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Non-Uniqueness of Leray–Hopf Solutions to the Unforced Three-Dimensional Navier–Stokes Equations

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Abstract

We prove that the incompressible Navier–Stokes equations in $\mathbb{R}^3$, without external force, admit a continuum of distinct suitable weak solutions in the Leray–Hopf class with the same initial datum. The datum is compactly supported, axisymmetric without swirl, and smooth away from the origin; it lies in $L^{3,\infty}$ and in L^q for every $q < 3$, but not in L^3. The solutions are axisymmetric without swirl and smooth at positive times, and they satisfy the local and global energy equalities. Any two of them separate in L^2 at the rate $t^{1/4+\sigma}$, for a fixed $0 < \sigma < 1/4$. The proof follows the program of Jia and Šverák, with a discretely self-similar background in place of a self-similar one. The background is an exact unforced solution built from a train of thin vortex rings, which we obtain by lifting a periodic row of planar vortices with Gaussian stream function. We construct a growing Floquet mode of its linearization directly, from the angular-two instability of the planar vortex, and conclude by a nonlinear correction and a localization of the datum.

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1 Introduction

We consider the incompressible Navier–Stokes equations in $\mathbb{R}^3$ without external force:

$$\partial_t u + \operatorname{div}(u \otimes u) - \Delta u + \nabla p = 0, \qquad \operatorname{div} u = 0, \qquad u(0) = u_0. \tag{1.1}$$

Here u is the velocity, p is the pressure, and the viscosity is normalized to one. For every divergence-free $u_0 \in L^2(\mathbb{R}^3)$, Leray [36] constructed a global weak solution that satisfies the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 \, ds \le \frac{1}{2}\|u_0\|_2^2,$$

and Hopf [25] gave the corresponding construction in bounded domains. We call such solutions Leray–Hopf solutions. Whether they are uniquely determined by their datum is a basic question that goes back to Leray's work. In this paper we prove that, without force, uniqueness fails in the axisymmetric class without swirl: Theorem A below gives a compactly supported datum from which a continuum of distinct Leray–Hopf solutions emanates. The proof is analytic. Recent results on the same question are discussed in Section 1.2.

1.1 Uniqueness and regularity

Uniqueness holds as long as one of the solutions is regular. If a Leray–Hopf solution lies in $L^r(0,T;L^q)$ with $2/r+3/q = 1$ and $3 < q \le \infty$, then every Leray–Hopf solution with the same datum coincides with it on $[0,T]$. This weak–strong uniqueness theorem is due to Prodi [39], Serrin [43] and Ladyzhenskaya [32]; Kozono and Sohr [31] treated the endpoint $L^\infty(0,T;L^3)$. Such regular solutions exist for data in the critical space L^3: Kato [29] constructed solutions for data in L^3, local in time for large data and global for small data. Koch and Tataru [30] obtained global solutions for small data in BMO^{-1}. Escauriaza, Seregin and Šverák [19] proved that Leray–Hopf solutions bounded in $L^\infty(0,T;L^3)$ are smooth. For a general Leray–Hopf solution only partial regularity is known. Scheffer [40] bounded the size of the singular set at the first singular time, and Caffarelli, Kohn and Nirenberg [10] proved that suitable weak solutions, those satisfying the local energy inequality, are smooth outside a closed set of zero one-dimensional parabolic Hausdorff measure. We refer to the monograph [34] for these results.

The critical spaces are those invariant under the scaling $u \mapsto \lambda u(\lambda x, \lambda^2 t)$ of (1.1). The datum of Theorem A lies in the weak Lebesgue space $L^{3,\infty}$ but not in L^3; near the origin it is of size $|x|^{-1}$. For such data the theory is incomplete. Seregin and Šverák [42] studied global weak solutions with large data in L^3, and Barker, Seregin and Šverák [3] constructed, for data in $L^{3,\infty}$, global weak solutions that are stable under weak-$*$ convergence of the data; their uniqueness is not known. For a scale-invariant datum it is natural to look for a self-similar solution $u(x,t) = t^{-1/2}U(x/\sqrt{t})$. Jia and Šverák [27] proved that every scale-invariant datum that is smooth away from the origin, of any size, gives rise to at least

one forward self-similar solution. Discretely self-similar data, invariant only under $u_0(x) \mapsto \lambda u_0(\lambda x)$ for one $\lambda > 1$, also admit discretely self-similar solutions: Tsai [44] constructed them for λ close to one, and Bradshaw and Tsai [6] and Chae and Wolf [12] for every $\lambda > 1$. Our solutions are built around a discretely self-similar solution of this kind.

In the axisymmetric class without swirl the theory is more complete. Ladyzhenskaya [33] and Ukhovskii and Yudovich [45] proved that smooth, sufficiently decaying axisymmetric data without swirl give global smooth solutions, and Leonardi, Málek, Nečas and Pokorný [35] relaxed the regularity of the data. Feng and Šverák [20] constructed solutions whose initial vorticity is a vortex ring, a measure supported on circles, and Gallay and Šverák [22] proved that the solution emanating from a circular vortex filament is unique. The datum of Theorem A is axisymmetric without swirl and smooth away from the origin. Theorem A therefore shows that a singularity at one point, of the critical size $|x|^{-1}$, is enough for uniqueness to fail in this class.

1.2 Non-uniqueness

Two methods have produced non-unique solutions of the equations of fluid mechanics. The first is convex integration. Buckmaster and Vicol [8] proved that weak solutions of the Navier–Stokes equations with finite kinetic energy are not unique. Buckmaster, Colombo and Vicol [7] constructed non-unique weak solutions that are smooth outside a set of singular times of Hausdorff dimension less than one, and Cheskidov and Luo [13] proved non-uniqueness on the torus in $L^p_t L^\infty_x$ for every $p < 2$, which is sharp in view of the Ladyzhenskaya–Prodi–Serrin criterion; we refer to the survey [9]. These solutions do not satisfy the energy inequality, so they are not Leray–Hopf solutions, and convex integration leaves the question open for Leray–Hopf solutions.

The second method is instability. In self-similar variables, an unstable eigenvalue of the linearization around a self-similar solution becomes, after the change back to physical variables, a second solution with the same datum. Vishik [46, 47] carried this out for the two-dimensional Euler equations with vorticity in L^p, with a force in $L^1_t(L^1 \cap L^p)_x$; a complete account is given by Albritton, Brué, Colombo, De Lellis, Giri, Janisch and Kwon [2]. Albritton, Brué and Colombo [1] adapted the method to the three-dimensional Navier–Stokes equations and proved non-uniqueness of Leray–Hopf solutions with a force; their unstable profile is a smooth compactly supported vortex ring whose cross-section is a modification of Vishik's unstable vortex. Castro, Faraco, Mengual and Solera [11] obtained sharp non-uniqueness with forcing for the α-SQG equations, which include the Euler equations, using smooth compactly supported unstable vortices. In all of these results the force is chosen so that an unstable profile is an exact solution of the forced equation. Without force, the background must be an exact solution, and constructing an exact solution whose linearization is provably unstable is the main difficulty; see the discussion in [15]. For the unforced two-dimensional Euler equations, Coiculescu [16] constructed non-unique solutions with vorticity in $L^1 \cap L^p$ by perturbing a stationary radial solution, with the spectral input verified by rigorous interval arithmetic. Choi [14] constructed two solutions with the same initial vorticity in every finite L^p, using a cascade of quadrupoles across infinitely many scales. For the Navier–Stokes equations, Coiculescu and Palasek [17] constructed data in BMO^{-1} with two distinct global solutions, both smooth at positive times, which shows that the small-data theorem of Koch and Tataru is sharp.

For the unforced Navier–Stokes equations, Jia and Šverák [28] proposed the following program. Take a scale-invariant datum and a forward self-similar solution emanating from it, which exists by [27]. If the linearization around this solution in self-similar variables has an unstable eigenvalue, then, under a further spectral condition, there are two distinct Leray–Hopf solutions with the same datum, and a localization of the datum makes it of finite energy. The spectral condition concerns a non-self-adjoint operator around a large solution that is not explicit. Guillod and Šverák [24] computed self-similar solutions for a one-parameter family of axisymmetric scale-invariant data and found a

symmetry-breaking bifurcation at large data, which would give non-uniqueness through this program.

Ionescu, Jia and Palasek [26] constructed numerically, with pointwise residuals of order 10^{-10}, unstable self-similar solutions in the axisymmetric class without swirl. Binz and Coiculescu [4] proved that, for sufficiently regular data, there are no nontrivial homothetic self-similar solutions in three dimensions.

Theorem A carries out a version of the program of Jia and Šverák in which the self-similar solution is replaced by a discretely self-similar one and the unstable eigenvalue by a Floquet exponent; both are constructed here. The proof uses no computer assistance, and the solutions have no swirl.

1.3 The main result

Write $L^2_{\mathrm{div}} = \{u \in L^2(\mathbb{R}^3;\mathbb{R}^3) : \operatorname{div} u = 0\}$ and $H^k_{\mathrm{div}} = H^k \cap L^2_{\mathrm{div}}$. The weak Lebesgue space $L^{3,\infty}$ has the quasinorm $\|f\|_{3,\infty} = \sup_{s>0} s|\{x : |f(x)| > s\}|^{1/3}$. We include the initial condition in the weak formulation: for a viscosity $\nu > 0$, a pair (u,p) solves the equation with datum u_0 on $[0,T)$ if, for every real $\varphi \in C_c^\infty(\mathbb{R}^3 \times [0,T);\mathbb{R}^3)$,

$$\int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \varphi + (u \otimes u) : \nabla \varphi + \nu u \cdot \Delta \varphi + p \operatorname{div} \varphi)\, dx\, dt + \int_{\mathbb{R}^3} u_0 \cdot \varphi(x,0)\, dx = 0. \tag{1.2}$$

Equation (1.1) is the case $\nu = 1$; a general viscosity is used for intermediate constructions. The main theorem of this paper is the following.

Theorem A. *There are $T > 0$, a nonzero compactly supported datum $u_0 \in L^2_{\mathrm{div}}$, and real solutions $(u^{(\alpha)}, p^{(\alpha)})$, $0 \le \alpha \le 1$, of the unforced equation with viscosity one and the same datum. The datum is smooth away from zero and satisfies*

$$u_0 \in L^{3,\infty}(\mathbb{R}^3) \cap \bigcap_{1 \le q < 3} L^q(\mathbb{R}^3), \qquad u_0 \notin L^3(\mathbb{R}^3).$$

Each solution is smooth at positive times and satisfies

$$u^{(\alpha)} \in C([0,T]; L^2_{\mathrm{div}}) \cap L^2(0,T;H^1),$$
$$u^{(\alpha)} \in C([\delta,T]; H^2_{\mathrm{div}}) \quad (0 < \delta < T), \qquad p^{(\alpha)} \in L^{5/3}(\mathbb{R}^3 \times (0,T)).$$

All solutions have the strong L^2 trace u_0 at zero and satisfy (1.2) with $\nu = 1$. For every $0 \le t_0 \le t_1 \le T$,

$$\frac{1}{2}\|u^{(\alpha)}(t_1)\|_2^2 + \int_{t_0}^{t_1} \|\nabla u^{(\alpha)}(t)\|_2^2\, dt = \frac{1}{2}\|u^{(\alpha)}(t_0)\|_2^2.$$

They also satisfy the local energy equality. Writing $u = u^{(\alpha)}$, $p = p^{(\alpha)}$ and $e = |u|^2/2$, for every real $\phi \in C_c^\infty(\mathbb{R}^3 \times [0,T))$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \phi\, dx\, dt = {} & \int_0^T \int_{\mathbb{R}^3} \left[e(\partial_t \phi + \Delta \phi) + (e+p)u \cdot \nabla \phi\right] dx\, dt \\ & + \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 \phi(x,0)\, dx. \end{aligned} \tag{1.3}$$

In particular, the solutions are suitable. Moreover,

$$\sup_{0 \le \alpha \le 1} \|u^{(\alpha)}\|_{L^\infty(0,T;L^{3,\infty})} < \infty, \qquad \sup_{0 \le \alpha \le 1} \|u^{(\alpha)}\|_{L^r(0,T;L^q)} < \infty$$

for every $1 \le r \le \infty$, $2 \le q \le \infty$ with $2/r + 3/q > 1$. There are constants $0 < c_* \le C_* < \infty$ and $0 < \sigma < 1/4$ such that, for every $\alpha, \alpha' \in [0,1]$ and $0 < t \le T$,

$$c_*|\alpha - \alpha'| t^{\sigma + 1/4} \le \|u^{(\alpha)}(t) - u^{(\alpha')}(t)\|_2 \le C_*|\alpha - \alpha'| t^{\sigma + 1/4}.$$

In particular,

$$\int_0^T \|u^{(\alpha)} - u^{(\alpha')}\|_2^2 \, dt \geq \frac{c_*^2 |\alpha - \alpha'|^2 T^{2\sigma+3/2}}{2\sigma + 3/2}.$$

Thus the family consists of pairwise distinct Leray–Hopf solutions. The datum and all solutions are axisymmetric without swirl: they are invariant under rotations about the x_3-axis, and their azimuthal components vanish.

The integrability in Theorem A is sharp on the scale of mixed Lebesgue spaces. Every solution of the family is bounded in $L^r(0,T;L^q)$ for all $1 \leq r \leq \infty$ and $2 \leq q \leq \infty$ with $2/r + 3/q > 1$. By the weak–strong uniqueness theorem [39, 43, 32], none of them lies in $L^r(0,T;L^q)$ with $2/r + 3/q = 1$ and $3 < q \leq \infty$, since otherwise all of them would coincide. Similarly, the uniform bound in $L^\infty(0,T;L^{3,\infty})$ cannot be improved to $L^\infty(0,T;L^3)$: the strong L^2 trace would then put u_0 in L^3.

The rate $t^{\sigma+1/4}$ has a simple origin. To leading order the difference of two solutions is a multiple of $t^{\sigma-1/2} b_*(x/\sqrt{t}, \log t)$, where b_* is the periodic profile of a Floquet mode with exponent σ (see Section 1.4); its L^2 norm is a multiple of $t^{\sigma+1/4}$.

1.4 The construction

We follow the program of Jia and Šverák [28]. In the similarity variables $y = x/\sqrt{t}$ and $\tau = \log t$, write $u(x,t) = t^{-1/2} U(x/\sqrt{t}, \log t)$. Equation (1.1) becomes

$$\partial_\tau U = \Delta U + \tfrac{1}{2} y \cdot \nabla U + \tfrac{1}{2} U - \mathbb{P} \operatorname{div}(U \otimes U),$$

where $\mathbb{P}$ is the Leray projection, and the initial time $t = 0$ becomes $\tau = -\infty$. Self-similar solutions are stationary in τ. Discretely self-similar solutions, those with $u(\lambda x, \lambda^2 t) = \lambda^{-1} u(x,t)$ for one $\lambda > 1$, are periodic in τ with period $2 \log \lambda$. Suppose that U_* is a periodic solution and that its linearization has a Floquet solution $e^{\sigma\tau} b_*$, with $\sigma > 0$ and b_* periodic. Since $e^{\sigma\tau} = t^\sigma$ vanishes at $t = 0$, the fields $U_* + \alpha e^{\sigma\tau} b_*$, $\alpha \in \mathbb{R}$, all carry the datum of U_*, and one expects to correct each of them to an exact solution. The difficulty is to find an exact unforced periodic solution whose linearization provably has such a mode. Proposition 2.1 in Section 2 states the background and the mode we construct. The construction has four steps.

Step 1: the planar vortex train and its Gaussian instability (Sections 3–4). The building block is the planar vortex with Gaussian stream function $\psi_0(x) = -2e^{-|x|^2/4}$, $x \in \mathbb{R}^2$. Its velocity $\nabla^\perp \psi_0 = e^{-|x|^2/4} x^\perp$ rotates, and its vorticity $(2 - |x|^2/2) e^{-|x|^2/4}$ changes sign and has zero integral: it is a shielded vortex. The instability of shielded vortices is classical in geophysical fluid dynamics [23]. We prove that the Euler equations linearized around ψ_0 have exactly one unstable pair of eigenvalues $\lambda, \bar{\lambda}$, with $\lambda = \gamma + i\omega$, $\gamma > 0$, and eigenfunctions in angular mode two (Proposition 4.1). The count uses the index theorem of Lin and Zeng [38] together with an explicit negative direction. Lin [37] developed related criteria for instability. With a small viscosity $\nu > 0$ the vortex diffuses: at time s its vorticity is $a^{-2} E_0(x/\sqrt{a})$, $a = 1 + \nu s$, where E_0 is the vorticity of ψ_0. The turnover clock $s/(1+\nu s)$ has total length ν^{-1}. In the heat variables the unstable evolution has the canonical normalization $N_\nu = e^{\lambda/\nu} \nu^{-\kappa}$, for a constant $\kappa \in \mathbb{C}$ (Proposition 4.5). A fixed canonical age $\theta = [\nu(1+\nu s)]^{-1} > 0$ corresponds to physical time $s = (\nu^2 \theta)^{-1} - \nu^{-1}$, of order ν^{-2}. The train is the exact solution of the planar Navier–Stokes equations with viscosity ν on the cylinder $(\mathbb{R}/c\mathbb{Z}) \times \mathbb{R}$, $c = \nu^{-2}$, whose initial stream function is the periodic sum $\sum_{n \in \mathbb{Z}} \psi_0(q - nc, \eta)$. It exists globally by the theorem of Gallay and Slijepčević [21]. Until the diffusing vortices overlap the train is close to the sum of independently evolving vortices, and afterwards it is a decaying shear flow plus a part that decays exponentially (Lemma 3.1).

Step 2: thin rings and the exact discretely self-similar background (Sections 5–7). In cylindrical coordinates (r, z) and for $R \geq 2$, use the logarithmic ring chart

$$q = R\log(r/R), \qquad \eta = Rz/r, \qquad s = R^2 t/r^2.$$

The stream function $r\psi(q, \eta, s)$ of the train defines an axisymmetric field without swirl. The planar vortex centered at $q = nc$ becomes a vortex ring of radius $r_n = Re^{nc/R}$ whose cross-section has size comparable to r_n/R; the rings accumulate at the origin and at infinity. The dilation $(r, z, t) \mapsto (\Lambda r, \Lambda z, \Lambda^2 t)$ with $\Lambda = e^{c/R}$ sends q to $q + c$ and fixes η and s, so the periodicity of the train becomes exact discrete self-similarity: translation becomes discrete scaling. The lifted field solves the Navier–Stokes equations up to a curvature residual, which is small when the rings are thin, $\epsilon = R^{-1} \to 0$ (Proposition 5.1). In similarity variables the residual is a small periodic force. Since the background must be an exact unforced solution, we remove it: a right inverse for the linearized periodic problem and a nonlinear correction give an exact periodic solution U_*. Its datum u_{dss} is discretely self-similar and of size $|x|^{-1}$, and U_* differs from the heat evolution of u_{dss} by a remainder in H^2 (Section 7). The three small parameters are tied together: $c = \nu^{-2}$ and $\epsilon = e^{-d/\nu}$ with d in a fixed compact interval.

Step 3: the direct three-dimensional Floquet construction (Sections 8–18). We fix the exact background and look for $0 < \sigma < 1/4$ and a nonzero periodic profile b_* solving (2.4). The mode comes from a loop through the rings. In similarity variables the rings move toward the origin and age as τ increases: rings far from the origin are young and still close to the inviscid Gaussian vortex, and rings near the origin are old and have diffused. A strain acting on a young ring excites its unstable angular-two mode, which grows by the factor N_ν during the life of the ring. Once the ring is old, the stress of the amplified perturbation generates, through the three-dimensional pressure, a strain on the younger rings. By discrete self-similarity this loop repeats in every period, and a Floquet mode is a loop that reproduces its input up to the factor $e^{-\sigma\mathcal{T}}$, where $\mathcal{T}$ is the period.

We find the exponent in the spirit of the Birman–Schwinger principle [5, 41]: instead of studying the spectrum of the full linearized operator, we reduce the Floquet problem to a scalar equation for the gain of one loop. We solve the forced linearized equation along the train with zero initial data and bounds polynomial in ν^{-1} (Section 9), and rewrite the Floquet problem as a fixed point problem for the return operator, which maps the strain received by young rings to the strain they return. In the joint limit $\epsilon, \nu \to 0$, and on a fixed space of smooth sources, this operator is a rank-one operator plus a remainder that tends to zero. Its nonzero eigenvalue, the gain of one loop for an exponent μ, is

$$2\epsilon^3 \nu^{5+2\mu}\,\mathrm{Re}\{\mathcal{C}(\mu)N_\nu\} + o(1),$$

where $\mathcal{C}$ is an explicit holomorphic function that does not vanish identically. The factor ϵ^3 measures the weak three-dimensional coupling of thin rings, and N_ν is the planar amplification. We choose $\nu_n \to 0$ so that the phase of $\mathcal{C}N_{\nu_n}$ is fixed and ϵ_n so that the gain is close to one, construct the background once at these parameters, and vary only μ in a window of width $|\log \nu_n|^{-1}$. Across the window the gain moves from about e to about e^{-1}. Its crossing of one gives a Floquet exponent σ_n of the exact three-dimensional operator, with its full pressure and the correction of Step 2 (Section 18). The planar response only supplies the limit of the return operator; the mode is constructed for the exact operator.

Step 4: nonlinear correction and localization (Section 19). With the background and the mode in hand we follow the instability and localization argument of Jia and Šverák [28], adapted to a profile periodic in τ. We replace the infinite-energy datum u_{dss} by a compactly supported divergence-free truncation that equals u_{dss} near the origin. The resulting common heat correction K satisfies $\|K(\tau)\|_{W^{2,\infty}} = O(e^{\tau/2})$. For each amplitude $\alpha \in [0, 1]$ we seek a profile $U_* + K + \alpha e^{\sigma\tau} b_* + z_\alpha$ with $\|z_\alpha(\tau)\|_{H^2} = O(e^{\beta\tau})$ as $\tau \to -\infty$, where $\sigma < \beta < 2\sigma < 1/2$. We choose β so that no Floquet multiplier

has modulus $e^{\beta\mathcal{T}}$ and split the periodic evolution at this rate. The resulting right inverse accounts for every faster unstable mode, so no maximal-growth assumption on σ is needed. Both forcing rates 2σ and $1/2$ exceed β, and a contraction gives the nonlinear corrections. Their Lipschitz dependence on α, uniformly on one interval $[0,T]$, gives the two-sided separation estimate of Theorem A.

Organization. Section 2 fixes conventions and states Proposition 2.1. Sections 3–19 carry out the four steps above, and the proof of Theorem A is completed at the end of Section 19.

Formalization. A Lean formalization of this work is forthcoming at `https://github.com/mcoiculescu/unforced-leray-hopf-public`.

Declaration on the use of large language models

This work was carried out with extensive use of the large language models Claude (Anthropic) and Codex (OpenAI). The models were used to search for the construction, to write and revise the proofs, and to check them.

While exploring the instability of an exact self-similar profile as a mechanism, among others, for nonuniqueness, GPT-6 Astra proposed using a planar unstable vortex as a building block to lift to an unstable profile. We found this approach promising and pushed the model to complete the construction of an exact discretely self-similar solution with a growing mode.

The argument first appeared in a draft of about 200 pages, written largely by the models, whose exposition was poor and highly redundant. We then realized that the construction does not need swirl; working in the axisymmetric class without swirl reduced the draft to about 150 pages. Finally, we replaced the original proof of the existence of the Floquet exponent by a more standard argument in the spirit of the Birman–Schwinger principle, which reduces the Floquet problem to a scalar equation for the gain of one loop (Section 1.4). This cut the length of the proof in half, to about 70 pages.

The authors take full responsibility for the content of this paper.

2 Conventions and the reduction to a growing mode

We use $(u\otimes v)_{ij}=u_iv_j$, $(\operatorname{div}T)_i=\partial_jT_{ij}$, $R_j=\partial_j(-\Delta)^{-1/2}$ and the Cartesian Leray projection $\mathfrak{P}=\mathbb{P}$. The pressure is the global choice $p=\sum_{i,k}R_iR_k(u_iu_k)$. A vector field is axisymmetric without swirl if it is invariant under rotations about the x_3-axis and its azimuthal component vanishes. For intermediate constructions the viscosity is $\nu>0$, and the initial condition is included through (1.2).

The construction uses the similarity variables $y=x/\sqrt{t}$, $\tau=\log t$. For a period $\mathcal{T}>0$, set

$$\mathcal{Y}_q=L^q([0,\mathcal{T}],d\tau/\mathcal{T};L^q(\mathbb{R}^3)),\qquad \mathcal{T}_0=\partial_\tau-\tfrac12 y\cdot\nabla.$$

Periodic spaces identify their endpoints. The analytic construction needed for localization is stated below.

Proposition 2.1. *There exist $\nu>0$, $\mathcal{T}>0$, and a nonzero real divergence-free field u_{dss} on $\mathbb{R}^3$, smooth away from zero, satisfying*

$$u_{\mathrm{dss}}(e^{\mathcal{T}/2}x)=e^{-\mathcal{T}/2}u_{\mathrm{dss}}(x),\qquad |D^\alpha u_{\mathrm{dss}}(x)|\le C_\alpha|x|^{-1-|\alpha|}\quad (x\neq0,\ |\alpha|\le4).\tag{2.1}$$

Set

$$H_\nu[u_{\mathrm{dss}}](y,\tau)=e^{\tau/2}[e^{\nu e^\tau\Delta}u_{\mathrm{dss}}](e^{\tau/2}y).$$

There is a real divergence-free $\mathcal{T}$-periodic distributional solution U_* of

$$\partial_\tau U_*=\nu\Delta U_*+\tfrac12 y\cdot\nabla U_*+\tfrac12U_*-\mathfrak{P}\operatorname{div}(U_*\otimes U_*),\qquad U_*=H_\nu[u_{\mathrm{dss}}]+r_{\mathrm{heat}},\quad r_{\mathrm{heat}}\in C_{\mathrm{per}}H^2(\mathbb{R}^3).\tag{2.2}$$

There are $0 < \sigma < 1/4$ and a nonzero real $\mathcal{T}$-periodic divergence-free field b_* satisfying

$$b_* \in \mathcal{Y}_{2,\mathrm{div}}, \qquad \nabla b_* \in \mathcal{Y}_2. \tag{2.3}$$

The second space is tensor-valued. The periodic field b_* is the Floquet profile: $e^{\sigma\tau} b_*$ solves the linearized equation and has multiplier $e^{\sigma\mathcal{T}} > 1$ over one period. The equation for the profile is

$$\begin{gathered} \sigma b_* + \mathcal{T}_0 b_* - \nu\Delta b_* - \tfrac{1}{2} b_* + \mathfrak{P}\,\mathrm{div}(U_* \otimes b_* + b_* \otimes U_*) = 0 \\ \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}/\mathcal{T}\mathbb{Z}). \end{gathered} \tag{2.4}$$

The fields u_{dss}, U_* and b_* are axisymmetric without swirl.

Proposition 19.1 in Section 19 shows that Proposition 2.1 implies Theorem A. Constants denoted by P_ν or $\mathcal{P}_\nu$ are fixed polynomials in ν^{-1} and the specified polynomial cell parameters. Each exponential loss is kept separately until the final parameter inequalities are imposed.

3 The exact periodic planar train

We start with an exact nonlinear solution on the periodic cylinder $(\mathbb{R}/c\mathbb{Z}) \times \mathbb{R}$. Its initial stream is a train of translated Gaussians; the independently evolving Gaussian sum is an approximation whose interactions must be estimated. The train bounds proved here and the Gaussian instability proved in Section 4 are used to construct the background and its growing mode.

Fix a finite collection of derivative orders, transverse weights and Lebesgue exponents, and fix $\rho > 1/2$. Throughout this section

$$c = \nu^{-\rho}, \qquad e_c = (\nu c^2)^{-1}, \qquad C_c = (\mathbb{R}/c\mathbb{Z}) \times \mathbb{R}, \qquad a(s) = 1 + \nu s, \qquad s_c = (c^2 - 1)/\nu.$$

Cell norms are unnormalized. A bar denotes the normalized horizontal average and a tilde its complement. The letter P denotes a polynomial in c, ν^{-1}, with degree depending only on the fixed collection; P_S may also depend polynomially on $1 + S$. All the smallness conditions below are satisfied by first fixing that collection and then decreasing e_c. In particular they hold for $c = \nu^{-2}$ and small ν.

Let $v = \nabla^\perp \psi$ be the exact cylinder Navier–Stokes solution with viscosity ν and initial stream

$$\psi(q, \eta, 0) = \sum_{n \in \mathbb{Z}} -2e^{-((q - nc)^2 + \eta^2)/4}.$$

This specifies the nonlinear train, rather than the sum of independently evolving vortices. The datum is divergence-free and belongs to $\mathrm{BUC} \cap H^k$ for every k, where BUC denotes the bounded uniformly continuous functions on C_c; its vorticity is in $L^1 \cap L^\infty$ and has zero integral. The cylinder existence theorem of Gallay and Slijepčević [21] therefore applies after interchanging the two coordinates and taking density one. It gives a unique global mild solution in BUC, smooth for positive time, with the bounded periodic pressure choice. It is also the finite-energy solution: on each finite interval the mild equation in $\mathrm{BUC} \cap L^2$ gives

$$\|v(t)\|_2 \le \|v(0)\|_2 + C\nu^{-1/2} \sup_{[0,t]} \|v\|_\infty \int_0^t (t - r)^{-1/2} \|v(r)\|_2 \, dr,$$

which continues L^2 membership. Approximation and transverse cutoffs then give the velocity and vorticity energy identities. The scalar vorticity equation preserves zero circulation and its L^1, L^∞ bounds. Uniqueness gives $v(-q, -\eta, s) = -v(q, \eta, s)$.

Lemma 3.1. *Put*

$$\psi^p(x,s) = -2a^{-1}e^{-|x|^2/(4a)}, \qquad V_c^p = \sum_n \nabla^\perp \psi^p(q - nc, \eta, s).$$

For every fixed j, with constants independent of c, ν,

$$\|D^j(v - V_c^p)(s)\|_2 \le C_j \nu^{-1} a^{-2-j/2} e^{-b_j c^2/a}, \tag{3.1}$$

$$\|D^j(v - V_c^p)(s)\|_\infty \le C_j \nu^{-1} a^{-(5+j)/2} e^{-b_j c^2/a}, \qquad 1 \le a \le c^2. \tag{3.2}$$

After overlap, set $X = q/c$, $Y = \eta/c$, $T = \nu(s - s_c)/c^2$, and write $c^3 v = b(Y,T)e_1 + W(X,Y,T)$, $\overline{W} = 0$. Then

$$\|\partial_Y^j b(T)\|_p \le C_{j,p}(1+T)^{-1-j/2+1/(2p)}, \qquad p = 2, \infty, \tag{3.3}$$

$$\|D^j W(T)\|_{2\cap\infty} \le C_j e^{-\lambda_j T}. \tag{3.4}$$

In particular, whenever $a(s_0) \le c^2$,

$$\|v(s)\|_2 \le C a(s)^{-1} \qquad (s \le s_c), \tag{3.5}$$

$$\|v(s)\|_2 \le C c^{-2}(s/s_c)^{-3/4} \qquad (s \ge s_c), \tag{3.6}$$

$$\int_{s_0}^\infty \sum_{j=1}^{k+1} \|D^j v(s)\|_\infty \, ds \le \frac{C_k}{\nu a(s_0)} + C_k e_c, \tag{3.7}$$

$$\int_{s_0}^\infty \|\widetilde{v}(s)\|_2 \, ds \le \frac{C}{\nu}\left(1 + \log\frac{c^2}{a(s_0)}\right). \tag{3.8}$$

Proof. The isolated vortex is an exact radial heat solution. Its periodization has residual $\operatorname{div} Q$, where $Q = \sum_{i\neq j} V_i \otimes V_j$. Completing squares in the Gaussian products and summing their separation index gives

$$\|D^j V_c^p\|_\infty \le C_j a^{-(3+j)/2}, \quad \|V_c^p\|_2 \le C a^{-1}, \quad \|D^j Q\|_2 \le C_j a^{-(5+j)/2} e^{-b_j c^2/a}.$$

Indeed the product is bounded by $Ca^{-3}e^{-b(i-j)^2c^2/a}$ times a Gaussian of width $\sqrt{a}$ centered at the midpoint; its cell L^2 norm contributes $C\sqrt{a}$. Fixed weights in units of $\sqrt{a}$ obey the same estimate.

Write $e = v - V_c^p$, use slow time $\sigma = \nu s$, and choose an integer $N \ge 3$ at least two above the required supremum order. For $0 < \delta \le (4N)^{-1}$ set $E_N^2 = \sum_{j=0}^N (\delta a)^j \|D^j e\|_2^2$. On the bootstrap interval $\|\nabla e\|_\infty \le a^{-2}$, differentiated energy and one integration by parts on the stress give

$$(E_N^2)' + \sum_{j=0}^N (\delta a)^j \|\nabla D^j e\|_2^2 \le \frac{C_N}{\nu a^2} E_N^2 + C_N \nu^{-2} a^{-5} e^{-b_N c^2/a}.$$

Here differentiated transport is controlled by $\|[D^j, e\cdot\nabla]e\|_2 \le C_j \|\nabla e\|_\infty \|D^j e\|_2$. Its constants are independent of the period: integration by parts gives $\|D^l f\|_{2m/l} \le C_m \|f\|_\infty^{1-l/m} \|D^m f\|_2^{l/m}$, and applying this to $f = \nabla e$ bounds each commutator product. The differentiated weights are absorbed by the preceding derivative's dissipation. Time-ordered Gronwall yields

$$E_N^2(a) \le C_N \nu^{-2} \int_1^a r^{-5} e^{-b_N c^2/r + C_N/(\nu r)} \, dr \le C_N \nu^{-2} a^{-4} e^{-b'_N c^2/a},$$

because e_c is small. In the last integral use $u = c^2/r$. The cylinder interpolation inequality

$$\|f\|_\infty \le C\|f\|_2^{1/2}\|D^2 f\|_2^{1/2} + Cc^{-1/2}\|f\|_2^{1/2}\|\nabla f\|_2^{1/2}$$

includes the one-dimensional mean term; its other term follows by Fourier inversion on nonzero horizontal modes. It proves the second estimate in (3.2), since $a^{1/4}c^{-1/2} \le 1$. Consequently $\|\nabla e\|_\infty \le Ce_c(c^2/a)e^{-bc^2/a}a^{-2} \le a^{-2}/2$. Continuity from the smooth datum closes the bootstrap to s_c.

At overlap, $U = c^3 v$ solves $U_T + e_c U \cdot \nabla U + \nabla \Pi = \Delta U$ on the unit cylinder, and (3.2) gives $\|U(0)\|_{H^N} \le C_N$. Fix $N \ge 3$ sufficiently above the entire required derivative budget for the Sobolev interpolations below. Differentiated transport gives

$$\frac{1}{2}\frac{d}{dT}\|U\|_{H^N}^2 + (1 - C_N e_c \|U\|_{H^N})\|\nabla U\|_{H^N}^2 \le 0.$$

Small e_c thus gives the uniform high-order norm directly. The exact fluctuation energy identity is

$$\frac{1}{2}\frac{d}{dT}\|W\|_2^2 + \|\nabla W\|_2^2 = -e_c \int W_1 W_2 b_Y.$$

The horizontal Poincare gap gives exponential L^2 decay; interpolation with the uniform high-order bound gives all required smooth decay.

The mean stream $S = \overline{\psi}$ is fixed by

$$S_s - \nu S_{\eta\eta} = g := \overline{v_1 v_2}, \qquad \overline{v}_1 = -S_\eta, \qquad \overline{v}_2 = 0. \tag{3.9}$$

The last equality follows from divergence and finite energy; uniqueness of the averaged velocity equation proves the preceding equality. Separate reflection symmetries of V_c^p imply $\overline{(V_c^p)_1 (V_c^p)_2} = 0$. Consequently

$$\|S(s_c) - \overline{\psi_c^p}(s_c)\|_1 \le C\nu^{-2}c^{-5} + C\nu^{-3}c^{-7}.$$

For example, the mixed term is bounded by $C\nu^{-2}c^{-1}\int_1^{c^2} a^{-3}e^{-bc^2/a}\,da$; the quadratic error has $\nu^{-3}a^{-4}$ in its place. Thus the rescaled mean stream function $\mathsf{S} = c^2 S$ satisfies $\|\mathsf{S}(0)\|_{L^1_Y} \le C$. It solves

$$\mathsf{S}_T - \mathsf{S}_{YY} = e_c \overline{W_1 W_2}, \qquad b = -\mathsf{S}_Y.$$

Split Duhamel at $T/2$. On its early half use the source L^1 bound and $\|\partial_Y^{j+1} G_T\|_p \le CT^{-1-j/2+1/(2p)}$; on its late half move derivatives onto the exponentially small source. This proves (3.4). Integrating the two regimes proves (3.8); in particular the pre-overlap error contributes at most $C_j\nu^{-2}c^{-j-3}$ to $\int \|D^j e\|_\infty$, and the post-overlap contribution is $C_j\nu^{-1}c^{-j-1}$. For smoothness statements at fixed parameters there is no restriction to this quantitative derivative budget. The resulting integrable strain and the tame higher-order inequality $d\|U\|_{H^k}/dT \le C_k e_c \|\nabla U\|_\infty \|U\|_{H^k}$ propagate every higher norm from its finite smooth overlap datum. All higher-order versions of the argument therefore hold at fixed parameters, without requiring a new smallness condition for each order. □

Lemma 3.2 (One weighted heat description at all ages)**.** *Every fixed mixed age/spatial derivative of* $(\widetilde{\psi}, \widetilde{v}, p)$ *obeys*

$$\|\langle\eta\rangle^m \partial_s^\ell D^j(\widetilde{\psi}, \widetilde{v}, p)(s)\|_{2\cap\infty} \le Pe^{-\kappa\nu s/c^2}. \tag{3.10}$$

The decaying pressure gauge is used here. The mean stream is even and, for every fixed N*, has the following expansion and remainder for nonnegative integers* m, b, ℓ *and* $1 \le p \le \infty$*:*

$$S(\eta, s) = \sum_{j=0}^{N} a_{2j}\partial_\eta^{2j} G_{\nu s}(\eta) + \mathcal{R}_N(\eta, s), \tag{3.11}$$

$$a_{2j} = \frac{\int \eta^{2j} S(\eta, 0)\,d\eta}{(2j)!} + \int_0^\infty \sum_{k+\ell=j} \frac{(-\nu t)^\ell}{\ell!(2k)!}\int \eta^{2k} g(\eta, t)\,d\eta\,dt, \tag{3.12}$$

$$\|\eta^m \partial_\eta^b \partial_s^\ell \mathcal{R}_N(s)\|_p \le Ps^{(m-b-2N-3)/2-\ell+1/(2p)}, \qquad s \ge 1. \tag{3.13}$$

In particular, replacing the remainder power by $(m-b-1)/2-\ell+1/(2p)$ bounds S itself with s replaced by $1+s$. On each $0\le s\le S_*$, all these fields and fixed derivatives have transverse tails

$$P_{S_*}\{e^{-bl/c}+e^{-bl^2/(C_0+\nu S_*)}\},\qquad |\eta|\ge C/\nu+2+l. \tag{3.14}$$

On every fixed slow-age interval $0\le s\le \Sigma/\nu$, put $\sigma=\nu s$ and let p_c^p be the sum of the isolated radial pressures in the decaying gauge. For $d=(v-V_c^p,\omega-\operatorname{curl}V_c^p,\nabla(p-p_c^p))$,

$$\sum_{\ell\le L,j\le k}\Big(\|\langle\eta\rangle^m\partial_\sigma^\ell D^j d\|_2+\|\partial_\sigma^\ell D^j d\|_\infty\Big)\le Pe^{C_\Sigma/\nu-b_\Sigma c^2}. \tag{3.15}$$

Proof. Lemma 3.1 gives $\int_0^\infty\|v_2(s)\|_\infty\,ds\le C/\nu$: integrate $a^{-3/2}$ and the comparison error before overlap, and $c^{-3}W_2$ afterwards. For the vorticity drift–diffusion equation, the Gaussian datum and the supersolutions

$$C\exp\{-\lambda(\eta-A(s))+(C_0+\nu s)\lambda^2\},\qquad A(s)=\int_0^s\|v_2(t)\|_\infty\,dt,$$

with both transverse signs, imply $|\omega|\le C\exp[-b(|\eta|-C/\nu)_+^2/(C_0+\nu s)]$. The comparison is justified by exhaustion with a positive quadratic barrier. Unit-patch interpolation with the fixed higher smooth bounds from Lemma 3.1 gives the same tail, with reduced exponent, for every required spatial derivative of ω.

For $k_n=2\pi n/c\ne0$, the stream inverse has kernel $-e^{-|k_n||\eta|}/(2|k_n|)$. Move higher derivatives to the source and sum modes using a fixed extra horizontal derivative. Splitting the convolution at distance $l/2$ proves (3.14) for the nonzero stream and velocity modes, with only polynomial costs in c. The zero mode has two decaying primitives:

$$\overline{v}_1(\eta)=\int_\eta^\infty\overline{\omega}(\xi)\,d\xi,\qquad S(\eta)=\int_\eta^\infty\overline{v}_1(\xi)\,d\xi.$$

Zero circulation makes the first primitive agree at both ends. Central oddness makes $\overline{v}_1$ odd and integrable, so its integral is zero and the second primitive also decays at both ends. It agrees with (3.9) and is even. These formulas prove the mean tails and polynomial weighted bounds for all fields at every age.

Interpolate a doubled polynomial weight with the unweighted exponential fluctuation bounds in (3.4); the resulting polynomial in $1+T$ is absorbed in a smaller exponential. For pressure, $\overline{p}=-\overline{v_2^2}$, and the pure mean tensor $\overline{v}_1^2e_1\otimes e_1$ has zero double divergence. Every other term in $-\Delta p=\partial_i\partial_j(v_iv_j)$ contains a fluctuation. The same inverse kernels prove (3.10) and (3.14) for pressure. The equations now give the age derivatives by finite induction: each fluctuation equation retains a fluctuation factor, and each differentiated vorticity equation retains a Gaussian vorticity factor.

It follows that $S(0)$ has every fixed spatial moment and g has every fixed mixed space/time moment, all bounded by P. Taylor-expand $G_{\nu s}(\eta-\xi)$ through order $2N+1$; evenness removes odd terms. In the source Duhamel integral, for $t\le s/2$, Taylor-expand jointly in (ξ,t) and retain $k+\ell\le N$. The identity $\partial_sG_{\nu s}=\nu\partial_\eta^2G_{\nu s}$ gives exactly the coefficients in (3.13). Integral Taylor remainders are controlled by the next even derivative and a fixed higher moment. For $t\ge s/2$, transfer derivatives to the exponentially decreasing source. Its tail, and the coefficient-integral tails, are exponential times a polynomial and obey the stated remainder. The same heat proof without subtraction gives the asserted bound for S.

Finally (3.2), interpolated with the just-proved doubled weights, proves fixed-band comparison for velocity and vorticity. The pressure difference has source tensor $Q+V_c^p\otimes e+e\otimes V_c^p+e\otimes e$. On each nonzero mode the pressure kernels are $k_n^2G_{k_n}$, $k_nG'_{k_n}$, and $G''_{k_n}=k_n^2G_{k_n}-\delta_0$ up to signs; their weighted L^1 norms cost only powers of c. The zero Fourier mode of the pressure difference is $-\overline{(Q+V_c^p\otimes e+e\otimes V_c^p+e\otimes e)_{22}}$. Weighted convolution proves its estimate, and the difference equations inductively give slow-time derivatives at polynomial cost in ν^{-1}. □

4 The Gaussian instability and canonical evolution

4.1 The Gaussian seed, its two couplings, and the complementary evolution

Use $u = \nabla^\perp \psi$, $\operatorname{curl} u = \Delta\psi$ and put

$$x = r^2/4, \quad \Omega = e^{-x}, \quad V_0(y) = \Omega y^\perp, \quad E_0 = (2-2x)e^{-x}, \quad k = E_0'/r = (x-2)e^{-x}, \quad G = (4\pi)^{-1}e^{-x}.$$

The Euler linearization is

$$\mathscr{L} f = -\Omega \partial_\vartheta f - (\nabla^\perp \Delta^{-1} f) \cdot \nabla E_0.$$

For $j \geq 0$, $M > 1$, let H^j_M be the space with norm $\sum_{|\alpha|\leq j} \|\langle y\rangle^M \partial^\alpha f\|_2$ and let $\mathcal{X}_{j,M} = \{f \in H^j_M : \int f = 0\}$. In angular sector two use

$$X_G = L^2((0,\infty), G^{-1} r\, dr) e^{2i\vartheta}, \qquad L_0 = -2i\Omega + 2ikR, \quad R = \Delta_2^{-1},$$

where the pairing is linear in its first argument and omits 2π. For a general planar function, $f_2(r) = (2\pi)^{-1} \int_0^{2\pi} f(r,\vartheta) e^{-2i\vartheta} d\vartheta$. The regular decaying Green inverse in angular sector $n \geq 1$ is

$$R_n w = -\frac{1}{2n}\left[r^{-n} \int_0^r s^{n+1} w(s)\, ds + r^n \int_r^\infty s^{1-n} w(s)\, ds \right]. \tag{4.1}$$

Proposition 4.1. *On every $\mathcal{X}_{j,M}$ the entire spectrum of $\mathscr{L}$ in $\{\Re z > 0\}$ is one algebraically simple conjugate pair $\lambda = \gamma + i\omega$, $\bar\lambda$, where $\gamma > 0$ and $-2 < \omega < 0$. Its eigenvectors are $q = h(r)e^{2i\vartheta}$ and $\bar q$. Every derivative of q is bounded by a polynomial times $e^{-r^2/4}$, $\int q = 0$, its velocity belongs to L^2, and*

$$M_q := \int_0^\infty r^3 h(r)\, dr \neq 0.$$

The same λ is the sole growing spectral value of L_0 on X_G. Its projection is $P = q\ell$, with

$$\ell(f) = \frac{1}{J_q} \int_0^\infty \frac{h(r) f_2(r)}{k(r)} r\, dr, \qquad J_q = \int_0^\infty \frac{h(r)^2}{k(r)} r\, dr \neq 0. \tag{4.2}$$

The quotients at $k = 0$ are removable. In particular, the harmonic-strain input $H = r^2 k(r) e^{2i\vartheta}$ satisfies $\ell(H) = M_q/J_q \neq 0$.

With $\mathscr{P}$ the projection onto the pair and $Q = I - P$,

$$\|e^{t\mathscr{L}}(I-\mathscr{P})\|_{H^j_M \to H^j_M} \leq C_{j,M,\eta} e^{\eta t}, \qquad \|e^{t(L_0-\lambda)}Q\|_{X_G \to X_G} \leq C_\eta e^{-(\gamma-\eta)t} \tag{4.3}$$

for every $\eta > 0$ and $t \geq 0$. Finally $D = \Delta + \frac{1}{2} y \cdot \nabla + 2$ is selfadjoint and nonpositive on X_G, with spectrum $\{0, -1, -2, \ldots\}$. Both q and the normalized Gaussian adjoint vector $q^\dagger$ belong to every $\mathcal{D}(D^m)$; thus DP and PD extend boundedly to X_G.

Proof. One negative direction. Set $a_* = e^{-2}$, $t = \log r$, and

$$v = -\frac{k}{\Omega - a_*} = \frac{x-2}{e^{x-2}-1} > 0, \qquad W = r^2 v = \frac{4x(x-2)}{e^{x-2}-1}, \qquad \mathcal{H}_n = -\partial_t^2 + n^2 - W.$$

At $x = 2$, define $v = 1$ and $W = 8$ by continuity. The bounded function W tends to zero at both ends, so $\mathcal{H}_n$ is selfadjoint on $H^2(\mathbb{R})$ and has essential spectrum $[n^2, \infty)$: multiplication by W is relatively compact by truncation and Rellich's theorem. Also $\sup W < 9$: for $y = x - 2$, the function $F(y) = 9(e^y - 1) - 8y - 4y^2$ satisfies $F(0) = 0$ and $F'(y) \geq -8\log(8/9) > 0$, so $9 - W = F(y)/(e^y - 1) > 0$; at $y = 0$ the inequality follows from $W = 8$.

There is a negative trial in sector two. For $\phi = r^2 e^{-r^2/8}$ direct integration gives

$$\int_0^\infty (|\phi'|^2 + 4r^{-2}|\phi|^2 - v|\phi|^2) r\,dr = 48 - 32\int_0^\infty \frac{x-2}{e^{x-2}-1} x^2 e^{-x}\,dx.$$

For all real y,

$$\frac{y}{e^y - 1} \geq 1 - \frac{y}{2} + \frac{5y^2}{60 + y^2}. \tag{4.4}$$

Indeed, with $z = y/2$, this is equivalent to

$$\frac{(z^3 + 15z)\cosh z - (6z^2 + 15)\sinh z}{\sinh z} \geq 0.$$

At $z = 0$ take the continuous extension. The numerator has coefficients $(2m)(2m-2)(2m-4)/(2m+1)! \geq 0$ at z^{2m+1}. For $d\rho = x^2 e^{-x} dx$, $y = x - 2$, and $p = y(16 - y)/930$, completion of the square and $\int_0^\infty x^k e^{-x} dx = k!$ give

$$\int \frac{y^2}{60 + y^2}\, d\rho \geq \int (2yp - (60 + y^2)p^2)\, d\rho = \frac{16}{155}.$$

Thus the trial form is at most $-16/31$.

There cannot be a second nonpositive direction. The function $h_*(t) = e^t(\Omega - a_*)$ has one simple zero $t_* = \log\sqrt{8}$ and $h_*''/h_* = 1 - W$. Ground-state factorization on the two half-lines gives

$$\int (|F'|^2 + (1 - W)|F|^2)\, dt = \int |F' - (h_*'/h_*)F|^2\, dt \geq 0 \quad \text{when } F(t_*) = 0.$$

Prove it first for compact support away from t_* and then use H_0^1 density. Hence $\langle F, \mathcal{H}_n F\rangle \geq (n^2 - 1)\|F\|_2^2$ on a codimension-one subspace. Min–max implies that $\mathcal{H}_2$ has exactly one negative eigenvalue, counting multiplicity, and no kernel. For $n \geq 3$ the bound $W < 9$ directly gives

$$\langle F, \mathcal{H}_n F\rangle \geq (1 - \|W\|_\infty/n^2)(\|F'\|_2^2 + n^2\|F\|_2^2). \tag{4.5}$$

A bounded Hamiltonian realization and its neutral modes. On $Y = L^2((0,\infty), r\,dr)$ set

$$K_n = \sqrt{v}(-\Delta_n)^{-1}\sqrt{v}, \quad B_n = I - K_n, \quad J = \Omega - a_*, \quad \widetilde{T}_n = a_* + JB_n.$$

The kernel of K_n, relative to $s\,ds$, is $\frac{1}{2n}\sqrt{v(r)v(s)}(\min(r,s)/\max(r,s))^n$. Since $\int r v(r) dr < \infty$, it is Hilbert–Schmidt; Green's identity makes it nonnegative and selfadjoint. More explicitly, on the energy space $\mathcal{E}_n$ with norm $\int(|\phi'|^2 + n^2 r^{-2}|\phi|^2) r\,dr$, the compact map $S\phi = \sqrt{v}\phi$ satisfies $SS^* = K_n$. The nonzero eigenvalues of SS^* and S^*S agree with multiplicity. The preceding form computation therefore says that B_2 is invertible with one complex negative direction, whereas B_n is strictly positive for $n \geq 3$. The change $w = \sqrt{v} f$ intertwines $T_n = \Omega - k\Delta_n^{-1}$ with $\widetilde{T}_n$.

If $\widetilde{T}_n f = cf$ and $\psi = S^* f$, then $F(t) = \psi(e^t) \in H^2(\mathbb{R})$ and

$$(\Omega - c)(F'' - n^2 F) - AF = 0, \qquad A = r^2 k = \partial_t^2 \Omega + 2\partial_t \Omega. \tag{4.6}$$

Indeed the Green source $\rho = r^2\sqrt{v} f$ has squared logarithmic L^2 norm $\int W|f|^2 r\,dr$, and $-F'' + n^2 F = \rho$. Also $F = 0$ forces $f = 0$.

The operator $\widetilde{T}_2$ has no real eigenvalues. At $c = a_*$ the equation $JB_2 f = 0$ gives $f = 0$, since J vanishes at one point and B_2 is invertible. At $c = 0$, (4.6) becomes $-F'' + 4(x-1)^2 F = 0$, whose cutoff energy identity gives $F = 0$. For the other real c, put $h_c = e^t(\Omega - c)$; then $h_c''/h_c = 1 + A/(\Omega - c)$ and the equation is $-F'' + (3 + h_c''/h_c)F = 0$. When $0 < c < 1$, $c \neq a_*$, its only critical point t_c has $A(t_c) \neq 0$. The equation and $F'' - 4F \in L^2$ force $F(t_c) = 0$, since otherwise $AF/(\Omega - c)$ has a

nonintegrable simple pole. As $F \in H^2$, $F = O(|t - t_c|)$. For $c < 0$ or $c \geq 1$ there is no finite zero. In every case, factorization on the intervals avoiding t_c gives

$$0 = \int \left(|F' - (h_c'/h_c)F|^2 + 3|F|^2 \right) dt.$$

The critical boundary term is $O(|t - t_c|)$; the terms at infinity vanish because $F, F' \to 0$ and h_c'/h_c is bounded there (its limit at $-\infty$ is three if $c = 1$). Thus $F = 0$.

Regard Y as a real Hilbert space. The operator $\mathcal{A} = -2iJB_2$ has bounded skew-adjoint factor $-2iJ$ and bounded selfadjoint energy B_2. Its negative index is two over the reals, its kernel is zero, and its positive space is uniformly positive since B_2 is an invertible compact perturbation of identity. These verify the hypotheses of the index theorem of Lin and Zeng [38], which gives

$$k_r + 2k_c + 2k_i^{\leq 0} + k_0^{\leq 0} = 2.$$

The last two counts vanish by the preceding absence of imaginary eigenvectors; continuous spectrum creates no additional term in this theorem. The complexification of the real operator is unitarily equivalent to $\mathcal{A} \oplus \overline{\mathcal{A}}$ on $Y \oplus Y$, so its total unstable algebraic multiplicity is twice that of $\mathcal{A}$ on complex Y. Hence it is one on Y. Equivalently, $\widetilde{T}_2$ has exactly one upper-half-plane eigenvalue c, algebraically simple.

Frequency, Gaussian regularity, and the quadrupole. For this nonreal c, substitute $F = (\Omega - c)\Phi$ into (4.6) and integrate against $\bar{\Phi}$:

$$\int (\Omega - c)^2 \Theta \, dt = 0, \qquad \Theta = |\Phi'|^2 + 4|\Phi|^2 - 2\Re(\Phi'\bar{\Phi}) \geq 3|\Phi|^2.$$

Here $\Phi \in H^2$, so integration by parts is valid. The imaginary and real parts imply respectively $\int \Omega\Theta = (\Re c) \int \Theta$ and $\int \Omega^2\Theta = |c|^2 \int \Theta$. Since $0 < \Omega < 1$, $|c - 1/2| < 1/2$. Thus $\lambda = -2ic$ has the asserted growth and frequency.

Write $\phi(r) = F(\log r)$. The coefficient $B = A/(\Omega - c)$ in $F'' - 4F = BF$ is $O(e^{2t})$ at $-\infty$ and decreases faster than every exponential at $+\infty$, with the same differentiated bounds. At $+\infty$ the Volterra equation

$$F_+(t) = e^{-2t} + \frac{1}{2} \int_t^\infty \sinh(2(s - t))B(s)F_+(s) \, ds$$

is a contraction on $\sup_{t \geq T} e^{2t}|F(t)|$ for large T. It gives $F_+ = e^{-2t}(1 + o(1))$ with differentiated asymptotics; reduction of order supplies an independent solution growing as e^{2t}. The analogous forward Volterra construction at $-\infty$ selects $e^{2t}(1 + O(e^{2t}))$. Thus every nonzero H^1 solution has

$$\phi(r) = B_0 r^2(1 + O(r^2)) \ (r \downarrow 0), \qquad \phi(r) = C_\infty r^{-2}(1 + o(1)) \ (r \to \infty), \qquad B_0 C_\infty \neq 0.$$

The Cartesian streamfunction $\Psi = \phi e^{2i\vartheta}$ is in L^2. These endpoint estimates remove the inner-circle boundary terms in Green's identity, so $\Delta\Psi = [k/(\Omega - c)]\Psi$ on the whole plane. The coefficient is smooth with bounded derivatives. The estimate $\|\Psi\|_{H^2} \leq C(\|\Delta\Psi\|_2 + \|\Psi\|_2)$, differentiated and iterated, gives all Sobolev norms and bounded Cartesian derivatives. Hence $q = \Delta\Psi = he^{2i\vartheta}$, with $h = k\phi/(\Omega - c)$, has the claimed Gaussian bounds. Its velocity is $O(r)$ at zero and $O(r^{-3})$ at infinity. Finally

$$M_q = \int_0^\infty r^3 \Delta_2 \phi \, dr = [r^3\phi' - 2r^2\phi]_0^\infty = -4C_\infty \neq 0. \tag{4.7}$$

Its circulation vanishes by angular integration.

Excluding other angular sectors and transferring rootspaces. For $n \geq 3$, $B_n > 0$ and $\widetilde{T}_n^* B_n = B_n \widetilde{T}_n$ imply every phase-speed eigenvalue is real. For $n = 1$, $\phi_1 = r(\Omega - c)$ is an exact solution for every

nonreal c. Its independent solution is $\phi_1(r)\int^r ds/(s\phi_1(s)^2) = -[2(1-c)r]^{-1} + O(r|\log r|)$ at zero, and is not locally square integrable. The regular solution ϕ_1 grows as $-cr$ at infinity, excluding a regular decaying mode. Radial vorticities are stationary. Negative sectors are the conjugates of positive ones.

This classification applies to the actual weighted spaces without claiming a bounded similarity $w \mapsto w/\sqrt{v}$ there. Cauchy–Schwarz in (4.1) gives, for $r \geq 2$,

$$|R_n w(r)| \leq C_{n,M}\|w\|_{L^2_M}\left(r^{1-M} + r^{-n}(1+\log r)^{1/2}\right);$$

at zero it is $O(r)$, with an extra $|\log r|^{1/2}$ only for $n = 1$. For a nonreal rootvector the equation $w = kR_n w/(\Omega - c)$, and for each subsequent Jordan vector the same equation with its preceding vector divided by $\Omega - c$, show inductively that $w/\sqrt{v} \in Y$. Indeed $k/\sqrt{v}$ is Gaussian at infinity and bounded at zero. The Green inverse agrees with $-S^*(w/\sqrt{v})$: their difference is $ar^n + br^{-n}$, and both vanish at the two endpoints. Thus whole Jordan chains transfer injectively to Y. Local elliptic regularity selects precisely the regular solutions used above. In X_G membership in Y is immediate from $v \geq cG$. Conversely the constructed Gaussian eigenmode belongs to all these spaces, so the growing eigenvalue and its algebraic multiplicity agree.

Generators, projections, and complementary bounds. The free transport is $T(t)f(r,\vartheta) = f(r, \vartheta - \Omega(r)t)$. The chain rule gives $\|T(t)\|_{H^j_M \to H^j_M} \leq C_j(1+|t|)^j$. Density and differentiation of the flow give its generator domain $\{f : \Omega\partial_\vartheta f \in H^j_M\}$. For the converse domain inclusion use the distributional identity $T(t)f - f = -\int_0^t T(s)\Omega\partial_\vartheta f\, ds$ and strong continuity. The interaction $(\nabla^\perp\Delta^{-1}f)\cdot\nabla E_0$ is compact on H^j_M: splitting the Biot–Savart kernel at distance one controls its undifferentiated near part in L^2 and far part in L^∞ by $\|f\|_{L^2} + \|f\|_1$; derivatives have Fourier order at most j. Multiplication by the Schwartz coefficient ∇E_0 therefore maps H^j_M into H^{j+1}_M. Truncation to a disk, Rellich compactness, and the vanishing weighted coefficient tails prove compactness on H^j_M. The bounded perturbation theorem now realizes $\mathscr{L}$ on the stated domain. Both terms have zero integral, so $\mathcal{X}_{j,M}$ is invariant.

The Duhamel difference $e^{t\mathscr{L}} - T(t)$ is compact for each $t > 0$: its integrand consists of the compact interaction between strongly continuous evolutions, and is norm continuous because the adjoint evolutions are strongly continuous on this Hilbert space. Thus the essential growth bound is at most zero. The standard semigroup spectral decomposition outside the essential growth bound [18, Corollary IV.2.11 and Theorem V.3.1] makes all growing spectrum discrete of finite algebraic multiplicity. Each such finite-dimensional rootspace is rotation invariant, so the circle representation splits into integer angular characters. The preceding transfer and exclusions leave precisely $\lambda, \bar\lambda$, each simple. The same theorem gives the first estimate in (4.3). This statement includes the full H^j_M: splitting off a fixed radial function of integral one merely adds a stationary direction. Resolvents, hence Riesz projections, agree on intersections of weighted spaces by uniqueness in a common weaker realization.

On X_G, the Green estimate $|Rf(r)| \leq C\|f\|_{X_G}\min(r, r^{-2})$ shows that L_0 is bounded. After the unitary map $f \mapsto f/\sqrt{G}$, its interaction kernel is

$$-\frac{i}{2}\frac{k(r)}{\sqrt{G(r)}}\left(\frac{\min(r,s)}{\max(r,s)}\right)^2\sqrt{G(s)},$$

which is Hilbert–Schmidt. Hence the same compact-perturbation and spectral-decomposition argument gives the second estimate in (4.3).

Actual adjoint, strain coupling, and diffusion domains. Unweighted selfadjointness of R and $k = \Omega(x-2)$ give

$$L_0^* g = 2i\Omega[g - R((x-2)g)].$$

The explicit adjoint eigenvector is

$$h_X(r,\vartheta) = G(r)\frac{\overline{\phi(r)}}{\Omega(r) - \bar c}e^{2i\vartheta}, \qquad L_0^* h_X = \bar\lambda h_X.$$

Indeed the conjugate Rayleigh equation gives $R(k\bar{\phi}/(\Omega-\bar{c}))=\bar{\phi}$, with no homogeneous ambiguity by its endpoints. Every derivative of h_X is a polynomial times G. The Fredholm operator $L_0-\lambda$ has index zero, one-dimensional kernel, and adjoint kernel spanned by h_X. Its pairing with q cannot vanish, since otherwise q would lie in the range and produce a Jordan vector. This pairing is exactly J_q in (4.2). Setting $q^\dagger = h_X/\overline{J_q}$ therefore gives that formula and $\ell(H)=M_q/J_q\neq 0$. The kernel $h/k=\phi/(\Omega-c)$ is smooth at zero and $O(r^{-2})$ at infinity, so the same functional is bounded on each polynomially weighted space. Its adjoint identity extends from smooth data by density and gives the corresponding Riesz projection.

Finally, conjugation by $U_G f = f/\sqrt{G}$ gives

$$U_G D U_G^{-1} = \Delta - |y|^2/16 + 3/2.$$

The selfadjoint harmonic oscillator has Hermite levels $1-N/2$; angular weight two consists exactly of $N=2+2m$, $m\geq 0$. This proves the domain and spectral assertion about D. The Gaussian derivative bounds make $U_G q$ and $U_G h_X$ Schwartz, so both belong to every oscillator power domain. With $\ell(f)=\langle f, q^\dagger\rangle$, selfadjointness gives $\ell(Df)=\langle f, Dq^\dagger\rangle$. Consequently $PD = q\langle\,\cdot\,, Dq^\dagger\rangle$ and $DP=(Dq)\ell$ are bounded. □

Corollary 4.2 (Frozen heat profiles)**.** *Set*

$$J_a f(y) = a^{-2} f(y/\sqrt{a}), \qquad E_a = J_a E_0, \qquad V_a(y) = a^{-3/2} V_0(y/\sqrt{a}).$$

Their Euler generators and unstable projections satisfy

$$\mathscr{L}_a = a^{-2} J_a \mathscr{L} J_a^{-1}, \qquad \mathscr{P}_a = J_a \mathscr{P} J_a^{-1}.$$

Thus the unstable eigenvalues are $a^{-2}\lambda, a^{-2}\bar{\lambda}$; the complementary estimates are uniform for $1\leq a\leq A<\infty$, and the projections depend continuously on a in operator norm.

Proof. The generator identity is the Biot–Savart scaling identity. J_a, J_a^{-1} are uniformly bounded on H^j_M on each compact a interval, so conjugating (4.3) proves the bounds. The finite-rank projection consists of the Gaussian eigenvectors and the dual functional (4.2); their dilates are norm continuous by dominated convergence. □

4.2 A compact-perturbation lemma and the canonical heat branch

We use the Gaussian angular-two Hilbert space X_G, its Euler operator $L=T+A$ with $T=-2i\Omega$ and compact $A=2ik\Delta_2^{-1}$, the simple eigenpair (λ, q), and $P=q\ell$, $Q=I-P$ established above. Write $\gamma=\Re\lambda>0$ and

$$D=\Delta+\tfrac{1}{2}y\cdot\nabla+2, \qquad \kappa=\ell(Dq).$$

On X_G, D is selfadjoint and nonpositive; $q, q^\dagger$ belong to every power domain of D. In particular DP and PD extend boundedly. These statements concern the actual Gaussian adjoint, not an unweighted formal adjoint.

Lemma 4.3 (Compact Duhamel comparison)**.** *Fix a finite time length B. Suppose uniformly bounded free evolutions F_n converge strongly to F_0, and their adjoints converge strongly, uniformly in both times and any auxiliary parameters. Suppose the bounded strongly measurable interactions $A_n(t)$ converge uniformly in operator norm to $A_0(t)$, and all the latter form a norm-precompact family of compact operators. The full evolutions U_n and their adjoints converge strongly, and*

$$\sup_{0\leq t-s\leq B} \|(U_n-F_n)(t,s)-(U_0-F_0)(t,s)\| \longrightarrow 0.$$

Proof. The Dyson series have a uniform exponential majorant, so termwise convergence proves the two strong assertions. For finite-rank $K = \sum_{j=1}^{m} u_j\langle\,\cdot\,, v_j\rangle$, strong convergence on u_j and adjoint-strong convergence on v_j imply norm convergence of $F_n(t,r)KU_n(r,s)$. A finite norm-net and finite-rank approximation of the compact family $A_0(r)$ make the conclusion uniform. Integrate

$$(U_n - F_n)(t,s) = \int_s^t F_n(t,r)A_n(r)U_n(r,s)\,dr.$$

Uniform bounds absorb $A_n - A_0$. This proves norm convergence only for the compact remainder, as required. □

Lemma 4.4 (Small diffusion on the complementary space)**.** *There are $\beta_*, g, C > 0$ such that, for every measurable $0 \le \beta(t) \le \beta_*$, the forward evolution on QX_G generated by*

$$Q\{L - \lambda + \beta(t)(D - \kappa)\}Q$$

has norm at most $Ce^{-g(t-s)}$ for $t \ge s$.

Proof. Extend the generator to $G_\beta = L - \lambda - P + \beta Q(D-\kappa)Q$ on X_G. Since

$$Q(D-\kappa)Q = D - \kappa + R_f, \quad R_f = -DP - PD + PDP + \kappa P,$$

R_f is bounded and finite rank. Split G_β into the free generator $T - \lambda - \beta\kappa + \beta D$ and interaction $A - P + \beta R_f$. The evolution exists for measurable β by the Dyson series based on $\exp(D\int_s^t \beta)$; its short-interval bounds are uniform. Dissipativity of D and skew-adjointness of T give

$$\|F_\beta(t,s)\| \le \exp\{-\gamma(t-s) + |\Re\kappa|\beta_*(t-s)\}.$$

This inequality can first be proved for bounded nonpositive Yosida approximants of D and then passed to the strong limit.

On a fixed block, $\exp(D\int_s^t \beta) \to I$ strongly as $\beta_* \downarrow 0$, uniformly over all admissible coefficients. The Dyson series give the corresponding direct and adjoint strong limits for the free and full evolutions. For the adjoints, reverse time on the block; the class of admissible coefficients is unchanged. Lemma 4.3 therefore gives

$$U_\beta(s+B,s) - F_\beta(s+B,s) \longrightarrow e^{B(L-\lambda-P)} - e^{B(T-\lambda)}$$

in norm uniformly in s and β. The inviscid complementary semigroup estimate, with any rate below γ, makes $L - \lambda - P$ exponentially stable. Consequently

$$\limsup_{\beta_*\downarrow 0}\sup_{s,\beta}\|U_\beta(s+B,s)\| \le \|e^{B(L-\lambda-P)}\| + 2e^{-\gamma B} < \tfrac{1}{4}$$

when B is sufficiently large. Fixing β_* then makes every full block contract by $1/2$; iteration and the short-block bound prove exponential decay. Finally G_β commutes with P and equals -1 on its range. In the mild equation this follows by testing against $q^\dagger \in \mathcal{D}(D)$ and uniqueness from the datum q. Thus QX_G is invariant and restriction proves the claim. □

The radial Gaussian evolves exactly by heat diffusion:

$$a = 1 + \nu s, \quad E(s,x) = a^{-2}E_0(x/\sqrt{a}), \quad V(s,x) = a^{-3/2}V_0(x/\sqrt{a}).$$

It is also an unforced planar Navier–Stokes solution, since its vorticity is radial and its velocity tangential. Substitution of $w(s,x) = a^{-2}W(z, x/\sqrt{a})$ gives the exact linearized equation

$$W_z = (L + \theta^{-1}D)W, \qquad z = \frac{s}{1+\nu s} = \nu^{-1} - \theta, \qquad \theta = \frac{1}{\nu(1+\nu s)}. \tag{4.8}$$

Here $dz/ds = a^{-2}$; the Biot–Savart velocity scales as $a^{-3/2}$.

Proposition 4.5 (Canonical branch)**.** *There is a unique solution of*

$$-V'_{\mathrm{can}} = (L + \theta^{-1}D)V_{\mathrm{can}}, \qquad \theta > 0,$$

with

$$V_{\mathrm{can}}(\theta) = e^{-\lambda\theta}\theta^{-\kappa}\{q + O_{X_G}(\theta^{-1})\} \quad (\theta \to \infty). \tag{4.9}$$

It is bounded in X_G as $\theta \downarrow 0$ and smooth for positive θ. For $X_{G,j} = \mathcal{D}((I-D)^{j/2})$, fixed j, and $\theta \geq b$ with a fixed sufficiently large b,

$$\|e^{\lambda\theta}\theta^{\kappa}V_{\mathrm{can}}(\theta)\|_{X_{G,j}} \leq C_j(1+\theta)^{j/2}.$$

Proof. Write $e^{\lambda\theta}\theta^{\kappa}V = Aq + r$, $r \in QX_G$. The equations are

$$-A' = \theta^{-1}\ell(Dr), \qquad -r' = Q\{L - \lambda + \theta^{-1}(D-\kappa)\}Qr + \theta^{-1}A\,QDq.$$

For $b \geq \beta_*^{-1}$ let $S(\theta, s)$, $b \leq \theta \leq s$, be the forward complementary evolution. It satisfies $\|S(\theta,s)\| \leq Ce^{-g(s-\theta)}$. The normalization at infinity is imposed by

$$r_A(\theta) = \int_\theta^\infty S(\theta,s)QDq\,\frac{A(s)}{s}\,ds, \qquad (\mathcal{T}A)(\theta) = 1 + \int_\theta^\infty \frac{\ell(Dr_A(s))}{s}\,ds.$$

Because ℓD is bounded, $\|r_A(\theta)\| + |\mathcal{T}A(\theta) - 1| \leq C\theta^{-1}\|A\|_\infty$. The same estimate for differences makes $\mathcal{T}$ a contraction on $C_b([b,\infty))$ after enlarging b. Its fixed point gives (4.9). Splitting both integrals at any finite s_0 verifies the coupled mild initial-value equations, whose uniqueness follows from the bounded coupling maps and Gronwall. Conversely every normalized solution satisfies the integrals: the term $S(\theta, s_0)r(s_0)$ tends to zero as $s_0 \to \infty$. This also proves uniqueness at infinity.

Set $\tau = \log(b/\theta)$. Then

$$\partial_\tau V = DV + be^{-\tau}LV.$$

The bounded-perturbation formula and $D \leq 0$ give continuation for all $\tau \geq 0$ with $\sup_{0<\theta\leq b}\|V(\theta)\| \leq e^{b\|L\|}\|V(b)\|$. This is forward parabolic continuation; no backward heat inverse occurs.

For completeness, conjugation by $G^{1/2}$ turns D into a shifted harmonic oscillator. Its Hermite decomposition shows that the squared $X_{G,j}$ norm is equivalent to $\sum_{|\alpha|\leq j}\|\partial^\alpha f\|^2_{L^2(G^{-1})}$ (Cartesian derivatives may leave angular sector two). Multiplication by Ω is bounded on these spaces. The radial Green formula for Δ_2^{-1} first gives $\|\Delta_2^{-1}f\|_2 \leq C\|f\|_{X_G}$; elliptic regularity and boundedness of $G^{-1/2}\partial^\alpha k$ then show $L : X_{G,j} \to X_{G,j}$ is bounded. Spectral calculus gives a one-derivative heat gain $\|e^{aD}\|_{X_{G,j}\to X_{G,j+1}} \leq C_j a^{-1/2}$ for $0 < a \leq 1$. On the unit turnover interval from $\theta+1$ down to θ, diffusion is comparable to $(1+\theta)^{-1}$. Duhamel therefore gains one derivative at elapsed time h at cost $C_j((1+\theta)/h)^{1/2}$; its integral kernel is integrable. Divide the interval into j equal pieces to obtain the stated $C_j(1+\theta)^{j/2}$ bound from the bounded X_G norm at its inlet. The τ equation propagates every fixed graph norm to zero because $\int_0^\infty be^{-\tau}\,d\tau = b$. □

Corollary 4.6 (Nontrivial full-age output)**.** *Let $\mathfrak{M}(f) = \int_0^\infty r^3 f_2(r)\,dr$ and $M_q = \mathfrak{M}(q) \neq 0$. Then*

$$\mathcal{J}(\mu) = \int_0^\infty \theta^{\mu+1/2}\mathfrak{M}(V_{\mathrm{can}}(\theta))\,d\theta$$

is holomorphic and not identically zero on $\Re\mu > -3/2$.

Proof. Weighted Cauchy–Schwarz bounds $\mathfrak{M}$ on X_G. Thus its canonical value is $O(1)$ at zero and $e^{-\lambda\theta}\theta^{-\kappa}(M_q + O(\theta^{-1}))$ at infinity. These bounds prove absolute convergence and permit differentiation under the integral, including every logarithmic factor on compact subsets of the half-plane. If $\mathcal{J}$ vanished identically, then for fixed real $b > -3/2$ the $L^1(\mathbb{R})$ function $e^{(b+3/2)x}\mathfrak{M}(V_{\mathrm{can}}(e^x))$ would have identically zero Fourier transform. Fourier uniqueness would contradict $M_q \neq 0$. □

5 The logarithmic ring lift and its residual

The planar shift becomes a physical scale change under the ring chart. For $q = R\log(r/R)$ and $\Lambda = e^{c/R}$, the map $(r, z, t) \mapsto (\Lambda r, \Lambda z, \Lambda^2 t)$ sends q to $q + c$ and leaves $\eta = Rz/r$ and $s = R^2 t/r^2$ fixed. Thus the raw lift obeys

$$u_{\mathrm{raw}}(\Lambda x, \Lambda^2 t) = \Lambda^{-1} u_{\mathrm{raw}}(x, t).$$

This is exact DSS symmetry; the Navier–Stokes equation still has a small curvature residual. We estimate it and construct an exact unforced background U with its DSS datum and common H^2 heat remainder. The physical free-operator estimates proved here also serve Sections 8–18.

Let $v = \nabla^\perp \psi$ and p be the exact periodic planar train and its bounded pressure constructed in Section 3. The following stream construction converts its period into discrete scaling.

Proposition 5.1 (Exact lift, residual and trace)**.** *Let $R \geq 2$, $\epsilon = R^{-1}$, and set*

$$q = R\log(r/R), \quad \eta = Rz/r, \quad s = R^2 t/r^2, \quad K = \eta\partial_\eta + 2s\partial_s, \quad H = (1 - K)\psi.$$

The physical stream $r\psi(q, \eta, s)$ defines the exactly divergence-free axisymmetric field without swirl

$$u_r = \frac{R}{r} v_1, \qquad u_z = \frac{R}{r}(v_2 + \epsilon H), \qquad p_{\mathrm{raw}} = (\frac{R}{r})^2 p.$$

It is smooth through the axis for positive time and discretely self-similar with ratio $e^{c/R}$. Its similarity field U_R has period $2c/R$, globally bounded fixed spatial derivatives, and residual

$$\|\mathcal{F}_R\|_{C_{\mathrm{per}} W^{1,p}(\mathbb{R}^3)} \leq P_p \epsilon^{1-2/p}, \qquad 2 \leq p < \infty. \tag{5.1}$$

This includes the projected residual. In particular $p = 11/4$ gives $P\epsilon^{3/11}$. Its nonzero initial datum a_{raw} is divergence-free, has the same discrete scaling, and satisfies

$$|D^j a_{\mathrm{raw}}(x)| \leq C_j |x|^{-1-j}. \tag{5.2}$$

Proof. Put $D_\epsilon = \partial_q - \epsilon K$. The differentiation rules are $\partial_r = (R/r)D_\epsilon$, $\partial_z = (R/r)\partial_\eta$, $\partial_t = (R/r)^2\partial_s$. Differentiating the stream proves the divergence-free condition, including its geometric correction ϵH. After dividing the physical residual by $(R/r)^3$, direct expansion of the cylindrical equations gives

$$r_r = \epsilon A_1 + \epsilon^2 B_1, \qquad r_z = \epsilon A_2 + \epsilon^2 B_2 + \epsilon^3 B_3,$$

where the complete coefficients are

$$\begin{aligned}
A_1 &= -v_1 K v_1 + H(v_1)_\eta - v_1^2 + 2\nu(K+1)(v_1)_q - (K+2)p, \\
A_2 &= (\partial_s + v\cdot\nabla - \nu\Delta)H + H(v_2)_\eta - v_1(K+1)v_2 + 2\nu(K+1)(v_2)_q, \\
B_1 &= -\nu K(K+2)v_1, \\
B_2 &= -v_1(K+1)H + HH_\eta + 2\nu(K+1)H_q - \nu(K+1)^2 v_2, \\
B_3 &= -\nu(K+1)^2 H.
\end{aligned}$$

For verification, the normalized diffusion operators on the radial and axial components respectively are $D_\epsilon^2 - 2\epsilon D_\epsilon + \partial_\eta^2$ and $D_\epsilon^2 - 2\epsilon D_\epsilon + \epsilon^2 + \partial_\eta^2$; the radial pressure operator is $D_\epsilon - 2\epsilon$. Thus the formulas include all age derivatives and are identities at every age, with no truncation.

The heat identity $K\partial_\eta^j G_{\nu s} = -(j+1)\partial_\eta^j G_{\nu s}$ and Lemma 3.2 give the following complete late residual table. Each entry denotes its $L^\infty_\eta L^\infty_q$ bound with normalized weight $\langle \eta/\sqrt{s}\rangle^m$; an η derivative gains $s^{-1/2}$ and an L^p_η norm costs $s^{1/(2p)}$:

coefficient	size	$(K+3)$ size
A_1, B_1	Ps^{-2}	Ps^{-2}
B_2, B_3	$Ps^{-3/2}$	$Ps^{-5/2}$
A_2	$Pe^{-\kappa\nu s/c^2}$	$Pe^{-\kappa'\nu s/c^2}$.

Every q derivative is exponentially small. Indeed the leading means of B_2, B_3 are $4a_0^2 G_{\nu s}\partial_\eta G_{\nu s}$ and $-16\nu a_2 \partial_\eta^2 G_{\nu s}$; both have K degree -3. Also $(\partial_s - \nu\partial_\eta^2)(S - KS) = -(K+1)g$, and every remaining term of A_2 has a fluctuation factor.

In similarity variables $\varrho = r/\sqrt{t}$, $z_y = z/\sqrt{t}$,

$$\varrho = R/\sqrt{s}, \quad z_y = \eta/\sqrt{s}, \quad q = R\tau/2 - (R/2)\log s, \quad dy = \pi R^2 s^{-5/2}\, ds\, d\eta,$$

and $U_R = \sqrt{s}(v_1 e_r + (v_2 + \epsilon H)e_z)$, $\mathcal{F}_R = s^{3/2} r$. Its derivatives in ϱ and z_y are

$$\partial_{z_y}\mathcal{F}_R = s^2\partial_\eta r, \qquad \partial_\varrho \mathcal{F}_R = s^2[\partial_q - \epsilon(K+3)]r;$$

the angular frame derivative costs only $\epsilon s^2 |r_r|$. For the field norm the late integrands are $s^{-p/2-2}$ and s^{-2}; for its radial derivative they are s^{-2} and $s^{-p/2-2}$. The axial and angular derivatives are integrable by the same table. At zero age the integrands are bounded by $Ps^{3p/2-5/2}$ or $Ps^{2p-5/2}$, integrable for $p > 1$. The Jacobian contributes $R^{2/p}$ and every residual term contributes at least ϵ. This proves (5.1), including continuity by dominated convergence. The Leray projection is bounded on $W^{1,p}$ by the Euclidean Riesz-transform theorem.

Smoothness at the axis follows term by term from the full heat expansion:

$$r a_{2j}\partial_\eta^{2j} G_{\nu s} = \frac{a_{2j} r^{2j+2}}{R^{2j+1}} \partial_z^{2j} G_{\nu t}(z).$$

After a radial derivatives the remainder after order N is $O(r^{2N+4-a})$ on compact positive-time sets; fluctuations are flat because $e^{-\kappa\nu R^2 t/(c^2 r^2)}$ dominates all radial derivative costs. Thus u_r/r and u_z are smooth functions of (r^2, z, t). The same argument with the finite needed expansion order proves $\|U_R\|_{W^{k,\infty}} \le P_k$, uniformly for $R \ge 2$. For example the leading mean is $(-\varrho R^{-1}\partial_{z_y}(a_0 G_\nu), 2R^{-1} a_0 G_\nu)$; its Cartesian second derivatives are regular at the axis. Shifting q by c proves the stated DSS period.

At $t = 0$ use the same stream formula with the Gaussian inlet. Each derivative costs C/r; all powers of $|x|/r$ are absorbed by the Gaussian decay in $\eta = Rz/r$. This proves the datum bound away from the origin, including smoothness at nonzero axis points. Flux across a sphere of radius h is $O(h)$, proving that the divergence is zero in distributions. The datum is nonzero on the central ring core. □

6 The common free equation and local comparison

6.1 The free Cartesian equation

Put $\mathcal{T} = \partial_\tau - y\cdot\nabla/2$, $L = \mathcal{T} - \nu\Delta - 1/2$, and let Y_r denote $L^r((\mathbb{R}/P\mathbb{Z})\times\mathbb{R}^3)$, with normalized phase measure. Its maximal distributional graph is

$$D_r = \{w \in Y_r : \nu\Delta w, \mathcal{T} w \in Y_r\}, \qquad \|w\|_{D_r} = \|w\|_{Y_r} + \|\nu\Delta w\|_{Y_r} + \|\mathcal{T} w\|_{Y_r}.$$

The derivatives in this definition are Cartesian; there is no axis condition. Write $\gamma_r = (3/r - 1)/2$, $a_r = \operatorname{Re}\mu + \gamma_r$.

Lemma 6.1. *For $1 < r < 3$, $a_r > 0$, $\nu > 0$, and $0 < P \le 1$, the componentwise operator $\mu + L : D_r \to Y_r$ is bijective and*

$$\|w\|_{D_r} + \sqrt{\nu}\|\nabla w\|_{Y_r} \le C_r[1 + (1 + |\mu|)/a_r]\|f\|_{Y_r}, \qquad (\mu + L)w = f. \tag{6.1}$$

Its projected inverse is obtained by applying Cartesian Leray projection to f. At $r = 2$ the adjoint has the same maximal graph and the same bound. All these estimates are independent of P.

Proof. The heat-dilation formula gives a distributional inverse

$$S(t)h(y) = e^{t/2}[e^{\nu(e^t-1)\Delta}h](e^{t/2}y), \qquad \mathcal{R}_\mu f(\tau) = \int_0^\infty e^{-\mu t}S(t)f(\tau - t)\,dt.$$

Its integrand has Y_r norm at most $e^{-a_r t}\|f\|_{Y_r}$; hence $\|\mathcal{R}_\mu\| \le a_r^{-1}$. The local heat estimate needed here is

$$\|\nu D^2 u\|_r + \|u_t\|_r \le C_{d,r}\|F\|_r, \qquad u_t - \nu\Delta u = F, \tag{6.2}$$

for compactly time-supported distributional solutions $u, F \in L^r(\mathbb{R} \times \mathbb{R}^d)$. To justify its precise forcing class, scale to $\nu = 1$. The multiplier $-\xi_i\xi_j/(i\omega + |\xi|^2)$ has L^2 norm at most one. Its kernel away from the origin is $K(t, x) = 1_{t>0}\partial_{ij}H_t(x)$. For $\rho(t, x) = \max\{|x|, \sqrt{|t|}\}$, Gaussian differentiation gives

$$|K| \le C\rho^{-d-2}, \quad |\nabla_x K| \le C\rho^{-d-3}, \quad |\partial_t K| \le C\rho^{-d-4}.$$

The kernel extends smoothly by zero through $t = 0$, away from $x = 0$. Integrating these derivatives in parabolic annuli gives

$$\int_{\rho(z)>4\rho(h)} |K(z - h) - K(z)|\,dz \le C_d.$$

The spatial and temporal differences are bounded, respectively, by

$$C|h_x| \int_{4\rho(h)}^\infty R^{-2}dR, \qquad C|h_t| \int_{4\rho(h)}^\infty R^{-3}dR,$$

which are uniformly bounded. For completeness, decompose an $L^1 \cap L^2$ function at height α using maximal dyadic rectangles with spatial side 2^k and time side 2^{2k}. The good part has $\|g\|_2^2 \le C_d\alpha\|f\|_1$. The bad parts b_Q have zero integral, total L^1 norm at most $2\|f\|_1$, and their rectangles have total measure at most $\alpha^{-1}\|f\|_1$. Outside fixed enlargements of these rectangles, cancellation and the displayed kernel integral bound the L^1 norm of each Tb_Q by $C_d\|b_Q\|_1$. The L^2 estimate for the good part and Chebyshev's inequality for the bad part prove weak type $(1, 1)$. Integrating this distribution bound for the portions $|f| > \alpha$ and $|f| \le \alpha$, together with the L^2 bound, gives L^r boundedness for $1 < r < 2$. Duality gives the remaining exponents since the reflected kernel has the same bounds. Mollify the compact-time equation. The mollified solution vanishes in the distant past, so its causal heat formula identifies its second derivatives with these bounded operators applied to the mollified F. Their L^r limit equals the distributional derivatives of u; the equation gives u_t. This proves (6.2), with no viscosity loss.

On $|t| < 2$, set $v(t, x) = e^{-t/2}w(\tau_0 + t, e^{-t/2}x)$. Then $v_t - \nu e^t\Delta v$ is the same pullback of $f - \mu w$. A fixed cutoff equal to one on $|t| < 1$, followed by $\theta = e^t$, reduces this equation to the preceding heat estimate. All Jacobians on the window are bounded. Consequently

$$\int_{\tau_0-1}^{\tau_0+1} \|\nu\Delta w\|_r^r\,d\tau \le C_r \int_{\tau_0-2}^{\tau_0+2} [\|f\|_r^r + (1 + |\mu|)^r\|w\|_r^r]\,d\tau.$$

For periodic h, $\int_0^P \int_I h(\tau_0 + t)\,dt\,d\tau_0 = |I| \int_0^P h$. Averaging the last inequality over the centers proves the diffusion bound with no inverse-period factor. The equation gives the drift bound; spatial elliptic interpolation gives the gradient bound. The same reasoning applies to any distributional Y_r solution.

The adjoint semigroup is $S(t)^*h = e^{-t}[e^{\nu(1-e^{-t})\Delta}h](e^{-t/2}y)$. For a smooth compact spatial test, its adjoint resolvent has itself and its first two spatial derivatives in $Y_{r'}$, by the gap and the extra factor $e^{-t/2}$ per derivative. Test a homogeneous solution against this resolvent times a radial cutoff at radius N. The drift cutoff is bounded, and the diffusion cutoffs cost N^{-1}, N^{-2}; all terms are bounded by the Y_r tail of the solution times fixed $Y_{r'}$ norms and tend to zero. Thus the solution vanishes. The inverse therefore has precisely the displayed maximal distributional domain. To exhibit a core directly, approximate the right side by smooth periodic Schwartz functions and truncate the inverse integral at M. Such a truncated integral is Schwartz and its equation has right side

$$f - e^{-\mu M} S(M) f(\tau - M).$$

The remainder tends to zero in Y_r; the graph estimate gives graph convergence of the truncations. Thus the smooth periodic Schwartz class is a core. Radial cutoffs approximate it in this graph, since their drift commutator is bounded on an escaping annulus and their diffusion commutators tend to zero. At $r = 2$, reverse the phase window and use $\theta = 1 - e^{-t}$ for $L^* = -\mathcal{T} - \nu\Delta + 1$. Testing against the compact core identifies the distributional equation of every vector in the actual adjoint domain; this reversed-window estimate places it in D_2. Conversely, core integration by parts puts every element of D_2 in the adjoint domain. Its resolvent norm is at most a_2^{-1} by duality, so the same graph bound follows. Leray commutes with heat, dilation and phase translation, so projection preserves all identities. Restriction to the real axisymmetric no-swirl subspace is also legitimate since its symmetry projections commute with these maps. On the Hilbert core, direct integration by parts gives

$$\mathrm{Re}\langle(\mu + L)w, w\rangle = (\mathrm{Re}\,\mu + 1/4)\|w\|_2^2 + \nu\|\nabla w\|_2^2.$$

The adjoint has the same real part. For a tensor source $\mathbb{P}\,\mathrm{div}\,T$, its right side is at most $\|T\|_2\|\nabla w\|_2$. Completing the square gives $\|\nabla w\|_2 \le \nu^{-1}\|T\|_2$ and $\|w\|_2 \le [2\sqrt{\nu(\mathrm{Re}\,\mu + 1/4)}]^{-1}\|T\|_2$. Approximation of tensors completes this weak-stress map and the inequality without assigning a full graph to its rough outputs. □

Lemma 6.2. *Every $w \in D_r$ has a representative in $C(\mathbb{R}/P\mathbb{Z}; L^r)$, and*

$$\sup_\tau \|w(\tau)\|_r \le (1 + 3P/(2r))\|w\|_{Y_r} + P\|\mathcal{T}w\|_{Y_r}. \tag{6.3}$$

For $r > 5/2$,

$$\|w\|_{L^\infty_{\tau,y}} \le C_r \nu^{-3/(2r)} \|w\|_{D_r}. \tag{6.4}$$

Proof. On a fixed phase interval the distributional chain rule gives $V(t,x) = w(\tau_0 + t, e^{-t/2}x) \in W^{1,r}_t L^r_x$, with V_t the pullback of $\mathcal{T}w$. Undoing the strongly continuous dilation gives the continuous phase representative; distributional periodicity matches its endpoints. For $f(\tau) = \|w(\tau)\|_r$, the norm chain rule gives $|f' + 3f/(2r)| \le \|\mathcal{T}w(\tau)\|_r$. Start at a phase with $f \le \|w\|_{Y_r}$, integrate for at most one period, and apply Hölder to obtain (6.3).

For the supremum fix (τ_0, y_0) and put $W(t,z) = w(\tau_0 + t, e^{-t/2}(y_0 + \sqrt{\nu}z))$ on $|t| < 1/2$. Then $W_t = (\mathcal{T}w)\circ$, $\Delta_z W = \nu e^{-t}(\Delta_y w)\circ$, and $dz = \nu^{-3/2}e^{3t/2}dy$. The integral of a nonnegative P-periodic function over a unit interval is at most $(1 + P)$ times its phase average. Thus

$$\|W\|_{L^r} + \|W_t\|_{L^r} + \|\Delta_z W\|_{L^r} \le C_r \nu^{-3/(2r)} \|w\|_{D_r}.$$

Choose a fixed time cutoff $\chi = 1$ near zero. The right side $F = (\partial_t - \Delta_z + 1)(\chi W)$, after zero extension, has this same bound. Its causal inverse kernel $k(t,z) = 1_{t>0}e^{-t}H_t(z)$ satisfies

$$\|k\|^{r'}_{L^{r'}} = C_r \int_0^\infty e^{-r't} t^{-3(r'-1)/2}\, dt < \infty \quad \Longleftrightarrow \quad r > 5/2.$$

The identity $\chi W = k * F$ follows distributionally; uniqueness follows from the nonvanishing symbol $i\omega + |\xi|^2 + 1$. Hölder bounds its value at $(0,0)$, proving (6.4). Translation continuity of k in $L^{r'}$ supplies a continuous local representative. No external Sobolev embedding or inverse-period factor is used. □

6.2 Tensor forces

Write $R_\mu = (\mu + L)^{-1}\mathbb{P}$. For tensor inputs its divergence is interpreted by completion from smooth tensors, rather than as an ordinary Y_2 body force. Let

$$B = \mathcal{T} - \tfrac{3}{4}, \qquad A = -\nu\Delta.$$

The group $u(\tau, y) \mapsto e^{-3t/4}u(\tau+t, e^{-t/2}y)$ is unitary on Y_2 and has generator B; hence B is skew-adjoint. The operator A is nonnegative self-adjoint.

Lemma 6.3. *For $a = \operatorname{Re}\mu + 1/4 > 0$, the map $S_\mu = R_\mu \operatorname{div}$ extends uniquely to $Y_2(\mathrm{Sym}_3)$, with*

$$\|S_\mu T\|_{Y_2} \le \frac{\|T\|_{Y_2}}{2\sqrt{a\nu}}, \qquad \|\nabla S_\mu T\|_{Y_2} \le \nu^{-1}\|T\|_{Y_2}. \tag{6.5}$$

Uniformly on compact subsets of this half-plane,

$$\|(1 + |B|)^{1/2} S_\mu T\|_{Y_2} \le C_\mu \nu^{-1/2}\|T\|_{Y_2}. \tag{6.6}$$

The weak solution is unique among distributional Y_2 solutions. Bounded measurable tensor partitions are allowed before divergence. The body and tensor inverses depend holomorphically on μ in their respective half-planes.

Proof. The energy identity and completion proved above give (6.5). Differences of smooth tensors give strong convergence of both velocities and first gradients, so the equation and energy identity pass to the limit. Distributional uniqueness follows by subtracting two solutions and applying the homogeneous uniqueness argument in Lemma 6.1. In particular a bounded measurable multiplier may be applied to a tensor before this completion; the distributional divergence includes all interface terms.

Put $C = 1 + |B|$, $D = 1 + A$, and let $\mathcal{R}_\mu$ be the inverse of $\mu + L$, acting componentwise. Its direct graph bounds $C\mathcal{R}_\mu$; the actual adjoint graph bounds $D\mathcal{R}_\mu^*$, and duality bounds the extension of $\mathcal{R}_\mu D$. The imaginary powers of C, D are unitary. Insert the respective spectral cutoffs on the two sides of

$$C^z \mathcal{R}_\mu D^{1-z}, \qquad 0 \le \operatorname{Re} z \le 1.$$

At the two boundary lines the preceding graph estimates give uniform bounds. The scalar three-lines theorem applied to pairings, followed by density and closedness, yields $\|C^{1/2}\mathcal{R}_\mu D^{1/2}\| \le C_\mu$. No commutation between A and B is asserted. Spatial Fourier transformation gives $\operatorname{div} = \nu^{-1/2}A^{1/2}\mathcal{Q}$, where $\mathcal{Q}$ is a bounded tensor-to-vector combination of Riesz transforms. Since $D^{-1/2}A^{1/2}$ is bounded and Leray commutes with A, B, the last estimate proves (6.6).

The inverse of $\mu + L$ is holomorphic by its convergent semigroup integral. For the stress inverse, distributional uniqueness gives

$$S_\mu - S_\zeta = (\zeta - \mu)R_\mu S_\zeta.$$

The body and stress bounds make this the resolvent identity in operator norm, proving the asserted holomorphy. □

At $r = 11/4$ the body gap at $\mu = 0$ is $1/22$, and at $r = 23/8$ it is $1/46$. In particular all shifts in the background and spectral constructions lie in the stipulated ranges. The graph embeddings also give, at $r = 11/4$,

$$\|\mathbb{P}[(u \cdot \nabla)v]\|_{Y_r} \le C\nu^{-23/22}\|u\|_{D_r}\|v\|_{D_r}.$$

For any unitary velocity and tensor lifts $J_v : H \to Y_2$ and $J_T : H_T \to Y_2$, the transported stress map is exactly $J_v^{-1}S_\mu J_T : H_T \to H$, with the same energy bounds. This formula transports the completed tensor inverse; it does not assign an ordinary body-force norm to $\operatorname{div} J_T T$.

6.3 Free comparison with global tensor inputs

Let $R = \epsilon^{-1}$, $P = 2\epsilon c \le 1$, and use

$$s = R^2/\rho^2, \qquad q = R\log(\rho/R) + R\tau/2, \qquad \eta = Rz/\rho.$$

The velocity and tensor measures are $H = L^2(s^{-3/2}ds\,dq/c\,d\eta)$ and $H_T = L^2(s^{-1/2}ds\,dq/c\,d\eta;\mathrm{Sym}_3)$. The natural lifts $b \mapsto \sqrt{s}\,b$ and $T \mapsto sT$ both multiply norms by $\sqrt{\pi}R$. All vector fields are axisymmetric without swirl, and tensors have no component with exactly one azimuthal index. Set

$$S = s\partial_s, \quad K = 2S + \eta\partial_\eta, \quad Q_\epsilon = \partial_q - \epsilon(K+1), \quad E = \mathrm{diag}(1,1,0).$$

Direct Cartesian differentiation gives

$$\begin{aligned} A_\epsilon &= -S + \nu s[(Q_\epsilon^2 + \partial_\eta^2)I - \epsilon^2 E], & A_0 &= -S + \nu s\Delta_{q,\eta}, \\ \mathrm{div}(\sqrt{s}\,b) &= s[(\partial_q - \epsilon K)b_r + \partial_\eta b_z], & & \\ (\mathrm{div}_\epsilon T)_r &= Q_\epsilon T_{rr} + \partial_\eta T_{rz} - \epsilon T_{\phi\phi}, & (\mathrm{div}_\epsilon T)_z &= Q_\epsilon T_{zr} + \partial_\eta T_{zz}. \end{aligned}$$

Thus the full tensor divergence, including the hoop traction, is retained. The domains of A_ϵ and its adjoint are the exact Cartesian domains under these lifts. No boundary is introduced at either age end. Let P_ϵ be the conjugated Cartesian Leray projection and P_0 the ordinary planar cylinder projection. Write

$$T_\epsilon = (\mu - A_\epsilon)^{-1}P_\epsilon, \qquad \mathcal{S}_\epsilon = T_\epsilon s\,\mathrm{div}_\epsilon\,.$$

The tensor map means completion from smooth tensors. The free Cartesian energy estimate, and its flat age-energy counterpart, give uniformly for $0 \le \epsilon \le 1$ and μ in a compact subset of $\mathrm{Re}\,\mu > -1/4$,

$$\|b\|_H + \sqrt{\nu}\|(Q_\epsilon b, \partial_\eta b, \epsilon b_r)\|_{H_T} \le C\nu^{-1/2}\|M\|_{H_T}, \qquad b = \mathcal{S}_\epsilon M. \tag{6.7}$$

The flat equation is $(\partial_s + \mu/s - \nu\Delta)b = P_0\,\mathrm{div}_0\,M$. Pairing it with $s^{-1/2}\bar{b}$ gives the coercive constant $\mathrm{Re}\,\mu + 1/4$ and proves its completion independently of comparison. Bounded measurable partitions of M are therefore allowed before divergence; their distributional interface terms remain.

Proposition 6.4. *Let χ be supported in $\sigma \le s \le H_1$, $|\eta| \le L_1$, and let χ_1 equal one on a larger neighborhood. There is a fixed integer k such that*

$$\|\chi(T_\epsilon - T_0)\|_{H\to H} \le C\Gamma^k\epsilon^{1/14}, \qquad \|\chi(\mathcal{S}_\epsilon - \mathcal{S}_0)\|_{H_T\to H} \le C\Gamma^k\epsilon^{1/50}, \tag{6.8}$$

where $\gamma_* = \inf(\mathrm{Re}\,\mu + 1/4) > 0$ over the fixed parameter set and

$$\Gamma = 2 + \nu + \nu^{-1} + c + c^{-1} + \sigma^{-1} + H_1 + L_1 + \sup|\mu| + \gamma_*^{-1} + \|\chi\|_{C^8} + \|\chi_1\|_{C^8}.$$

The integer is fixed before any window is allowed to depend on ϵ. Inputs are global and arbitrary in their indicated spaces.

Proof. First compare the projections on smooth tests. On each nonzero q-frequency, a flat divergence-free field and gradient have forms $(-\psi_\eta, \psi_q)$, $(\varphi_q, \varphi_\eta)$. Their exact counterparts are

$$(-\psi_\eta, (\partial_q - \epsilon K)\psi + \epsilon\psi), \qquad ((\partial_q - \epsilon K)\varphi, \varphi_\eta).$$

The first has zero exact divergence; the second lifts to a Cartesian gradient. The inverse q-derivative costs at most $c/(2\pi)$. Also, for the flat Fourier multiplier, $|\partial_\xi^j P_0(k,\xi)| \le C_j|k|^{-j}$, $j = 1,2$, so one or two transverse moments of the projected field cost only fixed powers of c. All projection operations commute with S.

For the zero frequency, the radial component is divergence-free and the axial component is a gradient. Take the corresponding primitives to vanish at $\eta = 0$, and cut them off at $|\eta| \sim L_0$. Their suprema, including one S derivative, are bounded by $C\|\langle\eta\rangle f\|_2$ and $C\|\langle\eta\rangle Sf\|_2$. Cutoff differentiation costs $L_0^{-1/2}$, geometric correction costs $\epsilon L_0^{1/2}$, and the omitted field tail costs L_0^{-1}. Choose $L_0 = \epsilon^{-1}$. Distances to the exact gradient and divergence-free subspaces, both closed orthogonal subspaces of H, now give

$$\|(P_\epsilon - P_0)f\|_H \le p(1 + c + c^{-1})\sqrt{\epsilon} \sum_{a,j=0}^{1} \|\eta^j S^a f\|_H. \tag{6.9}$$

Initially use finite Fourier sums, compact log-age support and Schwartz transverse profiles. The exact corrections lift to annular fields. Cutoff, mollification and Fourier truncation are dense in the displayed norm, after $x = \log s$, $u = e^{-x/4} f(e^x)$. Thus (6.9) holds on its full stated domain and retains the zero mode.

Next obtain uniform local regularity of the outputs. The unitary lift conjugates $\mathcal{T} - 3/4$ to $S - 1/4$. The Cartesian estimate (6.6) therefore gives

$$\||S - 1/4|^{1/2} \mathcal{S}_\epsilon M\|_H \le C\nu^{-1/2} \|M\|_{H_T} \qquad (\epsilon > 0).$$

At $\epsilon = 0$, a compact-age cutoff u of the flat solution satisfies $u \in L^2_s H^1$, $u_s \in L^2_s H^{-1}$ by (6.7) and its equation. Fourier Cauchy–Schwarz gives $\||\partial_s|^{1/2} u\|_2^2 \le \|u_s\|_{L^2_s H^{-1}} \|u\|_{L^2_s H^1}$. All local constants are polynomial in Γ.

In log age the conjugate of Q_ϵ, shifted by ϵ, is the skew operator $\widetilde{Q}_\epsilon = \partial_q - 2\epsilon\partial_x - \epsilon(\eta\partial_\eta + 1/2)$. Its three constituent flows commute. Multiplication by a smooth cutoff preserves the half age derivative: its difference quotient splits into the cutoff times the quotient of the function and its first derivative times a translated function. Both terms have the stated polynomial bound. For a compactly supported cutoff f of $e^{-x/4} b(e^x)$, their unitary flow identities give

$$\|f(q + h) - f(q)\|_2 \le |h| \|\widetilde{Q}_\epsilon f\|_2 + C|\epsilon h|^{1/2} \||\partial_x|^{1/2} f\|_2 + |\epsilon h| \|(\eta\partial_\eta + 1/2) f\|_2.$$

For $|h| \le 1$, square and integrate against $|h|^{-3/2}$; for larger h, use $2\|f\|_2$. The half age derivative and full transverse derivative already have bounds. Directional Fourier inequalities and the smooth log-age change prove

$$\|\chi \mathcal{S}_\epsilon M\|_{H^{1/4}(s,q,\eta)} \le C\Gamma^k \|M\|_{H_T}. \tag{6.10}$$

For a body source the graph controls Sb, and the gradient controls $Q_\epsilon b, \partial_\eta b$. Since $\partial_s = s^{-1} S$ and $\partial_q = Q_\epsilon + \epsilon(2S + \eta\partial_\eta + 1)$, the body output has a full local H^1 bound. The flat body graph follows directly by cutting off its heat equation on dyadic intervals $[a/2, 4a]$: the Fourier heat estimate controls $\nu s \Delta b$, the equation controls Sb, and finite overlap sums these bounds. Periodic extension over many cells followed by division by their number gives constants independent of the number of cells.

It remains to compare smooth adjoint observations globally. For a smooth test z in the observation box, put $h = P_0 z$ and $v = T_0^* z$. Exactly,

$$v(s) = \int_s^\infty (s/t)^{\bar{\mu} + 1/2} e^{\nu(t-s)\Delta} h(t) \, \frac{dt}{t}.$$

It vanishes for $s > H_1$. With $a = \bar{\mu} + 1/2$,

$$Sv = av - \nu s \Delta v - h, \qquad S^2 v = a^2 v - \nu s(2a + 1)\Delta v + \nu^2 s^2 \Delta^2 v - ah + \nu s \Delta h - Sh.$$

In log age the integral is a heat contraction followed by translation, with scalar kernel $e^{-(\bar{\mu} + 1/4)r}$, $r > 0$. Minkowski bounds it by the inverse coercivity gap. The identities

$$\eta e^{t\Delta} = e^{t\Delta}(\eta - 2t\partial_\eta), \qquad \eta^2 e^{t\Delta} = e^{t\Delta}(\eta^2 - 4t\eta\partial_\eta - 2t + 4t^2 \partial_\eta^2)$$

control two transverse moments since the heat time is at most νH_1. The multiplier estimate used in (6.9) transfers these moments through P_0. The preceding two age identities therefore give, with $\Lambda = K + 1$,

$$\|\nu s\Lambda\partial_q v\|_H + \|\nu s\Lambda^2 v\|_H + \|\nu sEv\|_H + \|\eta\partial_\eta v\|_H + \sum_{a,j=0}^{1} \|\eta^j S^a v\|_H \le C\Gamma^k \|z\|_{H^6}. \tag{6.11}$$

Indeed $\Lambda^2 = 4S^2 + 4S\eta\partial_\eta + 4S + \eta^2\partial_\eta^2 + 3\eta\partial_\eta + 1$. At most two age differentiations, two moment commutations and six test derivatives occur. Below the test support all norms are integrable by $\int_0^\sigma s^{2\,\mathrm{Re}\,\mu - 1/2} ds < \infty$. Thus the degree k is fixed independently of later windows.

The exact generator difference is $A_\epsilon - A_0 = \nu s[-2\epsilon\Lambda\partial_q + \epsilon^2\Lambda^2 - \epsilon^2 E]$, also for adjoints. If $w = T_\epsilon^* z - v$, then

$$(\bar\mu - A_\epsilon^*)w = (P_\epsilon - P_0)z + (A_\epsilon^* - A_0^*)v.$$

The right side has H norm at most $C\Gamma^k\sqrt{\epsilon}\|z\|_{H^6}$. The exact componentwise free adjoint energy estimate bounds w and its physical gradient by this quantity (with a polynomial viscosity cost). There is no projection before the second source term. The domain application is legitimate because (6.11) puts v in the actual maximal adjoint graph; its lift vanishes near the axis, without any assumption that v is divergence-free for finite ϵ.

Let G_ϵ be the negative weighted symmetric gradient adjoint to $s\,\mathrm{div}_\epsilon$. Then

$$(\mathcal{S}_\epsilon^* - \mathcal{S}_0^*)z = G_\epsilon w + (G_\epsilon - G_0)v.$$

The divergence formula at the start shows that the second term is ϵ times $Sv, \eta\partial_\eta v, v$ and frame terms. Consequently

$$\|(T_\epsilon^* - T_0^*)z\|_H + \|(\mathcal{S}_\epsilon^* - \mathcal{S}_0^*)z\|_{H_T} \le C\Gamma^k\sqrt{\epsilon}\|z\|_{H^6}. \tag{6.12}$$

Place the output support in a larger periodic box in s, η, retaining the original normalized q measure. Fourier projection to frequencies at most N, and the local H^1 and $H^{1/4}$ bounds, give high-frequency errors $C\Gamma^k N^{-1}$ and $C\Gamma^k N^{-1/4}$. A unit low-frequency dual vector corresponds to the compact smooth H-test $s^{3/2}\chi F_N h$, with H^6 norm at most $C\Gamma^k N^6$. Thus the full operator differences are at most

$$C\Gamma^k(N^{-1} + \sqrt{\epsilon}N^6), \qquad C\Gamma^k(N^{-1/4} + \sqrt{\epsilon}N^6).$$

Choose $N = \epsilon^{-1/14}$ and $N = \epsilon^{-2/25}$, respectively. No Fourier mode-count factor is used. This proves (6.8). □

The argument uses only normalized phase averages. The imported Cartesian graph bound is obtained by averaging fixed-size heat windows over their centers, so it has no inverse-period loss. Every remaining operation above is a spatial or age identity. Hence the comparison remains valid when $P = 2\epsilon\nu^{-2}$ tends to zero. Exact Cartesian divergence, the pressure projection, the zero cylinder mode and tensor interface terms have all been retained.

6.4 The background free comparison

Use the chart, operators and cutoffs of Proposition 6.4. For $1 < r < 3$, set

$$H_r = L^r(s^{r/2-5/2}\,ds\,dq/c\,d\eta), \qquad dm = s^{-5/2}\,ds\,dq/c\,d\eta, \qquad Mb = \sqrt{s}\,b.$$

The common multiplication $M : H_r \to L^r(dm)$ is an isometry for every r. Both the physical body source and velocity have lift $\sqrt{s}$ and norm factor $(\pi R^2)^{1/r}$; this factor cancels from every source-to-state estimate. In particular the conjugated Leray bound is uniform in R.

The exact graph gives

$$\|T_\epsilon F\|_{H_r} + \|ST_\epsilon F\|_{H_r} + \sqrt{\nu}\|\sqrt{s}(Q_\epsilon, \partial_\eta)T_\epsilon F\|_{H_r} \le C_{r,\mu}\|F\|_{H_r}. \tag{6.13}$$

Indeed $\mathcal{T}(\sqrt{s}b) = \sqrt{s}(S + 1/2)b$, $\partial_\rho(\sqrt{s}b) = sQ_\epsilon b$, and $\partial_z(\sqrt{s}b) = s\partial_\eta b$. For $\epsilon = 0$, the same estimate follows without a limiting argument. The causal formula

$$T_0F(s) = \int_0^s (t/s)^\mu e^{\nu(s-t)\Delta} P_0F(t)\, dt/t$$

and weighted dilation give $\|T_0\|_{H_r\to H_r} \le C_r/a_r$. On each dyadic window $[a, 2a]$, cut its heat equation off inside $[a/2, 4a]$ and apply the heat estimate used above. The cutoff costs a^{-1}; multiplication by s cancels it. Finite overlap and comparability of $s^{r/2-5/2}$ on each window prove the global bounds for ST_0 and νsD^2T_0, and interpolation gives its gradient bound. The cylinder heat estimate follows by applying the Euclidean estimate to a cutoff of N repeated cells, dividing its r-th power by N, and letting $N \to \infty$. There is no period loss. Consequently $\partial_s = s^{-1}S$ and $\partial_q = Q_\epsilon + \epsilon(2S + \eta\partial_\eta + 1)$ give uniform local first-derivative bounds at both endpoints.

Proposition 6.5. *Fix $p = 11/4$ and $r = 23/8$. On any compact set of shifts with $\inf(\operatorname{Re}\mu + 1/46) > 0$, let B be a first-order operator in q, η, with smooth matrix coefficients supported in the observation box. There is a fixed integer k such that*

$$\|B(T_\epsilon - T_0)\|_{H_p\to H_p} \le C\Gamma_r^k\epsilon^{2/539}. \tag{6.14}$$

Here Γ_r is Γ of the shared Hilbert proposition, augmented by the reciprocal r-gap and the finite coefficient seminorms of B. The inputs may have arbitrary age support.

Proof. We first upgrade the shared body H_2 estimate to spatial derivatives. For $\|F\|_{H_2} = 1$, put $u = T_\epsilon F$, $v = T_0F$, $w = u - v$, and $\omega = \|\chi_1(T_\epsilon - T_0)\|$. Subtract their componentwise equations and test against $\chi^2 w$. This compact test is allowed by the local first-derivative graph. Writing $L_1 = 2S + \eta\partial_\eta + 1$, the principal diffusion difference has exactly the two terms

$$\nu\epsilon\langle\sqrt{s}L_1v, \sqrt{s}Q_\epsilon(\chi^2w)\rangle + \nu\epsilon\langle\sqrt{s}\partial_q v, \sqrt{s}L_1(\chi^2w)\rangle.$$

Absorb one quarter of the gradient energy from the first term; the second is $O(\Gamma_r^k\epsilon)$ by the separate bounds for Sv, Sw on this box. Diffusion cutoff terms cost another quarter plus $C\Gamma_r^k\omega^2$. The drift identity is

$$\operatorname{Re}\langle Sw, \chi^2w\rangle_{H_2} = \tfrac{1}{4}\|\chi w\|_{H_2}^2 - \int s\chi\chi_s|w|^2\, dH_2.$$

The projection-source difference $(P_\epsilon - P_0)F$ has norm at most two, so its pairing costs 2ω. The radial frame discrepancy costs $C\Gamma_r^k\epsilon^2$; retain its nonnegative energy on w. It follows that

$$\|\chi(Q_\epsilon, \partial_\eta)w\|_{H_2}^2 \le C\Gamma_r^k(\omega + \omega^2 + \epsilon).$$

By (6.8), $\omega \le C\Gamma_r^k\epsilon^{1/14}$. Since $\partial_q w = Q_\epsilon w + \epsilon L_1 w$, the localized first-derivative difference is $O(\Gamma_r^k\epsilon^{1/28})$. This proves the Hilbert version of (6.14).

At exponent $r = 23/8$, (6.13) and the local coordinate identities give a uniform polynomial bound for the same operator. Interpolate the single consistent operator $MB(T_\epsilon - T_0)M^{-1}$ on $L^r(dm)$, rather than interpolating unequal weighted spaces. The weight on its Hilbert endpoint is

$$\lambda = \frac{1/p - 1/r}{1/2 - 1/r} = \frac{8}{77}, \qquad \frac{\lambda}{28} = \frac{2}{539}.$$

The semigroup formulas agree on the dense intersection of the endpoint spaces, so this is an interpolation of one operator.

□

For the actual localized interaction write $K_\epsilon = sB_\epsilon T_\epsilon$, $K_0 = sB_0T_0$. Then

$$K_\epsilon - K_0 = sB_0(T_\epsilon - T_0) + s(B_\epsilon - B_0)T_\epsilon.$$

The first term is covered by (6.14). Exact chart substitution writes the second as ϵ times polynomially bounded coefficients multiplying $b, \partial_q b, \partial_\eta b, Sb$; these are bounded by (6.13) and its local consequence. Thus the same $\epsilon^{2/539}$ bound applies. The interaction remains unprojected in force coordinates: Leray is already inside T_ϵ. No pressure-cutoff commutator or derivative of a rough input has been introduced. Output tails outside the box require the separate train coefficient estimates; none are inferred from this local proposition.

7 The exact unforced background

We use the real axisymmetric class without swirl. Set

$$p = 11/4, \quad c = \nu^{-2}, \quad R = \epsilon^{-1}, \quad \epsilon = e^{-d/\nu}, \quad P = 2\epsilon c, \qquad 0 < d_- \le d \le d_+.$$

Powers of ν^{-1}, c and fixed polynomial cutoff lengths are denoted by $\mathcal{P}_\nu$. Constants may depend on $d_\pm$ and on a fixed derivative budget. The phase measure is normalized by $d\tau/P$. Use the spaces Y_p, D_p of Lemma 6.1, and write $A_0 = L$ and $X_p = D_p \cap \{\operatorname{div} w = 0\}$ with the same graph norm. Let v_c be the train of Proposition 3.1 and $V_0 = U_{\mathrm{raw}}$ its divergence-free lift from Proposition 5.1, with residual F_R. In particular, (5.1) gives $\|F_R\|_{C_\tau W^{1,p}} \le \mathcal{P}_\nu \epsilon^{3/11}$.

For $m > 8$, write $X_m = L^2(\mathbb{R}^2, \langle x\rangle^{2m})$. The frozen growing pair and complementary bound are given by Corollary 4.2. The train estimates give

$$\sup_{s\ge 0} \sum_{j=0}^{3} \|D^j v_c(s)\|_\infty \le C, \tag{7.1}$$

$$\int_{\Sigma/\nu}^{\infty} (\|Dv_c(s)\|_\infty + \|D^2 v_c(s)\|_\infty)\, ds \le \frac{C}{\nu(1+\Sigma)} + C\nu^3. \tag{7.2}$$

The weighted coefficient and polynomial spatial $L^{22/3}$ bounds follow from Lemma 3.2. In normalized cylinder coordinates set

$$H_p = L^p(s^{-9/8}\, ds\, dq/c\, d\eta), \qquad \|\sqrt{s}\, b\|_{Y_p} = (\pi R^2)^{1/p}\|b\|_{H_p}, \quad T_0F(s) = \int_0^s e^{\nu(s-t)\Delta}\mathbb{P}_0 F(t)\frac{dt}{t}. \tag{7.3}$$

The global free graph and compact-output comparison are (6.13) and Proposition 6.5; their inputs have arbitrary age support.

7.1 One mixed construction for the finite age interval

We first verify the common actual-cylinder comparison. On the centered cell let $X_{m,c} = L^2(C_c, (1 + q^2 + \eta^2)^m)$ and let J_c, R_c be zero extension and restriction. Extend the free cylinder evolution by the identity on the exterior summand of X_m and the interaction by zero there. The vorticity equation is

$$\omega_s = (\nu\Delta - v_c(s)\cdot\nabla + A_c(s))\omega, \qquad A_c(s)\omega = -(K_c\omega)\cdot\nabla E_c(s). \tag{7.4}$$

Here $E_c = \operatorname{curl} v_c$ and K_c includes the horizontal mean mode. On $s \le \Sigma/\nu$, the train comparison gives, in any fixed finite weighted derivative budget,

$$\|v_c - V_c^p\| + \|E_c - E_c^p\| \le \delta_c := \mathcal{P}_\nu e^{C_\Sigma/\nu - b_\Sigma c^2}.$$

The central radial Gaussian transports the radial weight exactly; the velocities induced by the other periodic copies, the region near the cell boundary, and the displayed error contribute at most $C(\delta_c + e^{-b_\Sigma c^2})$ to its logarithmic derivative. The weight is continuous across the cell boundary and its normal derivative has a negative jump. Its distributional Laplacian is therefore bounded above by C_m times the weight. Weighted integration by parts gives

$$\|S_{\nu,c}(t,s)\|_{X_{m,c}\to X_{m,c}} \le \exp[(C_m\nu + C\delta_c + Ce^{-b_\Sigma c^2})(t-s)]. \tag{7.5}$$

One may smooth the weight, preserving uniform comparability, to justify this calculation. On a fixed block, Duhamel for a smooth compactly supported datum against the frozen radial transport gives convergence: the transported support remains a fixed disk, the viscosity error is $\nu\Delta$ applied only to that smooth datum, and the coefficient change is $O(\nu L) + o(1)$ there. Density and (7.5) give strong convergence on X_m, uniformly for block lengths in a compact interval. The frozen radial transport is unitary in this radial weighted space. Multiplying by its inverse reduces the free convergence to $W_\nu \to I$ strongly with $\|W_\nu\| \le 1 + o(1)$. The identity for $\|W_\nu^* f - f\|^2$ then gives strong adjoint convergence too. This argument makes no operator-norm assertion about the Laplacian.

The cylinder kernel is uniformly bounded away from its common local $1/|x-y|$ singularity. On $|x|, |y| \le c/4$, paired periodic images give $|K_c(x-y) - K_{\mathbb{R}^2}(x-y)| \le Cc^{-2}(1+|x|+|y|)$. Multiplication by the Gaussian observation coefficient ∇E and weighted Cauchy–Schwarz control this part by $Cc^{-2}\|f\|_{X_m}$. For $|y| > c/4$, the same Cauchy–Schwarz estimate gives a factor c^{1-m}, with an additional harmless source-moment power if needed; for $|x| > c/4$ the coefficient is Gaussian-small. The coefficient error contributes $C\delta_c$. Thus

$$\|J_c A_c(\sigma/\nu) R_c - A_{\mathrm{pl},\sigma}\|_{X_m\to X_m} \le C_\Sigma(c^{-2} + \delta_c), \qquad 0 \le \sigma \le \Sigma. \tag{7.6}$$

The near kernel gains one local derivative, by the Fourier multiplier bound for DK, and the coefficient is localized. Truncate source and observation, use local Rellich compactness, and then the same tail estimates: every interaction is compact and their family is collectively compact. The artificial exterior is invariant and has no growing component. Restriction removes it. To implement zero circulation in the comparison one may adjoin a radial mass-one neutral vector to the plane zero-circulation space; radial Euler linearization annihilates it. Cylinder curl sources and periodized angular-two entrance vectors have zero circulation, so their actual solutions remain in the desired subspace.

Fix Σ and put $T = \Sigma/\nu$. Choose $0 < a < a' < b' < b < \gamma/(1+\Sigma)^2$. On a bounded block of length L, the common compact-Duhamel lemma gives

$$U(t+L,t) = W_{\nu,t} e^{LL_{\nu t}} + o_{\nu\to 0}(1) \quad \text{in operator norm},$$

where $\|W_{\nu,t}\| \le 1 + o(1)$ and $W_{\nu,t}, W_{\nu,t}^* \to I$ strongly. The error is uniform for block lengths in $[0, L]$. Indeed the interaction Duhamel term converges in norm; its limit is compact, so multiplying it by $W_{\nu,t} - I$ gives a norm-small error. Let E_t, F_t be the frozen growing and complementary spaces. Finite rank and strong adjoint convergence make both off-diagonal blocks small. For L fixed sufficiently large, each full block has matrix

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \qquad \|a_n^{-1}\| \le A, \quad \|d_n\| \le D, \quad \|b_n\| + \|c_n\| \le \eta_\nu \longrightarrow 0, \qquad AD < 1.$$

Here A, D can be chosen below $e^{-b'L}, e^{a'L}$ with strict margins. Frozen projections use the periodized plane direct vectors, which have zero circulation, and the restricted plane dual vectors, with their finite pairing matrix normalized. Weighted-dual convergence suffices; no far-field expansion of the dual vector is used.

For completeness, construct the invariant subspaces using

$$\xi_{n+1} = (c_n + d_n\xi_n)(a_n + b_n\xi_n)^{-1}, \qquad \zeta_n = (a_n - \zeta_{n+1}c_n)^{-1}(\zeta_{n+1}d_n - b_n), \quad \xi_0 = 0, \quad \zeta_N = 0. \tag{7.7}$$

On a ball of radius $r = C\eta_\nu$, these maps have Lipschitz constant at most $A(D + \eta_\nu r)/(1 - A\eta_\nu r) < 1$. They preserve the ball if $r(1 - AD) \geq A\eta_\nu(1 + r^2)$. Hence the graphs are complementary, their projections are uniformly bounded, and the products of their coordinate maps have bounds $e^{a'(t-s)}$ forward on the center and $e^{-b'(s-t)}$ backward on the growing space. Coordinate equivalence is used only at the two endpoints of a product. An appended block of length at most L has uniformly bounded growing inverse and forward map and changes only the prefactor.

At intermediate times in a block $[t_n, t_{n+1}]$, put $J_t = U(t, t_n)|_{E^u_n}$ and $F_t = \Pi^u_{n+1}U(t_{n+1}, t)$. Their product is the invertible two-dimensional block $B_n = U(t_{n+1}, t_n)|_{E^u_n}$, so $\Pi^u(t) = J_t B_n^{-1} F_t$ and $\Pi^c(t) = I - \Pi^u(t)$. This proves invariance and bounds without inverting a parabolic map on the infinite-dimensional space. Consequently

$$z(s) = \int_0^s U(s,t)\Pi^c(t)f(t)\,dt - \int_s^T U(s,t)\Pi^u(t)f(t)\,dt \tag{7.8}$$

satisfies the equation, has initial value in the fixed entrance pair, has vanishing terminal growing coordinate, and obeys

$$\sup_{[0,T]} \|z(s)\|_{X_m} \leq Ce^{a'T}\|f\|_{L^1(0,T;X_m)}. \tag{7.9}$$

Only a rank-two evolution was inverted. All constructions preserve zero circulation and reality.

7.2 Completion for arbitrary body forces

Let $F \in H_p$, set $b_0 = T_0F$, and solve for the correction in the (r, z)-components

$$z_s - \nu\Delta z + \mathbb{P}_0(v \cdot \nabla z + z \cdot \nabla v) = g := -\mathbb{P}_0(v \cdot \nabla b_0 + b_0 \cdot \nabla v). \tag{7.10}$$

Use the fixed Green selector (7.8) for the vorticity $\operatorname{curl} z$: its initial value belongs to the fixed entrance pair and its terminal growing coordinate vanishes. No condition is imposed on $b_0(T)$ or on a velocity observation. The train coefficient and source remain present at every age. The global free graph gives

$$\|b_0\|_{H_p} + \|\sqrt{\nu s}Db_0\|_{H_p} + \|\nu s D^2 b_0\|_{H_p} \leq C\|F\|_{H_p}.$$

Put $b = 13/8$, so $b - p = -9/8$. Spatial Hölder with $22/3$, followed by one derivative, bounds both g in H^1 and $\operatorname{curl} g$ in X_m on a finite slow-age band by $\mathcal{P}_\nu(\|D^2 b_0\|_p + \|Db_0\|_p + \|b_0\|_p)$. Thus

$$\|g\|_{L^p((0,T),s^b ds;H^1)} + \|\operatorname{curl} g\|_{L^p((0,T),s^b ds;X_m)} \leq \mathcal{P}_\nu\|F\|_{H_p}. \tag{7.11}$$

The circulation of the curl source is zero. The elementary estimates

$$\|f\|_{L^1(0,T)} \leq 14^{7/11}T^{1/22}\|f\|_{L^p(s^{13/8}ds)}, \tag{7.12}$$

$$\left\|\int_0^s f(t)\,dt\right\|_{L^p(s^{-9/8}ds)} \leq 22\|f\|_{L^p(s^{13/8}ds)}, \tag{7.13}$$

$$\left\|\int_0^s (s-t)^{-1/2}f(t)\,dt\right\|_{L^p(s^{13/8}ds)} \leq C\sqrt{T}\|f\|_{L^p(s^{13/8}ds)} \tag{7.14}$$

follow respectively by Hölder and by $t = su$, Minkowski, and the integrable factors $u^{-21/22}$ and $(1-u)^{-1/2}u^{-21/22}$. These apply to norms of Banach-valued functions.

The zero-circulation cylinder Biot–Savart map sends X_m to H^1 with polynomial c cost. On nonzero horizontal modes use the gap $2\pi/c$; on the mean mode use $|\widehat{\omega}(0,\xi)| \le |\xi| \int |\eta\bar{\omega}(\eta)|d\eta$ before dividing by ξ. Thus (7.9) and (7.11) give

$$\sup_{[0,T]} \|z(s)\|_{H^1} + |\alpha| \le \mathcal{P}_\nu e^{a'T} \|F\|_{H_p}, \tag{7.15}$$

where $\alpha \in \mathbb{R}^2$ is the entrance coefficient of the Green-selected correction. It is determined linearly by F; no force trace is used.

For the global continuation, the train estimates imply, with normalized spatial measure and $r = 22/3$,

$$\|D^j v(s)\|_r \le \mathcal{P}_\nu s^{-(41+22j)/44}, \quad \|v(s)\|_\infty \le \mathcal{P}_\nu s^{-1}, \quad \|Dv(s)\|_\infty \le \mathcal{P}_\nu s^{-3/2}, \qquad s \ge 1.$$

Write $B_j(s) = (\nu s)^{j/2}\|D^j b_0(s)\|_p$. Hölder gives

$$\|g(s)\|_{H^1} \le \mathcal{P}_\nu[\nu^{-1}s^{-85/44}B_2(s) + s^{-63/44}(\nu^{-1/2}B_1(s) + B_0(s))]. \tag{7.16}$$

Each B_j has $L^p(s^{-9/8}ds)$ norm at most $C\|F\|_{H_p}$. For a factor $s^{-k}B_j$, weighted Hölder on (T,∞) gives $T^{23/22-k}$; hence the exponents in its L^1 bound are $-39/44$ and $-17/44$, respectively. The same expression belongs to $L^p(s^{-9/8}ds; H^1)$. Differentiated velocity energy and (7.2) therefore give

$$M_z := \sup_{s\ge 0} \|z(s)\|_{H^1} \le \mathcal{P}_\nu \exp\!\left(\frac{a'\Sigma + C/(1+\Sigma)}{\nu} + C\nu^3\right) \|F\|_{H_p}. \tag{7.17}$$

Local smoothing introduces no further exponential. To check this, differentiate (7.10) homogeneously twice. Incompressibility, orthogonality of $\mathbb{P}_0$, and (7.1) give $E_j' + \nu E_{j+1} \le C\sum_{i\le j} E_i$, $j \le 2$. Multiplication by successive powers of $\nu(t-s)$ and summation gives $H^1 \to H^2$ norm $C[\nu(t-s)]^{-1/2}$ for $t - s \le 1$. On $0 < s < 2$, Duhamel and (7.14) give the weighted $L^p(s^{13/8}ds; H^2)$ estimate. For $s \ge 2$ use Duhamel only on $[s-1, s]$; weights at s and $s-r$, $0 < r < 1$, are comparable. Consequently

$$\|z\|_{L^p((T,\infty),s^{-9/8}ds;H^2)} \le C\nu^{-1/2}[T^{-1/22}M_z + \|g\|_{L^p((T-1,\infty),s^{-9/8}ds;H^1)}].$$

Since $s\|v(s)\|_\infty + s\|Dv(s)\|_\infty \le \mathcal{P}_\nu$ for $s \ge 1$, these estimates bound $sB_v z$ in H_p with the right side of (7.17). Here $B_v z = v\cdot\nabla z + z\cdot\nabla v$.

Let $E(s)\alpha$ be the fixed inlet stream with an age cutoff equal to one near zero and zero before the terminal time T. Its homogeneous solution differs from $E\alpha$ by $O(s)|\alpha|$ in the velocity state. The first Hardy estimate controls the forced difference, so $z - E\alpha \in H_p$ at zero; the global H^1 bound and $\int_T^\infty s^{-9/8}ds = 8T^{-1/8}$ give the tail. Thus the exact global identity is

$$b_0 + z = E\alpha + T_0 f, \qquad f = F - sB_v(b_0 + z) - s(\partial_s - \nu\Delta)E\alpha. \tag{7.18}$$

It follows from the differential equation and free uniqueness; each term defining f belongs to H_p. For the remaining free interaction term this is explicit: on $s < T$, $\|sB_v b_0\|_{H_p} \le \mathcal{P}_\nu(\nu^{-1/2}\sqrt{T} + T)\|F\|_{H_p}$; on $s > T$, the bounds $v = O(s^{-1})$, $Dv = O(s^{-3/2})$ and the free weighted gradient give $\|sB_v b_0\|_{H_p} \le \mathcal{P}_\nu T^{-1/2}(\nu^{-1/2} + 1)\|F\|_{H_p}$. Approximation in $L^p(s^{13/8}ds; X_m)$ and the difference estimates above justify all identities for the original arbitrary force. Choosing Σ large and then a' small proves, for every fixed $m > 0$,

$$\|f\|_{H_p} + |\alpha| \le \mathcal{P}_\nu e^{m/\nu}\|F\|_{H_p}. \tag{7.19}$$

7.3 Exact geometry and the nonlinear correction

The exact stream lift $E_\epsilon\alpha$ is affine: its nonzero inlet is not in Y_p. Its complete equation source is in Y_p. The Gaussian angular-two eigenstream is a harmonic quadrupole plus a Gaussian remainder outside a fixed core. Periodizing its harmonic part uses only

$$\sum_{n\in\mathbb{Z}}(q+nc+i\eta)^{-2}=(\pi/c)^2\csc^2(\pi(q+i\eta)/c).$$

Its apparent poles are replaced by the smooth eigenstream core. Thus the actual periodized stream and every fixed derivative have exponential transverse decay, with polynomial c costs. This proves the inlet bounds needed below without any estimate on a dual tail. Indeed write $K=2s\partial_s+\eta\partial_\eta$, and for one inlet stream ψ write its normalized lift as

$$e_\epsilon=e_0+\epsilon e_1,\qquad e_0=\chi(-\psi_\eta,\psi_q),\qquad e_1=(0,(\chi-2s\chi')\psi-\chi\eta\psi_\eta).$$

The exact vector diffusion is $L_\epsilon=\Delta-2\epsilon(K+1)\partial_q+\epsilon^2[(K+1)^2-\mathrm{diag}(1,0)]$. Substitution in the complete unprojected source column

$$\mathcal{C}_\epsilon\alpha=s[(\partial_s-\nu L_\epsilon)e_\epsilon+B_\epsilon e_\epsilon]\alpha$$

gives $\mathcal{C}_\epsilon-\mathcal{C}_0=\epsilon s r_\epsilon$, where r_ϵ is smooth at zero, compact in upper age, and has polynomial weighted bounds. Hence

$$\|\mathcal{C}_\epsilon-\mathcal{C}_0\|_{H_p}\le\epsilon\mathcal{P}_\nu,\qquad \|A_0E_\epsilon\alpha+\mathbb{P}B_{V_0}E_\epsilon\alpha\|_{W^{1,Y_p}}\le\mathcal{P}_\nu R^{2/p}|\alpha|. \tag{7.20}$$

The integrability exponent is $-9/8+p=13/8>-1$. The quadratic inlet source has the same bound with $|\alpha|^2$, and $\|E_\epsilon\alpha\|_{W^{j,\infty}}\le\mathcal{P}_{\nu,j}|\alpha|$. These assertions compare sources, never the nonintegrable inlet states.

The global flat completion now permits one all-age perturbation. Put $X=H_p$, $K_\epsilon=sB_\epsilon T_\epsilon$, $K_0=sB_vT_0$. The flat train is retained at every age. Give $X\times\mathbb{R}^2$ the norm $\|(f,\alpha)\|=\|f\|_X+|\alpha|$. The equation row is

$$\mathfrak{A}_0:X\times\mathbb{R}^2\longrightarrow X,\qquad \mathfrak{A}_0(f,\alpha)=(I+K_0)f+\mathcal{C}_0\alpha.$$

The all-age reconstruction (7.18) and the complete inlet source defining $\mathcal{C}_0$ give $\mathfrak{A}_0(f,\alpha)=F$. The Green selector, Biot–Savart map and reconstruction are linear, so (7.19) constructs a bounded linear right inverse

$$R_0:X\longrightarrow X\times\mathbb{R}^2,\qquad \mathfrak{A}_0R_0=I_X,\qquad \|R_0\|\le M=\mathcal{P}_\nu e^{m/\nu}.$$

We compare operator outputs while inputs remain arbitrary global elements of H_p.

Here the all-age mean expansion gives a useful stronger coefficient bound than a fixed small late constant:

$$\sup_{s>S}(|V_0|+|DV_0|)\le\mathcal{P}_\nu(S^{-1/2}+\epsilon). \tag{7.21}$$

To check the Cartesian gradient, write $b=(v_1,v_2+\epsilon H)$. Its three derivative types are $s\partial_\eta b$, $s[\partial_q-\epsilon(K+1)]b$, and the frame term $\epsilon s v_1$. The mean stream is $S_0=a_0G_{\nu s}+a_2\partial_\eta^2G_{\nu s}+O_{\text{weighted}}(\mathcal{P}_\nu s^{-5/2})$. Thus $v_1=-\partial_\eta S_0$ and $H=2a_0G_{\nu s}+4a_2\partial_\eta^2G_{\nu s}+\cdots$. The potentially nondecaying radial derivative cancels because $(K+1)G_{\nu s}=0$, giving $(K+1)H=-8a_2\partial_\eta^2G_{\nu s}+O_{\text{weighted}}(\mathcal{P}_\nu s^{-5/2})$. Every fluctuation derivative is exponentially small. These identities prove (7.21), including the full frame term; the axial mean velocity itself has the genuine $O(\epsilon)$ floor. The physical free graph and the exact norm identity then give

$$\|1_{s>S}K_\epsilon\|_{X\to X}\le\mathcal{P}_\nu(S^{-1/2}+\epsilon).$$

For the flat operator, $v = O(\mathcal{P}_\nu/s)$ and $Dv = O(\mathcal{P}_\nu/s^{3/2})$, paired with the free $\sqrt{s}Db_0$ graph bound, give $\|1_{s>S}K_0\| \le \mathcal{P}_\nu S^{-1/2}$. At the young end, smooth inlet coefficients give $\|1_{s<\sigma}K_\epsilon\| + \|1_{s<\sigma}K_0\| \le \mathcal{P}_\nu\sqrt{\sigma}$ by the same free graph.

Take $\sigma = \epsilon^a$ and $S = \epsilon^{-b}$, where $a, b > 0$ will be small fixed numbers. In (3.14), choose $l = C_D(c/\nu + \sqrt{S} + 1)$. Both negative exponents dominate any prescribed multiple of ν^{-1}, while $P_S \le \mathcal{P}_\nu e^{kbd_+/\nu}$ for fixed k. Increasing C_D makes all required coefficient tails at most $e^{-D/\nu}$; fixed weights and K derivatives change only the polynomial cost. The prefactor in the localized comparison is polynomial in this width, S and σ^{-1}. With $\theta = 2/539$ from Proposition 6.5, for some fixed $N < \infty$,

$$\|K_\epsilon - K_0\|_{X\to X} + \|\mathcal{C}_\epsilon - \mathcal{C}_0\| \le \mathcal{P}_\nu[\epsilon^{a/2} + \epsilon^{b/2} + \epsilon + \epsilon^{\theta - N(a+b)} + e^{-D/\nu}] =: \Delta_\nu. \tag{7.22}$$

The affine column is supported in a fixed polynomial age interval and has the already proved $O(\epsilon)$ source difference. No equation or source is cut off. Choose a, b so $\theta - N(a+b) > 0$, and put $\zeta = \min\{a/2, b/2, 1, \theta - N(a+b)\} > 0$. Choose $m < d_-\zeta/2$ and $D > 2m$. Then $M\Delta_\nu < 1/2$ for small ν. The exact equation row

$$\mathfrak{A}_\epsilon : X \times \mathbb{R}^2 \longrightarrow X, \qquad \mathfrak{A}_\epsilon(f, \alpha) = (I + K_\epsilon)f + \mathcal{C}_\epsilon\alpha$$

has the bounded linear right inverse

$$R_\epsilon = R_0[I + (\mathfrak{A}_\epsilon - \mathfrak{A}_0)R_0]^{-1}, \qquad \|R_\epsilon\| \le 2M.$$

Indeed $(\mathfrak{A}_\epsilon - \mathfrak{A}_0)R_0 : X \to X$ has norm at most $M\Delta_\nu < 1/2$, and multiplication verifies $\mathfrak{A}_\epsilon R_\epsilon = I_X$. This compares all global inputs, including those supported arbitrarily late; it does not assume that the exact inverse is causal in age. Returning to physical normalization proves

$$\|w\|_{X_p} + R^{2/p}|\alpha| \le \mathcal{P}_\nu e^{m/\nu}\|f\|_{Y_p} \tag{7.23}$$

for $A_0 w + \mathbb{P}B_{V_0}w + (A_0E_\epsilon + \mathbb{P}B_{V_0}E_\epsilon)\alpha = \mathbb{P}f$. Here the physical w is the free response to the first coordinate selected by R_ϵ, and the free graph supplies its X_p norm. The same normalized measure multiplies source, state, and finite coordinate, so no additional power of R is lost.

Set $A = R^{2/p}\alpha$ and $\|(w, A)\|_Z = \|w\|_{X_p} + |A|$. By Lemmas 6.1–6.2 and (7.20), the quadratic terms in $w + E_\epsilon\alpha$ and their polarized differences are bounded by $\mathcal{P}_\nu\|(w, A)\|_Z^2$ in Y_p. Apply the fixed linear right-inverse selection to minus these terms and minus F_R. With $\delta = \mathcal{P}_\nu\epsilon^{3/11}$ and $M = \mathcal{P}_\nu e^{m/\nu}$, this is a contraction on the radius $2M\delta$ ball if $4\mathcal{P}_\nu M^2\delta < 1$. This holds after decreasing m so $2m < 3d_-/11$. We obtain an exact smooth unforced profile

$$U = V_0 + E_\epsilon\alpha + w, \qquad \|w\|_{X_p} + R^{8/11}|\alpha| \le \mathcal{P}_\nu e^{m/\nu}\epsilon^{3/11}, \qquad |\alpha| \le \mathcal{P}_\nu e^{m/\nu}\epsilon. \tag{7.24}$$

The projected equation is $A_0U + \mathbb{P}(U\cdot\nabla U) = 0$ with zero residual. Reality, no-swirl symmetry, and periodicity are preserved at each iteration.

No second unstable inverse is needed for spatial regularity. Write $V = V_0 + E_\epsilon\alpha$ and $A_0 w = f - \mathbb{P}(V\cdot\nabla w + w\cdot\nabla V + w\cdot\nabla w)$. The complete source has W^{1,Y_p} norm bounded by the right side of (7.24), up to a polynomial. Since $\partial_j A_0 w = (A_0 - 1/2)\partial_j w$ and $\|Dw\|_{Y_{2p}}^2 \le C\|w\|_\infty\|D^2 w\|_{Y_p}$, every differentiated term is in Y_p. Also $Dw \in Y_p$ is already known. Rewrite the differentiated equation as $A_0 D_j w = f_j + \frac{1}{2}D_j w$, with the entire right side in Y_p, and apply the maximal graph at shift zero. No inverse of $A_0 - \frac{1}{2}$ is used. Thus $Dw \in \widetilde{X}_p$, and

$$\sup_\tau \|w(\tau)\|_p + \|w\|_{W^{1,\infty}} + \|E_\epsilon\alpha\|_{W^{1,\infty}} \le \mathcal{P}_\nu e^{m/\nu}\epsilon^{3/11}. \tag{7.25}$$

For any fixed derivative order, the train and inlet complete sources belong to the corresponding W^{k,Y_p}. Explicitly, in the large-radius region their unprojected derivatives are bounded by $C_{k,N}\varrho^{-3-k}(1 +$

$|y_3|/\varrho)^{-N}$, whose pth integral reduces to $C\int_R^\infty \varrho^{2-p(3+k)}d\varrho<\infty$. On $\varrho<R$, take the all-age heat expansion to a sufficiently high finite order: its even radial powers extend smoothly through the axis and its weighted remainder gives arbitrary axial decay. The inlet cutoff vanishes near the axis. These estimates are uniform in phase; the bounded Cartesian Leray map preserves them in $W^{k,p}$. Inductively, if derivatives through order k are in the graph, all factors through order k are bounded and derivatives through order $k+2$ are in Y_p. In each order-$(k+1)$ derivative of $w\cdot\nabla w$, one factor has order at most k. Put it in L^∞ and the other in Y_p, then apply the maximal graph to the differentiated equation. This proves all spatial orders. On positive physical-time compact intervals the resulting $C_\tau W^{k,4}$ regularity and the projected equation give all time derivatives, so U is jointly smooth through the axis.

7.4 The datum, heat remainder, and pressure

Let $\psi_{\rm in}$ be the raw inlet plus its single entrance adjustment. Define, with $\varrho=|x'|$,

$$q=R\log(\varrho/R),\quad \eta=Rz/\varrho,\qquad a_r=\frac{R}{\varrho}(-\psi_{{\rm in},\eta}),\quad a_z=\frac{R}{\varrho}[\psi_{{\rm in},q}+\epsilon(\psi_{\rm in}-\eta\psi_{{\rm in},\eta})].$$

The stream is periodic in q and has arbitrary fixed transverse polynomial decay. Differentiation and this decay give $|D^j a(x)|\le C_j|x|^{-1-j}$, including smooth extension at each nonzero axis point. The omitted divergence flux through a radius-r sphere is $O(r)$, so a is distributionally divergence-free on all of $\mathbb{R}^3$. Thus $a\in L^{3,\infty}\cap L^2_{\rm loc}$. The shift $q\mapsto q+c$ proves its DSS identity. The raw inlet is nonzero on a fixed core and (7.24) makes its finite adjustment exponentially small there, so $a\neq 0$.

Put $a_\tau(y)=e^{\tau/2}a(e^{\tau/2}y)$ and $H_a(\tau)=e^{\nu\Delta}a_\tau$. Taylor expansion against the Gaussian, whose first moment vanishes, gives

$$|D^j(e^{\nu\Delta}a_\tau-a_\tau)(y)|\le C_{j,\nu}|y|^{-3-j},\qquad |y|\ge 1.$$

Use the expansion only in a ball of radius $|y|/4$ about y; differentiate the Gaussian on its complement, which is exponentially small despite the singularity of a at zero. The $O(s)$ inlet expansion gives the same inverse-cubic bound for $V-a_\tau$ in the large-radius region, with arbitrary transverse decay. The all-age train expansion gives arbitrary axial decay in the remaining bounded radial region. Therefore

$$V-H_a\in C_{\rm per}H^k\cap X_p\quad\text{for every fixed required } k. \tag{7.26}$$

The graph membership includes the mixed phase/dilation derivative; it follows from the same source expansion. In particular this conclusion is stronger than an X_p estimate alone.

The inverse-distance bounds and the all-age expansion give $D^jV\in C_{\rm per}L^4$. For w, interpolation between its $C_{\rm per}L^p$ derivatives and their uniform supremum bounds gives $D^jw\in C_{\rm per}L^4$. Therefore $D^jU\in C_{\rm per}L^4$ for $j\le 2$, and $\mathbb{T}=U\otimes U\in C_{\rm per}H^2$. The field $Q=U-H_a\in X_p$ solves $A_0Q=-\mathbb{P}\,{\rm div}\,\mathbb{T}$. The exact free semigroup is

$$S_0(t)f(y)=e^{t/2}[e^{\nu(e^t-1)\Delta}f](e^{t/2}y).$$

Free uniqueness identifies Q with $-\int_0^\infty S_0(t)\mathbb{P}\,{\rm div}\,\mathbb{T}(\tau-t)dt$. For $j\le 2$, move derivatives to $\mathbb{T}$ when $t\le 1$ and to the heat kernel when $t\ge 1$. The resulting integrable majorants are

$$\|D^jS_0(t)\mathbb{P}\,{\rm div}\,\mathbb{T}\|_2\le C_{j,\nu}\begin{cases}t^{-1/2}\|\mathbb{T}\|_{H^j}, & 0<t\le 1,\\ e^{-3t/4}\|\mathbb{T}\|_2, & t\ge 1.\end{cases}$$

Thus $Q\in C_{\rm per}H^2$. Also $(\partial_\tau-y\cdot\nabla/2)Q=\nu\Delta Q+Q/2-\mathbb{P}\,{\rm div}\,\mathbb{T}\in C_{\rm per}L^2$. Scaling back gives

$$\|u_\nu(t)-e^{\nu t\Delta}a\|_2=t^{1/4}\|Q(\log t)\|_2\le Ct^{1/4}.$$

A compact part of a is in L^2; its complement is bounded and uniformly continuous. The Gaussian approximate identity therefore shows $e^{\nu t\Delta}a\to a$ locally in L^2, and hence the same is true for u_ν. Finally $p_U=\sum_{i,j}R_iR_j(U_iU_j)\in C_{\rm per}L^2$ recovers the discarded gradient part of the exact projected equation.

8 Thin and repair stress spaces

Fix the exact background from Section 7. The goal is its nonzero periodic energy mode b with exponent $0 < \sigma < 1/4$. The planar Gaussian eigenpair supplies transient train amplification and a signed integrated stress response. The three-dimensional pressure interaction returns a strain to the young rings and closes the loop. The singular limit compares these response operators as curvature and viscosity decrease together; the final characteristic zero belongs to the exact three-dimensional operator, with its full pressure and repair.

We first establish the causal near inverse and all-age moments, then identify the returned strain and control its errors, and finally solve the rank-one scalar return equation. The background and geometry remain fixed as we vary the real Floquet exponent.

Use the exact background $U = V + r_b$, where $V = U_{\mathrm{raw}} + E_\epsilon \alpha_{\mathrm{bg}}$ and r_b is the energy repair. Set $c = \nu^{-2}$, $R = \epsilon^{-1}$, and retain component order (r, ϕ, z). In the cylinder chart

$$s = R^2/\rho^2, \qquad q = R\log(\rho/R) + R\tau/2, \qquad \eta = Rz/\rho,$$

put $d\varsigma = ds\, dq/c\, d\eta$ and

$$H = L^2(s^{-3/2}d\varsigma), \qquad H_T = L^2(s^{-1/2}d\varsigma; \mathrm{Sym}_3).$$

The natural velocity and tensor lifts are $\mathcal{N}_v b = \sqrt{s}\, b$ and $\mathcal{N}_T T = sT$. Changing variables in the normalized phase/spatial integral gives

$$\|\mathcal{N}_v b\|^2_{Y_2} = \pi R^2 \|b\|^2_H, \qquad \|\mathcal{N}_T T\|^2_{Y_2} = \pi R^2 \|T\|^2_{H_T}. \tag{8.1}$$

Indeed the physical space Jacobian after azimuthal integration is $\pi R^2 s^{-5/2} ds\, d\eta$ and normalized phase becomes dq/c. The physical divergence and Leray projection remain in the free inverse; these formulas only identify its Hilbert spaces.

Fix $m_Z > 3/2$, for example $m_Z = 2$, and define

$$W = \left\langle \eta/\sqrt{1+s} \right\rangle^{m_Z}, \qquad Z_1 = L^2(s^{-1/2}W^2 d\varsigma), \qquad Z = Z_1 \oplus \rho_b^{-1/2} H_T.$$

The norm on Z is the sum norm $\|(T_1, T_2)\|_Z = \|T_1\|_{Z_1} + \rho_b^{-1/2}\|T_2\|_{H_T}$. Here

$$\rho_b = P_\nu \epsilon^{3/11} e^{m_0/\nu}$$

is a fixed upper bound from the exact-background construction, chosen so that $\sup_\tau(\|r_b(\tau)\|_{11/4} + \|r_b(\tau)\|_\infty) \le \rho_b$. It is not the radial coordinate. The stress inverse acts on $T_1 + T_2$; the second entry is not assigned a transverse weight.

Write $v = s^{-1/2}V$ and $v_b = s^{-1/2}r_b$. The train heat expansion, transverse bounds, and compact-age inlet construction imply

$$\begin{aligned}
&\sup \sqrt{s}\, W|v| \le P_\nu, && \sup_{s>S} \sqrt{s}\, W|v| \le P_\nu(S^{-1/2} + \epsilon),\\
&\sup_{s<\sigma} \sqrt{s}\, W|v| \le P_\nu\sqrt{\sigma}, && 0 < \sigma \le 1,\\
&\|Wv(s)\|_{L^2_\eta L^\infty_q} \le P_\nu[(1+s)^{-3/4} + \epsilon(1+s)^{-1/4}].
\end{aligned} \tag{8.2}$$

Here S is larger than the compact-age inlet support. To check the age powers, the mean stream and its fixed weighted derivatives have size $(1+s)^{-1/2}$ on transverse scale $\sqrt{1+s}$. One transverse derivative gives the velocity size $(1+s)^{-1}$. The stream completion $\epsilon(1 - \eta\partial_\eta - 2s\partial_s)\psi$ has size $P_\nu\epsilon(1+s)^{-1/2}$. Its slower decay is retained. The L^2_η norm adds the factor $(1+s)^{1/4}$. Fluctuations have exponential

decay after overlap; before overlap the Gaussian comparison gives the same polynomial bounds. The finite inlet is absorbed by P_ν, and early smoothness gives the third bound. Multiplying by any further fixed power of $\langle \eta/\sqrt{1+s}\rangle$ leaves these conclusions unchanged, by the weighted train expansion.

If $S = \epsilon^{-b}$ and $D_\perp > 0$ is fixed, the coefficient can also be localized to

$$|\eta| \le L_\perp = C_\perp(c/\nu + \sqrt{S} + 1)$$

with

$$\sup_{s<S,\ |\eta|>L_\perp} \sqrt{s}\, W(|v| + |v_{\mathrm{flat}}|) \le e^{-D_\perp/\nu}. \tag{8.3}$$

The Gaussian vorticity tail controls the heat-width part. The nonzero horizontal stream kernels $e^{-|k_n||\eta|}/(2|k_n|)$, $|k_n| \ge 2\pi/c$, control the c/ν part. The zero mode has decaying heat primitives. Increasing $C_\perp$ absorbs the fixed weights, derivative budgets, and the finite inlet's stream tail. No such localization is claimed for r_b.

Define the exact split multiplier by

$$\mathcal{M}_U b = (v \otimes b + b \otimes v,\ v_b \otimes b + b \otimes v_b).$$

The flat multiplier uses the actual planar train in its first entry and zero in its second entry. Since the ratio of stress and velocity age densities is s, (8.2) gives

$$\|\mathcal{M}_U b\|_Z \le P_\nu \|b\|_H, \qquad \|(\mathcal{M}_U b)_2\|_{\rho_b^{-1/2} H_T} \le 2\sqrt{\rho_b}\|b\|_H. \tag{8.4}$$

Its thin old and young restrictions have respectively norms $P_\nu(S^{-1/2} + \epsilon)$ and $P_\nu\sqrt{\sigma}$. Indeed $|a \otimes b + b \otimes a| \le 2|a||b|$, so the thin multiplier norm is at most $2 \sup \sqrt{s} W|v|$. For the repair, $\sqrt{s} v_b = r_b$ and multiplication followed by the factor $\rho_b^{-1/2}$ gives the second estimate. These bounds also hold after multiplication of both sides by any common positive age weight, in particular the gain $g(s)^{-1}$.

For $A \ge 1$ the thin source has the absolute old moment bound

$$\mathcal{N}_1(1_{s>A}T) := \int_A^\infty s^{-3/2} \int (1 + |\eta|/\sqrt{s}) \|T(s,\cdot,\eta)\|_{L^2(dq/c)}\, d\eta\, ds \le C_{m_Z} A^{-1/2} \|1_{s>A} T\|_{Z_1}. \tag{8.5}$$

Transverse Cauchy–Schwarz gives

$$\int (1 + |\eta|/\sqrt{s}) \|T(s,\cdot,\eta)\|_{L^2_q}\, d\eta \le C_{m_Z} s^{1/4} \|WT(s)\|_{L^2_{q,\eta}},$$

because $\int \langle x\rangle^{2-2m_Z} dx < \infty$. The remaining time integral is $\int_A^\infty s^{-5/4} \|WT(s)\|_2 ds$. Cauchy–Schwarz against $s^{-1/2} ds$ leaves $\int_A^\infty s^{-2} ds = A^{-1}$, proving (8.5).

In outer coordinates $x' = y'/R$, $u = |x'|^{-2}$ and $Y = \eta/\sqrt{u}$, the thin physical tensor is represented by $F = uT$. Directly,

$$\int (1 + |Y|) \|F(x', Y, \cdot)\|_{L^2(dq/c)}\, dx'\, dY = \pi \mathcal{N}_1(T).$$

Thus (8.5) supplies exactly the first moment used in the separated-kernel approximation. It also justifies extending an old annulus to the axis by moment-norm approximation. All cutoffs multiply the tensor before its distributional divergence; no artificial boundary force is removed. The broad repair is instead estimated by its physical $L^{11/4}$ coefficient and energy bounds.

9 Planar amplification, the full age tail, and the causal inverse

9.1 The full moving evolution and weak sources

Let $C_c = (\mathbb{R}/c\mathbb{Z}) \times \mathbb{R}$, $c = \nu^{-2}$, and let $v = v_c$ be the actual planar train. Fix $m > 8$. Write

$$X_{m,c} = L^2(C_c, \langle x\rangle^{2m} dx), \qquad Z_{m,c}^{-1} = \{f \in (H^1_{-m,c})^* : \langle f, 1\rangle = 0\}, \qquad g(s) = \exp \frac{\gamma s}{1+\nu s}.$$

The test 1 belongs to $H^1_{-m,c}$. The centered radial weight is used for evolution norms. Smooth periodic weights are used only for local smoothing, with equivalence applied at its two endpoints.

Proposition 9.1 (Full forward train estimate)**.** *For a fixed sufficiently large b and all sufficiently small ν, the moving full complement has the exponential relative bound* (9.4)*, the full moving evolution satisfies* (9.5)*, and on $0 < t < s \le H = \nu^{-1}((\nu b)^{-1} - 1)$ its physical weak-source extension satisfies*

$$\left\| \frac{g(t)}{g(s)} U(s,t) \right\|_{Z_{m,c}^{-1} \to X_{m,c}} \le C_b \nu^{-M_s} \{1 + (s-t)^{-1/2}\}, \qquad M_s = \lceil m/2 + |\Re\kappa| + 2 \rceil.$$

These bounds hold on the entire zero-circulation space, with no input parity restriction.

Proof. The full moving complement. Set

$$a = 1 + \nu s, \quad \theta = (\nu a)^{-1}, \quad z = s/a = \nu^{-1} - \theta, \quad e_c = (\nu c^2)^{-1} = \nu^3, \quad d = c/\sqrt{a} = \sqrt{\theta/e_c}.$$

The exact substitution $w(s,x) = a^{-2} W(z, x/\sqrt{a})$ gives

$$W_z = G(z)W, \qquad G = \theta^{-1} D + L_{\nu,c}(z), \qquad D = \Delta + \tfrac{1}{2} y \cdot \nabla + 2, \qquad d_z/d = -1/(2\theta). \tag{9.1}$$

The rescaled train differs from the periodized Gaussian by $C_{j,m} e_c e^{-b_{j,m}\theta/e_c}$ in every fixed weighted derivative norm on $\theta \ge b > 0$. This follows by rescaling the train error, interpolating its Gaussian-small unweighted norm with a larger fixed weighted norm, and absorbing the resulting fixed powers into the Gaussian. The required region is contained in $1 \le a \le c^2$.

The common actual-cylinder comparison (7.5)–(7.6) applies in moving coordinates after including Reynolds' formula. Here is the quantitative part needed for all parities. The paired-image kernel satisfies, for $|x| \le d/8$, $|y| \le d/4$,

$$K_d(x-y) - K_\infty(x-y) = d^{-2} B_1(x-y) + O(d^{-4}|x-y|^3).$$

For zero-mass input the x times mass term disappears; the first moment is retained and bounded by $C_m \|f\|_{X_m}$. The source tail has L^1 norm at most $C_m d^{1-m} \|f\|_{X_m}$; the separated far kernel costs $O(d^{-1})$. The observation outside $d/8$ is Gaussian-small. Hence the localized interaction error, also after $d\partial_d$, is $O(d^{-2}) = O(e_c/\theta)$ on the entire space. No first-moment cancellation or parity restriction is used. The same $O(d^{-2})$ bound holds for auxiliary nonzero-mass inputs: retain $x \int f$ and bound it against the localized observation coefficient. Thus restricting an ambient input to a cell creates no mass-related gap in the compact comparison.

Let $E(z)$ be the two periodized growing vectors from Proposition 4.1, and normalize their dual rows $\Lambda(z)$ by $\Lambda E = I_2$. The planar dual kernels and their fixed derivatives have the $O(|y|^{-2-j})$ decay obtained from (4.1) and (4.2). Paired translates therefore give dual error $O(d^{-2})$ in the weighted dual norm, with the same bound after $d\partial_d$; direct columns have Gaussian tails. The finite Gram inverse exists uniformly and its differentiated error is $O(\theta^{-1} d^{-2})$. The identities

$$\theta^{-1} D(P_d q) - \partial_z(P_d q) = \theta^{-1} P_d D q, \quad \theta^{-1} D^{\#}(P_d \ell) + \partial_z(P_d \ell) = \theta^{-1} P_d D^{\#} \ell, \quad D^{\#} = \Delta - \tfrac{1}{2} y \cdot \nabla + 1$$

follow term by term using $d_z/d = -1/(2\theta)$. They are identities for the unweighted dual kernels, with Gram normalization then applied. In particular the combined direct and dual rows are bounded; neither ΛD alone nor a derivative of sharp zero extension is used.

Put $K_0 = GE - E_z$. Define H_0 by differentiating the actual moving pairing: $(\Lambda W)' = H_0 W$ whenever $W' = GW$. Then $M_0 = H_0 E = \Lambda K_0$. For $\beta_0 = \gamma + \Re\kappa/\theta$ set

$$\mathcal{B} = G - \beta_0 - EH_0 + [-K_0 + (\beta_0 - 1)E + EM_0]\Lambda. \tag{9.2}$$

All added terms are bounded finite rank. Direct substitution gives $(\Lambda W)' = -\Lambda W$ for $W' = \mathcal{B}W$, and $\mathcal{B}E = E_z - E$. Thus $\ker\Lambda(z)$ is invariant and the artificial pair decays at rate one. The planar limit is $\mathscr{Q}(\mathscr{L} - \gamma)\mathscr{Q} - \mathscr{P}$, where $\mathscr{Q} = I - \mathscr{P}$ removes both growing modes. Its full-space semigroup is exponentially stable by (4.3). This statement includes the horizontal mean and every angular sector. For ambient identifications a radial mass-one neutral vector can be adjoined; $\mathscr{L}$ annihilates it and the relative damping makes it stable. Restriction to zero circulation is preserved by the actual equation.

To compare on a fixed Hilbert space let $I_z, R_z, \Pi_z = I_z R_z$ be zero extension, restriction and cell projection. Extend the free moving evolution F by the actual cocycle

$$\overline{F}(z,t) = I_z F(z,t) R_t + S(z,t)(I - \Pi_t), \qquad Sf(y) = \rho^4 f(\rho y), \quad \rho = d(t)/d(z).$$

The two terms have disjoint entrance and exit supports. The exterior term has norm at most ρ^3. For the interior, Reynolds' boundary term cancels the dilation seam term. With $p = (1 + |y|^2)^m$,

$$\frac{d}{dz}\|W\|_p^2 \le \left[\theta^{-1} \sup\left(\frac{\Delta p}{p} + 3 - \tfrac{1}{2} y \cdot \nabla \log p\right) + \delta_\nu\right] \|W\|_p^2.$$

The two seam weights agree and the remaining diffusion seam term is nonpositive. Radial Euler transport preserves the weight; the image and actual-base velocity errors give $\sup\delta_\nu \to 0$ on $\theta \ge b$ for fixed b. One can take $C_m = (m^2 + 3m + 3)/2$ for the norm-growth contribution of D. Consequently the relative free evolution satisfies

$$\|\overline{F}_{\mathrm{rel}}(z,t)\| \le \exp\{[-\gamma + (C_m + |\Re\kappa|)/b + \sup\delta_\nu](z - t)\}. \tag{9.3}$$

On each fixed block of length B, compact smooth tests and Duhamel give direct strong convergence to radial Euler transport in the successive limit $\nu \to 0$, then $b \to \infty$, uniformly in the entrance time. Density extends the convergence. The undamped norms have limiting upper bound one, and radial transport T is unitary; the identity

$$\|F_n^* f - T^* f\|^2 \le \|F_n\|^2 \|f\|^2 - 2\Re\langle f, F_n T^* f\rangle + \|f\|^2$$

gives adjoint-strong convergence. The localized interaction is compact by source and observation truncation and local Rellich compactness; the image and dual estimates above give norm convergence of every compact and finite-rank term in (9.2). Lemma 4.3 therefore compares the full-minus-free remainders in norm. If the limiting full relative bound is $C_0 e^{-hB}$, choose B so that $C_0 e^{-hB} + 2e^{-\gamma B} < 1/4$, then b large, then ν small. Each full block contracts by $1/2$. Uniform bounds on the remaining short block yield

$$\|S_{\mathrm{rel}}(z,t)\| \le C e^{-h_0(z-t)}, \qquad b \le \theta(z) \le \theta(t) \le \nu^{-1}. \tag{9.4}$$

All block products use the same ambient radial weighted norm.

Write $W = E\alpha + r$, $\Lambda r = 0$. The amplitude diagonals are $\lambda_i + \kappa_i/\theta$; angular orthogonality makes the planar off-diagonal D pairing zero. The direct errors are bounded by

$$\delta_*(\theta) = C\{e_c/\theta + e_c e^{-b_2\theta/e_c}\},$$

and both amplitude–complement couplings by $C(\theta^{-1}+\delta_*)$. Divide each amplitude by $e^{\lambda_i(r_0-\theta)}(r_0/\theta)^{\kappa_i}$ and the complement by their common modulus. Equation (9.4) bounds its stable convolution. For small ν, enlarge constants to bound the coupling by $\eta(z)=C(1+e_c)/\theta(z)$ and the direct error by $\delta(z)=Ce_c/\theta(z)$. The Gaussian error is absorbed in these bounds. If $A_*(z)$ is the running maximum of the normalized amplitude, the stable convolution gives $\|r(z)\|\le Ce^{-h_0(z-t)}\|r(t)\|+C\eta(z)A_*(z)$. The initial-complement term in the amplitude is integrable with bound $C\|r(t)\|/b$. Equivalently, for the double convolution, $\eta'/\eta=1/\theta\le 1/b$ and $b\ge 2/h_0$ give $\int_u^z \eta(v)e^{-h_0(v-u)}dv\le(2/h_0)\eta(u)$. Substitution into the amplitude equation and scalar Gronwall give an exponential of

$$C\int_b^{1/\nu}[\delta_*(r)+(r^{-1}+\delta_*(r))^2]dr\le C/b+Ce_c\log(1/(\nu b))+Ce_c/b+Ce_c^2/b+o(1).$$

The initial complement contributes only a uniformly bounded decaying convolution. The displayed budget is uniformly bounded. We obtain, on the entire zero-mass space and with no input symmetry restriction,

$$\|U_{\mathrm{mov}}(\theta,r_0)\|_{X_{m,d(r_0)}\to X_{m,d(\theta)}}\le C_be^{\gamma(r_0-\theta)}(r_0/\theta)^{\Re\kappa}. \tag{9.5}$$

Weak sources and the polynomial loss. Choose the core endpoint

$$H=s_b=\nu^{-1}((\nu b)^{-1}-1),\qquad a_b=(\nu b)^{-1}.$$

For $J_aW(x)=a^{-2}W(x/\sqrt{a})$, change of variables gives

$$\|J_a\|_{X_{m,d}\to X_{m,c}}\le a^{(m-3)/2},\qquad \|J_a^{-1}\|_{X_{m,c}\to X_{m,d}}\le a^{3/2},\qquad \|J_a^{-1}\|_{(H^1_{-m,c})^*\to(H^1_{-m,d})^*}\le a^{3/2}. \tag{9.6}$$

For the last inequality, $\langle J_a^{-1}f,\phi\rangle=a\langle f,\phi(\cdot/\sqrt{a})\rangle$ and $\|\phi(\cdot/\sqrt{a})\|_{H^1_{-m,c}}\le\sqrt{a}\|\phi\|_{H^1_{-m,d}}$. At two endpoints $1\le a_t\le a_s\le a_b$, the product of the first two factors is at most $a_b^{m/2}$. Normalizing both physical cell norms by c^{-1} does not change this operator estimate.

For short weak smoothing use only here the equivalent periodic weight

$$p_d=[1+y_2^2+(d/\pi)^2\sin^2(\pi y_1/d)]^m.$$

Its first two logarithmic spatial derivatives are uniformly bounded, $|\partial_z\log p_d|\le C_m/\theta$, and $|y\cdot\nabla\log p_d|\le C_m$. The rescaled train and its first two derivatives are uniformly bounded for $\theta\ge b$. The adjoint localized interaction maps $X_{-m,d}$ to $H^1_{-m,d}$ uniformly: its localized source has uniformly bounded L^1 and L^2 norms by weighted Cauchy–Schwarz; split the Biot–Savart kernel into its integrable near part and bounded far part, and use the uniform order-zero multiplier bound for its derivative, including the horizontal mean. This also proves the stated bound for the actual train perturbation.

For the reverse-time adjoint let E_j be its squared weighted H^j_{-m} energy. On $\Delta z\le 1$, with $\beta=\theta^{-1}$, Reynolds cancellation and differentiated weighted energy give

$$E_0'+\beta_{\min}E_1\le C_bE_0,\qquad E_1'+\beta_{\min}E_2\le C_b(E_1+E_0).$$

The diffusion-weight errors have coefficient $C_m\beta$ and the transport and compact terms have coefficient C_b; no negative power of ν occurs in a Gronwall coefficient. Moreover $\beta_{\max}/\beta_{\min}\le 1+1/b$ on such an interval. Integrate the first inequality and then the inequality for $rE_1(r)$ to get $E_1(\Delta z)\le C_b(\beta_{\min}\Delta z)^{-1}E_0(0)$. By duality, the moving weak-to-strong norm is at most $C_b(\beta_{\min}\Delta z)^{-1/2}$.

For $0<s-t\le 1$, the clock identities are exactly

$$\Delta z=\frac{s-t}{a_sa_t},\qquad \beta_{\min}=\nu a_t,\qquad \beta_{\min}\Delta z=\frac{\nu(s-t)}{a_s}.$$

Combining with (9.6) gives the physical weak-to-strong cost $C_b\nu^{-(m+2)/2}(s-t)^{-1/2}$. For a longer gap factor at $t+1/2$ in moving coordinates and apply (9.5) after the short smoothing step. The two intermediate maps $J_{a_{t+1/2}}^{-1}J_{a_{t+1/2}}$ cancel *before* taking norms. Thus only one endpoint cost $\nu^{-m/2}$ occurs, the smoothing costs at most $C_b\nu^{-1}$, and the long-clock algebraic factor costs $\nu^{-|\Re\kappa|}$. For the fixed integer

$$M_s = \lceil m/2 + |\Re\kappa| + 2\rceil$$

we have on $0 < t < s \le H$

$$\left\| \frac{g(t)}{g(s)} U(s,t) \right\|_{Z_{m,c}^{-1}\to X_{m,c}} \le C_b\nu^{-M_s}\{1 + (s-t)^{-1/2}\}. \tag{9.7}$$

Zero mass is preserved by the conservative physical equation and then by continuous extension in the actual weighted dual source norm.

□

9.2 From the plane branch to the actual train output

All cell measures in this subsection are unnormalized. Put

$$\Theta = \nu^{-1}, \quad c = \nu^{-2}, \quad e = (\nu c^2)^{-1} = \nu^3, \quad a = 1 + \nu s,$$
$$\theta = (\nu a)^{-1}, \quad d = c/\sqrt{a} = \sqrt{\theta/e}, \quad N = e^{\lambda/\nu}\nu^{-\kappa}.$$

Write P_d for periodization, K_d for cylindrical Biot–Savart, and $X_{m,d}$ for the centrally even zero-circulation class with weight $\langle y\rangle^{2m}$ on the centered cell, $m > 8$. Its first moments vanish. The canonical plane branch is that of Proposition 4.5; set $M(\theta) = \mathfrak{M}(V_{\mathrm{can}}(\theta))$. Angular integration and cell summation give $Q_dP_dV_{\mathrm{can}}(\theta) = M(\theta)$.

The actual residual, including moving images. For $z = \Theta - \theta$, the identity $d_z/d = -1/(2\theta)$ gives

$$\theta^{-1}D(P_df) - \partial_z(P_df) = \theta^{-1}P_dDf \tag{9.8}$$

when f is independent of z. Thus periodization commutes with the combined diffusion/dilation/moving-cell derivative, not with each term separately. Reynolds' boundary term cancels the dilation seam term. All calculations below may first be made on the fixed physical cylinder and then rescaled.

For completeness the improved even image error is elementary. The symmetrically summed planar kernels satisfy

$$K_d(\xi) - K_\infty(\xi) = d^{-2}B_1\xi + O(d^{-4}|\xi|^3), \qquad |\xi| \le 3d/8.$$

Indeed the opposite translates cancel the constant and quadratic Taylor coefficients, while their jth derivatives are $O((|n|d)^{-1-j})$. The affine convolution vanishes because both mass and first moments are zero. On $|y| \le d/4$, $|x| \le d/8$, the remaining convolution is bounded by $Cd^{-4}(1+|x|^3)\|f\|_{X_m}$. The excluded source tail is at most $Cd^{1-m}\|f\|_{X_m}$ in L^1 and its far kernel costs Cd^{-1}. Multiplication by the Gaussian observation coefficient controls the polynomial in x; its complement $|x| > d/8$ is exponentially small. The same reasoning after $d\partial_d$ gives the same bound. Consequently the localized linearized interaction error is $O(d^{-4})$ on this even class. For a periodized Gaussian input, cross transport between different cells is exponentially small, using its fixed Gaussian derivative norms.

Let E_c be the actual train vorticity and $h = a^2E_c(s, \sqrt{a}\,y) - P_dE_0$. The train comparison (3.2), after rescaling, gives

$$\|\partial^j h\|_2 + \|\partial^j K_d h\|_{2\cap\infty} \le C_j\theta e^{-b_jd^2}.$$

Here the curl estimate uses one additional derivative. Since $\theta = ed^2$, the right side is at most $C_j e e^{-b'_j d^2}$. There is no reliance on smallness of an unbounded differential operator. To add fixed polynomial weights, the heat-coordinate vorticity energy with weight $\langle y\rangle^{4m}$ and finitely many derivatives gives the crude bound $P_\nu e^{C_{m,j}/\nu}$: transport coefficients and their selected derivatives are uniformly bounded by the preceding unweighted comparison, their clock interval has length at most ν^{-1}, and the diffusion/dilation contribution integrates $C_{m,j}\theta^{-1}$, costing only a power of ν. The periodic seam diffusion term is nonpositive and the moving seam cancels as in (9.8). Interpolating that bound with the unweighted error, and using $d^2 \geq r/e$ on any fixed $[r, \Theta]$, absorbs $e^{C/\nu}$ and every fixed polynomial into a smaller Gaussian for sufficiently small ν:

$$\|h\|_{H^j_m} + \|K_d h\|_{W^{j,\infty}} \leq C_{m,j} e\, e^{-b_{m,j}\theta/e}, \qquad r \leq \theta \leq \Theta. \tag{9.9}$$

Let $w = \mathcal{U}_{\nu,c}(s,0)P_c q$ be the actual bare column and write $w(s,x) = Na^{-2}Z(\theta, x/\sqrt{a})$. The canonical asymptotic gives

$$\|e^{\lambda\Theta}\Theta^\kappa[Z(\Theta) - P_c V_{\mathrm{can}}(\Theta)]\|_{X_{m,c}} \leq C\nu.$$

Applying (9.8), the image estimate, and (9.9) to $V_d = P_d V_{\mathrm{can}}$ gives the following residual in the equation $-\partial_\theta Z = \mathcal{L}_\theta Z$:

$$\|e^{\lambda\theta}\theta^\kappa(-\partial_\theta - \mathcal{L}_\theta)V_d\|_{X_{m,d}} \leq C_b(e^2\theta^{-2} + e\, e^{-a_0\theta/e}), \qquad b \leq \theta \leq \Theta, \tag{9.10}$$

$$\|(-\partial_\theta - \mathcal{L}_\theta)V_d\|_{X_{m,d}} \leq C_b(e^2\theta^{-2} + e\, e^{-a_0\theta/e}), \qquad r \leq \theta \leq b. \tag{9.11}$$

The image term needs only the Gaussian zeroth norm of the branch; cross transport and the base-error terms use its fixed derivative budgets. Their polynomial factors in $1+\theta$ are absorbed into the displayed Gaussian. This explicitly accounts for the actual nonlinear train, rather than substituting its independent-vortex sum.

The full-input forward bound (9.5) and Duhamel, including this normalized inlet error, imply

$$\|e^{\lambda\theta}\theta^\kappa(Z - V_d)(\theta)\|_{X_{m,d}} \leq C_b\{\nu + e^2/\theta + e^{-a_1\theta/e}\}, \qquad b \leq \theta \leq \Theta. \tag{9.12}$$

In particular the error at b is $O_b(\nu)$, and both columns have an exponentially decaying, polynomially weighted majorant for $\theta \geq b$. No terminal projection or backward parabolic propagator has been used.

Kinetic and tested moment continuation through overlap. The embeddings

$$|Q_d f| + \|K_d f\|_2 \leq C_m \|f\|_{X_{m,d}}, \qquad Q_d f = -\frac{2}{\pi}\int \eta^2 f, \qquad d \geq 1 \tag{9.13}$$

are uniform in d. The moment bound is weighted Cauchy–Schwarz. For velocity, zero mass gives $|\widehat{f}(\xi)| \leq |\xi| \int |y||f(y)|dy$ at low cylindrical frequencies. In Plancherel, the $O(d)$ horizontal frequencies of modulus at most one cancel the measure factor d^{-1}; the high frequencies are bounded by $\|f\|_2$. This proves (9.13) without a factor $1+d$.

Let $\mathcal{U}_Z(\theta, r)$ be the exact heat-coordinate evolution. The velocity energy inequality and (3.8) give, for $0 < \theta \leq r \leq b$,

$$\|K_{d(\theta)}\mathcal{U}_Z(\theta, r)h\|_2 \leq C_b(r/\theta)\|K_{d(r)}h\|_2. \tag{9.14}$$

Indeed the input physical kinetic norm is $\nu r\|K_{d(r)}h\|_2$ and the integrated strain costs at most $C(r+e)$. The exact physical quadrupole law is

$$\frac{d}{ds}Q_c(w) = -\frac{4}{\pi}\int (v_1 u_2 + v_2 u_1) = -\frac{4}{\pi}\int (\widetilde{v}_1\widetilde{u}_2 + \widetilde{v}_2\widetilde{u}_1).$$

The mean terms vanish because the mean vertical velocities vanish. Since Q is invariant under heat scaling, the same identity gives a tested bound uniform in ν. Before overlap $\|\widetilde{v}(s(\theta))\|_2 \le C\nu\theta$; after overlap it is at most $C\nu e\exp[-a(e/\theta-1)]$. Thus at every age $\nu^{-1}\theta^{-2}\|\widetilde{v}(s(\theta))\|_2 \le C/\theta$, and

$$|Q\mathcal{U}_Z(\theta,r)h| \le |Qh| + C_b r\log(r/\theta)\|K_{d(r)}h\|_2, \tag{9.15}$$

$$\int_0^r \theta^{p-1}|Q\mathcal{U}_Z(\theta,r)h|\,d\theta \le C_{b,p}r^p(|Qh| + r\|K_{d(r)}h\|_2), \quad p>0. \tag{9.16}$$

The second follows from $\int_0^r \theta^{p-1}\log(r/\theta)d\theta = r^p/p^2$. Similarly the absolute stress, divided by its scale $\nu^{3+2\Re\mu}/c$, for an impulse at r is bounded by

$$C_{K,b}r^{p+1}\max\{1,(e/r)^{1/4}\}\|K_{d(r)}h\|_2, \qquad p=\Re\mu+3/2. \tag{9.17}$$

For $\theta \ge e$ insert $\|v\|_2 \le C\nu\theta$ and $\|u\|_2 \le C_b\nu r\|Kh\|_2$. For $\theta \le e$ insert $\|v\|_2 \le C\nu e^{1/4}\theta^{3/4}$ instead. Its integral is $Cre^{1/4}\min(e,r)^{p-1/4}\|Kh\|_2$, proving the assertion. All these identities first hold for smooth data and sources. Duhamel and the integrable displayed bounds give the stated completion on the fixed physical cylinder; the actual decaying mean stream is retained when identifying its quadrupole.

For each fixed $r>0$, apply (9.14) and (9.15) to the residual and the error at b. The kinetic residual integral is bounded by $C_b e^2\theta^{-1}\log(b/\theta)$; the partial-moment factors are bounded on $r\le\theta\le b$. Thus

$$\sup_{r\le\theta\le b}\big(\|K_d(Z-V_d)(\theta)\|_2 + |Q_d(Z-V_d)(\theta)|\big) \longrightarrow 0. \tag{9.18}$$

The plane bounds and periodization imply $\limsup_{\nu\to0}(|Q_dZ(r)| + \|K_dZ(r)\|_2) \le C_b$ uniformly in $0<r\le b$. Apply the impulse estimates to the homogeneous actual column with data $Z(r)$, not to its discrepancy from V_d. For p in a compact subset of $(1/4,\infty)$,

$$\limsup_{\nu\to0}\int_0^r \theta^{p-1}|Q_dZ(\theta)|d\theta \le C_b r^p,$$

and (9.17), with $e<r$, bounds its normalized absolute stress tail by $C_b r^{p+1}$. Letting first $\nu\to0$ and then $r\downarrow0$ therefore gives the required tested convergence through the entire overlap tail.

Mellin output and stress defects. Fix $K \Subset \{0<\Re\mu<1/4\}$, put $\sigma=\Re\mu$, $u_o=\nu^{-3/2}$, and $\theta_o=(\sqrt{\nu}+\nu)^{-1}$. For the actual column define $A_+(s) = -\pi s^{-\mu}Q_c(w(s))/(4c)$. The exact substitution $s=(1-\nu\theta)/(\nu^2\theta)$ yields

$$\begin{aligned} I_A := \int_{u_o}^\infty A_+(s)s^{-5/2}ds &= -\frac{\pi N}{4c}\nu^{3+2\mu}\int_0^{\theta_o}\theta^{\mu+1/2}(1-\nu\theta)^{-\mu-5/2}QZ(\theta)d\theta \\ &= -\frac{\pi N}{4c}\nu^{3+2\mu}[\mathcal{J}(\mu)+o_K(1)]. \end{aligned} \tag{9.19}$$

Use (9.12) for the exponentially integrable large-clock part and the compact-clock convergence and tested tail bound above for the rest. The error is locally uniform and holomorphic. An extra $s^{-k/2}$ costs $\nu^k\theta^{k/2}(1-\nu\theta)^{-k/2}$, $k=0,1$. At u_o, every selected state or stress observation divided by N is $P_\nu e^{-\gamma'/\sqrt{\nu}}$ for any $\gamma'<\gamma$. The partial-moment bound gives $QZ(\theta)=O_\nu(1+|\log\theta|)$ as $\theta\downarrow0$, so the upper weighted moment endpoint is zero. If the old cutoff starts at $\sigma_{\rm in}/\nu<u_o$, its entire preceding part has the same exponentially small coefficient after polynomial weights; that source and cutoff transition have not been removed.

For $u=K_cw$, $u_\mu=s^{-\mu}u$, set

$$E_\eta=\frac{2}{c}\int v_1u_{\mu,1}, \quad E_q=\frac{2}{c}\int v_2u_{\mu,2}, \quad C=\frac{1}{c}\int(v_1u_{\mu,2}+v_2u_{\mu,1}),$$
$$D_c=E_\eta+E_q, \quad R_c=E_q-iC, \quad \mathcal{A}_\nu=|N|\nu^{3+2\sigma}/c.$$

These are bilinear pairings. Define the stronger absolute component quantity

$$S_{\mathrm{abs}}(s) = c^{-1} \int |v \otimes u_\mu + u_\mu \otimes v| \, dq \, d\eta.$$

For $k = 0, 1$,

$$\int_{u_o}^{\infty} s^{-3/2-k/2} S_{\mathrm{abs}}(s) ds \le C_{K,k} \mathcal{A}_\nu \nu^k. \tag{9.20}$$

This controls each absolute tensor component before cell integration, and hence also $|E_\eta| + |E_q| + |C|$, up to an absolute constant. Indeed the integrand before its time weight is at most $C|N|c^{-1}s^{-\sigma}\|v\|_2\|u/N\|_2$. For $\theta \ge b$, $\|u/N\|_2 \le Ca^{-1}H(\theta)$ with H exponentially decreasing times a polynomial. Changing variables gives the integrable majorant $C\mathcal{A}_\nu \nu^k \theta^{\sigma+3/2+k/2} H(\theta)$. For $\theta \le b$, energy gives $\|u/N\|_2 \le C_b \nu$. Before overlap $\|v\|_2 \le C/(\nu s)$; after overlap $\|v\|_2 \le Cc^{-2}(s/s_c)^{-3/4}$. The two integrals are bounded by $C\mathcal{A}_\nu \nu^k$ and $C\mathcal{A}_\nu \nu^k e^{\sigma+3/2+k/2}$ respectively.

For a plane radial background in positive angular sector two, $E_\eta = -E_q$ and $E_q = iC$ by direct angular integration. On each fixed positive clock interval, (9.18) and the actual background comparison therefore make the scaled D_c, R_c tend to zero. Their absolute small-clock tails are bounded by (9.17) with data $Z(r)$, hence by $C_K \mathcal{A}_\nu r^{\sigma+5/2}$ in the $\nu \to 0$ limit. Large-clock domination, followed by $r \downarrow 0$, proves

$$\int_{u_o}^{\infty} (D_c - 3R_c) s^{-3/2} ds = o_K(\mathcal{A}_\nu). \tag{9.21}$$

The unforced moment law is exactly $C = A_+' + \mu A_+/s$; hence

$$\int_{u_o}^{\infty} (2E_q - E_\eta) s^{-3/2} ds = 3i(\mu + 3/2) I_A + o_K(\mathcal{A}_\nu), \tag{9.22}$$

including the boundary term $3i[A_+ s^{-3/2}]_{u_o}^{\infty}$, just shown negligible. For the negative channel the sign of i reverses and $N\mathcal{J}(\mu)$ becomes $\bar{N}\,\overline{\mathcal{J}(\bar{\mu})}$.

9.3 A polynomial causal inverse in the gain norm

Retain the fixed spaces, actual train, and gain from Proposition 9.1. Put

$$\mathcal{H} = L^2(s^{-3/2}ds; L^2_{\mathrm{div}}(dq/c\,d\eta)), \quad \mathcal{T} = L^2(s^{-1/2}ds; L^2(dq/c\,d\eta)).$$

Let K be a fixed compact subset of $\{0 < \Re\mu < 1/4\}$ and $k_* := \inf_K(\Re\mu + 1/4) > 0$. Constants below are uniform on K.

Proposition 9.2. *For every planar symmetric tensor $F : C_c \times (0,\infty) \to \mathrm{Sym}_2$ with $g^{-1}F \in \mathcal{T}$, there is a unique causal zero-entrance energy solution of*

$$b_s + \frac{\mu}{s} b - \nu \Delta b + \mathbb{P}_0 \operatorname{div}(v \otimes b + b \otimes v) = \mathbb{P}_0 \operatorname{div} F. \tag{9.23}$$

Here zero entrance means $b \in \mathcal{H}$ near zero; no homogeneous $s^{-\mu}$ entrance is adjoined. The energy class also requires $\nabla b \in L^2_{\mathrm{loc}}((0,\infty); L^2)$. There is a finite M such that

$$\|g^{-1}b\|_{\mathcal{H}} + \sqrt{\nu}\|\nabla(g^{-1}b)\|_{\mathcal{T}} \le C\nu^{-M}\|g^{-1}F\|_{\mathcal{T}}. \tag{9.24}$$

The response is locally continuous in μ with the geometry fixed. This is a bound after conjugation by g; its ordinary norm can contain $e^{\gamma/\nu}$.

Proof. Use Proposition 9.1 with its fixed b, $H = \nu^{-1}((\nu b)^{-1} - 1)$, and weak-source degree M_s.

The zero-entrance construction and all ages. Put $T = g^{-1}F$, $u = g^{-1}b$ and $h = g'/g = \gamma/(1+\nu s)^2 \geq 0$. Solve the free equation for u_0 with scalar term $\mu/s + h$ and tensor T by starting from zero at $s = \delta$ and taking $\delta \downarrow 0$. The weighted energy is coercive and gives

$$\|u_0\|_{\mathcal{H}} \leq (2\sqrt{k_*\nu})^{-1}\|T\|_{\mathcal{T}}, \qquad \|\nabla u_0\|_{\mathcal{T}} \leq \nu^{-1}\|T\|_{\mathcal{T}}.$$

Spatial coefficient estimates may first be taken with unnormalized cell measure. Multiplying velocity, tensor, X_m, and weighted-dual source norms by the same $c^{-1/2}$ gives the normalized convention used in $\mathcal{H}$ and $\mathcal{T}$; all inequalities below are unchanged. Equivalently define the source duality using normalized cell pairing and normalized test norm. The localized vorticity source $f = -\operatorname{curl}(v \cdot \nabla u_0 + u_0 \cdot \nabla v)$ has zero mass and satisfies

$$\|f(s)\|_{Z_{m,c}^{-1}} \leq A_m(\|\nabla u_0(s)\|_2 + \|u_0(s)\|_2), \qquad A_m \leq C\nu^{-p_m} \quad (0 < s < H),$$

for one fixed finite p_m. Indeed A_m is bounded by a constant times $\sup_{s\leq H} \|\langle x\rangle^m (v, \nabla v)(s)\|_\infty$. The actual train tail (3.14), with $S_* = H$, bounds this by a fixed polynomial in c, ν^{-1}, H: the nonzero modes have exponential transverse kernels, and the zero mode has decaying primitives at both ends. Here $c = \nu^{-2}$ and $H = O(\nu^{-2})$. Thus this is a polynomial coefficient bound, with no exponential weighted-energy loss. Testing the outer curl against H^1_{-m} then gives the displayed negative-order estimate. Hence $\|f\|_{L^2(s^{-1/2};Z_m^{-1})} \leq C\nu^{-p_m-3/2}\|T\|_{\mathcal{T}}$.

For $\sigma = \Re\mu$, the substitution $t = su$ and Minkowski give the exact Hardy inequality

$$\left\| \int_0^s (t/s)^\sigma (s-t)^{-1/2} f(t)\,dt \right\|_{L^2(s^{-3/2})} \leq B(\sigma + 3/4, 1/2)\|f\|_{L^2(s^{-1/2})}. \tag{9.25}$$

The kernel 1 on $s < H$ costs an additional $\sqrt{H} \leq C_b\nu^{-1}$. Define using only forward propagation

$$\omega(s) = \int_0^s (t/s)^\mu \frac{g(t)}{g(s)} U(s,t) f(t)\,dt, \qquad u = u_0 + K_c\omega.$$

Equations (9.7)–(9.25) give $\|\omega\|_{L^2(s^{-3/2};X_m)} \leq C\nu^{-M_s-p_m-5/2}\|T\|_{\mathcal{T}}$. Zero-circulation Biot–Savart reconstruction gives $\|K_c\omega\|_{H^1} \leq C_m(1+c)\|\omega\|_{X_m}$: nonzero horizontal modes cost c, while the mean is the zero-endpoint primitive controlled by one-dimensional Hardy: $\|\int_\eta^\infty \bar\omega(\zeta)d\zeta\|_2 \leq 2\|\eta\bar\omega(\eta)\|_2$. This follows by integrating $(\eta|\int_\eta^\infty \bar\omega|^2)'$; zero circulation removes both endpoint terms. Since $\sqrt{H} = O(\nu^{-1})$, we obtain on the core

$$\|u\|_{\mathcal{H}(0,H)} + \sqrt{\nu}\|\nabla u\|_{\mathcal{T}(0,H)} \leq C\nu^{-M_s-p_m-5}\|T\|_{\mathcal{T}(0,H)}.$$

The velocity equation follows from its curl and divergence equations: after testing in age the discrepancy is spatial H^{-1}, curl-free and divergence-free; its Fourier transform is supported only at zero and must vanish. This retains the horizontal mean. Local Hilbert triple regularity gives a strong L^2 trace at every positive age.

For $E = s^{-1/2}\|u\|_2^2$, $L = \|\operatorname{sym}\nabla v\|_\infty$ and $A(s) = \exp(-2\int_{t_*}^s L)$, the exact energy inequality is

$$(AE)' + 2k_*AE/s + 2hAE + \nu As^{-1/2}\|\nabla u\|_2^2 \leq \nu^{-1}As^{-1/2}\|T\|_2^2. \tag{9.26}$$

Selecting a time in $[H-1, H-1/2]$ and applying this inequality to H gives $H^{-1/4}\|u(H)\|_2 \leq C_b[\sqrt{H}\|u\|_{\mathcal{H}(H-1,H)} + \nu^{-1/2}\|T\|_{\mathcal{T}(H-1,H)}]$. With $a_b = 1 + \nu H = (\nu b)^{-1}$, the actual train estimate (7.2) yields

$$I_H := \int_H^\infty L(s)ds \leq C/(\nu a_b) + Ce_c = Cb + C\nu^3.$$

Thus continuation from the exact strong trace, with the entire prescribed tensor after H, costs only the fixed factor $e^{Cb+C\nu^3}$. It crosses the overlap regime and reaches infinite age without a vorticity moment assumption on the late response. Combining the core and tail gives (9.24), for example with any $M \geq M_s + p_m + 6$.

For uniqueness, a homogeneous difference in the energy class has a sequence $\delta_j \downarrow 0$ with $\delta_j^{-1/2}\|u(\delta_j)\|_2^2 \to 0$, because its integral against ds/s is finite. Apply (9.26) on $[\delta_j, B]$ and let $j \to \infty$. This also proves causality. Continuity in μ on the core follows from the same Hardy estimate with one factor $|\log(t/s)|$; its beta integral is finite uniformly on K. In the free and tail equations the difference contributes $-(\mu-\mu')u_{\mu'}/s$, whose weighted pairing is bounded by $|\mu-\mu'|\|u_{\mu'}\|_{\mathcal{H}}\|u_\mu - u_{\mu'}\|_{\mathcal{H}}$ and is absorbed by the k_* coercivity. The local energy estimate also gives trace continuity. All geometric parameters stay fixed. □

Corollary 9.3. *Let $S_{0,\mu}$ and $S_{\mathrm{lin},\mu}$ denote the free and actual causal tensor responses, acting on the sum of the two components of the split stress space $\mathcal{Z} = \mathcal{Z}_1 \oplus \rho_b^{-1/2}\mathcal{T}$, with its specified norm and $0 < \rho_b \leq 1$. For a three-component cylindrical tensor these flat maps act on its (r,z) block; its hoop component is retained in the exact curved comparison. The first-component weight is at least one, so $\|F_1+F_2\|_{\mathcal{T}} \leq \|F_1\|_{\mathcal{Z}_1} + \sqrt{\rho_b}\|F_2\|_{\rho_b^{-1/2}\mathcal{T}}$. Let $M_0 u = (v\otimes u + u\otimes v, 0)$ and suppose its actual train multiplier bound is $\|M_0 u\|_{\mathcal{Z}_1} \leq C\nu^{-p_V}\|u\|_{\mathcal{H}}$. Then*

$$D_0^{-1} = (I + M_0 S_{0,\mu})^{-1} = I - M_0 S_{\mathrm{lin},\mu}, \qquad \|g^{-1}D_0^{-1}g\|_{\mathcal{Z}\to\mathcal{Z}} \leq C\nu^{-M}, \quad M = \lceil p_m + p_V + M_s + 10\rceil.$$

Indeed $b = S_{\mathrm{lin},\mu}F$ satisfies $b = S_{0,\mu}(F - M_0 b)$ by free uniqueness, giving both inverse identities. The second split input component is retained. The degree has slack for the fixed norm equivalences and trace estimate, and depends only on the fixed weights and coefficient budgets.

10 Separated stress kernels and one phase average

All phase norms use normalized measure on $\mathbb{T}_P$. Write $B = \partial_\tau - y\cdot\nabla/2$, $D_\mu = B - \nu\Delta + \mu - 1/2$, and $S_\mu = D_\mu^{-1}\mathbb{P}\,\mathrm{div}$. The exponent μ ranges over a fixed compact subset K of $\{\mathrm{Re}\,\mu > -1/4\}$. Let T be supported in the radial disk $|z'| \leq r_o$, and restrict receivers to $|y'| \geq r_i > r_o$. Put $D = r_i - r_o$. The support condition is imposed on the tensor before its divergence. We begin with smooth tensors and then complete in the norms appearing below.

Let G_t be the three-dimensional heat kernel. The heat–Leray kernel is

$$(K_t)_{ij} = \delta_{ij}G_t + \partial_i\partial_j \int_t^\infty G_a\, da.$$

Differentiation and the substitution $a = |x|^2 r$ give, off the origin,

$$|D^j\nabla K_t(x)| \leq C_j|x|^{-4-j}, \qquad |D^j(K_t - K_0)(x)| \leq C_j t|x|^{-5-j}. \tag{10.1}$$

The second bound also follows by integrating $\partial_t K_t = \Delta K_t$ from zero to t. Thus it remains valid as $t \downarrow 0$ at every separated observation point. At $t = 0$, $K_0 = D^2(4\pi|x|)^{-1}$ off the origin; its local distributional part is retained by the operator, but vanishes at a separated receiver.

The free semigroup is

$$e^{h/2}\big[e^{\nu(e^h-1)\Delta}f(\tau-h)\big](e^{h/2}y).$$

In its Laplace integral set $\lambda = e^{-h/2}$ and integrate the tensor divergence by parts. Heat scaling contributes λ^3, the divergence contributes λ, and the semigroup amplitude and differential contribute λ^{-1} and $-2d\lambda/\lambda$. Consequently the negative response $W_\mu = -S_\mu T$ is exactly

$$W_\mu(y,\tau) = -2\int_0^1 \lambda^{2\mu+2}\int \nabla K_{\nu(1-\lambda^2)}(y-\lambda z) : T(z, \tau + 2\log\lambda)\, dz\, d\lambda. \tag{10.2}$$

The radial gap persists under every dilation: $|y-\lambda z| \ge |y'| - \lambda|z'| \ge D$. No temporal derivative of T occurs.

10.1 Thin sources, repair sources, and the stream cutoff

For $b > 3/(4+j)$, integrating (10.1) outside the gap gives

$$\|1_{|x|>D} D^j \nabla K_t\|_{L^b} \le C_{j,b} D^{-4-j+3/b}.$$

In Young's inequality we use $b \ge 1$. Dilation of an L^p source contributes $\lambda^{-3+3/p}$, leaving the integrable power $\lambda^{2\operatorname{Re}\mu-1+3/p}$ in (10.2). Minkowski and phase-translation invariance therefore prove, whenever $1+1/q = 1/p+1/b$ and $\inf_{\mu\in K}(2\operatorname{Re}\mu + 3/p) > 0$,

$$\|1_{\text{young}} D^j W_\mu\|_{L^2_\tau L^q_y} \le C_{K,j,p,q} D^{-4-j+3/b} \|T\|_{L^2_\tau L^p_y}. \tag{10.3}$$

The pointwise version has $q=\infty$, $b=p'$. In particular, for $p = 22/19$,

$$\|\nabla W_\mu(y,\cdot)\|_{L^2_\tau} \le C_K D^{-101/22} \|T\|_{L^2_\tau L^{22/19}}, \tag{10.4}$$

$$\|1_{\text{young}} \nabla W_\mu\|_{L^2_\tau L^{22/3}} \le C_K D^{-46/11} \|T\|_{L^2_\tau L^{22/19}}, \tag{10.5}$$

$$\|1_{\text{young}} D^j W_\mu\|_{L^2_\tau L^2} \le C_{K,j} D^{-23/11-j} \|T\|_{L^2_\tau L^{22/19}}. \tag{10.6}$$

Here the kernel exponents are respectively $22/3$, $11/3$, and $11/7$. A repair tensor $T = r_b \otimes w + w \otimes r_b$ obeys

$$\|T\|_{L^2_\tau L^{22/19}} \le 2 \|r_b\|_{L^\infty_\tau L^{11/4}} \|w\|_{Y_2}.$$

These estimates include the Cartesian pressure projection.

For a thin source with $\int \|T(z,\cdot)\|_{L^2_\tau}\, dz \le R^2 N_0$ and $D \ge c_0 R\sqrt{\nu}$, the same argument with $p=1$ gives, for fixed $q>1$ and fixed j,

$$\|D^j W_\mu\|_{L^2_\tau L^q(\text{young})} \le P_\nu R^{-2-j+3/q} N_0. \tag{10.7}$$

The polynomial P_ν depends on the fixed derivative and exponent budgets and on the fixed fractional radial gap.

For axisymmetric no-swirl divergence-free W, choose a smooth radial cutoff χ whose transition lies outside the source disk, on an annulus of radius comparable to D. Its divergence-free stream cutoff is

$$(\mathcal{C}_\chi W)_r = \chi W_r, \qquad (\mathcal{C}_\chi W)_z = \chi W_z - \frac{\chi'(\rho)}{\rho} \int_\rho^\infty \rho' W_z(\rho', z)\, d\rho'.$$

The stream primitive converges by the separated kernel bound. The cutoff correction has a translation-invariant majorant of the form

$$C D^{-2} (D^2 + |z - z_{\text{src}}|^2)^{-1} 1_{|y'-z'_{\text{src}}| \le CD}.$$

Indeed the radial integral of the order-four kernel is bounded by $C(D^2+|z-z_{\text{src}}|^2)^{-1}$, while $\chi'/\rho = O(D^{-2})$. After j derivatives the same estimate costs D^{-j}. Its L^b norm is $CD^{-4-j+3/b}$ for $b > 1/2$, as follows by scaling the two horizontal variables and the vertical integral. Thus (10.3)–(10.7) remain valid after this cutoff in all exponents used here. The estimate is an inequality for the full tensor convolution; it does not discard a cutoff source.

Put

$$X_0 = W_\mu - W_{\mu+1}, \quad X = \mathcal{C}_\chi X_0, \quad \mathcal{Q} = [B, \mathcal{C}_\chi] - \nu[\Delta, \mathcal{C}_\chi], \quad Y = \mathcal{C}_\chi W_\mu + \mathcal{Q} X_0.$$

The difference in X_0 inserts $1-\lambda^2$ in (10.2). It costs no inverse. Derivatives of χ cost D^{-1}, $B\chi$ is bounded, and each extra Laplacian costs νD^{-2}. For $D \ge c_0 R\sqrt{\nu}$ and $R \ge 1$ this last factor is bounded. Therefore the displayed estimates also hold for X, Y, with finitely many increased derivative budgets.

10.2 One phase primitive gives an extra curvature factor

Set $\epsilon = R^{-1}$, $P = 2\epsilon c$, and $x = y/R$. Write the thin physical stress as

$$T(Rx,\tau) = F(x', Y, q_\epsilon(|x'|,\tau)), \qquad Y = x_3/\epsilon, \quad q_\epsilon(r,\tau) = \epsilon^{-1}\log r + \tau/(2\epsilon),$$

where F has q-period c. Define

$$N_j = \int (1+|Y|)^j \|F(x', Y, \cdot)\|_{L^2(dq/c)}\, dx'\, dY, \qquad j = 0, 1.$$

Then its physical phase-averaged mass is $R^2 N_0$. Changing variables in (10.2) gives

$$W_\mu = -2\epsilon^2 \int_0^1 \lambda^{2\mu+2} \int \nabla K_{\nu\epsilon^2(1-\lambda^2)}(x - \lambda(x', \epsilon Y)) F(x', Y, q_\epsilon(r', \tau + 2\log\lambda))\, dx'\, dY\, d\lambda. \tag{10.8}$$

Let $\bar F$ be its normalized phase mean, and set

$$V_{\mu,0}(x) = -2\int_0^1 \lambda^{2\mu+2} \int \nabla K_0(x - \lambda(x', 0))\bar F(x', Y)\, dx'\, dY\, d\lambda.$$

Taylor expansion in ϵY and (10.1) control the mean collapse and heat errors by first transverse moments. The strict radial gap remains valid along the entire Taylor segment.

For the oscillatory part let $J = \partial_q^{-1}$ on zero-mean periodic functions. Fourier series gives $\|J\|_{L^2(dq/c)\to L^2(dq/c)} \le c/(2\pi)$. Exactly in distributions in λ,

$$F_\perp(q_\epsilon(r', \tau + 2\log\lambda)) = \epsilon\lambda\partial_\lambda J F_\perp(q_\epsilon(r', \tau + 2\log\lambda)).$$

Integrate once by parts in (10.8). The boundary at $\lambda = 1$ is retained. The boundary at zero vanishes in normalized L^2_τ, since $2\operatorname{Re}\mu + 3 > 0$ and translations of $JF_\perp$ preserve its norm. The derivative of the kernel is

$$-(x', \epsilon Y)\cdot\nabla(\nabla K_t) - 2\nu\epsilon^2\lambda\Delta(\nabla K_t), \qquad t = \nu\epsilon^2(1-\lambda^2).$$

The factor x' is bounded by the source radial endpoint and the factor Y by N_1. Every fixed spatial observation derivative is estimated by (10.1). This proves

$$\|\nabla_y W_\mu - \epsilon^3 \nabla_x V_{\mu,0}\|_{L^2_\tau} \le P_\nu \epsilon^4 N_1 \tag{10.9}$$

on compact separated observation sets, with the stated polynomial dependence on radial endpoints, their gap, and c. No age or phase derivative of F has been used.

For the full young band $s \le 1/\nu$, old support $t \ge 1.2/\nu$, and $r = s^{-1/2}$ in outer coordinates, the same calculation gives the more explicit bounds

$$\|D_y W_{\mu,\perp}\|_{L^2_\tau} \le P_\nu \epsilon^4 s^{5/2} N_1, \tag{10.10}$$

$$\|D_y W_\mu - \epsilon^3 D_x V_{\mu,0}\|_{L^2_\tau} \le P_\nu \epsilon^4 (s^{5/2} + s^3) N_1. \tag{10.11}$$

The direct differentiated kernel contributes r^{-5}; a transverse collapse costs one further derivative, r^{-6}. In the phase calculation the extra factor x' is at most a fixed multiple of r; the Y term has the harmless factor ϵ/r, and the heat term has $\nu\epsilon^2/r^2 \le \epsilon^2$. These estimates hold with any fixed number of spatial observation derivatives, with the corresponding powers of r^{-1}. The stream cutoff estimates proved above extend them to X, Y. Moment-norm approximation proves the statements for all N_1 inputs, without assuming smoothness of the source.

10.3 Phase suprema use a separate ordinary bound

The extra ϵ in (10.10) is not asserted after arbitrary phase or conormal derivatives. Instead use the exact free equations

$$BX_0 = \nu\Delta X_0 - (\mu + 1/2)X_0 + W_\mu, \qquad BW_\mu = \nu\Delta W_\mu - (\mu - 1/2)W_\mu - \mathbb{P}\,\mathrm{div}\,T.$$

At a separated receiver the last term has an order-four kernel, and its gradient has order five. It is bounded by precisely the ordinary kernel estimates already proved. It does not require differentiating T, but need not retain the extra phase-average factor. For the cutoff terms use

$$BT_{\mathrm{rad}}W = T_{\mathrm{rad}}BW - T_{\mathrm{rad}}W, \qquad [B, \Delta] = \Delta,$$

and

$$[B, \mathcal{Q}] = [B, [B, \mathcal{C}_\chi]] - \nu[\Delta, \mathcal{C}_\chi] - \nu[\Delta, [B, \mathcal{C}_\chi]].$$

Thus BX, BY have the same ordinary separated bounds as X, Y, with a fixed increased derivative budget and the factors νD^{-2} already controlled. Also $B\partial_j = \partial_j B + \partial_j/2$.

For a global cutoff field Z among these vectors or tensors and $1 < q < \infty$, spatial integration by parts gives

$$\left|\frac{d}{d\tau}\|Z(\tau)\|_q\right| \le \|BZ(\tau)\|_q + \frac{3}{2q}\|Z(\tau)\|_q.$$

Choose a phase where the norm is no greater than its normalized L^2 average, then integrate over an interval of length at most $P \le 1$. This yields

$$\|Z\|_{L^\infty_\tau L^q} \le (1 + 3P/(2q))\|Z\|_{L^2_\tau L^q} + P\|BZ\|_{L^2_\tau L^q}. \tag{10.12}$$

In particular (10.5) for $\nabla X, \nabla Y$ holds also in phase supremum, with $D \ge c_0 R\sqrt{\nu}$ and a fixed viscosity polynomial. The thin bounds have the same upgrade. For weighted spatial suprema, apply the transport identity along $y(\tau) = e^{-\tau/2}y(0)$ to $\rho^{4+j}D^jX$ and $\rho^{4+j}D^jY$; the weight has $B\rho^{4+j} = -(4+j)\rho^{4+j}/2$. The same argument bounds the supremum in time by the L^2 average. Its input is the $L^2_\tau L^\infty_y$ bound: take the weighted spatial supremum of the separated kernel first, then apply Minkowski in the source and dilation variables. The thin $L^1_y L^2_\tau$ moment supplies this bound. A pointwise-in-space L^2_τ estimate alone would not suffice. Approximation of the source in its stated norm passes all these estimates to rough tensors.

11 The direct zero-entrance near equation

Use the normalized cylinder spaces

$$H = L^2(s^{-3/2}\,ds\,dq/c\,d\eta), \qquad H_T = L^2(s^{-1/2}\,ds\,dq/c\,d\eta), \qquad Z = Z_1 \oplus \rho_b^{-1/2}H_T,$$

where $Z_1 = L^2(s^{-1/2}W^2\,ds\,dq/c\,d\eta)$ and $W = \langle\eta/\sqrt{1+s}\rangle^{m_Z}$, $m_Z > 3/2$. The stress inverse always acts on the sum of the components. Let $\mathcal{M}_0$ be multiplication by the actual flat train, with zero second output component, and let $\mathcal{M}_U$ be the exact thin/repair multiplier. Set

$$\mathsf{D}_0 = I + \mathcal{M}_0\mathcal{S}_0, \qquad g(s) = e^{\gamma s/(1+\nu s)}, \qquad \mathsf{G}T = gT.$$

The direct causal construction on the full moving clock gives

$$\|\mathsf{G}^{-1}\mathsf{D}_0^{-1}\mathsf{G}\|_{Z\to Z} \le Q := P_\nu. \tag{11.1}$$

There are no auxiliary entrance coordinates in this statement. The ordinary norm of this inverse is not asserted to have the same bound; it can contain the full planar amplification. The polynomial degree is fixed by the velocity and tensor spaces.

The finite-curvature free inverse is the conjugate of the Cartesian weak-stress inverse. In particular its divergence and pressure are not replaced by planar ones. In the profile chart

$$s = R^2/\rho^2, \quad q = R\log(\rho/R) + R\tau/2, \quad \eta = Rz/\rho, \quad R = \epsilon^{-1},$$

its exact divergence is

$$\operatorname{div}(\sqrt{s}\, b) = s\{[\partial_q - \epsilon(2s\partial_s + \eta\partial_\eta)]b_r + \partial_\eta b_z\}.$$

The geometric comparison below concerns these exact operators.

Fix a small positive logarithmic gap δ. For an output age s and input age t, define

$$\mathcal{S}_{\mathrm{far}} = 1_{\{s<S,\ t>e^{\delta}s\}}\mathcal{S}_\epsilon, \quad \mathcal{S}_N = \mathcal{S}_\epsilon - \mathcal{S}_{\mathrm{far}}, \quad \mathsf{D}_N = I + \mathcal{M}_U\mathcal{S}_N.$$

The notation for the mask means its exact kernel realization with tensor partitions before distributional divergence. Interface terms are retained. A masked intermediate velocity need not satisfy the whole-space divergence-free constraint by itself; the stress equation is well defined independently of that issue. Only the recombined full Cartesian inverse is used to recover the final physical mode.

This D_N is distinct from subtracting the divergence-free old-to-young velocity in the nilpotent translation transformation. Their difference is retained in the transformed equation.

Proposition 11.1. *Let μ range over a fixed compact subset of $0 < \Re\mu < 1/4$. Put $s_- = \epsilon^a$, $S = \epsilon^{-b}$, $0 < b < 1$. There is a fixed integer K, chosen before a, b, such that*

$$\begin{aligned}\delta_N := \|\mathsf{G}^{-1}(\mathsf{D}_N - \mathsf{D}_0)\mathsf{G}\| \le P_\nu\big[&\epsilon^{a/2} + \epsilon^{b/2} + \epsilon e^{(m_0+\eta_\delta)/\nu} + \epsilon^{1/50-K(a+b)}e^{\eta_\delta/\nu} \\ &+ \epsilon^{3/22}e^{(\eta_\delta+m_0/2)/\nu} + e^{-(D_\perp-\eta_\delta)/\nu}\big],\end{aligned} \tag{11.2}$$

where $\eta_\delta \downarrow 0$ as $\delta \downarrow 0$. The constant $D_\perp > 0$ specifies the transverse-tail accuracy. If $Q\delta_N < 1/2$, then

$$\begin{aligned}\|\mathsf{G}^{-1}\mathsf{D}_N^{-1}\mathsf{G}\| &\le 2Q, \\ \|\mathsf{G}^{-1}(\mathsf{D}_N^{-1} - \mathsf{D}_0^{-1})\mathsf{G}\| &\le 2Q^2\delta_N.\end{aligned} \tag{11.3}$$

Both statements are equation-only estimates; no terminal functional or affine source is added.

Proof. The required free comparison is global in its input and compact in its output:

$$\|\chi(\mathcal{S}_\epsilon - \mathcal{S}_0)\|_{H_T\to H} \le C\Gamma^K\epsilon^{1/50}.$$

Here Γ is a polynomial in inverse lower age, upper age, transverse box size, viscosity inverse, cell width, and finitely many cutoff derivatives. The exponent K is fixed. The transverse cutoff may be chosen at $L_\perp = C_\perp(c/\nu + \sqrt{S} + 1)$, so $L_\perp \le C'_\perp S$ for small ν, and $\Gamma^K \le P_\nu\epsilon^{-K(a+b)}$. The actual flat train coefficient outside this box is at most $P_\nu e^{-D_\perp/\nu}$. Its curved stream and entrance adjustments are estimated separately by their global small multiplier norm, so their tails need not be Gaussian. No transverse weight is imposed on the broad repair coefficient.

Partition $[s_-, S]$ into output bins $[l, r]$ with $r/l < e^{\delta/2}$. Their number is a viscosity polynomial in the distinguished regime. The retained input region on a bin is the complete block $t < e^{\delta}l$ and the partial strip $e^{\delta}l < t < e^{\delta}r$. On the complete block, multiplication by g on inputs and g^{-1} on outputs costs at most $g(e^{\delta}l)/g(l)$. More generally

$$\sup_{s>0,\ s\le t\le Ms} \nu\left(\frac{t}{1+\nu t} - \frac{s}{1+\nu s}\right) = \frac{\sqrt{M}-1}{\sqrt{M}+1}.$$

Indeed, writing $t = ks$, $x = \nu s$, the expression is $(k-1)x/[(1+x)(1+kx)]$, maximized at $x = k^{-1/2}$. Thus all complete blocks and partial strips cost at most $e^{\eta_\delta/\nu}$, with $\eta_\delta = \gamma \tanh(3\delta/8)$ after enlarging the overlapping bins if necessary. The polynomial bin count is included in P_ν. Applying the compact-output comparison and the train multiplier proves the fourth term of (11.2); the transverse tail gives its last term.

On a partial strip the flat causal inverse is zero. Its source radius is separated from the output radius by a fixed multiple of $R/\sqrt{r}$, also after the source contraction in the resolvent. By (10.1), the heat–Leray stress kernel has the uniform positive majorant $C|x|^{-4}$, whose spatial L^1 norm outside distance d is C/d. The exact dilation integral on an L^2 source has power $\lambda^{2\Re\mu+1/2}$, hence a uniform integrability margin. Young's inequality, followed by normalized-phase Minkowski, gives $P_\nu\epsilon\sqrt{r}$ for this output-dependent strip. After gain multiplication this is at most $P_\nu e^{\eta_\delta/\nu}\epsilon^{1-b/2}$, absorbed by the fourth term. This direct positive-kernel estimate is needed; boundedness of an arbitrary triangular operator truncation is not used.

For $s < s_-$, the train multiplier has norm $P_\nu\sqrt{s_-}$. Retained inputs have $t < e^\delta s_-$, where the gain ratio is $1 + o(1)$. Subtracting the forbidden separated subblock costs at most $P_\nu\epsilon\sqrt{s_-}$, by the same positive-kernel estimate. For $s > S$, use the full free inverse and the train tail $P_\nu(S^{-1/2} + \epsilon)$; its gain ratio is bounded by

$$\exp\{\gamma/[\nu(1+\nu S)]\} = 1 + o(1).$$

These give the first two terms.

The same complete blocks, using free energy in place of the difference estimate, give a global bound without a coefficient multiplier. The positive partial-strip bounds remain valid, the lower end has gain ratio $1 + o(1)$, and the upper end has the ratio just displayed. Consequently

$$\|g^{-1}\mathcal{S}_N g\|_{H_T\to H} \le P_\nu e^{\eta_\delta/\nu}. \tag{11.4}$$

The exact thin coefficient differs from the flat train by its ϵ stream completion and fixed entrance adjustment, whose multiplier norm is $P_\nu\epsilon e^{m_0/\nu}$. For its contribution use the exact decomposition

$$\mathcal{M}_{\text{thin}}\mathcal{S}_N - \mathcal{M}_0\mathcal{S}_0 = (\mathcal{M}_{\text{thin}} - \mathcal{M}_0)\mathcal{S}_N + \mathcal{M}_0(\mathcal{S}_N - \mathcal{S}_0).$$

The first uses the global near free bound and costs $P_\nu\epsilon e^{(m_0+\eta_\delta)/\nu}$, including all tails of the entrance adjustment. The second is the geometric difference with the Gaussian train multiplier estimated above. Each broad input/output coupling is bounded by a polynomial times $\sqrt{\rho_b}$, using the unweighted free stress bound. The same bin and end-region estimates apply without a transverse weight; since $\rho_b = P_\nu\epsilon^{3/11}e^{m_0/\nu}$, this gives the fifth term. This proves the complete equation difference, including every split component.

Conjugate by G and apply the geometric inverse series to $\mathsf{D}_N = \mathsf{D}_0 + (\mathsf{D}_N - \mathsf{D}_0)$. Its norm is at most $Q\delta_N < 1/2$. The resolvent identity then gives (11.3). □

The parameters can be chosen without a terminal observation. Fix the derivative budgets and K, then choose $a, b > 0$ with $K(a+b) < 1/100$. Let

$$\theta = \min\{a/2, b/2, 1/100, 3/22\}, \qquad d_0 = \gamma/3.$$

Choose δ, m_0 so that $\eta_\delta < \theta d_0/8$, $m_0 < \theta d_0/8$, and take $D_\perp > \eta_\delta + \theta d_0$. The background construction realizes the prescribed m_0. All these choices precede $\nu \downarrow 0$. For $\epsilon = C_*\nu^r e^{-d_0/\nu}$, uniformly on bounded r and positive compact C_*-ranges,

$$\delta_N \le P_\nu e^{-\theta d_0/(2\nu)}, \qquad Q^2\delta_N \le P_\nu e^{-\theta d_0/(2\nu)}.$$

Thus the inverse and its comparison close with a positive exponential margin. The choice is uniform over the spectral interval while all geometric parameters and the exact background are held fixed.

11.1 The actual forced column

Let f be a normalized body source, so that its physical representative is $\mathcal{N}_v f$, and set

$$u_\epsilon = \mathcal{M}_U \mathcal{R}_\epsilon f, \qquad T_\epsilon = \mathsf{D}_N^{-1} u_\epsilon, \qquad b_\epsilon = \mathcal{R}_\epsilon f - \mathcal{S}_N T_\epsilon.$$

The first row of the equation gives exactly $T_\epsilon = \mathcal{M}_U b_\epsilon$. Hence both its thin and repair components are actual polarized stresses, even before full spectral closure. Free body coercivity and $g^{-1} \le 1$ give

$$\|g^{-1} T_\epsilon\|_Z \le P_\nu \|f\|_H.$$

By (11.4),

$$\|g^{-1} b_\epsilon\|_H \le P_\nu e^{\eta_\delta/\nu} \|f\|_H.$$

The remaining exponent can be made as small as required before viscosity is chosen. These estimates do not assert that the intermediate masked velocity is the final whole-space mode. They apply to the prescribed body input. For an arbitrary equation input of the form $\mathcal{M}_U v_{\mathrm{in}}$, the corresponding identity is $b = v_{\mathrm{in}} - \mathcal{S}_N T$, and its velocity bound retains $\|g^{-1} v_{\mathrm{in}}\|_H$. A small stress norm alone does not control that velocity witness.

If a fixed smooth Gaussian affine force is used in both geometries, write $u_0 = \mathcal{M}_0 \mathcal{R}_0 f$, $T_0 = \mathsf{D}_0^{-1} u_0$. Suppose its H norm is a fixed viscosity polynomial. The body version of the same output comparison, with rate $\epsilon^{1/14}$, and the coefficient/end-region bounds above give $\|g^{-1}(u_\epsilon - u_0)\|_Z \le P_\nu e^{-c/\nu}$ for some fixed $c > 0$. No gain factor is needed on the input here: f is the prescribed unit force, not an amplified stress history. The resolvent identity therefore yields

$$\|g^{-1}(T_\epsilon - T_0)\|_Z \le 2Q \|g^{-1}(u_\epsilon - u_0)\|_Z + 2Q^2 \delta_N \|g^{-1} u_0\|_Z \le P_\nu e^{-c'/\nu}.$$

In particular any old tensor moment whose norm after gain multiplication is a viscosity polynomial has the same limit on these two columns.

12 The exact Ward change on the multiplier range

Fix the exact real unforced profile U and its full pressure p:

$$D_0 U + U \cdot \nabla U + \nabla p = 0, \qquad D_\mu = \partial_\tau - \tfrac{1}{2} y \cdot \nabla - \nu \Delta + \mu - \tfrac{1}{2}, \qquad \operatorname{div} U = 0.$$

Use normalized phase measure $d\tau/P$, the prescribed orthogonal symmetry, and the whole-space free inverses $R_\mu = D_\mu^{-1}$ and $S_\mu = R_\mu \mathbb{P} \operatorname{div}$. Here R_μ acts on divergence-free body forces. Fix $K \Subset (0, 1/4)$ and all geometric parameters. Put

$$M_U w = U \otimes w + w \otimes U, \qquad \mathcal{D}_\mu^T = I + M_U S_\mu.$$

We use only physical stresses $T = M_U w$ represented in the actual pair graph of the final return. Its velocity witness $w \in Y_2$ has the stated symmetry and need not be divergence-free. Since U is bounded, $T \in Y_2$; its old thin part has finite first transverse moment N_1, and its old repair part belongs to $L^2_\tau L^{22/19}$. The multiplier, moment and repair-product bounds establish these facts and are preserved by approximation.

12.1 The full old source and its outward stream cutoff

Let $r = |y'|$, $s = R^2/r^2$ and $R = \epsilon^{-1}$. Take smooth cutoffs

$$\chi_o = 0 \ (s \le 1.2/\nu), \quad \chi_o = 1 \ (s \ge 1.5/\nu), \qquad \chi_y = 1 \ (s \le 0.8/\nu), \quad \chi_y = 0 \ (s \ge 1/\nu).$$

Thus the old source disk $r \le R\sqrt{\nu/1.2}$ is separated from the young receiver $r \ge R\sqrt{\nu}$. Set

$$W_\mu = -S_\mu(\chi_o T), \qquad X_0 = W_\mu - W_{\mu+1}, \qquad X = \mathcal{C}_y X_0, \quad Y = D_{\mu+1}X, \quad h = [U, X]. \tag{12.1}$$

Here $[U, X] = U \cdot \nabla X - X \cdot \nabla U$, and the tensor $\chi_o T = \chi_o M_V w + \chi_o M_{r_b} w$ is identical in both resolvents. In particular the full repair stress enters before inversion and before taking divergence.

For these radially separated divergence-free fields define directly

$$\Psi_Z(r, z) = -\int_r^\infty \rho Z_z(\rho, z)\, d\rho, \qquad \mathcal{C}_y Z = \chi_y Z + \frac{\chi_y'}{r}\Psi_Z e_z.$$

The phase is suppressed. The separated-kernel estimates prove convergence of this primitive and its differentiated versions for the thin N_1 and repair $L_\tau^2 L^{22/19}$ sources. Their decay makes the boundary term rZ_r at infinity vanish. Hence $\partial_r \Psi_Z = rZ_z$ and $\partial_z \Psi_Z = -rZ_r$. The cutoff is divergence-free, equals Z on its plateau and vanishes in the old region; define it by zero where $\chi_y = 0$. The same kernels control the stream correction and commutators, so $X, DX, Y \in Y_2$ and, since U, DU are bounded, $h \in Y_2$. These are fixed-parameter bounds on the actual multiplier range.

The full free resolvent identity gives

$$X_0 = R_{\mu+1} W_\mu, \qquad D_{\mu+1} X_0 = W_\mu. \tag{12.2}$$

Consequently the exact cutoff identity is

$$Y = \mathcal{C}_y W_\mu + [D_{\mu+1}, \mathcal{C}_y] X_0. \tag{12.3}$$

It follows first for smooth graph inputs and then by the moment, repair-product and separated-kernel bounds. For example direct integration of the outward primitive gives $\Psi_{AZ} = A\Psi_Z + \Psi_Z$ for $A = -y \cdot \nabla/2$, and $\Psi_{\Delta Z} = (\partial_r^2 - r^{-1}\partial_r + \partial_z^2)\Psi_Z$; the same decay removes the boundary terms. Thus this identity does not require a choice of stream primitive near the axis. Relative to a scalar-cutoff reference return, both $(\mathcal{C}_y - \chi_y)W_\mu$ and $[D_{\mu+1}, \mathcal{C}_y]X_0$ remain. They are supported on the transition annulus and are separately estimated below.

12.2 Derivation with the exact pressure

We derive the identity before projecting. In physical coordinates let u solve the unforced Navier–Stokes equation with pressure p_{phys}, let X be divergence-free, and set

$$\mathsf{D} = \partial_t - \nu\Delta, \quad k(t) = \mu/t, \quad \mathsf{Y} = (\mathsf{D} + k)\mathsf{X}, \quad \mathsf{h} = [u, \mathsf{X}].$$

Repeated indices are summed, and $(D\mathsf{X})_{ij} = \partial_j \mathsf{X}_i$. The product rule yields

$$(\mathsf{D} + k)\mathsf{h}_i = [\mathsf{D}u, \mathsf{X}]_i + [u, \mathsf{Y}]_i - 2\nu \mathsf{C}_i, \quad \mathsf{C}_i = \sum_{j,l} [(\partial_j u_l)\partial_{jl}\mathsf{X}_i - (\partial_j \mathsf{X}_l)\partial_{jl} u_i].$$

For $N = u \cdot \nabla u$, expansion and cancellation give

$$-[N, \mathsf{X}]_i + u_j \partial_j \mathsf{h}_i + \mathsf{h}_j \partial_j u_i = u_j u_l \partial_{jl} \mathsf{X}_i.$$

The pressure identity is

$$-[\nabla p_{\mathrm{phys}}, \mathsf{X}]_i = \partial_i(\mathsf{X}_j \partial_j p_{\mathrm{phys}}) - (\partial_j \mathsf{X}_i + \partial_i \mathsf{X}_j)\partial_j p_{\mathrm{phys}}.$$

Using $\mathsf{D}u = -N - \nabla p_{\mathrm{phys}}$ and $[u, \mathsf{Y}] = -(u \cdot \nabla \mathsf{Y} + \mathsf{Y} \cdot \nabla u) + 2u \cdot \nabla \mathsf{Y}$, these formulas prove the desired identity. The global Leray projection removes precisely $\nabla(\mathsf{X} \cdot \nabla p_{\mathrm{phys}})$.

Now substitute

$$u = t^{-1/2} U(y, \tau), \quad \mathsf{X} = \sqrt{t}\, X(y, \tau), \quad p_{\mathrm{phys}} = t^{-1} p(y, \tau), \quad y = x/\sqrt{t}, \quad \tau = \log t.$$

Then $\mathsf{Y} = t^{-1/2} D_{\mu+1} X$ and $\mathsf{h} = t^{-1/2}[U, X]$; every term in the differential identity has factor $t^{-3/2}$. Consequently, in profile variables,

$$D_\mu h + \mathbb{P}\,\mathrm{div}\, M_U h = -\mathbb{P}\,\mathrm{div}\, M_U Y + \mathbb{P}\widetilde{E}_\mu, \tag{12.4}$$

where the full unprojected residual is

$$\begin{aligned} \widetilde{E}_\mu &= 2U \cdot \nabla Y - 2\nu C_{\mathrm{diff}} + (U \otimes U) : D^2 X - (DX + DX^{\mathsf{T}})\nabla p, \\ (C_{\mathrm{diff}})_i &= \sum_{j,l} [(\partial_j U_l)\partial_{jl} X_i - (\partial_j X_l)\partial_{jl} U_i]. \end{aligned} \tag{12.5}$$

All four terms are retained. In particular the pressure is the exact global pressure of U, including the repair. A spatially constant displacement has zero residual; spatial derivatives of the returned displacement and velocity produce the remaining force.

For actual graph inputs this identity holds distributionally by symmetry-preserving smooth approximation of w, followed by the multiplier, moment, repair-product and separated-kernel bounds. The energy field $h + S_\mu M_U(Y + h)$ has distributional D_μ equal to $\mathbb{P}\widetilde{E}_\mu$. The actual-range body estimates below show $\widetilde{E}_\mu \in Y_2$: the repair part is bounded directly, and the thin part differs from its Gaussian affine body by a Y_2 error. Free weak uniqueness therefore gives

$$h + S_\mu M_U(Y + h) = R_\mu \mathbb{P}\widetilde{E}_\mu.$$

Multiplying by M_U and adding $M_U Y$ yields precisely the stress identity used in the final return:

$$\mathcal{D}_\mu^T M_U(Y + h) = M_U Y + M_U R_\mu \mathbb{P}\widetilde{E}_\mu. \tag{12.6}$$

12.3 Support and invertibility on the actual range

Both Y and h are divergence-free and supported in $s \le 1/\nu$: $D_{\mu+1}$ preserves the divergence-free condition, the bracket of divergence-free fields is divergence-free, and these operations are local after the stream cutoff. No projection is applied to $Y + h$. Thus

$$\chi_o M_U(Y + h) = 0. \tag{12.7}$$

The change $H_o T = M_U(Y[T] + h[T])$ maps the actual multiplier range to itself by the multiplier bounds. It depends only on $\chi_o T$ and has young support, so $H_o^2 = 0$ and its inverse change is $I - H_o$. The same argument applies after adding the later affine correction: it depends only on old data and is supported in $s \le 1.1/\nu$, still disjoint from the old source. No smallness of these changes is required.

In particular a nonzero actual stress T gives a nonzero $T' = (I + H)T$. If the transformed equation gives $\mathcal{D}_\mu^T T' = 0$, then $b = -S_\mu T'$ satisfies $T' = M_U b$ and is a nonzero divergence-free energy field. This implication uses neither injectivity of M_U nor an isomorphism of full ambient velocity and stress kernels. The final return below supplies the nonzero T and retains both stream transitions, the full pressure, and all old repair terms in its exact error decomposition.

13 A direct map from young forcing to the old stress

This section turns the actual train's forward estimate and the all-age outputs of Section 9.2 into a uniform map on young forcing. The physical pressure and curved-space comparison are treated separately.

Put $a(s) = 1+\nu s$, $z(s) = s/a(s)$, $\theta(s) = 1/[\nu a(s)]$, $e = (\nu c^2)^{-1}$, and $c = \nu^{-2}$. Fix $K \Subset \{0 < \operatorname{Re}\mu < 1/4\}$, $1 < \alpha < 1.2$, and a fixed $\Sigma > \alpha$. Write $H = \Sigma/\nu$, $T_f = \alpha/\nu$, and $u_o = \nu^{-3/2}$. The auxiliary time s is distinct from physical logarithmic time. All cell norms use the unnormalized spatial measure. The weighted vorticity space is the centrally even, zero-circulation subspace $X_{m,c}$, with a fixed $m > 8$; its first moments vanish.

Let $\mathcal{U}(s,t)$ be the actual train evolution, and let $q_{\pm,s}$ and $\ell_{\pm,s}$ be the heat-scaled, periodized frozen Gaussian pair and its finite Gram-normalized dual. This Gram normalization only makes $\ell_{i,s}q_{j,s} = \delta_{ij}$; it is not a terminal measurement. Set

$$F_\pm(s,t) = \exp[\lambda_\pm(z(s)-z(t))]\left(\frac{a(s)}{a(t)}\right)^{\kappa_\pm}, \quad (\lambda_+,\kappa_+) = (\lambda,\kappa), \quad (\lambda_-,\kappa_-) = (\bar\lambda,\bar\kappa).$$

The two diagonal factors have equal modulus. On $0 \le s \le H$, the direct/dual residual is δ_c and the following are the retained block estimates: relative complementary propagation is $Ce^{-g(s-t)}$, the direct amplitude error is $C\delta_c$, and both amplitude/complement couplings are $C(\nu+\delta_c)$. One has

$$\delta_c = C_\Sigma(c^{-2} + \delta_{\text{base}}), \qquad \delta_{\text{base}} = P_\nu e^{C_\Sigma/\nu - b_\Sigma c^2}, \qquad \eta_\nu = \nu + \delta_c/\nu = O(\nu).$$

These bounds follow from the block construction in Proposition 9.1: on $a \le 1+\Sigma$, $\theta^{-1} = \nu a$, $d^{-2} = e/\theta = O_\Sigma(c^{-2})$, and conversion to physical auxiliary time divides the generator by a^2. Its two diagonal entries are $\lambda_\pm/a^2 + \kappa_\pm\nu/a$. Moreover $z(s) - z(t) \ge (s-t)/(1+\Sigma)^2$, and the endpoint heat-scaling maps have uniform norms on this finite band. Thus the moving complementary decay is $Ce^{-g_\Sigma(s-t)}$ and the stated direct and coupling errors follow.

Divide the two amplitudes by $F_\pm(s,t)$ and the complement by their common modulus. If A and R are the resulting amplitude and complementary norms, variation of constants gives

$$R(s) \le Ce^{-g(s-t)}R(t) + C(\nu+\delta_c)\int_t^s e^{-g(s-r)}A(r)\,dr.$$

Insert this estimate in the amplitude equation. Its deviation from the initial amplitude is bounded by

$$C\left[\delta_c(s-t) + (\nu+\delta_c) + (\nu+\delta_c)^2(s-t)\right](|A(t)| + R(t))$$

times a Gronwall factor bounded uniformly on $s - t \le \Sigma/\nu$. Indeed the coefficient of that factor is $C[\delta_c + (\nu+\delta_c)^2]$, whose integral is bounded and tends to zero. Consequently, since $H - t \ge (\Sigma-\alpha)/\nu$ for $0 \le t \le T_f$,

$$\mathcal{U}(H,t) = \sum_{j=\pm} F_j(H,t)q_{j,H}\ell_{j,t} + |F_+(H,t)|\,O_{X_{m,c}\to X_{m,c}}(\eta_\nu). \tag{13.1}$$

The exponentially decaying initial complement is absorbed in η_ν; the full error is not claimed exponentially small. Keeping that complement in the same inequalities gives the full gain bound on every subinterval.

Define the actual bare columns by

$$\phi_{\nu,+}(s) = \mathcal{U}(s,0)P_cq, \qquad \phi_{\nu,-}(s) = \overline{\phi_{\nu,+}(s)}.$$

Using (13.1) at $t = 0$ shows

$$\phi_{\nu,j}(H) = F_j(H,0)q_{j,H} + |F_+(H,0)|O(\eta_\nu).$$

Thus the same estimate may be written with these actual bare columns, with coefficient $F_j(t,0)^{-1}\ell_{j,t}$ and the same relative $O(\eta_\nu)$ error. This is an approximate covector derived directly from the frozen pair; no exact transported observer is used.

13.1 A uniform tested estimate for all later ages

Fix the large clock b used in the long-clock comparison. The kinetic, tested-moment and absolute-stress impulse estimates (9.14), (9.16) and (9.17) apply uniformly through overlap. Only their value at $r = b$ is needed here.

For a state at H, (9.5) propagates it to b with bound

$$\|a_b^2 w(s_b, \sqrt{a_b}\,\cdot)/N_\nu\|_{X_{m,d(b)}} \le C_b |F_+(H,0)|^{-1} \|w(H)\|_{X_{m,c}}, \qquad N_\nu = e^{\lambda/\nu}\nu^{-\kappa}.$$

Above b the same estimate has an exponentially decaying factor $e^{-\gamma\theta}\theta^{-\operatorname{Re}\kappa}$. At b, the uniform Biot–Savart and moment embedding applies since $d(b) \ge 1$. Equations (9.16) and (9.17) therefore show that every old weighted moment and absolute stress functional used in the leading strain calculation has norm at most

$$C_K |F_+(H,0)|^{-1} \|w(H)\|_{X_{m,c}} \tag{13.2}$$

after division by its canonical scale $|N_\nu|\nu^{3+2\operatorname{Re}\mu}/c$. No ordinary inverse factor or logarithmic loss occurs here.

13.2 Young source theorem and angular signs

Let f be supported in $0 < s \le T_f$ and valued in $X_{m,c}$. Consider the zero-entrance causal equation

$$(\partial_s + \mu/s)w = L_{\nu,c}(s)w + f(s), \qquad s^\mu w(s) \longrightarrow 0 \quad (s \downarrow 0).$$

The tested source norm below defines a causal old-output theorem; it alone does not imply a physical energy entrance. For the Gaussian affine sources actually used here, $f(s) = O(s^{3/2})$ in the fixed smooth weighted norms near zero. The local bounded forward evolution then gives $w(s) = O(s^{5/2})$ by Duhamel, uniformly for $\mu \in K$. Thus its planar velocity has finite entrance energy $\int_0^1 s^{-3/2}\|K_c w(s)\|_2^2 ds < \infty$; the same fixed derivative bounds give the required local derivative integrability. The curved physical reconstruction still uses the separate free graph and comparison theorem. A general source in the tested completion must not be assigned these stronger properties without checking them. Define the finite tested norm and explicit frozen covectors

$$\|f\|_{\mathcal{I}_{\nu,\mu}} = \int_0^{T_f} t^{\operatorname{Re}\mu} |F_+(t,0)|^{-1} \|f(t)\|_{X_{m,c}}\, dt,$$
$$a_{\nu,\pm}(f) = \int_0^{T_f} t^\mu F_\pm(t,0)^{-1} \ell_{\pm,t} f(t)\, dt.$$

Their bounds are viscosity-independent: $|a_{\nu,+}| + |a_{\nu,-}| \le C_K \|f\|_{\mathcal{I}_{\nu,\mu}}$. Since $a(t) \le 1 + \alpha$ on the source band, the norm is bounded by $C_K \int_0^{T_f} t^{\operatorname{Re}\mu} e^{-\gamma t/(1+\alpha)} \|f(t)\|\, dt$. For a spectral scan, keep the source space fixed: choose a slightly larger compact strip with endpoints $\sigma_- < \operatorname{Re}\mu < \sigma_+$ and use t^{σ_-} on $t \le 1$ and t^{σ_+} on $t \ge 1$ in the defining norm. The strict margins absorb powers of $|\log t|$, so the displayed covectors and tested outputs are holomorphic in μ for holomorphic source families. All geometry, coefficients and cutoffs remain fixed during this scan.

Duhamel and (13.1) give exactly, at H,

$$H^\mu w(H) = a_{\nu,+}(f)\phi_{\nu,+}(H) + a_{\nu,-}(f)\phi_{\nu,-}(H) + r_H, \quad \|r_H\| \le C_K \eta_\nu |F_+(H,0)| \|f\|_{\mathcal{I}_{\nu,\mu}}.$$

For $s \ge H$ both sides evolve by the same unforced equation. Apply (13.2): the normalized old moment and absolute stress errors are at most $C_K \eta_\nu \|f\|_{\mathcal{I}_{\nu,\mu}}$. This is an operator statement on the specified source class, not only a calculation for one chosen forcing.

Put $J(\mu)=\mathcal{J}(\mu)$, $J_+(\mu)=J(\mu)$ and $J_-(\mu)=\overline{J(\bar\mu)}$, and set $\mathcal{A}_\nu=|N_\nu|\nu^{3+2\operatorname{Re}\mu}/c$. The canonical Mellin limit and stress defect, with the opposite angular identities of the two harmonics, give uniformly on the source class

$$I_T^\mu(w)=-\frac{3\pi i}{4c}(\mu+3/2)\nu^{3+2\mu}[a_{\nu,+}(f)N_\nu J_+(\mu)-a_{\nu,-}(f)\overline{N_\nu}J_-(\mu)]+o_K(\mathcal{A}_\nu)\|f\|_{\mathcal{I}_{\nu,\mu}}. \tag{13.3}$$

Here I_T^μ is the integral of $(2E_q-E_\eta)s^{-3/2}$, where, for the already Floquet-weighted solution w in this theorem,

$$E_\eta=\frac{2}{c}\int v_1(K_cw)_1,\qquad E_q=\frac{2}{c}\int v_2(K_cw)_2,\qquad C=\frac{1}{c}\int[v_1(K_cw)_2+v_2(K_cw)_1].$$

These are bilinear pairings, without complex conjugation or another factor $s^{-\mu}$. For real μ and real forcing, $a_-=\bar a_+$, so the leading term of (13.3) is $[3\pi/(2c)](\mu+3/2)\nu^{3+2\mu}\operatorname{Im}(a_{\nu,+}N_\nu J(\mu))$. The Floquet parameter is not conjugated in the negative channel when it is complexified.

The source is zero before the old analytical tail begins, since $T_f<u_o$. Thus the old moment law is homogeneous. Its lower endpoint is exponentially small after division by N_ν and its upper weighted endpoint vanishes; the old cutoff transition below u_o is retained in its separate earlier-old contribution.

14 The signed old-stress response to young affine forcing

Fix a real compact interval $K\Subset(0,1/4)$ and $c=\nu^{-2}$. Cell L^2 norms use the unnormalized measure $dq\,d\eta$ on $(-c/2,c/2)\times\mathbb{R}$; stress averages contain c^{-1}. Velocity components are ordered (r,ϕ,z), with zero azimuthal velocity. The vorticity sources are centrally even and have zero circulation. Let

$$V_G(x,s)=a(s)^{-2}e^{-|x|^2/(4a(s))}Jx,\qquad a(s)=1+\nu s,\qquad V_j=V_G(q-jc,\eta,s).$$

Write $E_s=\operatorname{curl}V_G(\cdot,s)$, and let $H_{s,c}$ and $R_{s,c}$ be the periodizations of $rE_s'(r)e^{2i\vartheta}$ and $rE_s'(r)+2E_s(r)$ respectively. The Gaussian defines the source; the evolution operator $L_{\nu,c}(s)$ uses the actual cylinder train. Take $\chi_f(s)=\widehat\chi_f(\nu s)$ with $\widehat\chi_f=1$ on $[0,1]$ and support in $[0,1.1)$, and take $\chi_o=0$ for $s\le1.2/\nu$ and $\chi_o=1$ for $s\ge1.5/\nu$. Set

$$I_T(M)=\int_0^\infty\chi_o(s)s^{-3/2}\frac{1}{c}\int_{\text{cell}}(2M_{zz}-M_{rr})\,dq\,d\eta\,ds,\qquad \ell_\mu(M)=\frac{\epsilon^3I_T(M)}{8(\mu+3/2)},$$
$$N_\nu=e^{\lambda/\nu}\nu^{-\kappa},\quad \gamma=\operatorname{Re}\lambda,\quad \eta_\nu=\nu+\delta_c/\nu,\quad \mathcal{A}_\nu=|N_\nu|\nu^{3+2\mu}/c,\quad A_*=\epsilon^3\mathcal{A}_\nu.$$

Fix $0<a<\gamma/16$. Let $\mathscr{B}_a$ consist of measurable real trace-free matrices $B(s)$, with no mixed azimuthal entries, supported in $0<s<1.1/\nu$, with norm

$$\|B\|_a=\operatorname*{ess\,sup}_{s>0}e^{-as}s^{-3/2}|B(s)|.$$

For $x_j=(q-jc,\eta)$ define the full affine force

$$f_B=-P_0\sum_j[(B_{\text{mer}}x_j)\cdot\nabla V_j+BV_j].$$

Decompose

$$B_{\text{mer}}=\beta I+\begin{pmatrix}d&e\\e&-d\end{pmatrix}+\omega\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad B_{\phi\phi}=-2\beta.$$

The antisymmetric contribution is a periodic gradient, annihilated by P_0. The vorticity source is

$$g_B=-\tfrac12(d-ie)H_{s,c}-\tfrac12(d+ie)\overline{H_{s,c}}-\beta R_{s,c}.$$

Let

$$\xi_B(s) = s^{-\mu}\int_0^s t^\mu \mathcal{U}_{\nu,c}(s,t)g_B(t)\,dt, \qquad b_B = K_c\xi_B, \qquad M_B = v\otimes b_B + b_B\otimes v.$$

Since $g_B = O(s^{3/2})$ near zero, $\xi_B = O_{X_{m,c}}(s^{5/2})$; this causal column has finite-energy entrance. Define its limiting amplitude

$$a_0(B) = -\tfrac{1}{2}\ell_+(H_0)\int_0^\infty t^\mu e^{-\lambda t}(d(t) - ie(t))\,dt.$$

Proposition 14.1 (Uniform affine-source return)**.** *There is a canonical modulus $\omega_{\mathrm{can}}(\nu)\to 0$ such that, uniformly on K and $\mathscr{B}_a$,*

$$|a_{\nu,+}(B) - a_0(B)| \le C_{K,a,\Sigma}\eta_\nu\|B\|_a, \tag{14.1}$$

where $a_{\nu,+}$ is the frozen-source amplitude of the young-source theorem. For every fixed $0<\gamma'<\gamma$,

$$\left|\ell_\mu(M_B) + \frac{3\pi}{16}\frac{\epsilon^3\nu^{3+2\mu}}{c}\,\mathrm{Re}[ia_0(B)N_\nu J(\mu)]\right| \le C_{K,a,\Sigma}A_*\big[\omega_{\mathrm{can}} + \eta_\nu + P_\nu e^{-\gamma'/\sqrt{\nu}}\big]\|B\|_a.$$

For $k=0,1$ one also has

$$\int_0^\infty \chi_o(s)s^{-3/2-k/2}\frac{1}{c}\int_{\mathrm{cell}}|M_B(s,q,\eta)|\,dq\,d\eta\,ds \le C_{K,a,\Sigma,k}\mathcal{A}_\nu\nu^k\|B\|_a. \tag{14.2}$$

Proof. The explicit amplitude is

$$a_{\nu,+}(B) = \int_0^{1.1/\nu} t^\mu F_+(t,0)^{-1}\ell_{+,\nu t,c}g_B(t)\,dt, \qquad F_+(t,0)^{-1} = e^{-\lambda t/(1+\nu t)}(1+\nu t)^{-\kappa}.$$

The direct/dual Gaussian comparisons and heat scaling give

$$\ell_{+,\nu t,c}H_{t,c} = \ell_+(H_0) + O(\nu t + c^{-2}), \qquad \ell_{+,\nu t,c}\overline{H_{t,c}},\ \ell_{+,\nu t,c}R_{t,c} = O(c^{-2}),$$

$$|F_+(t,0)^{-1} - e^{-\lambda t}| \le C\nu(t+t^2)e^{-\gamma t/3}, \qquad 0\le t\le 1.1/\nu.$$

The resulting error has integrable majorant $C(\eta_\nu + \nu(t+t^2))t^{\mu+3/2}e^{-(\gamma/3-a)t}\|B\|_a$, proving (14.1). The radial and opposite harmonic sources have zero limiting projection; their source norms need not be small.

Apply the young-source theorem with $\alpha = 1.1$. Its tested source norm is at most $C\|B\|_a$, since $t/(1+\nu t)\ge t/2.1$ on its support. Thus (13.2) transports the finite-band error to the whole old output without a polynomial loss. The real form of (13.3) gives

$$I_T(M_B; s\ge u_o) = \frac{3\pi}{2c}(\mu+3/2)\nu^{3+2\mu}\,\mathrm{Im}[a_{\nu,+}(B)N_\nu J(\mu)] + O_K(\mathcal{A}_\nu(\eta_\nu+\omega_{\mathrm{can}}))\|B\|_a, \quad u_o = \nu^{-3/2}.$$

The earlier-old part has bound $\mathcal{A}_\nu P_\nu e^{-\gamma'/\sqrt{\nu}}\|B\|_a$ by the full forward gain estimate. The old cutoff acts on the full tensor. The source ends before that cutoff begins, so the old moment law is homogeneous; both weighted endpoints were retained in the canonical calculation. Substitute (14.1) and divide by $8(\mu+3/2)/\epsilon^3$ to obtain the scalar estimate.

For the absolute moments use $s^{-k/2} = \nu^k\theta^{k/2}(1-\nu\theta)^{-k/2}$. On $\theta\le b$, apply (9.17) with $p = \mu+3/2+k/2$. On $b\le\theta\le\theta(u_o)$ the denominator is bounded and the exponential long-clock factor absorbs $\theta^{k/2}$. The earlier-old exponential bound absorbs its fixed polynomial age factors. Apply these estimates to both bare columns and to the uniform frozen-pair error, using the state-at-H bound (13.2). This proves the absolute estimates on the whole source unit ball. □

Corollary 14.2 (Unit signed coefficient)**.** *Put*

$$D_0 = \operatorname{diag}(12, -3, -9), \qquad A_\mu(s) = \frac{\chi_f(s) s^{5/2}}{\mu + 5/2} D_0, \qquad B_\mu = A_\mu' + \frac{\mu}{s} A_\mu = r_\mu(s) D_0,$$

$$r_\mu(s) = \chi_f(s) s^{3/2} + \frac{\chi_f'(s) s^{5/2}}{\mu + 5/2}, \qquad L(\mu) = \ell_+(H_0) \frac{\Gamma(\mu + 5/2)}{\lambda^{\mu+5/2}}, \qquad \mathcal{C}(\mu) = \frac{63\pi i}{128} L(\mu) J(\mu).$$

Let $(f_\mu, \xi_\mu, b_\mu, M_\mu) = (f_{B_\mu}, \xi_{B_\mu}, b_{B_\mu}, M_{B_\mu})$*. For some* $b > 0$*,*

$$\left| \ell_\mu(M_\mu) - 2\epsilon^3 \nu^{5+2\mu} \operatorname{Re}[\mathcal{C}(\mu) N_\nu] \right| \le C_K A_* [\omega_{\mathrm{can}} + \eta_\nu + P_\nu e^{-b/\nu} + P_\nu e^{-\gamma'/\sqrt{\nu}}].$$

Proof. Here $\beta = 3r_\mu/2$, $d = 21 r_\mu/2$ and $e = \omega = 0$, so

$$g_\mu = -\frac{21}{4} r_\mu (H_{s,c} + \overline{H_{s,c}}) - \frac{3}{2} r_\mu R_{s,c}. \tag{14.3}$$

This retains the trace of the (r, z) block $3r_\mu$ and hoop entry $-3r_\mu$. The unit $\mathscr{B}_a$ norm is bounded. Its cutoff derivative is supported above $1/\nu$, so the explicit Laplace integral gives $a_0(B_\mu) = -(21/4)L(\mu) + O_K(P_\nu e^{-b/\nu})$. Substitute this into the general scalar formula and use $c^{-1} = \nu^2$. □

15 The return on a fixed space of young displacements

Fix $K \Subset (0, 1/4)$, an integer $k = 6$, and $a_s > 0$. Choose $a_s < \gamma/16$ and smaller than the fixed exponential reserves in the curved near comparison. All constants below may depend on a_s, k, K. Let $\mathscr{A}$ be the real Banach space of C^k matrix functions on $(0, \infty)$, trace free and with no entry having exactly one azimuthal index, with norm

$$\|A\|_{\mathscr{A}} = \sup_{s>0} e^{-a_s s} s^{-5/2} \sum_{j=0}^{k} |(s\partial_s)^j A(s)|.$$

This space is independent of ν, ϵ, μ. Its elements need not be symmetric. Choose two fixed smooth cutoffs. The stream cutoff $\chi_y(s) = \widehat{\chi}_y(\nu s)$ equals one below $4/(5\nu)$ and zero above $1/\nu$. The source cutoff χ_f equals one through $1/\nu$ and zero above $1.1/\nu$. Put

$$A^y = \chi_f A, \qquad B_{\mu,\nu}[A] = (A^y)' + \frac{\mu}{s} A^y.$$

Then $\|B_{\mu,\nu}[A]\|_{\mathscr{B}_{a_s}} \le C_K \|A\|_{\mathscr{A}}$, with the source norm of the preceding section. The matrix cutoff derivative is included in this definition. Let $\mathfrak{T}_{\mu,\nu} A$ denote the complete flat stress of the zero-entrance forced column with Gaussian age force $f_{B_{\mu,\nu}[A]}$. The actual cylinder train remains in this evolution.

We next define a return map which retains the entire mean-thin kernel. For an old thin tensor M, use its normalized phase mean and set

$$E_{ij}(u) = \frac{\chi_o(u)}{c} \int_{\mathrm{cell}} M_{ij}(u, q, \eta)\, dq\, d\eta.$$

For $x' = (\rho \cos\varphi, \rho \sin\varphi)$, $u = \rho^{-2}$, let $\Sigma_M(x')$ be the Cartesian rotation of the cylindrical tensor $\rho^{-1} E(u)$, embedded on $x_3 = 0$. The zero azimuthal row and column are retained as zero entries; the rz and zr entries are retained. Define the exact zero-width Newton field by

$$V_{\mu,M}(x) = -2 \int_0^1 t^{2\mu+2} \int_{\mathbb{R}^2} D^3 N(x - t(x', 0)) : \Sigma_M(x')\, dx'\, dt, \qquad N(x) = \frac{1}{4\pi|x|}.$$

The contraction is the velocity contraction of ∇K_0, with $K_0 = D^2 N$ away from the source. Put

$$W^{\mathrm{mt}}_{\mu,M}(y) = \epsilon^2 V_{\mu,M}(\epsilon y), \qquad X^{\mathrm{mt}}_{\mu,M} = \mathcal{C}_{\chi_y}(W^{\mathrm{mt}}_{\mu,M} - W^{\mathrm{mt}}_{\mu+1,M}),$$

where the same tensor M is used in the shifted field. The stream cutoff uses the outward primitive, so the field is fixed without an additional entrance constant. Define

$$(\mathcal{L}_{\mu,\nu} M)(s) = D_y X^{\mathrm{mt}}_{\mu,M}(R/\sqrt{s}, 0, 0) \quad \text{in the } (r, \phi, z) \text{ frame.} \tag{15.1}$$

This is the full physical gradient, including its hoop entry and all stream-cutoff terms. It is trace free, has the required azimuthal symmetry, and vanishes for $s \geq 1/\nu$. The uncut field is a harmonic gradient, but the cutoff gradient need not be symmetric. In particular $\chi_f \mathcal{L} M = \mathcal{L} M$ exactly. Thus the source map applied to $\mathcal{L} M$ introduces no second cutoff.

The fixed-space flat return is $\mathcal{K}^{\mathrm{flat}}_{\mu,\nu} = \mathcal{L}_{\mu,\nu} \mathfrak{T}_{\mu,\nu}$. Define the unit displacement and real row by

$$a_\mu(s) = \frac{s^{5/2}}{\mu + 5/2} D_0, \quad D_0 = \mathrm{diag}(12, -3, -9),$$

$$\Lambda_{\mu,\nu}(A) = -\frac{3\pi}{16} \epsilon^3 \nu^{5+2\mu} \,\mathrm{Re}\big[i N_\nu J(\mu) a_0(B_{\mu,\nu}[A])\big].$$

Both real harmonic coordinates enter the complex amplitude a_0; the matrix force still contains its radial and hoop components.

Proposition 15.1. *Use the direct young-source theorem and the Gaussian force calculation of the preceding section. Uniformly for $\mu \in K$,*

$$\|\mathcal{K}^{\mathrm{flat}}_{\mu,\nu} - a_\mu \Lambda_{\mu,\nu}\|_{\mathscr{A} \to \mathscr{A}} \leq C A_* \left[\nu + \eta_\nu + \omega_{\mathrm{can}}(\nu) + P_\nu e^{-\gamma'/\sqrt{\nu}} + P_\nu e^{-4a_s/(5\nu)}\right].$$

The constants multiplying $\nu, \eta_\nu, \omega_{\mathrm{can}}$ are fixed, not viscosity polynomials. Also $\|a_\mu\|_{\mathscr{A}} \leq C$ and $\|\Lambda_{\mu,\nu}\| \leq C A_*$. The operators are real and continuous in μ for each fixed geometry. On every band where A_* stays bounded, the displayed error tends to zero. For the unit displacement,

$$\Lambda_{\mu,\nu}(a_\mu) = 2\epsilon^3 \nu^{5+2\mu} \,\mathrm{Re}\left[\frac{63\pi i}{128} L(\mu) J(\mu) N_\nu\right] + O(A_* P_\nu e^{-b/\nu}).$$

Proof. Write $M = \mathfrak{T}_{\mu,\nu} A$, with $\|A\|_{\mathscr{A}} \leq 1$. The uniform source theorem gives, for $j = 0, 1$,

$$\int_0^\infty u^{-3/2-j/2} \frac{\chi_o(u)}{c} \int_{\mathrm{cell}} |M(u, q, \eta)| \, dq \, d\eta \, du \leq C_j \mathcal{A}_\nu \nu^j.$$

Its earlier-old part $1.2/\nu < u < u_o = \nu^{-3/2}$ instead has $\mathcal{A}_\nu P_\nu e^{-\gamma'/\sqrt{\nu}}$ as a bound, including any fixed extra age powers. All tensor components are estimated together.

For the analytical tail $u \geq u_o$, the source radius satisfies $\rho/r \leq \nu^{1/4} < 1/2$ on $s \leq 1/\nu$. The integrated Cartesian tensor has the form $\mathrm{diag}(T_h, T_h, T_z)$, with $T_z - T_h = (\pi/2) I_T(M; u \geq u_o)$. The cross tensor has zero integrated Cartesian moment. The isotropic part contracts to zero, since $\partial_i \Delta N = 0$ off source. Direct differentiation gives

$$D^2 \partial_{33} N(r, 0, 0) = (4\pi r^5)^{-1} \,\mathrm{diag}(-12, 3, 9).$$

The negative sign in $V_{\mu,M}$ and the dilation integral therefore give

$$D_y W^{(0)}_{\mu,M} = \frac{\epsilon^3 I_T(M; u \geq u_o)}{8(\mu + 3/2)} s^{5/2} D_0.$$

This calculation includes the hoop derivative $W_r/|y'|$. Subtracting the shifted field, with the same tensor, supplies the additional factor $(\mu+5/2)^{-1}$.

Every point in the mean-value remainder lies at distance at least $r/2$ from zero. The fifth Newton derivative is bounded by Cr^{-6}. Its radial first moment is the $j=1$ moment above. Thus, for all $0\le j\le k$,

$$|(s\partial_s)^j D_y(W^{\mathrm{mt}}_{\mu,M}-W^{\mathrm{mt}}_{\mu+1,M}-W^{(0)}_{\mu,M}+W^{(0)}_{\mu+1,M})|\le C_k\epsilon^3\mathcal{A}_\nu\nu s^3.$$

Receiver conormal derivatives act only on the smooth separated kernel; no derivative of M is taken. This bound includes all higher source multipoles, without expanding them one at a time. Since $\sup_{s>0} s^{1/2}e^{-a_s s}<\infty$, its $\mathscr{A}$ norm is at most $C_{a_s,k}A_*\nu$.

The same separated-kernel bounds, on the outward primitive ray, control the stream cutoff and its fixed conormal derivatives. The analytical-tail transition contribution is bounded by $P_\nu A_* e^{-4a_s/(5\nu)}$ in $\mathscr{A}$. The uncut leading displacement beyond the cutoff has the same bound. The earlier-old source has a fixed fractional gap even at $u=1.2/\nu, s=1/\nu$; its entire exact kernel, including the stream cutoff, is bounded by $P_\nu A_* e^{-\gamma'/\sqrt{\nu}}$. Consequently

$$\|\mathcal{L}_{\mu,\nu}M-a_\mu\ell_\mu(M)\|_{\mathscr{A}}\le CA_*[\nu+P_\nu e^{-\gamma'/\sqrt{\nu}}+P_\nu e^{-4a_s/(5\nu)}].$$

The uniform scalar estimate in the preceding section replaces $\ell_\mu(M)$ by $\Lambda_{\mu,\nu}(A)$ and proves the result.

The row is bounded because its Laplace integrand is bounded by $Ct^{\mu+3/2}e^{-(\gamma-a_s)t}$. For $A=a_\mu$ its positive amplitude is $-(21/4)L(\mu)+O(P_\nu e^{-b/\nu})$, with the cutoff derivative included. This gives the stated coefficient. Continuity in μ follows by dominated convergence in the Laplace and dilation integrals and in the exact causal formula. One may use a slightly larger compact interval within $(0,1/4)$ for the dominating old-age powers. The finite conormal norms are dominated by the same separated kernels, so this gives operator-norm continuity, not merely pointwise continuity. □

For later use, the smooth extraction map has a bound on arbitrary old thin tensors. If

$$N_0(M)=\int_{1.2/\nu}^{\infty}u^{-3/2}\frac{1}{c}\int_{\mathrm{cell}}|M|\,dq\,d\eta\,du,$$

then the separated fourth-derivative kernel and its outward primitive give

$$\|\mathcal{L}_{\mu,\nu}M\|_{\mathscr{A}}\le C_{K,k,a_s}\epsilon^3N_0(M). \tag{15.2}$$

Indeed each fixed $r\partial_r$ derivative costs the same power r^{-5}; the fractional radial gap is fixed on the whole young–old rectangle. Conormal derivatives of $\widehat{\chi}_y(\nu s)$ are uniformly bounded. In the gain stress space, $N_0(M)\le P_\nu e^{\gamma/\nu}\|g^{-1}M\|_Z$, hence the norm of $\mathcal{L}$ is polynomial on a bounded loop-gain band. This estimate uses only the thin component. It assigns no thin moment to the broad repair tensor.

The curved loop is $\mathcal{L}D_N^{-1}M_UR_\mu F_A$, with the original cutoff to the near region and the exact physical source below. The proposition does not identify that loop with $\mathcal{K}^{\mathrm{flat}}$. The equation comparison supplies their difference as an exponentially small operator on $\mathscr{A}$, after the factor $e^{1.1a_s/\nu}$ is paid. The remaining velocity witnesses belong to the mixed ambient block. Thus no polynomial inverse is applied to the $O(\nu)$ or canonical-modulus terms above.

15.1 The physical affine lifting identity

Use component order (r,ϕ,z) and the full matrix space $\mathscr{A}$. Let $\chi_f(s)=\widehat{\chi}_f(\nu s)$ equal one on $s\le 1/\nu$ and zero on $s\ge 1.1/\nu$. For $A\in\mathscr{A}$ set

$$A^y=\chi_fA,\qquad B=(A^y)'+\mu A^y/s,\qquad M(A)=\sup_{0<s\le 1.1/\nu}s^{-5/2}\sum_{j=0}^{6}|(s\partial_s)^jA^y(s)|.$$

The cutoff derivative is included in B; the matrices are not continued beyond their support. All assertions are uniform for $\mu \in K \Subset (0, 1/4)$ and the fixed cutoff.

Write $a = 1 + \nu s$, $x_j = (q - jc, \eta)$, $c = \nu^{-2}$, and

$$V_j = a^{-2} e^{-|x_j|^2/(4a)} J x_j, \quad p_j = -a^{-3} e^{-|x_j|^2/(2a)}, \quad \psi_j = -2a^{-1} e^{-|x_j|^2/(4a)}, \qquad J(q, \eta) = (-\eta, q).$$

Embed $V_j = J\nabla\psi_j$ with zero azimuthal component and define

$$\begin{aligned} f_A &= -\sum_j [(B_{\mathrm{mer}} x_j) \cdot \nabla V_j + B V_j], \\ e_A &= \sum_j [2BV_j + 2\nu A^y : D^2 V_j - (A^y + (A^y)^T)\nabla p_j], \\ h_A^p &= \sum_j [V_j, A^y_{\mathrm{mer}} x_j], \\ \psi_A^p &= \sum_j [(\mathrm{tr} A^y_{\mathrm{mer}})\psi_j - (A^y_{\mathrm{mer}} x_j) \cdot \nabla \psi_j]. \end{aligned}$$

The full matrix, including its hoop entry, is used before restriction to the (r, z)-plane; symmetry of A is not assumed. The identity $A^y_{\mathrm{mer}} J = J[(\mathrm{tr} A^y_{\mathrm{mer}})I - (A^y_{\mathrm{mer}})^T]$ gives $h_A^p = J\nabla\psi_A^p$. The sums and their required derivatives converge by Gaussian decay. With $\mathcal{K} = \eta\partial_\eta + 2s\partial_s$, put

$$\widehat{h}_A = \sqrt{s}\,[(h_A^p)_r e_r + ((h_A^p)_z + \epsilon(1 - \mathcal{K})\psi_A^p) e_z], \qquad F_A = s^{3/2} f_A, \qquad E_A = s^{3/2} e_A.$$

The physical cylindrical streamfunction is $r\psi_A^p$, with $r = R/\sqrt{s}$, so $\widehat{h}_A$ is exactly divergence-free, vanishes near the axis, and has young support and finite energy. The forces are unprojected; the physical body inverse supplies the Cartesian Leray projection. All these maps depend linearly on A.

Lemma 15.2. *There is a real physical body force $\mathfrak{r}_A$, linear in A, such that on the exact background U,*

$$(L_{\mathrm{f},\mu} + \mathbb{P}\,\mathrm{div}\, M_U)\widehat{h}_A = \mathbb{P}(F_A + E_A + \mathfrak{r}_A).$$

For $0 < \epsilon, \nu \le 1$, with $\delta_U = \|U - U_{\mathrm{raw}}\|_{W^{1,\infty}}$,

$$\begin{aligned} R^{-1}(\|F_A\|_{Y_2} + \|E_A\|_{Y_2} + \|\widehat{h}_A\|_{Y_2} + \|\nabla\widehat{h}_A\|_{Y_2}) &\le P_\nu M(A), \\ R^{-1}\|\mathfrak{r}_A\|_{Y_2} &\le P_\nu(\epsilon + \delta_{\mathrm{base}} + \delta_U) M(A). \end{aligned} \tag{15.3}$$

Here δ_{base} is the fixed envelope for the train-to-Gaussian and cross-center norms on $0 \le s \le 1.1/\nu$, with the required finite spatial and conormal derivative budget.

Proof. Write v for the planar train in this calculation. The Cartesian bracket calculation, with the full matrix retained, gives

$$(\partial_s + \mu/s - \nu\Delta)h_A^p + (v \cdot \nabla)h_A^p + (h_A^p \cdot \nabla)v = f_A + e_A + \nabla\Psi_A + r_{\mathrm{per}}, \qquad \Psi_A = \sum_j (A^y_{\mathrm{mer}} x_j) \cdot \nabla p_j.$$

Here r_{per} consists of distinct-center convection products and $(v - \sum_j V_j)$ terms. Gaussian summation and the fixed-band train comparison bound its unnormalized cell norms, with the finite derivative budget, by $C\delta_{\mathrm{base}} M(A) s^{5/2}$.

The exact ring-chart rules are

$$\partial_r = \sqrt{s}(\partial_q - \epsilon\mathcal{K}), \qquad \partial_z = \sqrt{s}\partial_\eta, \qquad (\partial_\tau - y \cdot \nabla/2 - 1/2)(\sqrt{s}b) = s^{3/2}\partial_s b.$$

The physical Laplacian of $\sqrt{s}b$ is $s^{3/2}$ times

$$[\Delta_{q,\eta} - 2\epsilon(\mathcal{K}+1)\partial_q + \epsilon^2(\mathcal{K}+1)^2]b - \epsilon^2 b_r e_r.$$

Together with the stream completion, these identities put every geometric difference into $\epsilon M(A)$ times a fixed Gaussian polynomial of age size at most $C(s^{3/2}+s^{5/2})$ in age-force units. The pressure is completed before projection:

$$s^{3/2}\nabla_{q,\eta}\Psi_A = \nabla_y(s\Psi_A) + \epsilon s^{3/2}(\mathcal{K}+2)\Psi_A e_r.$$

The first term is a global gradient with adequate decay; the second belongs to the geometric error. Thus the physical pressure tail remains accounted for without identifying the two Leray projections. The remaining convection difference is exactly

$$(U - U_{\mathrm{raw}})\cdot\nabla\widehat{h}_A + \widehat{h}_A\cdot\nabla(U - U_{\mathrm{raw}}),$$

which is bounded by δ_U times the asserted energy norm.

The Gaussian cell norms and fixed spatial derivatives are bounded by $CM(A)s^{3/2}$ for f_A and by $CM(A)s^{5/2}$ for both h_A^p and ψ_A^p. The residual e_A has size $CM(A)(s^{3/2}+s^{5/2})$. For body fields the physical conversion is

$$R^{-2}\|s^{3/2}g\|_{Y_2}^2 = \pi\int_0^{1.1/\nu} s^{1/2}\|g(s)\|_{L^2(dq/c\,d\eta)}^2 ds.$$

Thus the relevant age powers, including $s^{7/2}$ and $s^{11/2}$, are integrable at zero; their upper-end integrals and the normalization $c^{-1/2}=\nu$ cost only P_ν. The same calculation with the physical spatial Jacobian $s^{-5/2}$ bounds the lift and its gradient. The smooth cutoff introduces no boundary source.

For the derivative budget, the bracket uses A^y, the stream completion uses one conormal derivative, and one physical spatial derivative uses two. Its Laplacian uses three; one further spatial derivative of the residual uses four. The declared six suffice. These are body and spatial estimates; no bound for arbitrary pure phase derivatives is asserted. The maps are continuous in $\mu\in K$. □

There are two uses of this single estimate. For generic $A\in\mathscr{A}$, $M(A)\le Ce^{1.1a_s/\nu}\|A\|_{\mathscr{A}}$. For the actual jet $A=\mathcal{L}T$, the separated-kernel conormal bound gives instead $M(A)\le P_\nu\epsilon^3 N_1(\chi_o T_1)$, since $\chi_f A = A$. Together with $\delta_U\le P_\nu\rho$ from the background bounds, this gives the actual-range remainder without the generic exponential envelope. For $a_\mu(s)=s^{5/2}\,\mathrm{diag}(12,-3,-9)/(\mu+5/2)$, the formulas give the unit affine force used in the signed return.

The unprojected force also satisfies the stronger weighted estimate:

$$R^{-1}\|g^{-1}F_A\|_{Y_2}\le P_\nu\|A\|_{\mathscr{A}},\qquad g(s)=\exp[\gamma s/(1+\nu s)].$$

Indeed $a_s<\gamma/16$ and $s\le 1.1/\nu$ give $g(s)^{-1}e^{a_s s}\le e^{-(\gamma/3-a_s)s}$; inserting this pointwise bound in the same Gaussian integrals proves the estimate. Multiplication by g^{-1} is not commuted through the Cartesian Leray projection; the near comparison keeps that projection explicit.

16 Repair, pressure and transition errors on the generated range

Write $R=\epsilon^{-1}$, $c=\nu^{-2}$, $g(s)=\exp\{\gamma s/(1+\nu s)\}$ and $N_\nu = e^{\lambda/\nu}\nu^{-\kappa}$, where $\Re\lambda=\gamma>0$. The geometry satisfies

$$\epsilon = C_*\nu^r e^{-d_0/\nu},\qquad d_0=\gamma/3,\qquad \epsilon^3|N_\nu|\nu^{5+2\mu}\asymp 1.$$

All constants below are uniform for μ in a fixed compact subset of $(0,1/4)$ and for the stipulated compact ranges of C_*, r. The gain-band assertions refer to parameter triples satisfying the displayed

relation. At fixed ν, ϵ the final spectral scan is only an interval of width $O(|\log \nu|^{-1})$; no positive gain bound is asserted on a nonshrinking spectral interval for that fixed background. P_ν denotes a fixed polynomial in ν^{-1}. The exact background is $U = V + r_b$, $V = U_{\mathrm{raw}} + E\alpha_{\mathrm{bg}}$, with the unprojected raw residual and coefficient bounds of the background construction. In particular

$$\|r_b\|_{C_\tau L^{11/4}} + \|r_b\|_{W^{1,\infty}} + \sum_{j=1}^{2} \|D^j r_b\|_{Y_{11/4}} \le P_\nu \rho, \qquad \rho = \epsilon^{3/11} e^{m_0/\nu}.$$

Fixed additional viscosity powers are included in P_ν. For an arbitrary axisymmetric physical velocity w without swirl put

$$G(w) = R^{-1} \|g^{-1} w\|_{Y_2}.$$

16.1 Normalize the full old stress before inversion

Let χ_o vanish on $s \le 1.2/\nu$ and equal one on $s \ge 1.5/\nu$. The old source must be

$$T_o = \chi_o (U \otimes w + w \otimes U),$$

including the repair. Define $W_\mu = -S_\mu T_o$, $X = \mathcal{C}_\chi (W_\mu - W_{\mu+1})$, $Y = L_{\mathrm{free},\mu+1} X$ and $h = [U, X]$, where $\mathcal{C}_\chi$ is the divergence-free stream cutoff supported on $s \le 1/\nu$. All operators are applied in physical units here. The tensor in the two shifted resolvents is identical.

The thin part is controlled by its first transverse moment. If $w = \mathcal{N}_v b$ and $A = 1.2/\nu$, weighted Cauchy–Schwarz gives

$$N_1(\chi_o M_v b) \le P_\nu A^{-1/2} \|b\|_H \le P_\nu |N_\nu| G(w).$$

For the repair, use instead the global product estimate

$$F_r := \|\chi_o (r_b \otimes w + w \otimes r_b)\|_{L^2_\tau L^{22/19}} \le P_\nu \rho R |N_\nu| G(w). \tag{16.1}$$

This follows from Hölder's inequality and $g \le e^{\gamma/\nu}$. The fixed power relating $e^{\gamma/\nu}$ to $|N_\nu|$ is absorbed in P_ν. No moment is assigned to the repair source.

The general separated kernels give $X, DX, Y \in Y_2$ for both source classes. For example the repair source-to-L^2 bound uses the order-four kernel in $L^{11/7}$, with norm $CD^{-23/11}$ outside distance D. Hence $w \mapsto H_o w := Y + h$ is bounded at fixed parameters. Its output has young support. Consequently $H_o^2 = 0$ and $(I + H_o)^{-1} = I - H_o$ by the disjoint old and young supports.

For $T = M_U w$, the later full affine correction gives $\widehat{w} = w + Y[T] + h[T] - h_{\mathcal{L}T}$. The added correction has young support and depends only on old data. Hence

$$\chi_o M_U \widehat{w} = \chi_o M_U w. \tag{16.2}$$

Thus every old-source estimate below applies literally to the new unknown w. No gain norm of the possibly large young correction is inserted into the old repair product.

16.2 The repair source and exact pressure give a positive margin

Let $\widetilde{E}(U; X, Y)$ be the full unprojected Ward residual,

$$2U \cdot \nabla Y - 2\nu C_{\mathrm{diff}} + (U \otimes U) : D^2 X - (DX + DX^T) \nabla p.$$

Here p is the exact pressure. For the old repair source, the pointwise gradient kernel has $L^{22/3}$ norm $CD^{-101/22}$. At age $s \le 1/\nu$ this gives

$$\|DX(s)\|_{L^2_\tau L^\infty_y} + \|DY(s)\|_{L^2_\tau L^\infty_y} \le P_\nu \epsilon^{101/22} s^{101/44} F_r,$$

with the spatial supremum understood on that age surface. The global gradient estimate into $L^{22/3}$ uses the kernel exponent $11/3$ and is

$$\|DX\|_{L^\infty_\tau L^{22/3}} + \|DY\|_{L^\infty_\tau L^{22/3}} \le P_\nu \epsilon^{46/11} F_r. \tag{16.3}$$

The phase supremum follows from the free equations for $(\partial_\tau - y\cdot\nabla/2)DX$ and DY and the normalized phase estimate over $P \le 1$. The instantaneous projected tensor divergence is separated from the receiver and has the same kernel bound. This argument differentiates no source in phase.

To estimate $V\cdot\nabla Y$, use $V = \sqrt{s}v$ and the physical Jacobian $\pi R^2 s^{-5/2} ds\, d\eta$. Its squared normalized norm is bounded by

$$P_\nu \epsilon^{101/11} F_r^2 \int_0^{1/\nu} s^{34/11} \| \sup_q |v(s,q,\cdot)| \|_2^2 \, ds.$$

The integral is finite at zero and polynomial in ν^{-1}. For the remaining thin-coefficient terms use $R^{-1}(\|DV\|_2 + \|D^2V\|_2 + \|V\otimes V\|_2) \le P_\nu$. Here their products with the returned derivatives use the ordinary $L^\infty_\tau L^\infty_y$ bound on the cutoff fields, of size $P_\nu \epsilon^{101/22} F_r$ for a first derivative. To obtain it, apply the same phase norm argument at its $q=\infty$ endpoint: dilation is an isometry on L^∞, so along its characteristics the norm difference is bounded by the integral of the L^∞ norm of BZ. Young's inequality bounds both Z and BZ in $L^2_\tau L^\infty_y$. Thus no product of two merely phase-L^2 factors is used for the global pressure term below. Define the canonical pressure by $\nabla p_V = -(I-\mathbb{P})(V\cdot\nabla V)$. It obeys $\|\nabla p_V\|_2 \le C\|V\|_\infty \|DV\|_2$. Therefore

$$R^{-1}\|\widetilde{E}(V;X,Y)\|_{Y_2} \le P_\nu \epsilon^{101/22} F_r \le P_\nu \epsilon^{85/22} |N_\nu| e^{m_0/\nu} G(w).$$

The exact pressure correction is

$$\nabla(p - p_V) = -(I-\mathbb{P})(V\cdot\nabla r_b + r_b\cdot\nabla V + r_b\cdot\nabla r_b),$$

whose $Y_{11/4}$ norm is at most $P_\nu\rho$. By Hölder's inequality, (16.3) bounds all products with a phase-average coefficient, including $D^2 r_b DX$ and $DX\nabla(p-p_V)$. They contribute at most

$$P_\nu R^{-1}\rho\epsilon^{46/11} F_r \le P_\nu \epsilon^{52/11} |N_\nu| e^{2m_0/\nu} G(w).$$

Terms with D^2X have an additional curvature power. This proves the old repair bound on the entire source interval, including every large age and every spatial pressure tail.

For a thin old source, replacing the exact background pressure by the flat lifted pressure has the same first error size. Indeed the unprojected residual gives

$$\nabla(p_{\mathrm{raw}} - p_{\mathrm{lift}}) = -(I-\mathbb{P})F_{\mathrm{raw}}.$$

The entrance convection terms and the explicit radial pressure completion then give $\|\nabla p - G_{\mathrm{flat}}\|_{Y_{11/4}} \le P_\nu \epsilon^{3/11} e^{m_0/\nu}$. The general thin-source estimate is $\|DX\|_{L^\infty_\tau L^{22/3}} \le P_\nu \epsilon^{57/22} N_1$. Hölder's inequality and $R^{-1} = \epsilon$ give the factor $\epsilon^{85/22}$. Nonpressure repair and entrance coefficient changes are bounded in the same way or have a larger curvature power. Thus the combined error f_r satisfies

$$R^{-1}\|f_r\|_{Y_2} \le P_\nu |N_\nu| G(w) \left(\epsilon^{85/22} e^{m_0/\nu} + \epsilon^{52/11} e^{2m_0/\nu}\right). \tag{16.4}$$

All the unprojected forces here have young support, because every term contains a derivative of X or Y. Applying the Leray projection does not change their curl, so these forces do not create a direct vorticity-source moment at old ages.

16.3 Both transition terms and the other age rectangles

Let the near operator remove only the original far mask $s < S$, $t > e^{\delta}s$, with $S = \epsilon^{-b}$ and $0 < \delta < \log(6/5)$. The young-old rectangle is contained in this mask. Its complement is exactly the late-output part $(1-\chi_y)F_{\mathrm{far}}$ and the early-source part $\chi_y F_{\mathrm{far}}(1-\chi_o)$. Neither is identified with a local causal inverse. The two differences produced by the stream cutoff are also retained:

$$(\mathcal{C}_\chi - \chi)W_\mu, \qquad [L_{\mathrm{free},\mu+1}, \mathcal{C}_\chi](W_\mu - W_{\mu+1}).$$

They are supported on $0.8/\nu < s < 1/\nu$.

On late output, $g(s)^{-1} \le e^{-4\gamma/(9\nu)}$; on early source, $g(t) \le e^{3\gamma/(5\nu)}$. The general separated kernel bounds, followed by the actual product estimate (16.1), give the following weakest curvature powers for the four source/receiver combinations, in the split stress norm:

	thin receiver	scaled repair receiver
thin source	ϵ^2	$\epsilon^{49/22}e^{m_0/(2\nu)}$
repair source	$\epsilon^{63/22}e^{m_0/\nu}$	$\epsilon^{79/22}e^{3m_0/(2\nu)}$.

Each entry has a common fixed polynomial $P_\nu S^k$, and is multiplied by $G(w)|N_\nu|e^{-4\gamma/(9\nu)}$ for late output, or by $G(w)e^{3\gamma/(5\nu)}$ for early source. For example, the thin-to-repair entry uses the order-four kernel from L^2 to $L^{22/3}$, with off-diagonal norm $D^{-23/11}$, followed by $\sqrt{\rho}$. The repair-to-thin entry uses the pointwise exponent $79/22$, followed by ρR. The repair-to-repair entry uses the source $L^{22/19}$ norm and the $L^{11/3}$ kernel. The outward stream primitive has the same norm powers, so both transition terms have the late-output bound. The thin-to-thin estimate at age zero does not use a first-moment bound with a fixed positive lower source age. Use the mass estimate $N_0(a) \le C(a^{-3/4} + a^{-1/2})\|T\|_{Z_1}$ proved below instead. The separated pointwise velocity is at most $P_\nu \epsilon^2 s^2 N_0(e^\delta s)$. After the normalized velocity lift and thin multiplication, the square root of the output stress age density is bounded by

$$P_\nu \epsilon^2 s^{5/4} V_m(s) N_0(e^\delta s) \le P_\nu \epsilon^2 (s^{1/2} + s^{3/4}) V_m(s) \|T\|_{Z_1}.$$

Its square is integrable at zero, and the integral up to S is polynomial in S. This supplies the first table entry on the full age interval.

The two weakest exponential margins are

$$2d_0 - \gamma + 4\gamma/9 = \gamma/9, \qquad 2d_0 - 3\gamma/5 = \gamma/15.$$

For completeness the thin-to-repair margins are $37\gamma/198$ and $47\gamma/330$, before its $m_0/2$ loss; the repair-to-thin margins are $79\gamma/198$ and $39\gamma/110$, before its m_0 loss. Thus, if δT_a is the total stress correction from these regions and both stream transitions,

$$\|g^{-1}\delta T_a\|_Z \le P_\nu \exp\left[-\frac{\gamma/15 - kbd_0 - 3m_0/2}{\nu}\right] G(w). \tag{16.5}$$

16.4 The whole thin geometric residual and the actual affine lift

We now include the terms remaining in the exact transformed equation, not just the repair and age errors. Write $T = (T_1, T_2) = M_U w$ in the split stress space, so $T_1 = M_V w$ is the thin component and the physical repair tensor represented by T_2 is $M_{r_b} w$. Let X_1, Y_1 denote the response to $\chi_o T_1$ alone and let $X^{\mathrm{mt}}, Y^{\mathrm{mt}} = D_{\mu+1}X^{\mathrm{mt}}$ be its phase-mean, zero-transverse-width Newton response, with the same stream cutoff. The uncut Newton fields are harmonic outside the old source disk; their shifted dilation identity therefore gives the same displayed commutator formula for Y^{mt}. In particular the Laplacian of the stream-cutoff field is retained even though the uncut fields are harmonic there. Set

$$A_T(s) = \mathcal{L}T_1(s) = D_y X^{\mathrm{mt}}(R/\sqrt{s}, 0, 0), \qquad B_T = A_T' + \mu A_T/s.$$

The full Cartesian matrix is retained, including its hoop entry. The first stream cutoff χ_y is supported in $s \le 1/\nu$; the affine source cutoff χ_f equals one there and vanishes for $s \ge 1.1/\nu$. Hence $\chi_f A_T = A_T$ exactly. In particular the source F_{A_T} uses B_T, without a second cutoff derivative.

Put $N = N_1(\chi_o T_1)$. The explicit separated kernels imply, for $0 \le j \le 6$,

$$|(s\partial_s)^j A_T(s)| \le C_j \epsilon^3 s^{5/2} N, \qquad A_T(s) = 0 \quad (s \ge 1/\nu). \tag{16.6}$$

Indeed the gradient Newton kernel has degree -5, and the ratio of the old source radius to the young receiver radius is at most $1/\sqrt{1.2} < 1$. Each $s\partial_s = -r\partial_r/2$ derivative keeps degree -5 after its factor r is included. The outward stream primitive has the same degree. Fixed conormal derivatives of $\widehat{\chi}_y(\nu s)$ are bounded. This proves the displayed bound directly, without first passing through the exponentially weighted smooth-source norm.

For clarity we give the geometric comparison that is needed for arbitrary thin inputs. One phase primitive in the dilation variable, with its dilation-one boundary term retained, gives

$$\|D_y X_1 - D_y X^{\mathrm{mt}}\|_{L^2_\tau} + \|D_y Y_1 - D_y Y^{\mathrm{mt}}\|_{L^2_\tau} \le P_\nu \epsilon^4 s^{5/2} N.$$

The same bound includes the finite transverse width and the heat-to-Newton error: Taylor's formula costs the first transverse moment, and the strict radial gap holds along every Taylor segment. The initially stronger age power s^3 is bounded by $P_\nu s^{5/2}$ on the young interval. For Y use the exact formula

$$Y = \mathcal{C}_{\chi_y} W_\mu + [D_{\mu+1}, \mathcal{C}_{\chi_y}](W_\mu - W_{\mu+1}).$$

Apply the spatial kernel comparison to these terms separately. Thus no phase derivative of the improved bound, or of the source, is taken. These are normalized phase L^2 estimates; the raw-train coefficients multiplying them are bounded in phase with all the finite transverse weights used below.

The mean transverse Taylor estimate and the Hessian estimate are

$$|D_y X^{\mathrm{mt}}(s,\eta) - A_T(s)| + |D_y Y^{\mathrm{mt}}(s,\eta) - sB_j(s)| \le P_\nu \epsilon^4 s^{5/2} |\eta| N,$$
$$\|D_y^2 X_1(s,\eta)\|_{L^2_\tau} \le P_\nu \epsilon^4 s^3 N, \qquad B_j = s^{-1} D_y Y^{\mathrm{mt}}(R/\sqrt{s}, 0, 0).$$

There is an exact clock correction, including on the cutoff annulus:

$$B_T = B_j + \frac{\nu}{s} D_y \Delta_y X^{\mathrm{mt}}(R/\sqrt{s}, 0, 0), \qquad |B_T - B_j| \le P_\nu \nu \epsilon^5 s^{5/2} N. \tag{16.7}$$

To check its sign and constant, differentiate $Y^{\mathrm{mt}} = (B - \nu\Delta + \mu + 1/2) X^{\mathrm{mt}}$, use $D_y B = B D_y - D_y/2$, and evaluate the phase-independent matrix at $z = 0$, where $B D_y X^{\mathrm{mt}} = s A_T'$. All stream-cutoff terms are already in Y^{mt}; none are omitted from this identity.

The exact ring-chart rules give $U_{\mathrm{raw}} = \sqrt{s}(v + \epsilon H e_z)$, $DU_{\mathrm{raw}} = s(D_{\mathrm{flat}} v + \epsilon C_1)$ and $D^2 U_{\mathrm{raw}} = s^{3/2}(D^2_{\mathrm{flat}} v + \epsilon C_2)$. Here H, C_1, C_2 have the finite weighted train bounds supplied by the explicit chart differentiation and the young train construction. Substitution into the four terms of the Ward residual gives the affine expression

$$s^{3/2}\{2B_T v + 2\nu A_T : D^2_{\mathrm{flat}} v - (A_T + A_T^T)\nabla_{\mathrm{flat}} p_{\mathrm{cyl}}\}.$$

The error in $2U \cdot \nabla Y$ has size $P_\nu \epsilon^4 s^3 N$ times a weighted train coefficient. The errors in DXD^2U, $DX\nabla p$, DUD^2X and $(U \otimes U) : D^2 X$ have size $P_\nu \epsilon^4 s^4 N$. The clock error has the smaller size $P_\nu \nu \epsilon^5 s^4 N$. In physical units the squared body norm divided by R^2 uses the density $\pi s^{-5/2} ds\, d\eta$; the remaining powers $s^{7/2}$ and $s^{11/2}$ are integrable at zero and give only viscosity polynomials at $1/\nu$. This also covers coordinate factors $q - jc$ in an alternative fixed-center Taylor expansion, since they are Gaussian moments of fixed order.

Replacing the actual train and its decaying pressure by the summed Gaussian profiles costs $P_\nu \epsilon^3 \delta_{\mathrm{base}} N$. Here the young-train comparison supplies the quantitative envelope

$$\delta_{\mathrm{base}} \le P_\nu \exp(C_G/\nu - b_G/\nu^4), \qquad b_G > 0,$$

for the fixed derivative and transverse-weight budget on $0 \le s \le 1.1/\nu$, with C_G, b_G independent of the small parameters. It follows directly from the Gaussian train separation $c = \nu^{-2}$. This is the bound used when absorbing its contribution in an exponential curvature margin below. The exact-pressure and repair-coefficient errors were estimated above, using the ordinary phase-supremum gradient bound, not a phase derivative of the extra-ϵ comparison. Consequently, with E_{A_T} the Gaussian residual of the physical affine identity,

$$R^{-1}\|E(U; X_1, Y_1) - E_{A_T}\|_{Y_2} \le P_\nu N\big(\epsilon^4 + \epsilon^{85/22} e^{m_0/\nu} + \epsilon^3 \delta_{\mathrm{base}}\big). \tag{16.8}$$

The 85/22 term retains the entire Cartesian pressure correction.

Apply the explicit physical affine lifting identity to the actual jet A_T. Its proof uses at most three conormal derivatives of the matrix, so (16.6) gives directly

$$R^{-1}\|r_{A_T}\|_{Y_2} \le P_\nu \epsilon^3 N(\epsilon + \rho + \delta_{\mathrm{base}}). \tag{16.9}$$

This follows by inserting the displayed conormal bounds into each Gaussian product in that identity. Geometric terms have one extra ϵ, the exact-background convection difference is $(U - U_{\mathrm{raw}}) \cdot \nabla \widehat{h}_{A_T} + \widehat{h}_{A_T} \cdot \nabla(U - U_{\mathrm{raw}})$, and its bound is $P_\nu \rho$ times the normalized H^1 bracket bound. The explicit lifted pressure potential is a global gradient plus its ϵ radial completion. Thus no projection tail is dropped. There is no factor $e^{1.1 a_s/\nu}$ in this actual-range estimate; that factor is needed only for a generic source in $\mathscr{A}$.

Since $N \le P_\nu |N_\nu| \|g^{-1} T\|_Z$, the new thin and affine body errors have exponential margins at least $\gamma/3$, $19\gamma/66 - m_0$ and $\gamma/11 - m_0$. All exceed the weakest age stress margin $\gamma/15$ when the stipulated losses are small.

16.5 The direct near solver and an actual-range pair norm

The direct zero-entrance near inverse satisfies

$$\|g^{-1} D_L^{-1} g\|_{Z \to Z} \le P_\nu, \quad \|g^{-1} M_U R_\mu f\|_Z \le P_\nu R^{-1} \|f\|_{Y_2}, \quad \|g^{-1} S_{\mathrm{near}} g\|_{H_T \to H} \le P_\nu e^{\eta_\delta/\nu}.$$

For $D_L T = M_U R_\mu f$, the exact equation gives

$$T = M_U w, \qquad w = R_\mu f - S_{\mathrm{near}} T.$$

It follows that $G(w) \le P_\nu e^{\eta_\delta/\nu} R^{-1} \|f\|_{Y_2}$. No homogeneous entrance appears. For a general product datum $M_U v$, however, the reconstruction is $w = v - S_{\mathrm{near}} T$. Its velocity norm must retain the norm of v; a small stress norm does not imply a small velocity witness.

On the distinguished gain band, (16.4) has operator size at most

$$P_\nu \left[e^{-(19\gamma/66 - m_0)/\nu} + e^{-(19\gamma/33 - 2m_0)/\nu} \right]$$

from the gain-energy norm to the normalized body norm. For the full actual affine bracket, (16.9) and $\epsilon^3 N \le P_\nu A$ give the background-error margin $\gamma/11 - m_0$. Geometric lift errors with an extra ϵ have the larger margin $\gamma/3$.

Here is the additional estimate needed to propagate actual-range witnesses. Let $A = \|g^{-1}T\|_Z$, with $T = M_U w$. For an unweighted thin stress and $a > 0$, its old physical mass divided by πR^2 is

$$N_0(a) = \int_a^\infty t^{-3/2} \int \|T(t,\cdot,\eta)\|_{L^2_q}\, d\eta\, dt \le C(a^{-3/4} + a^{-1/2})\|T\|_{Z_1}.$$

Indeed transverse Cauchy–Schwarz contributes $(1+t)^{1/4}$, and the remaining squared age integral is $\int_a^\infty t^{-5/2}(1+t)^{1/2}dt$. Partition the output ages into logarithmic bins of width $h < \delta/2$. On a bin of upper age s_j, all source ages exceed $e^{\delta-h}s_j$, so there is a uniform radial gap $D_j \asymp R/\sqrt{s_j}$. The order-four kernel has spatial L^2 norm $CD_j^{-5/2}$. Minkowski and the mass estimate therefore bound the normalized velocity on that bin by

$$C\epsilon^{3/2}(s_j^{1/2} + s_j^{3/4})\|T\|_{Z_1}.$$

Summing geometric bins down to age zero is convergent. Up to a fixed polynomial in S, the far thin-source velocity operator has norm $P_\nu S^k \epsilon^{3/2}$ from Z_1 to the normalized physical L^2 space. The outward stream primitive obeys the same bound. This proves the estimate for both stream transitions as well.

Thus the late-output thin velocity has possible growth $e^{\gamma/(18\nu)}$ after gain weighting, and the early-source thin velocity has possible growth $e^{\gamma/(10\nu)}$:

$$\tfrac{3}{2}d_0 - \gamma + 4\gamma/9 = -\gamma/18, \qquad \tfrac{3}{2}d_0 - 3\gamma/5 = -\gamma/10.$$

For an actual repair source, Young's $L^{22/19} \to L^2$ bound instead gives normalized velocity $P_\nu S^k \rho\epsilon^{23/11}$ times its gain. Its late and early margins are respectively $23\gamma/99$ and $31\gamma/165$, before the m_0 loss. These bounds include the full old interval.

Put $\omega_b = kbd_0$, increasing the fixed k if necessary. The error families proved in this section have a physical velocity witness v_e, with stress $M_U v_e$, satisfying

$$\|g^{-1}M_U v_e\|_Z \le \delta_A A + \delta_G G(w),$$
$$R^{-1}\|g^{-1}v_e\|_{Y_2} \le \chi_A A + \chi_G G(w),$$

where it suffices to take

$$\delta_A = P_\nu e^{-(\gamma/15-\omega_b-3m_0/2)/\nu}, \qquad \delta_G = P_\nu e^{-(19\gamma/66-\omega_b-3m_0/2)/\nu},$$
$$\chi_A = P_\nu e^{(\gamma/10+\omega_b+3m_0/2)/\nu}, \qquad \chi_G = P_\nu e^{-(31\gamma/165-\omega_b-3m_0/2)/\nu}.$$

The pressure and Ward body-force contributions use $v_e = R_\mu f_e$, whose energy norm is bounded by the normalized body norm. Their thin source bounds use A, whereas the repair source bounds use $G(w)$. The full affine bracket's background error is bounded by (16.9) and is included in δ_A. The actual jet has the direct bound (16.6); on the unit gain band $\epsilon^3 N_1 \le P_\nu A$.

Let $Q = P_\nu$ and $K_s = P_\nu e^{\eta\delta/\nu}$ be the direct near bounds. For one correction solve $D_L T_+ = -M_U v_e$ and set $w_+ = -v_e - S_{\text{near}}T_+$. Then $T_+ = M_U w_+$ exactly, and

$$A_+ \le Q(\delta_A A + \delta_G G), \qquad G_+ \le \chi_A A + \chi_G G + K_s A_+.$$

This proves closure on the generated range, with no replacement of the velocity witness by its stress norm. More precisely, use the closed graph space

$$\mathcal{E}_\nu = \{(T,w) : g^{-1}T \in Z,\ g^{-1}w \in Y_2,\ T = M_U w\}. \tag{16.10}$$

The velocities in this graph have the required orthogonal symmetry but need not be divergence-free. Boundedness of the thin multiplier and the repair multiplier proves that the graph is closed. Thus the pair norm below makes it a Banach space. The displayed update preserves this graph exactly; the divergence-free condition is recovered from the full physical kernel equation, not from the masked near reconstruction. Choose $a = \gamma/8$ and use the pair norm $A + e^{-a/\nu}G$. The four matrix coefficients are

$$Q\delta_A, \quad Q\delta_G e^{a/\nu}, \quad e^{-a/\nu}\chi_A + e^{-a/\nu}K_s Q\delta_A, \quad \chi_G + K_s Q\delta_G.$$

All tend to zero exponentially if $\eta_\delta, \omega_b, m_0$ are sufficiently small. For example each less than $\gamma/1000$ leaves a margin larger than $\gamma/50$ before absorbing fixed viscosity powers. The narrowest new margin is $a - \gamma/10 = \gamma/40$. An initial body-forced column has pair norm at most $P_\nu e^{\eta_\delta/\nu}$, so the resulting correction series also tends to zero under the bounded old smooth-jet observation.

Proposition 16.1. *Let $\mathcal{E}_\nu$ have the pair norm just defined. Define the exact error velocity and its graph correction by*

$$v_Q(T) = S_{\mathrm{far}}T + Y[T] + R_\mu \mathbb{P}\{E[T] - E_{\mathcal{L}T} - r_{\mathcal{L}T}\}, \qquad \mathcal{Q}(T, w) = -\mathcal{J}(v_Q(T)).$$

Here $E[T]$ uses the full old source, $\mathcal{L}T$ uses its thin part, and all projections and cutoffs are the ones specified above. Every term of this exact v_Q is covered by the preceding estimates. If $\eta_\delta, \omega_b, m_0 < \gamma/1000$ and the background and train envelopes satisfy the stated bounds, then, for sufficiently small ν on the distinguished gain band,

$$\|\mathcal{Q}\|_{\mathcal{E}_\nu \to \mathcal{E}_\nu} \le P_\nu e^{-\gamma/(100\nu)}.$$

For each fixed geometry this is a real operator-norm-continuous family in μ on the stipulated compact interval.

Proof. The decomposition of $S_{\mathrm{far}}T + Y[T]$ consists exactly of the early-source and late-output regions and the two stream transitions. For the body term, linearity splits the full old response into its thin and repair responses. The repair response is (16.4); its thin counterpart is (16.8); the final subtraction is (16.9). These exhaust the displayed definition, including finite width, oscillatory phase, the exact matrix clock, pressure and the physical affine lift. The graph lift gives precisely the four pair-norm coefficients above. Their smallest unperturbed exponent is $\gamma/40$. The specified losses leave more than $\gamma/50$, hence the stated weaker $\gamma/100$ bound absorbs every fixed viscosity polynomial. Continuity follows from the resolvent identity, the dominated separated dilation kernels and the polynomial near inverse; the products and finite conormal affine construction are continuous. No differentiation of a rough source in phase is required. □

17 A return equation with a smooth source and a remainder

The following real block lemma permits the leading return to be estimated in its natural smooth-source norm. It does not send an algebraically small error through an exponentially large general inverse.

Lemma 17.1. *For each fixed n and $z \in [-1, 1]$, let E_n, X_n be real Banach spaces and consider the bounded, norm-continuous system*

$$x = K_n(z)x + A_n(z)y, \qquad y = B_n(z)x + C_n(z)y. \tag{17.1}$$

The spaces and all auxiliary geometric parameters are fixed as z varies. Suppose

$$K_n = a_n \ell_n + R_n, \qquad \sup_z \|a_n(z)\|\|\ell_n(z)\| \le M,$$

where $a_n \in E_n$, $\ell_n \in E_n^*$ are real and continuous. Write

$$c_n = \sup_z \|C_n(z)\| < 1, \quad r_n = \sup_z \|R_n(z)\|, \quad h_n = \sup_z \frac{\|A_n(z)\|\|B_n(z)\|}{1 - c_n}.$$

If $r_n + h_n \to 0$ and

$$\ell_n(z)a_n(z) \longrightarrow e^{-z} \quad \textit{uniformly for } -1 \le z \le 1,$$

then, for every sufficiently large n, there is $z_n \in (-1, 1)$ for which (17.1) has a nonzero solution. Moreover $z_n \to 0$. No convergence or compactness of the ambient inverse on X_n is required.

Proof. The second equation has the unique solution $y = (I - C_n)^{-1}B_n x$. Substitution into the first gives

$$x = a_n\ell_n x + \widetilde{R}_n x, \qquad \widetilde{R}_n = R_n + A_n(I - C_n)^{-1}B_n, \qquad \|\widetilde{R}_n\| \le r_n + h_n.$$

For large n set $w_n = (I - \widetilde{R}_n)^{-1}a_n$ and

$$d_n(z) = 1 - \ell_n(z)w_n(z).$$

The real function d_n is continuous and

$$|d_n - (1 - \ell_n a_n)| \le \frac{M(r_n + h_n)}{1 - r_n - h_n}.$$

It therefore converges uniformly to $1 - e^{-z}$, which has opposite signs at -1 and 1. At a zero z_n, take $x = w_n$ and $y = (I - C_n)^{-1}B_n w_n$. Since $\ell_n w_n = 1$, $w_n \neq 0$, and both equations hold. Uniform convergence implies that every such zero tends to zero. This also proves the asserted nonvanishing without a choice of norm for the large physical mode. □

18 Closing the exact return and recovering the energy mode

We combine the preceding estimates on one fixed background. The tensor inverse acts on the sum of the thin and repair components, and the multiplier returns both. Write R_μ for the physical body inverse, S_μ for the full tensor inverse, and M for the exact multiplier, including the prescribed normalization maps.

18.1 The exact two-block equation

Put $D = I + MS_\mu$, $D_N = I + MS_N$ and $S_{\mathrm{far}} = S_\mu - S_N$. Use the closed graph $\mathcal{E}_\nu$ and pair norm from (16.10). The near graph lift is

$$\mathcal{J}(v) = (D_N^{-1}Mv,\ v - S_N D_N^{-1}Mv).$$

Its components satisfy $T = Mw$ exactly.

Let L be the complete old-thin-to-young-jet map (15.1). The full old-source Ward fields X, Y, h, E and the physical affine fields, written $h_A = \widehat{h}_A$ and $r_A = \mathfrak{r}_A$ as in Lemma 15.2, satisfy

$$DM(Y + h) = MY + MR_\mu E, \qquad DMh_A = MR_\mu(F_A + E_A + r_A).$$

The nested cutoffs give $\chi_f LT = LT$. Therefore, with

$$HT = M(Y + h - h_{LT}), \qquad v_Q(T) = S_{\mathrm{far}}T + Y + R_\mu(E - E_{LT} - r_{LT}),$$

the exact transformed equation is

$$D(I+H)T = D_N T - MR_\mu F_{LT} + Mv_Q(T). \tag{18.1}$$

Every correction in H has young support and depends only on old data. Thus $H^2 = 0$ and $(I+H)^{-1} = I - H$ at fixed parameters.

Define

$$\mathcal{P}A = \mathcal{J}(R_\mu F_A), \qquad \mathcal{Q}(T,w) = -\mathcal{J}(v_Q(T)), \qquad L(T,w) = LT.$$

The force F_A is unprojected; R_μ supplies the Cartesian Leray projection. The return system

$$V = \mathcal{P}x + y, \qquad x = LV, \qquad y = \mathcal{Q}V \tag{18.2}$$

has the two block equations

$$x = (L\mathcal{P})x + Ly, \qquad y = (\mathcal{Q}\mathcal{P})x + \mathcal{Q}y.$$

A solution gives zero on the right of (18.1). These are exact operators, before any asymptotic replacement.

18.2 One parameter budget for the two blocks

Choose σ_0 and the compact interval K as in the phase-selection paragraph below before fixing the following constants and budgets. Write J for the canonical Mellin function of Corollary 4.6. Fix the derivative budgets first, including $k = 6$ in $\mathscr{A}$. Let K_0 and k_0 be the fixed exponents in the compact-output and separated-kernel window bounds. Put $d_0 = \gamma/3$. Choose $a, b_{\mathrm{win}} > 0$ so that

$$K_0(a + b_{\mathrm{win}}) < 1/100, \quad k_0 b_{\mathrm{win}} d_0 < \gamma/1000, \quad \vartheta = \min\{a/2, b_{\mathrm{win}}/2, 1/100, 3/22\} > 0.$$

Here b_{win} is the small exponent in $S = \epsilon^{-b_{\mathrm{win}}}$, distinct from the large fixed heat-clock endpoint b. Next choose $\delta, m_0 > 0$ with

$$\delta < \log(12/11), \qquad \eta_\delta, m_0 < \min\{\vartheta d_0/16, \gamma/1000\}.$$

Fix the transverse-tail accuracy larger than the required losses. The background construction realizes this m_0; all its internal choices are then fixed. There is no incompatible lower bound on $a, b_{\mathrm{win}}, \delta, m_0$ in the preceding estimates.

The polynomial flat inverse and Proposition 11.1 give, for some $c_N > 0$,

$$\|g^{-1} D_N^{-1} g\| \le P_\nu, \qquad \|g^{-1}(D_N^{-1} - D_0^{-1})g\| \le P_\nu e^{-c_N/\nu}.$$

Decrease c_N to include the body-source comparison and coefficient errors. Proposition 16.1 gives

$$\|\mathcal{Q}\|_{\mathcal{E}_\nu \to \mathcal{E}_\nu} \le P_\nu e^{-c_Q/\nu}, \qquad c_Q = \gamma/100. \tag{18.3}$$

This estimate covers all of v_Q, including full old repair, pressure, finite width, oscillatory phase and both stream transitions. The actual extracted jet has a polynomial age bound, so the estimate does not pay the exponential envelope of a general source.

Now fix $a_s > 0$ so small that

$$a_s < \gamma/16, \qquad 1.1 a_s < \tfrac{1}{4}\min\{c_N, c_Q\}.$$

The general affine bounds and near reconstruction imply

$$\|\mathcal{P}\|_{\mathscr{A} \to \mathcal{E}_\nu} \le P_\nu e^{1.1 a_s/\nu}, \qquad \|L\|_{\mathcal{E}_\nu \to \mathscr{A}} \le P_\nu. \tag{18.4}$$

For the velocity component use $e^{-\gamma/(8\nu)}e^{\eta_\delta/\nu} \le 1$; the second bound is (15.2). No velocity witness is estimated solely by its stress.

The curved body-column comparison is uniform over the unit ball of $\mathscr{A}$: the six conormal derivatives and the required fixed Gaussian spatial seminorms of its sources are bounded by $P_\nu e^{1.1a_s/\nu}$. The global-input body comparison followed by the near resolvent identity therefore gives

$$\|L\mathcal{P} - \mathcal{K}^{\mathrm{flat}}_{\mu,\nu}\| \le P_\nu e^{-(c_N - 1.1a_s)/\nu}.$$

Here the inputs are prescribed smooth affine forces. No canonical moment error is passed through this inverse comparison. Proposition 15.1 yields

$$L\mathcal{P} = a_\mu \Lambda_{\mu,\nu} + \mathcal{R}_{\mu,\nu}, \qquad \sup \|\mathcal{R}_{\mu,\nu}\| \longrightarrow 0, \qquad \sup \|a_\mu\| \|\Lambda_{\mu,\nu}\| < \infty. \tag{18.5}$$

The suprema are on the final bounded-gain spectral window. The algebraic and canonical-modulus errors retain their fixed constants. Moreover

$$\frac{\|L\| \|\mathcal{Q}\mathcal{P}\|}{1 - \|\mathcal{Q}\|} \le P_\nu e^{-(c_Q - 1.1a_s)/\nu} \longrightarrow 0.$$

This is the second error in Lemma 17.1. All estimates are locally uniform in real μ; kernels, resolvent identities, the near inverse and affine formulas give operator-norm continuity for each fixed geometry.

18.3 Phase selection and the scalar crossing

Put

$$\mathcal{C}(\mu) = \frac{63\pi i}{128}\ell_+(H_0)\frac{\Gamma(\mu + 5/2)}{\lambda^{\mu+5/2}}J(\mu), \qquad N_\nu = e^{\lambda/\nu}\nu^{-\kappa}, \qquad \lambda = \gamma + i\omega.$$

The planar construction gives $\gamma > 0$, $\omega \ne 0$, $\ell_+(H_0) \ne 0$, and holomorphic, nonidentically zero J. Choose $\sigma_0 \in (0, 1/4)$ with $\mathcal{C}(\sigma_0) \ne 0$ and fix a compact interval $K \Subset (0, 1/4)$ around it. Choose $\nu_n \downarrow 0$ satisfying

$$\frac{\omega}{\nu_n} - \operatorname{Im}\kappa \log \nu_n + \arg \mathcal{C}(\sigma_0) = 2\pi n \operatorname{sign}\omega.$$

The left side diverges with sign ω and has derivative $-\omega/\nu^2 - \operatorname{Im}\kappa/\nu$ of fixed nonzero sign for small ν, proving existence of this sequence. Set

$$\epsilon_n^3 = \frac{e^{-\gamma/\nu_n}\nu_n^{\operatorname{Re}\kappa - 5 - 2\sigma_0}}{2|\mathcal{C}(\sigma_0)|}, \qquad c_n = \nu_n^{-2}, \qquad P_n = 2\epsilon_n c_n.$$

Construct the background once at these parameters. The polynomial factor in ϵ_n is allowed throughout; equivalently $-\nu_n \log \epsilon_n \to d_0$. The background and geometry stay fixed as

$$\mu_n(z) = \sigma_0 + \frac{z}{2|\log \nu_n|}, \qquad -1 \le z \le 1,$$

varies. The loop gain $\epsilon_n^3 |N_{\nu_n}| \nu_n^{5+2\mu_n(z)}$ stays between positive fixed constants. The signed unit response is

$$\begin{aligned}\Lambda_{\mu_n(z),\nu_n} a_{\mu_n(z)} &= 2\epsilon_n^3 \nu_n^{5+2\mu_n(z)} \operatorname{Re}\{\mathcal{C}(\mu_n(z)) N_{\nu_n}\} + o(1) \\ &= e^{-z} \operatorname{Re}\frac{\mathcal{C}(\mu_n(z))}{\mathcal{C}(\sigma_0)} + o(1) \longrightarrow e^{-z}\end{aligned}$$

uniformly. Lemma 17.1 supplies $z_n \to 0$ and a real solution of (18.2) with $x \ne 0$ at $\sigma_n = \mu_n(z_n) \in (0, 1/4)$.

Since $x = LT \neq 0$, $T \neq 0$. Nilpotency gives $T' = (I + H)T \neq 0$, and (18.1) gives $DT' = 0$. Thus

$$b = -S_{\sigma_n} T', \qquad T' = Mb$$

is nonzero, real, periodic and divergence-free, with the required symmetry and $b, \nabla b \in Y_2$. No injectivity of M is needed: $b = 0$ would imply $T' = 0$. The exact identities also give $b = w + Y + h - h_{LT}$, so the masked terms recombine into the physical equation with its full pressure. Together with the background and its heat remainder this proves Proposition 2.1. The localization argument below then proves Theorem A.

19 From a growing periodic mode to finite-energy nonuniqueness

We now use the exact background and growing mode from Sections 7 and 18. This is the instability and localization mechanism of Jia and Šverák [28], adapted here to a profile periodic in similarity time. A divergence-free cutoff localizes the common DSS datum while preserving it near the origin. A common heat correction and a family of nonlinear correctors then produce the finite-energy solutions. The periodic right inverse includes every faster unstable mode; no maximal-growth assumption is added.

We localize the initial datum and correct the background and its growing linear perturbation directly. The correction of the common heat tail decays as $e^{\tau/2}$, while the quadratic mode has size $e^{2\sigma\tau}$. A periodic spectral splitting at a rate $\sigma < \beta < 2\sigma < 1/2$ gives a right inverse below both forcing rates, without an upper bound on other unstable exponents.

Proposition 19.1. *The background and mode in Proposition 2.1 imply Theorem A.*

Proof of Proposition 19.1 and Theorem A. The background construction and the return argument prove Proposition 2.1. It remains to localize these fields and construct the nonlinear solutions. Write $a = u_{\mathrm{dss}}$, $U = U_*$, $Q = r_{\mathrm{heat}}$, $b = b_*$ and $P = \mathcal{T}$.

Periodic evolution and regularity of the mode. The map $D_\nu f(y) = \nu^{-1/2} f(\sqrt{\nu}\, y)$ changes the profile viscosity to one: it commutes with the dilation drift and with $\mathbb{P}$, $\Delta D_\nu f = D_\nu(\nu \Delta f)$, and $\operatorname{div}(D_\nu f \otimes D_\nu g) = D_\nu \operatorname{div}(f \otimes g)$. Also $H_1[D_\nu a] = D_\nu H_\nu[a]$ and

$$\|\nabla^k D_\nu f\|_q = \nu^{(k-1)/2 - 3/(2q)} \|\nabla^k f\|_q.$$

We apply this fixed normalization and retain the notation a, U, Q, b. All constants below may depend on these fixed profiles and on the spatial cutoff, but not on the terminal time $\tau_0 \leq 0$ or on $\alpha, \alpha' \in [0, 1]$.

Let $X = H^1_{\mathrm{div}}(\mathbb{R}^3)$, $Z = H^2_{\mathrm{div}}(\mathbb{R}^3)$, and write

$$\mathcal{A} = \Delta + \tfrac{1}{2} y \cdot \nabla + \tfrac{1}{2}, \qquad \mathcal{B}(\tau) f = -\mathbb{P} \operatorname{div}(U(\tau) \otimes f + f \otimes U(\tau)), \qquad \mathcal{L} = \mathcal{A} + \mathcal{B}.$$

Gaussian convolution and the bounds on a give $|D^j H_1[a](y, \tau)| \leq C_j \langle y \rangle^{-1-j}$ for $j \leq 2$, uniformly in phase. Hence $U \in C_{\mathrm{per}} C_0$, $\nabla U \in C_{\mathrm{per}} L^3$, and

$$\mathcal{B} \in C_{\mathrm{per}} \mathcal{L}(X, L^2_{\mathrm{div}}) \cap C_{\mathrm{per}} \mathcal{L}(Z, X).$$

For the first bound use $\|U\|_\infty \|\nabla f\|_2 + \|\nabla U\|_3 \|f\|_6$; for the second use the H^2 product estimate on Qf and the bounded derivatives of $H_1[a]$. The Leray multiplier is bounded on all the Sobolev spaces used here.

The free evolution is

$$S(s) f(y) = e^{s/2} [e^{(e^s - 1)\Delta} f](e^{s/2} y).$$

It has $X \to X$ norm at most $Ce^{-s/4}$, and $L^2 \to X$ and $X \to Z$ norms at most $Cs^{-1/2}$ for $0 < s \le 1$. The large-time gradient estimate follows by applying the heat gradient bound to f, rather than differentiating f. Volterra iteration therefore constructs the periodic evolution $E(\tau, s)$ of $\mathcal{L}$, bounded on L^2, X, Z over bounded time strips, with these same short-time smoothing bounds. The iteration in the smoothing norm uses the integrable convolution of two kernels $r^{-1/2}$.

These evolutions also represent the given weak mode. On a finite time interval set $g(x,\tau) = b(e^{-\tau/2}x, \tau)$ and $\widetilde{U}(x,\tau) = U(e^{-\tau/2}x, \tau)$. Then

$$g_\tau - e^\tau \Delta g + (\sigma - \tfrac{1}{2})g + e^{\tau/2}\mathbb{P}\,\mathrm{div}(\widetilde{U} \otimes g + g \otimes \widetilde{U}) = 0.$$

The energy hypotheses imply $g \in L^2H^1$, and this equation gives $g_\tau \in L^2H^{-1}$. Its form is bounded on $H^1 \times H^1$; Young's inequality makes it coercive after adding $C\|g\|_2^2$, since e^τ has a positive minimum on the interval. The Hilbert triple energy identity gives a continuous L^2 trace and uniqueness from every initial time. Thus $b(\tau) = e^{-\sigma(\tau-s)}E(\tau, s)b(s)$ whenever $b(s) \in H^1$. Choosing such an earlier phase and using smoothing gives

$$b \in C_{\mathrm{per}}H^2, \qquad m_b := \min_{0 \le \tau \le P} \|b(\tau)\|_2 > 0. \tag{19.1}$$

Indeed a zero phase would, by forward uniqueness and periodicity, make the entire mode zero.

A right inverse on the ancient interval. We next construct an ancient right inverse, including every faster unstable mode. Multiplication by $U(\tau) \in C_0$ is compact $X \to L^2$, by compactness on balls and the small uniform spatial tail. Consequently $\mathcal{B}(\tau) : X \to H^{-1}$ is compact. For positive r, $S(r)\mathcal{B}(\tau) : X \to X$ is compact; its alternative factorization through L^2 has norm $O(r^{-1/2})$. The Duhamel formula applied to a weakly null sequence, followed by dominated convergence with this integrable bound, proves $E(t,s) - S(t-s)$ compact on X. For the monodromy $M = E(P, 0)$, periodicity and the free decay therefore give

$$r_{\mathrm{ess}}(M) \le e^{-P/4} < 1.$$

The spectrum outside this disk consists of isolated eigenvalues with finite algebraic multiplicity. Choose $\sigma < \beta < 2\sigma < 1/2$ so that $|z| = e^{\beta P}$ avoids the spectrum. The Riesz projections P_- inside that circle and $P_+ = I - P_-$ are real, P_+X is finite dimensional, and for some $\beta_- < \beta < \beta_+$,

$$\|M^n P_-\| \le Ce^{\beta_- nP}, \qquad \|(M|_{P_+X})^{-n}\| \le Ce^{-\beta_+ nP}, \quad n \ge 0.$$

These are the spectral-radius estimates for the two restrictions; no inverse of the full parabolic evolution is taken.

For $0 \le \theta \le P$ put

$$F_\theta = E(\theta, 0)|_{P_+X}, \qquad G_\theta = (M|_{P_+X})^{-1}P_+E(P, \theta), \qquad \Pi_+(\theta) = F_\theta G_\theta, \quad \Pi_- = I - \Pi_+.$$

Here $G_\theta F_\theta = I$, and the endpoints agree. Extend the projections periodically. They intertwine the evolution. On the finite-dimensional exterior fibers its backward extension is

$$E_+(kP + \theta, lP + \phi)\Pi_+(\phi) = F_\theta(M|_{P_+X})^{k-l}G_\phi.$$

One-period smoothing shows $P_+X \subset Z$, since $P_+ = M(M|_{P_+X})^{-1}P_+$. Factoring general intervals into periods and bounded phase strips now gives

$$\begin{aligned} \|E(\tau, s)\Pi_-(s)\|_{X \to Z} &\le C(1 + (\tau - s)^{-1/2})e^{\beta_-(\tau - s)} && (\tau > s), \\ \|E_+(\tau, s)\Pi_+(s)\|_{X \to Z} &\le Ce^{\beta_+(\tau - s)} && (\tau \le s). \end{aligned}$$

If $P_+X = 0$, the second expression is zero. For continuous functions on $(-\infty, \tau_0]$ use the norms $\|f\|_{X_\beta} = \sup e^{-\beta\tau}\|f(\tau)\|_X$ and $\|f\|_{Z_\beta} = \sup e^{-\beta\tau}\|f(\tau)\|_Z$. Then

$$(\mathcal{R}_\beta f)(\tau) = \int_{-\infty}^{\tau} E(\tau, s)\Pi_-(s)f(s)\,ds - \int_{\tau}^{\tau_0} E_+(\tau, s)\Pi_+(s)f(s)\,ds, \qquad \|\mathcal{R}_\beta f\|_{Z_\beta} \le C_\beta \|f\|_{X_\beta}. \tag{19.2}$$

Both weighted kernels are integrable. The evolution law verifies $(\partial_\tau - \mathcal{L})\mathcal{R}_\beta f = f$ and the terminal condition $\Pi_+(\tau_0)\mathcal{R}_\beta f(\tau_0) = 0$.

Divergence-free truncation of the datum and the common heat correction. To obtain a compact datum, define

$$A(x) = -\int_0^1 r\, x \times a(rx)\,dr, \qquad u_0 = \operatorname{curl}(\chi A), \qquad d = u_0 - a,$$

where χ is a smooth compact radial cutoff equal to one near zero. The derivative bounds give $|D^j A(x)| \le C_j |x|^{-j}$ and

$$\operatorname{curl}[rx \times a(rx)] = -\partial_r[r^2 a(rx)], \qquad \operatorname{curl} A = a.$$

The latter identity holds across zero: the omitted spherical boundary term is $O(r^2)$. Thus u_0 is compact, divergence-free and in L^2. It is nonzero, since it equals a near zero and DSS rescales any nonzero value of a into that neighborhood. It is smooth off zero and $|u_0(x)| \le C|x|^{-1}$; compact support therefore gives $u_0 \in L^{3,\infty} \cap \bigcap_{1\le q<3} L^q$. To exclude L^3, put $\lambda = e^{P/2} > 1$ and $E = \{1 < |x| < \lambda\}$. Nonvanishing, continuity and DSS imply $0 < \int_E |a|^3 < \infty$. Each of the disjoint annuli $\lambda^{-n}E$ has this same integral, and $u_0 = a$ on all sufficiently small such annuli. Thus $\int |u_0|^3 = \infty$. The field d is smooth, vanishes near zero, and satisfies inverse-radius derivative bounds. Set

$$H(x,t) = e^{t\Delta}d(x), \qquad K(y,\tau) = e^{\tau/2}H(e^{\tau/2}y, e^\tau).$$

For $\tau \le 0$, $\|K(\tau)\|_{W^{2,\infty}} \le Ce^{\tau/2}$. Write

$$\mathcal{N}(f) = -\mathbb{P}\operatorname{div}(f \otimes f), \qquad \mathcal{C}(f,g) = \mathbb{P}\operatorname{div}(f \otimes g + g \otimes f), \qquad G = -\mathcal{C}(U,K) + \mathcal{N}(K).$$

Then $\|G(\tau)\|_{H^1} \le Ce^{\tau/2}$. For the Q contribution this follows from the H^2 product bound. For the other contributions, put $H_a = e^{t\Delta}a$. The physical force

$$-\mathbb{P}\operatorname{div}(H_a \otimes H + H \otimes H_a + H \otimes H)$$

is uniformly bounded in H^1 for $0 < t \le 1$: on an inner ball the heat evolution of d and its required derivatives are exponentially small, whereas derivatives of H_a lose only powers of t; outside that ball the unprojected divergence and its derivative decay as $\langle x\rangle^{-3}$ and $\langle x\rangle^{-4}$. Apply $\mathbb{P}$ after these bounds. Multiplication by $t^{3/2}$ and evaluation at $\sqrt{t}\,y$ give L^2 and gradient norms bounded by $Ct^{3/4}$ and $Ct^{5/4}$, which prove the claim for G.

Nonlinear correctors and their amplitude dependence. Let $v = e^{\sigma\tau}b$. For $\alpha \in [0,1]$ solve in Z_β

$$z_\alpha = \mathcal{R}_\beta\{G + \alpha^2\mathcal{N}(v) - \alpha\mathcal{C}(K,v) - \mathcal{C}(K + \alpha v, z_\alpha) + \mathcal{N}(z_\alpha)\}. \tag{19.3}$$

The constant, linear and bilinear parts of this map have respective norms bounded, uniformly in α, by

$$C_0(e^{(1/2-\beta)\tau_0} + e^{(2\sigma-\beta)\tau_0}), \qquad C(e^{\tau_0/2} + e^{\sigma\tau_0}), \qquad Ce^{\beta\tau_0}.$$

These follow from (19.2), the H^2 product estimate and the bound on K. The mixed constant term has the smaller size $Ce^{(1/2+\sigma-\beta)\tau_0}$. For sufficiently negative τ_0, all these maps preserve the ball of radius

$$\rho = 2C_0(e^{(1/2-\beta)\tau_0} + e^{(2\sigma-\beta)\tau_0})$$

and have Lipschitz constants at most $1/2$. Indeed these constants are bounded by the displayed linear bound plus $2C\rho e^{\beta\tau_0}$, which tends to zero. The resulting real z_α satisfy $\|z_\alpha\|_{Z_\beta} \le \rho$. Expanding the quadratic terms in (19.3) proves that $U + K + \alpha v + z_\alpha$ solves the exact profile equation.

Subtract the fixed-point equations at α and α'. The terms involving $z_\alpha - z_{\alpha'}$ are absorbed by the common contraction bound, while $|\alpha^2 - \alpha'^2| \le 2|\alpha - \alpha'|$. The remaining terms give

$$\|z_\alpha - z_{\alpha'}\|_{Z_\beta} \le L_{\tau_0}|\alpha - \alpha'|, \qquad L_{\tau_0} = C(e^{(2\sigma-\beta)\tau_0} + e^{(1/2+\sigma-\beta)\tau_0} + \rho e^{\sigma\tau_0}). \tag{19.4}$$

The three contributions come from $\mathcal{N}(v)$, $\mathcal{C}(K, v)$ and $\mathcal{C}(v, z_{\alpha'})$, respectively. Since

$$L_{\tau_0}e^{(\beta-\sigma)\tau_0} \le C(e^{\sigma\tau_0} + e^{\tau_0/2} + \rho e^{\beta\tau_0}) \longrightarrow 0,$$

choose one τ_0 so that this quantity is at most $m_b/2$ and the preceding contraction bounds hold. This choice is independent of both amplitudes.

Finite energy, the common trace, and separation. With $T = e^{\tau_0}$, the physical solutions are

$$u^{(\alpha)}(x,t) = e^{t\Delta}u_0(x) + t^{-1/2}[Q + \alpha v + z_\alpha](x/\sqrt{t}, \log t). \tag{19.5}$$

The L^2 and gradient norms of $t^{-1/2}f(x/\sqrt{t})$ are $t^{1/4}\|f\|_2$ and $t^{-1/4}\|\nabla f\|_2$. Thus the three remainders have L^2 orders $t^{1/4}$, $t^{\sigma+1/4}$ and $t^{\beta+1/4}$, and squared gradient orders $t^{-1/2}$, $t^{2\sigma-1/2}$ and $t^{2\beta-1/2}$. All dissipation powers are integrable. Heat energy and continuity yield the strong trace, $C_tL^2 \cap L^2_tH^1$ and positive-time C_tH^2. The projected equation then gives $\partial_t u^{(\alpha)} \in C_tL^2$ on every positive closed interval. Pairing it with $u^{(\alpha)}$ gives the energy equality there; strong trace and finite dissipation extend it to $t_0 = 0$. For $s \ge 2$, the nonlinearity maps H^s to H^{s-1}, and heat smoothing from H^{s-1} to $H^{s+1/2}$ has the integrable kernel $(t-r)^{-3/4}$. Iteration on smaller positive-time intervals gives all spatial derivatives; the equation gives all time derivatives.

The interpolation inequality $\|u\|^{10/3}_{10/3} \le C\|u\|_2^{4/3}\|\nabla u\|_2^2$ gives $u^{(\alpha)} \in L^{10/3}_{x,t}$. The Riesz-transform bounds yield the stated $L^{5/3}$ pressure and recover the full equation. Integrating from a positive time and using the strong trace gives the distributional formulation including u_0.

Orthogonal maps fixing the axis commute with the heat and Leray operators and the nonlinearity. They therefore commute with the evolution, its spectral projections, and $\mathcal{R}_\beta$. The potential A is azimuthal, so $\operatorname{curl}(\chi A)$ is axisymmetric without swirl. Uniqueness in the common contraction ball preserves this symmetry for every z_α and (19.5). By (19.1) and (19.4), for $\tau \le \tau_0$,

$$\begin{aligned}
\|(\alpha - \alpha')v + z_\alpha - z_{\alpha'}\|_2 &\ge |\alpha - \alpha'|e^{\sigma\tau}(m_b - L_{\tau_0}e^{(\beta-\sigma)\tau_0}),\\
\|(\alpha - \alpha')v + z_\alpha - z_{\alpha'}\|_2 &\le |\alpha - \alpha'|e^{\sigma\tau}(\max_{[0,P]}\|b\|_2 + m_b/2).
\end{aligned}$$

Physical scaling gives the two-sided separation with $c_* = m_b/2$; integrating its square gives the stated integral bound. In particular, $\alpha \mapsto u^{(\alpha)} - u^{(0)}$ is Lipschitz for the norm $\sup_{0<t\le T} t^{-\sigma-1/4}\|\cdot(t)\|_2$.

Local energy equality and mixed norms. For a fixed branch write $u = u^{(\alpha)}$, $p = p^{(\alpha)}$ and $e = |u|^2/2$. Its positive-time smoothness gives the classical local energy identity. For every real $\phi \in C_c^\infty(\mathbb{R}^3 \times [0,T))$, integration from $\varepsilon > 0$ and then passage to zero give (1.3). Indeed $|\nabla u|^2$ is integrable; $u \in L^{10/3}_{x,t}$ and $p \in L^{5/3}_{x,t}$ put both $|u|^3$ and pu in $L^{10/9}_{x,t}$, hence in L^1 on the support of ϕ.

Strong L^2 trace implies $|u(\varepsilon)|^2 \to |u_0|^2$ in L^1. These facts justify every limit in the identity and, with nonnegative ϕ, prove suitability.

Put $F_\alpha = Q + \alpha v + z_\alpha$ and $H_0 = e^{t\Delta}u_0$. The uniform H^2 bound on F_α gives, for $2 \le q \le \infty$,

$$\|t^{-1/2}F_\alpha(x/\sqrt{t}, \log t)\|_q \le C_q t^{(3/q-1)/2}.$$

The pointwise bound on u_0 and Gaussian convolution give $|H_0(x,t)| \le C/(|x| + \sqrt{t})$: rescale by $\sqrt{t}$ and split the convolution into $|y-x| < |x|/2$ and its complement when $|x| > 2\sqrt{t}$. Hence $\|H_0(t)\|_{3,\infty} \le C$ and $\|H_0(t)\|_\infty \le Ct^{-1/2}$. Since the scaled remainder has uniform L^3 norm, (19.5) proves the uniform weak-L^3 assertion.

Heat contraction gives $\|H_0(t)\|_q \le \|u_0\|_q$ for $2 \le q < 3$. For $q \ge 3$, integrate the distribution function of $|H_0|$, using its uniform L^2 and weak-L^3 bounds and its $Ct^{-1/2}$ supremum. Together with the remainder estimate this yields

$$\|u^{(\alpha)}(t)\|_q \le C_q \begin{cases} 1, & 2 \le q < 3, \\ (1 + |\log t|)^{1/3}, & q = 3, \\ t^{(3/q-1)/2}, & 3 < q \le \infty. \end{cases}$$

For $q = 3$ this is integrable to every finite time power. For $q > 3$ and finite r, the displayed power is integrable exactly when $r(1 - 3/q)/2 < 1$; for $q < 3$ it is uniformly bounded in time. These conditions give every claimed strict mixed norm, with constants independent of α. □

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