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Bures Geometry of Projection States

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Abstract

We study the Bures metric on states obtained by conditioning a faithful normal state on nonzero projections of a von Neumann algebra. The metric determines the reference state, projection probabilities, orthogonality, and order. For algebras with no type I_2 direct summand, surjective isometries are exactly those induced by reference-state-preserving normal Jordan *-isomorphisms; the same conclusion holds for $M_2(\mathbb{C})$. A semifinite tracial version determines the trace up to a single positive factor. In finite tracial algebras, we characterize optimal finite projective measurements and express their fidelity deficit as a sum of squared L^2 errors. Spectral measurements with at most $n+1$ outcomes approximate fidelity with error $O(n^{-2})$; the interval grid is minimax optimal, and the order is sharp within spectral measurements. We compare the fidelity defect of a common commutative context with approximation by commuting projections, obtain sharp spectral rounding constants, and solve both optimization problems in the algebra generated by two projections. The space of normalized finite-trace projection states is complete; in a finite tracial algebra its rectifiable components are unitary orbits, and within each orbit the intrinsic Bures distance is the root mean square of the principal angles. Finally, a fidelity bound on local unitary orbits characterizes tracial reference states.

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1 Introduction

A projection in a von Neumann algebra represents an event in noncommutative probability and a point in a Grassmannian. Its order and orthogonality encode algebraic information, while a state assigns its probability. These viewpoints originate in quantum logic and continuous geometry [8, 26]. We study the Bures metric on the states obtained by conditioning a fixed state on nonzero projections. We ask which algebraic data this metric determines, how finite measurements attain or approximate fidelity, and what intrinsic length it induces on the projection family.

If ω is a faithful normal state on a von Neumann algebra M, its projection-conditioned family is

$$\mathcal{L}(M,\omega) = \left\{ \omega_p : \ \omega_p(x) = \frac{\omega(pxp)}{\omega(p)}, \quad 0 \neq p \in \mathcal{P}(M) \right\}.$$

We use root fidelity F, so the Bures metric is $d_B^2 = 2 - 2F$. When the reference state is a trace τ, we write $\varphi_p(x) = \tau(px)/\tau(p)$. The resulting formula

$$F(\varphi_p, \varphi_q) = \frac{\tau(|pq|)}{\sqrt{\tau(p)\tau(q)}} \tag{1.1}$$

links fidelity to the principal angles of a pair of projections. The reconstruction argument begins with two metric identities: distance $\sqrt{2}$ detects orthogonality, and fidelity with the reference state recovers projection probabilities. These identities also hold for nontracial faithful normal reference states. In the semifinite tracial setting, finite-trace projections determine the full projection lattice and all trace ratios.

1.1 Related work

The operator-algebraic theory of Bures distance and transition probability was developed by Bures, Uhlmann, Araki–Raggio, and Alberti–Uhlmann [1, 6, 10, 25]. In particular, (1.1) follows directly from the tracial density formula. Li–Pereira–Plosker give its finite-dimensional principal-angle interpretation for normalized projections [18, Theorem 4.7]. Araki–Raggio studied reconstruction of transition probability from projection-valued measures [6]. Fuchs–Caves developed the finite-dimensional optimal-measurement formulation [15]. Alberti–Uhlmann give the finite-projective reconstruction for arbitrary normal states [1, Corollary 2(2)]; we recall its variational proof in theorem 2.1.

Characterizing the measurements that attain fidelity requires control of the kernels of the states. Chen–Zhu study fidelity-optimal measurements for singular density operators, including minimal optimal measurements and support-compatible constructions [11]. For projection states in finite tracial algebras, we give an equality criterion and an exact quadratic expression for the measurement deficit. These lead to finite spectral approximation bounds. A continuous-spectrum example distinguishes exact attainment in the spectral algebra from exact attainment in the ambient algebra. The three-effect construction below is the operator-algebraic extension of the familiar unambiguous subspace-discrimination measurement [7].

Metric rigidity of states and projections has several related forms. Molnár–Timmermann classify Bures isometries of quantum state spaces [20]; Molnár treats density spaces in C^*-algebras with faithful traces [19]. Mori studies projection lattices with the operator norm [21], and Qian–Wang–Wu–Yuan study maps preserving the tracial transition quantity $\tau(pq)$ in semifinite factors [23]. The metric in this paper is the Bures metric on the smaller family $\mathcal{L}(M,\omega)$, with a specified conditioning rule and an intrinsically recognizable reference state. Once orthogonality has been recovered, Dye's theorem supplies the Jordan extension [12].

The intrinsic geometry uses the established two-projection decomposition and direct rotations [3, 17]. Andruchow–Recht establish L^2 minimizing geodesics and study smooth connectivity of projection orbits in type II_1 algebras [4]. Grassmann connections and their relation to trace metrics are discussed in [5]. We compute the metric derivative of the restricted Bures distance for L^2-differentiable curves of projections with fixed trace and determine all rectifiable components. This identifies the intrinsic length induced by the ambient state-space metric.

1.2 Main results

Reference-state rigidity. For every faithful normal ω,

$$F(\omega_p, \omega) = \sqrt{\omega(p)}, \qquad F(\omega_p, \omega_q) = 0 \iff pq = 0.$$

The reference state is the unique point with no point at distance $\sqrt{2}$. These observations recover the full projection order and all values $\omega(p)$ from the unpointed metric. If neither algebra has a type I_2 direct summand, every surjective isometry is induced by a unique normal Jordan *-isomorphism J satisfying $\nu \circ J = \omega$ (theorem 3.4). We give a separate proof for $M_2(\mathbb{C})$. For faithful normal semifinite traces, orthogonal closure of the finite-trace projections recovers the full projection lattice, and the metric determines the trace up to a single global factor (theorem 3.8).

Optimal measurements and finite outcomes. In a finite tracial algebra, writing $pq = v|pq|$, theorem 4.2 characterizes every attaining finite projective measurement through v and the support defect $p \wedge q^\perp$. The measurement deficit is a sum of squared L^2 errors. An exact finite measurement whose atoms commute with q exists precisely when $|pq|$ has finite spectrum (theorem 4.7); a three-atom example shows that general exact measurements can coexist with a continuous angle spectrum. For $\alpha = \tau(p) \geq \beta = \tau(q)$, we construct at most $n+1$ outcomes with

$$0 \leq \mathrm{BC}_{\mathcal{E}_n}(p, q) - F(\varphi_p, \varphi_q) \leq \frac{\sqrt{\beta/\alpha}}{2n(n+1)}. \tag{1.2}$$

The spectral endpoints $j(j+1)/[n(n+1)]$ solve a fixed-grid minimax problem exactly. A scalar quantization formula and a uniform spectral example give the matching order n^{-2} for spectral measurements (propositions 5.4 and 5.7 and theorem 5.8).

Common contexts and commuting approximation. For a finite tuple of projections $\mathbf{p} = (p_1, \ldots, p_m)$, let $\Delta_M(\mathbf{p})$ be the least total weighted radial fidelity defect of a common abelian context, and let $D_M(\mathbf{p})$ be the least sum of squared L^2 distances to a commuting projection tuple. We prove

$$\frac{D_M(\mathbf{p})}{1+\sqrt{2}} \leq \Delta_M(\mathbf{p}) \leq \sqrt{m D_M(\mathbf{p})}. \tag{1.3}$$

The underlying rounding constant $1+\sqrt{2}$ is sharp; both powers in this comparison are optimal uniformly over finite tracial algebras. There are also commutator bounds, continuity estimates, and a finite context approximation theorem. For $N = W^*(p, q)$, put $S = |p - q|$ and $C = |p + q - 1|$. These operators are central in N, and

$$\Delta_N(p, q) = \frac{1}{2}\tau(S + C - 1), \qquad D_N(p, q) = \tau\big(1 - \max\{S, C\}\big). \tag{1.4}$$

These exact formulas optimize within the algebra generated by the pair (theorem 6.9).

Intrinsic geometry. The normalized finite-trace projection states form a complete Bures metric space. In a finite algebra, its rectifiable components are exactly the level sets of the centre-valued trace T. On the orbit $T(p) = T(q)$ with $t = \tau(p)$, the metric tensor and the intrinsic length distance are

$$g_p(X,Y) = \frac{\operatorname{Re}\tau(XY)}{2t}, \qquad d_{\mathrm{len}}(p,q) = \left(\frac{\tau(\Theta^2)}{t}\right)^{1/2}, \quad \Theta = \arccos((pqp)^{1/2}) \in pMp. \tag{1.5}$$

The length formula holds for all rectifiable projection curves. It yields $d_B \le \arccos F \le d_{\mathrm{len}}$, with equality in the second inequality exactly when Θ is a scalar multiple of p (theorem 7.9 and corollary 7.10).

Geometric characterization of the trace. For a faithful normal state ω, traciality is equivalent to

$$F(\omega_p, \omega_q) \le \sqrt{\frac{\omega(q)}{\omega(p)}} \qquad (0 \ne p, q \in \mathcal{P}(M)). \tag{1.6}$$

It suffices to test this bound for $q = e^{itx}pe^{-itx}$ near $t = 0$, for every self-adjoint x and every nonzero p (theorem 8.2). Together with reference-point recovery, this makes traciality an invariant of the unpointed metric space. A faithful nontracial qubit example shows both local failure of the bound and equality without projection inclusion.

Section 2 fixes the fidelity conventions and recalls commutative reconstruction. Section 3 proves rigidity. Sections 4 and 5 treat exact and approximate finite measurements. Section 6 develops common contexts, Section 7 determines intrinsic length, and Section 8 characterizes traciality.

2 States and measurements

Let M be a von Neumann algebra, with projection lattice $\mathcal{P}(M)$. The support of a normal positive functional φ is denoted by $s(\varphi)$. All states and traces in the paper are normal. A trace is faithful unless stated otherwise. In a finite tracial algebra we normalize $\tau(1) = 1$ and write $\|a\|_2 = \tau(a^*a)^{1/2}$. For a semifinite trace, this notation is used on $L^2(M,\tau)$.

We use the *root fidelity* convention. Thus the transition probability in the terminology of [25] is F^2, and

$$d_B(\varphi,\psi)^2 = 2 - 2F(\varphi,\psi), \qquad d_{\mathrm{ang}}(\varphi,\psi) = \arccos F(\varphi,\psi). \tag{2.1}$$

Fidelity is the supremum of the absolute inner products of unit vectors implementing the two states in common representations. Its variational formula [1, Corollary 2(5)] is

$$F(\varphi,\psi) = \frac{1}{2}\inf_{a \in M_+^\times}\big(\varphi(a) + \psi(a^{-1})\big), \tag{2.2}$$

where $M_+^\times$ denotes the positive invertible elements. We use the standard facts

$$F(\varphi,\psi) = 0 \quad \Longleftrightarrow \quad s(\varphi)s(\psi) = 0, \tag{2.3}$$

and that restriction to a unital von Neumann subalgebra increases fidelity. These statements, including the variational formula and their relation to Bures distance, are developed in [1, 10, 25]. If $\varphi(a) = \tau(h_\varphi a)$ and $\psi(a) = \tau(h_\psi a)$ have densities in $L^1(M,\tau)_+$, then

$$F(\varphi,\psi) = \tau\big(|h_\varphi^{1/2} h_\psi^{1/2}|\big). \tag{2.4}$$

The formula applies to faithful normal semifinite traces; see [1, 6, 13].

For a faithful state ω and $0 \neq p \in \mathcal{P}(M)$, define its normalized projection conditioning by

$$\omega_p(a) = \frac{\omega(pap)}{\omega(p)}, \qquad \mathcal{L}(M,\omega) = \{\omega_p : 0 \neq p \in \mathcal{P}(M)\}. \tag{2.5}$$

When $\omega = \tau$ is tracial, we write $\varphi_p = \tau_p$; its density is $p/\tau(p)$. For a semifinite trace the same definition is used precisely when $0 < \tau(p) < \infty$. For nonzero finite-trace projections p, q, the conditional probability is $\varphi_p(q) = \tau(pq)/\tau(p)$, whereas their root fidelity is $\tau(|pq|)/\sqrt{\tau(p)\tau(q)}$. The trace and the normalized centre-valued trace provide the dimension data used below; both are invariant under Murray–von Neumann equivalence.

A *finite projective measurement* is a family $\mathcal{E} = \{e_1, \dots, e_m\}$ of nonzero mutually orthogonal projections whose sum is 1. Its sums form a finite Boolean subalgebra of $\mathcal{P}(M)$. We call a unital abelian von Neumann subalgebra a *commutative context*; a finite context is generated by a finite projective measurement. The resulting classical quantities are

$$\mathrm{BC}_{\mathcal{E}}(\varphi,\psi) = \sum_i \sqrt{\varphi(e_i)\psi(e_i)}, \tag{2.6}$$

$$d_{H,\mathcal{E}}(\varphi,\psi)^2 = \sum_i \big(\sqrt{\varphi(e_i)} - \sqrt{\psi(e_i)}\big)^2 = 2 - 2\,\mathrm{BC}_{\mathcal{E}}(\varphi,\psi). \tag{2.7}$$

Alberti–Uhlmann's reconstruction formula [1, Corollary 2(2)] expresses fidelity as an infimum over finite projective measurements. We recall its proof from the variational formula.

Theorem 2.1 (Commutative reconstruction)**.** *For any normal states φ, ψ on a von Neumann algebra,*

$$F(\varphi,\psi) = \inf_{\mathcal{E}} \mathrm{BC}_{\mathcal{E}}(\varphi,\psi), \qquad d_B(\varphi,\psi) = \sup_{\mathcal{E}} d_{H,\mathcal{E}}(\varphi,\psi), \tag{2.8}$$

where $\mathcal{E}$ ranges over all finite projective measurements.

Proof. For a fixed measurement, apply (2.2) to $a = \sum_i t_i e_i$, with $t_i > 0$. The scalar identity

$$\inf_{t>0} \frac{ut + v/t}{2} = \sqrt{uv} \qquad (u, v \geq 0)$$

shows that $F(\varphi,\psi) \leq \mathrm{BC}_{\mathcal{E}}(\varphi,\psi)$. This includes zero probabilities by taking the corresponding scalar infima as limits.

Conversely, choose a positive invertible a for which the expression in (2.2) is within ε of F. Bounded spectral calculus approximates a in norm by a positive invertible finite-spectrum element $b = \sum_i t_i e_i$, and simultaneously approximates a^{-1} by b^{-1}. Choosing this approximation sufficiently close gives

$$\mathrm{BC}_{\mathcal{E}}(\varphi,\psi) \leq \frac{1}{2}\big(\varphi(b) + \psi(b^{-1})\big) < F(\varphi,\psi) + 2\varepsilon.$$

Letting $\varepsilon \downarrow 0$ proves the fidelity identity; (2.7) then gives the distance identity. □

For an abelian context A in a finite tracial algebra, $E_A : M \to A$ denotes the unique trace-preserving normal conditional expectation. We use its positivity, A-bimodularity, and orthogonal projection property on $L^2(M,\tau)$. These facts, and the projection comparison theory used below, can be found in [24].

3 Metric reconstruction and rigidity

Let M be a nonzero von Neumann algebra with a faithful normal state ω. In $\mathcal{L}(M,\omega)$, the largest possible Bures distance detects orthogonality, while distances from ω determine projection probabilities. We use these facts to recover the projection lattice and apply Dye's theorem. The first two subsections require neither finiteness nor traciality.

3.1 The reference state and projection order

The radial identity below holds for any normal state, which allows successive conditioning. It specializes the inner-derived fidelity identity in [1, Theorem 1]; we give a direct proof.

Lemma 3.1. *Let φ be a normal state on a von Neumann algebra, and let p be a projection with $\varphi(p) > 0$. Define $\varphi_p(x) = \varphi(pxp)/\varphi(p)$. Then*

$$F(\varphi_p, \varphi) = \sqrt{\varphi(p)}. \tag{3.1}$$

If φ is faithful, then the support projection of φ_p is p. In particular, the map $p \mapsto \omega_p$ is injective.

Proof. Choose a standard representation of the algebra and a unit vector ξ implementing φ. The unit vector

$$\eta = \frac{p\xi}{\sqrt{\varphi(p)}}$$

implements φ_p. The common-representation characterization of fidelity gives

$$F(\varphi_p, \varphi) \geq |\langle \eta, \xi \rangle| = \sqrt{\varphi(p)}.$$

Restriction to the abelian algebra generated by p gives the probability vectors $(1,0)$ and $(\varphi(p), 1-\varphi(p))$, with any zero atom omitted. Monotonicity of fidelity under this measurement yields the reverse inequality. This proves (3.1).

Suppose now that φ is faithful. We have $\varphi_p(p) = 1$. If a projection e satisfies $\varphi_p(e) = 1$, then

$$0 = \varphi\big(p(1-e)p\big).$$

Faithfulness implies $p(1-e)p = 0$, and hence $(1-e)p = 0$. Thus $p \leq e$, proving that the support of φ_p is p. Equal states have equal supports, which proves injectivity. □

Proposition 3.2. *The metric space $(\mathcal{L}(M,\omega), d_{\mathrm{B}})$ determines its reference state, the projection order, and the values of ω on projections. More precisely:*

(i) *$\omega = \omega_1$ is the unique point of $\mathcal{L}(M,\omega)$ having no point at distance $\sqrt{2}$.*

(ii) *For nonzero projections p, q,*

$$p \perp q \quad \Longleftrightarrow \quad d_{\mathrm{B}}(\omega_p, \omega_q) = \sqrt{2}.$$

(iii) *If*

$$\mathcal{O}(p) = \{r \in \mathcal{P}(M) \setminus \{0\} : r \perp p\},$$

then

$$p \leq q \quad \Longleftrightarrow \quad \mathcal{O}(q) \subseteq \mathcal{O}(p). \tag{3.2}$$

(iv) *For every nonzero projection p,*

$$\omega(p) = \left(1 - \frac{1}{2} d_{\mathrm{B}}(\omega_p, \omega)^2\right)^2. \tag{3.3}$$

After adjoining a zero element, these data determine all meets, joins, and orthocomplements in $\mathcal{P}(M)$.

Proof. By Lemma 3.1, $F(\omega, \omega_q) = \sqrt{\omega(q)} > 0$ for every nonzero q. If $p \neq 1$, then $1 - p \neq 0$, and (2.3) gives

$$d_{\mathrm{B}}(\omega_p, \omega_{1-p}) = \sqrt{2}.$$

This proves the first assertion. The support statement in Lemma 3.1 and (2.3) prove the second.

If $p \leq q$, every projection orthogonal to q is orthogonal to p, so the forward implication in (3.2) holds. Conversely, if $q \neq 1$, apply the assumed inclusion to $1 - q$. It follows that $p \perp 1 - q$, and hence $p \leq q$. If $q = 1$, the conclusion holds directly.

Equation (3.3) follows from (3.1). A complete lattice is determined by its order, so arbitrary meets and joins are recovered as greatest lower bounds and least upper bounds. The orthocomplement of p is the greatest element orthogonal to p, including the adjoined zero element when $p = 1$. □

The first assertion concerns an attained distance. In a diffuse algebra, distances from ω can have supremum $\sqrt{2}$ without any point attaining that value.

3.2 Jordan rigidity

The variational formula (2.2) gives fidelity invariance under Jordan isomorphisms and will prove the converse direction of the classification.

Lemma 3.3. *Let $J : M \to N$ be a normal Jordan *-isomorphism. For normal states α, β on N,*

$$F(\alpha \circ J, \beta \circ J) = F(\alpha, \beta).$$

If $\omega = \nu \circ J$, then

$$\nu_{J(p)} \circ J = \omega_p \tag{3.4}$$

for every nonzero projection $p \in M$.

Proof. A unital Jordan *-isomorphism preserves positivity and continuous functional calculus for self-adjoint elements. In particular, it maps $M_+^\times$ bijectively onto $N_+^\times$, and

$$J(a^{-1}) = J(a)^{-1} \qquad (a \in M_+^\times).$$

Substitution in (2.2) proves the fidelity identity.

Jordan *-isomorphisms also preserve the quadratic product:

$$J(pxp) = J(p)J(x)J(p).$$

Consequently,

$$\nu_{J(p)}(J(x)) = \frac{\nu(J(pxp))}{\nu(J(p))} = \frac{\omega(pxp)}{\omega(p)} = \omega_p(x),$$

which proves (3.4). □

Theorem 3.4. *Let M, N be von Neumann algebras having no direct summand of type I_2, and let ω, ν be faithful normal states on M, N, respectively. For a bijection*

$$\Phi : \mathcal{L}(M, \omega) \longrightarrow \mathcal{L}(N, \nu),$$

the following conditions are equivalent:

(i) *Φ is an isometry for the Bures distance.*

(ii) *There is a normal Jordan *-isomorphism $J : M \to N$ such that*

$$\nu \circ J = \omega, \qquad \Phi(\omega_p) = \nu_{J(p)} \quad (p \in \mathcal{P}(M) \setminus \{0\}).$$

The Jordan *-isomorphism in the second condition is unique.

Proof. Suppose first that Φ is a surjective isometry. By Proposition 3.2, $\Phi(\omega) = \nu$. The injective parametrizations by supports define a bijection

$$T : \mathcal{P}(M) \setminus \{0\} \longrightarrow \mathcal{P}(N) \setminus \{0\}, \qquad \Phi(\omega_p) = \nu_{T(p)}.$$

It preserves orthogonality in both directions. Set $T(0) = 0$. Equation (3.2) shows that T is an order isomorphism of the full projection lattices. It therefore preserves arbitrary joins and meets, and it preserves orthocomplements by the second assertion of Proposition 3.2.

Dye's extension theorem now gives a Jordan *-isomorphism $J : M \to N$ whose restriction to projections is T [12]. This map is normal: its projection map preserves arbitrary increasing joins. The same theorem gives uniqueness; it also follows by norm approximation of self-adjoint elements by finite linear combinations of spectral projections.

The radial identity yields

$$\nu(J(p)) = F(\nu_{J(p)}, \nu)^2 = F(\omega_p, \omega)^2 = \omega(p)$$

for every projection p. Norm approximation by spectral step functions extends this equality to all elements of M, proving $\nu \circ J = \omega$.

Conversely, suppose such a J is given. Equation (3.4) and Lemma 3.3 imply

$$F(\nu_{J(p)}, \nu_{J(q)}) = F(\omega_p, \omega_q)$$

for all nonzero p, q. Since J maps projections bijectively onto projections, the indicated map of conditioned states is a surjective Bures isometry. □

The metric recovers the reference state and projection orthogonality; Dye's theorem then extends the induced projection map. This separates the metric step from the algebraic extension and relates the result to Bures isometries of full state spaces [20].

Proposition 3.5. *For $M = N = M_2(\mathbb{C})$, the equivalence and uniqueness in Theorem 3.4 hold for any two faithful states ω, ν.*

Proof. Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices, and write the rank-one projections as

$$p(u) = \frac{1}{2}\left(1 + \sum_{j=1}^{3} u_j \sigma_j\right), \qquad u \in S^2 \subset \mathbb{R}^3.$$

For a rank-one projection p, the identity $pxp = \mathrm{Tr}(px)p$ gives

$$\omega_p(x) = \mathrm{Tr}(px),$$

independently of the faithful state ω. Hence

$$F(\omega_{p(u)}, \omega_{p(v)})^2 = \mathrm{Tr}(p(u)p(v)) = \frac{1 + u \cdot v}{2}. \tag{3.5}$$

A surjective isometry fixes the reference point in the sense that $\Phi(\omega) = \nu$, and it induces a bijection $f : S^2 \to S^2$. Equation (3.5) implies

$$f(u) \cdot f(v) = u \cdot v.$$

Let O be the real 3×3 matrix whose columns are $f(e_1), f(e_2), f(e_3)$, where e_j are the standard coordinate vectors. These columns are orthonormal. Taking inner products with them gives $f(u) = Ou$ for every $u \in S^2$.

Define the complex-linear map

$$J\left(z_0\, 1 + \sum_{j=1}^{3} z_j \sigma_j\right) = z_0\, 1 + \sum_{j=1}^{3} (Oz)_j \sigma_j.$$

The Pauli anticommutator relations

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} 1$$

show directly that J is a Jordan *-isomorphism. It implements the induced projection map. The radial identity then gives $\nu \circ J = \omega$, exactly as in Theorem 3.4. The converse follows from Lemma 3.3, and uniqueness follows because projections span $M_2(\mathbb{C})$. □

3.3 Semifinite traces

For a semifinite trace, the identity may have infinite trace and hence may not define a normalized projection state. Finite-trace projections nevertheless determine the full projection lattice through orthogonality. Trace ratios on nested projections then determine the trace up to a single global positive factor.

Let (M, τ) be a nonzero semifinite von Neumann algebra with faithful normal semifinite trace. Put

$$V_\tau(M) = \{p \in \mathcal{P}(M) : 0 < \tau(p) < \infty\}.$$

For $p \in V_\tau(M)$, the functional

$$\varphi_p^\tau(x) = \frac{\tau(pxp)}{\tau(p)}$$

is a normal state with support p. Its density in $L^1(M, \tau)$ is $p/\tau(p)$, so

$$F(\varphi_p^\tau, \varphi_q^\tau) = \frac{\tau(|pq|)}{\sqrt{\tau(p)\tau(q)}}. \tag{3.6}$$

Here $pq \in L^1(M, \tau)$, since $p, q \in L^2(M, \tau)$. Denote the resulting state family by

$$\mathcal{L}_{\mathrm{f}}(M, \tau) = \{\varphi_p^\tau : p \in V_\tau(M)\}.$$

Lemma 3.6. *Every projection $e \in M$ is the supremum of its finite-trace subprojections:*

$$e = \bigvee \{p \in V_\tau(M) : p \le e\}. \tag{3.7}$$

The finite-trace projections below e, together with zero, are directed by their joins. In particular, the join of two finite-trace projections has finite trace.

Proof. If $r \neq 0$ is a projection, faithfulness and semifiniteness give a nonzero element x with $0 \le x \le r$ and $\tau(x) < \infty$. For a suitable $\varepsilon > 0$, the spectral projection $f = 1_{[\varepsilon,\infty)}(x)$ is nonzero and satisfies

$$f \le r, \qquad \tau(f) \le \varepsilon^{-1}\tau(x) < \infty.$$

If the supremum on the right of (3.7) were strictly smaller than e, the complementary part inside e would contain such an f, which is impossible.

For projections p, q, the polar decomposition of $(1-p)q$ identifies $q - (p \wedge q)$ and $(p \vee q) - p$ as Murray–von Neumann equivalent projections. When p, q have finite trace, it follows that

$$\tau(p \vee q) = \tau(p) + \tau(q) - \tau(p \wedge q) \le \tau(p) + \tau(q) < \infty.$$

This also proves directedness. □

Lemma 3.7. *For $e \in \mathcal{P}(M)$, set*

$$L_\tau(e) = \{p \in V_\tau(M) : p \le e\}.$$

For a subset $S \subseteq V_\tau(M)$, define its orthogonal set by

$$S^\perp = \{p \in V_\tau(M) : pq = 0 \text{ for every } q \in S\}.$$

Then

$$S^{\perp\perp} = L_\tau\left(\bigvee_{p \in S} p\right). \tag{3.8}$$

Consequently, $e \mapsto L_\tau(e)$ identifies $\mathcal{P}(M)$ with the subsets $C \subseteq V_\tau(M)$ satisfying $C = C^{\perp\perp}$. Under this identification, inclusion is the projection order and $C \mapsto C^\perp$ is orthocomplementation.

Let (N, σ) be another von Neumann algebra with faithful normal semifinite trace. If

$$T : V_\tau(M) \longrightarrow V_\sigma(N)$$

is a bijection preserving orthogonality in both directions, it extends uniquely to a complete orthocomplemented lattice isomorphism $\widehat{T} : \mathcal{P}(M) \to \mathcal{P}(N)$, determined by

$$L_\sigma(\widehat{T}(e)) = T[L_\tau(e)], \qquad \widehat{T}(e) = \bigvee_{\substack{p \in V_\tau(M) \\ p \le e}} T(p). \tag{3.9}$$

Proof. Let $a = \bigvee_{p \in S} p$. Orthogonality to every member of S is equivalent to orthogonality to a, and hence

$$S^\perp = L_\tau(1-a).$$

By Lemma 3.6, the supremum of $L_\tau(1-a)$ is $1-a$. Taking orthogonal sets once more proves (3.8).

It follows that every set fixed by double orthogonality has the form $L_\tau(e)$. Equation (3.7) shows that this representation is unique and that

$$e \le f \quad \Longleftrightarrow \quad L_\tau(e) \subseteq L_\tau(f).$$

The identity $L_\tau(e)^\perp = L_\tau(1-e)$ gives orthocomplementation. Intersections give meets of closed sets, and double orthogonal closure of unions gives their joins.

Since T is a bijection preserving orthogonality,

$$T[S^\perp] = T[S]^\perp$$

for every $S \subseteq V_\tau(M)$. It therefore induces a bijection of the double-orthogonality closed subsets, preserving their order and orthocomplementation. This defines $\widehat{T}$ and proves (3.9). For $p \in V_\tau(M)$,

$$L_\tau(p) = \{p\}^{\perp\perp},$$

so $\widehat{T}(p) = T(p)$. Finally, any complete lattice extension must satisfy the join formula in (3.9), proving uniqueness. □

Theorem 3.8. *Let M, N be nonzero von Neumann algebras having no direct summand of type I_2, and let τ, σ be faithful normal semifinite traces on M, N. A bijection*

$$\Phi : \mathcal{L}_{\mathrm{f}}(M,\tau) \longrightarrow \mathcal{L}_{\mathrm{f}}(N,\sigma)$$

is a Bures isometry if and only if there are a normal Jordan *-isomorphism $J : M \to N$ and a constant $c > 0$ such that

$$\sigma(J(x)) = c\,\tau(x) \qquad (x \in M_+) \tag{3.10}$$

as an equality of extended positive values, and

$$\Phi(\varphi_p^\tau) = \varphi_{J(p)}^\sigma \qquad (p \in V_\tau(M)).$$

Both J and c are unique.

Proof. Suppose that Φ is a surjective isometry. Supports give a bijection $T : V_\tau(M) \to V_\sigma(N)$. By (3.6), or by (2.3), it preserves orthogonality in both directions. Lemma 3.7 extends it to a complete orthocomplemented lattice isomorphism $\widehat{T} : \mathcal{P}(M) \to \mathcal{P}(N)$. Dye's theorem yields the unique normal Jordan *-isomorphism J satisfying $J(e) = \widehat{T}(e)$ on projections.

For finite-trace projections $0 < p \le r$, formula (3.6) gives

$$F(\varphi_p^\tau, \varphi_r^\tau)^2 = \frac{\tau(p)}{\tau(r)}.$$

The corresponding formula for $J(p) \le J(r)$, together with the isometry assumption, implies

$$\frac{\sigma(J(p))}{\tau(p)} = \frac{\sigma(J(r))}{\tau(r)}.$$

If $p, q \in V_\tau(M)$, their join $r = p \vee q$ has finite trace by Lemma 3.6. Applying the preceding equality to both $p \le r$ and $q \le r$ shows that

$$\frac{\sigma(J(p))}{\tau(p)} = \frac{\sigma(J(q))}{\tau(q)}.$$

There is therefore a single constant $c > 0$ with $\sigma(J(p)) = c\tau(p)$ on all of $V_\tau(M)$.

For an arbitrary projection e, the directed finite-trace subprojections in (3.7), normality of J, and normality of the traces give

$$\sigma(J(e)) = \sup_{\substack{p \le e \\ \tau(p)<\infty}} \sigma(J(p)) = c \sup_{\substack{p \le e \\ \tau(p)<\infty}} \tau(p) = c\tau(e).$$

Zero is allowed in these suprema. Increasing spectral step approximations of a bounded positive element extend the equality from projections to every $x \in M_+$, proving (3.10).

Conversely, let J, c satisfy (3.10). This equality and its inverse show that J maps $V_\tau(M)$ bijectively onto $V_\sigma(N)$. For $x \ge 0$, preservation of the quadratic product gives

$$\varphi^{\sigma}_{J(p)}(J(x)) = \frac{\sigma(J(pxp))}{\sigma(J(p))} = \frac{\tau(pxp)}{\tau(p)} = \varphi^{\tau}_{p}(x).$$

For general x, this identity follows by linearity from positive elements of the finite-trace corner pMp. Lemma 3.3 proves that the resulting bijection is a Bures isometry.

The projection map uniquely determines J. For any fixed $p \in V_\tau(M)$, the identity

$$c = \frac{\sigma(J(p))}{\tau(p)}$$

then proves uniqueness of the scale. □

Remark 3.9. Multiplying a semifinite trace by a positive scalar leaves every normalized projection state unchanged. Theorem 3.8 shows that this is the only freedom in the trace. A finite-trace join compares any two finite-trace projections, so the scale is global even when the centre is nontrivial. When $\tau(1) = \sigma(1) = 1$, equation (3.10) forces $c = 1$.

4 Optimal finite projective measurements

Throughout this section, (M, τ) is a finite von Neumann algebra with a faithful normal tracial state. For nonzero projections $p, q \in M$, put

$$\alpha = \tau(p), \qquad \beta = \tau(q), \qquad \varphi_p(a) = \frac{\tau(pa)}{\alpha}, \qquad \varphi_q(a) = \frac{\tau(qa)}{\beta}.$$

For a finite projective measurement $\mathcal{E} = \{e_1, \dots, e_m\}$, abbreviate its classical overlap as

$$\mathrm{BC}_{\mathcal{E}}(p, q) = \sum_{i=1}^{m} \sqrt{\varphi_p(e_i)\varphi_q(e_i)}.$$

The equality cases in the measurement bound are determined by the polar decomposition of pq and the part of p orthogonal to q.

4.1 Fidelity and equality conditions

The fidelity formula for normalized projections is a specialization of (2.4); its finite-dimensional relation to principal angles is given in [18, Theorem 4.7].

Proposition 4.1. *For nonzero projections p, q,*

$$F(\varphi_p, \varphi_q) = \frac{\tau(|pq|)}{\sqrt{\alpha\beta}}, \qquad |pq| = (qpq)^{1/2}.$$

In particular, $F(\varphi_p, \varphi_q) = 0$ if and only if $pq = 0$.

Proof. The respective densities are p/α and q/β, whose positive square roots are $p/\sqrt{\alpha}$ and $q/\sqrt{\beta}$. Substitution into $F(\varphi, \psi) = \tau(|h_\varphi^{1/2} h_\psi^{1/2}|)$ gives the formula. The assertion about vanishing follows from faithfulness of τ. □

The next theorem characterizes all finite projective measurements attaining fidelity, including when the projection states have nontrivial kernels. It uses the polar decomposition in M; for finite-dimensional classifications of optimal measurements for singular states, see [11].

Theorem 4.2 (Equality criterion)**.** *Write*

$$pq = v|pq|, \qquad r = vv^* = s(pqp), \qquad p_0 = p - r = p \wedge q^\perp.$$

For a finite projective measurement $\mathcal{E} = \{e_i\}_{i=1}^m$, set

$$a_i = \tau(pe_i), \qquad b_i = \tau(qe_i), \qquad e_0 = \sum_{b_i = 0} e_i, \qquad K = \sum_{b_i > 0} \sqrt{\frac{a_i}{b_i}}\, e_i.$$

Then

$$F(\varphi_p, \varphi_q) \le \mathrm{BC}_{\mathcal{E}}(p, q).$$

The following conditions are equivalent:

(i) $\mathrm{BC}_{\mathcal{E}}(p, q) = F(\varphi_p, \varphi_q)$;

(ii) *for every i with $b_i > 0$,*

$$e_i p_0 = 0, \qquad e_i v = \sqrt{\frac{a_i}{b_i}}\, e_i q;$$

(iii)

$$p_0 \le e_0, \qquad (1 - e_0)v = Kq.$$

Moreover, $e_0 \le q^\perp$, $K \ge 0$, $Ke_0 = 0$, and K has finite spectrum.

Proof. Since $v^* p = v^*$, the polar decomposition gives $v^* q = v^* pq = |pq|$. Therefore

$$\begin{aligned}
\tau(|pq|) &= \sum_i \mathrm{Re}\, \tau(v^* e_i q) \\
&\le \sum_i |\tau(v^* e_i q)| \\
&\le \sum_i \|e_i v\|_2 \, \|e_i q\|_2 \\
&= \sum_i \sqrt{\tau(r e_i) b_i} \\
&\le \sum_i \sqrt{a_i b_i}.
\end{aligned}$$

Dividing by $\sqrt{\alpha\beta}$ proves the inequality.

Every difference in this chain is a sum of nonnegative terms. For an index with $b_i > 0$, equality in the last inequality is equivalent to $\tau(p_0 e_i) = 0$. Since $\tau(p_0 e_i) = \|e_i p_0\|_2^2$, faithfulness gives $e_i p_0 = 0$. Equality in the preceding two inequalities then says that the $L^2(M,\tau)$ vectors $e_i v$ and $e_i q$ are proportional by a nonnegative scalar. Their norms determine the proportionality constant as $\sqrt{a_i/b_i}$. This includes $a_i = 0$, when $e_i v = 0$. For $b_i = 0$, the vector $e_i q$ vanishes and the corresponding terms are all zero. This proves the equivalence of the first two conditions.

The equality $b_i = 0$ implies $e_i q = 0$, so $e_0 \le q^\perp$. Summing the identities in the second condition gives the third condition. Conversely, multiplication of the third condition on the left by each e_i with $b_i > 0$ recovers the second condition. The remaining assertions follow from the definition of K. □

Extending the polar partial isometry to a unitary turns the measurement deficit into a sum of squared L^2 errors.

Proposition 4.3 (Measurement deficit)**.** *Let $U \in M$ be any unitary extending the polar partial isometry of pq, so that $pq = U|pq|$. With the notation of Theorem 4.2,*

$$\begin{aligned}\sqrt{\alpha\beta}\,\big(\mathrm{BC}_{\mathcal{E}}(p,q) - F(\varphi_p,\varphi_q)\big)\\ = \frac{1}{2}\sum_{a_i b_i > 0} \sqrt{\frac{b_i}{a_i}}\,\left\| e_i p U - \sqrt{\frac{a_i}{b_i}}\, e_i q \right\|_2^2 .\end{aligned} \tag{4.1}$$

Proof. The initial and final projections of v are equivalent. In a finite von Neumann algebra their complements are equivalent as well, so v extends to a unitary U. Set $Z_i = e_i p U$ and $Y_i = e_i q$. Then

$$\|Z_i\|_2^2 = a_i, \qquad \|Y_i\|_2^2 = b_i, \qquad \sum_i \operatorname{Re}\tau(Z_i^* Y_i) = \tau(|pq|).$$

For $a_i b_i > 0$, expansion of the square gives

$$\frac{1}{2}\sqrt{\frac{b_i}{a_i}}\,\left\| Z_i - \sqrt{\frac{a_i}{b_i}} Y_i \right\|_2^2 = \sqrt{a_i b_i} - \operatorname{Re}\tau(Z_i^* Y_i).$$

If $a_i b_i = 0$, both terms on the right vanish. Summation proves the identity. □

Corollary 4.4. *Suppose $L > 0$ and $a_i/b_i \le L$ whenever $a_i b_i > 0$. Then*

$$\sum_{a_i b_i > 0} \left\| e_i p U - \sqrt{\frac{a_i}{b_i}}\, e_i q \right\|_2^2 \le 2\sqrt{\alpha\beta L}\,\big(\mathrm{BC}_{\mathcal{E}}(p,q) - F(\varphi_p,\varphi_q)\big).$$

Proof. In (4.1), each coefficient $\sqrt{b_i/a_i}$ is at least $L^{-1/2}$. □

4.2 Support-compatible measurements

The spectral algebra of $|pq|$, together with $q^\perp$, preserves the fidelity of the pair. In this algebra, finite exact attainment reduces to a finite-spectrum condition.

Proposition 4.5 (Abelian reduction)**.** *Let $x = |pq|$ and*

$$A_{p,q} = W^*(x, q^\perp).$$

Then

$$F(\varphi_p|_{A_{p,q}}, \varphi_q|_{A_{p,q}}) = F(\varphi_p, \varphi_q).$$

Moreover,

$$F(\varphi_p, \varphi_q) = \inf_{\mathcal{E}\subseteq\mathcal{P}(A_{p,q})} \mathrm{BC}_{\mathcal{E}}(p,q) = \inf_{\mathcal{E}\subseteq\mathcal{P}(M)} \mathrm{BC}_{\mathcal{E}}(p,q),$$

where both infima range over finite projective measurements with sum 1*.*

Proof. The algebra $qA_{p,q}$ is generated by x with unit q, and $q^\perp A_{p,q} = \mathbb{C}q^\perp$. For $b \in qA_{p,q}$,

$$\tau(pb) = \tau(qpqb) = \tau(x^2 b), \qquad \tau(qb) = \tau(b).$$

Thus the restrictions of the two states to $qA_{p,q}$ have densities x^2/α and q/β relative to τ. The restriction of φ_q to $q^\perp A_{p,q}$ is zero. The commutative fidelity formula consequently gives $\tau(x)/\sqrt{\alpha\beta}$.

To approximate this fidelity by finite measurements, take a finite spectral partition of x in the corner qMq, with nonzero projections f_j summing to q, and adjoin $q^\perp$ when it is nonzero. The resulting measurement satisfies

$$\mathrm{BC}_{\mathcal{E}}(p,q) = \frac{1}{\sqrt{\alpha\beta}} \sum_j \sqrt{\tau(x^2 f_j)\tau(f_j)}. \tag{4.2}$$

If the spectral intervals have length at most δ, both $\tau(xf_j)/\tau(f_j)$ and $\sqrt{\tau(x^2 f_j)/\tau(f_j)}$ lie in the same interval. Hence

$$0 \le \mathrm{BC}_{\mathcal{E}}(p,q) - \frac{\tau(x)}{\sqrt{\alpha\beta}} \le \delta\sqrt{\frac{\beta}{\alpha}}.$$

Letting δ tend to zero, and applying the lower bound in Theorem 4.2 to arbitrary projective measurements, proves both infimum identities. □

Definition 4.6. A finite projective measurement $\mathcal{E} = \{e_i\}$ is *q-compatible* if $e_i q = q e_i$ for every i. It *refines* q if each e_i is dominated by either q or $q^\perp$.

Theorem 4.7 (Finite exact measurements compatible with a support)**.** *Let $x = |pq|$. For a q-compatible measurement $\mathcal{E} = \{e_i\}$, write*

$$f_i = e_i q, \qquad g_i = e_i q^\perp.$$

It attains fidelity if and only if, for each i with $f_i \neq 0$, there is a scalar $\lambda_i \ge 0$ such that

$$x f_i = \lambda_i f_i, \qquad p g_i = 0.$$

Consequently, the following are equivalent:

(i) *x has finite spectrum;*

(ii) *some finite projective measurement refining q attains fidelity;*

(iii) *some finite q-compatible projective measurement attains fidelity;*

(iv) *some finite projective measurement in $A_{p,q}$ attains fidelity.*

Proof. First suppose a measurement refines q. For an atom $f_i \leq q$, the relation $qv = x$ gives $f_i v = f_i x$, and $f_i p_0 = 0$ holds automatically. Theorem 4.2 therefore says that the measurement attains fidelity exactly when $f_i x = \lambda_i f_i$ for every such atom. Taking adjoints gives the equivalent relation $x f_i = \lambda_i f_i$.

For a general q-compatible measurement, split each e_i into its nonzero parts f_i and g_i. The refined overlap is no larger than the original overlap. For an index with $f_i \neq 0$, the difference between the corresponding unnormalized summands is

$$\sqrt{\big(\tau(pf_i) + \tau(pg_i)\big)\tau(f_i)} - \sqrt{\tau(pf_i)\tau(f_i)}.$$

It vanishes precisely when $\tau(pg_i) = 0$, equivalently $pg_i = 0$. The characterization follows by combining this observation with the preceding paragraph and the lower bound by fidelity.

If a finite q-compatible measurement attains fidelity, its nonzero f_i sum to q and make x scalar on each summand. Thus x has finite spectrum. Conversely, its finite spectral resolution in qMq, together with $q^\perp$ when nonzero, supplies an attaining measurement in $A_{p,q}$ refining q. Every measurement in $A_{p,q}$ is q-compatible, which completes the equivalence. □

4.3 Exact measurements with continuous spectrum

Proposition 4.8. *Suppose a finite-spectrum positive element $K \in M$ satisfies $p = KqK$. Then the spectral projections corresponding to all eigenvalues of K, including 0 when present, form a finite projective measurement attaining fidelity.*

Proof. Write $K = \sum_i \lambda_i e_i$, with $\lambda_i \geq 0$. Traciality gives

$$\tau(pe_i) = \lambda_i^2 \tau(qe_i).$$

Moreover,

$$qpq = qKqKq = (qKq)^2,$$

and $qKq \geq 0$, so $|pq| = qKq$. It follows that

$$\sum_i \sqrt{\tau(pe_i)\tau(qe_i)} = \sum_i \lambda_i \tau(qe_i) = \tau(Kq) = \tau(|pq|).$$

□

Example 4.9 (A continuous spectrum with three exact outcomes)**.** Let

$$M = L^\infty([0,1])\overline{\otimes}M_3(\mathbb{C}), \qquad \tau(a) = \int_0^1 \frac{\mathrm{Tr}(a(t))}{3}\, dt,$$

and define unit vectors

$$\xi_t = \begin{pmatrix} 2\sqrt{t/5} \\ \sqrt{1-t} \\ \sqrt{t/5} \end{pmatrix}, \qquad \eta_t = \begin{pmatrix} \sqrt{t/5} \\ \sqrt{1-t} \\ 2\sqrt{t/5} \end{pmatrix}.$$

Set $q(t) = \xi_t \xi_t^*$ and $p(t) = \eta_t \eta_t^*$. Then $\alpha = \beta = 1/3$ and

$$\langle \xi_t, \eta_t \rangle = 1 - \frac{t}{5}, \qquad |pq|(t) = \left(1 - \frac{t}{5}\right) q(t).$$

Thus $|pq|$ has spectrum $\{0\} \cup [4/5, 1]$. Nevertheless, the three constant coordinate projections yield

$$\big(\varphi_p(e_i)\big)_{i=1}^3 = \left(\frac{1}{10}, \frac{1}{2}, \frac{2}{5}\right), \qquad \big(\varphi_q(e_i)\big)_{i=1}^3 = \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{10}\right),$$

and hence

$$\mathrm{BC}_{\mathcal{E}}(p, q) = \frac{9}{10} = \int_0^1 \left(1 - \frac{t}{5}\right) dt = F(\varphi_p, \varphi_q).$$

This also follows from Proposition 4.8, since $p = KqK$ for $K = \mathrm{diag}(1/2, 1, 2)$.

The construction occurs inside every type II_1 factor. Indeed, choose three equivalent orthogonal projections of trace $1/3$, identify the resulting matrix decomposition, and place a diffuse abelian algebra in its first diagonal corner. Theorem 4.7 shows that these three exact outcomes cannot be q-compatible.

4.4 Three positive effects

A finite positive operator-valued measure (POVM) is a family of positive elements of M summing to 1. For projection states, three effects always suffice to attain fidelity. On each generic two-dimensional block, the construction is the equal-prior unambiguous discrimination measurement. Bergou–Feldman–Hillery study its assembly for equal-dimensional subspaces [7]; the two-projection representation [17] gives the form below for arbitrary ranks and angle spectra.

Proposition 4.10. *For every pair of nonzero projections $p, q \in M$, there is a POVM $(E_p, E_q, E_?)$ with at most three nonzero effects such that*

	E_p	E_q	$E_?$
φ_p	$1 - c/\alpha$	0	c/α
φ_q	0	$1 - c/\beta$	c/β

where $c = \tau(|pq|)$.

Its classical overlap is $c/\sqrt{\alpha\beta} = F(\varphi_p, \varphi_q)$.

Proof. Decompose the algebra generated by p, q into the four commuting parts $p \wedge q$, $p \wedge q^\perp$, $p^\perp \wedge q$, $p^\perp \wedge q^\perp$ and its remaining part. On the remaining part, the two-projection representation has the form

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}, \qquad S = (1 - C^2)^{1/2},$$

where C is a positive contraction in an abelian von Neumann algebra. Expressions involving C without a matrix display mean its diagonal copy in both corners. Define there

$$E_p = (1 + C)^{-1}(1 - q), \qquad E_q = (1 + C)^{-1}(1 - p), \qquad E_? = 1 - E_p - E_q.$$

The first two effects are positive. The third is

$$E_? = \frac{C}{1 + C} \begin{pmatrix} 1 + C & S \\ S & 1 - C \end{pmatrix} \geq 0,$$

because the displayed matrix is the product of the column $(\sqrt{1 + C}, \sqrt{1 - C})^{\mathsf{T}}$ and its adjoint. Direct multiplication gives

$$pE_q = 0, \quad qE_p = 0, \quad pE_?p = Cp, \quad qE_?q = Cq.$$

The two diagonal corners have the same trace, and $|pq| = Cq$ on this part. Consequently both $\tau(pE_?)$ and $\tau(qE_?)$ equal its contribution to $\tau(|pq|)$.

On $p \wedge q$, assign the identity to $E_?$; on $p \wedge q^\perp$, assign it to E_p; on $p^\perp \wedge q$, assign it to E_q; and on $p^\perp \wedge q^\perp$, assign it to $E_?$. The resulting three effects are positive and sum to 1. Their traces against p and q are respectively $(\alpha - c, 0, c)$ and $(0, \beta - c, c)$, which proves the assertion. □

Remark 4.11. The effects in Proposition 4.10 need not be projections, so they need not generate a Boolean context. An attaining projective measurement satisfies the criterion in Theorem 4.2; Theorem 4.7 gives the criterion for q-compatible measurements.

5 Finite spectral approximation

Spectral measurements reduce the fidelity deficit to the gap between the root mean square and the mean on each spectral cell. A sharp interval estimate yields a universal finite-outcome bound, while scalar quantization describes the optimum for a fixed distribution.

Let p, q be nonzero projections in a finite tracial von Neumann algebra, and put $\alpha = \tau(p)$, $\beta = \tau(q)$ and $x = |pq| \in qMq$. The spectral distribution of x with respect to the normalized trace of qMq is the probability measure

$$\mu(B) = \frac{\tau(1_B^q(x))}{\beta}, \qquad B \subseteq [0,1] \text{ Borel},$$

where the superscript indicates spectral calculus in the corner with unit q. For a finite measurable partition $\pi = \{B_1, \dots, B_m\}$, discard the μ-null cells and take the spectral projections $e_i = 1_{B_i}^q(x)$, together with $q^\perp$ when nonzero. Denote the resulting projective measurement by $\mathcal{E}_\pi$.

Write $w_i = \mu(B_i)$ and let $\mathbb{E}_i$ denote conditional expectation with respect to the normalized restriction of μ to B_i. Formula (4.2) becomes

$$\begin{aligned} \mathrm{BC}_{\mathcal{E}_\pi}(p,q) - F(\varphi_p, \varphi_q) &= \sqrt{\frac{\beta}{\alpha}}\, D_\mu(\pi), \\ D_\mu(\pi) &= \sum_i w_i \left(\sqrt{\mathbb{E}_i(t^2)} - \mathbb{E}_i(t) \right). \end{aligned} \tag{5.1}$$

The quantity $D_\mu(\pi)$ is nonnegative and depends only on the spectral distribution and the partition.

5.1 An exact interval bound

Lemma 5.1. *Let $0 \le l \le u$, and let X be a random variable taking values in $[l, u]$. Then*

$$0 \le \sqrt{\mathbb{E}(X^2)} - \mathbb{E}(X) \le G(l,u) := \frac{(u-l)^2}{4(u+l)},$$

with $G(0,0) = 0$. For $l < u$, the upper bound is attained by a distribution supported on the two endpoints, with

$$\mathbb{P}(X = u) = \frac{u + 3l}{4(u+l)}, \qquad \mathbb{P}(X = l) = \frac{3u + l}{4(u+l)}.$$

Proof. The lower bound is Cauchy–Schwarz. Since $(X - l)(X - u) \le 0$,

$$\mathbb{E}(X^2) \le (l+u)m - lu, \qquad m = \mathbb{E}(X).$$

Consequently the gap is at most $\sqrt{(l+u)m-lu}-m$ for $m\in[l,u]$. If $l<u$, this concave function reaches its maximum where $\sqrt{(l+u)m-lu}=(l+u)/2$. Substitution gives $(u-l)^2/[4(u+l)]$. The stated endpoint distribution has precisely this mean and makes the preceding quadratic inequality an equality. The case $l=u$ is immediate. □

Theorem 5.2 (A universal outcome bound)**.** *After interchanging p and q if necessary, suppose $\alpha\geq\beta$. For every integer $n\geq 1$, there is a projective measurement $\mathcal{E}_n$ in $A_{p,q}$ with at most $n+1$ atoms such that*

$$0\leq \mathrm{BC}_{\mathcal{E}_n}(p,q)-F(\varphi_p,\varphi_q)\leq \frac{\sqrt{\beta/\alpha}}{2n(n+1)}\leq \frac{1}{2n(n+1)}. \tag{5.2}$$

One may take the spectral intervals with endpoints

$$t_j=\frac{j(j+1)}{n(n+1)},\qquad j=0,\ldots,n, \tag{5.3}$$

and adjoin $q^\perp$ when nonzero.

Proof. Choose disjoint intervals with the endpoints in (5.3); endpoints common to two intervals may be assigned to either one. For consecutive endpoints,

$$t_{j+1}-t_j=\frac{2(j+1)}{n(n+1)},\qquad t_{j+1}+t_j=\frac{2(j+1)^2}{n(n+1)}.$$

Thus

$$G(t_j,t_{j+1})=\frac{1}{2n(n+1)}\qquad (0\leq j<n).$$

Apply Lemma 5.1 to the conditional distribution on each nonnull cell. Since the cell masses sum to 1, $D_\mu(\pi)\leq 1/[2n(n+1)]$. Equation (5.1) proves the result. □

Corollary 5.3. *For the measurement in Theorem 5.2,*

$$0\leq d_B(\varphi_p,\varphi_q)^2-d_{H,\mathcal{E}_n}(\varphi_p,\varphi_q)^2\leq\frac{\sqrt{\beta/\alpha}}{n(n+1)}.$$

In particular,

$$0\leq d_B(\varphi_p,\varphi_q)-d_{H,\mathcal{E}_n}(\varphi_p,\varphi_q)\leq\frac{(\beta/\alpha)^{1/4}}{\sqrt{n(n+1)}}.$$

Proof. The difference of the squared distances is twice the overlap deficit. For $a\geq b\geq 0$, $(a-b)^2\leq a^2-b^2$. □

The grid in (5.3) minimizes the worst-case deficit when the interval endpoints are chosen before the spectral distribution.

Proposition 5.4 (Optimal fixed interval grids)**.** *For $0=t_0<t_1<\cdots<t_n=1$, let $\pi_{\boldsymbol{t}}$ be its interval partition, with any fixed convention at common endpoints. Then*

$$\inf_{\boldsymbol{t}}\sup_{\mu\in\mathcal{P}([0,1])} D_\mu(\pi_{\boldsymbol{t}})=\frac{1}{2n(n+1)}. \tag{5.4}$$

Here $\mathcal{P}([0,1])$ denotes the Borel probability measures on $[0,1]$. The unique minimizing set of endpoints is (5.3).

Proof. For a fixed grid, Lemma 5.1 gives

$$\sup_{\mu} D_\mu(\pi_{\boldsymbol{t}}) = \max_{1\le j\le n} G(t_{j-1}, t_j).$$

For the reverse inequality implicit in this equality, choose a cell with the largest G and a two-point distribution approaching its maximizing endpoint distribution from within that cell. This argument is unaffected by the convention for common endpoints.

Put $\delta = \max_j G(t_{j-1}, t_j) > 0$. The inequality $G(l, u) \le \delta$, for $0 \le l < u$, is equivalent to

$$\sqrt{2u + \delta} \le \sqrt{2l + \delta} + 2\sqrt{\delta}.$$

Applying this to consecutive intervals and adding yields

$$\sqrt{2 + \delta} \le (2n + 1)\sqrt{\delta},$$

and hence $\delta \ge 1/[2n(n + 1)]$. The grid (5.3) attains this value by the computation in Theorem 5.2.

If equality holds in the minimax bound, every step in the preceding sum is an equality. Therefore $\sqrt{2t_j + \delta} = (2j + 1)\sqrt{\delta}$ for each j, which gives exactly (5.3). □

Remark 5.5. A partition chosen for a particular μ can improve on the fixed-grid bound. Measurements outside the spectral algebra are governed by the equality conditions of Theorem 4.2.

5.2 Variance and scalar quantization

Proposition 5.6 (Variance identity)**.** *For every finite measurable partition π,*

$$D_\mu(\pi) = \sum_i \frac{w_i \operatorname{Var}_i(t)}{\sqrt{\mathbb{E}_i(t^2)} + \mathbb{E}_i(t)}. \tag{5.5}$$

A summand with zero denominator is defined as zero. If μ is supported on $[a, b] \subset (0, 1]$, then

$$\frac{1}{2b} \sum_i w_i \operatorname{Var}_i(t) \le D_\mu(\pi) \le \frac{1}{2a} \sum_i w_i \operatorname{Var}_i(t). \tag{5.6}$$

Proof. For $m_i = \mathbb{E}_i(t)$ and $r_i = \sqrt{\mathbb{E}_i(t^2)}$,

$$r_i - m_i = \frac{r_i^2 - m_i^2}{r_i + m_i} = \frac{\operatorname{Var}_i(t)}{r_i + m_i}$$

whenever $r_i + m_i > 0$. If the denominator is zero, $t = 0$ almost surely on the cell and its contribution vanishes. On $[a, b]$, the denominator lies between $2a$ and $2b$. □

The variance identity compares spectral approximation with squared-error quantization. Optimizing the original deficit leads to the following scalar cost, whose minimizing cells for fixed positive centers are intervals.

Proposition 5.7. *For $n \ge 1$, define*

$$D_n(\mu) = \inf_{\pi:\, |\pi| \le n} D_\mu(\pi),$$

where the infimum ranges over finite measurable partitions modulo μ-null cells. Then

$$D_n(\mu) = \inf_{r_1,\dots,r_n>0} \int_{[0,1]} \min_{1\le i\le n} \frac{(t-r_i)^2}{2r_i}\, d\mu(t). \tag{5.7}$$

For fixed distinct positive values $r_1 < \cdots < r_n$, the minimizing cells are intervals separated by the thresholds $\sqrt{r_i r_{i+1}}$. Consequently, the infimum defining $D_n(\mu)$ is unchanged if it is restricted to interval partitions chosen as a function of μ.

Proof. For $z \ge 0$,

$$\sqrt{z} = \inf_{r>0} \left(\frac{z}{2r} + \frac{r}{2} \right).$$

Applying this separately on each cell gives

$$D_\mu(\pi) = \inf_{r_1,\dots,r_m>0} \sum_i \int_{B_i} \frac{(t-r_i)^2}{2r_i}\, d\mu(t).$$

The formula remains valid for a cell on which $t = 0$ almost surely, by letting its r_i decrease to zero. For fixed r_i, the minimum over measurable assignments is obtained pointwise by assigning t to an index minimizing the integrand. This proves (5.7).

For $0 < r < s$,

$$\frac{(t-r)^2}{2r} - \frac{(t-s)^2}{2s} = \frac{s-r}{2rs}(t^2 - rs).$$

Thus the two costs cross at $t = \sqrt{rs}$. Ordered centers give interval cells, with arbitrary assignments on their common boundaries. Optimizing the centers, or approximating their infimum, proves the final assertion. □

Theorem 5.8 (A sharp approximation order)**.** *Suppose μ is the uniform probability measure on $[a, b]$ with $0 < a < b \le 1$. For every measurable partition π with at most n nonnull cells,*

$$D_\mu(\pi) \ge \frac{(b-a)^2}{24b\, n^2}. \tag{5.8}$$

Consequently, every associated spectral measurement satisfies

$$\mathrm{BC}_{\mathcal{E}_\pi}(p, q) - F(\varphi_p, \varphi_q) \ge \sqrt{\frac{\beta}{\alpha}}\, \frac{(b-a)^2}{24b\, n^2}.$$

Together with Theorem 5.2, this proves that the order n^{-2} cannot be improved uniformly within finite spectral measurements.

Proof. Let $\ell = b - a$ and put $c_i = \mathbb{E}_i(t)$. The sum of the conditional variances is

$$V(\pi) := \sum_i w_i \operatorname{Var}_i(t) = \frac{1}{\ell} \sum_i \int_{B_i \cap [a,b]} (t - c_i)^2\, dt.$$

Replacing the assigned center at each point by a nearest one gives

$$V(\pi) \ge \frac{1}{\ell} \int_a^b \min_i (t - c_i)^2\, dt.$$

Order the distinct centers and intersect their one-dimensional Voronoi cells with $[a,b]$. These are intervals $J_1,\dots,J_k$, with $k\le n$, whose lengths ℓ_i sum to ℓ. For every interval J of length h and every real c,

$$\int_J (t-c)^2\,dt \ge \frac{h^3}{12},$$

as follows by centering the interval at its midpoint and expanding the square. Therefore

$$V(\pi) \ge \frac{1}{12\ell}\sum_{i=1}^{k}\ell_i^3 \ge \frac{\ell^2}{12k^2} \ge \frac{\ell^2}{12n^2}.$$

The middle inequality is Jensen's inequality for the convex function s^3. Combining this estimate with the lower bound in (5.6) proves (5.8). Equation (5.1) gives the assertion about measurement overlaps. □

Example 5.9. The spectral distributions in Theorem 5.8 occur for projection pairs. Equip

$$M = L^\infty([a,b])\overline{\otimes}M_2(\mathbb{C})$$

with the product of normalized Lebesgue measure and the normalized matrix trace, and set

$$p(t)=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad q(t)=\begin{pmatrix}t^2&t\sqrt{1-t^2}\\t\sqrt{1-t^2}&1-t^2\end{pmatrix}.$$

Then $\alpha=\beta=1/2$ and $|pq|(t)=tq(t)$. Its distribution in qMq is uniform on $[a,b]$, so the lower bound applies to every finite measurable partition of this spectral variable.

Remark 5.10. The spectral restriction is essential: Example 4.9 has a continuous nonzero angle spectrum but admits an exact three-atom projective measurement outside the spectral algebra.

6 Common contexts and commuting projections

A common commutative context increases the fidelities between the trace state and a finite family of projection states. Conditional expectation expresses this increase in terms of positive contractions. Rounding these contractions to projections relates the fidelity defect to approximation by commuting projections in the tracial L^2 norm.

Throughout this section, (M,τ) is a finite von Neumann algebra with faithful normal tracial state, and $\|x\|_2=\tau(x^*x)^{1/2}$. For a unital von Neumann subalgebra $N\subseteq M$, let $E_N:M\to N$ be the trace-preserving conditional expectation. We use its extension as the orthogonal projection from $L^2(M,\tau)$ onto $L^2(N,\tau)$.

6.1 The radial fidelity defect

We compare three quantities: the weighted radial fidelity defect $\eta_N(p)$, the squared distance $V_N(p)$ to the subalgebra, and the squared distance $R_N(p)$ to its projections.

Definition 6.1. For $p\in\mathcal{P}(M)$ and a unital von Neumann subalgebra $N\subseteq M$, define

$$\eta_N(p):=\tau\big(E_N(p)^{1/2}-E_N(p)\big),\qquad V_N(p):=\|p-E_N(p)\|_2^2. \tag{6.1}$$

We also write

$$R_N(p):=\inf_{e\in\mathcal{P}(N)}\|p-e\|_2^2.$$

These quantities are defined at $p=0$ as well as at nonzero projections.

Proposition 6.2. *Let $p \neq 0$ and $r = \tau(p)$. Then*

$$\eta_N(p) = \sqrt{r}\big[F(\varphi_p|_N, \tau|_N) - F(\varphi_p, \tau)\big]. \tag{6.2}$$

Moreover, $\eta_N(p) \geq 0$, and if $N \subseteq N_1$, then $\eta_{N_1}(p) \leq \eta_N(p)$.

Proof. The density of $\varphi_p|_N$ relative to $\tau|_N$ is $E_N(p)/r$. Consequently,

$$F(\varphi_p|_N, \tau|_N) = \frac{\tau(E_N(p)^{1/2})}{\sqrt{r}}.$$

Since $F(\varphi_p, \tau) = \sqrt{r}$ and $\tau(E_N(p)) = r$, this gives (6.2). Positivity follows from $0 \leq E_N(p) \leq 1$. The monotonicity assertion follows from monotonicity of fidelity under restriction. Equivalently, it follows by applying operator concavity of the square root to $E_N(E_{N_1}(p)) = E_N(p)$ and then taking the trace. □

Theorem 6.3 (Variance and spectral rounding)**.** *For every projection p and every unital von Neumann subalgebra N, put $a = E_N(p)$ and $e = 1_{[1/2,1]}(a)$. Then*

$$V_N(p) = \tau(a - a^2), \tag{6.3}$$

$$R_N(p) = \|p - e\|_2^2 = \tau\big(\min\{a, 1-a\}\big), \tag{6.4}$$

$$\tfrac{1}{2}V_N(p) \leq \eta_N(p) \leq \sqrt{V_N(p)}, \tag{6.5}$$

$$R_N(p) \leq (1+\sqrt{2})\eta_N(p), \qquad R_N(p) \leq 2V_N(p). \tag{6.6}$$

In particular, $\eta_N(p) = 0$ if and only if $p \in N$. The constants 2 in $V_N(p) \leq 2\eta_N(p)$ and $1+\sqrt{2}$ in (6.6) are optimal. The constant 1 in the upper bound of (6.5), and the constant 2 in $R_N(p) \leq 2V_N(p)$, are also optimal.

Proof. Orthogonality of conditional expectation gives

$$\|p - a\|_2^2 = \tau(p) - \tau(a^2) = \tau(a - a^2).$$

For a projection $f \in N$, traciality and the defining property of E_N give

$$\|p - f\|_2^2 = \tau(a) + \tau\big((1-2a)f\big).$$

If $h = h^* \in N$ and f is a projection, then $\tau(hf) \geq -\tau(h_-)$, with equality for the negative spectral projection of h, allowing either choice on its kernel. Applying this observation to $h = 1 - 2a$ proves (6.4) and the optimality of e.

For $0 \leq t \leq 1$, let $g(t) = \sqrt{t} - t$. The scalar inequalities

$$\begin{aligned} t - t^2 &\leq 2g(t), & g(t)^2 &\leq t - t^2, \\ \min\{t, 1-t\} &\leq (1+\sqrt{2})g(t), & \min\{t, 1-t\} &\leq 2(t - t^2). \end{aligned} \tag{6.7}$$

imply the stated bounds by functional calculus and Cauchy–Schwarz for τ. For completeness, the first inequality follows from $t - t^2 = \sqrt{t}(1+\sqrt{t})g(t)$, and the second from $(1-\sqrt{t})^2 \leq 1 - t$. For the third inequality, the ratio of its left side to $g(t)$ is

$$\frac{\sqrt{t}}{1-\sqrt{t}} \quad (0 < t \leq 1/2), \qquad 1 + \frac{1}{\sqrt{t}} \quad (1/2 \leq t < 1).$$

It is maximized at $t = 1/2$, where it equals $1 + \sqrt{2}$. The last inequality is immediate on the two intervals separated by $1/2$. The characterization of zero defect follows from (6.5) and (6.3).

To verify the constants, take $N = \mathbb{C}1$, so $a = r1$ when $\tau(p) = r$. As $r \uparrow 1$,

$$\frac{V_N(p)}{\eta_N(p)} = \sqrt{r}(1 + \sqrt{r}) \longrightarrow 2,$$

whereas $\eta_N(p)/\sqrt{V_N(p)} \to 1$ as $r \downarrow 0$. Such traces occur for projections in a diffuse finite abelian algebra. At $r = 1/2$,

$$R_N(p) = \tfrac{1}{2}, \qquad V_N(p) = \tfrac{1}{4}, \qquad \eta_N(p) = \tfrac{\sqrt{2}-1}{2},$$

which gives equality in both rounding bounds. This last example already occurs for a rank-one projection in $M_2(\mathbb{C})$ with normalized trace. □

6.2 Finite common contexts

For a tuple $\mathbf{p} = (p_1, \dots, p_m)$ of projections, define

$$\Delta_M(\mathbf{p}) := \inf_{\substack{A \subseteq M \text{ unital} \\ A \text{ abelian}}} \sum_{j=1}^{m} \eta_A(p_j), \tag{6.8}$$

$$D_M(\mathbf{p}) := \inf_{\substack{e_1,\dots,e_m \in \mathcal{P}(M) \\ [e_i,e_j]=0 \text{ for all } i,j}} \sum_{j=1}^{m} \|p_j - e_j\|_2^2. \tag{6.9}$$

The infimum in (6.8) is taken over von Neumann subalgebras. A finite-dimensional choice of A is equivalent to a finite Boolean context. Approximation by commuting operators in normalized Hilbert–Schmidt norm has been studied in [14, 16]. The quantity Δ_M measures the least total radial fidelity defect of a common context; D_M measures the best commuting projection approximation.

Proposition 6.4 (Finite approximation of a common context)**.** *Let $A \subseteq M$ be unital and abelian. For each integer $n \geq 1$, there is a finite-dimensional unital subalgebra $B \subseteq A$ with at most n^m atoms such that*

$$0 \leq \sum_{j=1}^{m} \eta_B(p_j) - \sum_{j=1}^{m} \eta_A(p_j) \leq \frac{m}{2n(n+1)}. \tag{6.10}$$

Consequently, Δ_M is unchanged if its defining infimum is restricted to finite Boolean contexts.

Proof. Put $a_j = E_A(p_j)$ and $t_k = k(k+1)/[n(n+1)]$, $0 \leq k \leq n$. Partition $[0,1]$ into n intervals with endpoints t_k^2, choosing the intervals disjoint and including 1. Let B be generated by the corresponding spectral projections of all the a_j. Their joint refinement has at most n^m nonzero atoms. On an atom f of B, use the normalized trace $\tau(\,\cdot\, f)/\tau(f)$ to regard $X = a_j^{1/2} f$ as a scalar random variable. Its values lie in some $[t_{k-1}, t_k]$, and

$$\sqrt{\frac{\tau(a_j f)}{\tau(f)}} - \frac{\tau(a_j^{1/2} f)}{\tau(f)} = \sqrt{\mathbb{E}_f(X^2)} - \mathbb{E}_f(X) \leq \frac{1}{2n(n+1)}$$

by Lemma 5.1 and the grid calculation in Theorem 5.2. Multiply by $\tau(f)$ and sum over the atoms. Since $E_B(p_j) = E_B(a_j)$, this gives

$$0 \leq \eta_B(p_j) - \eta_A(p_j) \leq \frac{1}{2n(n+1)}.$$

Summation over j proves (6.10). □

Theorem 6.5 (Common contexts and commuting approximation)**.** *For every finite tuple of projections,*

$$\frac{D_M(\mathbf{p})}{1+\sqrt{2}} \le \Delta_M(\mathbf{p}) \le \sqrt{m D_M(\mathbf{p})}. \tag{6.11}$$

The following conditions are equivalent:

(i) *the projections $p_1, \dots, p_m$ commute;*

(ii) $D_M(\mathbf{p}) = 0$*;*

(iii) $\Delta_M(\mathbf{p}) = 0$*;*

(iv) *one finite Boolean context has zero defect for every p_j.*

If the p_j are nonzero, these conditions are also equivalent to the existence of a single abelian subalgebra whose restriction preserves all pairwise fidelities in $\{\tau, \varphi_{p_1}, \dots, \varphi_{p_m}\}$.

Proof. For an abelian A, round each $E_A(p_j)$ at $1/2$ as in Theorem 6.3. The resulting projections lie in A and commute. Thus

$$D_M(\mathbf{p}) \le (1+\sqrt{2}) \sum_j \eta_A(p_j).$$

Taking the infimum gives the first inequality in (6.11). Conversely, let $e_1, \dots, e_m$ be commuting projections and let $A = W^*(e_1, \dots, e_m)$. Orthogonal projection in L^2 and (6.5) give

$$\sum_j \eta_A(p_j) \le \sum_j \|p_j - E_A(p_j)\|_2 \le \sum_j \|p_j - e_j\|_2 \le \sqrt{m \sum_j \|p_j - e_j\|_2^2}.$$

Taking the infimum proves the second inequality.

If $D_M(\mathbf{p}) = 0$, choose commuting projection tuples $\mathbf{e}^{(k)}$ with $e_j^{(k)} \to p_j$ in L^2. Multiplication is continuous in L^2 on operator-norm bounded sets, so $[p_i, p_j] = 0$. Conversely, commuting p_j generate an abelian algebra with at most 2^m atoms, and all its defects vanish. The inequalities (6.11) now give all four equivalences.

For the last assertion, preservation of the fidelities with τ and (6.2) imply that each p_j belongs to the common algebra. They therefore commute. If they commute, their densities and the density 1 of τ all belong to $W^*(p_1, \dots, p_m)$, where the trace formula for fidelity gives preservation of every pair. □

6.3 Commutators and continuity

Proposition 6.6 (Commutator bounds)**.** *If $m \ge 2$, then*

$$\Delta_M(\mathbf{p}) \ge \frac{1}{4(m-1)} \sum_{1 \le i < j \le m} \|[p_i, p_j]\|_2^2. \tag{6.12}$$

For two projections p, q, put

$$b = pqp + (1-p)q(1-p), \qquad A_p = W^*(p, b).$$

The algebra A_p is abelian, and

$$\Delta_M(p, q) \le \eta_{A_p}(q) = \tau(\sqrt{b}) - \tau(q) \le \|[p, q]\|_2. \tag{6.13}$$

Proof. For a positive contraction c and $x \in L^2(M,\tau)$,

$$\|[c,x]\|_2 = \|[c-\tfrac{1}{2}1,x]\|_2 \le 2\|c-\tfrac{1}{2}1\|\|x\|_2 \le \|x\|_2. \tag{6.14}$$

Fix an abelian A and write $a_j = E_A(p_j)$ and $V_j = V_A(p_j)$. Since the a_j commute,

$$[p_i,p_j] = [p_i-a_i,p_j] + [a_i,p_j-a_j],$$

so (6.14) gives

$$\|[p_i,p_j]\|_2^2 \le 2(V_i+V_j).$$

Summing over the pairs and using $V_j \le 2\eta_A(p_j)$ yields

$$\sum_{i<j}\|[p_i,p_j]\|_2^2 \le 2(m-1)\sum_j V_j \le 4(m-1)\sum_j \eta_A(p_j).$$

Taking the infimum over A proves (6.12).

For the upper bound, p and b commute. Pinching by p is the trace-preserving conditional expectation onto $\{p\}'\cap M$ and sends q to b. Since A_p is contained in this subalgebra and contains b, we have $E_{A_p}(q) = b$ and $E_{A_p}(p) = p$. The two off-diagonal corners give

$$\|q-b\|_2^2 = 2\|pq(1-p)\|_2^2 = \|[p,q]\|_2^2.$$

Equation (6.5) proves (6.13). □

Proposition 6.7 (L^2 continuity)**.** *For every unital abelian $A \subseteq M$ and projections p, q,*

$$|\eta_A(p) - \eta_A(q)| \le \|p-q\|_2. \tag{6.15}$$

Consequently, if $\mathbf{p} = (p_1,\dots,p_m)$ and $\mathbf{q} = (q_1,\dots,q_m)$, then

$$|\Delta_M(\mathbf{p}) - \Delta_M(\mathbf{q})| \le \sum_{j=1}^m \|p_j - q_j\|_2, \tag{6.16}$$

$$|\sqrt{D_M(\mathbf{p})} - \sqrt{D_M(\mathbf{q})}| \le \left(\sum_{j=1}^m \|p_j-q_j\|_2^2\right)^{1/2}. \tag{6.17}$$

Proof. Put $a = E_A(p)$ and $c = E_A(q)$. Complete positivity of E_A gives

$$\begin{pmatrix} a & E_A(pq) \\ E_A(qp) & c \end{pmatrix} \ge 0.$$

Since A is abelian, $|E_A(pq)| \le \sqrt{ac}$. Taking traces yields

$$\|\sqrt{a}-\sqrt{c}\|_2^2 = \tau(p)+\tau(q)-2\tau(\sqrt{ac}) \le \tau(p)+\tau(q)-2\tau(pq) = \|p-q\|_2^2.$$

The elements $\sqrt{a}$ and $\sqrt{c}$ commute and lie between 0 and 1. Hence $\|1-\sqrt{a}-\sqrt{c}\| \le 1$, and

$$|\eta_A(p)-\eta_A(q)| = \left|\tau\big((\sqrt{a}-\sqrt{c})(1-\sqrt{a}-\sqrt{c})\big)\right| \le \|\sqrt{a}-\sqrt{c}\|_2.$$

This proves (6.15). Summation and the uniformity in A give (6.16). Finally, $\sqrt{D_M}$ is the distance in $L^2(M,\tau)^m$ to the set of commuting projection tuples, and distance to a fixed subset is 1-Lipschitz. □

6.4 Two-projection algebras

In $W^*(p,q)$, the two-projection decomposition reduces both optimization problems to a measurable family of two-dimensional problems. We first solve the matrix problem, then integrate its minima over the central angle distribution.

Lemma 6.8. *Let $M = M_2(\mathbb{C})$ with normalized trace, let p be the projection onto $(1,0)$, and let q_θ be the projection onto $(\cos\theta, \sin\theta)$, where $0 \le \theta \le \pi/2$. Then*

$$\Delta_M(p, q_\theta) = \tfrac{1}{2}(\cos\theta + \sin\theta - 1), \tag{6.18}$$

$$D_M(p, q_\theta) = 1 - \max\{\cos\theta, \sin\theta\}. \tag{6.19}$$

The infimum in (6.18) is attained by either $W^*(p)$ or $W^*(q_\theta)$.

Proof. The endpoint cases are commuting pairs, so assume $0 < \theta < \pi/2$. An abelian algebra can be enlarged to a maximal abelian algebra without increasing the defect. In the Bloch representation, write $u, v \in S^2$ for the vectors of p, q_θ, and $n \in S^2$ for its measurement axis. The sum of the two defects is

$$\tfrac{1}{2}\big(h(n\cdot u) + h(n\cdot v) - 2\big), \qquad h(s) = \sqrt{\frac{1+s}{2}} + \sqrt{\frac{1-s}{2}}.$$

The function h decreases with $|s|$. Projecting n onto the plane spanned by u, v and normalizing increases the absolute values of both inner products. If the projection is zero, any axis in that plane has no larger cost. Thus a minimizing measurement may be chosen in the real plane of the two defining vectors. If its first vector has angle ϕ, the cost becomes

$$\tfrac{1}{2}\big(L(\phi) + L(\phi - \theta) - 2\big), \qquad L(t) = |\cos t| + |\sin t|.$$

Between consecutive points of $(\pi/2)\mathbb{Z} \cup (\theta + (\pi/2)\mathbb{Z})$, the sum of the two L terms has strictly negative second derivative. Its minimum is therefore at an endpoint. At every such endpoint its value is $1 + \cos\theta + \sin\theta$. This proves (6.18) and its attainment statement.

For the second formula, commuting rank-one projections in M_2 are equal or complementary. In the equal case, minimizing over the common rank-one projection gives $1-\cos\theta$; in the complementary case it gives $1-\sin\theta$. One way to see these values is to use the Bloch vectors: the two minimizations maximize $n\cdot(u+v)$ and $n\cdot(u-v)$, respectively, and $|u+v| = 2\cos\theta$, $|u-v| = 2\sin\theta$. If either approximating projection is 0 or 1, its squared distance to a rank-one projection is $1/2$. Since $1 - \max\{\cos\theta, \sin\theta\} \le 1 - 1/\sqrt{2} < 1/2$, such a choice cannot improve the minimum. This proves (6.19). □

Theorem 6.9 (Principal-angle formulas)**.** *Let $p, q \in \mathcal{P}(M)$ and $N = W^*(p,q)$, with the restricted trace. Set*

$$S = |p - q|, \qquad C = |p + q - 1|.$$

Then $S, C \in Z(N)$, $S^2 + C^2 = 1$, and

$$\Delta_N(p,q) = \tfrac{1}{2}\tau(S + C - 1), \tag{6.20}$$

$$D_N(p,q) = \tau\big(1 - \max\{S, C\}\big). \tag{6.21}$$

The maximum is taken in the commutative algebra $Z(N)$. The infimum in (6.20) is attained by

$$A_p = W^*\big(p, pqp + (1-p)q(1-p)\big),$$

and also by the corresponding algebra obtained by exchanging p, q. In particular,

$$\Delta_M(p,q) \le \tfrac{1}{2}\tau(S + C - 1). \tag{6.22}$$

Proof. Direct multiplication shows that $z = (p-q)^2$ commutes with both p and q, and $(p+q-1)^2 = 1-z$. Thus $S = \sqrt{z}$ and $C = \sqrt{1-z}$ are central. The two-projection decomposition [17] separates the commuting part, where $p, q \in \{0, 1\}$ on each central component, from a generic part of the form $M_2(Z)$ with Z abelian. On the generic part,

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}, \qquad C, S \geq 0, \quad C^2 + S^2 = 1. \tag{6.23}$$

The trace on this part is the integral of normalized matrix traces against a finite measure on the spectrum of the central angle $\theta = \arcsin S$.

Given an abelian $A \subseteq N$, put $B = W^*(A, Z(N))$. This algebra remains abelian and its defect is no larger. Its conditional expectation is $Z(N)$-linear. On the generic part, it is therefore determined by the images of the four matrix units and acts fiberwise as a trace-preserving conditional expectation onto an abelian subalgebra of $M_2(\mathbb{C})$. Indeed, positivity, idempotence, bimodularity on its range, and commutativity of the range all pass to the matrix fibers. Lemma 6.8, followed by $\eta_A \geq \eta_B$, gives

$$\eta_A(p) + \eta_A(q) \geq \tfrac{1}{2}\tau(S + C - 1).$$

The commuting components contribute zero to the right side. For $b = pqp + (1-p)q(1-p)$, the identities

$$b = C^2 p + S^2(1-p), \qquad \sqrt{b} = Cp + S(1-p)$$

show that the algebra A_p attains the two-dimensional minimum on every generic fiber and has zero defect on the commuting part. This proves (6.20), including attainment.

Any commuting pair of projections in N gives commuting pairs in its matrix fibers. The second assertion of Lemma 6.8 therefore proves the lower bound in (6.21). The minimizing pairs in that lemma can be chosen measurably in θ: for $\theta \leq \pi/4$, take both projections onto the angle bisector; for $\theta > \pi/4$, take a complementary pair using the bisector between the range of p and the range of $1-q$. Together with the original commuting projections on the remaining components, these choices define commuting projections in N and attain the lower bound. Finally, $A_p \subseteq M$ is admissible for Δ_M, giving (6.22). □

Remark 6.10. The formulas optimize over $W^*(p, q)$. The attaining algebra A_p may be infinite dimensional, but Proposition 6.4 gives the same infimum over finite Boolean contexts in $W^*(p, q)$. Whether enlarging the ambient algebra can decrease $\Delta_M(p, q)$ remains open here.

6.5 Optimal comparison exponents

Proposition 6.11. *The power* $1/2$ *in the upper comparison* $\Delta_M(\mathbf{p}) \leq \sqrt{mD_M(\mathbf{p})}$ *cannot be increased, even for pairs in* $M_2(\mathbb{C})$. *The power* 1 *in the lower comparison* $\Delta_M(\mathbf{p}) \geq D_M(\mathbf{p})/(1+\sqrt{2})$ *cannot be decreased in an estimate uniform over finite tracial algebras. Likewise, for pairs of projections, the powers* 2 *and* 1 *in*

$$\tfrac{1}{4}\|[p, q]\|_2^2 \leq \Delta_M(p, q) \leq \|[p, q]\|_2$$

are optimal as uniform powers near commuting pairs.

Proof. For the rank-one pair of Lemma 6.8, as $\theta \downarrow 0$ we have

$$\Delta_M(p, q_\theta) = \tfrac{1}{2}\theta + O(\theta^2), \qquad D_M(p, q_\theta) = \tfrac{1}{2}\theta^2 + O(\theta^4), \qquad \|[p, q_\theta]\|_2 = \sin\theta\cos\theta.$$

Consequently, an upper bound $\Delta_M \le K D_M^{\beta}$ with $\beta > 1/2$, or $\Delta_M \le K\|[p,q]\|_2^{\gamma}$ with $\gamma > 1$, cannot hold with a uniform finite constant K.

For the lower powers, take $M = M_2(\mathbb{C}) \oplus \mathbb{C}$ with

$$\tau_\varepsilon(x \oplus \lambda) = \varepsilon \operatorname{tr}_2(x) + (1-\varepsilon)\lambda, \qquad 0 < \varepsilon < 1,$$

and let $P = p \oplus 1$, $Q = q_{\pi/4} \oplus 1$. These projections generate M. Indeed, $(P-Q)^2 = \frac{1}{2}1_{M_2} \oplus 0$ distinguishes its two central components, and the two rank-one projections generate the matrix component. Theorem 6.9 gives

$$\Delta_M(P,Q) = \varepsilon\frac{\sqrt{2}-1}{2}, \qquad D_M(P,Q) = \varepsilon\left(1 - \frac{1}{\sqrt{2}}\right), \qquad \|[P,Q]\|_2^2 = \frac{\varepsilon}{4}.$$

As $\varepsilon \downarrow 0$, these identities exclude any uniform positive lower bound $\Delta_M \ge k D_M^{\alpha}$ with $\alpha < 1$, or $\Delta_M \ge k\|[P,Q]\|_2^{\gamma}$ with $\gamma < 2$. These examples establish optimality of the exponents. Optimal constants in the tuple and commutator comparisons are not determined here. □

7 Metric topology and intrinsic length

Throughout this section, (M,τ) is finite and $\tau(1) = 1$, except where a semifinite trace is explicitly allowed. We identify a nonzero projection with its normalized projection state and write $d(p,q) = d_B(\varphi_p, \varphi_q)$. The centre-valued trace of a finite von Neumann algebra is denoted by T, normalized by $T(1) = 1$. Thus p and q are unitarily conjugate exactly when $T(p) = T(q)$. For the trace identities and projection comparison, see [24]. We first compare d with the tracial L^2 metric. A quadratic bound on changes in T then forces every rectifiable curve to remain in one unitary orbit, where the metric derivative determines the intrinsic length.

7.1 Metric comparison and completeness

Proposition 7.1. *Let p, q be nonzero finite-trace projections in a semifinite tracial von Neumann algebra. Put $\alpha = \tau(p)$, $\beta = \tau(q)$, $f = \tau(|pq|)/\sqrt{\alpha\beta}$, and $d^2 = 2 - 2f$. Then*

$$f \le \sqrt{\frac{\min(\alpha,\beta)}{\max(\alpha,\beta)}}, \tag{7.1}$$

$$\|p-q\|_2^2 \le 2\max(\alpha,\beta)(1-f^2), \qquad d^2 \le \frac{\|p-q\|_2^2}{\sqrt{\alpha\beta}}. \tag{7.2}$$

When $f > 0$,

$$|\log\alpha - \log\beta| \le -2\log f. \tag{7.3}$$

Consequently, the Bures and L^2 metrics are bi-Lipschitz equivalent on every set $\{p : a \le \tau(p) \le b\}$ with $0 < a \le b < \infty$.

Proof. Write $c = \tau(|pq|)$ and $b_0 = \tau(pq) = \tau(|pq|^2)$. The positive contractions $|pq| \le q$ and $|qp| \le p$ have equal traces, so $c \le \min(\alpha,\beta)$. This proves (7.1). Cauchy–Schwarz in both corners gives $c^2 \le \min(\alpha,\beta) b_0$. Therefore

$$\|p-q\|_2^2 = \alpha + \beta - 2b_0 \le 2\max(\alpha,\beta)(1-f^2).$$

Since $b_0 \le c$ and $\alpha + \beta \ge 2\sqrt{\alpha\beta}$,

$$\|p - q\|_2^2 \ge 2\sqrt{\alpha\beta} - 2c = \sqrt{\alpha\beta}\, d^2.$$

Taking logarithms in (7.1) proves (7.3). Finally, $1 - f^2 \le 2(1 - f) = d^2$ gives the asserted metric comparison, with constants depending only on a, b. □

Theorem 7.2 (Completeness)**.** *The space of normalized finite-trace projection states in a faithful normal semifinite tracial von Neumann algebra is complete for the Bures metric.*

Proof. Let (p_n) be Bures-Cauchy and set $\alpha_n = \tau(p_n)$. For sufficiently large m, n, their fidelity is positive and tends uniformly to one. Equation (7.3) shows that $(\log \alpha_n)$ is Cauchy. Hence $\alpha_n \to \alpha \in (0, \infty)$. By (7.2), (p_n) is Cauchy in $L^2(M, \tau)$. Let a be its L^2 limit. A uniformly operator-norm-bounded L^2 limit lies in M with the same operator-norm bound. For completeness, this follows by taking a weak-star convergent subnet of the bounded sequence and testing against $L^1 \cap L^2$, which is dense in both L^1 and L^2; the weak-star and L^2 limits then agree. Thus $a = a^*$ and $\|a\| \le 1$. Multiplication is continuous in L^2 on uniformly bounded sets, so $p_n^2 = p_n$ implies $a^2 = a$. Moreover, $\tau(a) = \|a\|_2^2 = \lim_n \alpha_n = \alpha > 0$. The second inequality in (7.2) now gives $d(p_n, a) \to 0$. □

7.2 Rectifiable components

The centre-valued trace varies by at most a constant times the square of the Bures distance. Summing this estimate along a rectifiable curve forces the centre-valued trace to remain constant.

Lemma 7.3. *For nonzero projections in a finite tracial algebra,*

$$\|T(p) - T(q)\|_{L^1(Z(M),\tau)} \le \alpha + \beta - 2\tau(|pq|) \le \max(\alpha, \beta) d(p, q)^2. \tag{7.4}$$

In particular, $|\tau(p) - \tau(q)| \le \max(\alpha, \beta) d(p, q)^2$.

Proof. The centre-valued trace of $|pq|$ equals that of $|qp|$, by polar decomposition and the tracial property of T. Hence $T(|pq|) \le T(p), T(q)$. In the commutative algebra $Z(M)$ this implies

$$|T(p) - T(q)| \le T(p) + T(q) - 2T(|pq|).$$

Applying τ proves the first inequality. Suppose $\alpha \ge \beta$ and write $c = \tau(|pq|) \le \beta$. The difference between the final upper bound and the middle expression in (7.4) is

$$\alpha - \beta - 2c\big(\sqrt{\alpha/\beta} - 1\big) \ge (\sqrt{\alpha} - \sqrt{\beta})^2 \ge 0.$$

The scalar statement follows from $\tau \circ T = \tau$. □

Theorem 7.4 (Rectifiable components)**.** *Two normalized projection states in a finite tracial algebra can be joined by a continuous curve of finite Bures length if and only if their projections are unitarily conjugate. Equivalently, the rectifiable components are the level sets of T. For a finite factor they are the sets of fixed scalar trace.*

Proof. Let $p(t)$ be a continuous curve of finite Bures length L. For any partition of a subinterval, (7.4) gives

$$\sum_j \|T(p(t_j)) - T(p(t_{j-1}))\|_1 \le \sum_j d(p(t_j), p(t_{j-1}))^2 \le L \max_j d(p(t_j), p(t_{j-1})).$$

Uniform continuity and arbitrarily fine partitions imply that $T(p(t))$ is constant. Conversely, $T(p) = T(q)$ gives $q = upu^*$ for a unitary u. Bounded Borel functional calculus supplies $h = h^* \in M$ with $u = e^{ih}$. The path $e^{ith}pe^{-ith}$ is operator-norm smooth and has constant positive trace. Equation (7.2) gives it finite Bures length. □

Example 7.5. In $M = L^\infty([0,1])$ with the integral trace, let $p_t = 1_{[0,t]}$ for $t \in [a,b] \subset (0,1]$. Then

$$d(p_s, p_t)^2 = 2 - 2\sqrt{\frac{\min(s,t)}{\max(s,t)}}.$$

This is a nonconstant continuous path. Every nonconstant subpath has infinite length by theorem 7.4, since unitary orbits in an abelian algebra are singletons.

7.3 Metric derivatives and the Bures tensor

On a unitary orbit $\mathcal{O}_p$ with $\tau(p) = t > 0$, the bounded tangent vectors are the self-adjoint operators X satisfying

$$pXp = (1-p)X(1-p) = 0. \tag{7.5}$$

Proposition 7.6. *Let q_h be projections of trace t with $(q_h - p)/h \to X$ in L^2 as $h \to 0$. Then X satisfies (7.5) in L^2, and*

$$\lim_{h\to 0} \frac{d(p,q_h)}{|h|} = \frac{\|X\|_2}{\sqrt{2t}}. \tag{7.6}$$

Consequently, the weak Riemannian tensor on every operator-norm smooth unitary orbit is

$$g_p(X,Y) = \frac{\operatorname{Re}\tau(XY)}{2t}. \tag{7.7}$$

Every Bures-rectifiable curve, parametrized by arclength, is L^2-absolutely continuous, and its Bures length is

$$L_B(p(\cdot)) = \frac{1}{\sqrt{2t}} \int \|\dot{p}(s)\|_2 \, ds. \tag{7.8}$$

Proof. Put $D_h = (q_h - p)/h$. If a uniformly bounded family converges in L^2, its left multiplication operators converge strongly on L^2. More explicitly, for $Y \in L^2$, approximate Y by a bounded element and use $\|(q_h - p)Y\|_2 \le \|(q_h - p)(Y - Y_0)\|_2 + \|(q_h - p)Y_0\|_2$. It follows that $\|(q_h - p)D_h\|_2 \to 0$. Differentiating $q_h^2 = q_h$ in L^2 therefore yields $pX + Xp = X$, which is equivalent to (7.5).

Let $C_h = (pq_hp)^{1/2}$ in the corner pMp. The exact identity

$$d(p,q_h)^2 - \frac{\|p - q_h\|_2^2}{2t} = \frac{\tau((p - C_h)^2)}{t} \tag{7.9}$$

follows by expansion. Since $0 \le C_h \le p$,

$$0 \le \tau((p - C_h)^2) \le \tau((p - C_h^2)^2) \le \tau((p - q_h)^4).$$

Here $p - C_h^2 = p(p - q_h)^2 p$, and compression is contractive for the L^2 norm. Finally,

$$h^{-2}\tau((p - q_h)^4) = \|(q_h - p)D_h\|_2^2 \longrightarrow 0.$$

Dividing (7.9) by h^2 proves (7.6); polarization gives (7.7).

An arclength parametrization is Bures-Lipschitz and has constant centre-valued trace by theorem 7.4. Equation (7.2) makes it Lipschitz into the Hilbert space $L^2(M,\tau)$, hence differentiable almost everywhere. Formula (7.6) identifies its metric derivative. Integrating the metric derivative gives (7.8), by the length formula for absolutely continuous metric curves [2, Section 1.1]. □

With the usual symmetric logarithmic derivative convention, the quantum Fisher metric is $4g$. Thus the normalization in (7.7) agrees with the second-order expansion of $d_B^2 = 2 - 2F$; see [9, 22].

7.4 The intrinsic distance

A direct rotation gives a path between projections with the same centre-valued trace [3, 17]. To prove minimality among all Bures-rectifiable paths, we send each projection path to a unitary path by multiplying its symmetry by the initial symmetry. The unitary L^2 length bound then gives the required lower bound. This recovers the Grassmann length studied in [4], with the Bures normalization and the full class of rectifiable curves. For related connections and conjugate-point geometry, see [5].

Lemma 7.7 (Unitary L^2 distance)**.** *If $u(s)$ is an L^2-absolutely continuous unitary path with $u(0) = 1$, then*

$$\| \operatorname{Arg}(u(1))\|_2 \le \int_0^1 \|\dot u(s)\|_2 \, ds, \tag{7.10}$$

where Arg *takes values in $[-\pi, \pi]$ and the choice at -1 has no effect on the norm. Equality is attained by $u(s) = \exp(is \operatorname{Arg}(u(1)))$.*

Proof. Let $r(e^{i\theta}) = |\theta|$ for $-\pi \le \theta \le \pi$. Choose smooth, strictly positive periodic functions r_ε converging uniformly to r and satisfying $|r_\varepsilon'| \le 1$. Such functions are obtained by circular convolution followed by addition of a positive constant tending to zero. Write $K = -iu^*\dot u$, a self-adjoint L^2 element for almost every s. Tracial differentiation of smooth functions of a unitary gives

$$\frac{d}{ds}\tau(r_\varepsilon(u)^2) = 2\tau\big(r_\varepsilon(u) r_\varepsilon'(u) K\big).$$

This identity follows first for trigonometric polynomials by the product rule and cyclicity of the trace. For a smooth periodic function f, $\sum_{n\in\mathbb{Z}} |n\widehat{f}(n)| < \infty$; the corresponding derivative series is dominated in L^1 along the path by this constant times $\|\dot u(s)\|_2$. Fourier approximation therefore proves the identity also for L^2-absolutely continuous unitary paths. Cauchy–Schwarz gives

$$\left| \frac{d}{ds}\tau(r_\varepsilon(u)^2)^{1/2} \right| \le \|K\|_2 = \|\dot u\|_2.$$

Integrate and let $\varepsilon \downarrow 0$ to obtain (7.10). The stated exponential path has constant L^2 speed $\| \operatorname{Arg}(u(1))\|_2$. □

Lemma 7.8 (Direct rotation)**.** *If $T(p) = T(q)$, there is $Z \in M$ with*

$$Z^* = -Z, \qquad pZp = (1-p)Z(1-p) = 0, \qquad \|Z\| \le \frac{\pi}{2}, \qquad e^Z p e^{-Z} = q.$$

For $\Theta = \arccos((pqp)^{1/2}) \in pMp$ one has

$$\|Z\|_2^2 = 2\tau(\Theta^2), \qquad (2q-1)(2p-1) = e^{2Z}. \tag{7.11}$$

Proof. Decompose the pair by the four projections $e_{11} = p \wedge q$, $e_{10} = p \wedge q^\perp$, $e_{01} = p^\perp \wedge q$, and $e_{00} = p^\perp \wedge q^\perp$, and their common orthogonal complement. On the latter, the two-projection decomposition has the form

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}, \qquad S = (1 - C^2)^{1/2}.$$

The block identification is implemented by the polar decomposition of $(1-p)qp$ on its full supports; in particular it is internal to M. Set $\theta = \arccos C$ and $Z = \left(\begin{smallmatrix} 0 & -\theta \\ \theta & 0 \end{smallmatrix}\right)$ on this part. On $e_{11} + e_{00}$ set $Z = 0$. Equality of the centre-valued traces and equivalence of the two generic blocks give $e_{10} \sim e_{01}$. If $v^*v = e_{10}$ and $vv^* = e_{01}$, put $Z = (\pi/2)(v - v^*)$ on their sum. Direct exponentiation verifies all asserted properties. Each nonzero angle occurs once on the p side and once on its paired complement, giving the trace identity. Finally $2p - 1$ anticommutes with Z, which gives the second identity in (7.11). □

Theorem 7.9 (Intrinsic Bures distance)**.** *Let d_{len} be the extended length distance obtained by taking the infimum of Bures lengths of continuous rectifiable projection paths. If $T(p) \neq T(q)$, then $d_{\mathrm{len}}(p, q) = \infty$. If $T(p) = T(q)$ and $t = \tau(p)$, then*

$$d_{\mathrm{len}}(p, q) = \left[\frac{\tau(\Theta^2)}{t}\right]^{1/2}, \qquad \Theta = \arccos((pqp)^{1/2}) \in pMp. \tag{7.12}$$

A minimizing path is $p(s) = e^{sZ} p e^{-sZ}$ with Z as in lemma 7.8.

Proof. The first assertion is theorem 7.4. For a rectifiable path with endpoints p, q, take its arclength parametrization and rescale the parameter to $[0, 1]$. A constant path requires no rescaling. Define $u(s) = (2p(s) - 1)(2p - 1)$. Then $u(0) = 1$, $u(1) = (2q - 1)(2p - 1)$, and $\|\dot{u}(s)\|_2 = 2\|\dot{p}(s)\|_2$ almost everywhere. By lemmas 7.7 and 7.8,

$$\int \|\dot{p}(s)\|_2 \, ds \geq \tfrac{1}{2} \| \operatorname{Arg}(e^{2Z}) \|_2 = \|Z\|_2.$$

The last equality follows from $\|Z\| \leq \pi/2$, including the endpoint spectral values. Formula (7.8) gives the required lower bound. On the rotation path, $\|\dot{p}(s)\|_2 = \|[Z, p]\|_2 = \|Z\|_2$ because Z is codiagonal. It therefore attains the bound, and (7.11) yields (7.12). □

Corollary 7.10 (Three distances)**.** *For $T(p) = T(q)$,*

$$d_B(\varphi_p, \varphi_q) \leq \arccos F(\varphi_p, \varphi_q) \leq d_{\mathrm{len}}(p, q). \tag{7.13}$$

The second inequality is an equality if and only if Θ is a scalar multiple of p. In $M_n(\mathbb{C})$, for rank-k projections with principal angles $\theta_1, \ldots, \theta_k$,

$$d_B^2 = 2 - \frac{2}{k} \sum_{j=1}^{k} \cos\theta_j, \qquad \arccos F = \arccos\left(\frac{1}{k} \sum_{j=1}^{k} \cos\theta_j\right), \qquad d_{\mathrm{len}}^2 = \frac{1}{k} \sum_{j=1}^{k} \theta_j^2.$$

Proof. The first inequality is $2\sin(u/2) \leq u$ for $0 \leq u \leq \pi/2$. The function $s \mapsto \cos\sqrt{s}$ is strictly convex on $[0, \pi^2/4]$. Apply scalar Jensen to the spectral distribution of Θ^2 under the normalized corner trace τ/t:

$$F(\varphi_p, \varphi_q) = \frac{\tau(\cos\Theta)}{t} \geq \cos\left(\sqrt{\frac{\tau(\Theta^2)}{t}}\right).$$

Strict convexity and faithfulness give the equality condition. The matrix formulas follow from the eigenvalues of Θ. □

8 Characterizing tracial reference states

Let ω be a faithful normal state on M. The radial identity of Lemma 3.1 recovers $\omega(p)$ from the metric. We now characterize traciality by comparing the fidelity of ω_p, ω_q with the square root of the ratio of these radial values. Along a local unitary orbit $p_t = e^{itx}pe^{-itx}$, we have $1 - F(\omega_p, \omega_{p_t}) = O(t^2)$. The fidelity bound therefore forces the first variation of $\omega(p_t)$ to vanish.

8.1 Nested projections

Proposition 8.1. *If $0 \neq q \leq p$, then*

$$F(\omega_p, \omega_q) = \sqrt{\frac{\omega(q)}{\omega(p)}}. \tag{8.1}$$

If ω is tracial, the equality in (8.1) also implies $q \leq p$.

Proof. The state ω_p assigns probability $\omega_p(q) = \omega(q)/\omega(p)$ to q. Since $pq = qp = q$, its conditioning by q satisfies

$$(\omega_p)_q(x) = \frac{\omega(pqxqp)/\omega(p)}{\omega(q)/\omega(p)} = \omega_q(x).$$

Apply Lemma 3.1 to the normal state ω_p and the projection q.

If ω is tracial, equality in (8.1) means $\omega(|pq|) = \omega(q)$. Since $0 \leq |pq| \leq q$, faithfulness gives $|pq| = q$, hence $qpq = q$ and $q \leq p$. □

For nontracial states, equality in (8.1) need not imply inclusion; Example 8.4 gives a qubit instance. Requiring the corresponding upper bound on every local unitary orbit, however, characterizes traces.

8.2 The fidelity bound and traciality

Theorem 8.2. *Let M be a von Neumann algebra with faithful normal state ω. The following conditions are equivalent.*

(i) *ω is a trace.*

(ii) *For all nonzero projections $p, q \in M$,*

$$F(\omega_p, \omega_q) \leq \sqrt{\frac{\omega(q)}{\omega(p)}}. \tag{8.2}$$

(iii) *For every nonzero projection p and every self-adjoint $x \in M$, there is an $\varepsilon > 0$ such that, with*

$$p_t = e^{itx}pe^{-itx},$$

one has

$$F(\omega_p, \omega_{p_t}) \leq \sqrt{\frac{\omega(p_t)}{\omega(p)}} \qquad (|t| < \varepsilon). \tag{8.3}$$

Consequently, condition (8.2) need only be tested on pairs of unitarily conjugate projections.

Proof. Suppose first that ω is tracial. The density formula for projection states gives

$$F(\omega_p, \omega_q) = \frac{\omega(|pq|)}{\sqrt{\omega(p)\omega(q)}}.$$

Since $|pq| = (qpq)^{1/2} \le q$, this proves (8.2). Thus the first condition implies the second, and the second immediately implies the third.

Assume the third condition. Fix $p \neq 0$ and $x = x^*$, and write

$$r(t) = \omega(p_t), \qquad r = r(0) > 0.$$

Choose a standard representation and a unit vector ξ implementing ω. For sufficiently small $|t|$, the vectors

$$\eta_t = \frac{p_t \xi}{\sqrt{r(t)}}$$

are well-defined unit vectors implementing ω_{p_t}. The map $t \mapsto p_t$ is differentiable in operator norm. Since $r(t)$ stays bounded away from zero near $t = 0$, there is a constant $C_0 < \infty$ such that

$$\|\eta_t - \eta_0\| \le C_0 |t|$$

for sufficiently small $|t|$. The common-representation characterization of fidelity therefore gives

$$\begin{aligned} F(\omega_p, \omega_{p_t}) &\ge |\langle \eta_0, \eta_t \rangle| \\ &\ge \operatorname{Re}\langle \eta_0, \eta_t \rangle \\ &= 1 - \frac{1}{2}\|\eta_t - \eta_0\|^2 \ge 1 - \frac{C_0^2}{2} t^2. \end{aligned}$$

Combining this estimate with (8.3) yields

$$\sqrt{\frac{r(t)}{r}} \ge 1 - \frac{C_0^2}{2} t^2$$

on both sides of $t = 0$. The function on the left is differentiable and equals 1 at zero. Dividing the corresponding difference inequality by positive and negative t shows that its derivative at zero vanishes. Hence

$$0 = r'(0) = i\,\omega(xp - px).$$

Thus $\omega([x, p]) = 0$ for every self-adjoint x and every projection p. If $y = y^* \in M$, approximate y in operator norm by real finite linear combinations of its spectral projections. Continuity of ω gives $\omega([x, y]) = 0$. Complex linearity then yields $\omega(ab) = \omega(ba)$ for all $a, b \in M$, proving that ω is tracial. □

Corollary 8.3. *Traciality of the reference state is determined by the unpointed metric space $(\mathcal{L}(M, \omega), d_{\mathrm{B}})$. In particular, a surjective Bures isometry between two such spaces preserves whether their reference states are traces, without a restriction on type I_2 summands.*

Proof. Proposition 3.2 identifies the reference point $o = \omega$ from the metric alone. For $u \in \mathcal{L}(M, \omega)$, define its radial value by

$$R_o(u) = \left(1 - \frac{1}{2} d_{\mathrm{B}}(u, o)^2\right)^2.$$

It is positive and satisfies $R_o(\omega_p) = \omega(p)$. Condition (8.2) is therefore the metric condition

$$1 - \frac{1}{2} d_{\mathrm{B}}(u, v)^2 \le \sqrt{\frac{R_o(v)}{R_o(u)}} \qquad (u, v \in \mathcal{L}(M, \omega)).$$

The equivalence in Theorem 8.2 proves the assertion. □

8.3 Equality without inclusion

Example 8.4. Let $M = M_2(\mathbb{C})$, let Tr be the ordinary matrix trace, and set

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \qquad \omega(a) = \mathrm{Tr}(\rho a).$$

The eigenvalues of ρ are $3/4$ and $1/4$, so ω is faithful. Let

$$p = |e_1\rangle\langle e_1|, \qquad v_t = \cos t\, e_1 - \sin t\, e_2, \qquad q_t = |v_t\rangle\langle v_t|.$$

Conditioning a faithful state on a rank-one projection produces the corresponding pure state. Consequently, for $0 \le t \le \pi/4$,

$$F(\omega_p, \omega_{q_t}) = \cos t, \qquad \omega(p) = \frac{1}{2}, \qquad \omega(q_t) = \frac{1}{2} - \frac{1}{4}\sin(2t).$$

For $0 < t < \pi/4$,

$$F(\omega_p, \omega_{q_t})^2 - \frac{\omega(q_t)}{\omega(p)} = \sin t\,(\cos t - \sin t) > 0,$$

so the tracial upper bound fails along this unitary orbit.

At $t = \pi/4$, write

$$q = q_{\pi/4} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\omega(q) = \frac{1}{4}, \qquad F(\omega_p, \omega_q) = \frac{1}{\sqrt{2}} = \sqrt{\frac{\omega(q)}{\omega(p)}}.$$

Nevertheless, p and q are distinct rank-one projections, so $q \not\leq p$. The order reconstruction (3.2) applies to this example without change.

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