Formalization of Robust Strictly Positive Real Synthesis for Fourth-Order Interval Polynomial Families

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Abstract

That every robustly Hurwitz-stable fourth-order interval polynomial family admits a common strictly positive real (SPR) numerator was conjectured by Anderson et al. (1990) and proved by Yu and Wang (2001). In this paper, we present a complete formalization of this classical synthesis theorem in the Lean 4 interactive theorem prover.

Keywords: strictly positive real synthesis; interval polynomial family; Hurwitz stability; proof reconstruction; formal verification; Lean 4

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1 Introduction

1.1 Background and the Classical Fourth-Order Synthesis Theorem

Strictly positive real (SPR) transfer functions play a fundamental role in passivity analysis, absolute stability, adaptive control, and output-error system identification [4, 12]. When the denominator coefficients vary independently within closed uncertainty intervals, the robust SPR synthesis problem asks for a single fixed real numerator polynomial that renders the transfer function strictly positive real for every admissible denominator in the family and at all real frequencies. Although Kharitonov's theorem [6] reduces robust Hurwitz stability of an interval polynomial family to four vertex polynomials, and robust SPR analysis for a prescribed numerator can similarly be reduced to finite vertex combinations [1, 10], the synthesis problem is existential: the numerator coefficients must be fixed once and for all before the plant varies over the interval box and the frequency varies over the real line.

For fourth-order robustly Hurwitz-stable interval polynomial families, Anderson et al. [1] conjectured in 1990 that a common SPR numerator always exists, proposing an argument based on linear programming. After Betser and Zeheb [2] demonstrated via counterexample that the proposed linear-programming condition was not necessary, Yu and Wang [11] established a complete constructive proof of Anderson's claim in 2001 (see also [10, 13, 12]). In the classical proof [11, 12], each fourth-order Hurwitz denominator $a(s)$ determines a convex feasible region $\Omega^a = \Omega^a_e \cup \Omega^a_t \subset \mathbb{R}^2$ bounded by an ellipse and tangent lines in the space of cubic numerator parameters (x, y). Nonemptiness of the four-vertex intersection $\bigcap_{i=1}^{4} \Omega^{a_i}$ is proved by intersecting chords connecting tangency points—splitting into cases according to whether the odd-part root endpoints coincide or are strictly separated—followed by two sequential small-parameter perturbations justified by compactness infima on bounded frequency intervals.

1.2 Formalization Obstacles, Proof Reconstruction, and Related Work

Our goal in this work is to construct a complete, kernel-verified formal proof of the fourth-order interval SPR synthesis theorem in the Lean 4 theorem prover [5] using the mathlib library [9]. As witnessed across modern formalized mathematics, paper proofs that rely on geometric region intersections, implicit nondegeneracy conventions, and qualitative "sufficiently small parameter" arguments often require structural reformulation into explicit algebraic invariants and quantitative bounds before they can be checked effectively by a proof assistant kernel. For instance, Dahmen, Hölzl, and Lewis [3], in formalizing the solution to the cap set problem in Lean, systematically separated the reuse of mathlib's algebraic hierarchy from the combinatorics reformulations needed for dependent type theory. Mahboubi and Sibut-Pinote [8], in their formal proof of the irrationality of $\zeta(3)$, combined *a posteriori* verification of expanded polynomial identities with explicit quantitative estimates to replace qualitative asymptotic steps. Likewise,

Li and Paulson [7], in formalizing the Budan–Fourier theorem and polynomial root counting in Isabelle/HOL, demonstrated both the substantial infrastructure required for general complex/real root counting via Sturm chains and Cauchy indices and the utility of Taylor shifts in root isolation.

Specifically, mechanizing the classical fourth-order interval SPR synthesis proof directly faces three main formalization obstacles: bridging the semantic open left-half-plane root predicate to fourth-order Routh–Hurwitz inequalities, constructing a common cubic candidate without branching on whether coefficient intervals degenerate to single points, and eliminating nonconstructive compactness infima and sequential small-parameter limits while respecting the strict quantifier order "there exists one numerator for all box members and all real frequencies." To address these challenges, we develop a formalization-oriented proof reconstruction (Sections 3–5): a degree-six horizontal translation polynomial bridges semantic Hurwitz stability directly to the third Routh–Hurwitz determinant and the two-vertex positive root band; a chord intersection anchored at the strictly separated roots of the worst-case even part provides a degeneracy-uniform cubic candidate with piecewise square-plus-linear lower bounds and a closed-form positive margin over the entire frequency half-axis; and a single explicit perturbation parameter simultaneously restores zero-frequency strictness and lifts the numerator to a monic quartic polynomial. The complete proof chain is verified by the Lean 4 kernel within 598 lines of code using only the three standard foundational axioms.

2 Formal Specification and Proof Overview

2.1 Mathematical Statement and Quantifier Structure

A real polynomial $a(s) \in \mathbb{R}[s]$ is *Hurwitz stable* if all of its complex roots lie in the open left half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, and a polynomial family is *robustly Hurwitz stable* if every member is Hurwitz stable. Following the equal-degree frequency-domain SPR convention [12], for a fourth-order Hurwitz-stable denominator $a(s)$ and a monic fourth-order real numerator $b(s)$, the transfer function $b(s)/a(s)$ is strictly positive real on the imaginary axis if

$$\operatorname{Re} \frac{b(i\omega)}{a(i\omega)} > 0 \qquad (\forall\, \omega \in \mathbb{R}). \tag{2.1}$$

Theorem 2.1 (Fourth-Order Interval SPR Synthesis [1, 11])**.** *Let $l_j \le u_j$ $(1 \le j \le 4)$ be real interval endpoints, and consider the fourth-order monic independent closed interval polynomial family*

$$K = \{\, a(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 : l_j \le a_j \le u_j,\ 1 \le j \le 4 \,\}. \tag{2.2}$$

If K is robustly Hurwitz stable, then there exists a single monic real polynomial $b(s)$ of degree four such that

$$\operatorname{Re} \frac{b(i\omega)}{a(i\omega)} > 0 \qquad (\forall\, a \in K,\ \forall\, \omega \in \mathbb{R}). \tag{2.3}$$

In the logical prefix $\forall K,\ \exists b,\ \forall a \in K,\ \forall \omega \in \mathbb{R}$ of Theorem 2.1, the quantifier order (uniform witness constraint) governs the entire formalization: the common numerator $b(s)$ and all intermediate construction parameters (the cubic coefficients x, y, the uniform frequency margin m, and the perturbation parameter ε) must be defined solely in terms of the eight box endpoints (l_j, u_j), prior to introducing $a \in K$ or $\omega \in \mathbb{R}$.

Setting the squared frequency $t = \omega^2 \ge 0$, decompose any denominator $a \in K$ and candidate real numerator $b(s)$ into their even and odd parts along the imaginary axis:

$$a(i\omega) = E_a(t) + i\omega O_a(t), \qquad E_a(t) = t^2 - a_2 t + a_4, \qquad O_a(t) = a_3 - a_1 t, \tag{2.4}$$

and $b(i\omega) = E_b(t) + i\omega O_b(t)$. Define the real frequency pairing polynomial

$$Q_{a,b}(t) \coloneqq \operatorname{Re}\big[b(i\omega)\overline{a(i\omega)}\big] = E_b(t)E_a(t) + t\, O_b(t)O_a(t). \tag{2.5}$$

Because every Hurwitz-stable $a \in K$ is nonvanishing on the imaginary axis, $|a(i\omega)|^2 = E_a(\omega^2)^2 + \omega^2 O_a(\omega^2)^2 > 0$ and $\mathrm{Re}[b(i\omega)/a(i\omega)] = Q_{a,b}(\omega^2)/|a(i\omega)|^2$. Consequently, (2.3) holds for all $a \in K$ and $\omega \in \mathbb{R}$ if and only if $Q_{a,b}(t) > 0$ for all $a \in K$ and all $t \geq 0$.

2.2 Formal Encoding in Lean 4

In `Lean/SPR/Common/Hurwitz.lean` and `Lean/SPR/N4/Basic.lean`, the core data structures, predicates, and top-level synthesis target are formalized as follows (shown verbatim from the source files, omitting only docstrings and rendering mathematical symbols in standard notation):

```
def SPR.IsHurwitz (a : ℝ[X]) : Prop :=
  ∀ z : ℂ, (a.map (algebraMap ℝ ℂ)).IsRoot z → z.re < 0

structure Poly4 where (a1 a2 a3 a4 : ℝ)

def Poly4.poly (a : Poly4) : ℝ[X] :=
  X ^ 4 + C a.a1 * X ^ 3 + C a.a2 * X ^ 2 + C a.a3 * X + C a.a4

structure Box4 where
  lower : Poly4
  upper : Poly4
  ordered1 : lower.a1 ≤ upper.a1   ordered2 : lower.a2 ≤ upper.a2
  ordered3 : lower.a3 ≤ upper.a3   ordered4 : lower.a4 ≤ upper.a4

def Box4.Contains (K : Box4) (a : Poly4) : Prop :=
  K.lower.a1 ≤ a.a1 ∧ a.a1 ≤ K.upper.a1 ∧
  K.lower.a2 ≤ a.a2 ∧ a.a2 ≤ K.upper.a2 ∧
  K.lower.a3 ≤ a.a3 ∧ a.a3 ≤ K.upper.a3 ∧
  K.lower.a4 ≤ a.a4 ∧ a.a4 ≤ K.upper.a4

def Box4.RobustlyHurwitz (K : Box4) : Prop :=
  ∀ a, K.Contains a → SPR.IsHurwitz a.poly

def FrequencySPR (a b : ℝ[X]) : Prop :=
  ∀ w : ℝ, 0 < ((b.eval₂ (algebraMap ℝ ℂ) ((w : ℂ) * Complex.I)) /
    (a.eval₂ (algebraMap ℝ ℂ) ((w : ℂ) * Complex.I))).re

def RobustSPRStatement : Prop :=
  ∀ K : Box4, K.RobustlyHurwitz →
    ∃ b : ℝ[X], b.Monic ∧ b.natDegree = 4 ∧
      ∀ a : Poly4, K.Contains a → FrequencySPR a.poly b
```

2.3 Proof Overview and Dependency Architecture

The formal proof decomposes Theorem 2.1 into a five-stage unidirectional pipeline, illustrated in Figure 1. Table 1 summarizes the five core source files (598 lines in total) plus the public entry point `Lean/Main.lean`, their declaration counts, and their proof reconstructions relative to [11, 12].

3 Bridging Hurwitz Stability via Horizontal Translation

3.1 Classical Proof Requirement and Formalization Obligations

The starting point of the classical synthesis proof is that every monic Hurwitz-stable quartic polynomial $a(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$ satisfies the two necessary algebraic conditions

$$a_j > 0 \quad (1 \leq j \leq 4) \qquad \text{and} \qquad \Delta_3(a) := a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0. \tag{3.1}$$

In informal treatments, (3.1) is either cited from the general Routh–Hurwitz theorem or derived by pairing complex-conjugate and negative real roots to factor $a(s) = (s^2 + ps + q)(s^2 + rs + h)$ with $p, q, r, h > 0$, which expands to $\Delta_3(a) = pr\big[(q-h)^2 + (p+r)(ph+rq)\big] > 0$.

In an interactive theorem prover, however, the hypothesis is given semantically by the root predicate `SPR.IsHurwitz`$(a.\mathtt{poly})$ ($\mathrm{Re}\, z < 0$ for all complex roots). Formalizing either paper argument directly would impose a heavy auxiliary obligation:

(i) Formalizing the existence of a real factorization into two monic quadratic factors from the Fundamental Theorem of Algebra and complex conjugation in mathlib, and linking the open left-half-plane root condition to positivity of each quadratic factor's coefficients; or

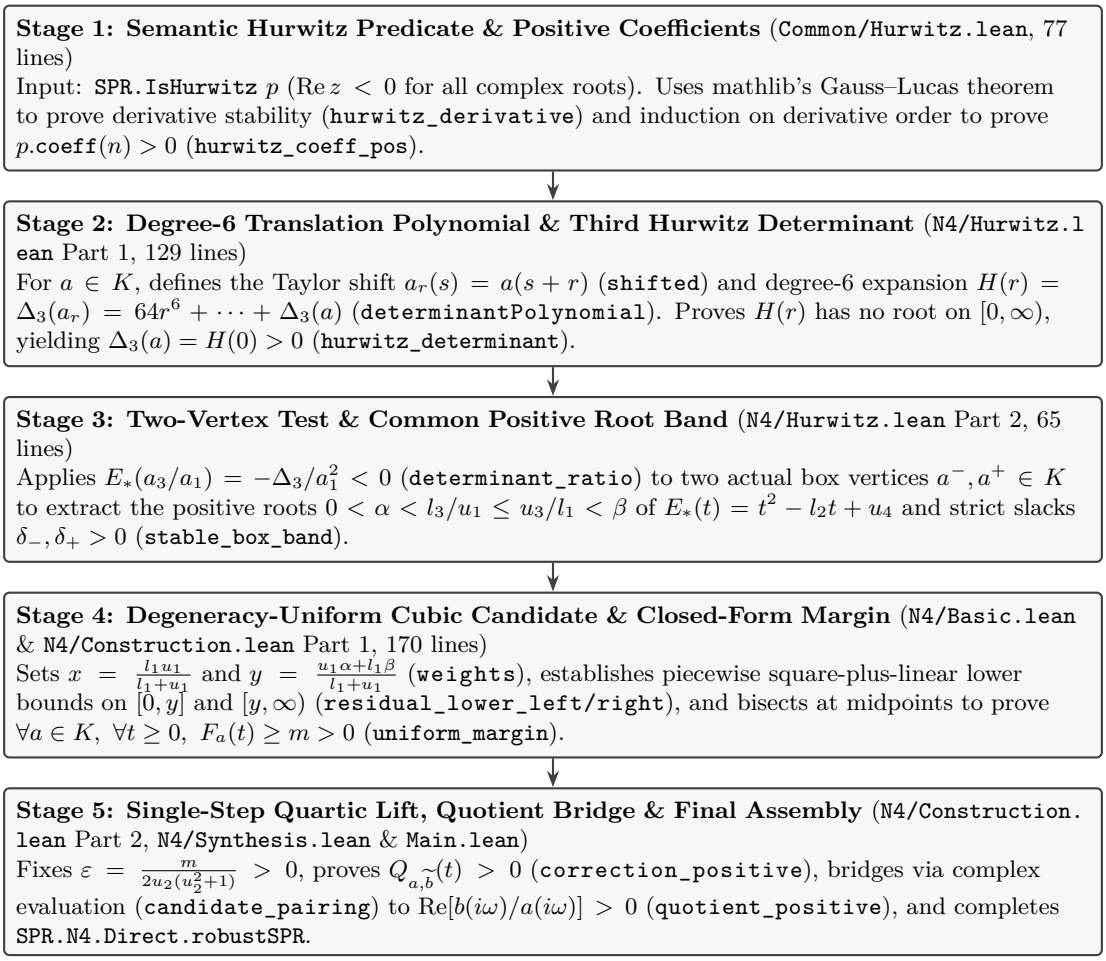


Figure 1: Formalization pipeline and declaration dependency graph for fourth-order interval SPR synthesis.

Table 1: Lean 4 module structure, code statistics, and proof reconstructions relative to [11, 12].

Module Path	Lines / Decls	Key Lean 4 Declarations	Proof Reconstruction vs. [11, 12]
Common/Hurwitz.lean	77 lines (1 def, 6 thms)	IsHurwitz, eval_pos, hurwitz_derivative, hurwitz_coeff_pos	Derivative induction via mathlib's Gauss–Lucas theorem instead of quadratic root pairing
N4/Basic.lean	80 lines (2 structs, 11 defs)	Poly4, Box4, weight, middle, margin, candidate, RobustSPRStatement	Non-strict bounds $l_j \le u_j$ preserving all degenerate boxes; parameters defined purely from K
N4/Hurwitz.lean	194 lines (3 defs, 12 thms)	shifted, determinantPolynomial, hurwitz_determinant, stable_box_band	Degree-6 translation polynomial $H(r) = \Delta_3(a(s+r))$ proving $\Delta_3(a) > 0$; two-vertex root band
N4/Construction.lean	173 lines (1 def, 7 thms)	weights, residual_lower_left, residual_lower_right, uniform_margin, correction_positive	Chords anchored at (α, β) avoiding degeneracy splits; explicit $m > 0$ and $\varepsilon > 0$ replacing limits
N4/Synthesis.lean & Main.lean	74 lines (3 thms) + 1 entry thm	candidate_pairing, quotient_positive, synthesis, SPR.N4.Direct.robustSPR	Deduces $a(i\omega) \neq 0$ and complex quotient positivity directly from positive real pairing

(ii) Developing a full complex half-plane root-counting theory via Cauchy indices, Sturm chains, or the argument principle, as carried out by Li and Paulson [7] in Isabelle/HOL.

Because fourth-order synthesis requires only the one-way necessary conditions (3.1), building general root-counting or conjugate-factorization machinery solely for degree four would disproportionately expand the formalization effort.

3.2 Gauss–Lucas Derivative Induction and the Degree-Six Translation Polynomial

To establish (3.1) directly from `SPR.IsHurwitz` with minimal infrastructure, we split the bridge into a degree-independent derivative induction and an *a posteriori* polynomial translation identity.

First, mathlib provides the real-root sign lemma `zero_lt_eval_of_roots_lt_of_leadingCoeff_nonneg` (if $p \in \mathbb{R}[X]$ has nonnegative leading coefficient and all real roots strictly below x, then $p(x) > 0$) and the complex Gauss–Lucas theorem `Polynomial.rootSet_derivative_subset_convexHull_rootSet` (the roots of p' lie in the convex hull of the roots of p). In `Lean/SPR/Common/Hurwitz.lean`, we combine the convexity of $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ with the Gauss–Lucas theorem to prove that differentiation preserves Hurwitz stability (`hurwitz_derivative`), and induct on the derivative order n to establish $n!\, p.\mathtt{coeff}(n) = (p^{(n)})(0) > 0$:

```
theorem hurwitz_coeff_pos {p : ℝ[X]} (hp : IsHurwitz p)
    (hlc : 0 < p.leadingCoeff) (n : ℕ) (hn : n ≤ p.natDegree) :
    0 < p.coeff n
```

Instantiating `hurwitz_coeff_pos` at $n = 3, 2, 1, 0$ immediately yields $a_1, a_2, a_3, a_4 > 0$ (`hurwitz_coefficients`).

Second, to prove $\Delta_3(a) > 0$ without factoring $a(s)$, we use the observation that when $a_1, a_3 > 0$, the vanishing of $\Delta_3(a)$ characterizes the presence of a purely imaginary root $s = i\sqrt{a_3/a_1}$. Consequently, if we apply a horizontal Taylor shift by $r \in \mathbb{R}$,

$$a_r(s) \coloneqq a(s + r) = s^4 + a_{r,1}s^3 + a_{r,2}s^2 + a_{r,3}s + a_{r,4}, \tag{3.2}$$

then any zero of $\Delta_3(a_r) = 0$ at $r \geq 0$ produces a purely imaginary root $i\omega_r$ of $a_r(s)$ and hence a complex root $z = r + i\omega_r$ of $a(s)$ in the closed right half-plane $\operatorname{Re} z = r \geq 0$, contradicting `SPR.IsHurwitz`$(a.\mathtt{poly})$.

Following the *a posteriori* algebraic verification paradigm [8], we define the expanded degree-six polynomial $H(r) \coloneqq \Delta_3(a_r)$ (`determinantPolynomial` in `Lean/SPR/N4/Hurwitz.lean`; see Appendix A.1, equation (A.2) for the full expansion):

$$H(r) = 64r^6 + 96a_1r^5 + (48a_1^2 + 32a_2)r^4 + \cdots + \Delta_3(a), \tag{3.3}$$

and verify the polynomial evaluation identity $H(r) = \Delta_3(a_r)$ (`determinantPolynomial_eval`) by a three-line `ring` proof. This yields a self-contained proof of Lemma 3.1.

Lemma 3.1 (Positivity of the Third Hurwitz Determinant, `hurwitz_determinant`)**.** *If a monic quartic polynomial $a(s) = s^4 + a_1s^3 + a_2s^2 + a_3s + a_4$ satisfies `SPR.IsHurwitz`$(a.\mathtt{poly})$, then $a_1, a_2, a_3, a_4 > 0$ and $\Delta_3(a) = a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$.*

Proof. Positivity of a_1, a_2, a_3, a_4 is given by `hurwitz_coefficients`. The polynomial $H(r)$ in (3.3) has degree six and positive leading coefficient $64 > 0$. Suppose $H(r) = 0$ for some $r \in \mathbb{R}$. If $r \geq 0$, then $a_1, a_2, a_3 > 0$ imply $a_{r,1} = a_1 + 4r > 0$ and $a_{r,3} = a_3 + 2a_2r + 3a_1r^2 + 4r^3 > 0$. Setting $\omega_r = \sqrt{a_{r,3}/a_{r,1}} > 0$, direct algebraic verification shows $a_r(i\omega_r) = 0$ (`determinant_zero_root`; see Appendix A.1), and the shift identity $a_r(z) = a(r+z)$ (`eval_shifted`) gives a root $z = r + i\omega_r$ of $a(s)$ with $\operatorname{Re} z = r \geq 0$, contradicting `SPR.IsHurwitz`$(a.\mathtt{poly})$. Thus every real root of $H(r)$ is strictly negative, and applying `zero_lt_eval_of_roots_lt_of_leadingCoeff_nonneg` at $r = 0$ yields $\Delta_3(a) = H(0) > 0$. □

3.3 Resulting Formal Interface: Two-Vertex Test and Common Positive Root Band

Because the lower endpoint tuple (l_1, l_2, l_3, l_4) belongs to K (`lower_contains`), Lemma 3.1 implies $0 < l_j \le u_j$ $(1 \le j \le 4)$. Set

$$l = l_1, \qquad u = u_1, \qquad A = l_2, \qquad C = u_4, \qquad U = u_2, \tag{3.4}$$

and define the worst-case even part across the box:

$$E_*(t) \coloneqq t^2 - At + C, \qquad \text{so that} \qquad E_a(t) \le E_*(t) \quad (\forall\, a \in K,\ \forall\, t \ge 0). \tag{3.5}$$

Evaluating the identity $E_a(a_3/a_1) = -\Delta_3(a)/a_1^2$ (`determinant_ratio`) at the two Kharitonov vertices (`aL` and `aR` in `stable_box_band`)

$$a^-(s) = s^4 + us^3 + As^2 + l_3 s + C \in K, \qquad a^+(s) = s^4 + ls^3 + As^2 + u_3 s + C \in K \tag{3.6}$$

packages Stage 2 into the root-band interface `stable_box_band` consumed by Stage 4:

Lemma 3.2 (Common Positive Root Band, `stable_box_band`)**.** *Suppose K is robustly Hurwitz stable. Then $D \coloneqq A^2 - 4C > 0$, and $E_*(t) = t^2 - At + C$ has two distinct positive roots*

$$\alpha = \frac{A - \sqrt{D}}{2} > 0, \qquad \beta = \frac{A + \sqrt{D}}{2} > \alpha, \tag{3.7}$$

satisfying $A = \alpha + \beta$, $C = \alpha\beta$, and

$$0 < \alpha < \frac{l_3}{u} \le \frac{u_3}{l} < \beta. \tag{3.8}$$

In particular, whether or not any intervals $[l_j, u_j]$ degenerate to single points, the two odd-endpoint slacks

$$\delta_- \coloneqq l_3 - u\alpha > 0, \qquad \delta_+ \coloneqq l\beta - u_3 > 0 \tag{3.9}$$

are strictly positive.

Proof. Applying Lemma 3.1 and $E_a(a_3/a_1) = -\Delta_3(a)/a_1^2$ to $a^-, a^+ \in K$ yields

$$E_*\left(\frac{l_3}{u}\right) = -\frac{\Delta_3(a^-)}{u^2} < 0, \qquad E_*\left(\frac{u_3}{l}\right) = -\frac{\Delta_3(a^+)}{l^2} < 0. \tag{3.10}$$

From $E_*(l_3/u) < 0$ we obtain $D = A^2 - 4C > (2l_3/u - A)^2 \ge 0$, which together with $A, C > 0$ gives $0 < \alpha < \beta$ and $E_*(t) = (t - \alpha)(t - \beta)$. Since $E_*(q) < 0 \iff \alpha < q < \beta$ and $0 < l_3/u \le u_3/l$, inequalities (3.10) immediately imply (3.8) and (3.9). □

4 Unified Cubic Candidate and Explicit Frequency Margin

4.1 Classical Proof Requirement and the Degeneracy Branching Obstacle

For a monic cubic candidate $b_0(s) = s^3 + xs^2 + ys$ without constant or quartic terms, we have $E_{b_0}(t) = -xt$ and $O_{b_0}(t) = y - t$, so the real frequency pairing $Q_{a,b_0}(t)$ factors through t:

$$F_a(t) \coloneqq \frac{Q_{a,b_0}(t)}{t} = a_1(t - y)t + a_3(y - t) - xE_a(t). \tag{4.1}$$

In the classical proof [11, 12], the set of (x, y) where $F_a(t) > 0$ for all $t \ge 0$ is described geometrically as $\Omega^a = \Omega_e^a \cup \Omega_t^a$, and nonemptiness of $\Omega^{a^-} \cap \Omega^{a^+}$ is proved by intersecting chords anchored at the odd-part roots $(0, l_3/u)$ and $(0, u_3/l)$. When $l = u$ and $l_3 = u_3$ (for example, when the first and third coefficient intervals degenerate to single points), l_3/u and u_3/l coincide, forcing the classical proof to split into two cases ($l_3/u = u_3/l$ vs. $l_3/u < u_3/l$) with different candidate formulas; moreover, once $(x, y) \in \bigcap_{i=1}^4 \Omega^{a_i}$ is chosen, positivity of $F_a(t)$ is used qualitatively via compactness on bounded frequency intervals.

In a formal proof, branching the definition of (x, y) on $l_3/u = u_3/l$ vs. $l_3/u < u_3/l$ would duplicate every subsequent algebraic identity and frequency bound across two candidate definitions, while invoking compactness infima would require formalizing joint continuity and asymptotic uniformity over $K \times [0, \infty)$.

4.2 Unified Chord Intersection Anchored at the Even-Part Roots

To eliminate case distinctions on interval degeneracy from the candidate definition, we observe in `Lean/SPR/N4/Basic.lean` and `Construction.lean` that while l_3/u and u_3/l may coincide, the two positive roots $\alpha < \beta$ of $E_*(t)$ guaranteed by Lemma 3.2 **are always strictly separated** (even for a singleton family, Hurwitz stability forces $D > 0$ and $\alpha < \beta$). We therefore anchor the two chords at $(0, \alpha), (u, \beta)$ and $(0, \beta), (l, \alpha)$:

$$uy - x\beta = (u - x)\alpha, \qquad ly - x\alpha = (l - x)\beta. \tag{4.2}$$

Solving (4.2) yields the single closed-form parameter pair (`weight` and `middle` in `Basic.lean`):

$$x = \frac{lu}{l + u}, \qquad y = \frac{u\alpha + l\beta}{l + u}, \qquad b_0(s) = s^3 + xs^2 + ys. \tag{4.3}$$

Lemma 4.1 (Algebraic Properties of the Unified Cubic Parameters, `weights`)**.** *For any real numbers satisfying $0 < l \le u$ and $0 < \alpha < \beta$, the parameters x, y in (4.3) satisfy (4.2) and*

$$0 < x < l \le u, \qquad \alpha < y < \beta. \tag{4.4}$$

Proof. Since $l, u > 0$ and $\alpha < \beta$, we have $l + u > 0$, $l - x = l^2/(l+u) > 0$, $u - x = u^2/(l+u) > 0$, $y - \alpha = l(\beta - \alpha)/(l + u) > 0$, and $\beta - y = u(\beta - \alpha)/(l + u) > 0$. Clearing denominators (`field_simp; ring`) verifies both identities in (4.2). □

4.3 Piecewise Square-Plus-Linear Lower Bounds and Closed-Form Margin

The chord equations (4.2) are designed so that when a_1, a_2, a_3, a_4 are replaced by their worst-case box endpoints on $[0, y]$ and $[y, \infty)$, respectively, the even-part term $-xE_*(t) = -x(t - \alpha)(t - \beta)$ combines with the leading odd term into an exact square around α or β, leaving the strictly positive linear slack $\delta_-(y - t)$ or $\delta_+(t - y)$ from Lemma 3.2.

Lemma 4.2 (Whole-Box Piecewise Square-Plus-Linear Lower Bounds, `residual_lower_left` and `residual_lower_right`)**.** *For every $a \in K$, the reduced pairing $F_a(t)$ in (4.1) satisfies*

$$F_a(t) \ge (u - x)(t - \alpha)^2 + \delta_-(y - t) \qquad (0 \le t \le y), \tag{4.5}$$

$$F_a(t) \ge (l - x)(t - \beta)^2 + \delta_+(t - y) \qquad (t \ge y). \tag{4.6}$$

The step-by-step algebraic derivation of Lemma 4.2 is given in Appendix A.2. Observe that in (4.5), as $t \in [0, y]$ approaches y, the linear slack $\delta_-(y - t)$ vanishes while the square term $(u - x)(t - \alpha)^2$ is bounded below by $(u - x)(y - \alpha)^2 > 0$; conversely, near $t = \alpha$, the linear slack is bounded below by $\delta_-(y - \alpha) > 0$. Bisecting $[\alpha, y]$ at $(\alpha + y)/2$ and $[y, \beta]$ at $(\beta + y)/2$ converts this complementarity into an explicit closed-form positive margin $m > 0$ over $[0, \infty)$ (`margin` in `Basic.lean` and `uniform_margin` in `Construction.lean`):

```
theorem uniform_margin {K : Box4} {α β : ℝ}
    (hl : 0 < K.lower.a1) (hab : α < β)
    (hA : K.lower.a2 = α + β) (hC : K.upper.a4 = α * β)
    (hlo : K.upper.a1 * α < K.lower.a3)
    (hhi : K.upper.a3 < K.lower.a1 * β) :
    let x := weight K.lower.a1 K.upper.a1
    let y := middle K.lower.a1 K.upper.a1 α β
    0 < margin K x y α β ∧
    ∀ a, K.Contains a → ∀ t, 0 ≤ t → margin K x y α β ≤ residual a x y t
```

Lemma 4.3 (Explicit Uniform Frequency Margin on $[0, \infty)$, `uniform_margin`)**.** *Set $d_- := y - \alpha > 0$ and $d_+ := \beta - y > 0$, and define the box-dependent constant*

$$m := \min\left\{ \frac{\delta_- d_-}{2}, \ \frac{(u - x)d_-^2}{4}, \ \frac{\delta_+ d_+}{2}, \ \frac{(l - x)d_+^2}{4} \right\} > 0. \tag{4.7}$$

Then for every $a \in K$ and every $t \ge 0$,

$$F_a(t) \ge m > 0 \qquad \text{and} \qquad Q_{a,b_0}(t) = t\, F_a(t) \ge mt. \tag{4.8}$$

Proof. By Lemmas 3.2 and 4.1, $\delta_\pm > 0$, $d_\pm > 0$, $u - x > 0$, and $l - x > 0$, so all four arguments of min in (4.7) are strictly positive and $m > 0$. Fix $a \in K$ and $t \geq 0$. For $0 \leq t \leq y$ (`left_margin`): if $t \leq (\alpha + y)/2$, then $y - t \geq d_-/2$ and (4.5) gives $F_a(t) \geq \delta_- d_-/2 \geq m$; if $(\alpha + y)/2 < t \leq y$, then $t - \alpha > d_-/2 > 0$ and (4.5) gives $F_a(t) \geq (u - x)d_-^2/4 \geq m$. Symmetrically, for $t \geq y$ (`right_margin`), splitting (4.6) at $(\beta + y)/2$ yields $F_a(t) \geq \delta_+ d_+/2 \geq m$ when $t \geq (\beta + y)/2$ and $F_a(t) \geq (l - x)d_+^2/4 \geq m$ when $y \leq t < (\beta + y)/2$. □

5 Single-Step Quartic Perturbation and Quotient Bridge

5.1 Classical Proof Requirement and the Single-Step Perturbation Budget

By (4.8), the cubic candidate $b_0(s) = s^3 + xs^2 + ys$ satisfies $Q_{a,b_0}(t) \geq mt > 0$ for all $t > 0$, but vanishes at $t = 0$ ($Q_{a,b_0}(0) = 0$) and has degree three rather than four. In [11, 12], this is resolved by two sequential perturbations: first adding a sufficiently small constant $\varepsilon_1 > 0$ to achieve strict positivity at $t = 0$, and then adding $\varepsilon_2 s^4$ with a second sufficiently small parameter $\varepsilon_2 > 0$.

To avoid nested existential limits and strictly respect the quantifier order $\forall K \to \exists b \to \forall a \in K \to \forall \omega \in \mathbb{R}$, we perturb the constant and quartic terms simultaneously by adding $\varepsilon(s^4 + 1)$. Since $(i\omega)^4 + 1 = t^2 + 1 > 0$ for $t = \omega^2$, we define the unnormalized and normalized quartic numerators (`candidate` in `Basic.lean`):

$$\widetilde{b}(s) \coloneqq b_0(s) + \varepsilon(s^4 + 1), \qquad b(s) \coloneqq \frac{\widetilde{b}(s)}{\varepsilon} = s^4 + \frac{1}{\varepsilon}s^3 + \frac{x}{\varepsilon}s^2 + \frac{y}{\varepsilon}s + 1. \tag{5.1}$$

For any $a \in K$, the real pairing of $\widetilde{b}(s)$ with $a(s)$ (`correctedPairing` in `Construction.lean`) is

$$Q_{a,\widetilde{b}}(t) = t\,F_a(t) + \varepsilon(t^2 + 1)E_a(t). \tag{5.2}$$

To choose a single $\varepsilon > 0$ depending only on K that works for all $a \in K$ and all $t \geq 0$, we bound the region where the perturbation term $\varepsilon(t^2 + 1)E_a(t)$ can be negative:

- At $t = 0$, $Q_{a,\widetilde{b}}(0) = \varepsilon E_a(0) = \varepsilon a_4 \geq \varepsilon l_4 > 0$ for any $\varepsilon > 0$;
- For $t > 0$ with $E_a(t) \geq 0$, the perturbation $\varepsilon(t^2 + 1)E_a(t) \geq 0$ reinforces $tF_a(t) \geq mt > 0$;
- For $t > 0$ with $E_a(t) = t^2 - a_2 t + a_4 < 0$, since $a_4 \geq l_4 > 0$ and $a_2 \leq u_2 = U$, we have $t^2 < t^2 + a_4 < a_2 t \leq Ut$. Dividing by $t > 0$ yields the a priori frequency bound $0 < t < U$, while dropping $t^2 + a_4 > 0$ gives $E_a(t) > -a_2 t \geq -Ut$.

Consequently, whenever $E_a(t) < 0$, t is confined to $(0, U)$ and $(t^2 + 1)E_a(t) \geq -U(U^2 + 1)t$, which is linear in t just like the lower bound mt for $tF_a(t)$. Setting

$$\varepsilon \coloneqq \frac{m}{2U(U^2 + 1)} > 0 \qquad (U = u_2 > 0) \tag{5.3}$$

ensures that the negative perturbation consumes at most half of the linear margin mt (`correction_positive`):

Lemma 5.1 (Strict Positivity of the Perturbed Quartic Pairing, `correction_positive`)**.** *With $m > 0$ defined by (4.7) and $\varepsilon > 0$ defined by (5.3), the perturbed pairing (5.2) satisfies $Q_{a,\widetilde{b}}(t) > 0$ for every $a \in K$ and every $t \geq 0$.*

Proof. For any $a \in K$, we have $a_4 \geq l_4 > 0$ and $a_2 \leq U$. (1) If $t = 0$, then $Q_{a,\widetilde{b}}(0) = \varepsilon a_4 > 0$. (2) If $t > 0$ and $E_a(t) \geq 0$, Lemma 4.3 gives $Q_{a,\widetilde{b}}(t) \geq mt > 0$. (3) If $t > 0$ and $E_a(t) < 0$, the bounds $0 < t < U$, $t^2 + 1 \leq U^2 + 1$, and $E_a(t) \geq -Ut$ give $(t^2 + 1)E_a(t) \geq -U(U^2 + 1)t$. Combining this with (4.8) and (5.3) yields

$$Q_{a,\widetilde{b}}(t) \geq mt - \varepsilon U(U^2 + 1)t = \frac{m}{2}t > 0.$$ □

5.2 Complex Quotient Bridge and Assembly of the Main Theorem

In the final module (`Lean/SPR/N4/Synthesis.lean`), the real inequality $Q_{a,\widetilde{b}}(\omega^2) > 0$ is connected to the complex quotient predicate $\texttt{FrequencySPR}(a.\texttt{poly}, b)$. In mathlib, complex division B/A is a total function (evaluating to 0 if $A = 0$) with real part $(B/A).\texttt{re} = (B.\texttt{re} \cdot A.\texttt{re} + B.\texttt{im} \cdot A.\texttt{im})/\texttt{Complex.normSq}(A)$. Rather than proving $a(i\omega) \neq 0$ separately from `SPR.IsHurwitz`, `Synthesis.lean` discharges both nonvanishing of the denominator and positivity of the quotient in two steps:

(i) `candidate_pairing`: Using the imaginary-axis evaluation decomposition `eval_imaginary` and $\varepsilon \neq 0$, `field_simp` and `ring` prove that for all $\omega \in \mathbb{R}$,

$$\varepsilon[\operatorname{Re} b(i\omega)\ \operatorname{Re} a(i\omega) + \operatorname{Im} b(i\omega)\ \operatorname{Im} a(i\omega)] = Q_{a,\widetilde{b}}(\omega^2). \tag{5.4}$$

(ii) `quotient_positive`: Since $\varepsilon > 0$ and $Q_{a,\widetilde{b}}(\omega^2) > 0$, (5.4) implies $\operatorname{Re} b(i\omega)\ \operatorname{Re} a(i\omega) + \operatorname{Im} b(i\omega)\ \operatorname{Im} a(i\omega) > 0$. If $a(i\omega)$ were zero, this inner product would vanish; hence $a(i\omega) \neq 0$, $|a(i\omega)|^2 = \texttt{Complex.normSq}(a(i\omega)) > 0$, and dividing by $|a(i\omega)|^2$ yields $\operatorname{Re}[b(i\omega)/a(i\omega)] > 0$.

The main theorem `synthesis` (exported as `SPR.N4.Direct.robustSPR` in `Lean/Main.lean`) assembles all five stages (shown verbatim from `Synthesis.lean`, omitting only the three lines checking $\varepsilon > 0$):

```
theorem synthesis (K : Box4) (hK : K.RobustlyHurwitz) :
    ∃ b :  ℝ[X], b.Monic ∧ b.natDegree = 4 ∧
      ∀ a :  Poly4, K.Contains a → FrequencySPR a.poly b := by
  obtain ⟨α, β, hα, hab, hA, hC, hlo, hhi, hl1, hl4, hU⟩ :=
    stable_box_band hK
  let x := weight K.lower.a1 K.upper.a1
  let y := middle K.lower.a1 K.upper.a1 α β
  let m := margin K x y α β
  obtain ⟨hm, hF⟩ := uniform_margin hl1 hab hA hC hlo hhi
  let U := K.upper.a2
  let ε := m / (2 * U * (U ^ 2 + 1))
  ...
  refine ⟨(candidate x y ε).poly, (poly_degree _).2, (poly_degree _).1, ?_⟩
  intro a ha
  apply quotient_positive hε
  intro t ht
  exact correction_positive hm hU hε hsmall ha.2.2.2.1
    (hl4.trans_le ha.2.2.2.2.2.2.1) ht (hF a ha t ht)
```

Remark 5.2 (Paper-Level Supplement on the Hurwitz Stability of the Constructed Numerator). As noted in Section 2.2, the formalized theorem `SPR.N4.Direct.robustSPR` verifies the equal-degree imaginary-axis predicate `FrequencySPR`. Mathematically, the monic quartic numerator $b(s)$ in (5.1) is itself Hurwitz stable, as can be seen in two ways: (a) *Analytic route*: Fix any $a \in K$. Since $a(s)$ is Hurwitz stable, $f(s) = b(s)/a(s)$ is analytic on $\operatorname{Re} s \geq 0$, satisfies $\lim_{|s|\to\infty} f(s) = 1$, and has $\operatorname{Re} f(i\omega) > 0$ on $\mathbb{R}$; applying the harmonic minimum principle on large right-half disks shows $\operatorname{Re} f(s) > 0$ on $\operatorname{Re} s \geq 0$, so $b(s)$ has no zeros in $\operatorname{Re} s \geq 0$. (b) *Algebraic route*: For $b(s) = s^4 + \varepsilon^{-1}s^3 + (x/\varepsilon)s^2 + (y/\varepsilon)s + 1$, all four coefficients are positive and $\Delta_3(b) = (xy - \varepsilon y^2 - \varepsilon)/\varepsilon^3$; replacing ε in (5.3) by $\varepsilon^* = \min\{\frac{m}{2U(U^2+1)}, \frac{xy}{2(y^2+1)}\} > 0$ guarantees $\Delta_3(b) > 0$ directly while preserving Lemma 5.1. Neither argument is encoded as a separate Lean declaration in the current repository.

6 Illustrative Examples, Limitations, and Reproducibility

6.1 Nondegenerate Box and Completely Degenerate Singleton Examples

We illustrate how the formalized formulas operate identically on a full-width four-interval box and on a completely degenerate singleton family.

Example 1 (Nondegenerate Four-Interval Box). Consider the robustly Hurwitz-stable fourth-order interval family K defined by

$$a_1 \in [2,3], \qquad a_2 \in [5,6], \qquad a_3 \in [4,6], \qquad a_4 \in [3,4].$$

Here $l = 2$, $u = 3$, $A = 5$, $C = 4$, and $U = 6$. The worst-case even part $E_*(t) = t^2 - 5t + 4 = (t-1)(t-4)$ has discriminant $D = 9$ and positive roots $\alpha = 1$, $\beta = 4$, with $1 = \alpha < l_3/u = 4/3 < u_3/l = 3 < 4 = \beta$ and endpoint slacks $\delta_- = 4 - 3 = 1 > 0$, $\delta_+ = 8 - 6 = 2 > 0$. Equations (4.3) and (4.7) give

$$x = \frac{6}{5}, \qquad y = \frac{11}{5}, \qquad d_- = \frac{6}{5}, \qquad d_+ = \frac{9}{5}, \qquad m = \min\left\{\frac{3}{5}, \frac{81}{125}, \frac{9}{5}, \frac{81}{125}\right\} = \frac{3}{5}.$$

With $2U(U^2+1) = 444$, equation (5.3) gives $\varepsilon = \frac{3/5}{444} = \frac{1}{740}$, and (5.1) yields the exact integer-coefficient monic common SPR numerator

$$b(s) = s^4 + 740s^3 + 888s^2 + 1628s + 1. \tag{6.1}$$

Example 2 (Completely Degenerate Singleton Family). Consider the singleton family $K = \{(s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1\}$, where all four intervals have zero width: $l = u = 4$, $A = U = 6$, $l_3 = u_3 = 4$, and $l_4 = C = 1$. Here the odd-part root endpoints $l_3/u = u_3/l = 1$ coincide. Nevertheless, $E_*(t) = t^2 - 6t + 1$ has $D = 32 > 0$ and strictly separated roots $\alpha = 3 - 2\sqrt{2} < \beta = 3 + 2\sqrt{2}$. Evaluating (4.3)–(5.3) without modification gives $x = 2$, $y = 3$, $\delta_\pm = 8(\sqrt{2} \pm 1) > 0$, $d_\pm = 2\sqrt{2}$, $m = \min\{8(2-\sqrt{2}),\ 4,\ 8(2+\sqrt{2}),\ 4\} = 4$, $\varepsilon = 4/444 = 1/111$, and

$$b(s) = s^4 + 111s^3 + 222s^2 + 333s + 1. \tag{6.2}$$

While $b(s) = (s+1)^4$ trivially satisfies $b(i\omega)/a(i\omega) \equiv 1 > 0$ for this singleton, Example 2 shows that the degeneracy-uniform formulas produce a valid witness without branching on interval widths.

6.2 Conservative Margins and Practical Control Limitations

As Examples 1 and 2 illustrate, the constructed monic numerators have relatively large intermediate coefficients ($1/\varepsilon \sim 10^2$–10^3). This is a direct consequence of three conservative bounding choices made to streamline formal verification:

(1) **Decoupled worst-case even and odd bounds**: Stages 3 and 4 bound $E_a(t)$ globally by $E_*(t) = t^2 - l_2 t + u_4$ while independently setting (a_1, a_3) to (u_1, l_3) on $[0, y]$ and (l_1, u_3) on $[y, \infty)$;

(2) **Midpoint bisection for the uniform margin**: To avoid formalizing constrained quadratic minimization and vertex-location case splits, Lemma 4.3 bounds the sum of the square and linear terms via midpoint bisection;

(3) **Cubic scaling in the perturbation denominator**: Lemma 5.1 bounds $(t^2+1)E_a(t)$ over $0 < t < U = u_2$ by $-U(U^2+1)t$, causing $\varepsilon^{-1} = 2U(U^2+1)/m$ to scale cubically with u_2.

Accordingly, our explicit formulas are designed to provide an algebraically transparent, kernel-checkable proof of universal mathematical existence, rather than to optimize controller gain or numerical conditioning. In practical control synthesis, once existence is guaranteed by Theorem 2.1, one can select well-conditioned numerator coefficients within the nonempty convex intersection $\bigcap_{i=1}^4 \Omega^{a_i}$ via linear matrix inequalities (LMIs) or convex optimization [12].

6.3 Library Reuse, Project Infrastructure, and Reproducibility

Following the reporting practice of [3], we distinguish three categories of formal components (all new code resides in the project namespace `SPR` and has not been upstreamed to mathlib, so we do not claim it as a mathlib contribution):

- **Reused mathlib infrastructure**: Univariate polynomials (`Polynomial`) and complex evaluation ($\texttt{eval}_2$), the complex Gauss–Lucas theorem (`rootSet_derivative_subset_convexHull_rootSet`), the real-root sign lemma (`zero_lt_eval_of_roots_lt_of_leadingCoeff_nonneg`), real square roots (`Real.sqrt`), complex norm-square positivity (`Complex.normSq_pos`), and arithmetic/ring automation (`ring`, `field_simp`, `linarith`, `nlinarith`, `positivity`).
- **Degree-independent project infrastructure (`Lean/SPR/Common/Hurwitz.lean`)**: Defines the semantic predicate `SPR.IsHurwitz` for polynomials of arbitrary degree and proves positivity on $[0, \infty)$ (`eval_pos`), stability of derivatives (`hurwitz_derivative`), and strict positivity of all coefficients (`hurwitz_coeff_pos`). This module places no restriction on polynomial degree and is reusable across higher-order SPR formalizations.
- **Fourth-order synthesis and axiom auditor (`Lean/SPR/N4/`, `Main.lean`, `Audit.lean`)**: The four modules in `Lean/SPR/N4/` (521 lines, 22 explicit theorem declarations, plus 1 entry theorem in `Main.lean`, generating 58 fourth-order theorem constants in the compiled environment) implement Stages 2–5. During `lake build`, `Lean/Audit.lean` traverses the environment's constant table via `Lean.collectAxioms` to verify that every theorem (including private helpers) depends only on Lean's three standard axioms (`propext`, `Classical.choice`, and `Quot.sound`), validated against a negative test injecting a forbidden axiom.

The formalization is locked to Lean 4.30.0 and mathlib v4.30.0. All source files, toolchain locks, and build instructions are publicly available at https://github.com/dgjszy/quartic-spr-proof, where running `lake build` replays the complete verification and axiom audit.

7 Conclusion

We have presented a formalization-oriented proof reconstruction and complete Lean 4 verification of the fourth-order interval SPR synthesis theorem of Yu and Wang [11]. By combining a degree-six horizontal translation polynomial for the third Hurwitz determinant, a degeneracy-uniform chord-intersection candidate, piecewise square-plus-linear lower bounds with an explicit frequency margin, and a quantifier-preserving single-step quartic perturbation, the formal proof covers all nondegenerate and degenerate fourth-order interval boxes and all real frequencies within 598 lines of kernel-checked Lean 4 code.

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A Algebraic Expansions and Piecewise Lower-Bound Details

A.1 Horizontal Translation Polynomial and Third Hurwitz Determinant

For a monic quartic polynomial $a(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$ and $r \in \mathbb{R}$, binomial expansion of the horizontally shifted polynomial $a_r(s) = a(s+r)$ (`shifted` in `Lean/SPR/N4/Hurwitz.lean`) gives the shifted coefficients

$$\begin{aligned} a_{r,1} &= a_1 + 4r, & a_{r,2} &= a_2 + 3a_1 r + 6r^2, \\ a_{r,3} &= a_3 + 2a_2 r + 3a_1 r^2 + 4r^3, & a_{r,4} &= a_4 + a_3 r + a_2 r^2 + a_1 r^3 + r^4. \end{aligned} \tag{A.1}$$

Substituting (A.1) into $\Delta_3(a_r) = a_{r,1}a_{r,2}a_{r,3} - a_{r,3}^2 - a_{r,1}^2 a_{r,4}$ and collecting powers of r yields the degree-six polynomial $H(r)$ (`determinantPolynomial`):

$$\begin{aligned} H(r) = {} & 64r^6 + 96a_1 r^5 + (48a_1^2 + 32a_2)r^4 + (8a_1^3 + 32a_1 a_2)r^3 \\ & + (4a_2^2 + 8a_1^2 a_2 + 4a_1 a_3 - 16a_4)r^2 + (2a_1 a_2^2 + 2a_1^2 a_3 - 8a_1 a_4)r + \Delta_3(a). \end{aligned} \tag{A.2}$$

In Lemma 3.1, if a monic quartic $p(s) = s^4 + p_1 s^3 + p_2 s^2 + p_3 s + p_4$ satisfies $p_1 > 0$, $p_3 > 0$, and $\Delta_3(p) = p_1 p_2 p_3 - p_3^2 - p_1^2 p_4 = 0$, then at $\omega = \sqrt{p_3/p_1} > 0$ ($\omega^2 = p_3/p_1$) its odd part vanishes, $O_p(\omega^2) = p_3 - p_1(p_3/p_1) = 0$, and its even part is

$$E_p(\omega^2) = \left(\frac{p_3}{p_1}\right)^2 - p_2\left(\frac{p_3}{p_1}\right) + p_4 = -\frac{p_1 p_2 p_3 - p_3^2 - p_1^2 p_4}{p_1^2} = -\frac{\Delta_3(p)}{p_1^2} = 0.$$

Hence $p(i\omega) = E_p(\omega^2) + i\omega O_p(\omega^2) = 0$ (`determinant_zero_root`). Applying this to $p = a_r$ ($r \geq 0$) gives $a(r + i\omega_r) = a_r(i\omega_r) = 0$.

A.2 Derivation of the Piecewise Square-Plus-Linear Bounds (Lemma 4.2)

Proof of Lemma 4.2. For any $a \in K$, we have $l \leq a_1 \leq u$, $A \leq a_2$, $l_3 \leq a_3 \leq u_3$, and $a_4 \leq C$. Since $x > 0$ (Lemma 4.1) and $t \geq 0$, we have for all $t \geq 0$

$$-xE_a(t) = -x(t^2 - a_2 t + a_4) \geq -x(t^2 - At + C) = -x(t-\alpha)(t-\beta).$$

(1) When $0 \leq t \leq y$, we have $(t-y)t \leq 0$ and $y - t \geq 0$, so $a_1(t-y)t \geq u(t-y)t$ and $a_3(y-t) \geq l_3(y-t)$. Substituting these into (4.1) gives

$$\begin{aligned} F_a(t) &\geq u(t-y)t + l_3(y-t) - x(t-\alpha)(t-\beta) \\ &= u(t-\alpha)(t-y) + u\alpha(t-y) + l_3(y-t) - x(t-\alpha)(t-\beta) \\ &= (t-\alpha)\big[u(t-y) - x(t-\beta)\big] + (l_3 - u\alpha)(y-t). \end{aligned}$$

By the first chord identity in (4.2), $uy - x\beta = (u-x)\alpha$, so

$$u(t-y) - x(t-\beta) = (u-x)t - (uy - x\beta) = (u-x)(t-\alpha),$$

which together with $\delta_- = l_3 - u\alpha$ yields (4.5) (`residual_lower_left`).

(2) Symmetrically, when $t \geq y$, we have $(t-y)t \geq 0$ and $y - t \leq 0$, so $a_1(t-y)t \geq l(t-y)t$ and $a_3(y-t) \geq u_3(y-t) = -u_3(t-y)$. Substituting these into (4.1) and applying the second chord identity $ly - x\alpha = (l-x)\beta$ gives

$$\begin{aligned} F_a(t) &\geq l(t-y)t - u_3(t-y) - x(t-\alpha)(t-\beta) \\ &= (t-\beta)\big[l(t-y) - x(t-\alpha)\big] + (l\beta - u_3)(t-y) = (l-x)(t-\beta)^2 + \delta_+(t-y), \end{aligned}$$

which establishes (4.6) (`residual_lower_right`). □

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