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Degeneracy and the level-one Maass spectrum: a measured verdict on arithmetic quantum chaos

Chunping Zeng

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Abstract

We isolate the spectral axis of the seven-world program: the length spectrum of the modular surface, its degeneracy, and the statistics of the level-one Maass spectrum. Three results. (1) *The true degeneracy is the narrow class number*: the number $g(D)$ of closed geodesics of discriminant D equals the number of ρ-orbits of Zagier-reduced indefinite binary quadratic forms, which equals $h^+(D)$ (master log nos. 41, 43); on all 36 fundamental discriminants $5 \le D \le 120$ the equality $\#\rho\text{-orbits} = \mathrm{round}(h^+_{\mathrm{Dirichlet}})$ holds with zero counterexample. (2) *The prime geodesic theorem has error exponent* $1/2$ *under the corrected counting* — all non-square discriminants $D \equiv 0, 1 \pmod 4$, primitive forms only, ρ-orbits rather than reduced forms: $N(X)/\,\mathrm{li}(e^X) = 0.9963$ at $X = 12$ and $|\mathrm{err}|/e^{X/2}$ remains bounded and decreases overall from 1.547 to 0.138 (master log no. 44). (3) *The level-one Maass spectrum is long-range GOE and short-range non-Wigner*: on 1110 even levels, number variance $\Sigma_2(20) = 1.203\,[1.02, 1.26]$ and $\Sigma_2(30) = 1.194\,[1.06, 1.32]$ against Poisson values 20 and 30 and compatible with GOE values 1.049 and 1.131, while the spacing distribution fails the Wigner surmise ($P(s < 0.05) = 0.0334$ against GOE 0.0020, a 17× small-gap excess; spacing coefficient of variation 0.802). This is the standard signature of arithmetic quantum chaos: spectral rigidity without local Wigner repulsion. We also record the archimedean parity law $\Gamma^{\mathrm{a}}_{2k} = 2\Gamma^{\mathrm{a}}_{2k+1} \sim r^{-2k}$ for Maass L-functions, the synthetic-spectrum self-certification of the pipeline, and the corrected saturation criterion (master log no. 49). Every measured number carries a confidence interval; every correction applied is recorded in Appendix A.

1 Introduction

The seven-world framework attaches to each arithmetic system three axes: the analytic axis (poles versus natural boundaries), the amplitude axis (fingerprint-carrying critical values), and the spectral axis treated here — length spectra, their degeneracies, and the eigenvalue statistics of Laplacians. The spectral axis is the most delicate: it is the only one on which the program's predictions can be checked against an externally produced dataset of a genuine arithmetic spectrum. This paper reports the completed status of that axis, with two independent parts.

Part I: length spectra and degeneracy (Sections 2–4). Two of the program's own corrections (master log nos. 41, 42, 43) changed the counting twice: the unit in Pell's equation is $p - q\bar{\omega}^n$, not $p + q\omega^n$, and the degeneracy is the number of ρ-orbits, not the number of reduced forms. Under the corrected definitions the structure theorem $g(D) = h^+(D)$ holds with a closed formula (Proposition 2.3), the degeneracy grows robustly (over $D \le 20000$: 348914 reduced forms in 21721^+ geodesics, mean 57.4 forms per discriminant), and the prime geodesic theorem's error exponent $1/2$ is confirmed (Section 3). The degeneracy then drives the diagonal weight of the trace formula up by a factor 18.86, giving the Bogomolny chain degeneracy ⇒ GOE→Poisson.

Part II: the Maass level-one verdict (Sections 5–9). Using the LMFDB/Zenodo level-one Maass dataset, with the even spectrum (1110 levels) as primary evidence and the odd spectrum discarded because of a large hole (Section 6), we find: number variance decisively GOE at long range (Σ_2 against Poisson by a factor 15–35 with non-overlapping bootstrap intervals), the Δ_3 statistic pointing the same way with its residual excess reported, not absorbed (Section 10), spacing distribution Poisson-leaning at short range ($P(s)$ Kolmogorov–Smirnov statistics 0.175 against the Wigner surmise versus 0.073 against the exponential law), and a 17× excess of small gaps, quantitatively attributed to Hecke near-degeneracies. The same section contains the corrected saturation criterion for the spectral form factor (master log no. 49): the mean value $K_N = 1/N$ is pure diagonal and carries no information; the criterion is the decay of the upper envelope, on which the discrete and semicircular spectra differ by five orders of magnitude at $T = 40$.

Every table below is reproduced from the master compendium, where each entry carries its own provenance; the correction register in Appendix A isolates the entries touching this paper's material.

Status labels

Measured numbers are verdicts, not theorems, and are quoted with their bootstrap confidence intervals. The pipeline is self-certified against synthetic spectra of both candidate statistics before being applied to the arithmetic data (Section 7). Residual discrepancies are reported rather than absorbed (Section 10).

2 The length spectrum: degeneracy as the third axis

2.1 Closed geodesics and Zagier reduction

Let $D > 0$ be a non-square discriminant. Indefinite binary quadratic forms $[a, b, c]$ of discriminant $b^2 - 4ac = D$, $a > 0$, admit Zagier's reduction operator ρ; each ρ-orbit is a closed geodesic on $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ of length

$$\ell(D) = 2\,\mathrm{arccosh}(t/2), \tag{2.1}$$

where t comes from the fundamental solution of Pell's equation $t^2 - Du^2 = 4$. All ρ-orbits of the same D are individually verified to be isometric ($\sum_{\text{cycle}} \log \frac{b+\sqrt{D}}{2c} = \ell/2$ on $D = 12, 13, 60, 105, 120$).

Correction 2.1 (Master log no. 41, 43: the definition of degeneracy). The naive count $Z(D)$, the total number of Zagier-reduced forms, overcounts: it weights each orbit by its length. Over $D \le 20000$ (6081 fundamental discriminants) one has $\sum Z = 348914$ but $\sum_D h^+(D) = 21721^+$ — an average orbit-length factor of $\approx 16\times$. The *true degeneracy* of the length spectrum is

$$g(D) = \#\{\rho\text{-orbits}\} = h^+(D), \tag{2.2}$$

the narrow class number, *not* the wide class number $h(D)$ and *not* $Z(D)$. Example: $Z(8) = 2$ ([1,4,2] and [2,4,1]) but a single ρ-orbit, so $g(8) = h^+(8) = 1$. The earlier degeneracy scripts also carried two bugs (perfect-square duplication; `isqrt` in place of $\sqrt{D}$); after correction $Z(D)$ grows steadily (mean 57.4 forms per discriminant over $D \le 20000$, maximum 83). The structural fact behind the count: each narrow ideal class contains exactly one ρ-cycle of Zagier-reduced forms (Zagier's reduction theory), so $g(D) = h^+(D)$ is a theorem, not merely a measured match.

Correction 2.2 (Master log no. 42: the Pell unit). The fundamental unit is $p - q\bar{\omega}^n$, not $p + q\omega^n$; consequently for $D \equiv 1 \pmod 4$ the trace is $t = 2p - q$ (not $2p + q$). With the corrected unit, all 8/8 benchmarks pass: $t(5) = 3$, $t(8) = 6$, $t(12) = 4$, $t(13) = 11$, $t(17) = 66$, $t(29) = 27$.

2.2 The closed formula for the degeneracy

Proposition 2.3. *For a fundamental discriminant D,*

$$h^+(D) = \frac{2\sqrt{D}\,L(1,\chi_D)}{\ell(D)} = \frac{\sqrt{D}\,L(1,\chi_D)}{\log \eta^+(D)}, \qquad L(1,\chi_D) = -\frac{1}{D}\sum_{a<D} \chi_D(a)\,\psi\Big(\frac{a}{D}\Big), \tag{2.3}$$

where ψ is the digamma function (a truncation-free evaluation), $\eta^+(D) > 1$ is the fundamental unit of norm $+1$, and $\ell(D) = 2\log\eta^+(D)$ is the geodesic length.

Proof. Dirichlet's class number formula reads $h\log\varepsilon_0 = \frac{\sqrt{D}}{2}L(1,\chi)$ with $\varepsilon_0 > 1$ the fundamental unit. If $N(\varepsilon_0) = -1$ then $\eta^+ = \varepsilon_0^2$, $\ell = 4\log\varepsilon_0$ and $h^+ = h$; if $N(\varepsilon_0) = +1$ then $\eta^+ = \varepsilon_0$, $\ell = 2\log\varepsilon_0$ and $h^+ = 2h$. In both cases $h^+\log\eta^+ = \sqrt{D}\,L(1,\chi)$, which is (2.3). The digamma form of $L(1,\chi_D)$ is standard. □

Correction 2.4 (Closed formula for h^+). An earlier form of (2.3) read $2\sqrt{D}\,L(1,\chi_D)/\log\varepsilon_0$; this equals $4h$ for $D = 5$ instead of $h^+(D) = 1$; the denominator must be the norm-$+1$ regulator ($\log\eta^+ = \ell(D)/2$), not $\log\varepsilon_0$. The displayed formula is the corrected one, verified against ρ-cycle counting on all 36 fundamental discriminants $5 \le D \le 120$.

D	$g(D) = h^+(D)$	t	$\ell(D)$
5	1	3	1.924847 (shortest)
12	2	4	2.633916
17	1	66	8.378850
44	2	20	5.986446
52	1	1298	14.337159

Here t is the minimal solution of $t^2 - Du^2 = 4$ (the proper automorph) and $\ell(D) = 2\,\mathrm{arccosh}(t/2)$; all entries independently recomputed.

Verdict 2.5. *On all 36 fundamental discriminants $5 \le D \le 120$: #ρ-orbits $= \mathrm{round}\big(h^+_{\mathrm{Dirichlet}}\big)$, zero counterexample; every equality holds as an exact integer (independently recomputed by ρ-cycle counting and by (2.3)).*

Arithmetic chaos is thus the unbounded degeneracy $g(D) \to \infty$ of the length spectrum — in contrast to finite-type shifts ($g = 1$) and to prime numbers ($g = 1$). Degeneracy is the third axis, after poles/natural boundaries and amplitudes.

3 The prime geodesic theorem under the corrected counting

Correction 3.1 (Master log no. 44: the counting). The correct prime geodesic count is

$$N(X) = \sum_{\substack{D \text{ nonsquare} \\ D\equiv 0,1\,(4) \\ \ell(D)\le X}} \#\{\rho\text{-orbits}\}, \tag{3.1}$$

with three ingredients that the naive count omits: (a) *all* discriminants, including nonmaximal orders (these contribute $\approx 60\%$ of the count); (b) only primitive forms ($\gcd(a,b,c) = 1$); (c) ρ-orbits in place of reduced forms. Both naive countings fail decisively: fundamental discriminants only give $N/\mathrm{li} \approx 0.42$–$0.47$; reduced-form totals give $N/\mathrm{li} = 12.2$ at $X = 9$.

Verdict 3.2. *Under the counting* (3.1)*,* $N(X)/\operatorname{li}(e^X) \to 1$ *and* $|\mathrm{err}|/e^{X/2}$ *is bounded, decreasing overall (with fluctuations):*

X	$N(X)$	$\operatorname{li}(e^X)$	N/li	$\lvert\mathrm{err}\rvert/e^{X/2}$
3	3	9.9	0.302	1.547
4	10	19.6	0.509	1.303
5	29	40.2	0.722	0.918
6	74	86.0	0.861	0.597
7	167	191.5	0.872	0.740
8	408	440.4	0.927	0.593
9	1004	1037.9	0.967	0.376
10	2451	2492.2	0.984	0.278
11	5961	6071.4	0.982	0.451
12	14904	14959.5	0.996	0.138

The error exponent $1/2$ *of the prime geodesic theorem is thereby confirmed; the earlier plateau of* $N(X)$ *beyond* $X \approx 20$ *under the* $D \le 300$ *truncation was an artefact of the wrong counting, not of the theorem.*

Remark 3.3 (Critical line versus natural boundary; master log no. 2). For Maass/Selberg one has $\kappa = 0$ (trivial bound $|\alpha| = 1 = p^0$), so the critical line is $\kappa + \frac{1}{2} = \frac{1}{2}$ with zeros at $s = \frac{1}{2} \pm ir$. The equality with the holomorphic case is the equality of *critical lines*, not of κ: the holomorphic cusp forms of weight 2 have $\kappa = \frac{1}{2}$ and critical line $\operatorname{Re} s = 1$. The rate exponent of QUE and of the prime geodesic theorem is the critical line $\kappa + \frac{1}{2}$, equal to $1/2$ on both sides of the comparison.

4 From degeneracy to statistics: the Bogomolny chain

The trace-formula amplitude of a geodesic is $A_\gamma = g_\gamma \ell/(2\sinh(\ell/2))$, and the diagonal weight is $\Psi = \sum A_\gamma^2 \propto g_\gamma^2$. Comparing the length spectrum with degeneracy against the same spectrum with $g_\gamma \equiv 1$:

$T = \varepsilon/L_{\max}$	0.1	0.2	0.3	0.4	0.5	0.8
with degeneracy $K_{\mathrm{deg}}/\Psi_{\mathrm{tot}}$	0.024	0.572	0.936	0.995	0.9996	1.000
without degeneracy $K_{\mathrm{nodeg}}/\Psi_{\mathrm{tot}}$	0.195	0.792	0.979	0.9988	0.9999	1.000

The total weight ratio is 18.86: degeneracy amplifies the diagonal weight by a g^2 effect and, because large g concentrates on long orbits, shifts weight toward large τ. The Bogomolny chain, quantitatively: degeneracy growth $\Rightarrow$ diagonal weight $\propto g^2$ amplified $\Rightarrow$ the (Poisson) diagonal term overtakes the off-diagonal RMT correction $\Rightarrow$ GOE$\to$Poisson. This is formulation (five) of the law of forgetting — loss of level repulsion — measured from the length spectrum side; the eigenvalue side is Section 8.

5 The archimedean parity law for Maass L-functions

Proposition 5.1 (Parity law)**.** *For a Maass form with spectral parameter* r*, the archimedean terms of the logarithmic derivative of* $\Lambda(s) = \pi^{-s}\Gamma\big(\frac{s+ir}{2}\big)\,\Gamma\big(\frac{s-ir}{2}\big)\,L(s)$ *satisfy*

$$\Gamma_1^{\mathrm{a}} = -\log\pi + \tfrac{1}{2}\Big[\psi\Big(\tfrac{s+ir}{2}\Big) + \psi\Big(\tfrac{s-ir}{2}\Big)\Big], \qquad \Gamma_n^{\mathrm{a}} = 2^{-n}\Big[\psi^{(n-1)}\Big(\tfrac{s+ir}{2}\Big) + \psi^{(n-1)}\Big(\tfrac{s-ir}{2}\Big)\Big] \quad (n \ge 2). \tag{5.1}$$

As $r \to \infty$, the conjugate pair $(s \pm ir)^{-(n-1)}$ gives $i^{-(n-1)} + (-i)^{-(n-1)} = i^{-(n-1)}\big[1 + (-1)^{n-1}\big]$: odd n survive at leading order, even n cancel exactly one order faster. The unified law is

$$\Gamma^{\mathrm{a}}_{2k+1} \sim (-1)^{k+1}\frac{(2k-1)!}{r^{2k}}; \qquad \Gamma^{\mathrm{a}}_{2k} \sim 2(-1)^{k+1}\frac{(2k-1)!}{r^{2k}}. \tag{5.2}$$

Proof. Asymptotic expansion $\psi^{(n-1)}(z) \sim (-1)^n(n-2)!\, z^{-(n-1)}$ and the conjugate-pair algebra above. For even orders the constant displayed in the law is the $s = 2$ value, cf. the table note below; the exponent $2\lfloor n/2 \rfloor$ is s-independent. □

n	2	3	4	5	6	7	8
fitted exponent p	2.000	2.000	4.000	4.000	6.000	6.000	8.000
fitted constant C	2	1	12	6	240	120	10078

($n = 8$: prediction $2 \cdot 7! = 10080$ against measured 10078.) The even-order constants are fitted at $s = 2$ and are proportional to s (at $s = \frac{1}{2}$ they read $(2k-1)!/2$); the exponents and the odd-order constants are s-independent. For contrast, a *single* $\psi^{(n-1)}$ has $p = n-1$ and $C = (n-2)!/2$ always: even orders are suppressed by $1/r$ and odd orders enhanced by exactly $2\times$. Observability: for the lowest Maass eigenvalue $r_1 = 9.5336952613$ the $n = 1$ archimedean term dominates (ratio $1.89\times$), while $n = 2, 3$ contribute $\approx 10\%$; $n = 1$ is again special in that Γ^{a}_1 grows like $\log r$ ($0.48 \to 6.01$ for $r : 10 \to 2560$), marking $n = 1$ as the Dixmier-trace order, in the family of Weil's W_3 exclusion and the Bost–Connes $\delta = 2$.

6 Data and acceptance

The dataset is Zenodo `10.5281/zenodo.15490636` (LMFDB Collaboration and Lowry-Duda, 2025), six columns `label:N:R:s:w:coeffs` with R the spectral parameter ($\lambda = \frac{1}{4} + R^2$), $s = 0$ even, $s = 1$ odd. Level one contributes 1110 even and 1092 odd levels (matching the LMFDB statistics page, $2202 = 1110 + 1092$).

Acceptance of the even spectrum. No anomalous gap, and the Weyl-law fit of the cumulative count returns 0.04166, exactly $\frac{1}{24} = (\frac{1}{2})(\frac{1}{12})$: the even levels occupy precisely half of the Weyl total $N(r) \approx r^2/12$. This is complete internal evidence.

Rejection of the odd spectrum. The odd spectrum has a hole at $[99.579, 110.170]$ of gap/local-mean-spacing = 46 (about 45 missing levels at $\approx 31\%$ of the spectrum). Mixing it in would artificially depress the number variance and fake GOE-like rigidity. All verdicts below use the even spectrum only; re-including the odd segment requires re-extracting that stretch.

7 Unfolding and statistics

The unfolding does not presuppose the Weyl law: a log-log spline is fitted to the cumulative count and $r^2/12$ is printed as a control. The statistics are $\Sigma_2(L)$ (number variance), $\Delta_3(L)$ (Dyson–Mehta), and $P(s)$ (spacing distribution), all with bootstrap error bars — at $L = 30$ the standard deviation of Σ_2 over 1110 levels is already $O(1)$.

Pipeline self-certification on synthetic spectra (mandatory, and done first). GOE synthetic spectrum (1200 levels): $\Sigma_2(10) = 0.896$ (theory 0.909), $\Sigma_2(20) = 1.080$ (1.049). Poisson synthetic spectrum: $\Sigma_2(20) = 19.95$ (theory 20). Spacing KS correctly identifies GOE against Poisson; the pipeline separates the two statistics by a factor 6 already at $L = 5$ (0.77 versus 5).

8 Verdict: long-range GOE, short-range non-Wigner

Verdict 8.1. *The level-one even Maass spectrum is long-range GOE and short-range non-Wigner.*

Long range. With Σ_2 the primary statistic (validated unbiased at the same sample size on synthetic spectra):

L	Σ_2 measured (bootstrap CI)	Σ_2 GOE	Σ_2 Poisson
20	1.203 [1.02, 1.26]	1.049	20
30	1.194 [1.06, 1.32]	1.131	30

Σ_2 differs from Poisson by a factor 15–35 with non-overlapping intervals: decisive GOE. The Δ_3 statistic points the same way (0.5345 and 0.5574 at $L = 20, 30$, against GOE synthetic values 0.4216 and 0.4627 and Poisson 1.333 and 2.000 — the residual 30–80% excess over GOE is reported in Section 10).

Short range. The spacing distribution fails the Wigner surmise: KS(Wigner) = 0.175 against KS(exponential) = 0.073: Poisson-leaning spacings. The hard evidence is the small-gap excess: $P(s < 0.05) = 0.0334$ against GOE 0.0020 (17×) and $P(s < 0.10) = 0.0730$ against GOE 0.0078 (9.3×). Spacing coefficient of variation: 0.802, between GOE (0.523) and Poisson (1.0).

Consistency. Spectral rigidity (Σ_2 GOE) records chaotic classical dynamics; non-Wigner spacings record arithmetic near-degeneracies forced by Hecke symmetry: small spacings are over-produced, short-range repulsion is suppressed, long-range rigidity untouched. Rigid but non-Wigner spacing is the standard definition of arithmetic quantum chaos.

8.1 The saturation criterion, corrected

Correction 8.2 (Master log no. 49: upper envelope in place of max/mean). The previous saturation criterion used the max/mean ratio, which fails: the semicircular form factor $\phi = J_1(2t)/t$ oscillates (ratio ~ 5 from Bessel oscillation), and the true spectrum's ratio $\approx 9.0 \approx \log 6000$ is the χ^2 extreme-value/mean ratio of a complex Gaussian — neither is structure. The corrected criterion is the *absolute decay of the upper envelope*, with the χ^2 baseline removed:

T	semicircle $\max K$	$1/(\pi T^3)$ theory	discrete $\max K$	ratio
20	3.97×10^{-5}	3.98×10^{-5}	1.000	$2.5 \times 10^4\times$
40	4.81×10^{-6}	4.97×10^{-6}	1.000	$2.1 \times 10^5\times$

The semicircle envelope decays like $1/(\pi T^3)$ (never saturates: the forgetting side); the discrete spectrum keeps $\max K = 1$ at recurrences (saturating: memory side). By $T = 40$ the two differ by five orders of magnitude: the criterion is decisive. For a finite spectrum, mean $K_N = 1/N$ is a pure diagonal floor (9.0×10^{-4} at $N \approx 1110$) and is shared by GOE, Poisson, and equispaced alike — it is *not* a criterion. The recurrence-peak scaling $P_N \sim N^{-\beta}$ separates the two regimes by a factor 27 ($\beta = 0.028$ equispaced against 0.76–0.82 random/chaotic); the full dictionary is in the companion paper on the law of forgetting [1], whose Section 12 closes this question for the genuine arithmetic spectrum.

9 Arithmetic reading: structure survives

Maass forms share with holomorphic cusp forms the SU(2) Sato–Tate semicircle: $\langle \lambda^m \rangle = 1, 2, 5, 14$ for $m = 2, 4, 6, 8$, matching the Catalan moments digit by digit; their central charges are identical to the holomorphic case, and the parity splitting lives only in the archimedean terms Γ_n^{a} of Proposition 5.1. The three-regime table:

regime	Σ_2 behavior	position
Poisson	$\sim L$	forgetting
GOE	$\sim \log L$	chaos, no arithmetic structure
Maass	GOE long range, non-Wigner short range	chaos + arithmetic structure

Maass is not an instance of forgetting but an instance of structure survival at the rigidity end; the direction of forgetting — pushing Σ_2 from 1.2 toward L — is exactly the loss-of-repulsion formulation of the law of forgetting.

10 Honest boundaries

(1) The odd spectrum is unused (hole at $[99.58, 110.17]$; re-extraction required). (2) The first Δ_3 implementation was biased (underestimating GOE by 45%, Poisson by 5%); with the standard integral definition recomputed, Δ_3 still sits 30–80% above GOE theory — the residual is not fully explained and is reported as such; Σ_2 is the primary criterion (validated unbiased on synthetic data at the same size). (3) $L \in [0, 30]$ is the reliable window. (4) Finite size compresses both ways (equispaced theory $\beta = 0$ measures 0.028; GOE/Poisson theory $\beta = 1$ measures 0.76–0.82), so the $27\times$ separation of the recurrence criterion is a conservative lower bound.

A Corrections applied in this paper

Each entry was found by the program's own checks; the master compendium records 69 entries in total, of which the following touch this paper's material.

#	error as made	correction as used here
2	Maass κ written as $1/2$	$\kappa = 0$; critical line $= \kappa + \frac{1}{2} = \frac{1}{2}$; equality with the holomorphic case is of critical lines
41	degeneracy taken as $Z(D)$, total reduced forms	$g(D) = \#\rho$-orbits $= h^+(D)$; $Z(D)$ overcounts by the orbit length (average $\approx 16\times$)
42	Pell unit $p + q\omega^n$	unit is $p - q\bar{\omega}^n$; for $D \equiv 1\,(4)$, $t = 2p - q$; 8/8 benchmarks pass after fix
43	two further degeneracy bugs (tight enumeration bound; adjacency by Zagier's ρ mis-stated)	corrected; on 36 fundamental discriminants $\#\rho$-orbits $= \mathrm{round}(h^+_{\mathrm{Dirichlet}})$, zero counterexample
44	prime geodesic theorem blocked by length-spectrum truncation	counting corrected (all discriminants, primitive forms, ρ-orbits): error exponent $1/2$ confirmed
49	saturation judged by max/mean ratio	criterion replaced by absolute upper-envelope decay + χ^2 baseline removal
57	valley search on heavy-tail means	naive branch ("any local minimum in window") fired on noise (false positive $(997, G_3, 21)$: SNR $70\times$, minimum drifts with p, expected false hits ≈ 2.0, observed 1, Poisson); upgraded criterion: depth $> 3\sigma_{\mathrm{mean}}$ *and* isolated; predictions written before the run so false positives are caught in the same round

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