borel. Squaring gives the amplitudes, and the derivative is twice the extended root times its derivative. This proves (10.100).

Finally $\Lambda\times\mathbb{R}$ is Borel in the ambient space. A Borel function on this Borel subspace, set equal to zero on its complement, is Borel. □

Fix η, J, σ and the common constants B_*, B_{cap} of Theorem 8.20, as in Section 8.10. The amplitude in (10.97) can be placed below any prescribed positive bound by decreasing a uniformly in the labels. Indeed, for $a\leq 1$,

$$n_2\leq a,\qquad n_3\leq a^{\ell}\leq a^4,\qquad 0<B_\nu\leq 800a.\tag{10.103}$$

The first bound uses $2\varepsilon\leq 1$. Choose also $a\leq 1/256$. The lower radius bound then gives $g_\nu\geq 4$, so

$$0<\xi_\nu<1,\qquad \xi_\nu'\geq\frac{1}{8\sqrt{3}},\qquad \Gamma_\nu(2/\xi_\nu(\rho))=G_\nu(\rho).\tag{10.104}$$

The last identity is (10.99), since $2/\xi_\nu=g_\nu$.

Theorem 10.35 (Statistical entrances for the first wall ordering)**.** *There is $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 3$, $\ell > 4$, and all sufficiently small $a > 0$, the following holds for each $0 \le \varepsilon \le 1/64$. For $\nu \in \Lambda$ define*

$$\mathcal{F}_\nu = \{\rho \in I : (B_\nu, G_\nu(\rho)) \notin \mathcal{S}_{\mathrm{stat}}\}. \tag{10.105}$$

Then $0 < B_\nu \le B_* \le B_{\mathrm{cap}}$, each $\mathcal{F}_\nu$ is Borel, and the family $\{(\nu,\rho) \in \Lambda \times I : \rho \in \mathcal{F}_\nu\}$ is Borel. For every Borel $A \subset I$ and nonnegative Borel weight $\omega \le M$ on A, with $M \ge 0$,

$$\int_{A\cap\mathcal{F}_\nu} \omega(\rho)\,d\rho \le \min\left\{\int_A \omega(\rho)\,d\rho, 640\sqrt{3}\,MB_\nu^\sigma\right\}. \tag{10.106}$$

For $\rho \in I \setminus \mathcal{F}_\nu$, the selected era digits and phases starting at the returned state $G_\nu(\rho)$ satisfy all six statistical limits of Theorem 2.4.

Proof. Take the common small-h threshold of the return, cone and phase estimates. For each h, q, ℓ take their common small-a threshold and require further

$$a \le \min\{1/256, B_*/800\}.$$

Equation (10.103) gives $0 < B_\nu \le B_*$. The graph is admissible at this amplitude by Proposition 10.33, and (10.104) verifies the phase and representation hypotheses of Corollary 8.29. That corollary gives the Borel excluded set and (10.106), because

$$\frac{80M}{1/(8\sqrt{3})}B_\nu^\sigma = 640\sqrt{3}\,MB_\nu^\sigma.$$

It also gives all six limits outside the excluded set for the same entrance state.

For the joint assertion, $\nu \mapsto B_\nu$ is Borel by its explicit formula, and $(\nu,\rho) \mapsto G_\nu(\rho)$ is Borel by Proposition 10.34. The family in the statement is the inverse image of the complement of the common Borel set $\mathcal{S}_{\mathrm{stat}}$ under $(\nu,\rho) \mapsto (B_\nu, G_\nu(\rho))$. □

This completes the entrance construction for positive shear under the wall bounds (10.91). The factor B_ν^σ retains both initial weak walls in the measure estimate, and joint measurability permits integration over their labels. The next section extends the construction to the other wall ordering by allowing an additional interval of slow decay.

11. Slow passage and both weak-wall orderings

The other ordering of the weak walls can require a longer wait before the next core. We first control the phase and its derivative during this slow interval. The endpoint then enters the core construction of Section 10. Identifying the two returns on their overlap gives one entrance map for both wall orderings. Reflection of the axes supplies the other sign of shear. The final statistical estimate thus covers either choice of the smaller wall, with constants independent of their ratio.

11.1. **Slow wall decay near a Taub point.** We now consider positive shear when n_2 is very small and n_3 is allowed to be of order εa. After the initial decay of N_1, the third wall can still be too large in projective coordinates. Positive shear makes this wall decrease at a rate of order a. Waiting for a time of order $a^{-1}\log(1/a)$ therefore supplies the additional decay. The initial size of n_2 must allow for its possible growth during the same interval.

The estimates below keep the three wall rates separate. Their integrals control the shear displacement and preserve the quadratic size of the projection height $1 + p$ throughout this longer interval. All times in this subsection are the original normalized time t of (2.2).

Write a state as $(p, s, n, x, y) = (p, s, N_1, N_2, N_3)$. For $0 < a \le 1/256$, consider the region

$$\begin{aligned} &G = 1, \qquad p \le -\frac{1}{2}, \qquad \frac{a}{2} \le s \le 3a, \\ &0 \le n \le a^2, \qquad 0 \le x \le a^2, \qquad 0 \le y \le \frac{a}{16}. \end{aligned} \tag{11.1}$$

The different bounds on x and y reflect their different roles: the wall that can grow starts at the smaller scale.

Lemma 11.1. *On* (11.1),

$$\begin{aligned} |Q(n,x,y)| &\le \frac{a^2}{128}, \qquad \frac{a^2}{5} \le 1-p^2 \le 10a^2, \\ -1 < p &\le -\frac{15}{16}, \qquad \frac{a^2}{10} \le 1+p \le 6a^2. \end{aligned} \tag{11.2}$$

Writing the wall equations as $\mathrm{D}_t\, n = \lambda_1 n$, $\mathrm{D}_t\, x = \lambda_2 x$, *and* $\mathrm{D}_t\, y = \lambda_3 y$, *their rates satisfy*

$$\lambda_1 \le -2, \qquad |\lambda_2| \le 6a, \qquad \lambda_3 \le -\frac{a}{2}. \tag{11.3}$$

The shear velocity satisfies

$$|\mathrm{D}_t\, s| \le 3(x^2+y^2) + \frac{a}{4} n. \tag{11.4}$$

Proof. Nonnegativity of the walls gives

$$|Q(n,x,y)| \le (n+x+y)^2, \qquad n+x+y \le 2a^2 + \frac{a}{16} \le \frac{9a}{128}.$$

This proves the first bound in (11.2). The identity $1-p^2 = s^2 + Q$ gives the next two bounds. Since $p<0$ and $a \le 1/256$, these imply $-1 < p \le -15/16$. Rationalizing

$$1+p = \frac{1-p^2}{1-p}, \qquad \frac{5}{3} \le 1-p \le 2,$$

proves the height bounds. This argument also proves (11.2) with the weaker initial assumption $p<0$ in place of $p \le -1/2$.

On $G=1$, the three rates are

$$\lambda_1 = -p^2 - s^2 + 2p, \qquad \lambda_2 = Q - (1+p) + \sqrt{3}s, \qquad \lambda_3 = Q - (1+p) - \sqrt{3}s.$$

The bounds just obtained, together with $3/2 \le \sqrt{3} \le 7/4$, yield (11.3).

For the shear equation, retain the factor s multiplying the strong-wall square. Directly from the curvature polynomial,

$$\begin{aligned} |Q(n,x,y)| &\le n^2 + x^2 + y^2 + 2n(x+y), \\ |-x^2+y^2+nx-ny| &\le x^2+y^2+n(x+y). \end{aligned}$$

Using $|s| \le 3a$, $\sqrt{3} \le 2$, and $6a \le 1$ in (2.2) gives

$$|\mathrm{D}_t\, s| \le 3\big(x^2+y^2+n(x+y)+an^2\big).$$

Finally, the wall bounds in (11.1) imply

$$3(x+y+an) \le 3\left(a^2 + \frac{a}{16} + a^3\right) \le \frac{a}{4},$$

which proves (11.4). □

Suppose an orbit remains in (11.1) for $0 \le t \le T$ and its initial walls are bounded by nonnegative numbers B_n, B_x, B_y. Scalar comparison gives

$$n(t) \le B_n e^{-2t}, \qquad x(t) \le B_x e^{6at}, \qquad y(t) \le B_y e^{-at/2}. \tag{11.5}$$

These comparisons also hold when a wall vanishes, by its multiplicative evolution equation. Inserting them into (11.4) and integrating yields

$$\begin{aligned} |s(t)-s(0)| \le E(t) := {} & \frac{3B_y^2}{a}(1-e^{-at}) + \frac{B_x^2}{4a}(e^{12at}-1) \\ & + \frac{aB_n}{8}(1-e^{-2t}). \end{aligned} \tag{11.6}$$

In particular,

$$E(t) \le \frac{3B_y^2}{a} + \frac{B_x^2}{4a}e^{12aT} + \frac{aB_n}{8} \quad (0 \le t \le T). \tag{11.7}$$

The decaying wall contributes at most $3B_y^2/a$ even when T is large. This is the bound that keeps the shear in its original annulus.

Proposition 11.2. *Let $0 < a \le 1/256$, $T \ge 0$, and suppose*

$$0 \le B_n < a^2, \qquad 0 \le B_x, \quad B_x e^{6aT} < a^2, \qquad 0 \le B_y < \frac{a}{16}. \tag{11.8}$$

Let $z_0 = (p_0, s_0, n_0, x_0, y_0)$ satisfy $G(z_0) = 1$, $p_0 < 0$, and

$$\frac{3a}{4} \le s_0 \le \frac{9a}{4}, \qquad 0 \le n_0 \le B_n, \quad 0 \le x_0 \le B_x, \quad 0 \le y_0 \le B_y. \tag{11.9}$$

Then the actual orbit $\phi_t z_0$ remains in (11.1) *for $0 \le t \le T$. Throughout this interval it satisfies* (11.2), (11.5), *and* (11.6).

Proof. The forward existence and wall-sign argument of Section 9.2 applies also to zero walls. Thus the actual solution exists throughout $[0, T]$, preserves the constraint, and has nonnegative walls. The assumptions imply $B_x < a^2$. The constraint calculation in Lemma 11.1, using $p_0 < 0$, gives $p_0 \le -15/16$. Hence the initial point satisfies strictly the six inequalities

$$p < -\frac{1}{2}, \qquad \frac{a}{2} < s < 3a, \qquad n < a^2, \qquad x < a^2, \qquad y < \frac{a}{16}.$$

On any initial interval where the corresponding non-strict inequalities hold, the preceding comparison estimates apply. The bound in (11.7) is at most

$$\frac{3a}{256} + \frac{a^3}{4} + \frac{a^3}{8} \le \frac{a}{8}.$$

Indeed, $B_y^2 \le a^2/256$ and $B_x^2 e^{12aT} \le a^4$, while $B_n \le a^2$. Consequently,

$$\frac{5a}{8} \le s(t) \le \frac{19a}{8},$$

strictly inside the prescribed shear interval. Equations (11.5) and (11.8) give strict upper bounds for all three walls, and the constraint gives $p(t) \le -15/16 < -1/2$. Every defining inequality has therefore improved strictly.

The times up to which the closed inequalities hold form a closed initial interval. If its endpoint were smaller than T, the strict improvements and continuity would extend it. The interval is thus all of $[0, T]$. Applying the comparison estimates there proves the result. □

Return to the small section and write its data as

$$z_\rho = \Xi_h(\rho, \zeta, \nu), \qquad a \le \rho \le 2a, \qquad t_0 = \frac{q}{2}\log(1/a), \qquad T = \frac{D}{a}\log(1/a). \tag{11.10}$$

Here ζ is the initially small second wall, whereas ν is the third wall that will decay during $[t_0, t_0 + T]$. The duration T is chosen in advance for the whole family. The identity

$$e^{caT} = a^{-cD} \qquad (c \in \mathbb{R}) \tag{11.11}$$

converts the wall estimates into powers of the radial scale.

Theorem 11.3. *There exists $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 2$, $D \ge 0$, and $P > 6D + 2$, the following holds for all sufficiently small $a > 0$, uniformly over*

$$0 \le \varepsilon \le \frac{1}{128}, \qquad 0 \le \zeta \le \min\{a^P, 2\varepsilon a\}, \qquad 0 \le \nu \le 2\varepsilon a, \qquad a \le \rho \le 2a. \tag{11.12}$$

Let

$$z(t) = \phi_{t_0+t} z_\rho = (p(t), s(t), n(t), x(t), y(t)), \qquad 0 \le t \le T.$$

This is the actual orbit $\phi_t(\phi_{t_0} z_\rho)$. It lies in (11.1)*, and*

$$n(t) \le h a^q e^{-2t}, \quad x(t) \le 2\zeta e^{6at}, \quad y(t) \le 4\varepsilon a e^{-at/2}. \tag{11.13}$$

Moreover,

$$\begin{aligned} |s(t) - s(0)| &\le 48\varepsilon^2 a + \zeta^2 a^{-12D-1} + \frac{h}{8} a^{q+1}, \\ \frac{a^2}{10} &\le 1 + p(t) \le 6a^2, \qquad 0 \le V(t) < 1. \end{aligned} \tag{11.14}$$

At the endpoint, the projective walls satisfy

$$\begin{aligned} 0 \le U(T) &\le 20 h a^{q-2}, \\ 0 \le V(T) &\le 40\zeta a^{-6D-2}, \\ 0 \le W(T) &\le 80\varepsilon a^{D/2-1}. \end{aligned} \tag{11.15}$$

Proof. Choose h_* as in Proposition 9.15. For sufficiently small a, that proposition places $z(0) = \phi_{t_0} z_\rho$ on $G = 1$ with $p(0) < 0$ and

$$\frac{3a}{4} \le s(0) \le \frac{9a}{4}, \qquad 0 \le n(0) \le h a^q, \qquad 0 \le x(0) \le 2\zeta, \qquad 0 \le y(0) \le 2\nu \le 4\varepsilon a.$$

Decrease the common threshold for a so that

$$a \le \frac{1}{256}, \qquad h a^{q-2} < \frac{1}{2}, \qquad 40 a^{P-6D-2} < \frac{1}{2}.$$

Both exponents are positive. With

$$B_n = h a^q, \qquad B_x = 2\zeta, \qquad B_y = 4\varepsilon a,$$

these inequalities give

$$B_n < a^2, \qquad B_x e^{6aT} = 2\zeta a^{-6D} < \frac{a^2}{40}, \qquad B_y \le \frac{a}{32} < \frac{a}{16}.$$

Also $T \ge 0$. Proposition 11.2 therefore applies to $z(0)$ on the entire interval $[0, T]$. Uniqueness of the normalized flow gives $\phi_t z(0) = \phi_{t+t_0} z_\rho$, so all conclusions concern the original small-section orbit.

The wall and height estimates follow immediately. Substitution in (11.7), using (11.11) with $c = 12$, gives the stated shear displacement. Throughout the interval, $x(t) < a^2/40$; hence

$$0 \le V(t) = \frac{\sqrt{3} x(t)}{1 + p(t)} \le \frac{20 x(t)}{a^2} < 1.$$

More generally, each nonnegative physical wall v with upper bound B satisfies

$$0 \le \frac{\sqrt{3} v}{1 + p} \le \frac{20 B}{a^2}.$$

At $t = T$, use $e^{-2T} \le 1$ for the strong wall and (11.11) with $c = 6$ and $c = -1/2$ for the other two walls. This proves (11.15). □

For $D > 2$, the last endpoint bound tends to zero, and the condition $P > 6D + 2$ makes the second one tend to zero as well. Thus the longer interval reduces both projective weak walls while keeping the shear of order a and the height of order a^2. The bound $V < 1$ also places this entire interval before any subsequent upward crossing of $V = 1$. To use the endpoint as a statistical entrance, we next control its dependence on the original shear coordinate ρ.

11.2. **Variation during the slow decay.** The interval of slow decay is long enough for the logarithmic wall derivatives to grow with time. Their effect on the shear derivative is nevertheless weighted by the small wall amplitudes. Two time moments of those amplitudes control this interaction: the zeroth moment bounds direct growth, and the first moment bounds the contribution of the growing wall derivatives.

We consider the family in Theorem 11.3, now with $\zeta, \nu > 0$. Write

$$z(t,\rho) = \phi_{t_0+t}\Xi_h(\rho,\zeta,\nu) = (p,s,n,x,y)(t,\rho), \qquad 0 \le t \le T.$$

Derivatives in ρ keep h, a, q, D, ζ, ν and t fixed; in particular, t_0 and T are fixed. We use D_t for differentiation in normalized time. The walls remain positive, so their logarithmic derivatives are defined. Set

$$S = \partial_\rho s, \qquad \ell = (\partial_\rho \log n, \partial_\rho \log x, \partial_\rho \log y), \qquad L = \|\ell\|_\infty. \tag{11.16}$$

Set

$$b = n^2 + x^2 + y^2 + n(x+y), \qquad J_0 = \int_0^T b(t)\,dt, \qquad J_1 = \int_0^T tb(t)\,dt. \tag{11.17}$$

We suppress ρ when discussing a fixed orbit.

Let $\delta = \partial_\rho$, $\chi = -p$, and $H = -x^2 + y^2 + nx - ny$. Differentiating the constraint and the two curvature polynomials gives

$$\begin{aligned} \delta p &= \frac{sS + \delta Q/2}{\chi}, \\ \delta Q &= 2\big((n-x-y)\delta n + (x-n-y)\delta x + (y-n-x)\delta y\big), \\ \delta H &= (x-y)\delta n + (n-2x)\delta x + (2y-n)\delta y. \end{aligned} \tag{11.18}$$

The constraint identity holds for nearby chart parameters and flow times, so its differential is the tangent constraint for the actual flow. Differentiating (2.2), and using (11.18) to eliminate δp, yields

$$\begin{aligned} \mathrm{D}_t S &= QS + s\delta Q + \sqrt{3}\,\delta H, \\ \mathrm{D}_t \ell_n &= (1+\chi^{-1})\delta Q + 2s\chi^{-1}S, \\ \mathrm{D}_t \ell_x &= (1-(2\chi)^{-1})\delta Q + (-s\chi^{-1} + \sqrt{3})S, \\ \mathrm{D}_t \ell_y &= (1-(2\chi)^{-1})\delta Q + (-s\chi^{-1} - \sqrt{3})S. \end{aligned} \tag{11.19}$$

In the last three equations the wall rate cancels when the quotient $\delta N_i/N_i$ is differentiated in time.

By definition, $|\delta n| \le nL$, $|\delta x| \le xL$, and $|\delta y| \le yL$. Nonnegativity of the walls and $(n+x+y)^2 \le 2b$ therefore imply

$$|Q| \le 2b, \qquad |\delta Q| \le 4bL, \qquad |\delta H| \le 2bL. \tag{11.20}$$

For example, each coefficient inside the parentheses in δQ has absolute value at most $n+x+y$. The bound for δH follows by bounding its three terms by $(x+y)nL$, $(n+2x)xL$, and $(2y+n)yL$.

On the region (11.1), the constraint gives $p \le -15/16$ and $|s| \le 1/8$. Consequently $\chi^{-1} \le 5/4$ and $|s|\chi^{-1} \le 1/4$. The coefficients of δQ in the last three rows of (11.19) have absolute value at most $9/4$, and the coefficients of S at most 2. Together with $\sqrt{3} \le 7/4$, this proves

$$|\mathrm{D}_t S| \le 2b|S| + 4bL, \qquad \|\mathrm{D}_t \ell\|_\infty \le 2|S| + 9bL. \tag{11.21}$$

Lemma 11.4. *Suppose $T \ge 0$ and an actual orbit remains in* (11.1) *for $0 \le t \le T$, with $0 < a \le 1/256$. Write $f_0 = n(0)^2 + n(0)(x(0) + y(0))$. Then*

$$J_0 \le \frac{2}{3} f_0 + \frac{y(0)^2}{a} + \frac{x(0)^2}{12a} e^{12aT}, \tag{11.22}$$

$$J_1 \le \frac{4}{9} f_0 + \frac{y(0)^2}{a^2} + \frac{T x(0)^2}{12a} e^{12aT}. \tag{11.23}$$

If also $T = (D/a)\log(1/a)$, $D \ge 0$, and $x(0)e^{6aT} \le a^2$, then

$$J_0 \le 4a, \qquad J_1 \le \frac{1}{256} + (2 + D)a. \tag{11.24}$$

Proof. The wall rates (11.3) show that

$$f = n^2 + n(x + y) \quad \text{satisfies} \quad \mathrm{D}_t f \le -\frac{3}{2} f.$$

Indeed, the rates of n^2, nx, and ny are respectively $2\lambda_1$, $\lambda_1 + \lambda_2$, and $\lambda_1 + \lambda_3$, all at most $-3/2$. Also $\frac{d}{dt} y^2 \le -a y^2$.

For any nonnegative differentiable g with $\mathrm{D}_t g \le -cg$ and $c > 0$, integration gives

$$\int_0^T g \, dt \le \frac{g(0)}{c}, \qquad \int_0^T t g \, dt \le \frac{g(0)}{c^2}.$$

For the second inequality, differentiate $(1 + ct)g(t)$ to obtain an upper bound $-c^2 t g(t)$, and use its nonnegative endpoint value. Apply these inequalities to f and y^2. The remaining square satisfies $x(t)^2 \le x(0)^2 e^{12at}$, so

$$\int_0^T x(t)^2 \, dt \le \frac{x(0)^2}{12a} e^{12aT}, \qquad \int_0^T t x(t)^2 \, dt \le T \frac{x(0)^2}{12a} e^{12aT}.$$

Since $b = f + y^2 + x^2$, these are (11.22) and (11.23).

For the simpler bounds, the initial region gives

$$f_0 \le 3a^3, \qquad \frac{y(0)^2}{a} \le \frac{a}{256}, \qquad \frac{y(0)^2}{a^2} \le \frac{1}{256}.$$

The additional growing-wall assumption gives

$$\frac{x(0)^2}{12a} e^{12aT} \le \frac{a^3}{12}.$$

Moreover, $a \log(1/a) \le 1$, and hence

$$T \frac{a^3}{12} = \frac{Da}{12} \, a \log(1/a) \le Da.$$

Insert these bounds in the two moment estimates and use $a^3 \le a$. This proves (11.24). □

The following estimate explains why the first time moment is sufficient. It applies directly to (11.21).

Lemma 11.5. *Let $T \ge 0$, let $S : [0, T] \to \mathbb{R}$ and $\ell : [0, T] \to \mathbb{R}^3$ be differentiable, and let $b \ge 0$ be continuous. Suppose* (11.21) *holds, with $L = \|\ell\|_\infty$. Put $L_0 = L(0)$ and define J_0, J_1 by* (11.17). *Then, with*

$$c = e^{9J_0}, \qquad B = (|S(0)| + 4cL_0J_0)e^{2J_0 + 8cJ_1}, \tag{11.25}$$

one has

$$|S(t)| \le B, \qquad L(t) \le c(L_0 + 2tB) \quad (0 \le t \le T). \tag{11.26}$$

In particular, if $|S(0)| \le 2$ and

$$9J_0 \le \log 2, \qquad 2J_0 + 16J_1 \le \frac{1}{2}, \qquad L_0 J_0 \le \frac{1}{16}, \tag{11.27}$$

then $|S(t)| \le 5$ *and* $L(t) \le 2L_0 + 20t$.

Proof. We use the integral norm comparison

$$\|u(t)\| \le \left(\|u(0)\| + \int_0^t e(v)\,dv\right) \exp\left(\int_0^t k(v)\,dv\right) \tag{11.28}$$

when $\|\mathrm{D}_t\, u\| \le k\|u\| + e$ and k, e are continuous and nonnegative. To verify it also at zeros of u, compare its norm with

$$\left(\|u(0)\| + \delta + \int_0^t e(v)\,dv\right) \exp\left(\int_0^t k(v)\,dv + \delta t\right), \qquad \delta > 0.$$

At any contact this function has strictly larger derivative than the permitted growth of the norm. Norm comparison followed by $\delta \downarrow 0$ proves (11.28).

Define a nondecreasing envelope

$$M(t) = |S(0)| + \int_0^t \left(2b(v)|S(v)| + 4b(v)L(v)\right) dv.$$

Then $|S(t)| \le M(t)$. Apply (11.28) to ℓ. For $0 \le v \le t$, $|S(v)| \le M(t)$, so

$$L(t) \le c\left(L_0 + 2\int_0^t |S(v)|\,dv\right) \le c(L_0 + 2tM(t)).$$

It follows that

$$\mathrm{D}_t\, M(t) \le (2 + 8ct)b(t)M(t) + 4cL_0 b(t).$$

A second application of (11.28), with $\int_0^T (2 + 8ct)b(t)\,dt = 2J_0 + 8cJ_1$, gives $M(t) \le B$. Substituting this into the preceding bound for L proves (11.26).

Under (11.27), $c \le 2$ and $e^{2J_0 + 8cJ_1} \le e^{1/2} < 2$. Also $|S(0)| + 4cL_0J_0 \le 5/2$. Thus $B \le 5$, which gives the last assertion. □

Theorem 11.6. *There are constants* $C \ge 32$ *and* $h_* > 0$ *with the following property. For every* $0 < h \le h_*$, $q > 2$, $D \ge 0$, *and* $P > 6D + 2$, *there is* $a_* > 0$ *such that for* $0 < a \le a_*$, *uniformly over*

$$\begin{gathered} 0 \le \varepsilon \le \frac{1}{128}, \qquad 0 < \zeta \le \min\{a^P, 2\varepsilon a\}, \\ 0 < \nu \le 2\varepsilon a, \qquad a \le \rho \le 2a, \end{gathered} \tag{11.29}$$

the actual flow and the derivatives in (11.16) *satisfy*

$$\begin{aligned} L_0 &\le 2 + 4Cq|\log a|, \\ |\partial_\rho s(t,\rho)| &\le 5, \\ L(t) &\le 2L_0 + 20t \le 4 + 8Cq|\log a| + 20t \quad (0 \le t \le T). \end{aligned} \tag{11.30}$$

In particular, for each physical wall,

$$|\partial_\rho N_i(t,\rho)| \le N_i(t,\rho)(2L_0 + 20t). \tag{11.31}$$

The moments satisfy both (11.24) *and* (11.27).

Proof. Take the constant C in the initial transport estimates large enough that $C \ge 32$, and decrease h_* to meet the hypotheses of Theorem 11.3 and Proposition 9.15. At $t = 0$ in the present coordinates, (9.79) gives

$$|S(0)| \le 2, \qquad |\ell_n(0)| \le 2, \qquad |\ell_x(0)|, |\ell_y(0)| \le 8Ct_0 = 4Cq|\log a|.$$

Here $a \le 1$ and $t_0 = (q/2)\log(1/a)$. This proves the claimed initial bound for L_0.

Theorem 11.3 keeps the actual orbit in (11.1) throughout $[0,T]$ and gives $x(0)e^{6aT} < a^2/40$. Lemma 11.4 therefore yields (11.24). Shrink a_* so that, in addition to the preceding scale conditions,

$$36a \le \log 2, \qquad (40+16D)a \le \frac{1}{4},$$
$$8a + 16Cq|a\log a| \le \frac{1}{16} \qquad (0 < a \le a_*).$$

Such a choice is possible because $a|\log a| \to 0$. The moment bounds give

$$2J_0 + 16J_1 \le \frac{1}{16} + (40+16D)a \le \frac{1}{2},$$

and

$$L_0 J_0 \le (2 + 4Cq|\log a|)4a = 8a + 16Cq|a\log a| \le \frac{1}{16}.$$

The remaining inequality $9J_0 \le \log 2$ follows from $J_0 \le 4a$.

The ordinary flow derivatives satisfy (11.21), so Lemma 11.5 proves (11.30). Multiplying the logarithmic bounds by the positive walls proves (11.31). All scale choices precede $\varepsilon, \zeta, \nu, \rho$, which gives the stated uniformity. □

The shear response remains bounded while the logarithmic wall response grows at most linearly with elapsed time. For the radial phase, we next retain the cancellation between the shear and weak-wall equations before integrating their response.

11.3. **Radial phase through the slow decay.** The coupling between the shear and the difference of the weak walls cancels from the evolution of their Euclidean radius. The remaining terms carry the small projection height or the wall amplitudes, and the same cancellation survives ordinary differentiation in the initial shear. This gives a quadratic value error and a linear derivative error over the entire interval of slow decay.

Use the actual family $z(t,\rho) = (p,s,n,x,y)(t,\rho)$ of Section 11.2, with $\zeta, \nu > 0$, and put

$$d = x - y, \qquad R = \sqrt{s^2 + d^2}, \qquad A = 1 + p, \qquad E = n + x^2 + y^2. \tag{11.32}$$

All derivatives in ρ keep the original parameters and both times t_0, T fixed. We first estimate the rate of R and its ordinary differential, then integrate them using the two time moments of E.

On the Einstein constraint, the equations for s and d take the form

$$\mathrm{D}_t\, s = Qs + \sqrt{3}(n - x - y)d, \qquad \mathrm{D}_t\, d = (Q - A)d + \sqrt{3}s(x+y).$$

The terms containing $s(x+y)d$ cancel in $s\,\mathrm{D}_t\, s + d\,\mathrm{D}_t\, d$. Since $s > 0$ on the region (11.1), we obtain the smooth identity

$$\mathrm{D}_t\, R = \frac{\mathcal{N}}{R}, \qquad \mathcal{N} = QR^2 - Ad^2 + \sqrt{3}\, nsd. \tag{11.33}$$

In particular, the small height A appears before any estimate is made on the weak-wall contribution.

Lemma 11.7. *There is an absolute constant $C_R \ge 1$ with the following property. Let $0 < a \le 1/256$ and let a state belong to* (11.1). *Let δ be a tangent direction to $G = 1$ such that, for some $L \ge 0$,*

$$|\delta s| \le 5, \qquad |\delta n| \le nL, \qquad |\delta x| \le xL, \qquad |\delta y| \le yL.$$

Then

$$\frac{a}{2} \le R \le 4a, \qquad |\,\mathrm{D}_t\, R| \le C_R a E, \qquad \left|\delta\left(\frac{\mathcal{N}}{R}\right)\right| \le C_R E(1 + aL). \tag{11.34}$$

Here $\mathrm{D}_t\, R$ denotes the radius rate in (11.33).

Proof. The regional bounds give $a/2 \le s \le 3a$, $x, y \le a$, and $|d| \le 2a$. Thus $a/2 \le R \le 4a$. Also

$$0 \le E \le 3a^2, \qquad n \le E, \qquad b \le 3E,$$

where b is defined in (11.17). Indeed, $n \le a^2 \le 1$ and $x + y \le 2$, so $n^2 + n(x + y) \le 3n$. The polynomial estimates (11.20) consequently give

$$|Q| \le 6E, \qquad |\delta Q| \le 12EL \le 36a^2 L.$$

Write $B = 1 + aL$. The ordinary differential of the radius is

$$\delta R = \frac{s\delta s + d\delta d}{R}.$$

Since $|\delta d| \le (x+y)L \le 2aL$, its numerator has absolute value at most $15a+4a^2L$. Division by $R \ge a/2$ therefore gives $|\delta R| \le 40B$. The tangent constraint gives

$$(-p)\delta p = s\delta s + \frac{1}{2}\delta Q.$$

Using $-p \ge 1/2$ yields

$$|\delta R| \le 40B, \qquad |\delta p| \le 40aB. \tag{11.35}$$

The numerator in (11.33) retains the factor E. In fact,

$$d^2 \le 2E, \qquad |d\delta d| \le (x+y)^2 L \le 2EL, \qquad 0 \le A \le 6a^2.$$

Together with $R \le 4a$, $n \le E$, and $|s| \le 3a$, these inequalities bound $|\mathcal{N}|$ by an absolute constant times a^2E. Its differential is

$$\begin{aligned} \delta\mathcal{N} = {} & (\delta Q)R^2 + 2Q(s\delta s + d\delta d) - (\delta p)d^2 - 2Ad\delta d \\ & + \sqrt{3}\big((\delta n)sd + n(\delta s)d + ns\delta d\big). \end{aligned} \tag{11.36}$$

Each term is bounded by an absolute constant times aEB. For the second term, use $|s\delta s + d\delta d| = |R\delta R| \le 160aB$. For the terms containing wall differentials, retain their wall factors and use $a^2EL \le aEB$; the term containing δp is controlled by (11.35) and $d^2 \le 2E$. These estimates account for all terms in (11.36).

Finally,

$$\delta\left(\frac{\mathcal{N}}{R}\right) = \frac{\delta\mathcal{N}}{R} - \frac{\mathcal{N}\delta R}{R^2}.$$

The lower bound $R \ge a/2$ and (11.35) prove both rate estimates after choosing one absolute constant C_R. □

The energy E differs from b only in how it treats the strong wall. Since $E \le b + n$ and $\mathrm{D}_t\, n \le -2n$, the dissipative moment estimate used in Lemma 11.4 gives

$$\begin{aligned} I_0 &:= \int_0^T E(t)\,dt \le J_0 + \frac{n(0)}{2} \le 5a, \\ I_1 &:= \int_0^T tE(t)\,dt \le J_1 + \frac{n(0)}{4} \le \frac{1}{256} + (3+D)a. \end{aligned} \tag{11.37}$$

Here we used (11.24) and $n(0) \le a^2$. These estimates hold uniformly for the actual family and are independent of how small either positive initial wall is relative to the other.

Proposition 11.8. *There exists $h_* > 0$ such that, for every $q > 2$ and $D \ge 0$, one can choose $K_{q,D} > 0$ with the following property. For every $0 < h \le h_*$ and $P > 6D + 2$, and for all sufficiently small $a > 0$, uniformly over* (11.29)*, the function*

$$F(\rho) = R(T, \rho) - R(0, \rho)$$

has an ordinary derivative at each $\rho \in [a, 2a]$ and

$$|F(\rho)| + a|F'(\rho)| \le K_{q,D}a^2. \tag{11.38}$$

The threshold for a may depend on h, q, D, P.

Proof. Use the thresholds of Theorem 11.6, and fix a member of the actual family. With $\delta = \partial_\rho$, that theorem gives

$$|\delta s| \le 5, \qquad L(t) \le 2L_0 + 20t, \qquad L_0 \le 2 + 4Cq|\log a|.$$

Its differential is tangent to $G = 1$, since the initial chart lies on the constraint and the flow preserves it. Lemma 11.7 therefore yields

$$|\partial_t R| \le C_R a E, \qquad |\partial_\rho(\mathcal{N}/R)| \le C_R E(1 + 2aL_0 + 20at). \tag{11.39}$$

These are derivatives of one ordinary smooth family. Indeed, the initial chart and the normalized flow are jointly C^2 near each (ρ, t) under consideration, and $s(t, \rho) > 0$ keeps the square root defining R smooth there. The constraint and (11.33) hold on the same ordinary neighborhood. Equality of mixed derivatives thus gives

$$\partial_t \partial_\rho R = \partial_\rho(\mathcal{N}/R). \tag{11.40}$$

In particular, the time derivative on the left is that of the actual initial response, including when ρ is an endpoint of $[a, 2a]$.

Integrate the first inequality in (11.39). By (11.37),

$$|F(\rho)| \le C_R a I_0 \le 5C_R a^2.$$

Using (11.40) for the second inequality gives

$$|F'(\rho)| \le C_R((1 + 2aL_0)I_0 + 20aI_1).$$

For $0 < a \le 1$, the inequality $a|\log a| \le 1$ implies $aL_0 \le 2 + 4Cq$. Also $I_1 \le 4 + D$ by (11.37). Consequently,

$$(1 + 2aL_0)I_0 + 20aI_1 \le (105 + 40Cq + 20D)a.$$

Thus (11.38) holds, for example, with $K_{q,D} = C_R(110 + 40Cq + 20D)$. The constants C and C_R are absolute; in particular this choice precedes h and P. □

Define the physical phase at the end of the slow interval by

$$f(\rho) = \frac{R(T, \rho)}{\sqrt{3}}, \qquad f_0(\rho) = \Phi_h(\rho, \zeta, \nu), \tag{11.41}$$

where Φ_h is the reference phase (9.21) on the original section. At the beginning of the slow interval, $R(0, \rho)/\sqrt{3}$ is exactly the physical phase after the initial decay that was estimated in Corollary 9.13.

Corollary 11.9. *There exists $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 2$, and $D \ge 0$, there is $K_{h,q,D} > 0$ with the following property. For every $P > 6D + 2$ and all sufficiently small $a > 0$, uniformly over* (11.29)*, the physical phase f has an ordinary derivative and satisfies*

$$\begin{aligned} |f - f_0| &\le K_{h,q,D} a^2, \qquad |f' - f_0'| \le K_{h,q,D} a, \\ \frac{1}{4\sqrt{3}} &\le f' \le \frac{2}{\sqrt{3}} + 1. \end{aligned} \tag{11.42}$$

The constant $K_{h,q,D}$ is chosen before P.

Proof. Let $g(\rho) = R(0, \rho)/\sqrt{3}$. Then the exact identity

$$f = \frac{F}{\sqrt{3}} + g$$

joins the two portions of the same orbit. Proposition 11.8 bounds the value and derivative of $F/\sqrt{3}$ by $K_{q,D}a^2$ and $K_{q,D}a$, respectively. Corollary 9.13 bounds $g - f_0$ and its derivative

by $K_h a^2$ and $K_h a$, with K_h chosen before q. The first two assertions follow with $K_{h,q,D} = K_{q,D} + K_h$.

Decrease h_* to be at most $1/16$ and then decrease the threshold for a so that $K_{h,q,D} a \le 1/(4\sqrt{3})$. The original reference slope satisfies

$$\frac{1}{2\sqrt{3}} \le f_0' \le \frac{2}{\sqrt{3}}$$

by Proposition 9.5. Combining this with the derivative error proves the last two bounds in (11.42). □

The radius has therefore retained the original phase and its positive response to the initial shear throughout the additional waiting time. The endpoint wall bounds (11.15) and this phase control provide the physical data for the subsequent projection and upward core entrance.

11.4. **Projection after the slow decay.** The physical phase at the end of the longer interval must now be expressed in the coordinates of the growing core. The projection divides by a height of order a^2. Its radial response is controlled because the strong wall and the decaying weak wall have both reached order a^2, and their ordinary derivatives have reached order a. These are precisely the scales needed to reuse the compact comparison of Section 9.4.

We keep the positive-wall family (11.29), with $q > 2$, $D > 2$, and $P > 6D + 2$. Write its endpoint as

$$z_T(\rho) = \phi_{t_0+T}\Xi_h(\rho, \zeta, \nu) = (p, s, n, x, y)(\rho), \qquad X_T(\rho) = (n, s, x, y)(\rho). \tag{11.43}$$

All primes in this subsection denote ordinary derivatives in ρ with h, a, q, D, ζ, ν fixed. In particular, the two times

$$t_0 = \frac{q}{2}\log\frac{1}{a}, \qquad T = \frac{D}{a}\log\frac{1}{a}$$

are constant along the family.

Proposition 11.10. *There are constants $C \ge 32$ and $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 2$, $D > 2$, and $P > 6D + 2$, the following holds for all sufficiently small $a > 0$, uniformly over* (11.29)*. The endpoint* (11.43) *belongs to* (11.1)*, has positive walls and positive height, and is ordinarily differentiable in ρ. With*

$$B = 13 + 16Cq + 40D, \tag{11.44}$$

its coordinates satisfy

$$\begin{gathered} 0 < n \le a^2, \qquad 0 < y \le a^2, \\ |s'| \le B, \qquad |x'| \le B, \qquad |n'| \le Ba, \qquad |y'| \le Ba. \end{gathered} \tag{11.45}$$

The scale threshold may depend on h, q, D, P.

Proof. Use the constant C and the thresholds of Theorem 11.6, decreasing h_* so that $h_* \le 1$ and Theorem 11.3 also applies. The actual endpoint then has all the asserted geometric and regularity properties. Its wall bounds give

$$n \le ha^q e^{-2T} \le a^q, \qquad y \le 4\varepsilon a^{1+D/2} \le a^{1+D/2}. \tag{11.46}$$

Both exponents are strictly greater than two. For any fixed $r > 2$, the elementary limit $a^{r-2}(1 + |\log a|) \to 0$ as $a \downarrow 0$ permits the common choice

$$a^q(1 + |\log a|) \le a^2, \qquad a^{1+D/2}(1 + |\log a|) \le a^2. \tag{11.47}$$

These inequalities already prove the two wall size bounds.

At the endpoint, let L be the logarithmic wall response from (11.16), and put $K_0 = 4 + 8Cq + 20D$. Since $aT = D|\log a|$ and $a \le 1$, (11.30) implies

$$aL \le K_0(1 + |\log a|).$$

Multiplying $|n'| \le nL$ by a and using (11.46) and (11.47) gives $a|n'| \le K_0 a^2$. The same argument applies to y. Thus $|n'|, |y'| \le K_0 a \le Ba$.

For x, the regional bound $x \le a^2$ gives

$$|x'| \le xL \le K_0 a(1 + |\log a|) \le 2K_0,$$

where $a|\log a| \le 1$ was used in the last step. Finally $|s'| \le 5$ by the same variation theorem. The identity $B = 2K_0 + 5$ proves the remaining bounds. □

Let $A = 1 + p$ at the endpoint, and use the projective coordinates (2.6). Define

$$\begin{aligned} M = \frac{\sqrt{3}s}{A}, \qquad V = \frac{\sqrt{3}x}{A}, \qquad k = \sqrt{(M-1)^2 + V^2}, \\ f_{\mathrm{p}} = \frac{2}{k}, \qquad g = \frac{\sqrt{s^2 + (x-y)^2}}{\sqrt{3}}. \end{aligned} \tag{11.48}$$

Here g is the physical phase already controlled in Corollary 11.9. The projection error admits the same scaled expression as in (9.85). We use maximum norms and their induced operator norms.

Proposition 11.11. *There is an absolute constant $C_{\mathrm{e}} \ge 1$ such that, after reducing the scale threshold in Proposition 11.10, the error $e = f_{\mathrm{p}} - g$ has an ordinary derivative and*

$$|e(\rho)| \le C_{\mathrm{e}} a^2, \qquad |e'(\rho)| \le 2C_{\mathrm{e}} Ba \quad (a \le \rho \le 2a). \tag{11.49}$$

The constants C_{e} and the additional scale restriction are independent of h, q, D, P.

Proof. Use the scaled function $F(u, z)$ from (9.84), now with the fixed compact set

$$\mathcal{K} = \{z : \|z\| \le 4,\ |z_1| \ge 1/2\}.$$

The proof of (9.86) gives constants $\delta > 0$ and $C_{\mathrm{e}} \ge 1$ on this set. In particular, the function is smooth near every (u, z) with $\|u\| \le \delta$ and $z \in \mathcal{K}$, and

$$\begin{aligned} |F(u,z)| &\le C_{\mathrm{e}}\|u\|, \\ |DF(u,z)(u',z')| &\le C_{\mathrm{e}}\|u'\| + C_{\mathrm{e}}\|u\|\|z'\|. \end{aligned} \tag{11.50}$$

The second term retains $\|u\|$ because $F(0, z)$ vanishes on the entire limiting slice.

At the actual endpoint take

$$u(\rho) = \left(a, \frac{n(\rho)}{a}, \frac{y(\rho)}{a}\right), \qquad z(\rho) = \left(\frac{s(\rho)}{a}, \frac{x(\rho)}{a}\right).$$

The regional bounds $a/2 \le s \le 3a$ and $x \le a^2$ place z in $\mathcal{K}$, while (11.45) gives

$$\|u\| \le a, \qquad \|u'\| \le B, \qquad \|z'\| \le \frac{B}{a}.$$

Choose $a \le \delta$. Since a is fixed under differentiation, (9.85) and (11.50) yield

$$|\mathcal{P}(X_T) - g| \le C_{\mathrm{e}} a^2, \qquad \left|\frac{d}{d\rho}(\mathcal{P}(X_T) - g)\right| \le aC_{\mathrm{e}}\left(B + a\frac{B}{a}\right) = 2C_{\mathrm{e}}Ba.$$

It remains to identify this smooth expression with the actual projective phase also in its derivative. At each initial point the chart radicand is positive, the total time $t_0 + T$ is nonnegative, and the endpoint has $p < 0$ and $1 + p > 0$. The three strict inequalities persist for nearby chart parameters. The constraint continues to hold for their actual flow values, so (9.83) gives $\mathcal{P}(X_T) = f_{\mathrm{p}}$ on an ordinary neighborhood in those parameters. The scaled expression is smooth on a neighborhood as well. Hence this identity transfers both the value and the ordinary derivative bounds to e, including at the endpoints of $[a, 2a]$. □

Theorem 11.12. *There exists $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 2$, and $D > 2$, one can choose $K > 0$ with the following property. For every $P > 6D + 2$ and all sufficiently small $a > 0$, uniformly over* (11.29)*, the actual phase f_{p} in* (11.48) *has an ordinary derivative and satisfies*

$$\begin{aligned} |f_{\mathrm{p}}(\rho) - \Phi_h(\rho,\zeta,\nu)| &\le Ka^2, \\ \left| f_{\mathrm{p}}'(\rho) - \frac{d}{d\rho}\Phi_h(\rho,\zeta,\nu) \right| &\le Ka, \\ \frac{1}{4\sqrt{3}} \le f_{\mathrm{p}}'(\rho) &\le \frac{2}{\sqrt{3}} + 1. \end{aligned} \tag{11.51}$$

The constant K is chosen before P and the weak-wall labels; the scale threshold may depend on h, q, D, P.

Proof. Choose h_* to meet the preceding propositions and Corollary 11.9, and require $h_* \le 1/16$. Let K_{r} be the constant in that corollary, and put

$$K_{\mathrm{p}} = 2C_{\mathrm{e}}B, \qquad K = K_{\mathrm{p}} + K_{\mathrm{r}}.$$

These constants are selected after h, q, D and before P. Since $B \ge 1$, Proposition 11.11 bounds both projection errors by $K_{\mathrm{p}}a^2$ and $K_{\mathrm{p}}a$, respectively. The identity

$$f_{\mathrm{p}} - \Phi_h = e + (g - \Phi_h)$$

and its ordinary derivative prove the first two lines of (11.51).

For the derivative bounds, compare directly with the original reference phase. By Proposition 9.5,

$$\frac{1}{2\sqrt{3}} \le \frac{d}{d\rho}\Phi_h(\rho,\zeta,\nu) \le \frac{2}{\sqrt{3}}.$$

Decrease the common scale threshold so that $Ka \le 1/(4\sqrt{3}) \le 1$. Combining these bounds with the derivative error gives the last line of (11.51). □

The extra waiting time has therefore preserved the reference phase through the subsequent projection. The projected state still has $V < 1$ by (11.14); its further approach to the growing core starts from this controlled phase.

11.5. **Projected directions after the slow decay.** The endpoint of the slow interval must also supply controlled ordinary derivatives in the projective coordinates. The denominator $A = 1 + p$ is of order a^2, so its relative derivative requires more information than a bound on the full physical tangent. The constraint expresses this derivative through the shear response and the product bL. The latter remains small because the wall energy absorbs the time growth of the logarithmic response.

We use the positive-wall family of Theorem 11.6 and its notation $z(t,\rho) = (p, s, n, x, y)$, b, L, and L_0. Throughout this subsection, a prime denotes the ordinary derivative in ρ, with h, a, q, D, ζ, ν and both times t_0, T fixed. Set

$$Y(t,\rho) = (M,U,V,W)(t,\rho) = \frac{\sqrt{3}}{A(t,\rho)}(s,n,x,y)(t,\rho). \tag{11.52}$$

The positive height, shear, and walls make all four coordinates positive.

Lemma 11.13. *Let $C \ge 32$ be the absolute constant in Theorem 11.6. For the same family and sufficiently small a, put $K = 13 + 20Cq + D$. Then, for $0 \le t \le T$,*

$$\begin{aligned} b(t) &\le 5a^2, \qquad tb(t) \le (3+D)a, \\ (L_0 + t)b(t) &\le Ka, \qquad b(t)L(t) \le 20Ka. \end{aligned} \tag{11.53}$$

Proof. The regional bounds and the wall rates give

$$n(t) \le a^2 e^{-2t}, \qquad x(t) \le a^2, \qquad y(t) \le a e^{-at/2}.$$

Since $n + x + y \le 3a$, it follows that

$$b(t) \le 3a^3 e^{-2t} + a^4 + a^2 e^{-at}. \tag{11.54}$$

For $a \le 1$, this is at most $5a^2$. The elementary bound $ue^{-u} \le 1$ for $u \ge 0$ gives

$$t(3a^3 e^{-2t}) \le \frac{3}{2} a^3, \qquad t(a^2 e^{-at}) \le a.$$

Also, since $t \le T = (D/a)\log(1/a)$ and $a \log(1/a) \le 1$,

$$ta^4 \le Da^2\, a \log(1/a) \le Da.$$

Multiplying (11.54) by t proves the second bound in (11.53).

By (11.30), $L_0 \le 2 + 4Cq|\log a|$. Thus

$$L_0 b(t) \le 5a^2(2 + 4Cq|\log a|) \le (10 + 20Cq)a.$$

Adding the estimate for $tb(t)$ gives $(L_0+t)b(t) \le Ka$. Finally, $L(t) \le 2L_0 + 20t \le 20(L_0+t)$, which proves the last bound. □

Proposition 11.14. *For each $q > 2$ and $D \ge 0$, there is a constant $C_{q,D} > 0$, independent of h, P and the weak-wall labels, such that the following holds under the thresholds of Theorem 11.6. Throughout $0 \le t \le T$, the projective family has ordinary derivatives and*

$$|(\log A)'| + |(\log M)'| \le \frac{C_{q,D}}{a}, \tag{11.55}$$

$$|(\log U)'| + |(\log V)'| + |(\log W)'| \le C_{q,D}\left(\frac{1}{a} + L_0 + t\right), \tag{11.56}$$

$$|M'| \le \frac{C_{q,D}}{a^2}. \tag{11.57}$$

Proof. The ordinary physical derivatives and the positive height give ordinary derivatives of the quotient in (11.52). Their logarithms satisfy the exact identities

$$\begin{aligned} (\log A)' &= \frac{p'}{A}, & (\log M)' &= \frac{s'}{s} - \frac{p'}{A}, \\ (\log U)' &= \frac{n'}{n} - \frac{p'}{A}, & (\log V)' &= \frac{x'}{x} - \frac{p'}{A}, \\ (\log W)' &= \frac{y'}{y} - \frac{p'}{A}. \end{aligned} \tag{11.58}$$

They hold on an ordinary neighborhood of each initial parameter, including the endpoints of $[a, 2a]$.

The polynomial bound (11.20) and Lemma 11.13 imply

$$|Q'| \le 4bL \le 80Ka.$$

Differentiating the constraint yields

$$(-p)p' = ss' + \frac{1}{2}Q'.$$

Use $-p \ge 1/2$, $s \le 3a$, and $|s'| \le 5$ to obtain $|p'| \le (30 + 80K)a$. Since $A \ge a^2/10$ and $s \ge a/2$,

$$\left|\frac{p'}{A}\right| \le \frac{300 + 800K}{a}, \qquad \left|\frac{s'}{s}\right| \le \frac{10}{a}.$$

The first two identities in (11.58) prove (11.55) with a constant depending only on q, D.

Each of $|n'/n|$, $|x'/x|$, and $|y'/y|$ is at most $L(t) \le 2L_0 + 20t$. Thus the sum of the last three logarithmic derivatives is at most

$$6L_0 + 60t + \frac{900 + 2400K}{a}.$$

Enlarging the same constant proves (11.56). Finally, the regional height and shear bounds imply $0 < M \le 60/a$. The identity $M' = M(\log M)'$ therefore gives (11.57), after one more enlargement of $C_{q,D}$. □

At the endpoint write

$$\begin{aligned} Y(\rho) &= Y(T,\rho) = (M,U,V,W)(\rho), \\ k(\rho) &= \sqrt{(M(\rho)-1)^2 + V(\rho)^2}, \qquad \Lambda = 1 + |\log a|. \end{aligned} \tag{11.59}$$

The parameter k is the radius of the growing core circle. It converts the preceding bounds in inverse powers of a into the natural scales of the core orbit.

Theorem 11.15. *There exists $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 2$, and $D > 2$, there is $B > 0$ with the following property. For every $P > 6D + 2$ and $0 < \delta \le 10^{-4}$, all sufficiently small $a > 0$ give, uniformly over* (11.29)*, an ordinarily differentiable endpoint family* (11.59) *such that*

$$\begin{aligned} &\frac{1}{24a} \le k \le \frac{64}{a}, \qquad k \ge 8, \qquad M > 1, \qquad 0 < V < 1, \\ &0 < U, W \le \delta, \qquad \Lambda(U+W) \le 1. \end{aligned} \tag{11.60}$$

The ordinary directions obey

$$|M'| \le Bk^2, \qquad |U'| \le B\Lambda k U, \qquad |V'| \le B\Lambda k V, \qquad |W'| \le B\Lambda k W. \tag{11.61}$$

The constant B is chosen before P and δ.

Proof. Choose h_* to meet the thresholds of the preceding results and keep $a \le 1/256$. The height and shear bounds give

$$\frac{1}{12a} \le M = \frac{\sqrt{3}s}{A} \le \frac{60}{a}. \tag{11.62}$$

In particular $M > 1$. The bound $0 < V < 1$ follows from Theorem 11.3 and positivity of the walls. For the core radius,

$$M - 1 \le k \le (M-1) + V < M.$$

Since $1/(24a) \ge 1$, the lower bound in (11.62) implies $k \ge 1/(24a)$. This is at least 8 when $a \le 1/256$; the upper bound gives $k \le 64/a$.

The endpoint wall powers (11.15) imply

$$U + W \le 20ha^{q-2} + 80a^{D/2-1}.$$

Both exponents are positive because $q > 2$ and $D > 2$. For every $r > 0$, $a^r(1 + |\log a|) \to 0$ as $a \downarrow 0$. We may therefore decrease the threshold for a so that

$$(20ha^{q-2} + 80a^{D/2-1})\Lambda \le \delta. \tag{11.63}$$

It follows that $\Lambda(U+W) \le \delta \le 1$ and, since $\Lambda \ge 1$, each of U, W is at most δ. Their positivity follows from the positive initial walls.

It remains to put the derivatives in the core scale. Let $N = 3 + 4Cq + D$, where C is the absolute constant in (11.30). The identities $aT = D|\log a|$ and $a \le 1$ give

$$\frac{1}{a} + L_0 + T \le \frac{N\Lambda}{a}.$$

Indeed, multiply by a and use $aL_0 \le 2a + 4Cqa|\log a| \le 2 + 4Cq|\log a|$. Proposition 11.14 therefore gives

$$|M'| \le \frac{C_{q,D}}{a^2}, \qquad |(\log U)'|, |(\log V)'|, |(\log W)'| \le \frac{C_{q,D}N\Lambda}{a}.$$

Use $1/a \le 24k$ and multiply the logarithmic estimates by their respective positive coordinates. Choosing B to dominate $24^2 C_{q,D}$ and $24C_{q,D}N$ proves (11.61). This choice depends only on the already fixed constants and q, D, so it can be made before P, δ and all subsequent scale choices. □

The logarithmic factor in the wall directions depends only on the radial scale a. The endpoint estimates therefore remain uniform as the two positive initial weak walls become arbitrarily unequal. The smallness in (11.63) preserves room for this logarithmic factor when the orbit is followed to the upward core entrance.

11.6. **The phase at the delayed core entrance.** At the end of the long decay interval, the active wall still lies below the core section. The remaining incoming passage brings it to $V = 1$. Its effect on the radius is controlled by the small transverse walls. These same walls absorb the logarithmic factor in their initial derivatives, so the phase remains close to the original section phase even when the active wall is arbitrarily small.

Write $\mathcal{P}$ for the first upward arrival of Proposition 6.1, and use $r(y) = H(M-1, V)$ for the core radius. We first give the direction estimate needed to apply this passage to the delayed family.

Lemma 11.16 (Weighted incoming directions)**.** *There are universal constants $A > 0$ and $0 < \delta \le 10^{-4}$ with the following property. Let $Y = (M, U, V, W)$ be an ordinarily differentiable curve at ρ, and write*

$$Y(\rho) = (m, u, v, w), \qquad k = r(Y(\rho)).$$

Suppose that $B \ge 0$, $\Lambda \ge 1$, and

$$k \ge 8, \qquad m \ge 1, \qquad 0 < u, w \le \delta, \qquad 0 < v < 1, \qquad \Lambda(u+w) \le 1,$$

$$|M'| \le Bk^2, \qquad |U'| \le B\Lambda k u, \quad |V'| \le B\Lambda k v, \quad |W'| \le B\Lambda k w. \tag{11.64}$$

Then $K = r \circ Y$ and $\kappa = r \circ \mathcal{P} \circ Y$ are ordinarily differentiable at ρ, and

$$|\kappa(\rho) - k| \le \frac{A}{k}, \qquad |\kappa'(\rho) - K'(\rho)| \le ABk. \tag{11.65}$$

Proof. Take the wall threshold from Proposition 10.24, and apply that proposition to $Y'(\rho)$ with direction constant $B\Lambda$. The M component satisfies the enlarged bound because $\Lambda \ge 1$. With

$$b = u + w, \qquad J = uv + v^2 b^2,$$

the proposition gives, for a universal $A_0 > 0$,

$$|\kappa - k| \le \frac{A_0}{k}(b^2 + J), \qquad |\kappa' - K'| \le A_0 B\Lambda(b^2 + kJ).$$

Now $0 \le b \le 1$ and $J \le b + b^2$. Moreover,

$$\Lambda b^2 \le 1, \qquad \Lambda J \le 2.$$

Consequently $b^2 + J \le 3$ and $\Lambda(b^2 + kJ) \le 1 + 2k \le 3k$. Taking $A = 3A_0$ proves both estimates. Differentiability follows from the ordinary local arrival map and the chain rule. □

Return to the positive shear family with

$$q > 2, \qquad D > 2, \qquad P > 6D + 2, \qquad t_0 = \frac{q}{2}\log(1/a), \quad T = \frac{D}{a}\log(1/a),$$
$$0 \le \varepsilon \le 1/128, \qquad 0 < \zeta \le \min(a^P, 2\varepsilon a), \qquad 0 < \nu \le 2\varepsilon a, \qquad a \le \rho \le 2a.$$

Let $Y(\rho)$ be the projective coordinates of $\phi_{t_0+T}(\Xi_h(\rho,\zeta,\nu))$, and put

$$k(\rho) = r(Y(\rho)), \qquad \kappa(\rho) = r(\mathcal{P}(Y(\rho))). \tag{11.66}$$

All parameters other than ρ, including the two elapsed times, remain fixed in derivatives.

Proposition 11.17 (Radius at the upward arrival)**.** *There is $h_* > 0$ such that, for each $0 < h \le h_*$, $q > 2$ and $D > 2$, there is a constant $C > 0$ with the following property. For every $P > 6D + 2$, all sufficiently small $a > 0$ satisfy the conclusions below uniformly over the stated labels and $\rho \in [a, 2a]$.*

The first upward arrival from $Y(\rho)$ exists. Its time and state are C^2 on an ordinary neighborhood of this input. The radii in (11.66) *are ordinarily differentiable and satisfy*

$$\frac{1}{48a} \le \kappa \le \frac{128}{a}, \qquad k \ge \frac{1}{48a},$$
$$|\kappa - k| \le Ca, \qquad |\kappa' - k'| \le \frac{C}{a}, \qquad |k'| \le \frac{6144}{a^2}. \tag{11.67}$$

The constant C is chosen before P.

Proof. Choose A, δ from Lemma 11.16. Theorem 11.15, applied with this δ, gives

$$k \ge 8, \qquad \frac{1}{24a} \le k \le \frac{64}{a}, \qquad M > 1, \qquad 0 < U, W \le \delta, \quad 0 < V < 1.$$

Thus Proposition 6.1 supplies the actual arrival and its local C^2 time and state maps. Throughout this incoming passage the radius stays within $k/100$ of k; in particular,

$$\frac{99}{100}k \le \kappa \le \frac{101}{100}k.$$

This proves the radius bounds in (11.67).

The same input theorem gives (11.64) with $\Lambda = 1 + |\log a|$ and a constant $B = B_{h,q,D} > 0$. The weighted estimate therefore yields

$$|\kappa - k| \le 24Aa, \qquad |\kappa' - k'| \le \frac{64AB}{a}.$$

Both constants are independent of P.

To bound k', use the projected phase $f = 2/k$ from Theorem 11.12. Its bounds give

$$f \ge a/32, \qquad |f'| \le 3.$$

Since $k = 2/f$ on an ordinary neighborhood of the parameter,

$$|k'| = \frac{2|f'|}{f^2} \le \frac{6144}{a^2}.$$

Taking $C = 24A + 64AB + 1$ completes the proof after the common restriction on a. □

Define the phase at this upward arrival by

$$\Phi(\rho) = \frac{2}{\kappa(\rho)}, \qquad f_0(\rho) = \Phi_h(\rho,\zeta,\nu). \tag{11.68}$$

The small radius displacement in the preceding proposition also controls its reciprocal derivative. The lower radius bounds are essential here: both reciprocal factors are of order a.

Theorem 11.18 (Phase and radial speed at the delayed entrance)**.** *For a sufficiently small universal $h_* > 0$ and each $0 < h \le h_*$, $q > 2$ and $D > 2$, there is $C = C_{h,q,D} > 0$ such that, for every $P > 6D + 2$ and all sufficiently small $a > 0$, the actual arrival above satisfies*

$$|\Phi - f_0| \le Ca^2, \qquad |\Phi' - f_0'| \le Ca,$$

$$\frac{1}{4\sqrt{3}} \le \Phi' \le \frac{2}{\sqrt{3}} + 1. \tag{11.69}$$

These are ordinary derivatives, and the estimates are uniform over the stated labels and $\rho \in [a, 2a]$. In addition,

$$\frac{1}{48a} \le \kappa \le \frac{128}{a}, \qquad \frac{1}{36864a^2} \le -\kappa' \le \frac{24576}{a^2}. \tag{11.70}$$

Proof. Write B for the constant in Proposition 11.17, and assume $a \le 1$. Set $e = \kappa - k$ and $f = 2/k$. The preceding proposition gives

$$|e| \le B, \qquad |e'| \le B/a, \qquad |k'| \le 6144/a^2, \qquad \kappa, k \ge 1/(48a).$$

The reciprocal identities

$$\Phi - f = -\frac{2e}{\kappa k}, \qquad (\Phi - f)' = -\frac{2e'}{\kappa^2} + \frac{2e(\kappa + k)k'}{k^2\kappa^2}$$

then imply

$$|\Phi - f| \le 2 \cdot 48^2 Ba^2, \qquad |\Phi' - f'| \le A_1 a, \qquad A_1 = 2 \cdot 48^2 B + 4 \cdot 6144 \cdot 48^3 B. \tag{11.71}$$

For the second term in the derivative, use $(\kappa + k)/(k^2\kappa^2) = 1/(k^2\kappa) + 1/(k\kappa^2) \le 2(48a)^3$.

Let K be the phase comparison constant in Theorem 11.12. Adding its value and derivative errors to (11.71) proves the first two bounds in (11.69) with $C = A_1 + K$. Choose a still smaller so that $Ca \le 1/(4\sqrt{3})$. The reference slope from Proposition 9.5 is

$$\frac{1}{2\sqrt{3}} \le f_0' \le \frac{2}{\sqrt{3}}.$$

It gives the asserted bounds on Φ'.

Finally, the radius bounds imply $a/64 \le \Phi \le 96a$, while the slope bounds imply $1/8 \le \Phi' \le 3$. Differentiating $\kappa = 2/\Phi$ gives

$$-\kappa' = \frac{2\Phi'}{\Phi^2}.$$

Substitution proves the two constants in (11.70). □

Thus the original shear coordinate parametrizes the actual upward entrance with a positive phase slope and a decreasing radius. The latter has the scale a^{-2} needed to compare transverse wall responses with radial motion through the core.

11.7. **Amplitudes and directions at the delayed entrance.** The upward arrival has a nonzero radial response. To continue through the core, its transverse amplitudes must also be small relative to the radius, and its tangent must lie in the core cone. Both properties follow from the wall powers accumulated during the slow decay. The bound $U + W = o(a)$ supplies the core amplitude condition, while the square roots of the walls absorb the logarithmic loss in their derivatives.

Keep the endpoint family $Y = (M, U, V, W)$ and its upward arrival from the preceding subsection. Write

$$Z = \mathcal{P} \circ Y = (M_+, U_+, 1, W_+), \qquad \kappa = r(Z), \qquad \Lambda = 1 + |\log a|. \tag{11.72}$$

The labels satisfy

$$\begin{gathered} 0 \le \varepsilon \le 1/128, \qquad 0 < \zeta \le \min(a^P, 2\varepsilon a), \qquad 0 < \nu \le 2\varepsilon a, \\ a \le \rho \le 2a, \qquad t_0 = \frac{q}{2}\log(1/a), \quad T = \frac{D}{a}\log(1/a). \end{gathered} \tag{11.73}$$

All derivatives are ordinary derivatives in ρ, with the other parameters fixed.

The endpoint estimates give

$$U \le 20ha^{q-2}, \qquad V \le 40\zeta a^{-6D-2}, \qquad W \le 80a^{D/2-1}. \tag{11.74}$$

In the last inequality we have used $\varepsilon \le 1$. The following two consequences have distinct uses.

Lemma 11.19. *For $q > 2$, $D > 2$, and $P > 6D + 2$, each $\delta > 0$ admits a sufficiently small scale threshold such that*

$$\Lambda\sqrt{U + W} \le \delta. \tag{11.75}$$

If $q > 3$ and $D > 4$, then each $\eta > 0$ also admits a sufficiently small threshold such that

$$U + VW \le \frac{\eta a}{128} \le \frac{\eta}{2k}, \qquad k = r(Y). \tag{11.76}$$

The thresholds are uniform over (11.73).

Proof. Put $b_a = 20ha^{q-2} + 80a^{D/2-1}$. For $h \ge 0$, squaring the nonnegative right-hand side verifies

$$\sqrt{b_a} \le (20h + 1)a^{(q-2)/2} + 80a^{D/4-1/2}.$$

Both exponents are positive. The limit $a^r(1 + |\log a|) \to 0$ for $r > 0$ therefore gives $\Lambda\sqrt{b_a} \le \delta$ for all sufficiently small $a > 0$. Since $U + W \le b_a$, this proves (11.75).

Under the stronger exponent assumptions,

$$\frac{b_a}{a} = 20ha^{q-3} + 80a^{D/2-2} \longrightarrow 0.$$

Thus $b_a \le \eta a/128$. Use $V < 1$ and $k \le 64/a$ from Theorem 11.15 to obtain (11.76). All bounds used to select the thresholds are independent of the labels. □

Theorem 11.20. *There is $h_* > 0$ such that, for each $0 < h \le h_*$, $q > 3$, $D > 4$, $P > 6D + 2$, $0 < \eta \le 10^{-4}$, and $\theta > 0$, all sufficiently small $a > 0$ have the following properties uniformly over* (11.73).

The actual upward arrival Z lies on the incoming boundary of the closed core region with reference radius κ, and

$$\begin{gathered} \kappa \ge 6, \qquad M_+ > 1, \qquad z(Z) = -\operatorname{arcosh}\kappa, \\ 0 < U_+ \le 20ha^{q-2}, \qquad 0 < W_+ \le 80a^{D/2-1}, \qquad U_+ + W_+ \le \frac{\eta}{\kappa}. \end{gathered} \tag{11.77}$$

For every real $d \le \min(q - 2, D/2 - 1)$,

$$U_+ + W_+ \le (20h + 80)a^d. \tag{11.78}$$

The curve Z is ordinarily differentiable, $\kappa' < 0$, and its tangent satisfies

$$|U_+'| \le \theta\sqrt{U_+}\,|\kappa'|, \qquad |W_+'| \le \theta\sqrt{W_+}\,|\kappa'|. \tag{11.79}$$

Proof. Take h_* small enough for the endpoint and upward arrival estimates. The endpoint has $k \ge 8$, $M > 1$, $0 < V < 1$, and $0 < U, W \le 10^{-4}$. Lemma 11.19 supplies $U + VW \le \eta/(2k)$. Lemma 10.19 now places Z on the incoming core boundary, with

$$\kappa \ge 6, \qquad U_+ \le U, \qquad W_+ \le VW, \qquad U_+ + W_+ \le \eta/\kappa.$$

It also gives the angle in (11.77). The walls are positive by their multiplicative equations. Combining these inequalities with (11.74) proves the amplitude bounds. For $0 < a \le 1$

and the stated d, both wall powers are at most a^d, proving (11.78) without any additional restriction depending on d.

For the tangent estimate, choose the universal wall threshold and constant C in Proposition 10.24. Theorem 11.15 supplies its direction hypotheses with constant $B\Lambda$: the M derivative bound may be enlarged by $\Lambda \geq 1$. The two output wall estimates give

$$|U'_+| \leq CB\Lambda k^2 \sqrt{UV}\sqrt{U_+},$$
$$|W'_+| \leq CB\Lambda k \sqrt{WV}\sqrt{W_+}.$$

Since $V < 1$ and $k \geq 8$, both square-root factors at the input are bounded by $\sqrt{U+W}$, and the second power of k may replace the first in the second line. The radial speed from Theorem 11.18 gives

$$k^2 \leq \frac{64^2}{a^2} \leq R_0|\kappa'|, \qquad R_0 = 64^2 \cdot 36864.$$

Apply (11.75) with $\delta = \theta/(R_0CB)$. Substitution proves both inequalities in (11.79). Ordinary differentiability follows by composition with the C^2 arrival map; the sign of κ' is already part of Theorem 11.18. □

The amplitude and direction thresholds may therefore be chosen together. Taking η to be a core threshold and $\theta = 1$ puts the delayed family directly in the domain of the full core return and its invariant cone.

11.8. **The return after the slow decay.** The delayed entrance now meets the full core hypotheses. We follow it through the core and the next connection, keeping the event times on the original normalized orbit. The long decay interval contains no $V = 1$ event, so these crossings can be identified relative to the original fast cutoff t_0. This identification also allows the core and connection derivative estimates to act on the same return curve.

Fix $0 < h \leq h_*$, $q > 3$, $D > 4$, and $P > 6D + 2$. All conclusions below hold for sufficiently small $a > 0$, uniformly over (11.73). Write

$$x_0 = \Xi_h(\rho, \zeta, \nu), \qquad s = t_0 + T, \qquad Y = \mathrm{proj}(\phi_s x_0), \qquad Z = \mathcal{P}(Y), \quad \kappa = r(Z). \tag{11.80}$$

The time s is fixed when the initial labels vary.

Define

$$t_+ = s + T_{\mathrm{in}}(Y), \qquad t_- = s + \tau_\downarrow(Y). \tag{11.81}$$

Here T_{in} is the first upward arrival time and $\tau_\downarrow$ is the first downward $V = 1$ time, both measured in original normalized time from the indicated state.

Proposition 11.21. *The time t_+ is the first $V = 1$ event after t_0 on the normalized orbit from x_0, and its active-wall rate is positive. The time t_- is the first such event after t_0 with negative active-wall rate. They satisfy*

$$t_0 < s < t_+ < t_-, \qquad t_- = t_+ + \tau_\downarrow(Z),$$
$$Z = \mathrm{proj}(\phi_{t_+} x_0), \qquad \mathcal{R}_\downarrow(Y) = \mathcal{R}_\downarrow(Z). \tag{11.82}$$

The downward time and state maps are C^2 on an ordinary neighborhood of Y. Their composition with the initial family is jointly C^2 in (ζ, ν, ρ).

Proof. Choose the amplitude threshold $\eta = 10^{-4}$ in Theorem 11.20 and its proof. The endpoint Y satisfies (10.55). Hence the incoming and downward passages exist, and Proposition 10.20 gives

$$\tau_\downarrow(Y) = T_{\mathrm{in}}(Y) + \tau_\downarrow(Z) > T_{\mathrm{in}}(Y) > 0$$

and the state identity, on an ordinary neighborhood.

The height at s is positive and $\phi_s x_0$ satisfies the constraint. Its inverse projective image is therefore exactly $\phi_s x_0$. The flow property gives

$$\Phi_\tau(Y) = \operatorname{proj}(\phi_{s+\tau} x_0), \tag{11.83}$$

along the incoming and core passages. The normalized states agree before projection, so their active-wall rates agree as well. This proves the orbit and time identities in (11.82).

On the preceding interval $[t_0, s]$, the active coordinate is strictly below one by Theorem 11.3. Thus no crossing has been omitted there. Any earlier candidate crossing with $t > s$ corresponds by (11.83) to the time $t - s$ from Y, with the same direction. Minimality of $T_{\mathrm{in}}(Y)$ proves the assertion for t_+; minimality of $\tau_\downarrow(Y)$ proves the assertion for t_-. The positive rate at t_+ follows from Proposition 6.1. We take $a < 1$, so $T > 0$ and $t_0 < s$.

Finally, the initial chart radicand is positive and the flow has an ordinary smooth neighborhood at the fixed time s. Its endpoint height is positive. Consequently Y is jointly C^2 in the three initial parameters. Composition with the ordinary downward time and state maps proves the asserted regularity. □

Write the downward output and the ensuing connection parameters as

$$(f, u, w) = \mathcal{R}_\downarrow(Y), \qquad \pi = (\mu, u, w), \qquad \mu = \sqrt{f^2 - 1} - 1. \tag{11.84}$$

At the first $W = 1$ event, the state in the original frame is $(-m_*, U_*, V_*, 1)$. The next incoming core state in the reflected frame is represented by

$$G = (g, \widehat{u}, \widehat{w}) = (H(m_* - 1, 1), U_*, V_*), \qquad t_{\mathrm{next}} = t_- + T_c(\pi). \tag{11.85}$$

Proposition 11.22. *For each $0 < \eta \le 10^{-4}$, a sufficiently small common scale threshold makes this connection well defined, with*

$$4 \le \mu, \qquad \kappa/2 \le \mu < \kappa, \qquad 0 < u \le \eta/\mu, \qquad 0 < w \le \eta,$$
$$u \le U_+ e^{-\kappa/12}, \qquad w \le W_+. \tag{11.86}$$

The actual state at t_- is $(-\mu, u, 1, w)$, and t_{next} is the first $W = 1$ time after t_-. Moreover,

$$\iota_+(G) = \mathsf{S} \operatorname{proj}(\phi_{t_{\mathrm{next}}} x_0). \tag{11.87}$$

All three coordinates of G are positive. The maps $\pi, G, 2/g, t_{\mathrm{next}}$ are jointly C^2 in (ζ, ν, ρ) on an ordinary neighborhood of each input under consideration.

Proof. Apply Theorem 11.20 with the stated η. At the upward arrival, $\kappa \ge 6$, $V_+ = 1$, $z(Z) = -\operatorname{arcosh} \kappa \le 0$, and

$$0 < U_+, V_+ W_+ = W_+ \le \eta/\kappa.$$

Proposition 10.15, with angle bound zero, applies from Z. It gives (11.86), the first connection event, and its positive output walls. By (11.82), its downward output is exactly (f, u, w) in (11.84). The identity $H(\mu + 1, 1) = f$ identifies the actual exit as $(-\mu, u, 1, w)$. Composing its connection with (11.83) proves (11.87).

The downward map is ordinarily C^2 near Y, and $f = H(\mu + 1, 1) > 1$. Thus the parameter change defining π is smooth there. The long connection has ordinary C^2 time and state maps at these parameters. Composition with the joint initial family gives the regularity of $\pi, G, t_{\mathrm{next}}$. The positive radius g gives the same regularity for $2/g$. □

The core direction estimate applies to the actual downward map because its identification with the full core return holds on an ordinary neighborhood. With $X = (\kappa, U_+, W_+)$, equation (10.75) gives

$$\mathcal{R}_\downarrow \circ Y = \mathcal{R} \circ X \quad \text{near each reference parameter.} \tag{11.88}$$

Here $Z_V = 1$ and $Z_M > 1$ persist on the neighborhood of the arrival map, and the full core and downward times solve the same transverse event equation. This is the ordinary identity needed when differentiating the return.

Proposition 11.23. *After a common decrease of the scale threshold, the ordinary derivatives of the actual family satisfy $\kappa', f', g' < 0$ and*

$$\left|\frac{f'}{\kappa'} - 1\right| \le \frac{1}{32\kappa}, \qquad \frac{3}{4} \le \frac{f'}{\kappa'} \le \frac{5}{4}, \tag{11.89}$$

$$|u'| \le \left(1 - \frac{1}{16\kappa}\right)\sqrt{u}\,|f'|, \qquad |w'| \le \left(1 - \frac{1}{16\kappa}\right)\sqrt{w}\,|f'|, \tag{11.90}$$

$$\left|\frac{g'}{f'} - 1\right| \le \frac{1}{4\mu}, \qquad \frac{3}{4} \le \frac{g'}{f'} \le \frac{5}{4}, \tag{11.91}$$

$$|\widehat{u}'| \le \sqrt{\widehat{u}}\,|g'|, \qquad |\widehat{w}'| \le \sqrt{\widehat{w}}\,|g'|. \tag{11.92}$$

Both reciprocal phases $2/f$ and $2/g$ have positive ordinary derivatives, and

$$\frac{2}{g} - \frac{2}{f} = -\frac{2(g-f)}{fg}, \qquad \frac{(2/g)'}{(2/f)'} = \left(\frac{f}{g}\right)^2 \frac{g'}{f'}. \tag{11.93}$$

Proof. Choose the amplitude threshold ε_c in Proposition 4.10. Theorem 11.20, with $\eta = \varepsilon_c$ and $\theta = 1$, gives

$$\kappa \ge 6, \qquad 0 < U_+, W_+ \le \varepsilon_c/\kappa, \qquad \kappa' < 0, \qquad |U_+'| \le \sqrt{U_+}|\kappa'|, \quad |W_+'| \le \sqrt{W_+}|\kappa'|.$$

Normalize the tangent by κ'. Its radial component is one, and it lies in the unit square-root cone. The core estimate, followed by (11.88), gives (11.89) and (11.90) by linearity. In particular $f' < 0$.

Choose also the long-connection threshold ε_ℓ in (5.66), and use it as η in Proposition 11.22. The output core cone is contained in the unit cone, since $1 - 1/(16\kappa) \le 1$. In connection coordinates put

$$\chi = \frac{\mu + 1}{H(\mu + 1, 1)}.$$

The identity $H(\mu + 1, 1) = f$ gives $\chi\mu' = f'$. Thus $(\mu', u', w')/f'$ has unit physical radial derivative and wall components bounded by $\sqrt{u}, \sqrt{w}$. Proposition 5.10 applies to this vector. Multiplication by $f' < 0$ proves (11.91) and (11.92), including $g' < 0$. The regularity proved above ensures that all these are derivatives of the actual return curve.

Finally $f > 1$ and $g > 0$. Differentiating their reciprocals gives the two positive derivatives and the ratio in (11.93); the value identity follows by subtraction. □

The return from the delayed family therefore reaches the next incoming section as a curve with decreasing radius and controlled transverse slopes. Its event selection and ordinary regularity are fixed along the same normalized orbit, allowing the remaining phase and amplitude estimates to be compared with the original small-section data.

11.9. **Phase and amplitude after the delayed return.** The long decay interval has brought the second weak-wall ordering to a complete core entrance. We now compare the next entrance with the original small-section data. The phase comparison uses the bounded radius displacement through one core and connection. For the amplitudes, the product of the two weak walls is the useful quantity: it is nonincreasing throughout the slow interval and controls both walls at the next entrance.

Use the actual return $G = (g, \widehat{u}, \widehat{w})$ and the upward radius κ from the preceding subsection. Put

$$\xi = \frac{2}{g}, \qquad \Phi = \frac{2}{\kappa}, \qquad F(\rho) = \Phi_h(\rho, \zeta, \nu). \tag{11.94}$$

Throughout this subsection $q > 3$, $D > 4$, $P > 6D + 2$, and the labels satisfy (11.73). Derivatives are ordinary derivatives in ρ, with all other parameters fixed. All scale thresholds are uniform over these labels.

Proposition 11.24. *There is $h_* > 0$ such that, for every $0 < h \le h_*$, $q > 3$ and $D > 4$, one can choose $C = C_{h,q,D} > 0$ before P with the following property. For every $P > 6D + 2$ and all sufficiently small $a > 0$, the actual return phase satisfies*

$$|\xi - F| \le Ca^2, \qquad |\xi' - F'| \le Ca, \qquad \frac{1}{8\sqrt{3}} \le \xi' \le C. \tag{11.95}$$

It is jointly C^2 in (ζ, ν, ρ) near each input.

Proof. Take the common thresholds for the delayed core and connection. Their ordinary regularity was proved in Proposition 11.22. At the upward arrival, $\kappa \ge 6$ and $U_+ + W_+ \le \varepsilon_0/\kappa$. The full-core and connection estimates used in (10.84) therefore apply here as well. With $b = U_+ + W_+^2$, they give

$$|g - (\kappa - 2)| \le \frac{64b}{\kappa}, \qquad 1 \le \kappa - g \le 3, \qquad \frac{\kappa}{2} \le g \le \kappa. \tag{11.96}$$

The input smallness makes $64b/\kappa \le 1$.

The two derivative ratios in Proposition 11.23 give

$$\begin{aligned} |g' - \kappa'| &\le \frac{|f'|}{4\mu} + \frac{|\kappa'|}{32\kappa} \\ &\le \left(\frac{5}{8} + \frac{1}{32}\right)\frac{|\kappa'|}{\kappa} \le \frac{|\kappa'|}{\kappa}. \end{aligned}$$

Here $|f'| \le (5/4)|\kappa'|$ and $\mu \ge \kappa/2$. Combining this with (11.70) yields

$$\kappa, g \ge \frac{1}{96a}, \qquad |g - \kappa| \le 3, \qquad |\kappa'| \le \frac{24576}{a^2},$$
$$|g' - \kappa'| \le \frac{1179648}{a}.$$

Apply the reciprocal identities (10.87), with κ in place of k. The common lower radius bound gives

$$\frac{1}{g\kappa} \le 96^2 a^2, \qquad \frac{g + \kappa}{g^2\kappa^2} \le 2 \cdot 96^3 a^3.$$

Consequently

$$|\xi - \Phi| \le 2 \cdot 96^2 \cdot 3a^2 \le Aa^2, \qquad |\xi' - \Phi'| \le Aa, \tag{11.97}$$

where the absolute constant

$$A = 2 \cdot 96^2 \cdot 1179648 + 4 \cdot 24576 \cdot 96^3 \cdot 3$$

is independent of all parameters.

Let $K = K_{h,q,D}$ be the comparison constant in Theorem 11.18. Adding its value and derivative errors to (11.97) gives the first two claims with constant $A + K$. Decrease the scale threshold so that

$$Aa \le \frac{1}{8\sqrt{3}}, \qquad Aa \le 1.$$

The bounds $1/(4\sqrt{3}) \le \Phi' \le 2/\sqrt{3} + 1$ then give

$$\frac{1}{8\sqrt{3}} \le \xi' \le \frac{2}{\sqrt{3}} + 2.$$

Thus $C = A + K + 2/\sqrt{3} + 2$ suffices, and is chosen before P. Joint regularity follows from the ordinary return already constructed. □

For $0 \le t \le T$, write

$$Y_t = (M_t, U_t, V_t, W_t) = \mathrm{proj}(\phi_{t_0+t} x_0), \qquad A_t = 1 + \Sigma_+(\phi_{t_0+t} x_0).$$

The height is positive on this interval. In the original normalized time, the product identity (2.9) reads

$$\frac{d}{dt}(V_t W_t) = -2A_t\big[1 + U_t(V_t + W_t - U_t)\big] V_t W_t. \tag{11.98}$$

The slow-decay bounds give

$$0 \le U_t \le 20 h a^{q-2} \le 1$$

for a sufficiently small a, uniformly in t. Since $V_t, W_t \ge 0$,

$$1 + U_t(V_t + W_t - U_t) \ge 1 - U_t^2 \ge 0.$$

It follows that $V_t W_t$ is nonincreasing. At $t = 0$, the fast-segment coordinate estimate (9.81) gives

$$0 \le V_T W_T \le V_0 W_0 \le \frac{256 \zeta \nu}{a^4}. \tag{11.99}$$

This argument also applies when either initial weak wall is zero, since it uses the product equation rather than logarithms.

Proposition 11.25. *For a sufficiently small universal $h_* > 0$, each $0 < h \le h_*$ and the stated q, D, P admit a common scale threshold such that*

$$\frac{1}{96a} \le g \le \frac{128}{a}, \qquad \widehat{u}, \widehat{w} > 0, \qquad \widehat{u} + \widehat{w} \le \frac{512 \zeta \nu}{a^4}. \tag{11.100}$$

These are the coordinates of the actual reflected entrance at $t_{\mathrm{next}} > t_0 + T$ in (11.87).

Proof. Use the amplitude threshold $\eta = \varepsilon_0$ in Proposition 11.22. The incoming passage and the core give, successively,

$$W_+ \le V_T W_T, \qquad w \le W_+.$$

The connection estimate (10.88) then yields

$$\widehat{u} + \widehat{w} \le 2w \le 2 V_T W_T \le \frac{512 \zeta \nu}{a^4}.$$

Both walls are positive along these passages. The radius bounds follow from (11.96) and $1/(48a) \le \kappa \le 128/a$. Finally, the downward arrival and the connection each take positive time after $t_0 + T$; their orbit identity was proved in Proposition 11.22. □

The delayed return thus has the same phase accuracy and the same product-dependent amplitude bound as the earlier weak-wall ordering. Together with its square-root tangent cone, these estimates provide the local data for placing both families on admissible statistical graphs.

11.10. **One return for both weak-wall orderings.** The two constructions differ in how long we wait before applying the core estimates. Their final event is selected in the same way: the first downward core crossing after the fast decay, followed by the next connection. We show that the additional waiting interval can be removed on an ordinary neighborhood of each initial point. This identifies the derivatives as well as the returned states and allows both weak-wall orderings to use one smooth map.

Recall that Φ_t denotes the projective expression of the flow in original normalized time. The following observation applies whenever an initial interval stays strictly below the active-wall section.

Lemma 11.26. *Let y have positive inverse-chart denominator and let $S \geq 0$. Suppose that the normalized orbit exists on $[0, S]$, with positive height and*

$$(\Phi_t(y))_V < 1 \qquad (0 \leq t \leq S).$$

Suppose also that, near $\Phi_S(y)$, the maps $\tau_\downarrow$ and $\mathcal{R}_\downarrow$ are C^2 and select the first downward event. Then, on an ordinary neighborhood of y,

$$\tau_\downarrow(z) = S + \tau_\downarrow(\Phi_S(z)), \qquad \mathcal{R}_\downarrow(z) = \mathcal{R}_\downarrow(\Phi_S(z)). \tag{11.101}$$

In particular both maps on the left are ordinarily C^2 there.

Proof. The inverse chart reconstructs the normalized state at time S. Uniqueness of the normalized flow gives

$$\Phi_t(\Phi_S(y)) = \Phi_{S+t}(y),$$

with the same active-wall rate at corresponding times. There is no downward crossing in $(0, S]$. Any later candidate crossing corresponds to one from $\Phi_S(y)$ at time reduced by S. Minimality of the latter event proves both identities at y.

For the neighborhood assertion, the normalized flow is jointly continuous in its initial point and time along the compact interval $[0, S]$. Positive height and strict inequality of the active wall persist on a common neighborhood of y, by a finite covering of this interval. The inverse-chart denominator remains positive as well. The endpoint map $z \mapsto \Phi_S(z)$ is ordinarily C^2 and maps a smaller neighborhood into the one where the first downward event is defined. The same minimality argument therefore proves (11.101) throughout that neighborhood. Its right-hand sides are C^2. □

The conclusion also permits a continuous choice of waiting time. If $S > 0$ and Θ is continuous at y with $\Theta(y) = S$, then, near y,

$$\begin{aligned} \tau_\downarrow(z) &= \Theta(z) + \tau_\downarrow(\Phi_{\Theta(z)}(z)), \\ \mathcal{R}_\downarrow(z) &= \mathcal{R}_\downarrow(\Phi_{\Theta(z)}(z)). \end{aligned} \tag{11.102}$$

Indeed, parametrize the varying interval by $t = \Theta(z)v$, $0 \leq v \leq 1$. Compactness and continuity preserve positive height and $V < 1$ on that interval, and its endpoint remains in the same event neighborhood. The preceding minimality argument applies. By comparison with the fixed choice S, both expressions in (11.102) are C^2 near y and have the same derivatives as the left-hand sides. No derivative of Θ is required.

For initial labels $v = (n_2, n_3, \rho)$ and a fixed time s, write

$$\begin{aligned} Y_s(v) &= \mathrm{proj}(\phi_s \Xi_h(\rho, n_2, n_3)), \\ (f_s, u_s, w_s) &= \mathcal{R}_\downarrow(Y_s(v)), \qquad \pi_s = (\sqrt{f_s^2 - 1} - 1, u_s, w_s), \\ G_s(v) &= \Psi(Y_s(v)) = (g_s, \widehat{u}_s, \widehat{w}_s), \qquad \xi_s(v) = 2/g_s(v), \\ t_s(v) &= s + \tau_\downarrow(Y_s(v)) + T_c(\pi_s(v)). \end{aligned} \tag{11.103}$$

These expressions are used wherever the downward crossing and its connection exist. The total time t_s is measured from the original small section.

Proposition 11.27. *For sufficiently small $h > 0$, $q > 3$, $D > 4$, $P > 6D + 2$, and sufficiently small $a > 0$, let*

$$t_0 = \frac{q}{2} \log(1/a), \qquad T = \frac{D}{a} \log(1/a).$$

At every input in (11.73)*, with $n_2 = \zeta$ and $n_3 = \nu$, the four identities*

$$\pi_{t_0} = \pi_{t_0+T}, \qquad G_{t_0} = G_{t_0+T}, \qquad \xi_{t_0} = \xi_{t_0+T}, \qquad t_{t_0} = t_{t_0+T} \tag{11.104}$$

hold on an ordinary open neighborhood in (n_2, n_3, ρ). All four maps are jointly C^2 there. Their values and ordinary radial derivatives therefore satisfy the delayed return estimates of Sections 11.8 and 11.9 with cutoff t_0.

Proof. At the reference input, the entire interval $[t_0, t_0 + T]$ has positive height and $V < 1$ by Theorem 11.3. Its endpoint satisfies (10.55) by Theorem 11.20 and its proof. Proposition 10.20 supplies an ordinary endpoint neighborhood on which the first downward event persists and is C^2. Apply Lemma 11.26 to Y_{t_0} with duration T.

The initial radicand and the height at t_0 are positive. They remain so on a neighborhood of the initial labels, where the fixed-time flow and its projection are smooth. On this neighborhood the exact flow identity is

$$\Phi_T(Y_{t_0}(v)) = Y_{t_0+T}(v).$$

Pulling (11.101) back to the labels gives equality of the downward states and a time difference exactly T. The state equality gives the first three identities in (11.104). For the total time the added T is absorbed into the cutoff:

$$t_0 + \tau_\downarrow(Y_{t_0}) + T_c(\pi_{t_0}) = t_0 + T + \tau_\downarrow(Y_{t_0+T}) + T_c(\pi_{t_0+T}).$$

The delayed maps are jointly C^2 by Proposition 11.22; the ordinary neighborhood identities transfer that regularity and all their radial derivatives to the maps at t_0. □

We can now replace the separate smallness conditions on n_2 and n_3 by a single condition on their minimum. Let

$$\begin{gathered} 0 \le \varepsilon \le 1/128, \qquad 0 < n_2, n_3 \le 2\varepsilon a, \qquad \min(n_2, n_3) \le a^P, \qquad \rho \in [a, 2a], \\ G = G_{t_0}, \quad \xi = 2/g, \quad t_{\mathrm{next}} = t_{t_0}, \quad F(\rho) = \Phi_h(\rho, n_2, n_3). \end{gathered} \tag{11.105}$$

Theorem 11.28. *There is $h_* > 0$ such that, for each $0 < h \le h_*$, $q > 3$ and $D > 4$, there is $C = C_{h,q,D} > 0$ with the following property. For every $P > 6D + 2$ and all sufficiently small $a > 0$, the same return* (11.103) *is defined throughout* (11.105)*. Its parameter, state, phase and total time are jointly C^2 near every input. It satisfies*

$$t_{\mathrm{next}} > t_0 \ge 0, \qquad \iota_+(G) = \mathsf{S}\,\mathrm{proj}(\phi_{t_{\mathrm{next}}}\Xi_h(\rho, n_2, n_3)), \tag{11.106}$$

$$\frac{1}{96a} \le g \le \frac{128}{a}, \qquad \widehat{u}, \widehat{w} > 0, \qquad \widehat{u} + \widehat{w} \le \frac{512 n_2 n_3}{a^4}, \tag{11.107}$$

$$|\xi - F| \le Ca^2, \qquad |\xi' - F'| \le Ca, \qquad \frac{1}{4\sqrt{3}} \le \xi' \le C, \tag{11.108}$$

$$g' < 0, \qquad |\widehat{u}'| \le \sqrt{\widehat{u}}\,|g'|, \qquad |\widehat{w}'| \le \sqrt{\widehat{w}}\,|g'|. \tag{11.109}$$

The constants and scale threshold are uniform in $\varepsilon, n_2, n_3, \rho$, and C is chosen before P.

Proof. If $n_3 \le a^P$, apply the positive shear return, phase, amplitude and cone estimates of Sections 10.5–10.7 with $\ell = P > 4$. Their radius band $[1/(64a), 64/a]$ is contained in the one stated here. If $n_3 > a^P$, the minimum condition gives $n_2 \le a^P$. Use the delayed return estimates and Proposition 11.27. In both cases the maps are precisely those in (11.103), on ordinary neighborhoods. Thus the two descriptions agree wherever their domains overlap, and the local regularity applies also at the boundary between the two cases.

Take the maximum of the two phase constants and the minimum of the scale thresholds. The first phase constant depends only on h, and the delayed one depends on h, q, D; neither depends on P. The orbit identity, amplitude and cone estimates now hold uniformly.

To obtain the stated phase lower bound, compare once with the original reference slope. Decrease h_* if necessary so that $h_* \le 1/16$. Proposition 9.5 gives $F' \ge 1/(2\sqrt{3})$ on the entire family. Choose a still smaller so that $Ca \le 1/(4\sqrt{3})$. Then

$$\xi' \ge F' - |\xi' - F'| \ge \frac{1}{2\sqrt{3}} - Ca \ge \frac{1}{4\sqrt{3}}.$$

This final restriction changes no constant. The preceding constructions give a positive initial radicand and a finite total return time, so the orbit in (11.106) is defined. □

The long waiting time has served to estimate one part of the initial family. The resulting return, its smooth dependence on the initial labels and its phase orientation are common to both parts.

11.11. **Statistical graphs for either small weak wall.** The common return has all the properties used to construct statistical entrance graphs: a decreasing output radius, a square-root tangent cone and a bound retaining the initial weak-wall product. We apply the same explicit graph construction to the enlarged label set. The uniform phase lower bound then converts the statistical loss back to the original shear coordinate with a single constant for both orderings.

Fix $h, q, D, P, a, \varepsilon$ under the thresholds of Theorem 11.28. Write

$$I = [a, 2a], \qquad t_0 = \frac{q}{2}\log(1/a),$$

$$\Lambda_c = \{\nu = (n_2, n_3) : 0 < n_2, n_3 \le 2\varepsilon a,\ \min(n_2, n_3) \le a^P\}, \tag{11.110}$$

$$G_\nu(\rho) = G_{t_0}(n_2, n_3, \rho), \qquad \xi_\nu(\rho) = 2/g_\nu(\rho), \qquad B_\nu = \frac{800 n_2 n_3}{a^4}.$$

In particular Λ_c is a Borel subset of $\mathbb{R}^2$.

Proposition 11.29. *Given any $B_* > 0$, a sufficiently small common scale threshold gives, for every $\nu \in \Lambda_c$, an admissible graph*

$$\Gamma_\nu(r) = (r, \widetilde{u}_\nu(r), \widetilde{w}_\nu(r))$$

at amplitude B_ν, with $0 < B_\nu \le B_*$. Its wall functions are positive and C^1 on $\mathbb{R}$, and

$$\widetilde{u}_\nu + \widetilde{w}_\nu \le B_\nu, \qquad |\widetilde{u}'_\nu| \le \sqrt{\widetilde{u}_\nu}, \qquad |\widetilde{w}'_\nu| \le \sqrt{\widetilde{w}_\nu}, \tag{11.111}$$

$$\Gamma_\nu(g_\nu(\rho)) = G_\nu(\rho), \qquad \widetilde{u}'_\nu(g_\nu(\rho)) = \frac{\widehat{u}'_\nu(\rho)}{g'_\nu(\rho)}, \qquad \widetilde{w}'_\nu(g_\nu(\rho)) = \frac{\widehat{w}'_\nu(\rho)}{g'_\nu(\rho)} \quad (\rho \in I). \tag{11.112}$$

The state $G_\nu(\rho)$ is jointly Borel on $\Lambda_c \times I$, and the four extended wall and wall-derivative functions are jointly Borel on $\Lambda_c \times \mathbb{R}$. Moreover,

$$0 < \xi_\nu(\rho) < 1, \qquad \xi'_\nu(\rho) \ge \frac{1}{4\sqrt{3}}, \qquad \Gamma_\nu(2/\xi_\nu(\rho)) = G_\nu(\rho). \tag{11.113}$$

Proof. For each label, $g'_\nu < 0$ on I. Construct its inverse on $[g_\nu(2a), g_\nu(a)]$ and the two radius graphs as in (10.95). The C^2 return gives the continuous one-sided endpoint derivatives in (10.96). Dividing (11.109) by $|g'_\nu|$ gives the unit square-root slope bounds for these graphs. Their sum is at most $512 n_2 n_3/a^4$.

Use the explicit extension of Lemma 8.13 with $\theta = 1$. It preserves values and derivatives on the closed interval, and its amplitude cost is $25/16$. Thus

$$\widetilde{u}_\nu + \widetilde{w}_\nu \le \frac{25}{16}\frac{512 n_2 n_3}{a^4} = B_\nu.$$

This proves the graph bounds and matching statements on the same extended graphs used in Section 10.8.

The measurability argument of Proposition 10.34 uses only a Borel label domain, a jointly Borel return, a strictly decreasing radius and positive C^1 wall graphs. All these conditions hold here. In detail, joint ordinary regularity makes the restricted map G continuous, and $(\nu, \rho) \mapsto (\nu, g_\nu(\rho))$ is an injective Borel map between standard Borel spaces. Its inverse is Borel on the varying radius intervals. The interior samples and adjacent secants in (10.102) recover measurable endpoint values and slopes. Applying the same explicit extension formulas proves joint measurability of the four extended functions.

For the amplitude threshold, $2\varepsilon \le 1$ gives $n_2, n_3 \le a$. At least one of them is at most $a^P \le a^4$ when $a \le 1$, since $P > 6D + 2 > 4$. In either case,

$$n_2 n_3 \le a^5, \qquad 0 < B_\nu \le 800a.$$

Require $a \le B_*/800$ and $a \le 1/384$. The radius bound $g_\nu \ge 1/(96a)$ then gives $g_\nu \ge 4$, hence $0 < \xi_\nu < 1$. The lower slope follows from Theorem 11.28. Finally $2/\xi_\nu = g_\nu$, so the value matching proves the last identity in (11.113). □

Fix admissible η, J, σ and the constants B_*, B_{cap} of Theorem 8.20. Use the same Borel statistical set $\mathcal{S}_{\mathrm{stat}}$ throughout the enlarged family. For $\nu \in \Lambda_c$ define

$$\mathcal{F}_\nu = \{\rho \in I : (B_\nu, G_\nu(\rho)) \notin \mathcal{S}_{\mathrm{stat}}\}. \tag{11.114}$$

Theorem 11.30 (Statistical entrances for both wall orderings)**.** *For sufficiently small $h > 0$, every $q > 3$, $D > 4$, $P > 6D + 2$, and all sufficiently small $a > 0$, the following holds uniformly for $0 \le \varepsilon \le 1/128$. One has $0 < B_\nu \le B_* \le B_{\mathrm{cap}}$ for all $\nu \in \Lambda_c$, and the family*

$$\{(\nu, \rho) \in \Lambda_c \times I : \rho \in \mathcal{F}_\nu\}$$

is Borel. For every Borel $A \subset I$ and nonnegative Borel weight $\omega \le M$ on A, $M \ge 0$,

$$\int_{A \cap \mathcal{F}_\nu} \omega(\rho)\, d\rho \le \min\left\{ \int_A \omega(\rho)\, d\rho, 320\sqrt{3}\, M B_\nu^\sigma \right\}. \tag{11.115}$$

Outside $\mathcal{F}_\nu$, the selected era digits and phases from the returned state $G_\nu(\rho)$ satisfy all six statistical limits of Theorem 2.4.

Proof. Apply Proposition 11.29 with the chosen B_*. The graph Γ_ν is admissible at B_ν, and (11.113) verifies the phase and representation hypotheses of Corollary 8.29 with $c = 1/(4\sqrt{3})$. Its weighted bound is

$$\min\left\{ \int_A \omega(\rho)\, d\rho, \frac{80M}{c} B_\nu^\sigma \right\},$$

which is (11.115). The same corollary gives all six limits for the actual state whenever $\rho \notin \mathcal{F}_\nu$.

The map $\nu \mapsto B_\nu$ is continuous, and G is jointly Borel. Hence the family of excluded sets is the inverse image of $\mathcal{S}_{\mathrm{stat}}^c$ under $(\nu, \rho) \mapsto (B_\nu, G_\nu(\rho))$. It is Borel, as is each of its sections $\mathcal{F}_\nu$. □

Both orderings now contribute to the same statistical family. The product $n_2 n_3$ remains in the loss bound, and the minimum with the original mass is retained for integration over the weak-wall labels.

11.12. **Both shear signs and their physical frames.** Changing the sign of the initial shear exchanges the roles of the two weak walls. The exact axis symmetry therefore transfers the preceding construction to negative shear. Keeping track of the frame is useful: the final reflection needed for positive shear is already supplied by the initial axis exchange for negative shear. Both families reach the same canonical core chart at a positive time on their original normalized orbit.

For $\varsigma \in \{1, -1\}$ set

$$x_\varsigma(\rho, n_2, n_3) = \Xi_h(\varsigma\rho, n_2, n_3), \qquad F_\varsigma(\rho) = \Phi_h(\varsigma\rho, n_2, n_3). \tag{11.116}$$

The axis exchange S of (2.4) fixes p, N_1, reverses s, and exchanges N_2, N_3. In projective coordinates it is

$$\mathsf{S}(M, U, V, W) = (-M, U, W, V).$$

The curvature polynomial is symmetric in its last two arguments. Thus the initial radicands agree and

$$x_-(\rho, n_2, n_3) = \mathsf{S}x_+(\rho, n_3, n_2), \qquad \phi_t x_-(\rho, n_2, n_3) = \mathsf{S}\phi_t x_+(\rho, n_3, n_2). \tag{11.117}$$

The maximal time domains agree as well, by equivariance and uniqueness of the normalized flow.

The same exchange preserves the reference phase as a function of the full initial data. From (9.21),

$$3\Phi_h(s,n_2,n_3)^2=[A(h)s+K(h)(n_2-n_3)]^2+C(h)^2(n_2-n_3)^2.$$

Both linear factors change sign when (s,n_2,n_3) is replaced by $(-s,n_3,n_2)$. Consequently

$$\Phi_h(-\rho,n_2,n_3)=\Phi_h(\rho,n_3,n_2) \tag{11.118}$$

as ordinary functions, so their ρ derivatives agree.

For $\nu=(n_2,n_3)\in\Lambda_c$, define

$$\nu^\varsigma=\begin{cases}(n_2,n_3), & \varsigma=1,\\ (n_3,n_2), & \varsigma=-1.\end{cases}\qquad G_\nu^\varsigma(\rho)=G_{\nu^\varsigma}(\rho),\quad \xi_\nu^\varsigma=2/g_\nu^\varsigma. \tag{11.119}$$

Let t_ν^ς be the total return time $t_{t_0}(\nu^\varsigma,\rho)$ from (11.103). The label set is invariant under the exchange, and $B_{\nu^\varsigma}=B_\nu$.

Proposition 11.31 (Entrances for both shear signs)**.** *Under the thresholds of Theorem 11.28, each $x_\varsigma(\rho,n_2,n_3)$ lies in $\mathcal{M}_{+++}$ and has the positive-time entrance*

$$\iota_+(G_\nu^\varsigma(\rho))=\begin{cases}\mathsf{S}\,\mathrm{proj}(\phi_{t_\nu^+}x_+), & \varsigma=1,\\ \mathrm{proj}(\phi_{t_\nu^-}x_-), & \varsigma=-1.\end{cases}\qquad t_\nu^\varsigma>t_0\ge 0. \tag{11.120}$$

The return parameters, state, phase and total time are jointly C^2 in (n_2,n_3,ρ) for each fixed sign. The radius, amplitude and tangent bounds (11.107) *and* (11.109) *hold for both signs. With the original signed reference F_ς, one also has*

$$|\xi_\nu^\varsigma-F_\varsigma|\le Ca^2,\qquad |(\xi_\nu^\varsigma)'-F_\varsigma'|\le Ca,\qquad \frac{1}{4\sqrt{3}}\le(\xi_\nu^\varsigma)'\le C. \tag{11.121}$$

All derivatives use the positive radial parameter ρ.

Proof. The case $\varsigma=1$ is Theorem 11.28. For $\varsigma=-1$, apply that theorem to (n_3,n_2). The domain conditions and the product n_2n_3 are unchanged. Its orbit identity, together with (11.117), reads

$$\iota_+(G_\nu^-)=\mathsf{S}\,\mathrm{proj}(\phi_{t_\nu^-}x_+(\rho,n_3,n_2))=\mathrm{proj}(\phi_{t_\nu^-}x_-).$$

This proves the frame in (11.120). The positive radicand, the constraint and the three positive walls put x_- in $\mathcal{M}_{+++}$, and the symmetry preserves the time domain.

The label exchange is linear. Composing the jointly C^2 positive-shear maps with it gives the stated regularity. It leaves ρ unchanged, so the radius direction and the two cone inequalities are transferred without a sign change in the radial derivative. Finally the reference identity (11.118) transfers both phase errors and the slope bound, with the same constant C. □

The statistical graphs and excluded sets have an equally simple description. For each sign use

$$\Gamma_\nu^\varsigma=\Gamma_{\nu^\varsigma},\qquad \mathcal{F}_\nu^\varsigma=\mathcal{F}_{\nu^\varsigma}. \tag{11.122}$$

These are Borel families. Their amplitude is B_ν, and the weighted estimate (11.115) holds on each radial interval with the same coefficient. This follows by applying Theorem 11.30 to ν^ς.

For a radial parameter outside $\mathcal{F}_\nu^\varsigma$, the returned state belongs to $\mathcal{S}_{\mathrm{stat}}$ at amplitude B_ν. Equation (11.120) identifies its normalized entrance with a positive-time state of the original orbit, in frame S for positive shear and the identity frame for negative shear. Corollary 8.27 therefore supplies successive complete eras on that orbit, with all six statistical limits and entrance times tending to infinity.

To integrate the radial estimates near vanishing weak walls, it is convenient to use logarithmic labels $\beta_i = -\log n_i$. On the open set where the radicand is positive, define

$$\mathcal{X}_h(s,\beta_2,\beta_3) = \Xi_h(s,e^{-\beta_2},e^{-\beta_3}). \tag{11.123}$$

This is the same small section, with inverse coordinates $(\Sigma_-,-\log N_2,-\log N_3)$. In particular

$$\mathcal{X}_h(\varsigma\rho,-\log n_2,-\log n_3) = x_\varsigma(\rho,n_2,n_3). \tag{11.124}$$

The axis exchange acts on this section by

$$\mathsf{s}(s,\beta_2,\beta_3) = (-s,\beta_3,\beta_2), \qquad \mathcal{X}_h\circ\mathsf{s} = \mathsf{S}\circ\mathcal{X}_h. \tag{11.125}$$

These identities follow directly from the chart formulas and $e^{\log n_i} = n_i$. They express both shear signs in one section chart while retaining the physical frame of each statistical entrance.

The entrance construction now covers both shear signs and both weak-wall orderings under the stated selection bounds. In every case the excluded parameter measure retains the small wall product. This is the information needed to integrate the estimate over wall labels and obtain finite section flux.

12. Section flux and capture

The statistical construction leaves an excluded set of finite section flux. We choose a density for which every positive-time return strictly increases flux. The recurrent points that stay excluded forever form a set of finite flux, mapped injectively into itself by the first return. Strict expansion forces this set to be null. This argument gives the capture theorem below. We establish its ingredients first in the region where shear dominates the weak walls.

12.1. **Flux through the small section.** The statistical estimate on a shear interval must now be integrated over the two weak walls. To compare repeated crossings we use a weighted flux of the normalized flow through the section. In logarithmic wall coordinates its density stays uniformly comparable to Lebesgue measure near the Taub point. This leaves two logarithmic factors to be paid for by the product in the statistical loss estimate.

For $x = (p,s,N_1,N_2,N_3)$, write the constraint as

$$\mathcal{C}(x) = p^2 + s^2 + Q(N_1,N_2,N_3).$$

For positive walls put

$$H(x) = (N_1N_2N_3)^{2/3}, \qquad \mathcal{W}(x) = N_1N_2N_3(1+4H(x)). \tag{12.1}$$

Let $\mathcal{V}$ be the standard volume form on $\mathbb{R}^5$. On $\mathcal{C}=1$ define the four-form

$$\Omega_x(v_1,v_2,v_3,v_4) = \frac{\mathcal{V}(x/2,v_1,v_2,v_3,v_4)}{\mathcal{W}(x)}, \qquad v_i\in\ker d\mathcal{C}_x. \tag{12.2}$$

The quadratic constraint gives $d\mathcal{C}_x(x/2)=1$. Thus $x/2$ can be replaced by any vector X satisfying $d\mathcal{C}_x(X)=1$: the difference is tangent, and five vectors in the four-dimensional tangent space have zero determinant. Contracting Ω with the normalized vector field F defines the section flux $\omega_x(v_1,v_2,v_3) = \Omega_x(F(x),v_1,v_2,v_3)$.

Use the logarithmic chart $\mathcal{X}_h$ from (11.123), on its open domain

$$\mathcal{D}_h = \{(s,\beta_2,\beta_3) : 1-s^2-Q(h,e^{-\beta_2},e^{-\beta_3}) > 0\}.$$

Here $p = -\sqrt{1-s^2-Q(h,e^{-\beta_2},e^{-\beta_3})} < 0$. Define

$$w_h(s,\beta_2,\beta_3) = \frac{p^2+s^2-2p}{2|p|\,[1+4(he^{-\beta_2-\beta_3})^{2/3}]}, \qquad d\mu_h = \mathbf{1}_{\mathcal{D}_h}w_h\,ds\,d\beta_2\,d\beta_3. \tag{12.3}$$

This is a positive continuous density on $\mathcal{D}_h$.

Proposition 12.1. *For $h > 0$, the pullback of the signed flux to this chart is*

$$\mathcal{X}_h^*\omega = -w_h\, ds \wedge d\beta_2 \wedge d\beta_3. \tag{12.4}$$

Consequently μ_h is its absolute flux measure and has the same null sets as Lebesgue measure restricted to $\mathcal{D}_h$.

Proof. Take $X = (2p)^{-1}\partial_p$ as the normalized transverse vector in (12.2). If $n_i = e^{-\beta_i}$, the three chart tangent vectors have the form

$$(*, 1, 0, 0, 0), \qquad (*, 0, 0, -n_2, 0), \qquad (*, 0, 0, 0, -n_3).$$

The determinant with columns X, F and these vectors is $-F_{N_1} n_2 n_3/(2p)$. By (2.2), $F_{N_1} = (-p^2 - s^2 + 2p)h$. Dividing by $\mathcal{W} = hn_2n_3(1 + 4H)$ gives $-w_h$, proving (12.4). The numerator $p^2 + s^2 - 2p$ and both denominator factors are strictly positive on $\mathcal{D}_h$. Integrating this positive density proves equivalence of the null sets. □

For use on a radial slice, abbreviate

$$w_h(\rho; n_2, n_3) = w_h(\rho, -\log n_2, -\log n_3), \qquad \mu_{h,n_2,n_3}(A) = \int_A w_h(\rho; n_2, n_3)\, d\rho. \tag{12.5}$$

Lemma 12.2. *Suppose $0 < h \le 1/8$, $0 < a \le h/8$, $0 \le \varepsilon \le 1/64$, and $0 < n_2, n_3 \le 2\varepsilon a$. Then for $\rho \in [a, 2a]$ the chart radicand is positive and*

$$\frac{1}{2} \le w_h(\rho; n_2, n_3) \le 3. \tag{12.6}$$

In particular $\mu_{h,n_2,n_3}(A) \le 3a$ for every measurable $A \subset [a, 2a]$.

Proof. We have $n_i \le h/8$, $|\rho| \le 1/8$ and $h + n_2 + n_3 \le 5h/4$. Since $|Q(h, n_2, n_3)| \le (h + n_2 + n_3)^2$, the radicand is at least $983/1024$. On its negative square-root branch,

$$p \le -15/16, \qquad p^2 + \rho^2 \le 2.$$

The uncorrected density is therefore

$$\frac{p^2 + \rho^2 - 2p}{2|p|} = 1 + \frac{p^2 + \rho^2}{2|p|} \in [1, 3].$$

Also $0 \le (hn_2n_3)^{2/3} \le h^2 \le 1/64$. Dividing by $1 + 4(hn_2n_3)^{2/3}$ proves (12.6). The interval has length a, which gives the last assertion. □

The reflection $\mathsf{s}(s, \beta_2, \beta_3) = (-s, \beta_3, \beta_2)$ preserves $\mathcal{D}_h$ and w_h. Its determinant is 1: reversing the shear and exchanging the two wall coordinates each contributes a minus sign. The change of variables formula consequently gives

$$\mu_h(\mathsf{s}E) = \mu_h(E), \qquad \mu_h(E \cup \mathsf{s}E) \le 2\mu_h(E) \tag{12.7}$$

for every Borel set E. The domain restriction in (12.3) is preserved as well.

The scale in the weak walls disappears from the logarithmic Jacobian. For $a > 0$ define

$$L_a(\rho, x, y) = (\rho, -\log(ax), -\log(ay)), \qquad x, y > 0. \tag{12.8}$$

This map is smooth and injective, with derivative $\operatorname{diag}(1, -1/x, -1/y)$ and determinant $1/(xy)$.

Let $\Lambda \subset (0, a)^2$ be Borel and let $\mathcal{F}_\nu \subset [a, 2a]$ have a jointly Borel membership relation on $\Lambda \times [a, 2a]$. The corresponding set on the logarithmic section is

$$E = \{(\rho, -\log n_2, -\log n_3) : \nu = (n_2, n_3) \in \Lambda,\ \rho \in \mathcal{F}_\nu\}. \tag{12.9}$$

Assume its chart radicand is positive. Then E is Borel and

$$\mu_h(E) = \int_0^1 \int_0^1 \mathbf{1}_\Lambda(ax, ay) \frac{\mu_{h,ax,ay}(\mathcal{F}_{(ax,ay)})}{xy}\, dy\, dx. \tag{12.10}$$

The integrand is taken to be zero outside the selected label set.

Indeed, the set

$$A = \{(\rho, x, y) : 0 < x, y < 1,\ (ax, ay) \in \Lambda, \rho \in \mathcal{F}_{(ax,ay)}\}$$

is Borel, and $E = L_a(A)$. A continuous injective map on this standard Borel domain has Borel image. Applying the Jacobian formula to L_a gives the integral over A with density $w_h(\rho; ax, ay)/(xy)$. Tonelli's theorem, with the radial coordinate integrated first, gives (12.10). All integrands are nonnegative, so this identity requires no prior finiteness assumption. Finiteness will follow from retaining the minimum in the radial statistical estimate.

12.2. **Finite flux of the excluded set.** There are two bounds for the excluded part of a radial interval: its total length and the statistical bound proportional to $(n_2 n_3)^\sigma$. Their minimum is integrable against the logarithmic wall measure. We first compute this integral, then apply it to the common statistical family and sum over all sufficiently small shear scales.

Lemma 12.3. *For $\ell, b, \sigma > 0$, put $z = \max\{\log(b/\ell), 0\}$. Then*

$$\int_0^1 \frac{\min\{\ell, bx^\sigma\}}{x}\, dx = \frac{\min\{\ell, b\}}{\sigma}(1 + z), \tag{12.11}$$

$$\int_0^1 \int_0^1 \frac{\min\{\ell, b(xy)^\sigma\}}{xy}\, dy\, dx = \frac{\min\{\ell, b\}}{\sigma^2}\left(1 + z + \frac{z^2}{2}\right). \tag{12.12}$$

Proof. The integrands are nonnegative. In one variable they are bounded above by $bx^{\sigma-1}$, which is integrable at zero. If $b \le \ell$, clipping never occurs, and direct power integration proves both formulas.

Suppose $b > \ell$, and set $c = (\ell/b)^{1/\sigma} \in (0, 1)$. For the first formula split at c. The integrand equals $bx^{\sigma-1}$ below c and ℓ/x above it. Hence the integral is $\ell/\sigma + \ell \log(1/c)$, giving (12.11).

For the second formula, integrate first in y and divide by x. The resulting function is

$$\begin{cases} (b/\sigma)x^{\sigma-1}, & 0 < x < c, \\ (\ell/\sigma - \ell \log c + \ell \log x)/x, & c < x < 1. \end{cases}$$

The first part has integral ℓ/σ^2; the second has integral $-(\ell/\sigma)\log c + (\ell/2)(\log c)^2$. Each is finite. Substituting $\log c = -z/\sigma$ proves (12.12). □

For $K \ge 1$ and $\sigma > 0$ define

$$\begin{aligned} \mathcal{L}(a; K, \sigma) &= \int_0^1 \int_0^1 \frac{\min\{a, Ka^{-2\sigma}(xy)^\sigma\}}{xy}\, dy\, dx, \\ C_{\log}(K, \sigma) &= \frac{1 + k + k^2/2}{\sigma^2}, \qquad k = \log K + 1 + 2\sigma. \end{aligned} \tag{12.13}$$

For $0 < a \le 1$, the preceding lemma gives

$$\mathcal{L}(a; K, \sigma) \le C_{\log}(K, \sigma)\, a(1 + |\log a|)^2. \tag{12.14}$$

To see the uniform constant, write $L = 1 + |\log a| \ge 1$. The logarithm in (12.12) is

$$z = \log K + (1 + 2\sigma)|\log a| \le kL.$$

It follows that $1 + z + z^2/2 \le (1 + k + k^2/2)L^2$, which proves (12.14).

Use the common label set Λ_c, amplitude B_ν and excluded set $\mathcal{F}_\nu$ from Section 11.11. Make their dependence on the scale explicit by setting

$$\begin{aligned} E_a^+ &= \{(\rho, -\log n_2, -\log n_3) : \nu \in \Lambda_c,\ \rho \in \mathcal{F}_\nu\}, \\ E_a &= E_a^+ \cup \mathsf{s}E_a^+, \qquad K_\sigma = 640\sqrt{3}\,800^\sigma. \end{aligned} \tag{12.15}$$

Here the return defining $\mathcal{F}_\nu$ uses the fixed time $t_0 = (q/2)\log(1/a)$. In particular both signs use the same statistical cap and the same set $\mathcal{S}_{\mathrm{stat}}$.

Theorem 12.4 (Finite flux on a shear layer)**.** *Fix admissible statistical parameters η, J, σ and their cap as in Theorem 11.30. There is $h_* > 0$ such that, for $0 < h \le h_*$, $q > 3$, $D > 4$ and $P > 6D+2$, all sufficiently small $a > 0$ have the following properties, uniformly for $0 \le \varepsilon \le 1/128$. The sets E_a^+ and E_a are Borel and*

$$\mu_h(E_a^+) \le 3C_{\log}(K_\sigma, \sigma)a(1 + |\log a|)^2, \tag{12.16}$$

$$\mu_h(E_a) \le 6C_{\log}(K_\sigma, \sigma)a(1 + |\log a|)^2 < \infty. \tag{12.17}$$

For $\nu \in \Lambda_c$, $\rho \in [a, 2a]$ and $\varsigma \in \{1, -1\}$, if

$$(\varsigma\rho, -\log n_2, -\log n_3) \notin E_a, \tag{12.18}$$

then the original orbit through $x_\varsigma(\rho, n_2, n_3)$ has successive complete eras satisfying all six statistical limits of Theorem 2.4, with entrance times tending to infinity.

Proof. Decrease h_* to at most $1/8$, and the scale threshold so that $a \le \min\{h/8, 1\}$. All labels satisfy $0 < n_i < a$, and Lemma 12.2 places the entire family inside $\mathcal{D}_h$. The joint Borel property from Theorem 11.30 and (12.9) show that E_a^+ is Borel. Its reflection and their union are Borel too.

On each radial interval use (11.115) with the actual density $w_h \le 3$. Together with the total mass bound $3a$, this gives

$$\mu_{h,n_2,n_3}(\mathcal{F}_\nu) \le 3\min\{a, 640\sqrt{3}\, B_\nu^\sigma\}.$$

Under $n_2 = ax$, $n_3 = ay$, the amplitude bound becomes

$$B_{(ax,ay)} = 800a^{-2}xy, \qquad 640\sqrt{3}\, B_{(ax,ay)}^\sigma = K_\sigma a^{-2\sigma}(xy)^\sigma.$$

Insert this in (12.10), and enlarge the nonnegative integral from the selected labels to the whole unit square. The result is $\mu_h(E_a^+) \le 3\mathcal{L}(a; K_\sigma, \sigma)$. Since $K_\sigma \ge 1$, inequality (12.14) proves (12.16). Reflection invariance (12.7) proves (12.17).

For positive shear, exclusion from E_a implies $\rho \notin \mathcal{F}_\nu$. For negative shear it implies $\rho \notin \mathcal{F}_{(n_3,n_2)}$, since

$$\mathsf{s}(\rho, -\log n_3, -\log n_2) = (-\rho, -\log n_2, -\log n_3).$$

These are exactly the two excluded sets in (11.122). The actual entrance and complete-era conclusions of Section 11.12 apply. □

For $0 < d \le 1$ let $a_j = d\,2^{-(j+1)}$, $j \ge 0$. The elementary geometric moments

$$\sum_{j\ge0} 2^{-(j+1)} = 1, \qquad \sum_{j\ge0}(j+1)2^{-(j+1)} = 2, \qquad \sum_{j\ge0}(j+1)^2 2^{-(j+1)} = 6$$

give

$$\begin{aligned}\sum_{j\ge0} a_j(1 + |\log a_j|)^2 &= d\big[(1 - \log d)^2 + 4(1 - \log d)\log 2 + 6(\log 2)^2\big] \\ &\le 11d(1 + |\log d|)^2.\end{aligned} \tag{12.19}$$

For the inequality use $0 < \log 2 \le 1 \le 1 - \log d$.

Corollary 12.5. *Under the choices of Theorem 12.4, take $d > 0$ below its scale threshold and $d \le 1$. For fixed $\varepsilon \in [0, 1/128]$, form each E_{a_j} with its own cutoff $(q/2)\log(1/a_j)$ and set $E_d = \bigcup_{j\ge0} E_{a_j}$. Then E_d is Borel and*

$$\mu_h(E_d) \le \sum_{j\ge0} \mu_h(E_{a_j}) \le 66C_{\log}(K_\sigma, \sigma)d(1 + |\log d|)^2 < \infty. \tag{12.20}$$

Every selected initial point from any of these layers outside E_d has the complete-era statistics stated in Theorem 12.4.

Proof. Each a_j lies below the common scale threshold. Countable subadditivity, (12.17) and (12.19) prove the flux bound. A point outside the union is outside its own layer's excluded set, so the last assertion follows from the one-layer theorem. □

The two logarithmic wall directions thus leave a finite total excluded flux after summation over the shear scales. The next task is dynamical: a positive-measure family that continues to cross the small section cannot remain confined to a finite-flux excluded set under strictly expanding returns.

12.3. **Strict expansion of the weighted volume.** Expanding section measures provide a useful route to almost-everywhere statements in homogeneous cosmology [4]. For the present argument the expansion must remain strict even where the curvature polynomial Q is negative. The factor $1 + 4H$ in (12.1) achieves this through a sharp algebraic lower bound for Q.

Lemma 12.6. *For $a, b, c > 0$,*

$$Q(a, b, c) \geq -3(abc)^{2/3}. \tag{12.21}$$

Consequently, on $\mathcal{M}_{+++}$ the function

$$\mathcal{D}(x) = \frac{2[Q(x) + 4H(x)]}{1 + 4H(x)} \geq \frac{2H(x)}{1 + 4H(x)} > 0 \tag{12.22}$$

is continuous and strictly positive.

Proof. By symmetry assume $a \geq b \geq c > 0$, and put $r = (abc)^{1/3}$. The identity

$$(a + b + c)^3 - 27abc = 9c\big[(a - b)^2 + (a - b)(b - c) + (b - c)^2\big] + (a + b - 2c)^3$$

gives $a + b + c \geq 3r$. A second identity is

$$\begin{aligned}(a + b + c)(Q + 3r^2) = (a - b)^2(a + b - c) + c(a - c)(b - c)\\ + 3r^2(a + b + c - 3r).\end{aligned}$$

Every term on the right is nonnegative. Dividing by $a + b + c > 0$ proves (12.21). Apply this with $a, b, c = N_1, N_2, N_3$ and $r^2 = H$. The denominator in (12.22) is positive, and $Q + 4H \geq H > 0$. □

Let F denote the polynomial vector field in (2.2). Its divergence in the five original coordinates is

$$\operatorname{div} F = 2Q - 3S^2, \qquad S^2 = p^2 + s^2. \tag{12.23}$$

Indeed, the first two diagonal derivatives are Q, Q, and the last three are the three wall rates, whose sum is $-3S^2$. Along a positive-wall solution,

$$(N_1N_2N_3)' = -3S^2N_1N_2N_3, \qquad H' = -2S^2H. \tag{12.24}$$

Hence the weight $\mathcal{W} = N_1N_2N_3(1 + 4H)$ satisfies

$$\frac{\mathcal{W}'}{\mathcal{W}} = -3S^2 - \frac{8S^2H}{1 + 4H}.$$

Using $S^2 + Q = 1$ yields the exact cancellation

$$\operatorname{div} F - \frac{\mathcal{W}'}{\mathcal{W}} = 2Q + \frac{8S^2H}{1 + 4H} = \mathcal{D}. \tag{12.25}$$

For $x \in \mathcal{M}_{+++}$ and $t \in I(x)$ write $P_t = D\phi_t(x)$, with derivative taken in the five original initial coordinates. Set

$$J(x, t) = \frac{\mathcal{W}(x)}{\mathcal{W}(\phi_t x)} \det P_t. \tag{12.26}$$

Proposition 12.7. *The multiplier in* (12.26) *is*

$$J(x,t) = \exp\left(\int_0^t \mathcal{D}(\phi_u x)\, du\right). \tag{12.27}$$

For every positive time t, $J(x,t) > 1$. On tangent vectors to the constraint, the four-form Ω and the flux form ω of Section 12.1 obey

$$\Omega_{\phi_t x}(P_t v_1, \dots, P_t v_4) = J(x,t)\Omega_x(v_1, \dots, v_4), \tag{12.28}$$

$$\omega_{\phi_t x}(P_t v_1, P_t v_2, P_t v_3) = J(x,t)\omega_x(v_1, v_2, v_3). \tag{12.29}$$

Proof. The ordinary flow differential satisfies $P_t' = DF(\phi_t x)P_t$ and $P_0 = \mathrm{Id}$. Differentiating the determinant by row multilinearity gives

$$(\det P_t)' = \operatorname{tr}(DF(\phi_t x)) \det P_t.$$

In this calculation, replacing a row by any other row makes its determinant zero; only the diagonal entries of DF contribute. Thus the identity also holds without assuming the matrix is invertible. Differentiating (12.26) and using (12.25) gives $J' = \mathcal{D}(\phi_t x)J$, $J(x,0) = 1$. The scalar integrating factor proves (12.27). On every compact positive-time interval the integrand is continuous and positive, so its integral is strictly positive.

To restrict this multiplier to the constraint, differentiate the ordinary flow identity $\mathcal{C} \circ \phi_t = \mathcal{C}$. It gives

$$d\mathcal{C}_{\phi_t x} P_t = d\mathcal{C}_x.$$

In particular $P_t(x/2)$ is a normalized transverse vector at $\phi_t x$, and $P_t v_i$ are tangent there. Use this transverse vector in (12.2). Multilinearity of the ambient determinant gives

$$\begin{aligned}\Omega_{\phi_t x}(P_t v_1, \dots, P_t v_4) &= \frac{\det P_t}{\mathcal{W}(\phi_t x)} \mathcal{V}(x/2, v_1, \dots, v_4) \\ &= J(x,t)\Omega_x(v_1, \dots, v_4).\end{aligned}$$

Finally, differentiating the flow identity in its time shift gives $P_t F(x) = F(\phi_t x)$. Insert $F(x)$ as the first tangent vector to obtain (12.29). □

The strict inequality is pointwise in the initial state and travel time. It will suffice for capture, even when the multipliers approach one along a sequence of crossings.

12.4. **Flux transported by actual arrivals.** An arrival time varies with the initial point. Its derivative adds a vector in the flow direction to the derivative of the fixed-time map. In the flux determinant this vector repeats the direction already contracted into the volume form, so its contribution vanishes. This observation transfers the multiplier of the preceding subsection to actual crossings.

Let τ be C^1 near $x \in \mathcal{M}_{+++}$, and suppose $\tau(x) \in I(x)$. On a smaller neighborhood the arrival map $A(y) = \phi_{\tau(y)} y$ is defined. The chain rule gives

$$DA(x)v = D\phi_{\tau(x)}(x)v + d\tau_x(v)F(A(x)). \tag{12.30}$$

The identity $\mathcal{C} \circ A = \mathcal{C}$ shows that $DA(x)$ preserves constraint tangent spaces. For tangent vectors v_1, v_2, v_3,

$$\omega_{A(x)}(DA(x)v_1, DA(x)v_2, DA(x)v_3) = J(x, \tau(x))\omega_x(v_1, v_2, v_3). \tag{12.31}$$

Indeed, after inserting (12.30), every term containing an added $F(A(x))$ vanishes by alternation. The remaining term is (12.29).

Suppose an arrival between the charts $\mathcal{X}_h$ and $\mathcal{X}_k$ has coordinate map R, so that on an open neighborhood of $y \in \mathcal{D}_h$,

$$\mathcal{X}_k(R(v)) = \phi_{\tau(\mathcal{X}_h(v))}\mathcal{X}_h(v), \qquad R(v) \in \mathcal{D}_k. \tag{12.32}$$

Assume $h, k > 0$ and R is C^1. Applying (12.31) to the three chart tangent vectors and using (12.4) gives

$$w_k(R(y))|\det DR(y)| = J(\mathcal{X}_h(y), \tau(\mathcal{X}_h(y)))w_h(y). \tag{12.33}$$

All factors on the right are positive. Thus $\det DR(y) \neq 0$, and the weighted coordinate Jacobian

$$j(y) = \frac{w_k(R(y))|\det DR(y)|}{w_h(y)} = \exp\left(\int_0^{\tau(\mathcal{X}_h(y))} \mathcal{D}(\phi_u \mathcal{X}_h(y))\, du\right) \tag{12.34}$$

is strictly greater than one whenever the arrival time is positive.

Proposition 12.8. *Let $h, k > 0$, $y \in \mathcal{D}_h$ and $t \in I(\mathcal{X}_h(y))$. Suppose the orbit at time t satisfies $N_1 = k$ and $p < 0$. There are an open neighborhood $U \subset \mathcal{D}_h$ of y, a C^1 arrival-time function τ near $\mathcal{X}_h(y)$, and a C^1 injective map $R : U \to \mathcal{D}_k$ such that $\tau(\mathcal{X}_h(y)) = t$ and* (12.32) *holds on U. In a neighborhood of (y, t) in the space of initial coordinates and time, the equation $N_1(\phi_u \mathcal{X}_h(v)) = k$ is equivalent to $u = \tau(\mathcal{X}_h(v))$.*

The function j in (12.34) *is positive and continuous on U. For every Borel $A \subset U$, the image $R(A)$ is Borel and*

$$\mu_k(R(A)) = \int_A j(y)\, d\mu_h(y). \tag{12.35}$$

In particular $\mu_k(R(A)) = 0$ if and only if $\mu_h(A) = 0$. If $t > 0$, U can be chosen so that all arrival times are positive, and then

$$0 < \mu_h(A) < \infty \quad \Longrightarrow \quad \mu_h(A) < \mu_k(R(A)). \tag{12.36}$$

Proof. At the given crossing,

$$\frac{d}{du} N_1(\phi_u \mathcal{X}_h(y))\Big|_{u=t} = (-p^2 - s^2 + 2p)k < 0.$$

The implicit function theorem applied to $N_1(\phi_u x) - k$ therefore provides a C^1 function $u = \tau(x)$ and the stated local uniqueness in initial state and time. Positivity of the walls and the constraint persist along the physical flow. At the endpoint, $p < 0$ persists by continuity. The inverse logarithmic coordinates consequently give a C^1 map R satisfying (12.32) on a neighborhood of y.

Equation (12.33) makes its derivative invertible. The inverse function theorem supplies a smaller open neighborhood on which R is injective. Restrict further to retain the domain, endpoint and event properties. If $t > 0$, continuity of $\tau \circ \mathcal{X}_h$ also retains positive arrival times.

The quotient defining j is continuous on this neighborhood, because R is C^1 and w_h, w_k are positive continuous densities on their respective chart domains. A Borel subset of U has Borel image under the continuous injective map R. The change of variables formula gives

$$\begin{aligned}\mu_k(R(A)) &= \int_A w_k(R(y))|\det DR(y)|\, dy \\ &= \int_A j(y) w_h(y)\, dy = \int_A j(y)\, d\mu_h(y).\end{aligned}$$

The positivity of j proves equivalence of the null sets. For positive-time arrivals, $j > 1$ throughout U. If $0 < \mu_h(A) < \infty$, strict monotonicity of the nonnegative integral gives $\int_A j\, d\mu_h > \int_A 1\, d\mu_h = \mu_h(A)$. □

This result concerns any transverse arrival on the negative p branch. To select the first visit to a prescribed Borel part of that section, we next organize these local maps into a countable family and use the order of their actual times.

12.5. **First returns to Borel parts of a section.** The regions selected by statistical estimates are Borel. We construct their first-arrival maps from the local crossings just proved. Local finiteness of transverse crossings guarantees that the earliest admissible time is attained; a countable collection of smooth arrivals then gives its measurability.

Let $h, k > 0$, let $B \subset \mathcal{D}_k$, and assume that for some $\epsilon > 0$,

$$p(\mathcal{X}_k(z)) \le -\epsilon \qquad (z \in B). \tag{12.37}$$

For $y \in \mathcal{D}_h$ and $I \subset [0, \infty)$ define the set of admissible hitting times

$$\mathcal{H}_{B,I}(y) = \{t \in I : N_1(\phi_t \mathcal{X}_h(y)) = k,\ p(\phi_t \mathcal{X}_h(y)) < 0,\ \mathcal{X}_k^{-1}(\phi_t \mathcal{X}_h(y)) \in B\}. \tag{12.38}$$

The inverse chart in this expression is used at the stated negative-p crossing, where it is defined.

Lemma 12.9. *For every $T \ge 0$, the set $\mathcal{H}_{B,I}(y) \cap [0, T]$ is finite. Every nonempty $\mathcal{H}_{B,I}(y)$ therefore has a least element.*

Proof. Forward existence gives a smooth orbit on $[0, T]$. The larger set

$$K = \{t \in [0, T] : N_1(\phi_t \mathcal{X}_h(y)) = k,\ p(\phi_t \mathcal{X}_h(y)) \le -\epsilon\}$$

is compact. At each of its points the derivative of N_1 is at most $-2\epsilon k < 0$. The derivative as a limit of secants makes each such crossing isolated in K. A compact discrete subset of $\mathbb{R}$ is finite, and $\mathcal{H}_{B,I}(y) \cap [0, T] \subset K$.

Choose one admissible time T. The finite nonempty set $\mathcal{H}_{B,I}(y) \cap [0, T]$ has a minimum. Every other admissible time is larger than T, so this is the minimum of the whole hitting set. □

Consider all pairs (y, t) with $y \in \mathcal{D}_h$, $t \in I(\mathcal{X}_h(y))$, and endpoint $N_1 = k$, $p < 0$. Proposition 12.8 gives each pair an open neighborhood in $\mathbb{R}^3 \times \mathbb{R}$ on which the crossing time is described by one smooth arrival. Second countability provides a countable subcover of this crossing relation. Write the corresponding local arrivals as

$$U_n, \qquad t_n(y) = \tau_n(\mathcal{X}_h(y)), \qquad R_n(y), \qquad n \ge 0. \tag{12.39}$$

Every crossing pair occurs in this family with its actual time: there is n with $y \in U_n$ and $t_n(y) = t$. This follows from local uniqueness in the space of initial coordinates and time.

Now suppose B and I are Borel. The eligible set for branch n is

$$A_n = \{y \in U_n : t_n(y) \in I,\ R_n(y) \in B\}.$$

It is Borel. Extend its time by infinity outside this set:

$$d_n(y) = \begin{cases} t_n(y), & y \in A_n, \\ +\infty, & y \notin A_n, \end{cases} \qquad d(y) = \inf_{n \ge 0} d_n(y). \tag{12.40}$$

These are Borel functions into $[0, \infty]$. The hitting domain is the Borel set

$$H_{B,I} = \{y \in \mathcal{D}_h : \mathcal{H}_{B,I}(y) \ne \varnothing\} = \bigcup_{n \ge 0} A_n = \{d < \infty\}. \tag{12.41}$$

Under (12.37), Lemma 12.9 and the covering property show that $d(y)$ is attained for every $y \in H_{B,I}$. It is precisely the least admissible time. Break ties by the least index, and define

$$P_n = A_n \cap \{d_n = d\} \cap \bigcap_{m < n} \{d_m \ne d\}. \tag{12.42}$$

The sets P_n form a disjoint Borel partition of $H_{B,I}$. On P_n put $R = R_n$ and $\tau = t_n = d$; extend both by zero outside the hitting domain. Then R and τ are Borel, and on $H_{B,I}$,

$$\tau(y) = \min \mathcal{H}_{B,I}(y), \qquad \mathcal{X}_k(R(y)) = \phi_{\tau(y)} \mathcal{X}_h(y). \tag{12.43}$$

Each piece uses a C^1 local arrival.

For returns to one section take $k = h$, $I = (0, \infty)$ and define

$$D_B = B \cap H_{B,(0,\infty)}. \tag{12.44}$$

Thus a point in D_B starts in B and has a later visit to B. The previously constructed R and τ select its first positive return.

Theorem 12.10. *Let $h > 0$ and let $B \subset \mathcal{D}_h$ be Borel and satisfy* (12.37) *for some $\epsilon > 0$. The set D_B is Borel, and its actual first-return map and time have Borel extensions R, τ to $\mathbb{R}^3$. The map R is injective on D_B. There is a Borel function $j : \mathbb{R}^3 \to [0, \infty)$ such that*

$$1 < j(y) = J(\mathcal{X}_h(y), \tau(y)) < \infty \qquad (y \in D_B), \tag{12.45}$$

and, for every Borel $A \subset D_B$, the image $R(A)$ is Borel and

$$\mu_h(R(A)) = \int_A j \, d\mu_h. \tag{12.46}$$

Proof. The Borel assertions for D_B, R, τ follow from the construction. Suppose $x, y \in D_B$ have the same returned point, with first-return times $s, t > 0$. The chart is injective, so the physical endpoints agree. If $s < t$, uniqueness and the flow identity give

$$\phi_{t-s}\mathcal{X}_h(y) = \mathcal{X}_h(x).$$

Since $x \in B$ and $0 < t - s < t$, this contradicts the minimality of t. The case $t < s$ is symmetric. Thus $s = t$, and injectivity of the fixed-time flow gives $x = y$.

On P_n, set j equal to the continuous weighted Jacobian of R_n from (12.34), and extend it by zero elsewhere. This gives a Borel function, with (12.45) on D_B. For a Borel $A \subset D_B$, put $A'_n = A \cap P_n$. These sets are disjoint and cover A; their images under R are disjoint by the injectivity just proved. Moreover, $R(A'_n) = R_n(A'_n)$ is Borel and

$$\mu_h(R(A'_n)) = \int_{A'_n} j \, d\mu_h$$

by (12.35). Countable additivity on the disjoint images and on the nonnegative integrals proves (12.46). □

For the small section used in the statistical construction, the algebraic estimate in Lemma 12.2 provides the uniform target bound $p \le -15/16$. Thus any Borel choice within that section has the first-return and measure properties above.

12.6. **Capture under recurrent expanding returns.** Strict expansion prevents a positive-measure family from remaining forever in a set of finite flux. To apply this principle to the Einstein flow, we first relate every iterate of the partial return map to its actual crossing time. This also identifies the recurrence condition needed for capture.

Let B, R, τ, D_B be as in Theorem 12.10. For $n \ge 0$ put

$$T_n(y) = \sum_{i=0}^{n-1} \tau(R^i y), \qquad T_0(y) = 0. \tag{12.47}$$

Every T_n is Borel. Whenever the first n returns are defined, the flow identity and (12.43) give inductively

$$\mathcal{X}_h(R^n y) = \phi_{T_n(y)}\mathcal{X}_h(y). \tag{12.48}$$

Each time increment is positive.

Proposition 12.11. *For $y \in B$, the following are equivalent: every iterate $R^n y$ lies in D_B; the original orbit has infinitely many positive visits to B; those visits occur at arbitrarily large times. Under these conditions, T_n is strictly increasing, $T_n \to \infty$, and*

$$\mathcal{H}_{B,(0,\infty)}(y) = \{T_{n+1}(y) : n \ge 0\}. \tag{12.49}$$

Thus these are all successive visits to B on the original orbit.

Proof. Suppose all iterates are in D_B. Equation (12.48) puts each T_n, $n \geq 1$, in the original hitting set, and the positive increments make these times distinct. If they were bounded, they would give infinitely many crossings in a bounded time interval, contrary to Lemma 12.9. Therefore $T_n \to \infty$.

More generally, the same local finiteness shows that an infinite positive hitting set is unbounded. Conversely, suppose the hitting times are unbounded. After any finite sequence of legal returns, (12.48) places the current point at time T_n on the original orbit. A visit at time $t > T_n$ becomes a positive visit at elapsed time $t - T_n$ from that point. The current point belongs to B and therefore lies in D_B. Induction shows $R^n y \in D_B$ for every n.

The same time shift proves that T_{n+1} is the least original hitting time strictly after T_n: any earlier candidate would contradict the first-return property at $R^n y$. For an arbitrary positive hit t, choose the least m with $t \leq T_m$. Such an m exists because $T_m \to \infty$, and $m \geq 1$. Then $T_{m-1} < t \leq T_m$, while minimality of the next hit gives $T_m \leq t$. Thus $t = T_m$, proving (12.49). □

The decisive measure argument has the following general form. Let (X, μ) be a measure space, let $R : X \to X$ and $j : X \to [0, \infty]$ be measurable, and let D, E be measurable sets. Suppose that $R(A)$ is measurable for every measurable $A \subset D$, and assume

$$\mu(E) < \infty, \qquad j(x) > 1 \quad (x \in D), \qquad \mu(R(A)) = \int_A j \, d\mu \tag{12.50}$$

for every measurable $A \subset D$.

Lemma 12.12 (Nullity under strict expansion)**.** *Under* (12.50)*, the set*

$$Z = \bigcap_{n \geq 0} (R^n)^{-1}(D \cap E) \tag{12.51}$$

is measurable and $\mu(Z) = 0$. Consequently, for almost every x, if $R^n x \in D$ for every n, then $R^n x \notin E$ for some n.

Proof. The countable intersection is measurable. It satisfies $Z \subset D \cap E$ and $R(Z) \subset Z$, hence has finite measure. If $\mu(Z) > 0$, then $j > 1$ on Z and strict monotonicity of the nonnegative integral gives

$$\mu(Z) = \int_Z 1 \, d\mu < \int_Z j \, d\mu = \mu(R(Z)) \leq \mu(Z),$$

a contradiction. Outside Z, an orbit whose iterates all belong to D must have an iterate outside E. □

Only strict inequality at each point enters this proof. The total measure of the return domain can be infinite, and the values of j may have infimum one.

We apply this principle to successive visits to the physical section.

Theorem 12.13 (Capture of recurrent points)**.** *Let $h > 0$ and let $B \subset \mathcal{D}_h$ be Borel with $p(\mathcal{X}_h(z)) \leq -\epsilon$ for some $\epsilon > 0$ and every $z \in B$. For every Borel $E \subset \mathbb{R}^3$ with $\mu_h(E) < \infty$, there is a Borel set $Z \subset D_B \cap E$, $\mu_h(Z) = 0$, such that the following holds. If $y \in B \setminus Z$ has infinitely many positive visits to B, then for some $n \geq 0$,*

$$R^n y \in B \setminus E, \qquad \phi_{T_n(y)} \mathcal{X}_h(y) = \mathcal{X}_h(R^n y). \tag{12.52}$$

All its positive visits are the successive crossings enumerated in (12.49)*.*

Proof. Use the Borel extensions and exact change of variables from Theorem 12.10. They satisfy (12.50) for $\mu = \mu_h$, $D = D_B$. Take Z from (12.51). Lemma 12.12 makes it null. Infinite visits imply that all iterates lie in D_B by Proposition 12.11. Outside Z, one of these iterates lies outside E, and (12.48) identifies it with the claimed state of the original orbit. □

The excluded sets of finite flux from Section 12.2 can now be used in this measure argument. The remaining geometric work is to cover the required initial positions by statistical entrance families and to obtain recurrent visits to their section regions. Once those conditions are established, the theorem turns finite excluded flux into almost-everywhere entrance to the statistical set.

12.7. **Covering the region of dominant shear.** The statistical construction on a shear layer required at least one weak wall to be smaller than a prescribed power of the layer scale. The remaining points have both weak walls bounded below. Their logarithmic coordinates therefore occupy a finite rectangle, whose area grows only quadratically in the logarithm of the scale. This gives a finite excluded flux on the entire region where shear dominates the weak walls.

For $y = (s, \beta_2, \beta_3)$ write $n_i = e^{-\beta_i}$. Given $a > 0$, $\varepsilon \geq 0$ and $P \geq 0$, set

$$\begin{aligned} \mathcal{L}^{\mathrm{s}}_a &= \{y : a \leq |s| \leq 2a, \quad n_2, n_3 \leq 2\varepsilon a\}, \\ \mathcal{U}^{\mathrm{s}}_a &= \{y \in \mathcal{L}^{\mathrm{s}}_a : n_2 > a^P,\ n_3 > a^P\}. \end{aligned} \tag{12.53}$$

Both sets are Borel. Outside $\mathcal{U}^{\mathrm{s}}_a$, a point of $\mathcal{L}^{\mathrm{s}}_a$ satisfies $\min\{n_2, n_3\} \leq a^P$ and hence belongs to one of the signed families of Section 11.12.

Lemma 12.14. *If $0 < h \leq 1/8$, $0 < a \leq h/8$, $0 \leq \varepsilon \leq 1/64$ and $P \geq 0$, then $\mathcal{L}^{\mathrm{s}}_a \subset \mathcal{D}_h$ and*

$$\mu_h(\mathcal{U}^{\mathrm{s}}_a) \leq 12P^2 a|\log a|^2. \tag{12.54}$$

Proof. The chart radicand and density depend on s only through s^2. Lemma 12.2, with $\rho = |s|$, therefore gives the chart inclusion and $w_h \leq 3$ on the whole layer.

Here $a \leq 1$ and $n_i \leq 2\varepsilon a \leq 1$. For a point in $\mathcal{U}^{\mathrm{s}}_a$, taking logarithms of $a^P < n_i \leq 1$ gives

$$0 \leq \beta_i \leq P|\log a|.$$

Thus

$$\mathcal{U}^{\mathrm{s}}_a \subset [-2a, 2a] \times [0, P|\log a|]^2.$$

This box has Lebesgue volume $4P^2 a|\log a|^2$. Multiplication by the density bound proves (12.54). □

For $0 < d \leq h/8$ let $a_j = d2^{-(j+1)}$. The same lemma and (12.19) imply

$$\mu_h\left(\bigcup_{j\geq 0} \mathcal{U}^{\mathrm{s}}_{a_j}\right) \leq \sum_{j\geq 0} \mu_h(\mathcal{U}^{\mathrm{s}}_{a_j}) \leq 132P^2 d(1 + |\log d|)^2. \tag{12.55}$$

The union is Borel.

For fixed statistical parameters, say that $x \in \mathcal{M}_{+++}$ has a statistical entrance if there are $t > 0$, an axis frame A, and $(B, Y) \in \mathcal{S}_{\mathrm{stat}}$ such that

$$\iota_+(Y) = \mathrm{proj}(\mathsf{A}\phi_t x). \tag{12.56}$$

Corollary 8.27 then provides successive complete eras on this orbit, all six statistical limits, and entrance times tending to infinity.

Define the full region of dominant shear by

$$\mathcal{B}^{\mathrm{s}}(d, \varepsilon) = \{y : 0 < |s| \leq d, \quad n_2, n_3 \leq \varepsilon|s|\}. \tag{12.57}$$

The half-open intervals $(a_j, 2a_j]$ partition $(0, d]$. Indeed, successive powers of $1/2$ place every $|s|/d \in (0, 1]$ in exactly one such interval; their endpoint convention prevents double counting. Consequently the Borel sets

$$\mathcal{B}^{\mathrm{s}}_j = \{y : a_j < |s| \leq 2a_j, \quad n_2, n_3 \leq \varepsilon|s|\} \tag{12.58}$$

partition $\mathcal{B}^{\mathrm{s}}(d, \varepsilon)$, and $\mathcal{B}^{\mathrm{s}}_j \subset \mathcal{L}^{\mathrm{s}}_{a_j}$.

Theorem 12.15. *Fix the statistical parameters and cap of Theorem 12.4. There is $h_* > 0$ such that, for $0 < h \le h_*$, $q > 3$, $D > 4$ and $P > 6D + 2$, there is $d_* > 0$ such that every $0 < d \le d_*$ and $0 \le \varepsilon \le 1/128$ admit a Borel set F_d^{s} with*

$$\mu_h(F_d^{\mathrm{s}}) \le \left[66C_{\log}(K_\sigma, \sigma) + 132P^2\right] d(1 + |\log d|)^2 < \infty, \tag{12.59}$$

Every $y \in \mathcal{B}^{\mathrm{s}}(d, \varepsilon) \setminus F_d^{\mathrm{s}}$ lies in $\mathcal{D}_h$, and $\mathcal{X}_h(y)$ has a statistical entrance with the chosen parameters and cap.

Proof. Choose the height and scale thresholds to satisfy Corollary 12.5 and Lemma 12.14, with $d \le h/8$. Let E_d be the excluded set of that corollary, and put

$$F_d^{\mathrm{s}} = E_d \cup \bigcup_{j \ge 0} \mathcal{U}_{a_j}^{\mathrm{s}}.$$

The Borel property and (12.59) follow from (12.20), (12.55) and subadditivity.

Take y outside this set and choose its annulus $\mathcal{B}_j^{\mathrm{s}}$. Since $y \notin \mathcal{U}_{a_j}^{\mathrm{s}}$, its wall labels satisfy $\min\{n_2, n_3\} \le a_j^P$, and both walls are at most $2\varepsilon a_j$. Write $\rho = |s|$ and $\varsigma = \operatorname{sign} s$. Then

$$y = (\varsigma\rho, -\log n_2, -\log n_3), \qquad \rho \in [a_j, 2a_j], \quad (n_2, n_3) \in \Lambda_c(a_j).$$

The small-section bound puts y in $\mathcal{D}_h$. Exclusion from E_d, together with the signed return identity (11.120) and its statistical conclusion, gives (12.56) at the positive actual return time. The frame is S for positive shear and the identity for negative shear. □

The remaining section points can have zero shear or shear comparable to a weak wall. For these points we vary a weak wall and the shear together. The resulting radial parameter retains a positive phase derivative even when the shear vanishes.

13. Angular directions and coverage of the section

The shear families cover the region where both weak walls are small compared with the shear. Near zero shear, we instead vary the larger weak wall while fixing the ratio of shear to that wall. These angular families retain a nonzero phase derivative at zero shear. Their returns and flux estimates complete the coverage in Theorem 13.15.

13.1. **Radial families through zero shear.** Vary the shear and the larger weak wall along the affine rays

$$Z_{\theta,\zeta}(\rho) = (\theta\rho, \rho, \zeta), \qquad |\theta| \le L, \quad a \le \rho \le 2a, \qquad 0 \le \zeta \le a/9, \tag{13.1}$$

where $L \ge 0$. The labels θ, ζ stay fixed when differentiating in ρ. Taking $\theta = 0$ gives the zero-shear family. The initial physical state is $\Xi_h(Z_{\theta,\zeta}(\rho))$.

Write $A = A(h)$, $C = C(h)$ and $K = K(h)$ for the transport coefficients of Section 9.1. The reference phase on (13.1) is

$$F_{\theta,\zeta}(\rho) = \Phi_h(Z_{\theta,\zeta}(\rho)) = \frac{1}{\sqrt{3}} \left|\rho(A\theta + K, C) - \zeta(K, C)\right|_2. \tag{13.2}$$

Here $|\cdot|_2$ denotes the Euclidean norm in two dimensions. The following elementary estimate explains why no division by the shear is needed.

Lemma 13.1. *Let $v, w \in \mathbb{R}^2$, $|v|_2 > 0$, $\rho > 0$ and $\zeta \ge 0$. If $3\zeta|w|_2 \le \rho|v|_2$, then $r(\rho) = |\rho v - \zeta w|_2$ is differentiable near ρ, and*

$$\frac{|v|_2}{2} \le r'(\rho) \le |v|_2. \tag{13.3}$$

Proof. Put $P = |v|_2$ and $Q = |w|_2$. Cauchy's inequality, applied after expanding the squared norm, gives

$$0 < \rho P - \zeta Q \le r \le \rho P + \zeta Q.$$

Thus the square root is smooth near the point and

$$r' = \frac{\rho P^2 - \zeta\langle v, w\rangle}{r}.$$

The numerator is at least $P(\rho P - \zeta Q)$. Since $\rho P \geq 3\zeta Q$, this is at least $P(\rho P + \zeta Q)/2 \geq Pr/2$. The upper bound follows from $\langle v, \rho v - \zeta w\rangle \leq Pr$. □

For $0 \leq h \leq 1/16$, the previously proved coefficient bounds imply

$$1 \leq A, C \leq 2, \qquad 0 \leq K \leq 1.$$

For $v = (A\theta + K, C)$ and $w = (K, C)$, it follows that

$$1 \leq |v|_2 \leq 2L + 3, \qquad |w|_2 \leq 3.$$

The condition $\zeta \leq a/9 \leq \rho/9$ verifies the hypothesis of Lemma 13.1. Consequently

$$\frac{1}{2\sqrt{3}} \leq F'_{\theta,\zeta}(\rho) \leq \frac{2L+3}{\sqrt{3}}. \tag{13.4}$$

This includes the direction $A\theta + K = 0$, in which the first component of the reference ray vanishes. The wall component C supplies its positive length.

On the families (13.1),

$$\|Z_{\theta,\zeta}(\rho)\| \leq 2(L+1)a, \qquad \varrho(Z_{\theta,\zeta}(\rho)) \geq a/2, \qquad \|Z'_{\theta,\zeta}\| \leq L+1. \tag{13.5}$$

The middle inequality follows from $\varrho(Z) \geq \rho - \zeta \geq 8a/9$; the stated weaker bound will be convenient. For $q > 2$, choose the common time

$$T = \frac{q}{2}\log(1/a), \qquad f(\rho) = \frac{\varrho(H_{h,T}(Z_{\theta,\zeta}(\rho)))}{\sqrt{3}}.$$

Proposition 13.2. *There is $h_* > 0$ such that, for each $0 < h \leq h_*$ and $L \geq 0$, one can choose $K_0 > 0$ before choosing $q > 2$. For all sufficiently small $a > 0$, uniformly over* (13.1)*, the actual orbit exists at T, f has an ordinary derivative on $[a, 2a]$, and*

$$\begin{aligned} &|f - F_{\theta,\zeta}| \leq K_0 a^2, \qquad |f' - F'_{\theta,\zeta}| \leq K_0 a, \\ &\frac{1}{4\sqrt{3}} \leq f' \leq \frac{2L+3}{\sqrt{3}} + 1. \end{aligned} \tag{13.6}$$

For $a \leq x \leq y \leq 2a$,

$$\frac{y-x}{4\sqrt{3}} \leq f(y) - f(x) \leq \left(\frac{2L+3}{\sqrt{3}} + 1\right)(y - x). \tag{13.7}$$

Proof. Apply Theorem 9.12 with $c_0 = 1/2$ and $C_0 = 2(L+1)$, using (13.5). Its error constant is chosen before q. Restricting the ordinary initial derivative to $Z'_{\theta,\zeta} = (\theta, 1, 0)$ multiplies its bound by at most $L + 1$. Enlarge the constant by this factor to obtain the first line of (13.6).

Reduce the scale threshold so that $K_0 a \leq 1/(4\sqrt{3}) \leq 1$. Comparison with (13.4) gives the derivative bounds. The mean value theorem on each subinterval gives (13.7). Ordinary differentiability at both endpoints follows from the ordinary derivative in the radial approximation theorem. □

The phase is therefore monotone throughout these families, including zero shear and the zero weak-wall boundary. We next express the same states in core coordinates. The positive active wall keeps the projection denominator away from zero after rescaling.

13.2. **Core coordinates in the angular region.** On the families of the preceding subsection, the second wall remains comparable to the radial scale throughout the initial decay. It gives a positive quadratic contribution to the constraint even when the shear vanishes. We use this contribution to control the height of the projective chart and to determine the initial core coordinates.

Keep θ and ζ fixed, and write

$$\begin{aligned} T &= \frac{q}{2}\log(1/a), \qquad x(\rho) = \phi_T \Xi_h(\theta\rho, \rho, \zeta), \\ x(\rho) &= (p, s, n, w_2, w_3), \qquad X(\rho) = (n, s, w_2, w_3), \qquad M_L = 8(L+1). \end{aligned} \tag{13.8}$$

All derivatives below are ordinary derivatives in ρ, with a, h, q, θ, ζ, and hence T, held fixed. Finite-dimensional norms are maximum norms.

Proposition 13.3. *There is a constant $C \geq 32$ such that, for $0 < h \leq 1/(4C)$, $L \geq 0$ and $q > 2$, the following holds for all sufficiently small $a > 0$. Uniformly for $|\theta| \leq L$, $0 \leq \zeta \leq a^2$ and $a \leq \rho \leq 2a$, the state in* (13.8) *exists, has ordinary derivatives, and satisfies*

$$\begin{aligned} &p \leq -15/16, \qquad 0 < n \leq ha^q, \qquad \|X\| \leq M_L a \leq 1/16, \\ &w_2 \geq a/2, \qquad 0 \leq w_3 \leq 2\zeta, \qquad n, w_3 \leq w_2/16, \\ &\|X'\| \leq 2(L+1), \qquad |n'| \leq \frac{2(L+1)n}{a}, \quad |w_3'| \leq \frac{8C(L+1)w_3}{a}. \end{aligned} \tag{13.9}$$

Its height $A = 1 + p$ has an ordinary derivative and obeys

$$\frac{a^2}{16} \leq A \leq 10 M_L^2 a^2, \qquad |A'| \leq 64 M_L (L+1) a. \tag{13.10}$$

Proof. Enlarge the common constant in the orbit, variation and weak-wall estimates of Sections 9.2–9.4. Set $b = M_L a$ and decrease the scale threshold so that

$$0 < a \leq 1/64, \qquad b \leq 1/16, \qquad CbT \leq 1/8, \qquad aT \leq 1.$$

These conditions hold for small a, since $a\log(1/a) \to 0$. The initial state satisfies $\|Z_{\theta,\zeta}(\rho)\| \leq 2(L+1)a < b/2$, and $\zeta \leq a/9$. The orbit and derivative estimates give

$$p \leq -15/16, \qquad 0 < n \leq he^{-2T} = ha^q, \qquad \|(s, w_2, w_3)\| \leq b, \qquad \|X'\| \leq 2(L+1).$$

Here the factor $L+1$ bounds the direction $(\theta, 1, 0)$. Also $n \leq a^2 \leq a \leq b$, so $\|X\| \leq b$.

By Lemma 9.14, $w_2 = \rho\mu_2$ and $w_3 = \zeta\mu_3$, with $1/2 < \mu_i < 2$. Thus $w_2 \geq a/2$ and $0 \leq w_3 \leq 2\zeta \leq 2a^2$. Together with $a \leq 1/64$, these inequalities give $n, w_3 \leq w_2/16$. The strong-wall derivative retains its factor n: $|n'| \leq 2(L+1)n \leq 2(L+1)n/a$. The initial third wall is constant along the curve, so (9.76) gives

$$|w_3'| \leq 8CT(L+1)w_3 \leq \frac{8C(L+1)w_3}{a}.$$

This proves (13.9).

For the height, use the defect $d = s^2 + Q(n, w_2, w_3)$ of (9.78). The wall ordering implies

$$Q(n, w_2, w_3) = (w_2 - n - w_3)^2 - 4nw_3 \geq \left(\frac{7w_2}{8}\right)^2 - 4\left(\frac{w_2}{16}\right)^2 = \frac{3w_2^2}{4}.$$

Writing $R = M_L a$, we therefore have

$$\frac{3a^2}{16} \leq d \leq s^2 + (n + w_2 + w_3)^2 \leq 10R^2 \leq \frac{1}{2}.$$

The constraint and $p < 0$ give $A = \mathcal{H}(X) = d/(1 + \sqrt{1-d})$. Its denominator lies between one and two, yielding the two bounds on A in (13.10). As in Section 9.4, $|Dd(X)v| \leq 32\|X\|\|v\|$, and hence

$$|A'| = \left|\frac{Dd(X)X'}{2\sqrt{1-d}}\right| \leq 32R\,2(L+1).$$

The initial chart radicand is strictly positive and the negative constraint branch persists on an ordinary neighborhood. Thus $A = \mathcal{H}(X)$ is a local identity, which also establishes the ordinary derivative asserted at the endpoints of the radial interval. □

Define the actual projective state and its core angle by

$$Y = (M, U, V, W) = \frac{\sqrt{3}}{A}(s, n, w_2, w_3), \qquad z = \operatorname{arsinh} \frac{1 - M}{V}. \tag{13.11}$$

Since $A > 0$ and $w_2 > 0$, these are smooth coordinate expressions on an ordinary neighborhood of each state.

Corollary 13.4. *Under the hypotheses of Proposition 13.3, and after decreasing the scale threshold, Y and z have ordinary derivatives and*

$$\begin{gathered} 2 \le V, \qquad \frac{1}{20M_L^2 a} \le V \le \frac{32M_L}{a}, \qquad 0 < U \le 32ha^{q-2}, \\ 0 \le W \le \frac{64\zeta}{a^2}, \qquad 0 \le VW \le \frac{2048M_L\zeta}{a^3}, \\ |z| \le \operatorname{arsinh}(4M_L). \end{gathered} \tag{13.12}$$

Proof. Divide (13.9) by the height bounds, using $1 \le \sqrt{3} \le 2$. In particular,

$$\frac{a/2}{10M_L^2 a^2} \le V \le \frac{2M_L a}{a^2/16} = \frac{32M_L}{a}.$$

The lower bound is at least two when $a \le 1/(40M_L^2)$. The estimates for U, W follow in the same way, and multiplication gives the one for VW.

Since $M_L a \le 1/16$, the upper height bound also gives $A \le 10(M_L a)^2 \le M_L a$. Therefore

$$\left|\frac{1 - M}{V}\right| = \frac{|A/\sqrt{3} - s|}{w_2} \le \frac{2M_L a}{a/2} = 4M_L.$$

Oddness and monotonicity of arsinh prove the angle bound. □

Thus the initial decay places these families at a large core radius and at an angle bounded in terms of L alone. The weak product VW retains the factor ζ. We next compare the reciprocal core radius with the reference phase, keeping its ordinary radial derivative.

13.3. **The projected phase across zero shear.** The reciprocal core radius depends on both shear and the active wall. Its comparison with the physical transverse radius can therefore be made uniformly across zero shear. The useful cancellation is the same one encountered in Section 9.4: after rescaling, the difference vanishes identically when the small wall amplitudes and the scale vanish. Here the active wall keeps the limiting state away from zero.

For $X = (n, s, w_2, w_3)$, let

$$\mathcal{E}(X) = \mathcal{P}(X) - \frac{\sqrt{s^2 + (w_2 - w_3)^2}}{\sqrt{3}},$$

where $\mathcal{P}$ is given by (9.82). We use maximum norms as above.

Lemma 13.5. *Fix $c > 0$ and $M_0 \ge c$. There are $\delta > 0$ and $B \ge 1$ such that, whenever*

$$\begin{gathered} 0 < a \le \delta, \qquad |s| \le M_0 a, \qquad ca \le w_2 \le M_0 a, \\ |n|, |w_3| \le \delta a, \end{gathered} \tag{13.13}$$

$\mathcal{E}$ is C^2 near X, and

$$|\mathcal{E}(X)| \le B(a^2 + |n| + |w_3|). \tag{13.14}$$

For every $D \geq 0$ and direction $v = (v_n, v_s, v_2, v_3)$ satisfying

$$|v_s|, |v_2| \leq D, \qquad |v_n| \leq D|n|/a, \qquad |v_3| \leq D|w_3|/a,$$

the ordinary differential obeys

$$|D\mathcal{E}(X)v| \leq 2BD\left(a + \frac{|n|}{a} + \frac{|w_3|}{a}\right). \tag{13.15}$$

Proof. Use the scaled function $F(u, z)$ of (9.84), now on the compact set

$$\mathcal{K}_{\mathrm{a}} = \{z : |z_1| \leq M_0,\ c \leq z_2 \leq M_0\}.$$

At $u = 0$, its three radicands are 1, $3(z_1^2 + z_2^2)$ and $z_1^2 + z_2^2$. They are strictly positive on this compact slice. Moreover $F(0, z) = 0$. Compact smoothness and the mean value theorem therefore give $\delta > 0$, $B \geq 1$ with

$$|F(u, z)| \leq B\|u\|, \qquad \|DF(u, z)\| \leq B,$$
$$|DF(u, z)(0, \delta z)| \leq B\|u\|\|\delta z\| \quad (z \in \mathcal{K}_{\mathrm{a}},\ \|u\| \leq \delta).$$

For the last bound, the tangential differential vanishes at $u = 0$; integrate its change in the normal direction, using a bound on D^2F on a compact neighborhood.

For fixed $a > 0$, put

$$u = (a, n/a, w_3/a), \qquad z = (s/a, w_2/a), \qquad b_a = a + |n|/a + |w_3|/a.$$

The domain conditions give $z \in \mathcal{K}_{\mathrm{a}}$ and $\|u\| \leq \delta$, while $\|u\| \leq b_a$. By (9.85), $\mathcal{E}(X) = aF(u, z)$. Its smoothness and (13.14) follow at once.

In direction v, the scaled directions are

$$\delta u = (0, v_n/a, v_3/a), \qquad \delta z = (v_s/a, v_2/a).$$

Separate these directions before estimating the differential:

$$|D\mathcal{E}(X)v| \leq aB\|\delta u\| + aB\|u\|\|\delta z\| \leq B(|v_n| + |v_3|) + BDb_a \leq 2BDb_a.$$

This is (13.15). □

For the state (13.11), define

$$f_{\mathrm{p}}(\rho) = \frac{2}{r(Y(\rho))}, \qquad r(Y) = \sqrt{(M-1)^2 + V^2}.$$

The estimate $V \geq 2$ keeps this radius positive throughout the radial interval.

Theorem 13.6. *There is $h_* > 0$ such that for every $0 < h \leq h_*$ and $L \geq 0$ one can choose $K > 0$ before choosing $q > 2$, with the following property. For all sufficiently small $a > 0$, uniformly for $|\theta| \leq L$, $0 \leq \zeta \leq a^2$ and $a \leq \rho \leq 2a$, f_{p} has an ordinary derivative and*

$$|f_{\mathrm{p}} - F_{\theta,\zeta}| \leq Ka^2, \qquad |f_{\mathrm{p}}' - F_{\theta,\zeta}'| \leq Ka,$$
$$\frac{1}{4\sqrt{3}} \leq f_{\mathrm{p}}' \leq \frac{2L+3}{\sqrt{3}} + 1. \tag{13.16}$$

Proof. Take the height threshold below those in Propositions 13.2 and 13.3, and below $1/16$. In Lemma 13.5, choose $c = 1/2$ and $M_0 = M_L$. Its constants δ, B depend only on L. Put $D = 8C(L + 1)$.

At time T, Proposition 13.3 gives $n \leq a^2$, $w_3 \leq 2a^2$ and all the directional bounds of the lemma with $v = X'$. Choose a so that $2a \leq \delta$. Then $e(\rho) = \mathcal{E}(X(\rho))$ has an ordinary derivative, and the lemma yields

$$|e| \leq 4Ba^2, \qquad |e'| \leq 8BDa.$$

Both are bounded by $K_{\mathrm{e}}a^2$ and $K_{\mathrm{e}}a$, respectively, with $K_{\mathrm{e}} = 8BD$ chosen before q.

The negative constraint branch and positive height persist on an ordinary neighborhood of each initial point. Equation (9.83) therefore gives the local identity

$$f_{\mathrm{p}}(\rho) = e(\rho) + \frac{\varrho(H_{h,T}(Z_{\theta,\zeta}(\rho)))}{\sqrt{3}}.$$

It holds for derivatives as well as values. For $a \le 1/9$, the condition $\zeta \le a^2$ implies $\zeta \le a/9$, so Proposition 13.2 applies to the second term. Set $K = K_{\mathrm{e}} + K_0$ to obtain the two comparison estimates.

Finally decrease the scale threshold so that $Ka \le 1/(4\sqrt{3})$. Comparing the total error directly with (13.4) proves the derivative bounds in (13.16). □

The actual reciprocal core radius is thus strictly increasing along every radial family, including $\theta = 0$ and $\zeta = 0$. The active wall supplies both the positive reference slope and the regularity of the projection. These estimates provide the initial radius and direction for the remaining core passage and the next entrance.

13.4. **The remaining core passage for angular families.** The active wall places the angular families inside a large core with a bounded initial angle. A definite part of the core passage therefore remains after the fixed time T. Its contraction makes the outgoing curves almost radial. We establish the initial directional estimates and then apply the general core results of Sections 10.1–10.4.

Throughout this subsection, retain

$$Z_{\theta,\zeta}(\rho) = (\theta\rho, \rho, \zeta), \qquad T = \frac{q}{2}\log\frac{1}{a}, \qquad M_L = 8(L+1), \qquad B_L = M_L + 1.$$

Write $Y = (M, U, V, W)$ for the actual projective state at T, and put

$$k = r(Y), \qquad p = VW, \qquad z = \operatorname{arsinh}\frac{1-M}{V}, \qquad Z_L = \operatorname{arsinh}(4M_L). \tag{13.17}$$

Primes always denote ordinary derivatives in ρ, with a, T, θ, ζ fixed. We use p for the wall product so that P can denote the exponent in the restriction $\zeta \le a^P$.

Proposition 13.7. *There is $h_* > 0$ such that, after fixing $0 < h \le h_*$ and $L \ge 0$, constants $K \ge 1$ and $D = 4B_L^2$ can be chosen before $q > 3$ and $P > 4$. For every $0 < \epsilon \le 10^{-4}$, all sufficiently small $a > 0$ have the following properties, uniformly for $|\theta| \le L$, $0 \le \zeta \le a^P$ and $\rho \in [a, 2a]$:*

$$k \ge \max\{6, 2\cosh Z_L\}, \qquad V \ge 2, \qquad -\operatorname{arcosh} k \le z \le Z_L,$$

$$0 < U, \quad 0 \le p, \quad U + p \le \epsilon/k, \tag{13.18}$$

$$\frac{2}{B_L a} \le k \le \frac{32}{a}, \qquad \frac{1}{Da^2} \le -k' \le \frac{512(2L+4)}{a^2}, \tag{13.19}$$

$$\frac{2}{Ka} \le k \le \frac{K}{a}, \qquad |k'| \le \frac{K}{a^2}, \qquad |U'| \le \frac{KU}{a}, \quad |p'| \le \frac{Kp}{a}, \quad |z'| \le \frac{K}{a}. \tag{13.20}$$

The state lies in the core region of Theorem 10.4. If $\zeta > 0$ then $p > 0$. If $\zeta = 0$ then $W = p = W' = p' = 0$.

Proof. First take the common height threshold from Sections 13.2 and 13.3. The affine reference phase obeys

$$a/8 \le F_{\theta,\zeta} \le M_L a.$$

For the lower bound, its second component has magnitude $C(\rho - \zeta) \ge a/2$. For the upper bound, use $\varrho(\Pi_h Z) \le 4\|Z\|$ and $\|Z\| \le 2(L+1)a$. Here a is small enough that $\zeta \le a^2 \le a/9$.

Let $x = 2/k$. Theorem 13.6, after decreasing the scale, gives

$$a/16 \le x \le B_L a, \qquad 1/8 \le x' \le 2L + 4.$$

The identities $k = 2/x$ and $-k' = 2x'/x^2$ prove (13.19). In particular k eventually exceeds both 6 and $2\cosh Z_L$.

We next control the other directions without dividing by a small wall. Put $D_0 = 2(L+1)$ and let $C \geq 32$ be the constant in Proposition 13.3. For the physical height A, the bounds in (13.10) imply

$$|A'| \leq 32 M_L D_0 a \leq F_0 A/a, \qquad F_0 = 512 M_L D_0.$$

Each physical wall $v = n, w_2, w_3$ satisfies $|v'| \leq E_0 v/a$ with $E_0 = 8C(L+1)$. For the active wall this follows from $|w_2'| \leq D_0$ and $w_2 \geq a/2$; the other two bounds were obtained with their wall factors in Proposition 13.3. Differentiating $\sqrt{3}v/A$ therefore gives, with $H_0 = E_0 + F_0$,

$$|U'| \leq H_0 U/a, \qquad |V'| \leq H_0 V/a, \qquad |W'| \leq H_0 W/a, \qquad |p'| \leq 2H_0 p/a.$$

For the angle use its physical expression $z = \operatorname{arsinh}(b/w_2)$, where $b = A/\sqrt{3} - s$. The same estimates, with $M_L a \leq 1/16$, give $|b| \leq 2M_L a$, $|b'| \leq 3D_0$, $a/2 \leq w_2 \leq M_L a$ and $|w_2'| \leq D_0$. Thus

$$|z'| \leq \left| \left(\frac{b}{w_2} \right)' \right| \leq \frac{20 M_L D_0}{a}.$$

Choose K at least $1 + B_L + 32 + 512(2L+4) + 3H_0 + 20 M_L D_0$. This proves (13.20), with constants independent of q, P.

It remains to check the core amplitude and geometry. Corollary 13.4 gives

$$U \leq 32 h a^{q-2}, \qquad p \leq 2048 M_L \frac{\zeta}{a^3}, \qquad |z| \leq Z_L. \tag{13.21}$$

Consequently

$$k(U+p) \leq 1024 h a^{q-3} + 65536 M_L a^{P-4} \longrightarrow 0.$$

Both exponents are positive, proving the required smallness. The relation $k = V\cosh z$ and $V \geq 2$ give $|z| \leq \operatorname{arcosh} k$. The initial core energy equals k^2 by definition, and $U, p \leq 10^{-4}/k$; these are precisely the initial-region conditions in (10.8). Finally positivity and vanishing of the physical wall multipliers give the assertions about ζ. The relative derivative bounds then give $W' = p' = 0$ when that wall vanishes. □

Let $\tau_\downarrow(Y)$ be the first positive $V = 1$ time from Y, measured in the original normalized time, and write

$$(f, u, w) = \mathcal{R}_\downarrow(Y).$$

The initial inequality $V \geq 2$ separates this event from the fixed time T.

Proposition 13.8. *Under the preceding thresholds, $\tau_\downarrow > 0$ exists, is a transverse downward crossing, and is locally C^2 in the initial state. Along each angular family,*

$$k/2 \leq f \leq 2k, \qquad |f - k| \leq 16K(U + ap^2),$$

$$|f' - k'| \leq 4C_p K^2 \frac{U + ap}{a^2}, \qquad -f' \geq \frac{1}{2Da^2}, \tag{13.22}$$

$$0 \leq u \leq U e^{-k/12}, \qquad 0 \leq w \leq p. \tag{13.23}$$

There is $C_{h,L} > 0$, chosen before q, P, such that, with

$$E_a = a^{(q-2)/2} + \sqrt{\zeta/a}, \tag{13.24}$$

$$|u'| \leq C_{h,L} E_a \sqrt{u}\, |f'|, \qquad |w'| \leq C_{h,L} E_a \sqrt{w}\, |f'|. \tag{13.25}$$

The factor $C_{h,L} E_a$ is uniformly smaller than any prescribed positive number when the scale is sufficiently small. On the original orbit, $T + \tau_\downarrow$ is the first $V = 1$ event strictly after T.

Proof. Apply Theorem 10.4 and Proposition 10.5 to (13.18). The ordinary radial inputs have just been checked, so Proposition 10.8 gives the first two radius estimates and the derivative error. Here C_p is the constant of (10.26). Since $U + p \le 1/k \le Ka/2$, taking $a \le 1/(8C_pK^3D)$ ensures $4C_pK^2(U + ap) \le 1/(2D)$. Theorem 10.13 now applies with $Z = Z_L$. It gives the negative radial speed and the amplitude bounds, as well as

$$|u'| \le C_U\sqrt{U}\sqrt{u}\,|f'|, \qquad |w'| \le C_P a\sqrt{p}\sqrt{w}\,|f'|,$$

where $C_U = 2D(K + 24A_m)$ and $C_P = 2DK(1 + B_m)$. Using (13.21),

$$C_U\sqrt{U} + C_P a\sqrt{p} \le C_U\sqrt{32h}\,a^{(q-2)/2} + C_P\sqrt{2048M_L}\sqrt{\zeta/a}.$$

This proves (13.25). Its uniform smallness follows from $\sqrt{\zeta/a} \le a^{(P-1)/2}$.

The original flow shifted by T has projective initial state Y. Uniqueness identifies it with the core trajectory throughout the passage. Since $V > 1$ before the downward exit, translating its first-event property gives the last assertion. The event identity holds on ordinary neighborhoods; the local time and output maps used for all derivatives are therefore the actual first-event maps. □

13.5. **Angular families at the next entrance.** The outgoing angular curves satisfy the amplitude and radial direction conditions of the long connections already studied. Their next entrance is therefore an actual event on the Einstein orbit. We follow its phase and amplitudes together: the phase remains a regular coordinate, while both returned walls are controlled by the original weak wall ζ.

We now require $\zeta > 0$. Fix h, L, q, P as in Section 13.4, and use the same notation for the core input and exit. Define

$$\begin{aligned} \pi &= (\mu, u, w), \qquad \mu = \sqrt{f^2 - 1} - 1, \\ G_{\theta,\zeta}(\rho) &= (g, \widehat{u}, \widehat{w}) = \mathcal{C}(\pi), \qquad \xi_{\theta,\zeta}(\rho) = 2/g, \\ t_{\mathrm{a}}(\theta, \zeta, \rho) &= T + \tau_{\downarrow}(Y) + T_c(\pi). \end{aligned} \tag{13.26}$$

Here T_c is the first $W = 1$ time from the downward core exit, and $\mathcal{C}$ expresses that exit in the reflected core frame.

Theorem 13.9. *There is $h_* > 0$ such that, for each $0 < h \le h_*$ and $L \ge 0$, a constant $C_{h,L} > 0$ can be chosen before $q > 3$ and $P > 4$. At all sufficiently small scales, uniformly for $|\theta| \le L$, $0 < \zeta \le a^P$ and $a \le \rho \le 2a$, the return* (13.26) *exists and has positive total time. The return state, phase and total time are jointly C^2 on a neighborhood of each such parameter triple. They satisfy*

$$\frac{1}{2B_L a} \le g \le \frac{64}{a}, \qquad \widehat{u}, \widehat{w} > 0,$$

$$\widehat{u} + \widehat{w} \le 4096 M_L \frac{\zeta}{a^3}, \tag{13.27}$$

$$g' < 0, \qquad |\widehat{u}'| \le \sqrt{\widehat{u}}\,|g'|, \qquad |\widehat{w}'| \le \sqrt{\widehat{w}}\,|g'|, \tag{13.28}$$

$$|\xi - F_{\theta,\zeta}| \le C_{h,L}a^2, \qquad |\xi' - F'_{\theta,\zeta}| \le C_{h,L}a,$$

$$\frac{1}{8\sqrt{3}} \le \xi' \le \frac{2L + 3}{\sqrt{3}} + 1. \tag{13.29}$$

For $y = (\theta\rho, -\log\rho, -\log\zeta)$, the orbit identity is

$$\iota_+(G_{\theta,\zeta}(\rho)) = \mathsf{S}\,\mathrm{proj}\,\phi_{t_{\mathrm{a}}}\mathcal{X}_h(y). \tag{13.30}$$

Proof. Choose the input amplitude threshold in Proposition 13.7 below the long-connection threshold (5.66). Proposition 10.15 gives

$$4 \le \mu, \qquad k/2 \le \mu < k, \qquad 0 < u \le \epsilon/\mu, \quad 0 < w \le \epsilon.$$

The preceding core passage and the connection are consecutive segments of the same orbit. Their composition proves (13.30), and both event times are positive.

For the cone, make $C_{h,L}E_a \le 1$ in (13.25). Since $f' < 0$, normalize the parameter direction by f' before applying Proposition 5.10. The connection's radial coordinate satisfies $H(\mu+1,1) = f$, so the normalized vector has unit incoming radial component. The two other components have absolute values at most $\sqrt{u}$ and $\sqrt{w}$. The proposition gives

$$\left|\frac{g'}{f'} - 1\right| \le \frac{1}{4\mu}, \qquad \frac{3}{4} \le \frac{g'}{f'} \le \frac{5}{4},$$

and the two unit square-root cone inequalities. Multiplying by $|f'|$ proves (13.28).

The connection also satisfies

$$\mu/2 \le g \le 2\mu, \qquad 0 < \widehat{u}, \widehat{w} \le w.$$

Indeed, in its exit band $\mu - 1/2 \le m_* \le \mu + 1/2$; apply $g = \sqrt{(m_*-1)^2+1}$ and $\mu \ge 4$. For the walls, write $\ell = \log(1/w) \ge 0$. The exponential estimates (5.12) give

$$\widehat{u} \le ue^{-\mu\ell/2} \le w, \qquad \widehat{w} \le we^{-2\ell/\mu} \le w,$$

using $u \le 1$ and $\mu/2 \ge 1$ in the first inequality. Combine these bounds with $w \le p$, (13.21) and (13.19) to obtain (13.27).

We turn to the phase comparison. Set $x = 2/k$ and

$$R = \xi - \frac{x}{1-x}, \qquad Q = \frac{x}{1-x} - x.$$

The initial data satisfy every hypothesis of Proposition 10.16. Thus, for constants depending only on the already chosen K,

$$|R| \le C_R a^2(U + ap^2), \qquad |R'| \le C'_R(U + ap). \tag{13.31}$$

The exponents $q > 3$, $P > 4$ give, for $a \le 1$,

$$U + p \le (32h + 2048M_L)a.$$

As $p \le 1$, the right sides of (13.31) are bounded by fixed multiples of a^2 and a. Also $0 < x \le 1/2$, $x \le B_L a$, and $0 < x' \le (2L+3)/\sqrt{3} + 1$. The elementary identities

$$Q = \frac{x^2}{1-x}, \qquad Q' = \frac{x(2-x)x'}{(1-x)^2}$$

give $|Q| \le 2B_L^2 a^2$ and $|Q'| \le 8B_L[(2L+3)/\sqrt{3}+1]a$. Since $\xi = x + Q + R$, adding Theorem 13.6 proves the two errors in (13.29). The lower slope follows by making $|\xi' - x'| \le 1/(8\sqrt{3})$ and using $x' \ge 1/(4\sqrt{3})$. For the upper slope, compare directly with $F' \le (2L+3)/\sqrt{3}$ and make the total derivative error at most one. All these constants are fixed before q, P; only the final scale depends on them.

Finally, the initial polynomial family and the flow at the fixed time T are jointly smooth on ordinary neighborhoods of their parameters. The constraint height is positive there. The first downward crossing has a C^2 event germ, since $V > 1$ initially and the exit is transverse. Uniqueness of the first event identifies that germ with the event in (13.26) on a whole neighborhood. The map $f \mapsto \sqrt{f^2-1} - 1$ is smooth there because $f > 1$. The connection time and state are C^2 at their inputs with strictly positive walls. Composition gives the asserted joint regularity, including at the radial endpoints. □

To exchange the two small walls, use the section reflection

$$\mathsf{s}(s, \beta_2, \beta_3) = (-s, \beta_3, \beta_2).$$

It exchanges the two weak walls and commutes with the exact flow through the axis reflection S. Consequently (13.30) also reads

$$\iota_+(G_{\theta,\zeta}(\rho)) = \operatorname{proj} \phi_{t_{\mathrm{a}}} \mathcal{X}_h(\mathsf{s}y). \tag{13.32}$$

Both original section points therefore reach the same core coordinates at the same positive time, with the indicated axis frames. This identity will allow one statistical family to cover both possible orders of the small walls.

13.6. **Statistics along angular families.** The phase at the next entrance is strictly increasing along the angular radial coordinate. We can therefore transfer the statistical estimate to this coordinate, including at zero shear. To do so, we first extend each returned curve to an admissible graph and keep track of its amplitude cost.

For fixed h, L, q, P, a as above, put

$$I = [a, 2a], \qquad \Lambda_a = [-L, L] \times (0, a^P], \qquad B_{a,\zeta} = 6400 M_L \frac{\zeta}{a^3}. \tag{13.33}$$

For $\ell = (\theta, \zeta) \in \Lambda_a$, abbreviate the returned state and phase by G_ℓ and ξ_ℓ.

Proposition 13.10. *There are positive C^1 functions $\widetilde{u}_\ell, \widetilde{w}_\ell : \mathbb{R} \to \mathbb{R}$ such that*

$$\widetilde{u}_\ell + \widetilde{w}_\ell \le B_{a,\zeta}, \qquad |\widetilde{u}'_\ell| \le \sqrt{\widetilde{u}_\ell}, \qquad |\widetilde{w}'_\ell| \le \sqrt{\widetilde{w}_\ell}. \tag{13.34}$$

The graph $\Gamma_\ell(r) = (r, \widetilde{u}_\ell(r), \widetilde{w}_\ell(r))$ agrees with the actual output:

$$\Gamma_\ell(g_\ell(\rho)) = G_\ell(\rho), \qquad \widetilde{u}'_\ell(g_\ell(\rho)) = \frac{\widehat{u}'_\ell(\rho)}{g'_\ell(\rho)}, \qquad \widetilde{w}'_\ell(g_\ell(\rho)) = \frac{\widehat{w}'_\ell(\rho)}{g'_\ell(\rho)} \tag{13.35}$$

for all $\rho \in I$. The two extended amplitudes and their derivatives are jointly Borel in (ℓ, r). For any prescribed $B_* > 0$, the scale can be decreased so that, on all labels and radial parameters,

$$0 < B_{a,\zeta} \le B_*, \qquad g_\ell \ge 4, \qquad 0 < \xi_\ell < 1, \qquad \Gamma_\ell(2/\xi_\ell) = G_\ell. \tag{13.36}$$

Proof. The radius g_ℓ is C^2 and strictly decreasing on I, with nonzero ordinary derivatives at its endpoints. It has a continuous inverse j_ℓ on $J_\ell = [g_\ell(2a), g_\ell(a)]$, whose derivative is $1/g'_\ell \circ j_\ell$, including one-sided derivatives at the two endpoints. Define on this interval

$$u_\ell = \widehat{u}_\ell \circ j_\ell, \qquad w_\ell = \widehat{w}_\ell \circ j_\ell.$$

The cone inequalities (13.28) give $|u'_\ell| \le \sqrt{u_\ell}$ and $|w'_\ell| \le \sqrt{w_\ell}$. The sum of their amplitudes is at most $4096 M_L \zeta / a^3$. Apply the explicit extension of Lemma 8.13 with cone constant one. It preserves values and derivatives on the closed interval, and multiplies the sum bound by $25/16$. Since $(25/16)4096 = 6400$, this proves (13.34) and (13.35).

For measurability, the actual return is jointly C^2 on ordinary neighborhoods by Theorem 13.9. Thus the endpoints of J_ℓ are Borel functions of ℓ. The map $(\ell, \rho) \mapsto (\ell, g_\ell(\rho))$ is an injective Borel map between standard Borel spaces, and its inverse on its image is Borel. The transported radius graphs are consequently jointly Borel. The interior-sampling formulas (10.101)–(10.102) apply to these varying intervals: both endpoints are Borel, the interval lengths are positive, and the one-sided derivatives are continuous. They give Borel endpoint values and slopes, as well as the clamped interior values and slopes. The explicit square-root extension uses only these data and Borel case distinctions. It follows that the extensions and their derivatives are jointly Borel, as claimed. This argument also permits extension by zero off the Borel label set Λ_a.

Finally, $P > 4$ and $a \le 1$ imply

$$B_{a,\zeta} \le 6400 M_L a^{P-3} \le 6400 M_L a.$$

Choose the scale so this is at most B_*, and so that $a \le 1/(8B_L)$. The lower radius bound in (13.27) gives $g_\ell \ge 4$. Taking its reciprocal and using (13.35) proves (13.36). □

Fix the statistical parameters

$$0 < \eta \le \delta, \qquad J \ge 20, \qquad 0 < \sigma < 1, \qquad 2J\sigma < 1,$$

where $\delta \le 1/40000$ is as in Theorem 8.20. Choose its B_*, B_{cap} and use that B_* in Proposition 13.10. Define

$$\mathcal{F}_\ell = \{\rho \in I : (B_{a,\zeta}, G_\ell(\rho)) \notin \mathcal{S}_{\mathrm{stat}}\}. \tag{13.37}$$

Theorem 13.11. *At all sufficiently small scales, the relation $\rho \in \mathcal{F}_\ell$ is jointly Borel on $\Lambda_a \times I$. For every Borel $A \subset I$, nonnegative Borel weight ω and constant $M_\omega \ge 0$ with $\omega \le M_\omega$ on A,*

$$\int_{A\cap\mathcal{F}_\ell} \omega \, d\rho \le \min\left\{ \int_A \omega \, d\rho, 640\sqrt{3} M_\omega B^\sigma_{a,\zeta} \right\}. \tag{13.38}$$

For $\rho \in I \setminus \mathcal{F}_\ell$, each of the two section points

$$y = (\theta\rho, -\log\rho, -\log\zeta), \qquad \mathsf{s}y$$

has a statistical entrance at the positive actual time t_{a}. Their complete eras obey all six limits of Theorem 2.4.

Proof. The pair $(B_{a,\zeta}, G_\ell(\rho))$ is jointly Borel, so (13.37) is the inverse image of the complement of the common Borel statistical set. The graph Γ_ℓ is admissible at amplitude $B_{a,\zeta} \le B_*$, and (13.36) identifies it with the actual entrance on I. Theorem 13.9 gives $\xi'_\ell \ge 1/(8\sqrt{3})$. Apply Theorem 8.30, or its single-fiber estimate (8.99), with this slope and the excluded phase length $80B^\sigma_{a,\zeta}$. The coefficient is $80/(1/(8\sqrt{3})) = 640\sqrt{3}$.

Outside $\mathcal{F}_\ell$, the returned state belongs to $\mathcal{S}_{\mathrm{stat}}$ by its definition. Equations (13.30) and (13.32) give the two actual statistical entrances. The complete-era and time-divergence conclusions follow from Corollary 8.27. □

In particular,

$$\lambda(\mathcal{F}_{\theta,\zeta}) \le \min\{a, 640\sqrt{3}B^\sigma_{a,\zeta}\}. \tag{13.39}$$

The minimum retains the total radial length, which will control the integral over the angular and weak-wall labels.

13.7. **Flux of the angular excluded set.** Changing from shear to angle removes one logarithmic factor from the section measure. The remaining factor comes from the smaller wall. We use the smaller of the radial loss bound and the interval length. This bound is integrable against the remaining factor, and the resulting flux is summable over all small radial scales.

For $a > 0$ introduce

$$\Xi_a(\rho,\theta,t) = (\theta\rho, -\log\rho, -\log(2at)), \qquad \rho, t > 0. \tag{13.40}$$

The physical weak-wall label is $\zeta = 2at$. The derivative matrix and its determinant are

$$D\Xi_a = \begin{pmatrix} \theta & \rho & 0 \\ -1/\rho & 0 & 0 \\ 0 & 0 & -1/t \end{pmatrix}, \qquad \det D\Xi_a = -1/t. \tag{13.41}$$

The factor ρ in the shear coordinate cancels the radial logarithmic derivative $1/\rho$ exactly. The map is injective: its second and third coordinates determine ρ, t, and its first then determines θ. Thus, for any Borel set A in the domain $\rho, t > 0$, its image is Borel and

$$\mathrm{Leb}(\Xi_a(A)) = \int_A \frac{1}{t} \, d\rho \, d\theta \, dt. \tag{13.42}$$

This follows from the injective Borel mapping theorem and the change of variables formula.

Define the excluded set for the first wall order by

$$E^+_a = \{(\theta\rho, -\log\rho, -\log\zeta) : (\theta,\zeta) \in \Lambda_a, \ \rho \in \mathcal{F}_{\theta,\zeta}\}. \tag{13.43}$$

When $a \le 1$ and $P > 4$, every such label has $0 < \zeta < 2a$. Consequently $E_a^+ = \Xi_a(A_a)$, where

$$A_a = \{(\rho, \theta, t) : |\theta| \le L,\ 0 < t < 1,\ (\theta, 2at) \in \Lambda_a,\ \rho \in \mathcal{F}_{\theta,2at}\}.$$

Theorem 13.11 makes A_a Borel, and hence E_a^+ is Borel.

Put

$$K_{\mathrm{a}}(L, \sigma) = 640\sqrt{3}\,[102400(L+1)]^{\sigma}, \qquad C_1(K, \sigma) = \frac{1 + \log K + 2\sigma}{\sigma}. \tag{13.44}$$

For $L \ge 0$ and $\sigma > 0$, $K_{\mathrm{a}} \ge 1$. Since $M_L = 8(L+1)$,

$$B_{a,2at} = 102400(L+1)a^{-2}t, \qquad 640\sqrt{3}B_{a,2at}^{\sigma} = K_{\mathrm{a}}a^{-2\sigma}t^{\sigma}. \tag{13.45}$$

For any $K \ge 1$, $\sigma > 0$ and $0 < a \le 1$, the one-variable identity (12.11) gives

$$\begin{aligned} \int_0^1 \frac{\min\{a, Ka^{-2\sigma}t^{\sigma}\}}{t}\,dt &= \frac{a}{\sigma}\big[1 + \log K + (1+2\sigma)|\log a|\big] \\ &\le C_1(K, \sigma)a(1 + |\log a|). \end{aligned} \tag{13.46}$$

Here $Ka^{-2\sigma} \ge a$. For the inequality, expand $(1 + \log K + 2\sigma)(1 + |\log a|)$; the difference from the preceding bracket is $2\sigma + \log K\,|\log a|$.

Proposition 13.12. *Choose the statistical parameters and sufficiently small scales as in Theorem 13.11. If also*

$$0 < h \le 1/8, \qquad a \le h/16, \qquad 2La \le 1/8,$$

then

$$\mu_h(E_a^+) \le 6LC_1(K_{\mathrm{a}}, \sigma)a(1 + |\log a|). \tag{13.47}$$

The Borel symmetric set $E_a = E_a^+ \cup \mathsf{s}E_a^+$ satisfies

$$\mu_h(E_a) \le 12LC_1(K_{\mathrm{a}}, \sigma)a(1 + |\log a|). \tag{13.48}$$

Every point of either selected wall order outside E_a has the statistical entrance of Theorem 13.11.

Proof. Use (13.42) and Tonelli's theorem, integrating first in ρ. For a valid label, (13.39) and (13.45) bound that integral by $\min\{a, K_{\mathrm{a}}a^{-2\sigma}t^{\sigma}\}/t$. For an invalid label the integral is zero. Enlarging the label domain to $[-L, L] \times (0, 1)$ therefore gives

$$\mathrm{Leb}(E_a^+) \le 2L \int_0^1 \frac{\min\{a, K_{\mathrm{a}}a^{-2\sigma}t^{\sigma}\}}{t}\,dt.$$

All integrals are nonnegative, so no finiteness assumption is needed for this calculation.

On E_a^+, $|s| \le 2La \le 1/8$ and both weak walls are at most $2a \le h/8$. The density calculation in Lemma 12.2 uses precisely these smallness bounds and gives $w_h \le 3$ on the chart. More generally, any Borel set satisfying these bounds has μ_h-measure at most three times its Lebesgue measure, since μ_h is restricted to $\mathcal{D}_h$. Multiplying the preceding integral by three and applying (13.46) proves (13.47).

Reflection preserves μ_h by (12.7), giving (13.48). If $y = (\theta\rho, -\log\rho, -\log\zeta)$ is selected, membership in E_a^+ is equivalent to $\rho \in \mathcal{F}_{\theta,\zeta}$, by injectivity of the coordinate map. Since $\mathsf{s}E_a = E_a$, a point outside E_a has its radial parameter outside the corresponding $\mathcal{F}_{\theta,\zeta}$ in either wall order. The two orbit identities of Section 13.5 finish the proof. □

For $a_j = d2^{-(j+1)}$ with $0 < d \le 1$, the elementary geometric sums used in (12.19) give

$$\sum_{j \ge 0} a_j(1 + |\log a_j|) = d(1 + |\log d| + 2\log 2). \tag{13.49}$$

Indeed, $|\log a_j| = |\log d| + (j+1)\log 2$, and $\sum 2^{-(j+1)} = 1$, $\sum (j+1)2^{-(j+1)} = 2$. Thus the Borel set $E_d^{\mathrm{a}} = \bigcup_{j\ge 0} E_{a_j}$ has finite flux

$$\mu_h(E_d^{\mathrm{a}}) \le 12LC_1(K_{\mathrm{a}},\sigma)d(1+|\log d| + 2\log 2). \tag{13.50}$$

Every scale uses its own fixed time $T_j = (q/2)\log(1/a_j)$ and the same statistical parameters and cap. A single sufficiently small upper bound on d makes all the preceding estimates valid at every a_j.

13.8. **Statistical entrances on the whole small section.** The angular families cover the region where shear is at most a fixed multiple of the larger weak wall. Points not selected by the weak-wall power condition occupy a finite logarithmic strip. Adding that strip to the excluded set gives coverage of the entire angular region. Together with the dominant-shear construction, this covers the whole small section.

As before, write $n_i = e^{-\beta_i}$. For $L \ge 0$, $a > 0$ and $P \ge 0$, define

$$\mathcal{U}_a^{\mathrm{a}} = \{y : a \le \max(n_2,n_3) \le 2a, \quad |s| \le L\max(n_2,n_3), \quad a^P < \min(n_2,n_3)\}. \tag{13.51}$$

This is a Borel set. Under $h \le 1/8$, $a \le h/16$ and $2La \le 1/8$, it has flux

$$\mu_h(\mathcal{U}_a^{\mathrm{a}}) \le 24LP\log 2\, a|\log a|. \tag{13.52}$$

To prove this, first suppose $n_2 \ge n_3$. Its coordinates lie in the box

$$[-2La, 2La] \times [-\log(2a), -\log a] \times [0, P|\log a|].$$

The first wall interval has logarithmic width $\log 2$; the lower bound on the other wall gives the last interval. If $n_3 \ge n_2$, interchange the two logarithmic coordinates. Each box has volume $4LP\log 2\, a|\log a|$. On $\mathcal{U}_a^{\mathrm{a}}$ both walls are at most $2a \le h/8$ and $|s| \le 1/8$, so the density estimate in the proof of Proposition 13.12 gives the factor three. Summing the two boxes proves (13.52), also when $L = 0$. At the dyadic scales,

$$\mu_h\left(\bigcup_{j\ge 0} \mathcal{U}_{a_j}^{\mathrm{a}}\right) \le 24LP\log 2\, d(|\log d| + 2\log 2). \tag{13.53}$$

The full angular region is

$$\mathcal{B}^{\mathrm{a}}(d,L) = \{y : \max(n_2,n_3) \le d, \quad |s| \le L\max(n_2,n_3)\}. \tag{13.54}$$

The larger wall is strictly positive, so the half-open annuli $a_j < \max(n_2,n_3) \le 2a_j$ partition this set. Outside $\mathcal{U}_{a_j}^{\mathrm{a}}$, the smaller wall is at most a_j^P. There are exactly the two representations needed for the angular families. If $n_2 \ge n_3$, set

$$\rho = n_2, \qquad \zeta = n_3, \qquad \theta = s/n_2.$$

Then $y = (\theta\rho, -\log\rho, -\log\zeta)$. If $n_3 \ge n_2$, set $\rho = n_3$, $\zeta = n_2$, $\theta = -s/n_3$; then $y = \mathsf{s}(\theta\rho, -\log\rho, -\log\zeta)$. In either case $|\theta| \le L$, $0 < \zeta \le a_j^P$ and $\rho \in [a_j, 2a_j]$. No division by the shear is used.

Theorem 13.13. *There are statistical thresholds $\delta > 0$ and, for each $0 < \eta \le \delta$, $J \ge 20$ and $0 < \sigma < 1$ with $2J\sigma < 1$, a cap $0 < B_{\mathrm{cap}} < 1$ and a height $h_* > 0$ with the following property. For $0 < h \le h_*$, $L \ge 0$, $q > 3$ and $P > 4$, every sufficiently small $d > 0$ admits a Borel set F_d^{a} satisfying*

$$\mu_h(F_d^{\mathrm{a}}) \le K_A(L)\, d(1+|\log d| + 2\log 2), \qquad K_A(L) = 12LC_1(K_{\mathrm{a}}(L,\sigma),\sigma) + 24LP\log 2. \tag{13.55}$$

Every $y \in \mathcal{B}^{\mathrm{a}}(d,L) \setminus F_d^{\mathrm{a}}$ belongs to $\mathcal{D}_h$, and $\mathcal{X}_h(y)$ has a statistical entrance with these parameters and cap.

Proof. Choose the statistical constants from Theorem 13.11. After fixing h, L, q, P, choose d below the common radial threshold, $h/16$, and $1/[16(L+1)]$. All dyadic scales then satisfy the conditions of the preceding flux estimates. Put

$$F_d^{\mathrm{a}} = E_d^{\mathrm{a}} \cup \bigcup_{j\ge 0} \mathcal{U}_{a_j}^{\mathrm{a}}.$$

The two unions are Borel. Add (13.50) and (13.53), enlarging the latter logarithmic factor by one, to obtain (13.55).

For a point outside F_d^{a}, choose the annulus of its larger wall and use the representation above. The actual angular-return theorem places the corresponding point in the constraint chart. The chart domain is preserved by s, so the same holds in the other wall order. Exclusion from E_{a_j} gives the actual statistical entrance by Proposition 13.12. □

The angular and shear constructions may initially select different upper amplitude caps. They nevertheless use the same return branches when their statistical parameters η, J, σ agree.

Lemma 13.14. *Fix η, J, σ. If $B_1 \le B_2$, then the survivor set and the common statistical set for cap B_1 are contained in their counterparts for cap B_2. A common initial pair (B, Y) has the same finite histories, digits and phases under either cap. In particular a statistical entrance for cap B_1 remains a statistical entrance for cap B_2.*

Proof. In the definition of a selected input, the cap appears only in the inequality $B \le B_{\mathrm{cap}}$. Increasing it enlarges the domain of each selected input and leaves the actual branch and amplitude update unchanged. Induction over a finite word therefore enlarges every prefix domain. Taking the union over words of each length and then the intersection over lengths proves the survivor inclusion.

For a common surviving pair, both words of any given length belong to the prefix domains for cap $\max(B_1, B_2)$. Uniqueness of the finite history makes those words equal. Their successive letters, actual return states and reciprocal radius phases are consequently equal. In particular their continued fraction codes are identical. Membership of that code in the fixed Gauss typical set proves the inclusion of $\mathcal{S}_{\mathrm{stat}}$. The time and axis frame of a statistical entrance do not change, giving the last assertion. □

Define

$$\mathcal{B}(d) = \{y : |s| \le d, \quad \max(n_2, n_3) \le d\}. \tag{13.56}$$

This is Borel, and

$$\mathcal{B}(d) \subset \mathcal{B}^{\mathrm{a}}(d, 128) \cup \mathcal{B}^{\mathrm{s}}(d, 1/128). \tag{13.57}$$

Indeed, either $|s| \le 128 \max(n_2, n_3)$, or $|s| > 128 \max(n_2, n_3) > 0$ and both weak walls are at most $|s|/128$. The two regions meet along their common boundary, as shown in Figure 2. The distinction is between directions in the same physical section; it determines which parameter supplies a nondegenerate phase derivative.

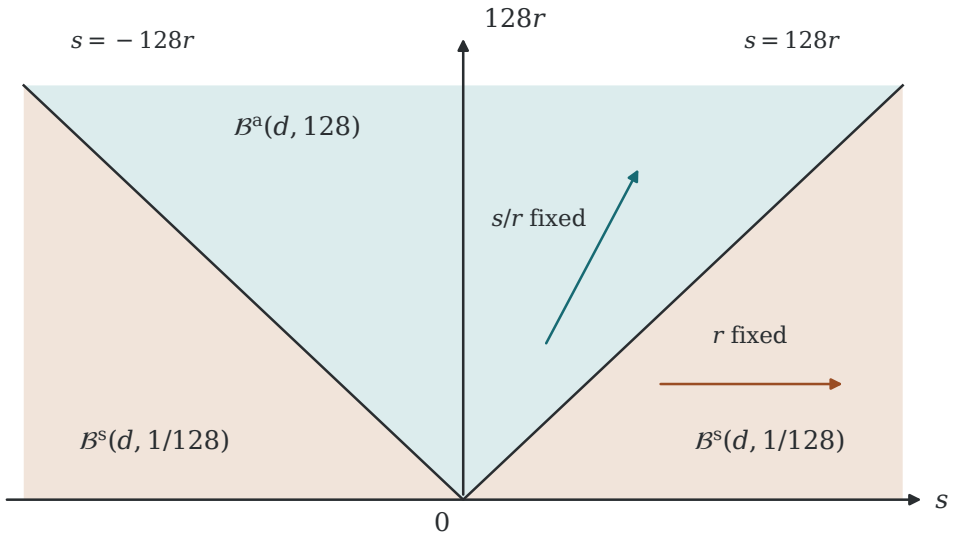


FIGURE 2. The two regions covering the small section. Write $r = \max(n_2, n_3) > 0$ and project the section coordinates to $(s, 128r)$. Above the two boundary rays, $|s| \le 128r$ gives the angular region $\mathcal{B}^{a}(d, 128)$; below them, $r \le |s|/128$ gives the shear region $\mathcal{B}^{s}(d, 1/128)$. Both include the boundary rays. The outer restrictions $|s| \le d$ and $r \le d$ are omitted to display the relative sizes. The arrows illustrate variation of s with both walls fixed and variation of the larger wall with s/r and the smaller wall fixed. The projection suppresses the smaller wall and its ordering; the construction treats both orderings.

Theorem 13.15 (Statistical entrances on the whole small section)**.** *There is* $0 < \delta \le 1/40000$ *such that, for every* $0 < \eta \le \delta$, $J \ge 20$ *and* $0 < \sigma < 1$ *with* $2J\sigma < 1$, *one can choose* $0 < B_{\mathrm{cap}} < 1$ *and* $h_* > 0$ *as follows. For every* $0 < h \le h_*$, $q > 3$, $D > 4$ *and* $P > 6D + 2$, *there are* $d_* > 0$ *and* $C > 0$ *such that every* $0 < d \le d_*$ *admits a Borel set* F_d *with*

$$\mu_h(F_d) \le Cd(1 + |\log d|)^2 < \infty. \tag{13.58}$$

Every $y \in \mathcal{B}(d) \setminus F_d$ *lies in* $\mathcal{D}_h$, *and* $\mathcal{X}_h(y)$ *has a statistical entrance for the common parameters and cap.*

Proof. Take the minimum of the angular and shear statistical thresholds. For the chosen η, J, σ, let their caps be B_A, B_S and set $B_{\mathrm{cap}} = \max(B_A, B_S) < 1$. Take the smaller height threshold and the smaller scale threshold from Theorems 13.13 and 12.15, with $L = 128$ and $\varepsilon = 1/128$. The exponent condition implies $P > 4$, as required for the angular part. Put $F_d = F_d^{a} \cup F_d^{s}$.

Let $K_S = 66C_{\log}(K_\sigma, \sigma) + 132P^2$ be the coefficient in (12.59). Since $\log 2 \le 1$,

$$1 + |\log d| + 2\log 2 \le 3(1 + |\log d|)^2.$$

Subadditivity gives (13.58) with $C = 3K_A(128) + K_S > 0$. For every point outside F_d, the cover (13.57) selects an angular or shear statistical entrance. Lemma 13.14 places it in the common cap without changing its orbit or statistical sequence. □

Theorem 13.15 supplies the finite-flux exceptional set needed for the capture argument of Section 12.6. It covers zero shear and arbitrarily unequal positive weak walls. We

next establish repeated visits in the finitely many axis frames and transfer the capture conclusion to arbitrary transverse sections.

14. Recurrence and global era statistics

It remains to establish the recurrence required for capture and to make one choice valid for all physical initial sections. Uniform avoidance of Taub would force bounded era digits, a null possibility by the local statistics. Deep Taub visits give repeated crossings of a fixed finite section family. We combine these crossings with capture and then transport a single exceptional set along the flow.

14.1. **Uniform Taub avoidance and bounded era lengths.** For sufficiently small transverse walls, uniform avoidance of the Taub points separates core radii from the ends of an era. We use this separation to continue its finite histories and bound their digits. The local coding estimates then give a null set of such core entrances.

Write $\mathcal{I}(r,u,w)$ for the physical state whose projective coordinates are $\iota_+(r,u,w)$. We say that a physical state has *uniform Taub avoidance at distance* ϵ if

$$\operatorname{dist}(\phi_t q, T_i) \geq \epsilon \qquad (t \geq 0,\ i = 1,2,3), \tag{14.1}$$

where distance is Euclidean in the normalized variables. This property is preserved by forward time shifts and by axis permutations.

Lemma 14.1. *There are $\delta > 0$ and $C \geq 1$ such that, for each $\epsilon > 0$, one can choose $0 < \rho \leq 1/16$ with the following property. If $Y = (r,u,w)$ satisfies*

$$u, w > 0, \qquad u + w \leq \frac{\delta}{1+r}, \qquad C(u+w) \leq \rho/4,$$

and $\mathcal{I}(Y)$ satisfies (14.1)*, then*

$$|r - 2n| > \rho/2 \qquad (n \geq 1). \tag{14.2}$$

Proof. First consider the single-wall orbit starting at radius two. On its descending part its projective state is

$$(1 - \sqrt{4 - v^2}, 0, v, 0), \qquad v \downarrow 0.$$

The inverse formula (2.7) shows that these states tend to T_2. Every positive v on this part is reached in finite positive original time. Choose one such time at which the distance is less than ϵ. Continuous dependence of the physical flow at this fixed time gives $0 < \rho \leq 1/16$ such that

$$|r-2| \leq \rho, \quad |u|, |w| \leq \rho \quad \Longrightarrow \quad \operatorname{dist}(\phi_T \mathcal{I}(r,u,w), T_2) < \epsilon. \tag{14.3}$$

This comparison uses a neighborhood of the initial state at the same fixed time T.

We next record the amplitude estimate that permits a finite sequence of reflections to reach this neighborhood. After decreasing δ, Lemma 8.1 applies whenever $r \geq 63/16$ and $b = u + w \leq \delta/(1+r)$. Its core exit radius obeys $|R - r| \leq 1/16$, so $R \geq 31/8$. If $G(Y) = (r', u', w')$ and $b' = u' + w'$, its conclusions give

$$3/2 \leq r - r' \leq 5/2, \qquad 0 < u' \leq u, \quad 0 < w' \leq w, \quad b' \leq e^{-c/r} b.$$

Since $1 - e^{-c/r} \geq (c/r)/(1 + c/r)$ and $r \geq 2$,

$$\frac{b}{r} \leq \frac{1 + c/2}{c}(b - b').$$

Thus the same lemma, with a larger $C \geq 1$, yields

$$|r' - (r-2)| \leq C(b - b'). \tag{14.4}$$

Each step is an actual positive-time passage, followed by an axis permutation, so it preserves (14.1).

Suppose now that $|r-2n| \le \rho/2$. Starting from $Y_0 = Y$, construct $Y_{j+1} = G(Y_j)$ for $j < n-1$, and put $b_j = u_j + w_j$. Induction using (14.4) gives

$$b_j \le b_0, \quad r_j \le r_0, \quad |r_j - (r_0 - 2j)| \le C(b_0 - b_j) \le \rho/4. \tag{14.5}$$

For each step that must be taken, $j \le n-2$, and hence $r_j \ge 4 - 3\rho/4 \ge 63/16$. Also $b_j \le \delta/(1+r_0) \le \delta/(1+r_j)$. These inequalities justify the induction. At $j = n-1$, (14.5) gives $|r_j - 2| \le 3\rho/4$, while $u_j, w_j \le b_0 \le \rho$. Equation (14.3) contradicts the avoidance property at this actual point of the orbit. The argument also covers $n = 1$, when no reflection is taken. □

For $N \in \mathbb{N}$ and $0 < \gamma < 1/4$, define the compact range of radii

$$\mathcal{K}_{N,\gamma} = \{(r,u,w) : 2 + 2\gamma \le r \le 2N, \quad \gamma \le r/2 - \lfloor r/2 \rfloor \le 1 - \gamma\}. \tag{14.6}$$

The transverse amplitudes are restricted separately.

Proposition 14.2. *For each $\epsilon > 0$, there are N, $0 < \gamma < 1/4$ and $B_0 > 0$ such that every $Y = (r,u,w)$ with*

$$r \ge 2 - \gamma/2, \qquad u, w > 0, \qquad u + w \le B_0,$$

whose physical state satisfies (14.1)*, belongs to $\mathcal{K}_{N,\gamma}$.*

Proof. For $r \ge 1$ and $0 \le u, w \le 1/100$, put $m = 1 + \sqrt{r^2 - 1}$ and $D_0 = 4 + m^2 + u^2 + w^2$. The denominator in (2.7) is $D = D_0 - 2(u + uw + w) \ge D_0/2 > 0$, and $D_0 \ge r^2$. Direct substitution gives

$$\operatorname{dist}(\mathcal{I}(Y), T_1)^2 = \frac{12 D_0}{D^2} \le \frac{48}{r^2}.$$

Thus avoidance implies $r < K := 8/\epsilon$.

Take ρ, C, δ from Lemma 14.1, and set

$$\gamma = \rho/4, \qquad N = \lceil K/2 \rceil, \qquad B_0 = \min\{1/100, \delta/(1+K), \rho/(4C)\}.$$

The assumed lower bound on r is greater than one, so the preceding estimate applies. The lemma excludes every positive even integer by a distance greater than $\rho/2$. At 2 the alternative below $2 - \rho/2$ is incompatible with $r \ge 2 - \rho/8$; hence $r \ge 2 + 2\gamma$. Dividing the separation inequalities by two separates $r/2$ from every positive integer by at least γ. Zero and the negative integers are already farther away. Applying these bounds to $\lfloor r/2 \rfloor$ and its successor proves (14.6). □

Lemma 14.3. *Fix the parameters of the selected era return, N, and $0 < \gamma < 1/4$. For sufficiently small $B_* > 0$, suppose $0 < B \le B_*$, $u, w > 0$, and $u + w \le B$. If the endpoint of every existing finite selected history from (B, Y) belongs to $\mathcal{K}_{N,\gamma}$, then (B, Y) survives indefinitely and all its selected digits are at most N.*

Proof. Choose $B_* \le B_{\mathrm{cap}}$, $B_* < 1$, below the era estimate threshold and such that

$$b^\sigma < \min\{\gamma/4, 1/(N+2)\} \qquad (0 < b \le B_*).$$

At an endpoint $X = (r,u,w) \in \mathcal{K}_{N,\gamma}$ with amplitude bound b, put $a = \lfloor r/2 \rfloor$ and $\beta = b^\sigma$. Then $1 \le a \le N$, $a - 1 < \lfloor 1/\beta \rfloor$, and the entrance inequalities (8.66) hold. The actual output phase y satisfies

$$|y - (r/2 - a)| < \beta/4.$$

Since $r/2 - a \in [\gamma, 1-\gamma]$, both $y - 2\beta$ and $1 - 2\beta - y$ exceed $7\gamma/16$. Thus the return is selected and its positive output walls have sum at most $b^{4/3}$.

Starting with the empty history, this extends a history of length n to one of length $n+1$. Its amplitude is $B_n = B^{(4/3)^n} \le B \le B_*$, and the assumed compactness applies at its endpoint. Histories therefore exist at every finite length. The definition (8.72) gives survival, and uniqueness of selected histories identifies these digits with the survivor digits. Their bound follows from $a \le N$ at every step. □

Proposition 14.4. *For each $\epsilon > 0$, there is $B_* > 0$ such that, for $0 < B \le B_*$, the set*

$$\mathcal{A}_{\epsilon,B} = \{(x,u,w) : 0 < x < 1,\ u,w > 0,\ u+w \le B,\ \mathcal{I}(2/x,u,w) \text{ satisfies (14.1)}\} \tag{14.7}$$

has three-dimensional outer Lebesgue measure zero.

Proof. Use Proposition 14.2 to choose N, γ. Choose B_* below its amplitude threshold and those for the actual era return and Lemma 14.3. Along every existing finite selected history the amplitude bound decreases. The actual orbit identity preserves avoidance, and the next radius is greater than two. Induction therefore places each endpoint in $\mathcal{K}_{N,\gamma}$. The lemma gives a survivor with bounded digits.

For any admissible graph, almost every survivor has a Gauss-typical code, by the coding measure estimate used in Theorem 8.20. Such a code cannot have bounded digits: for every fixed N, the limiting frequency of a digit at least $N+1$ is $\log(1+1/(N+1))/\log 2 > 0$. Consequently the bounded digit survivors have zero phase length on every admissible graph.

The set of bounded digit survivors in entrance state space is Borel. Indeed, on the Borel survivor set its defining condition is $\bigcup_N \bigcap_n \{a_n \le N\}$, and the digits are Borel by Proposition 8.18. Fixing positive u, w with $u + w \le B$ gives the admissible constant graph $r \mapsto (r, u, w)$. Its phase slice of this Borel set is null. All other slices are empty after imposing the amplitude conditions. Tonelli's theorem now gives a null set in (x, u, w) coordinates containing $\mathcal{A}_{\epsilon,B}$, as required. □

The nullity follows from the local statistics on continued survivors. The next two steps use product decay to find such entrances and transfer this nullity to initial sections.

14.2. **A growing wall away from Taub.** We now bring a physical orbit into the core coordinates of the preceding subsection. Throughout this step the orbit is assumed to stay at least $\epsilon > 0$ from every Taub point. Decay of the pairwise wall products will make the transverse core amplitudes as small as needed.

For $\Delta \ge 0$, write

$$\mathcal{P}_\Delta(q): \quad 4N_i(q)N_j(q) \le \Delta^2 \quad (i \ne j). \tag{14.8}$$

The constraint immediately gives, when $\Delta \le 1/4$,

$$p^2 + s^2 + N_1^2 + N_2^2 + N_3^2 \le 1 + \tfrac{3}{2}\Delta^2 \le 35/32, \qquad |p| < 17/16. \tag{14.9}$$

Lemma 14.5 (Leaving a dominant wall)**.** *Let $0 < \nu \le 1/4$ and $0 \le \Delta \le \nu/8$. If (14.8) holds on an interval of length $5/\nu^2$, then that interval contains a point at which all three walls are less than ν.*

Proof. If the walls are initially below ν, there is nothing to prove. Otherwise permute axes so that $n = N_1 \ge \nu$ initially, and write $b = N_2$, $c = N_3$. As long as $n \ge \nu$,

$$b + c \le \frac{\Delta^2}{2n} \le \frac{\nu}{128} \le \frac{n}{16}.$$

The first shear equation can be written exactly as

$$p' = (p-2)n^2 + (1+p)(b-c)^2 + (1-2p)n(b+c).$$

Using (14.9), the three terms are bounded above by $-15n^2/16$, $3n^2/256$, and $n^2/4$, respectively. Their sum is at most $-n^2/2 \le -\nu^2/2$. The bound $|p| < 17/16$ makes a stay of length $5/\nu^2$ impossible. There is therefore a time T in the interval with $n(T) < \nu$.

By continuity, between the initial time and T the wall takes the value $(\nu + n(T))/2 \in [\nu/2, \nu)$. At this time the product bounds give $b + c \le \Delta^2/\nu < \nu$. All three walls are then strictly below ν. □

Lemma 14.6 (One growing direction)**.** *For each $\epsilon > 0$, there are $0 < \nu_0 \le 1/4$ and $0 < c \le 1$ such that every constrained state with $0 \le N_i \le \nu_0$ and $\operatorname{dist}(q, T_i) \ge \epsilon$ has exactly one wall rate at least c; the other two are at most $-c$.*

Proof. Let $\lambda_i = N_i'/N_i$, interpreted by their polynomial formulas also when a wall vanishes. On the Kasner circle the rates are $-1 + 2p$, $-1 - p + \sqrt{3}s$, and $-1 - p - \sqrt{3}s$. The circle identity $p^2 + s^2 = 1$ shows that exactly one is positive and two are negative away from the three Taub points. For example, if $p > 1/2$ the first is positive and $3s^2 < (1 + p)^2$ makes the other two negative. If $-1 < p < 1/2$, the sign of s distinguishes the other two possibilities.

Define the continuous margin

$$m(q) = \max_i \min\big(\lambda_i(q), \{-\lambda_j(q) : j \ne i\}\big).$$

On the compact constrained set with $0 \le N_i \le 1$ and all Taub distances at least ϵ, the function $\max(m, N_1 + N_2 + N_3)$ is strictly positive. At a point where the wall sum is zero this follows from the Kasner calculation; elsewhere it follows from the sum itself. Choose a positive lower bound d, and put $\nu_0 = \min(1/4, d/6)$ and $c = \min(1, d)$. When $N_i \le \nu_0$, their sum is at most $d/2$, so $m \ge d$. This proves the assertion. If the compact set is empty, the assertion is vacuous and any positive sufficiently small constants suffice. □

Proposition 14.7 (A transverse fixed-height entrance)**.** *Choose ν_0, c as above. Let $0 < \nu \le \nu_0$ and $0 \le \Delta \le \nu/8$. If $q \in \mathcal{M}_{+++}$ satisfies uniform Taub avoidance and $\mathcal{P}_\Delta(\phi_t q)$ for all $t \ge 0$, then there are $s \in [0, 5/\nu^2]$, $T > s$, and an axis i such that all walls are less than ν at s, and T is their first subsequent arrival at height ν. Only the ith wall arrives, and*

$$N_i'(T) = \lambda_i(T)\nu \ge c\nu > 0.$$

Proof. Lemma 14.5 supplies s. Until the first height-ν arrival, Lemma 14.6 applies. Its growing index is constant on every connected time interval: a change would force one continuous rate to pass through zero, contrary to the gap. Hence one wall grows at least exponentially with rate c, and the other two decrease strictly.

There must be an arrival in finite time. Indeed, if no wall reached ν, the same index would remain growing for all future time and $N_i(t) \ge N_i(s)e^{c(t-s)}$ would contradict $N_i(t) < \nu$. Continuity of the maximum of the walls gives a first arrival, and the two decreasing walls remain strictly below their values at s. The rate gap persists at the endpoint, proving transversality. □

14.3. **Late entrances into a compact core range.** At the growing-wall entrance just obtained, the other walls are quadratically small in Δ. A projection preserving the dominant wall height puts this state near a compact family of exact single-wall solutions. Continuous dependence on initial data over a finite time interval then gives a core entrance on the original orbit.

Suppose $q = (p, s, n, b, c) \in \mathcal{M}_{+++}$, $0 < n \le 1/4$, and $\mathcal{P}_\Delta(q)$ holds with $\Delta \le n/8$. Define

$$\ell = \sqrt{\frac{1 - n^2}{p^2 + s^2}}, \qquad \bar{q} = (\ell p, \ell s, n, 0, 0). \tag{14.10}$$

The constraint gives

$$0 \le p^2 + s^2 - (1 - n^2) = 2n(b + c) - (b - c)^2 \le \Delta^2, \qquad 15/16 \le p^2 + s^2 \le 2.$$

Here nonnegativity follows from $b+c \le n/128 \le 2n$. Thus $0 \le \ell \le 1$, the projected state has constraint one, and $1-\ell \le 2\Delta^2$. Since $|p|, |s| \le 2$,

$$\begin{aligned} |\ell p - p|,\ |\ell s - s| &\le 4\Delta^2, \\ \operatorname{dist}(\bar q, q) &\le 2\Delta^2/n, \\ |\lambda_1(\bar q) - \lambda_1(q)| &\le 9\Delta^2. \end{aligned} \tag{14.11}$$

For the distance bound, each of the four nonzero coordinate errors is at most Δ^2/n, using $n \le 1/4$ and $b+c \le \Delta^2/(2n)$. For the last inequality use the exact identity

$$\lambda_1(\bar q) - \lambda_1(q) = p^2 + s^2 - (1-n^2) + 2(\ell p - p).$$

Suppose in addition that $\lambda_1(q) \ge c_0 > 0$, every Taub distance is at least ϵ, and

$$n^2 \le c_0/4, \quad \Delta^2 \le c_0/18, \quad n \le \epsilon^2/32, \quad 2\Delta^2/n \le \epsilon/2. \tag{14.12}$$

Then $\lambda_1(\bar q) \ge c_0/2$ and $\bar p > 1/2$. Order the two stable axes so that the transformed state is

$$z = \left(-\frac{\bar p + \sqrt{3}|\bar s|}{2}, \frac{\sqrt{3}\bar p - |\bar s|}{2}, 0, n, 0 \right).$$

Its height $A = 1 + z_p$ is positive. If its projective coordinates are $(M, 0, v, 0)$ and $k^2 = (M-1)^2 + v^2$, direct substitution gives

$$(M-1)A = 2\bar p - 1 > 0, \qquad k^2 A = 4A + 4\sqrt{3}|\bar s|. \tag{14.13}$$

In particular $k \ge 2$. Axis permutations preserve Euclidean distance. Since $\bar q$ is at least $\epsilon/2$ from all Taub points and $\operatorname{dist}(z, T_1)^2 = 2A$, we have $A \ge \epsilon^2/8$. Also $A \le 2$, and $(3 + M^2 + v^2)A = 6$. These formulas imply

$$2 \le k < 8/\epsilon, \qquad n/2 \le v \le 1/2, \qquad (M, 0, v, 0) = (1 + \sqrt{k^2 - v^2}, 0, v, 0). \tag{14.14}$$

For the upper bound on v, use $v = \sqrt{3}n/A$ and $n \le \epsilon^2/32$. For k, use $k^2 < 3 + M^2 + v^2$.

Lemma 14.8. *Fix K and $a > 0$. There are $\delta > 0$ and $C \ge 1$ such that every constrained physical state within Euclidean distance $e \le \delta$ of the reference family*

$$\mathcal{Q}(k, v) = \operatorname{proj}^{-1}(1 + \sqrt{k^2 - v^2}, 0, v, 0), \qquad 2 \le k \le K, \quad a \le v \le 1/2,$$

has a first positive upward $V = 1$ arrival in this frame, at a time $T \le C$. Its endpoint satisfies $A > 0$, $M > 1$,

$$|\sqrt{(M-1)^2 + 1} - k| \le Ce, \qquad N_2 \ge n_K := \frac{\sqrt{3}}{4 + (1+K)^2}. \tag{14.15}$$

The arrival is transverse, and its time and state are locally C^2 in ordinary physical initial coordinates.

Proof. Each reference solution reaches $V = 1$ with $M = 1 + \sqrt{k^2 - 1} > 1$. The entrance construction of Sections 6.4 and 6.5 gives a valid first arrival on an ordinary neighborhood of each reference initial state. Composing with the smooth projective coordinate maps gives C^2 time, projective endpoint, physical endpoint and radius there. The reference initial states form a compact set. Cover it by finitely many smaller balls whose doubled balls remain in these neighborhoods. Bound the derivatives of these four observables on the larger balls and apply the mean value estimate along line segments. A common positive radius and a common constant C then control their values and their differences from the reference values.

At the reference endpoint $M \ge 2$ and

$$N_2 = \frac{2\sqrt{3}}{4 + (1 + \sqrt{k^2 - 1})^2} \ge 2n_K.$$

Reduce δ until $C\delta \le \min(1/2, n_K)$. The physical wall lower bound and $M > 1$ follow. The actual first arrival, its direction, and positivity of A hold on the same neighborhoods by their construction. This proves all the assertions for one and the same event. □

Proposition 14.9. *Fix $\epsilon > 0$. There are N and $0 < \gamma < 1/4$ such that, for every $B > 0$, the following holds. If $q \in \mathcal{M}_{+++}$ eventually stays at least ϵ from all Taub points and*

$$N_i(\phi_t q)N_j(\phi_t q) \longrightarrow 0 \qquad (i \neq j), \tag{14.16}$$

then, after every prescribed time R, its orbit has a positive-time upward core entrance in some ordered frame, with

$$Y = (r, u, w) \in \mathcal{K}_{N,\gamma}, \qquad u, w > 0, \quad u + w \le B.$$

The physical core state satisfies uniform Taub avoidance at distance ϵ.

Proof. Choose N, γ, B_0 from Proposition 14.2. Choose the growth constants ν_0, c_0 of Lemma 14.6, and fix

$$0 < \nu \le \min\{\nu_0, c_0/4, \epsilon^2/32\}.$$

After a sufficiently late time $s_0 \ge \max(0, R)$, avoidance holds and $\mathcal{P}_\Delta$ holds forever, where $\Delta > 0$ may be chosen arbitrarily small before choosing s_0. Proposition 14.7 gives a later growing wall of height ν. Permute it to the first axis. For sufficiently small Δ, conditions (14.12) hold. The projection and ordering above put the actual state within $2\Delta^2/\nu$ of the compact reference family with $K = 8/\epsilon$ and $a = \nu/2$. Lemma 14.8 supplies the actual upward core, with a radius error at most $C_1\Delta^2$ and $N_2 \ge n_K$.

At this core, $V = 1$ means $A = \sqrt{3}N_2$. Consequently

$$u + w = \frac{N_1 + N_3}{N_2} \le \frac{\Delta^2}{2N_2^2} \le \frac{\Delta^2}{2n_K^2}.$$

Choose Δ further so that $C_1\Delta^2 \le \gamma/2$ and the last bound is at most $\min(B, B_0)$. The reference radius is at least two, so the actual radius is at least $2 - \gamma/2$. The actual endpoint inherits avoidance from the original orbit. Proposition 14.2 now places it in $\mathcal{K}_{N,\gamma}$. Both passages have positive elapsed time, so their final arrival is strictly later than R. All frames used are among the six fixed axis permutations. □

14.4. **Taub recurrence in transverse measure.** The late core entrance turns the null set of uniformly avoiding core states into a statement on every transverse section. We first explain the measure transfer, since an arbitrary union over arrival times would not preserve measure zero.

Lemma 14.10. *Let $C : U \to \mathcal{M}_{+++}$ be a C^1 transverse chart, with $U \subset \mathbb{R}^3$ open. Let Z be a null set in the coordinates of a core or fixed-wall section, in any ordered frame. The set of $y \in U$ whose orbit meets Z at a transverse arrival has outer Lebesgue measure zero. The arrival time may have either sign, provided it belongs to the maximal interval of the orbit.*

Proof. At any such pair (y_0, t_0), transversality at the target and the implicit function theorem give a local arrival time $t(y)$ and a C^1 map $R(y)$ to target coordinates. It is a local diffeomorphism. Indeed, differentiation of the actual arrival gives

$$D(C_{\text{target}} \circ R)[v] = D\phi_{t(y)}(C(y))DC[v] + F(\phi_{t(y)}C(y))Dt[v].$$

Contracting the constraint volume with F removes the second term. The remaining flux is a nonzero multiple of the initial transverse flux, by (12.29). Thus $\det DR \neq 0$, as in Proposition 12.8. A local C^1 inverse sends null sets to null sets, by covering its domain by countably many compact subsets on which it is Lipschitz. Hence $R^{-1}Z$ is locally null.

Cover the set of arrival pairs in $U \times \mathbb{R}$ by these open neighborhoods. Second countability gives a countable subcover. In each neighborhood the source coordinates of all relevant

pairs belong to the corresponding null set $R^{-1}Z$. Their countable union contains every source point under consideration. □

We will also use the four-dimensional version of this observation. If $C : U \subset \mathbb{R}^3 \to \mathcal{M}_{+++}$ is C^1 and $Z \subset U$ is null, then

$$\mathrm{Sat}(C(Z)) = \{\phi_t(C(z)) : z \in Z,\ t \in I(C(z))\} \tag{14.17}$$

is null in every regular four-dimensional chart of $\mathcal{M}_{+++}$. The relevant source points lie in $Z \times \mathbb{R}$, a four-dimensional null set by Tonelli. The orbit map is C^1 on its open domain. For any linear $P : \mathbb{R}^5 \to \mathbb{R}^4$, its composition with P therefore sends this null set to a null set, using a countable local Lipschitz cover. If C_4 is a regular four-dimensional chart, choose P to be a left inverse of DC_4 at a given point. Then $P \circ C_4$ is a local diffeomorphism, so the chart preimage of the saturation is locally null. A countable cover proves the assertion.

Proposition 14.11. *Fix $\epsilon > 0$. The set of $q \in \mathcal{M}_{+++}$ satisfying* (14.16) *and eventually remaining at distance at least ϵ from all Taub points is null on every C^1 transverse section and in every regular four-dimensional chart.*

Proof. Choose $B > 0$ so that Proposition 14.4 applies. Proposition 14.9 shows that each orbit in question reaches, in one of the six frames, a core whose phase coordinates (x, u, w) belong to $\mathcal{A}_{\epsilon,B}$. Its arrival is transverse.

The change from these coordinates to physical core coordinates is C^1 on $0 < x < 1$ wherever the inverse projective denominator is nonzero. The actual core points satisfy that condition, since $A > 0$. Hence their images in any core section coordinates form a null set. Apply Lemma 14.10 to each of the six frames. Their finite union proves the assertion on arbitrary transverse charts. For four-dimensional charts use (14.17) for the same six null sources. The inverse flow from the core entrance recovers the original point, so these saturations contain the whole set in the proposition. □

Let $\mathcal{G}$ consist of the Taub subspaces together with the eventually uniformly avoiding trajectories:

$$\begin{aligned}\mathcal{G} = \mathcal{T} \ \cup\ \{q \in \mathcal{M}_{+++} : &\text{ there exist } \epsilon > 0,\ R \geq 0 \text{ such that} \\ &\mathrm{dist}(\phi_t q, T_i) \geq \epsilon \text{ for every } t \geq R \text{ and } i\}.\end{aligned} \tag{14.18}$$

Theorem 14.12. *The set $\mathcal{G}$ is Borel and invariant throughout every maximal existence interval. It is null on every C^1 transverse section and in every regular four-dimensional chart. Every $q \in \mathcal{M}_{+++} \setminus \mathcal{G}$ satisfies* (14.16)*, lies outside all Taub subspaces, and has arbitrarily late visits to arbitrarily small neighborhoods of the Taub points.*

Proof. The Taub subspaces are closed relative to $\mathcal{M}_{+++}$ and invariant. For the first subspace, this also follows from $\mathcal{T}_1 = \{q : \mathsf{S}q = q\}$: commutation with the flow and uniqueness show that the fixed point condition holds at one time if and only if it holds at every existing time. The second set in (14.18) equals

$$\bigcup_{n,m\in\mathbb{N}} \bigcap_{t\geq m} \bigcap_{i=1}^{3} \{q \in \mathcal{M}_{+++} : \mathrm{dist}(\phi_t q, T_i) \geq 1/(n+1)\}.$$

For fixed n, m, the intersection is closed relative to $\mathcal{M}_{+++}$, because every positive-time flow map is continuous there. Only the two outer unions need be countable. This proves Borel measurability. The eventual condition is unchanged by any fixed shift of time within a maximal interval, proving invariance.

We verify nullity of $\mathcal{T}_1$ on a transverse chart C. At a point of this subspace, $s = 0$, $N_2 = N_3$ and $s' = 0$. If $D(s \circ C)$ vanished, every vector in the determinant defining the transverse flux, including $q/2$ and $F(q)$, would have zero s component. The flux would vanish, a contradiction. Thus each such point lies on a regular level of the scalar function $s \circ C$. The implicit function theorem and a countable cover make its zero set null in three

dimensions. Repeat in the other two frames. For four-dimensional nullity, parametrize the ambient first Taub subspace by $(p, n, b) \mapsto (p, 0, n, b, b)$. After any linear projection to $\mathbb{R}^4$, its image is the smooth image of a null coordinate hyperplane in $\mathbb{R}^4$. The same local inverse argument just used for (14.17) proves nullity in every regular chart. Axis permutations give the other two subspaces.

Outside $\mathcal{T}$, Proposition 2.2 supplies (14.16). For each n, apply Proposition 14.11 with $\epsilon = 1/(n+1)$. The countable union covers the eventually avoiding part of $\mathcal{G}$ outside $\mathcal{T}$, proving nullity. Finally, negating eventual avoidance says precisely that for every $\epsilon > 0$ and every R there are $t \geq R$ and i with $\operatorname{dist}(\phi_t q, T_i) < \epsilon$. □

Recurrence near Taub follows from product decay, the actual core construction, and the exclusion of bounded digit survivors.

14.5. **A fixed section before a deep Taub visit.** We locate a crossing of wall height h before a deep Taub visit by controlling a transverse radius backwards along the final part of the approach. The estimate remains valid when the two weak walls are equal or the shear vanishes.

Write $n = N_1$, $b = N_2$, $c = N_3$, and set

$$\mathfrak{r}^2 = s^2 + (b-c)^2, \qquad \mathcal{L} = \tfrac{1}{2}(\mathfrak{r}^2)' = ss' + (b-c)(b'-c'). \tag{14.19}$$

The zero set of $\mathfrak{r}^2$ is $\mathcal{T}_1$. On the constraint, substitution of the equations gives the polynomial identity

$$\begin{aligned}(1-p)\mathcal{L} = &-p(b-c)^2\mathfrak{r}^2 \\ &+ n\big[(n-2(b+c))((1-p)\mathfrak{r}^2 - (b-c)^2) + (1-p)\sqrt{3}s(b-c)\big].\end{aligned} \tag{14.20}$$

Lemma 14.13 (Backward control of the transverse radius)**.** *On an orbit interval where $p \leq -1/2$, $n \geq 0$ and $n^2+b^2+c^2 \leq 2$, the function $\mathfrak{r}^2 e^{-8n}$ is nondecreasing. If its endpoints are $a \leq b_0$ and $n(a) \leq h \leq 1/16$, then*

$$\mathfrak{r}^2(a) \leq 2\mathfrak{r}^2(b_0). \tag{14.21}$$

Proof. Put $A_0 = (1-p)\mathfrak{r}^2 - (b-c)^2 \geq 0$. The wall bound gives $b+c \leq 2$, so $n-2(b+c) \geq -4$. Also $\sqrt{3}s(b-c) \geq -\mathfrak{r}^2$, since the corresponding quadratic form is nonnegative. Therefore the bracket in (14.20) is at least $-5(1-p)\mathfrak{r}^2$. Its first term is nonnegative, and division by $1-p > 0$ yields $\mathcal{L} \geq -5n\mathfrak{r}^2$. The dominant wall rate satisfies

$$\lambda_1 = -p^2 - s^2 + 2p \leq -5/4.$$

Consequently

$$(\mathfrak{r}^2 e^{-8n})' = (2\mathcal{L} - 8\lambda_1 n\mathfrak{r}^2)e^{-8n} \geq 0.$$

Between the endpoints this gives $\mathfrak{r}^2(a) \leq \mathfrak{r}^2(b_0)e^{8(n(a)-n(b_0))}$. The exponential is at most $e^{1/2} \leq 2$, proving the bound. □

Proposition 14.14 (Crossing before a deep visit)**.** *Let $0 < h \leq 1/16$, $0 < d < h/8$, and $0 < \epsilon < d/4$. Suppose an orbit segment $[a, b_0]$ in $\mathcal{M}_{+++}$ satisfies*

$$\mathcal{P}_{d/4}(\phi_t q) \quad (a \leq t \leq b_0), \qquad \mathfrak{r}^2(\phi_a q) > 4\epsilon^2, \qquad \operatorname{dist}(\phi_{b_0} q, T_1) \leq \epsilon.$$

There is $c_0 \in (a, b_0)$ with

$$\begin{aligned} &N_1(c_0) = h, && p(c_0) < -1/2, && |s(c_0)| < d, \\ &N_2(c_0), N_3(c_0) < d, && \mathfrak{r}^2(c_0) \leq 2\mathfrak{r}^2(b_0), && N_1'(c_0) \leq -5h/4. \end{aligned} \tag{14.22}$$

Proof. The distance identity

$$\operatorname{dist}(q, T_1)^2 = (1+p)^2 + \mathfrak{r}^2 + n^2 + 2bc$$

implies at b_0 that $p < -1/2$, $n < h$, and $\mathfrak{r}^2 \leq \epsilon^2$. Consider the last closed interval $[c_0, b_0]$ on which $p \leq -1/2$ and $n \leq h$. More explicitly, apply continuity to $f(t) =$

$\max(p(t)+1/2, n(t)-h)$ and take the left endpoint of its final nonpositive interval. Either $c_0 = a$ or $f(c_0) = 0$.

Equation (14.9) bounds the sum of wall squares by two throughout the segment. Lemma 14.13 therefore gives

$$\mathfrak{r}^2(c_0) \le 2\mathfrak{r}^2(b_0) \le 2\epsilon^2.$$

The initial radius condition rules out $c_0 = a$. At c_0 the constraint reads

$$p^2 + \mathfrak{r}^2 + n^2 - 2n(b+c) = 1.$$

If $p = -1/2$, its left side is at most $1/4 + 2\epsilon^2 + h^2 < 1$, a contradiction. Thus $p < -1/2$ there, and the boundary condition forces $n = h$.

The radius bound gives $|s| < d$ and $(b-c)^2 \le 2\epsilon^2$. The product bound gives $4bc \le d^2/16$. Hence

$$(b+c)^2 = (b-c)^2 + 4bc \le 2\epsilon^2 + d^2/16 < d^2.$$

Both weak walls are less than d. Finally $n' = \lambda_1 h \le -5h/4$, which also makes this a transverse downward crossing of the fixed main wall. □

Corollary 14.15. *Suppose $q \in \mathcal{M}_{+++}$ has $\mathfrak{r}^2(q) > 0$, satisfies (14.16), and approaches T_1 arbitrarily closely at arbitrarily late times. For every $0 < h \le 1/16$, $0 < d < h/8$, $R \in \mathbb{R}$ and $\zeta > 0$, there is a time $c_0 > \max(R, 0)$ satisfying (14.22), with $\mathfrak{r}^2(c_0) < \zeta$.*

Proof. Choose $a > \max(R, 0)$ so late that $\mathcal{P}_{d/4}$ holds from a onwards. The radius at a is positive: its zero set is the fixed point set of S, and flow uniqueness makes that set invariant in both time directions. Choose

$$0 < \epsilon \le \min\{d/8, \mathfrak{r}(a)/4, \sqrt{\zeta}/4\}.$$

Take $b_0 \ge a + 1$ with $\operatorname{dist}(\phi_{b_0} q, T_1) < \epsilon$ and apply Proposition 14.14. Its radius estimate is at most $2\epsilon^2 < \zeta$, and its crossing time is greater than a. □

14.6. **Repeated visits to the small section family.** We use three copies of the small section, one for each choice of dominant wall. This finite family accommodates the Taub point visited by an orbit without changing the section height or its transverse size along that orbit.

For $i = 1, 2, 3$, let A_i be the axis permutation interchanging axis 1 with axis i, with A_1 the identity. These are among the six ordered frames and are involutions. Define

$$\mathcal{X}_{i,h}(y) = A_i^{-1}\mathcal{X}_h(y), \qquad E = \mathcal{B}(d) \cap \mathcal{D}_h, \qquad \Sigma_{h,d} = \bigcup_{i=1}^{3} \mathcal{X}_{i,h}(E), \tag{14.23}$$

where $\mathcal{B}(d)$ is (13.56). We retain the label i when using section coordinates, and give the disjoint coordinate space $\{1, 2, 3\} \times E$ the measure

$$\mu_{\mathrm{ax}}(H) = \sum_{i=1}^{3} \mu_h(H_i), \qquad H_i = \{y : (i, y) \in H\}. \tag{14.24}$$

For $d < h/8$ the physical images are disjoint: in the ith image the ith wall is h, while both other walls are less than h. Each individual chart is injective, so the labeled chart is injective as well.

Theorem 14.16 (Recurrence of the section family)**.** *Let $0 < h \le 1/16$ and $0 < d < h/8$. Every $q \in \mathcal{M}_{+++} \setminus \mathcal{G}$ meets $\Sigma_{h,d}$ at arbitrarily late times. More precisely, there is a fixed i such that, for every R and $\zeta > 0$, some $t > \max(R, 0)$ has*

$$\phi_t q = \mathcal{X}_{i,h}(s, \beta_2, \beta_3), \qquad |s| < d, \quad e^{-\beta_2}, e^{-\beta_3} < d,$$

and the squared transverse Taub radius in this frame is less than ζ. In particular, for μ_{ax}-almost every point of the labeled section family, the original flow has infinitely many future intersections with this family.

Proof. By Theorem 14.12, the orbit has pairwise product decay and arbitrarily late visits to the union of arbitrarily small Taub neighborhoods. At least one fixed Taub point has this property. Otherwise, for each of the three points one could choose a radius and a starting time after which its neighborhood is never visited. The minimum of the radii and the maximum of the starting times would contradict the visits to the union.

Use A_i to send this fixed Taub point to T_1. It preserves the Euclidean metric, permutes the wall products, and commutes with the flow. The initial transverse radius in this frame is positive, since $q \notin \mathcal{T}$. Corollary 14.15 gives the asserted crossings in that frame. Their negative shear coordinate p and the constraint reconstruct exactly the negative square root in $\mathcal{X}_h$. Taking $\beta_j = -\log N_j$ therefore gives the claimed chart coordinates.

Each $\mathcal{X}_{i,h}$ is a smooth transverse chart on $\mathcal{D}_h$. The pullback of $\mathcal{G}$ is null by Theorem 14.12, and remains null for μ_h, whose density is positive and finite there. Outside these three null sets, the first part gives unbounded future hitting times and hence infinitely many intersections. □

This supplies the recurrence required by the capture argument. The remaining step is to apply capture on the labeled section family, choose its successful core entrances measurably, and remove one invariant null set for all initial sections at once.

14.7. **Measurable statistical entrances on the whole section.** The local constructions give actual statistical entrances together with their times and axis frames. We now choose these data on the whole small section. Two details are useful: the initial amplitude can be made smaller than any prescribed positive bound, and all successful points use one Borel choice of entrance data.

Lemma 14.17 (Restriction of the amplitude cap)**.** *Suppose (B, Y) is a survivor with cap B_0, and $0 < B \le B_1 \le 1$. Then it is a survivor with cap B_1, with identical digits, states and phases. In particular, a statistical survivor remains statistical after this change of cap.*

Proof. The selected entrance and return conditions (8.66)–(8.67) depend on the cap only through the upper bound on the amplitude. At the end of a history of length n that amplitude is $B^{(4/3)^n} \le B \le B_1$. Thus every existing history for cap B_0 is also a history for cap B_1, with the same return states. This proves survival. Uniqueness of selected histories identifies the digits and phases. Equivalently, compare both histories under the larger cap, as in Lemma 13.14. □

Fix the era parameters in their allowed range. A statistical entrance datum at a physical state q is a tuple

$$(A, T, B, Y), \qquad T > 0, \quad (B, Y) \in \mathcal{S}_{\mathrm{stat}}, \qquad \iota_+(Y) = \mathrm{proj}(A\phi_T q), \tag{14.25}$$

where A is one of the six ordered frames.

Proposition 14.18. *There are $B_{\mathrm{cap}} \in (0, 1)$ and $h_* > 0$ such that, for each $0 < h \le h_*$ and each $b_* > 0$, one can choose $d > 0$ with $d < h/8$, a Borel set $F \subset \mathcal{B}(d)$ with $\mu_h(F) < \infty$, and a Borel function W assigning statistical entrance data to every $\mathcal{X}_h(y)$ with $y \in \mathcal{B}(d) \setminus F$. Its amplitude satisfies $B(y) \le b_*$.*

Proof. Fix the exponents $q = 50$, $D = 90$, $P = 550$ in the small-section constructions. They satisfy $q > 3$, $D > 4$, $P > 6D + 2$. Take $\varepsilon = 1/128$ and $L = 128$, and choose a common cap for the shear and angular statistical sets as in Theorem 13.15.

We specify a countable partition on which the actual entrance formulas can be used. In the open section

$$\mathcal{B}^\circ(d) = \{y : |s| < d,\ \max(n_2, n_3) < d\}, \qquad n_j = e^{-\beta_j},$$

use shear coordinates when $\max(n_2, n_3) \le \varepsilon |s|$, and angular coordinates otherwise. The radial variable is $r = |s|$ in the first case and $r = \max(n_2, n_3)$ in the second. Discard the

countable set of radial levels $r = d2^{-j-1}$, and use the unique open interval $(a, 2a)$ with $a = d2^{-j-1}$ containing r. In the shear part the sign of s distinguishes two pieces. In the angular part take the order $n_2 \geq n_3$ on one piece and $n_3 > n_2$ on the other. Their parameters are respectively

$$(r, \theta, \zeta) = (n_2, s/n_2, n_3), \qquad (r, \theta, \zeta) = (n_3, -s/n_3, n_2).$$

Here $|\theta| < L$. The negative shear piece uses the reflected positive shear family, including the exchange of its two wall labels. These are disjoint Borel pieces with explicit Borel inverses.

On each piece exclude points where the smaller-wall selection condition $\min(n_2, n_3) \leq a^P$ fails, or where the output pair (B, Y) is outside the common statistical set. These conditions are Borel by the joint regularity and measurability proved for the two families. An excluded point satisfying the selection condition belongs to the corresponding radial excluded set estimated in Sections 12.7 and 13.8; an unselected point belongs to the finite logarithmic regions estimated there. Cap monotonicity ensures the same containment when the common cap is larger than the cap used for either estimate.

Add the discarded radial levels and the outer boundary $\mathcal{B}(d) \setminus \mathcal{B}^\circ(d)$ to F. These sets are null in logarithmic coordinates. Each radial level is contained in the union of the four coordinate levels $s = \pm a$, $n_2 = a$, $n_3 = a$; the outer boundary is contained in the analogous levels with $a = d$. Thus they also have zero μ_h measure. The regional bounds give

$$\mu_h(F) \leq C_h d(1 + |\log d|)^2 < \infty.$$

Decrease d so that the whole section lies in $\mathcal{D}_h$ and all local constructions apply.

On a successful shear piece the initial amplitude bound is $800 n_2 n_3/a^4 \leq 800a$. On an angular piece it is $6400 M_L \zeta/a^3 \leq 51200(L+1)a$, with $M_L = 8(L+1)$. Since $a \leq d$, decreasing d once more makes both bounds at most b_*. This does not change the common cap.

The entrance time, state and amplitude are jointly Borel on each selected parameter domain; the frame takes one of finitely many values. Compose them with the explicit piece inverses and use the corresponding formula on each successful piece. A countable Borel partition makes the resulting function W Borel. Extend it by a fixed tuple elsewhere. The orbit identities for both shear signs and both wall orders give (14.25) at every successful point. □

The partition gives a Borel choice of entrance and preserves the return identities and measure estimates on each piece.

14.8. **Global capture and a common invariant exception.** Choose $h \leq 1/16$ and $d < h/8$ as in Proposition 14.18. In the labeled section family (14.23), let

$$\mathcal{F} = \{(i, y) : y \in E \cap F\}, \qquad \mathcal{H} = (\{1, 2, 3\} \times E) \setminus \mathcal{F}.$$

The excluded set has finite measure, since $\mu_{\mathrm{ax}}(\mathcal{F}) \leq 3\mu_h(F) < \infty$. Every point of $\mathcal{H}$ has the Borel statistical entrance obtained by conjugating $W(y)$ by its axis frame.

Lemma 14.19. *For μ_{ax}-almost every point of the labeled section family with infinitely many future intersections, some nonnegative time reaches $\mathcal{H}$.*

Proof. The first-return construction of Section 12.5 applies to this finite family. To check this explicitly, choose the source and target labels in each local arrival relation. There are only nine such pairs. The negative shear bound on every target is uniform, so the crossings have no finite-time accumulation, just as in Lemma 12.9. Their first positive arrival is attained. The countable event neighborhoods and the least-time selection therefore give a Borel first return R.

The physical section images are disjoint, so orbit uniqueness makes this labeled return injective. Axis permutations preserve the constraint volume and flux. On each local branch

its weighted Jacobian j is consequently the same positive-time multiplier as in (12.34), and $j > 1$. Partitioning the return domain into countably many Borel branch pieces gives

$$\mu_{\mathrm{ax}}(R(B)) = \int_B j \, d\mu_{\mathrm{ax}}$$

for every Borel subset B of that domain.

Apply the capture argument of Section 12.6 to $\mathcal{F}$. The set whose legitimate iterates all remain in $\mathcal{F}$ is forward invariant and has finite measure. Strictness of $j > 1$ forces its measure to be zero. Its finite backward iterates are also null by the change of variables formula. Finally, infinitely many actual intersections are equivalent to indefinitely continuing first returns: local finiteness orders the hitting times, and on a bounded interval permits only finitely many of them. Thus almost every recurrent point reaches the successful set, possibly at time zero. □

Lemma 14.20. *If $Z \subset \mathcal{D}_h$ is Borel and has zero three-dimensional Lebesgue measure, then $\mathrm{Sat}(\mathcal{X}_{i,h}(Z))$ is a Borel invariant subset of $\mathcal{M}_{+++}$, null on every transverse section and in every regular four-dimensional chart.*

Proof. The domain of the map

$$(y, t) \longmapsto \phi_t(\mathcal{X}_{i,h}(y))$$

is open, and the map is locally injective. At time zero its derivative has the three independent section tangents and the complementary flow direction; at any other existing time the flow transports this rank-four property. The inverse function theorem in constraint coordinates gives local injectivity. Cover the domain by countably many such open sets. On each, the image of its Borel intersection with $Z \times \mathbb{R}$ is Borel, by the Borel image theorem for continuous injective maps between standard Borel spaces. Their union is the full saturation, so it is Borel.

The maximal-flow composition and inverse identities show that membership in the saturation is unchanged at every existing time. It lies in $\mathcal{M}_{+++}$ because the physical phase space is invariant. Its transverse nullity follows from Lemma 14.10, applied to the reverse arrival from an arbitrary transverse section to $\mathcal{X}_{i,h}(Z)$. Its four-dimensional nullity is (14.17). □

Theorem 14.21 (Global statistical entrance)**.** *For every prescribed $b_* > 0$, there are a Borel invariant set $\mathcal{Z} \subset \mathcal{M}_{+++}$, containing $\mathcal{G}$ and null on all transverse sections and in all regular four-dimensional charts, and a Borel choice of statistical entrance data* (14.25) *for every $q \in \mathcal{M}_{+++} \setminus \mathcal{Z}$, with amplitude $B(q) \le b_*$.*

Proof. For a labeled section point $z = (i, y)$, consider the property that pairwise product decay along its orbit implies a nonnegative-time arrival in $\mathcal{H}$. Theorem 14.16 and Lemma 14.19 imply this property almost everywhere on the section family. For each label choose a Borel Lebesgue-null set $Z_i \subset E$ containing every point where it fails. Such a hull exists because the points without such an arrival have outer measure zero; zero flux and zero coordinate volume agree on $\mathcal{D}_h$, where w_h is positive. Define

$$\mathcal{Z} = \mathcal{G} \cup \bigcup_{i=1}^{3} \mathrm{Sat}(\mathcal{X}_{i,h}(Z_i)).$$

Theorem 14.12 and Lemma 14.20 give all its stated properties.

If $q \notin \mathcal{Z}$, Theorem 14.16 gives a future intersection $\phi_s q = \mathcal{X}_{i,h}(y)$, $y \in E$. This point is outside the saturation of Z_i, and hence $y \notin Z_i$. The orbit has product decay by Proposition 2.2, and this condition is unchanged by a finite time shift. By the defining property outside Z_i, a later nonnegative time reaches $\mathcal{H}$. Thus every $q \notin \mathcal{Z}$ reaches the same fixed successful target.

It remains to make the entrance choice Borel in q. The time-zero case is retained. For any physical point reaching $\mathcal{H}$, its nonnegative hitting times have an attained least element: in a compact time interval the uniformly transverse target family has only finitely many crossings. The local implicit arrival functions cover its spacetime hitting relation by countably many event neighborhoods. Restrict each branch to the Borel condition that its endpoint belongs to $\mathcal{H}$. Taking their infimum, and the first branch attaining it, gives a Borel least time $\tau(q) \geq 0$ and Borel arrival state $\phi_{\tau(q)}q$. This is the construction of Section 12.5 with an arbitrary physical source and the finite target family.

The labeled section chart is continuous, injective and Borel on $\{1,2,3\} \times E$; its inverse on its image is Borel. It therefore recovers a Borel pair $(i(q), y(q)) \in \mathcal{H}$ from the arrival state. At that point use $W(y(q))$ from Proposition 14.18. If it is (A_0, T_0, B, Y), set

$$A = A_0 A_{i(q)}, \qquad T = \tau(q) + T_0.$$

The product frame is one of the same six frames, and all these data are Borel. Commutation with the flow gives

$$\mathcal{I}(Y) = A_0 \phi_{T_0} \mathcal{X}_h(y(q)) = A_0 A_{i(q)} \phi_{\tau(q)+T_0} q.$$

Thus (14.25) holds. The total time is strictly positive because $T_0 > 0$, even when $\tau = 0$. The chosen initial amplitude is at most b_*. □

14.9. **Proof of the statistical theorem.** The whole-section estimate supplies finite excluded flux, recurrence supplies repeated visits, and strict expansion gives capture. Theorem 14.21 has combined these results into a measurable statistical entrance of arbitrarily small amplitude. We now join its selected returns into one infinite sequence of complete eras and transfer their statistics to the original trajectory.

Proof of Theorem 2.4. Choose $\eta > 0$ below the finitely many thresholds in the era, section and entrance results, with $\eta \leq 1/40000$. Set

$$J = 45, \qquad \sigma = 1/400.$$

These satisfy $J \geq 20$ and $2J\sigma < 1$. Choose a positive amplitude cap $b_* < 1$ below the thresholds for the Borel selected return, the actual event itineraries, the ordinary return-time functions and the absence of finite-time accumulation in Section 8.9. Theorem 14.21 gives one invariant null set $\mathcal{Z}$ and Borel entrance data (A, T, B, Y) on its complement, with $B \leq b_*$.

The statistical survivor in those data was obtained with the cap used by the section construction. Apply Lemma 14.17 to replace that cap by b_*. Its digits, states and phases are unchanged. The Borel map $q \mapsto (B(q), Y(q))$ takes values in this one Borel survivor set. Proposition 8.18 therefore gives Borel digits $a_n(q)$, states $Y_n(q)$ and phases $x_n(q)$.

Starting at the positive original time $T(q)$ in the frame $A(q)$, Propositions 8.22 and 8.25 assemble these states into one actual orbit with Borel frames and entrance times. Every selected return consists of its ordered cores, reflections and terminal connection, by Proposition 8.21. The times are positive and strictly increasing, and they tend to infinity by Theorem 8.24.

Delete the first selected era. Each remaining era then has its preceding terminal connection on this same trajectory, so it is complete in the sense of Definition 2.3. Corollary 8.27 gives the required sequences by setting $K_m = a_{m+1}$ and using the corresponding shifted states, frames and times. At every entrance the state is $\iota_+(Y_{m+1})$, and its radius is the first coordinate of Y_{m+1}. The selected entrance inequalities give

$$K_m = \left\lfloor \frac{Y_{m+1,0}}{2} \right\rfloor, \qquad Y_{m+1,0} > 2, \qquad x_m = \frac{2}{Y_{m+1,0}} \in (0,1).$$

These are precisely (2.13) and the physical observables (2.10).

The statistical entrance lies in the common statistical set, so its selected digits and phases satisfy all six limits in the theorem. A fixed shift preserves those limits, including the divergent nonnegative mean, by Lemma 8.26. The same limits thus hold for the complete eras just chosen.

Finally, $\mathcal{Z}$ contains $\mathcal{T}$ because it contains $\mathcal{G}$. Its invariance and its nullity in every transverse and four-dimensional chart are already part of Theorem 14.21. Every measure obtained by putting a nonnegative density on coordinate Lebesgue measure is absolutely continuous with respect to it. Hence the same $\mathcal{Z}$ is null for every smooth positive density in the statement. The statistical set was chosen using all finite cylinders and the single common Gauss-typical set, before the initial section or observation function was specified. This proves all assertions on the same complement of $\mathcal{Z}$. □

15. Consequences and further questions

The Gauss law describes a sequence of events on the exact vacuum Einstein trajectory. In particular, the long-era tail and the logarithmically growing truncated mean count actual curvature cores. Almost every physical initial condition, in transverse or four-dimensional measure, admits such an era tail. Short eras carry a substantial fraction of the frequency, while the rare long eras prevent the arithmetic mean from being finite.

The joint distributions describe correlations along each typical orbit. Corollary 2.5 gives the adjacent unit-era frequency $\log(10/9)/\log 2$, while the general cylinder formula determines every finite block. The same scalar distributions occur in the billiard description [5]; the present conclusion obtains them for the exact normalized flow with a common invariant exceptional set. The conclusion applies to every initial distribution that is absolutely continuous in local coordinates.

Two parts of the proof account for its range near the Taub points. Within an era, phase errors are summed against the loss of transverse amplitude, so the error bound does not grow with the number of cores. On the small section, shear and angular families cover complementary directions, including zero shear and arbitrarily unequal weak walls. Their remaining excluded set has finite flux. Strict expansion of that flux and actual recurrence then turn a local statistical construction into a full-measure statement.

The nearest further questions concern duration and fluctuations: statistics weighted by the time spent in cores and connections, and rates of convergence of the empirical frequencies.

The full axis sequence retains information discarded by the scalar era phase. Its joint statistics, together with extensions to matter models or nonzero cosmological constant, provide further directions for research. The exact event construction supplies observables for these questions.

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OpenAI AI tools were used to explore possible proof strategies, implement the Lean 4 formalization, and polish the manuscript. During formalization, these tools suggested proof approaches that facilitated formal verification. These approaches were incorporated into the manuscript to keep the written and formalized proofs consistent. The author checked the assumptions and conclusions of the formalization and takes responsibility for both the manuscript and the formalized results.

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Data availability

All mathematical results and proofs established in this article, starting from the normalized vacuum Bianchi IX equations, have been formalized in Lean 4. The formalization retains exactly two explicit literature hypotheses: Birkhoff's pointwise ergodic theorem in its probability-space form [11, Corollary 10.1.2], and the decay of curvature-wall products obtained from Brehm's attractor theorems [4, Theorems 2–3, pp. 44–45]. The complete source code and verification instructions are publicly available at `https://github.com/l46430640-del/ERA-STATISTICS-FORMAL`.

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