VERTEX SPLITTING, GLOBAL RIGIDITY AND NONDEGENERATE EQUILIBRIUM STRESSES

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Abstract. We prove Whiteley's conjecture that every nontrivial vertex split of a generically globally rigid graph preserves generic global rigidity in $\mathbb{R}^d$, for all $d \geq 1$. We give two proofs of the stronger assertion that the split graph is infinitesimally rigid at every generic coincident realisation of the two split vertices. Both proofs use maximum-rank equilibrium stress matrices and then apply the coincident-rigidity criterion of Cruickshank, Jackson and Tanigawa. The second construction also shows that every generically globally rigid graph admits a nondegenerate equilibrium stress of maximum rank at every generic realisation, establishing a conjecture of Jordán and Tanigawa.

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Statement on the use of artificial intelligence

The proofs of all the new results in this paper were discovered by ChatGPT 6.1 Sol, which also produced the first draft of the paper. The author directed ChatGPT to investigate the problem and contributed to the revision of the manuscript to its current form. This included requesting clarifications and rewrites of the proofs, and making decisions about the organisation and presentation of the paper. The author has checked the arguments presented here and takes full responsibility for the paper.

1. Introduction and statement of main results

Vertex splitting is a basic operation in rigidity theory. Whiteley proved that it preserves generic rigidity [12] under appropriate hypotheses. The corresponding question for global rigidity asks whether a vertex split preserves generic global rigidity in $\mathbb{R}^d$ when the two new vertices have $d-1$ common neighbours and each has degree at least $d+1$; see [3, Conjecture 16], [11, Conjecture 1], and [5]. We prove this assertion in every dimension $d \geq 1$. Our two proofs establish infinitesimal rigidity at a generic coincident realisation and then apply the criterion of Cruickshank, Jackson and Tanigawa [5]. We also prove the conjecture of Jordán and Tanigawa that generically globally rigid graphs admit nondegenerate equilibrium stresses [11, Conjecture 2].

Recent work of Cruickshank, Jackson and Tanigawa [6] establishes the vertex-splitting conclusion for graphs of simplicial k-circuits in $\mathbb{R}^{k+1}$, for $k \geq 2$, under $(k+2)$-connectivity. The general conjecture is also discussed in the survey [4]. Our results remove the restriction to these graph classes and provide an inductive operation for constructing further generically globally rigid graphs in every dimension.

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1.1. **Frameworks and stresses.** All graphs are finite and simple, and $d \geq 1$ is an integer. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, let $N_G(i)$ denote the neighbourhood of i, and put $\deg_G(i) = |N_G(i)|$. For $X \subseteq V(G)$, let $G[X]$ be the induced subgraph and let $G - X = G[V(G) \setminus X]$. We write $G - i$ for $G - \{i\}$, and $G - e$ for the graph obtained by deleting an edge e. Contracting an edge uv means identifying its endpoints, deleting the resulting loop, and replacing parallel edges by single edges. A graph is r-connected if it has at least $r+1$ vertices and remains connected after deleting fewer than r vertices.

A *framework* in $\mathbb{R}^d$ is a pair (G,p), where $p : V(G) \to \mathbb{R}^d$ is a realisation. Distinct vertices may have the same position. Two realisations p, q in $\mathbb{R}^d$ are *equivalent* if their edge lengths agree, and *congruent* if $\|p_i - p_j\| = \|q_i - q_j\|$ for every pair of vertices. The framework (G,p) is *rigid* if every sufficiently close equivalent realisation is congruent to p, and *globally rigid* if every equivalent realisation in $\mathbb{R}^d$ is congruent to p.

Position and velocity vectors are columns. We also write p and h for the configuration and velocity matrices whose rows are p_i^T and h_i^T. Their coordinate columns are $p^{(j)}$ and $h^{(j)}$. The stacked velocity is $\boldsymbol{h} = \mathrm{vec}(h^T)$, where vec stacks columns; equivalently, we stack the transposed rows of h.

The *rigidity matrix* $R(G,p)$ has one row for each edge and d columns for each vertex. The row for ij has $(p_i - p_j)^T$ in the columns of i, $(p_j - p_i)^T$ in the columns of j, and zero entries elsewhere. Thus

$$\big(R(G,p)\boldsymbol{h}\big)_{ij} = (p_i - p_j) \cdot (h_i - h_j).$$

An *infinitesimal motion* satisfies $R(G,p)\boldsymbol{h} = 0$. It is *trivial* if $h_i = a + Kp_i$ for some vector $a \in \mathbb{R}^d$ and skew-symmetric matrix K. A framework is *infinitesimally rigid* if all its infinitesimal motions are trivial. If p affinely spans $\mathbb{R}^d$, this is equivalent to $\operatorname{rank} R(G,p) = d|V(G)| - \binom{d+1}{2}$.

A realisation is *generic* if its coordinates are algebraically independent over $\mathbb{Q}$. A graph is *generically rigid*, respectively *generically globally rigid*, if its generic realisations are rigid, respectively globally rigid. At generic realisations, rigidity and infinitesimal rigidity are equivalent [1]. The maximum rank of the rigidity matrix determines generic rigidity, and global rigidity is also a generic graph property [8].

The space of *equilibrium stresses* of (G,p) is $\mathcal{S}(G,p) = \ker R(G,p)^T$. Thus a stress $\omega \in \mathbb{R}^{E(G)}$ satisfies

$$\sum_{j \in N_G(i)} \omega_{ij}(p_i - p_j) = 0 \qquad (i \in V(G)).$$

Its *stress matrix* Ω_ω is symmetric, with off-diagonal entries $-\omega_{ij}$ on edges and zero on nonedges, and diagonal entries $(\Omega_\omega)_{ii} = \sum_j \omega_{ij}$. It satisfies $\Omega_\omega \mathbf{1} = 0$ and $\Omega_\omega p = 0$, where $\mathbf{1}$ denotes the all-ones vector. If p affinely spans $\mathbb{R}^d$, every stress matrix has rank at most $|V(G)| - d - 1$. We call a stress attaining this bound a *maximum-rank stress*. For scalar vertex assignments x, y,

$$x^T \Omega_\omega y = \sum_{ij \in E(G)} \omega_{ij}(x_i - x_j)(y_i - y_j),$$

where each undirected edge is counted once.

We use the stress characterisation due to Connelly and to Gortler, Healy and Thurston.

Theorem 1.1 ([2, 8])**.** *Let G have $N \geq d+2$ vertices, and let p be a generic realisation in $\mathbb{R}^d$. Then G is generically globally rigid in $\mathbb{R}^d$ if and only if (G,p) has an equilibrium stress matrix of rank $N - d - 1$.*

Connelly proved sufficiency, and Gortler, Healy and Thurston proved necessity. For graphs on at most $d+1$ vertices, generic global rigidity is equivalent to completeness. We also use Hendrickson's necessary condition: a generically globally rigid graph on at least $d+2$ vertices is $(d+1)$-connected [9].

1.2. **Main results.** We state the main result in terms of an edge contraction $H = G/uv$. The degree and common-neighbour conditions below say precisely that the inverse operation is a nontrivial d-dimensional vertex split.

Theorem 1.2. *Let $d \geq 1$, let $uv \in E(G)$, and suppose that*

$$\deg_G(u) \geq d+1, \qquad \deg_G(v) \geq d+1, \qquad |N_G(u) \cap N_G(v)| = d-1.$$

If $H = G/uv$ is generically globally rigid in $\mathbb{R}^d$, then G is generically globally rigid in $\mathbb{R}^d$.

When $|V(H)| \geq d+2$, the degree conditions are also necessary for G to be generically globally rigid: Hendrickson's condition forces G to be $(d+1)$-connected, and hence to have minimum degree at least $d+1$. A size restriction is needed for this converse. Splitting a vertex of K_d with all $d-1$ neighbours common to the split vertices gives K_{d+1}; both graphs are generically globally rigid in $\mathbb{R}^d$, but the split vertices have degree d.

The assertion in dimension one is elementary, and the planar case was proved by Jordán and Szabadka [10]. The case $d \geq 3$ is the vertex splitting conjecture. Both proofs below establish the following stronger assertion.

Theorem 1.3. *Under the hypotheses of Theorem 1.2, let w be the contracted vertex of H, and let p be any generic realisation of H in $\mathbb{R}^d$. Define a realisation P of G by $P_u = P_v = p_w$ and $P_i = p_i$ for all other vertices. Then (G, P) is infinitesimally rigid.*

For a vertex w of a framework (H, p), a stress ω, and $Y \subseteq N_H(w)$, write

$$F_Y(w; \omega, p) = \sum_{i \in Y} \omega_{wi}(p_i - p_w). \tag{1.1}$$

A *proper d-subpartition* of $N_H(w)$ is a tuple $(X_1, \ldots, X_d)$ of pairwise disjoint nonempty subsets whose union is a proper subset of $N_H(w)$. Following Jordán and Tanigawa [11, Section 3], a stress is *nondegenerate* if the vectors

$$F_{X_1}(w; \omega, p), \ldots, F_{X_d}(w; \omega, p)$$

are linearly independent for every vertex w and every proper d-subpartition of its neighbourhood. When no such subpartition exists, the condition at that vertex is vacuous.

Theorem 1.4. *Let $d \geq 1$, and let H be generically globally rigid in $\mathbb{R}^d$. Every generic realisation (H, p) has a nondegenerate equilibrium stress ω such that*

$$\operatorname{rank} \Omega_\omega = \max\{0, |V(H)| - d - 1\}.$$

The existence of a nondegenerate stress is the conjecture of Jordán and Tanigawa [11, Conjecture 2]. They also proved [11, Theorem 4.1(1)] that a maximum-rank stress satisfying the appropriate nondegeneracy condition at the contracted vertex of $H = G/uv$ yields generic global rigidity of G under the degree and common-neighbour conditions of Theorem 1.2. Thus Theorem 1.4 supplies another deduction of Theorem 1.2 from their work.

Some formulations of vertex splitting allow more than $d-1$ common neighbours; see [4]. The following consequence covers that convention.

Corollary 1.5. *Let $d \geq 1$, let $uv \in E(G)$, and suppose that*

$$\deg_G(u) \geq d+1, \qquad \deg_G(v) \geq d+1, \qquad |N_G(u) \cap N_G(v)| \geq d-1.$$

If $H = G/uv$ is generically globally rigid in $\mathbb{R}^d$, then G is generically globally rigid in $\mathbb{R}^d$.

Proof. Put $D = N_G(u) \cap N_G(v)$. If $|D| = d-1$, apply Theorem 1.2. Suppose that $|D| \geq d$, and let w be the contracted vertex of H. If $|V(H)| \leq d+1$, then $d \leq |D| \leq |V(H)| - 1 \leq d$, so $|V(H)| = d+1$ and every other vertex is common to u, v. Generic global rigidity makes H complete; hence $G = K_{d+2}$, which is generically globally rigid.

Now suppose that $|V(H)| \geq d+2$. Hendrickson's condition gives $\deg_H(w) \geq d+1$. Write

$$A_0 = N_G(u) \setminus (D \cup \{v\}), \qquad B_0 = N_G(v) \setminus (D \cup \{u\}),$$

and choose $C_0 \subset D$ with $|C_0| = d-1$. Partition $D \setminus C_0 = D_u \sqcup D_v$ so that $A_0 \cup D_u$ and $B_0 \cup D_v$ are both nonempty. This is possible because $D \setminus C_0 \neq \varnothing$ and

$$|A_0| + |B_0| + |D \setminus C_0| = \deg_H(w) - (d-1) \geq 2.$$

If just one of A_0, B_0 is empty, assign a vertex of $D \setminus C_0$ to that side. If both are empty, that set has at least two vertices, one for each side. Distribute any remaining vertices arbitrarily.

Obtain a spanning subgraph G' by deleting vi for $i \in D_u$ and ui for $i \in D_v$. Then $G'/uv = H$, its common-neighbour set is C_0, and each split vertex has degree at least $d+1$. Theorem 1.2 makes G' generically globally rigid. Adding edges preserves generic global rigidity, so G has the same property. □

Section 2 gives the common setup and relates stress lifting to partial forces. Section 3 gives the first proof by lifting a stress and differentiating a local field of infinitesimal motions. Section 4 gives the second proof by an explicit deformation of a configuration and its stress matrix. Section 5 applies the same deformation to prove Theorem 1.4.

2. Coincident frameworks and partial forces

2.1. **Common setup.** Fix a graph G and an edge uv satisfying the degree and common-neighbour conditions of Theorem 1.2. Let $H = G/uv$, and let w be the contracted vertex. Put

$$C = N_G(u) \cap N_G(v), \qquad A = N_G(u) \setminus (C \cup \{v\}), \qquad B = N_G(v) \setminus (C \cup \{u\}).$$

Then $N_H(w) = A \sqcup C \sqcup B$, $|C| = d-1$, and $A, B \neq \varnothing$. Unless stated otherwise, write

$$n = |V(G)|, \qquad m = |V(H)| = n-1, \qquad k = d-1,$$

and let $C = \{c_1, \ldots, c_k\}$. In particular, $m \geq d+2$. Let $E : \mathbb{R}^{V(H)} \to \mathbb{R}^{V(G)}$ duplicate the coordinate at w into the coordinates at u, v, leaving all other coordinates unchanged. The same matrix acts on configuration matrices. The *coincidence locus* is

$$\mathcal{L} = \{P \in \mathbb{R}^{n \times d} : P_u = P_v\} = \{Ep : p \in \mathbb{R}^{m \times d}\}. \tag{2.1}$$

The framework (G, Ep) is *generic coincident* when p is generic. Infinitesimal rigidity uses the ordinary rigidity matrix: velocities at the coincident vertices are allowed to differ. We call G *uv-coincident rigid* if its generic coincident frameworks are infinitesimally rigid. Coincident rigidity was first studied by Fekete, Jordán and Kaszanitzky [7], who characterised this property in dimension two.

Lemma 2.1. *Every generic coincident realisation attains the maximum rank of $R(G,P)$ on $\mathcal{L}$. In particular, if G has an infinitesimally rigid coincident realisation that affinely spans $\mathbb{R}^d$, then G is uv-coincident rigid.*

Proof. The entries of $R(G,Ep)$ are polynomials with rational coefficients in the entries of p. A minor attaining the maximum rank is a nonzero polynomial, so it is nonzero at every generic p. For the last assertion, a nonzero minor of order $dn-\binom{d+1}{2}$ at the given framework remains nonzero at every generic coincident framework. Such frameworks affinely span $\mathbb{R}^d$, so this rank is equivalent to infinitesimal rigidity. □

We will use the following reduction from global rigidity to coincident rigidity.

Theorem 2.2 (Cruickshank, Jackson and Tanigawa [5, Theorem 1.2])**.** *Let $uv \in E(G)$ satisfy $|N_G(u) \cap N_G(v)| = d-1$, and suppose that $H = G/uv$ is generically globally rigid in $\mathbb{R}^d$. If G has an infinitesimally rigid realisation in $\mathbb{R}^d$ in which u and v coincide, then G is generically globally rigid in $\mathbb{R}^d$.*

2.2. **Stress contraction and the partial-force criterion.** For $P = Ep$, an equilibrium stress $\widetilde{\omega}$ on (G,P) contracts to an equilibrium stress $\pi(\widetilde{\omega})$ on (H,p). Stresses on edges away from w are unchanged, while

$$\pi(\widetilde{\omega})_{wi} = \begin{cases} \widetilde{\omega}_{ui}, & i \in A, \\ \widetilde{\omega}_{vi}, & i \in B, \\ \widetilde{\omega}_{ui} + \widetilde{\omega}_{vi}, & i \in C. \end{cases}$$

The stress on uv is discarded. Equilibrium at w follows by adding the equations at u, v; every other equilibrium equation is unchanged. The corresponding stress matrices satisfy

$$\Omega_{\pi(\widetilde{\omega})} = E^T \Omega_{\widetilde{\omega}} E. \tag{2.2}$$

An equilibrium stress $\widetilde{\omega}$ on (G,P) is a *lift* of a stress ω on (H,p) if $\pi(\widetilde{\omega}) = \omega$. We say that ω *lifts* when such an equilibrium stress exists. Let σ_{uv} denote the stress with value one on uv and zero on every other edge. Since $P_u = P_v$, it is an equilibrium stress of (G,P).

Proposition 2.3. *Let (H,p) be infinitesimally rigid, and suppose that the vectors $p_c - p_w$, $c \in C$, are linearly independent. Put $P = Ep$, choose a nonzero vector $\nu \in \operatorname{span}\{p_c - p_w : c \in C\}^\perp$, and define*

$$\Phi_p : \mathcal{S}(H,p) \longrightarrow \mathbb{R}, \qquad \Phi_p(\omega) = \nu \cdot F_A(w;\omega,p).$$

Then

$$\operatorname{im} \pi = \ker \Phi_p, \qquad \ker \pi = \mathbb{R}\sigma_{uv}. \tag{2.3}$$

Moreover, (G,P) is infinitesimally rigid if and only if Φ_p is not identically zero.

Proof. Put $L = \operatorname{span}\{p_c - p_w : c \in C\} = \nu^\perp$. Fix a stress ω on (H,p). In any lift, the stress on ua must be ω_{wa} for each $a \in A$, since a is adjacent to u but not to v. These edges contribute $F_A(w;\omega,p)$ to the equilibrium sum at u, written using the directions $P_i - P_u$. The stress on uv contributes zero because $P_u = P_v$. The remaining edges incident to u are uc, $c \in C$, whose stress values α_c must therefore satisfy

$$F_A(w;\omega,p) + \sum_{c \in C} \alpha_c (p_c - p_w) = 0. \tag{2.4}$$

Thus this equation is necessary for a lift to exist. It is equivalent to $F_A(w;\omega,p)\in L$, hence to $\Phi_p(\omega)=0$. It is also sufficient: retain the stresses on edges away from u,v, assign ω_{wa} to ua, ω_{wb} to vb, and assign $\alpha_c,\omega_{wc}-\alpha_c$ to uc,vc. Equation (2.4) gives equilibrium at u; equilibrium at v follows from the original equation at w. The common-neighbour forces are unchanged. The stress on uv is arbitrary. Independence of the common-neighbour directions makes the α_c unique, and also shows that a stress in $\ker\pi$ is supported on uv. This proves (2.3).

For the rigidity assertion, let h be an infinitesimal motion of (G,P). Subtracting the equations on uc and vc gives $h_u-h_v=s\nu$ for some scalar s. Define a velocity assignment x on H by $x_w=h_v$ and $x_i=h_i$ elsewhere, and define $b\in\mathbb{R}^{E(H)}$ by

$$b_{wa}=(p_a-p_w)\cdot\nu\quad(a\in A),\qquad b_{ij}=0\quad(ij\notin\{wa:a\in A\}).$$

For $a\in A$, the equation on ua gives

$$0=(p_w-p_a)\cdot(h_u-h_a)=(p_w-p_a)\cdot(x_w-x_a)-sb_{wa}.$$

The equations on vj, $j\in B$, give the unchanged equations on wj. For $c\in C$, the equation on vc gives the equation on wc, and the equation on uc agrees with it because $(p_c-p_w)\cdot\nu=0$. Edges away from u,v are unchanged, while the equation on uv is identically zero. Consequently,

$$R(H,p)\boldsymbol{x}=sb,\qquad\omega^Tb=\sum_{a\in A}\omega_{wa}(p_a-p_w)\cdot\nu=\Phi_p(\omega).$$

If Φ_p is not identically zero, choose a stress ω with $\Phi_p(\omega)\neq0$. Since $R(H,p)^T\omega=0$,

$$0=\omega^TR(H,p)\boldsymbol{x}=s\Phi_p(\omega),$$

so $s=0$. Thus $h_u=h_v$, and x is an infinitesimal motion of (H,p). Its infinitesimal rigidity gives $x_i=a+Kp_i$ for a vector a and a skew-symmetric matrix K. Duplicating the value at w shows that $h_i=a+KP_i$ on G, so every motion of (G,P) is trivial.

Conversely, if $\Phi_p\equiv0$, then $b\in(\ker R(H,p)^T)^\perp=\operatorname{im}R(H,p)$. Choose $\boldsymbol{x}$ with $R(H,p)\boldsymbol{x}=b$, and put $h_u=x_w+\nu$, $h_v=x_w$, and $h_i=x_i$ elsewhere. On ua, $a\in A$, the edge expression is $(p_w-p_a)\cdot(x_w-x_a+\nu)=b_{wa}-b_{wa}=0$. On uc, $c\in C$, the extra term involving ν is zero. All remaining edges other than uv inherit their zero edge expressions from $R(H,p)\boldsymbol{x}=b$, since b is zero on the corresponding edges; the equation on uv is automatic. Hence these velocities form an infinitesimal motion of (G,P). Every trivial motion has equal velocities at coincident vertices, whereas $h_u-h_v=\nu\neq0$. This motion is therefore nontrivial, which proves the converse. □

The following consequence is the stress-lifting statement used in the first proof. Neither it nor Proposition 2.3 requires global rigidity of the contracted graph.

Theorem 2.4 (Stress lifting)**.** *Let uv be an edge of a graph G with exactly $d-1$ common neighbours C, and let $H=G/uv$, with contracted vertex w. Suppose that (H,p) is infinitesimally rigid and that the vectors p_c-p_w, $c\in C$, are linearly independent. If (G,Ep) is not infinitesimally rigid, then every equilibrium stress of (H,p) lifts to an equilibrium stress of (G,Ep). For each stress being lifted, the values of its lift on all edges other than uv are uniquely determined, while its value on uv is arbitrary.*

Proof. Proposition 2.3 gives $\Phi_p\equiv0$, so $\operatorname{im}\pi=\mathcal{S}(H,p)$ and $\ker\pi=\mathbb{R}\sigma_{uv}$. □

3. First proof: stress lifting and infinitesimal motions

We first deal with the case $d = 1$. The hypotheses give $|V(H)| \geq 3$, so Hendrickson's condition implies that H is 2-connected. Hence $H - w$ is connected. Since A and B are nonempty, $G - uv$ is connected. At a generic coincident realisation in $\mathbb{R}$, every edge of $G - uv$ has nonzero length, so its motion equation forces the endpoint velocities to agree. Connectivity therefore makes every motion a translation. This proves Theorem 1.3 for $d = 1$, and Theorem 2.2 gives Theorem 1.2 in this case.

For the rest of this section let $d \geq 2$, so $k = d - 1 \geq 1$. Put $F = H - w$ and $W = V(H) \setminus (\{w\} \cup C)$, so $V(F) = C \sqcup W$. The proof uses a smooth field of infinitesimal motions near a maximum-rank point of the coincidence locus.

Lemma 3.1. *If $P \in \mathcal{L}$ attains the maximum rank of $R(G, P)$ on $\mathcal{L}$ and h is an infinitesimal motion of (G, P), then there is a smooth field of velocity matrices $Q \mapsto h(Q)$, defined in a neighbourhood of P in $\mathcal{L}$, such that $h(P) = h$ and*

$$R(G, Q)\boldsymbol{h}(Q) = 0, \qquad \boldsymbol{h}(Q) = \mathrm{vec}(h(Q)^T). \tag{3.1}$$

Proof. Let r_* be the maximum rigidity-matrix rank on $\mathcal{L}$. If P attains rank r_*, a nonzero minor remains nonzero in a neighbourhood of P in $\mathcal{L}$. The rank is at least r_* there, and cannot exceed r_*, so it is constant. If $r_* = 0$, the constant field $h(Q) = h$ suffices. Otherwise, after fixed row and column permutations, write

$$R(G, Q) = \begin{pmatrix} D(Q) & B(Q) \\ C(Q) & J(Q) \end{pmatrix}, \qquad \boldsymbol{h} = \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix},$$

where $D(Q)$ is an invertible $r_* \times r_*$ block near P, and the velocity coordinates have been permuted accordingly. Constant rank gives $J(Q) - C(Q)D(Q)^{-1}B(Q) = 0$. Keeping η_0 fixed, define

$$\boldsymbol{h}(Q) = \begin{pmatrix} -D(Q)^{-1}B(Q)\eta_0 \\ \eta_0 \end{pmatrix}.$$

This vector lies in the kernel of the permuted rigidity matrix and equals $\boldsymbol{h}$ at P. Undoing the column permutation and reshaping the vector into an $n \times d$ matrix gives the required smooth field. □

The field in Lemma 3.1 is defined on a neighbourhood in the full coincidence locus $\mathcal{L}$, with no restriction on the edge lengths of its placements.

3.1. A normalised coincident framework.

Lemma 3.2. *Assume the hypotheses of Theorem 1.2 with $d \geq 2$, and let $\widehat{q}$ be any generic realisation of $H = G/uv$. There is an invertible affine change of coordinates giving realisations q of H and $P = Eq$ of G with the following properties.*

(1) *The positions satisfy*

$$P_u = P_v = q_w = 0, \qquad P_{c_j} = q_{c_j} = \begin{pmatrix} e_j \\ 0 \end{pmatrix} \quad (1 \leq j \leq k), \tag{3.2}$$

where $e_1, \ldots, e_k$ are the standard basis vectors of $\mathbb{R}^k$. All coordinates at the vertices in W are algebraically independent over $\mathbb{Q}$.

(2) *The matrix $R(G, P)$ has maximum rank on $\mathcal{L}$.*

(3) *The framework (H, q) is infinitesimally rigid and has a stress matrix of rank $n - d - 2$, where $n = |V(G)|$.*

(4) *The sets A, B are nonempty, $F[W]$ is connected, and $\deg_F(c) \geq d$ for every $c \in C$.*

Proof. The degree assumptions give $|A| = \deg_G(u) - d \geq 1$ and $|B| = \deg_G(v) - d \geq 1$. Hence $\deg_H(w) = d - 1 + |A| + |B| \geq d + 1$, so H has at least $d + 2$ vertices. Hendrickson's condition implies that H is $(d + 1)$-connected. Deleting the d vertices $\{w\} \cup C$ therefore leaves the connected graph $F[W]$. Also, $\deg_F(c) = \deg_H(c) - 1 \geq d$ for $c \in C$.

Choose a vector $a \in \mathbb{Q}^d$ such that

$$T = [\, \widehat{q}_{c_1} - \widehat{q}_w \ \cdots \ \widehat{q}_{c_k} - \widehat{q}_w \ a \,]$$

is invertible, and set $q_i = T^{-1}(\widehat{q}_i - \widehat{q}_w)$. Such a rational vector exists because the first k columns are independent. This gives (3.2). Let K_0 be the field generated over $\mathbb{Q}$ by the original coordinates at $w, c_1, \ldots, c_k$. The original coordinates at W are algebraically independent over K_0, and the normalisation is an invertible affine substitution over K_0. The transformed coordinates at W remain algebraically independent over K_0, and hence over $\mathbb{Q}$.

An invertible linear change L of position vectors preserves the rank of the rigidity matrix: it is accompanied by the invertible change $h_i \mapsto L^{-T} h_i$ of velocities. Translation also preserves rank. Since an invertible affine change maps $\mathcal{L}$ onto itself, Lemma 2.1 gives the second assertion.

The original generic framework $(H, \widehat{q})$ is infinitesimally rigid and, by Theorem 1.1, has a stress matrix of rank $|V(H)| - d - 1 = n - d - 2$. An affine change preserves equilibrium stress matrices: such a matrix annihilates both $\mathbf{1}$ and the coordinate columns, and therefore also annihilates the transformed coordinate columns. The same matrix and its rank are thus available at q. Infinitesimal rigidity is preserved by the rank argument above. □

The full normalised configuration is not generic, since some of its coordinates are fixed. The conclusions of Lemma 3.2 specify exactly which genericity and rank properties are retained. We write $P_i = \binom{x_i}{t_i}$, with $x_i \in \mathbb{R}^k$ and $t_i \in \mathbb{R}$, for its horizontal and vertical position coordinates.

3.2. **The motion argument.** Assume the hypotheses of Theorem 1.2, and choose the normalised frameworks q, P given by Lemma 3.2, starting with an arbitrary generic realisation $\widehat{q}$ of H. In particular, $n \geq d + 3$, and P affinely spans $\mathbb{R}^d$.

Suppose, for a contradiction, that (G, P) is not infinitesimally rigid. All claims below are made under this assumption.

Claim 1. *There is a nontrivial infinitesimal motion h of (G, P) with a smooth local extension $Q \mapsto h(Q)$ on $\mathcal{L}$ satisfying* (3.1)*. Its relative velocity $z = h_u - h_v$ is nonzero and vertical.*

Proof. Choose a nontrivial infinitesimal motion h. By Lemma 3.2, P has maximum rank on $\mathcal{L}$. Lemma 3.1 therefore gives a smooth extension with $h(P) = h$. The equations on uc_j, vc_j imply $P_{c_j} \cdot z = 0$ for $1 \leq j \leq k$, so z is vertical by (3.2). If $z = 0$, the motion descends to (H, q), which is infinitesimally rigid, and is therefore trivial on G. Thus $z \neq 0$. □

Fix this motion and field. Let

$$e = e_u - e_v, \qquad \mathcal{A} = \operatorname{span}\{\mathbf{1}, P^{(1)}, \ldots, P^{(d)}\}, \qquad U = \operatorname{im} E = e^{\perp}.$$

Here e_u, e_v are standard basis vectors of $\mathbb{R}^n$. The space $\mathcal{A}$ has dimension $d + 1$, is contained in U, and consists of the scalar vertex assignments $f_i = b + a \cdot P_i$. Every equilibrium stress matrix of (G, P) annihilates $\mathcal{A}$.

Claim 2. *The framework (G,P) has an equilibrium stress matrix Γ such that*

$$\operatorname{rank}\Gamma = n-d-1, \qquad \ker\Gamma = \mathcal{A}. \tag{3.3}$$

Proof. By Lemma 3.2, (H,q) has a stress matrix Ω_H of rank $n-d-2$. Theorem 2.4 lifts it to a stress matrix Γ_0 of (G,P), and (2.2) gives $E^T\Gamma_0 E = \Omega_H$. Since E is an isomorphism from $\mathbb{R}^{V(H)}$ onto U, the radical of the bilinear form $x^T\Gamma_0 y$ restricted to U is

$$E(\ker\Omega_H) = E\big(\operatorname{span}\{\mathbf{1}, q^{(1)},\dots,q^{(d)}\}\big) = \mathcal{A}.$$

The equality for $\ker\Omega_H$ follows from its rank and the affine span of q. Thus the induced form on $U/\mathcal{A}$ is nondegenerate.

Changing only the stress on uv gives the family $\Gamma_\lambda = \Gamma_0 + \lambda ee^T$, $\lambda\in\mathbb{R}$. Every member annihilates $\mathcal{A}$. Set $\ell = n-d-2$, and choose $s_1,\dots,s_\ell \in U$ whose cosets form a basis of $U/\mathcal{A}$. Put $r = e/2$. Since $e^Te = 2$, we have $e^Tr = 1$, whereas $e^Ts_i = 0$. Hence $\overline{s}_1,\dots,\overline{s}_\ell,\overline{r}$ form a basis of $\mathbb{R}^n/\mathcal{A}$, where $\overline{x} = x+\mathcal{A}$.

The symmetric matrix Γ_λ induces a well-defined bilinear form on this quotient. With respect to the ordered basis $(\overline{s}_1,\dots,\overline{s}_\ell,\overline{r})$ of $\mathbb{R}^n/\mathcal{A}$, its matrix is

$$\begin{pmatrix} D & b \\ b^T & c+\lambda \end{pmatrix}, \tag{3.4}$$

where $D_{ij} = s_i^T\Gamma_0 s_j$, $b_i = s_i^T\Gamma_0 r$, and $c = r^T\Gamma_0 r$. The matrix D is invertible because it represents the form on $U/\mathcal{A}$. The added term contributes $\lambda(e^Tx)(e^Ty)$, so it changes only the bottom-right entry in (3.4). The determinant is

$$\det(D)\big(c+\lambda - b^TD^{-1}b\big).$$

Choose $\lambda \neq b^TD^{-1}b - c$, and set $\Gamma = \Gamma_\lambda$. The quotient form is nondegenerate, so $\ker\Gamma = \mathcal{A}$, proving (3.3). □

Fix a stress matrix Γ as in Claim 2, and let ω be its edge stress.

Claim 3. *For the motion h in Claim 1, there is a vertical vector $\eta\in\mathbb{R}^d$ such that*

$$\Gamma h = e\eta^T. \tag{3.5}$$

Consequently $h^{(j)}\in\mathcal{A}$ for $1\le j\le k$.

Proof. Take any $\delta P\in\mathcal{L}$ and set $P(\tau) = P+\tau\delta P$. For sufficiently small τ, this line is contained in the domain of the smooth field from Claim 1. Define

$$g = \frac{d}{d\tau}h(P(\tau))\Big|_{\tau=0}, \qquad \boldsymbol{g} = \operatorname{vec}(g^T).$$

Since the rigidity matrix depends linearly on the positions,

$$R(G,P(\tau)) = R(G,P) + \tau R(G,\delta P).$$

Differentiating (3.1) along this line gives

$$R(G,\delta P)\boldsymbol{h} + R(G,P)\boldsymbol{g} = 0.$$

Therefore, for each $ij\in E(G)$, we have

$$(\delta P_i - \delta P_j)\cdot(h_i-h_j) + (P_i-P_j)\cdot(g_i-g_j) = 0.$$

The two terms are the ordinary product rule applied to the position and velocity differences. The variation δP is arbitrary in the full space $\mathcal{L}$; the normalisation (3.2) is imposed only at the base point P. In particular, no preservation of edge lengths along $P(\tau)$ is assumed.

Multiplying each edge equation by ω_{ij} and summing over all edges gives

$$0 = \sum_{ij \in E(G)} \omega_{ij}(\delta P_i - \delta P_j) \cdot (h_i - h_j) + \sum_{ij \in E(G)} \omega_{ij}(P_i - P_j) \cdot (g_i - g_j).$$

The second sum is zero: regrouping it by vertices gives

$$\sum_{i \in V(G)} g_i \cdot \sum_{j \in N_G(i)} \omega_{ij}(P_i - P_j) = 0$$

by equilibrium of ω. For $n \times d$ matrices, recall the Frobenius inner product $\langle X, Y \rangle_F = \operatorname{tr}(X^T Y)$. Regrouping the remaining edge sum by vertices gives

$$0 = \sum_{ij \in E(G)} \omega_{ij}(\delta P_i - \delta P_j) \cdot (h_i - h_j) = \langle \delta P, \Gamma h \rangle_F. \tag{3.6}$$

The coordinate columns of a matrix in $\mathcal{L}$ may be chosen independently in $U = e^{\perp}$. A matrix is therefore orthogonal to $\mathcal{L}$ with respect to the Frobenius product precisely when each of its columns is a multiple of e. Since (3.6) holds for every $\delta P \in \mathcal{L}$, it follows that $\Gamma h = e\eta^T$ for some $\eta \in \mathbb{R}^d$.

The matrix $h^T \Gamma h$ is symmetric, and $h^T e = h_u - h_v = z$. Thus $z\eta^T$ is symmetric. Write $z = (0, \dots, 0, \zeta)^T$, where $\zeta \neq 0$ by Claim 1. For $1 \leq j \leq k$, symmetry gives

$$\zeta \eta_j = (z\eta^T)_{dj} = (z\eta^T)_{jd} = 0.$$

Hence η is vertical. Comparing horizontal columns in (3.5) gives $\Gamma h^{(j)} = 0$ for $1 \leq j \leq k$, and Claim 2 gives $h^{(j)} \in \mathcal{A}$. □

Claim 4. *After subtracting a trivial motion from h, the resulting infinitesimal motion has the form $\widetilde{h}_i = \binom{Sx_i}{r_i}$, where S is a symmetric $k \times k$ matrix with zero diagonal and $r_u - r_v \neq 0$. Moreover,*

$$(x_i - x_j)^T S(x_i - x_j) + (t_i - t_j)(r_i - r_j) = 0 \qquad (ij \in E(F)). \tag{3.7}$$

Proof. By Claim 3, for each $1 \leq j \leq k$ there are coefficients independent of the vertex such that

$$h^{(j)} = a_j \mathbf{1} + \sum_{\ell=1}^{k} M_{j\ell} P^{(\ell)} + \beta_j P^{(d)}.$$

Collecting them into $M \in \mathbb{R}^{k \times k}$ and $a, \beta \in \mathbb{R}^k$, we may write

$$h_i = \begin{pmatrix} Mx_i + \beta t_i + a \\ s_i \end{pmatrix}.$$

Here $P_i = \binom{x_i}{t_i}$ is the position of vertex i, and s_i is its vertical velocity. Set

$$K = \begin{pmatrix} (M - M^T)/2 & \beta \\ -\beta^T & 0 \end{pmatrix}.$$

The matrix K is skew-symmetric, so $i \mapsto \binom{a}{0} + KP_i$ is a trivial infinitesimal motion. Subtracting it gives

$$\widetilde{h}_i = h_i - \begin{pmatrix} a \\ 0 \end{pmatrix} - KP_i = \begin{pmatrix} Sx_i \\ r_i \end{pmatrix}, \qquad S = \tfrac{1}{2}(M + M^T), \quad r_i = s_i + \beta^T x_i.$$

Because $P_u = P_v$, this subtraction leaves the relative velocity unchanged. Thus $r_u - r_v \neq 0$. The edge equation on uc_j, together with (3.2), gives $S_{jj} = 0$. Finally, the equations on the edges of F are exactly (3.7). □

We now prove the rank lemma needed to apply (3.7). Its hypotheses concern only a graph with designated vertices and generic positions for the remaining vertices.

Lemma 3.3. *Let $k \geq 1$, and let F be a graph with vertex set $C \sqcup W$, where $C = \{c_1, \dots, c_k\}$ and $W \neq \varnothing$. Suppose that $F[W]$ is connected and $\deg_F(c_j) \geq k$ for every $c_j \in C$. Assign positions $(x_i, t_i) \in \mathbb{R}^k \times \mathbb{R}$ with $x_{c_j} = e_j$, $t_{c_j} = 0$, and with all coordinates at W algebraically independent over $\mathbb{Q}$. If a symmetric $k \times k$ matrix S with zero diagonal and scalars r_i, $i \in V(F)$, satisfy*

$$(x_i - x_j)^T S(x_i - x_j) + (t_i - t_j)(r_i - r_j) = 0 \qquad (ij \in E(F)), \tag{3.8}$$

then $S = 0$ and all r_i are equal.

Proof. Fix $i_0 \in W$ and impose $r_{i_0} = 0$. We call this an *anchor*: the equations depend only on differences of the r_i, so fixing one value removes the freedom to add a common constant. The equations form a linear system in the off-diagonal entries of S and the remaining r_i, with coefficients polynomial over $\mathbb{Q}$ in the free coordinates. It suffices to exhibit one specialisation at which this anchored system has trivial kernel. A nonzero minor using every column will then also be nonzero at algebraically independent coordinates.

We argue by induction on k. If $k = 1$, then $S = 0$. Choose pairwise distinct nonzero heights at the vertices of W. The equations on the connected graph $F[W]$ force the r_i on W to be equal. The vertex c_1 has a neighbour in W, whose edge equation gives the same value at c_1. The anchor makes this value zero.

Suppose $k \geq 2$. In $F - c_k$, every remaining designated vertex has degree at least $k - 1$, and the induced graph on W remains connected. Specialise $x_i = (x_i', 0)$ for $i \neq c_k$, with $x'_{c_j} = e_j \in \mathbb{R}^{k-1}$ for $j < k$, and retain $x_{c_k} = e_k$. Write

$$S = \begin{pmatrix} T & b \\ b^T & 0 \end{pmatrix}.$$

The equations on $F - c_k$ involve only T and the variables r_i, $i \neq c_k$. Let Y consist of the free coordinates (x_i', t_i), $i \in W$. By induction, a square minor $f(Y)$ using every column of the anchored coefficient matrix for $F - c_k$ is a nonzero polynomial. Whenever $f(Y) \neq 0$, these equations give $T = 0$ and $r_i = \rho$ for every $i \neq c_k$. The equation on an edge $c_k i$ then becomes

$$2b^T x_i' + t_i(r_{c_k} - \rho) = 0. \tag{3.9}$$

Suppose c_k has s neighbours among $c_1, \dots, c_{k-1}$. Their coefficient rows $(2x_i', t_i) = (2e_j, 0)$ are independent. Since $\deg_F(c_k) \geq k$, it has at least $k - s$ neighbours in W. Select $k - s$ of them. Their rows can be chosen arbitrarily in $\mathbb{R}^k$, so together with the fixed rows they can form a basis. The determinant $g(Y)$ of the resulting $k \times k$ matrix is therefore a nonzero polynomial.

Both f and g belong to the polynomial ring $\mathbb{Q}[Y]$, an integral domain. Their product is nonzero, so there is a real specialisation with $f(Y)g(Y) \neq 0$. The first factor gives $T = 0$ and the common value ρ; the second and (3.9) give $b = 0$ and $r_{c_k} = \rho$. Since $r_{i_0} = 0$, the full anchored system has trivial kernel at this specialisation. This proves the inductive step and the lemma. □

Claim 5. *The reduced motion in Claim 4 satisfies $S = 0$ and $r_u = r_v$.*

Proof. Lemma 3.2 gives that $F[W]$ is connected, $\deg_F(c) \geq d \geq k$ for $c \in C$, and the coordinates at W are algebraically independent. Thus Lemma 3.3, applied to (3.7), gives $S = 0$ and $r_i = \rho$ for every $i \in V(F)$.

Choose $a \in A$ and $b \in B$, which are nonempty by Lemma 3.2. Genericity of the free coordinates gives $t_a \neq 0$ and $t_b \neq 0$. The equations on ua and vb now yield $t_a(r_u - \rho) = 0$ and $t_b(r_v - \rho) = 0$, respectively. Hence $r_u = r_v = \rho$. □

Completion of the first proof. Claim 5 contradicts $r_u - r_v \neq 0$ in Claim 4. The normalised framework (G, P) is therefore infinitesimally rigid. Inverting the affine change preserves rigidity-matrix rank, so the coincident framework obtained from the arbitrary generic realisation $\widehat{q}$ is also infinitesimally rigid. This proves Theorem 1.3. The graph $H = G/uv$ is generically globally rigid, $|N_G(u) \cap N_G(v)| = d - 1$, and G has the required coincident infinitesimally rigid realisation. Theorem 2.2 now gives Theorem 1.2. □

4. Second proof: radial deformation of stresses

We give a second proof of Theorem 1.3. The construction in this section changes a realisation of the contracted graph together with its stress. It preserves the graph support and stress rank, while giving explicit control of the partial forces at the contracted vertex.

Lemma 4.1. *Let p be a realisation of a graph H on m vertices that affinely spans $\mathbb{R}^d$, and suppose that (H, p) has an equilibrium stress matrix Ω of rank $m - d - 1$. Fix $w \in V(H)$, put $X = [\,\mathbf{1}\ p\,]$, and let $c \in \mathbb{R}^{V(H)}$ satisfy*

$$X^T c = 0, \qquad \operatorname{supp}(c) \subseteq \{w\} \cup N_H(w).$$

For all sufficiently small real t, there are a realisation $q(t)$ of H and an equilibrium stress $\omega(t)$ of $(H, q(t))$, with stress matrix $\Omega(t)$, such that

1. *$q(0) = p$, $\Omega(0) = \Omega$, and both depend continuously on t;*
2. *$q_w(t) = p_w$, and $q_i(t) - p_w$ is a nonzero scalar multiple of $p_i - p_w$ for every $i \neq w$;*
3. *$q(t)$ affinely spans $\mathbb{R}^d$, and $\operatorname{rank}\Omega(t) = m - d - 1$;*
4. *for every $Y \subseteq N_H(w)$,*

$$F_Y(w; \omega(t), q(t)) = F_Y(w; \omega, p) - t\sum_{i \in Y} c_i(p_i - p_w), \tag{4.1}$$

where ω is the stress represented by Ω.

If (H, p) is infinitesimally rigid, then $(H, q(t))$ is infinitesimally rigid for all sufficiently small t.

Proof. Since $\Omega X = 0$, the rank hypothesis and affine spanning give

$$\ker\Omega = \operatorname{col} X, \qquad \operatorname{im}\Omega = \ker X^T.$$

The second equality uses symmetry of Ω. We can therefore choose $\eta \in \mathbb{R}^{V(H)}$ with $\Omega\eta = c$. Adding a constant vector to η, if necessary, we may also suppose that $\eta_w = 0$.

Let e_w be the coordinate vector indexed by w, and define

$$T_t = I + t\eta e_w^T, \qquad \lambda_i(t) = 1 - t\eta_i, \qquad D_t = \operatorname{diag}(\lambda_i(t)).$$

Since $e_w^T\eta = 0$, we have $T_t^{-1} = I - t\eta e_w^T$. For sufficiently small t, every $\lambda_i(t)$ is nonzero. Define

$$[\,\mathbf{1}\ q(t)\,] = D_t^{-1}T_t^{-1}X, \qquad \Omega(t) = D_tT_t^T\Omega T_tD_t. \tag{4.2}$$

The first column in the expression defining $q(t)$ is indeed $\mathbf{1}$, because $T_t^{-1}\mathbf{1} = \mathbf{1} - t\eta = D_t\mathbf{1}$. The position formula is

$$q_i(t) = \frac{p_i - t\eta_i p_w}{1 - t\eta_i}, \qquad q_i(t) - p_w = \frac{p_i - p_w}{\lambda_i(t)}. \tag{4.3}$$

In particular, $q_w(t) = p_w$.

The matrices T_t and D_t are invertible. Thus $[\,\mathbf{1}\ q(t)\,]$ has rank $d+1$, and congruence preserves the rank of Ω. Moreover,

$$\Omega(t)[\,\mathbf{1}\ q(t)\,] = D_t T_t^T \Omega X = 0.$$

This proves both equilibrium and the zero row-sum condition, provided that the graph support is preserved.

To verify the support, expand

$$T_t^T \Omega T_t = \Omega + t(e_w c^T + c e_w^T) + t^2(\eta^T \Omega \eta) e_w e_w^T. \tag{4.4}$$

Off-diagonal entries outside row and column w are unchanged. For $i \neq w$, the entry indexed by wi changes only if $c_i \neq 0$, in which case $wi \in E(H)$ by the support hypothesis. Multiplication by D_t preserves zero entries. Hence $\Omega(t)$ is the stress matrix of an equilibrium stress of $(H, q(t))$.

Since $\lambda_w(t) = 1$, equation (4.4) gives

$$\omega_{wi}(t) = \lambda_i(t)(\omega_{wi} - t c_i) \qquad (i \in N_H(w)).$$

Multiplying this identity by (4.3) and summing over Y gives (4.1). The formulas also imply $q(t) \to p$ and $\Omega(t) \to \Omega$ as $t \to 0$.

Finally, if (H, p) is infinitesimally rigid, a minor of $R(H, p)$ of size $dm - \binom{d+1}{2}$ is nonzero. It remains nonzero for all sufficiently small t, which proves the final assertion. □

Second proof of Theorem 1.3. Choose a generic realisation p of H. It is infinitesimally rigid. Since $A, B \neq \varnothing$ and $|C| = d-1$, the vertex w has at least $d+1$ neighbours, and hence $m \geq d+2$. By Theorem 1.1, (H, p) has a stress matrix Ω of rank $m - d - 1$.

Put

$$L = \operatorname{span}\{p_i - p_w : i \in C\}, \qquad 0 \neq \nu \in L^\perp.$$

Genericity gives $\dim L = d-1$. Choose $a \in A$ and $b \in B$. The $d+2$ points indexed by $S = \{w\} \cup C \cup \{a, b\}$ have a one-dimensional space of affine dependences. Choose a nonzero dependence c, extended by zero outside S. Every coefficient on S is nonzero, because every $d+1$ of these points are affinely independent. In particular,

$$c_a \neq 0, \qquad \nu \cdot (p_a - p_w) \neq 0. \tag{4.5}$$

Apply Lemma 4.1 with this dependence. For small t, each common-neighbour direction is rescaled by a nonzero factor, so $\operatorname{span}\{q_i(t) - p_w : i \in C\} = L$. The same normal vector ν therefore applies throughout the deformation. Since $\operatorname{supp}(c) \cap A = \{a\}$, equation (4.1) gives

$$\nu \cdot F_A(w; \omega(t), q(t)) = \nu \cdot F_A(w; \omega, p) - t c_a\, \nu \cdot (p_a - p_w).$$

By (4.5), the displayed partial force is an affine function of t with nonzero slope, so it has at most one zero. Choose $\varepsilon > 0$ so that every $\lambda_i(t)$ is nonzero and $(H, q(t))$ is infinitesimally rigid whenever $|t| < \varepsilon$. We may therefore choose t arbitrarily close to zero with $|t| < \varepsilon$ so that

$$\nu \cdot F_A(w; \omega(t), q(t)) \neq 0.$$

Proposition 2.3 implies that the coincident framework $(G, Eq(t))$ is infinitesimally rigid.

Lemma 2.1 now gives infinitesimal rigidity at every generic coincident realisation, proving Theorem 1.3. Theorem 1.2 follows from Theorem 2.2. □

5. Nondegenerate stresses

The same deformation proves Theorem 1.4. We first record the transfer from a regular realisation to a generic one, due to Jordán and Tanigawa [11, Lemma 3.1(a)]. We include the proof to specify the genericity needed below. A realisation is *regular* if its rigidity matrix attains the maximum rank over all realisations of the graph in $\mathbb{R}^d$.

Lemma 5.1. *Let H be a graph and $w \in V(H)$. Fix a proper d-subpartition $(X_1, \dots, X_d)$ of $N_H(w)$. If a regular realisation of H has an equilibrium stress nondegenerate with respect to this vertex and subpartition, then every generic realisation of H has such a stress.*

Proof. Let r be the maximum rank of $R(H,p)$, and let p^0 be the given regular realisation. Choose row and column index sets J, I of size r for which

$$\Delta(p) = \det\big(R(H,p)^T[J,I]\big)$$

is nonzero at p^0. The indices in I correspond to edges. Let $K = E(H) \setminus I$. On the set $\Delta(p) \neq 0$, write $a \in \mathbb{R}^K$ for the free stress coefficients and set

$$\omega_K = a, \qquad \omega_I = -\big(R(H,p)^T[J,I]\big)^{-1} R(H,p)^T[J,K]a.$$

Because the selected rows form a basis for the row space of $R(H,p)^T$, these equations parametrise its entire kernel. We may therefore write $\omega = B(p)a$, where the entries of $B(p)$ are rational functions, with rational coefficients, of the coordinates of p.

Substitute this parametrisation into

$$\det\big[F_{X_1}(w;\omega,p) \ \cdots \ F_{X_d}(w;\omega,p)\big].$$

Multiplication by $\Delta(p)^d$ gives a polynomial $N(p,a)$ with rational coefficients. At the given realisation and stress it is nonzero, so N is not the zero polynomial. At a generic realisation p, we have $\Delta(p) \neq 0$. Moreover, at least one coefficient of $N(p,a)$, viewed as a polynomial in a, is a nonzero polynomial in the coordinates of p, and hence does not vanish. Thus $N(p,a)$ is not identically zero as a function of a. Some real choice of a makes it nonzero and yields the required stress. □

Proof of Theorem 1.4. First suppose that H has $m \geq d+2$ vertices. Fix $w \in V(H)$ and a proper d-subpartition $(X_1, \dots, X_d)$ of $N_H(w)$. Choose a generic realisation p of H. Since H is generically globally rigid, (H,p) is infinitesimally rigid. By Theorem 1.1, it has a stress matrix Ω of rank $m-d-1$.

For each i, choose $a_i \in X_i$, and choose $b \in N_H(w) \setminus \bigcup_{i=1}^d X_i$. Let c be a nonzero affine dependence of the $d+2$ points indexed by $\{w, a_1, \dots, a_d, b\}$, extended by zero to all other vertices. Genericity implies that every coefficient on these vertices is nonzero. Lemma 4.1 gives

$$F_{X_i}(w;\omega(t),q(t)) = F_{X_i}(w;\omega,p) - tc_{a_i}(p_{a_i} - p_w) \qquad (1 \leq i \leq d).$$

The determinant of these d column vectors is a polynomial in t. Its coefficient of t^d is

$$(-1)^d \left(\prod_{i=1}^d c_{a_i}\right) \det[\, p_{a_1} - p_w \ \cdots \ p_{a_d} - p_w \,],$$

which is nonzero by genericity. We may therefore choose t arbitrarily close to zero so that $\omega(t)$ is nondegenerate with respect to the chosen vertex and subpartition and $(H, q(t))$ is infinitesimally rigid. Its rigidity matrix has rank $dm - \binom{d+1}{2}$, so $q(t)$ is regular. Lemma 5.1 now shows that every generic realisation has a stress nondegenerate with respect to this fixed vertex and subpartition.

Fix one generic realisation p. For each vertex and proper d-subpartition of its neighbourhood, the corresponding force determinant is a polynomial on the finite-dimensional real vector space $\mathcal{S}(H, p)$. The preceding argument shows that each of these polynomials is nonzero. There are only finitely many of them. Also, by Theorem 1.1, some stress matrix minor of size $m - d - 1$ is a nonzero polynomial on $\mathcal{S}(H, p)$. After choosing linear coordinates on this space, the product of all these polynomials is a nonzero polynomial. We can choose a stress at which the product does not vanish. This single stress is nondegenerate at every vertex and proper subpartition, and its stress matrix has rank $m - d - 1$.

If $m \leq d + 1$, generic global rigidity implies that H is complete. Every neighbourhood has at most d vertices, so there are no proper d-subpartitions. Nondegeneracy is vacuous in this case. The zero stress has rank $0 = \max\{0, m - d - 1\}$, as required. □

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