# Explicit lower bounds for the number of irrational even beta values

twinheterodox
Prepared with Codex

4 November 2026

**Abstract**

For every even integer $s \geq 160$, we prove that at least $\left\lfloor \pi\sqrt{2/135}\,\sqrt{s/\log s} \right\rfloor$ of the numbers $\beta(2), \beta(4), \ldots, \beta(s)$ are irrational, where $\beta$ is the Dirichlet beta function. We also give sharper finite endpoints for specified counts of irrational values.

hexagon:2611.00139v1

## 1 Introduction

For $u > 1$, the Dirichlet beta function is

$$\beta(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^u} = 1 - \frac{1}{3^u} + \frac{1}{5^u} - \cdots .$$

Its values at positive odd integers are nonzero rational multiples of powers of $\pi$, and hence transcendental. The arithmetic of its even values is much less understood.

For an even positive integer $s$, write

$$I(s) = \#\{j : 1 \leq j \leq s/2,\ \beta(2j) \notin \mathbb{Q}\}$$

for the number of irrational values among $\beta(2), \beta(4), \ldots, \beta(s)$.

In [4] we proved that the rational span of $\beta(2), \beta(4), \ldots, \beta(s)$ has dimension at least $0.21\sqrt{s/\log s}$ for sufficiently large even $s$. The same lower bound holds for $I(s)$; see Appendix B.

Here we show that an irrationality bound of the same shape and a better constant is valid from a small explicit value of $s$. In particular, Theorem 1.1 gives a stronger count for every even $s \geq 160$. We use the constant[1]

$$c = \pi\sqrt{\frac{2}{135}} = 0.3823824806363651\ldots .$$

All logarithms in this paper are natural.

**Theorem 1.1.** *For every even integer $s \geq 160$,*

$$I(s) \geq \left\lfloor c\sqrt{\frac{s}{\log s}} \right\rfloor . \tag{1.1}$$

For every even $s \geq 42\,170$, we will also prove the bound without the floor.

The construction gives more precise information at individual endpoints. For each column below, at least $B$ of the even beta values through $\beta(s)$ are irrational. For comparison, the last row shows the expression inside the floor in (1.1), rounded to two decimal places.

[1] Replacing $c$ by 0.358 gives a bound of the same shape for every even $s \geq 64$, giving one irrational value at 64, but only two rather than three at 364. For $64 \leq s \leq 158$, this follows from $I(64) \geq 1$ (Section 8) and $0.358\sqrt{158/\log 158} < 2$; for $s \geq 160$, use Theorem 1.1.

| $B$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $s$ | 160 | 364 | 588 | 1030 | 1296 | 1992 | 2502 | 3260 | 3828 |
| $c\sqrt{s/\log s}$ | 2.15 | 3.00 | 3.67 | 4.66 | 5.14 | 6.19 | 6.84 | 7.68 | 8.24 |

At $s = 160$ and $s = 364$, the expression inside the floor is $2.147\ldots$ and $3.004\ldots$ respectively. Hence the uniform bound (1.1) matches the counts in the table at these endpoints. At $s = 588$, the uniform bound gives $I(588) \geq 3$ after taking the floor of $3.67\ldots$, whereas applying the construction with parameters chosen for this endpoint gives $I(588) \geq 4$.

### 1.1 A finite endpoint for every requested count

We prove the main theorem by first giving an endpoint for every positive integer $B$. Let $\varphi$ be Euler's totient function, with $\varphi(1) = 1$.

**Theorem 1.2.** *For every integer $B \geq 1$, put*

$$D_B = \sum_{d=1}^{B} \varphi(d), \qquad s_B = 2\left\lceil D_B\left(33 + \frac{45}{2}\log B\right)\right\rceil. \tag{1.2}$$

*Then $I(s_B) \geq B$. The sharper endpoints in the preceding table also hold.*

The quantity $D_B$ counts the distinct reduced fraction classes needed for the $B$ grids in our construction. It grows quadratically in $B$; the endpoint has size $B^2 \log B$. Inverting this relation produces the square root in Theorem 1.1. The formula for $c$ is also of interest in its own right. The $\pi$ and the factor 3 in $135 = 3 \cdot 45$ come from $D_B \sim (3/\pi^2)B^2$, the asymptotic count of reduced fractions. The factor 45 comes from balancing denominator growth against decay, and the factor 2 from inverting the quadratic growth in $B$. Section 9 makes the inversion explicit, including the constant and the small endpoints.

### 1.2 Related constructions

Lai and Yu [3] obtained lower bounds of order $\sqrt{s/\log s}$ for the number of irrational odd zeta values, using rational functions whose zeros are arranged on rational grids. Kumar and Tosi [1, Theorem 1] adapt this approach to real Dirichlet characters. For the beta function, their stated asymptotic coefficient is $1.1510775\ldots$, with the starting endpoint depending on the chosen loss in that coefficient. Theorem 1.1 gives a fixed bound at every even endpoint from 160 onward.

The rational-function construction and coefficient elimination used here build on these methods. The separation of coefficients by an auxiliary prime has an antecedent in Lai and Sprang [2]. A positive Gram matrix supplies the needed rank for every $B$. We give the denominator, decay and prime-dilation arguments in full.

### 1.3 How the contradiction will arise

Fix $B$ and an even endpoint $s$. Suppose that fewer than $B$ of the listed beta values are irrational. We construct $B$ sequences of linear forms, indexed by positive odd integers $n$, of the shape

$$F_{n,b} = q_b(n) + \sum_{\substack{2 \leq i \leq s \\ i \text{ even}}} b^i \rho_i(n)\beta(i), \qquad 1 \leq b \leq B, \tag{1.3}$$

with rational coefficients. The coefficient column at $\beta(i)$ is a scalar multiple of the fixed vector $(1^i, 2^i, \ldots, B^i)^{\mathsf{T}}$. Fixed integer weights on the $B$ forms can therefore cancel any $B - 1$

specified beta terms at every index $n$. Choosing these terms to include the possible irrational values produces a sequence of rational numbers.

For example, when $B = 3$, the combination $15F_{n,1} - 6F_{n,2} + F_{n,3}$ cancels the coefficients at $\beta(2)$ and $\beta(4)$, since $15 - 6 \cdot 2^i + 3^i = 0$ for $i = 2, 4$. The same weights work for every $n$ because the factor $\rho_i(n)$ is shared by the forms.

We bound the growth of a common denominator for the coefficients and choose the parameters so that the forms decay faster than this denominator grows. The rational beta values that remain after cancellation contribute only a fixed additional denominator. Thus the fixed combination, after clearing all denominators, gives a sequence of integers tending to zero. To obtain a contradiction, we must show that infinitely many of these integers are nonzero.

The cancellation weights can be chosen with a nonzero last entry. Indeed, the first $B - 1$ coordinates of the chosen power vectors form a nonsingular generalized Vandermonde matrix (Section 7). We therefore arrange the constant vector at the first index to have only its last coordinate nonzero:

$$(q_1(1), \dots, q_B(1)) = \eta e_B, \qquad \eta \neq 0, \tag{1.4}$$

where $e_B = (0, \dots, 0, 1)$. The combined form then has a nonzero constant term at $n = 1$. A congruence at indices $n = p$, for arbitrarily large primes $p$, transfers this nonvanishing: after a suitable scaling, the constant term survives modulo $p$ and all remaining beta terms vanish. We obtain nonzero integers tending to zero, which is impossible.

The main construction is the choice of (1.4). We find $B$ polynomial multipliers whose constant vectors form a rational invertible matrix. Its entries are integrals of products of independent polynomials against one positive weight. This proves invertibility for every $B$ and allows us to prescribe the seed constant vector. The remaining sections check that these multipliers preserve the denominator and decay estimates needed for the contradiction.

## 2 Rational functions on the required grids

### 2.1 Zeros, poles and reflection

The $b$th form will be an alternating sum over the points $j/b$ with $j$ odd. We need numerator zeros at the initial points of every such grid, both to remove their contribution to the sum and to control the denominators of its rational constant. The signs on odd summation indices repeat after a shift by 4, so poles at $0, -4, \dots, -4n$ allow each partial-fraction sum to be shifted back to the beta series. The distinct fraction classes that occur for $1 \le b \le B$ are

$$\Theta = \left\{ \frac{a}{d} : 1 \le d \le B,\ 1 \le a \le d,\ (a, d) = 1 \right\}.$$

The integer class is represented by 1. Thus $|\Theta| = D_B$. For example, at $B = 3$ the classes are $1, 1/2, 1/3, 2/3$.

Fix an integer $r \ge 3$. If the rightmost zero at index $n$ is $rn$, reflection about $-2n$, the midpoint of the pole interval, puts the leftmost zero at $-(r+4)n$. The zero interval therefore has length $(2r+4)n$. Abbreviate

$$D = D_B, \qquad L = 2r + 4, \qquad S = s + 1, \qquad S > DL,$$

where $s$ is even. The integer $r$ controls the length of the initial zero interval. To normalize the denominators of each fraction class, put

$$A_d = d^L \prod_{p \mid d} p^{\lceil L/(p-1) \rceil}, \qquad K = \prod_{\theta \in \Theta} A_{\mathrm{den}(\theta)}, \tag{2.1}$$

where the first product is over primes and $\mathrm{den}(\theta)$ is the reduced denominator. The progression estimate in Section 4 explains this normalization.

For every positive odd integer $n$, define

$$P_n(t) = (t - rn) \prod_{\theta \in \Theta} \prod_{j=0}^{Ln-1} (t - rn + j + \theta), \tag{2.2}$$

$$C_n = 4^{Sn} K^n \frac{((n+1)!)^S}{(n!)^{DL}}, \qquad R_n(t) = \frac{C_n P_n(t)}{\prod_{h=0}^{n} (t+4h)^S}. \tag{2.3}$$

The zero set of $P_n$ consists of the required grid points in $[-(r+4)n, rn]$. All these zeros are simple. In particular, one zero cancels at every denominator point $0, -4, \ldots, -4n$, leaving pole order at most $s$.

Reflection about the midpoint $-2n$ preserves the zero set. The numerator has odd degree, so $P_n(-4n - t) = -P_n(t)$. Since $n$ is odd, the $n+1$ denominator factors give no sign change. Hence

$$R_n(-4n - t) = -R_n(t).$$

We will multiply $R_n$ by an integer polynomial $W_n$ that is also odd about $-2n$. The product $T_n = W_n R_n$ will then be reflection invariant. This symmetry is what removes the odd beta coefficients. The polynomial $W_n$ will have a fixed homogeneous degree $E < s$ in the variables $n, t$.

### 2.2 The criterion to be proved

To prove decay, we estimate the largest terms in the alternating sums of $T_n$. The rational number $u$ below bounds where their exponential size is largest, with distance from the last zero $rn$ measured in units of $n$. Once that location is bounded, the quantity $U$ bounds the fourth power of the exponential rate after clearing denominators. Thus the two inequalities locate the maximum and make the cleared forms tend to zero. The bound $\mathrm{e} < 68/25$ keeps the criterion rational.

**Proposition 2.1.** *With the preceding notation, suppose that a positive rational $u$ satisfies*

$$f(u) = \left(\frac{u+L}{u}\right)^{4D} \left(\frac{u+r}{u+r+4}\right)^{S} < 1, \tag{2.4}$$

$$U = \left(\frac{68}{25}\right)^{4S} K^4 (L+u)^{4DL} 4^{4S} \left(\frac{r^r}{(r+4)^{r+4}}\right)^{S} < 1. \tag{2.5}$$

*Then $I(s) \geq B$.*

Section 3 constructs the seed constant vector, Sections 4– 6 prove the arithmetic and analytic estimates, and Section 7 completes the contradiction. In Section 8 we choose parameters satisfying these inequalities uniformly in $B$.

## 3 Prescribing the constant vector by a Gram matrix

### 3.1 The seed constants as residues

At $n = 1$ the only poles are 0 and $-4$. This makes it possible to express the rational constants by integrals along the vertical line halfway between the poles. We first identify the rational kernels that occur in these integrals.

To write alternating sums over odd integers compactly, let $\chi_4(k)$ be 0 for even $k$, 1 for $k \equiv 1 \pmod 4$, and $-1$ for $k \equiv 3 \pmod 4$. Thus $\chi_4$ records the signs in the beta series. It is periodic with period 4 and multiplicative.

Let $T$ be a proper rational function with poles at $0, -4$, invariant under $t \mapsto -4 - t$, with partial fractions

$$T(t) = \sum_{i=1}^{s} \frac{a_i}{t^i} + \sum_{i=1}^{s} \frac{(-1)^i a_i}{(t+4)^i}.$$

In $\sum_{k\geq 1} \chi_4(k) T(k/b)$, shifting the second pole's sum by $4b$ leaves the signs unchanged and removes its first $4b$ terms. The rational constant is therefore

$$q_b = -\sum_{i=1}^{s} (-1)^i a_i b^i \sum_{k=1}^{4b} \frac{\chi_4(k)}{k^i}.$$

Introduce the kernel

$$H_b(t) = \sum_{\substack{1 \leq k < 4b \\ k \text{ odd}}} \frac{\chi_4(k)}{t + k/b}.$$

The coefficient of $t^{i-1}$ in $(t + k/b)^{-1}$ is $(-1)^{i-1}(b/k)^i$, so

$$q_b = \operatorname{Res}_{t=0} T(t) H_b(t). \tag{3.1}$$

On the midpoint line $t = -2 + iy$, pair the terms indexed by $k$ and $4b - k$. Their distances from the midpoint have opposite signs, as do their character weights. The result is a real rational function of $y^2$:

$$H_b(-2 + iy) = g_b(y^2), \qquad g_b(x) = -2 \sum_{j=1}^{b} (-1)^{b-j} \frac{(2j-1)/b}{x + ((2j-1)/b)^2}. \tag{3.2}$$

We will give these kernels a common positive denominator and use their numerators as multiplier polynomials. This makes the constant matrix an integral of products with one common weight.

*Example* 3.1. For $B = 2$,

$$g_1(x) = -\frac{2}{x+1}, \qquad g_2(x) = \frac{1}{x + 1/4} - \frac{3}{x + 9/4}.$$

With $Q(x) = (x+1)(4x+1)(4x+9)$, write $g_b = h_b/Q$. Then

$$h_1(x) = -2(4x+1)(4x+9), \qquad h_2(x) = 8(x+1)(3-4x).$$

The four entries of the constant matrix will be integrals of $h_1^2, h_1 h_2, h_2 h_1, h_2^2$ against the same positive weight. This is the reason for choosing the kernel numerators as multipliers.

### 3.2 Independent integer polynomials

Let $\mathcal{A}$ be the set of distinct positive pole distances:

$$\mathcal{A} = \left\{ \frac{2j-1}{b} : 1 \leq b \leq B,\ 1 \leq j \leq b \right\}, \qquad N = |\mathcal{A}|, \qquad J = N - 1.$$

For $a = v/d \in \mathcal{A}$, always use the reduced numerator and denominator. Define

$$Q(x) = \prod_{v/d \in \mathcal{A}} (d^2 x + v^2), \qquad h_b(x) = Q(x) g_b(x). \tag{3.3}$$

Then $Q(x) > 0$ for $x \geq 0$, and $h_b \in \mathbb{Z}[x]$ has degree at most $J$. Indeed, each summand of $g_b$ is of the form $\pm 2vd/(d^2x + v^2)$, whose denominator divides $Q$.

**Lemma 3.2.** *The polynomials $h_1, \dots, h_B$ are linearly independent over $\mathbb{R}$. Moreover,*

$$N = D + \sum_{\substack{d \le B \\ d \text{ even}}} \varphi(d) \le 2D. \tag{3.4}$$

*Proof.* In a nonzero relation among the $g_b$, choose the largest index $b$ with nonzero coefficient. The function $g_b$ has a pole at $x = -1/b^2$. Every earlier $g_a$ is regular there, since all its positive pole distances are at least $1/a > 1/b$. This is impossible. Multiplication by $Q$ preserves independence.

The elements of $\mathcal{A}$ are exactly the reduced fractions $v/d$ with $d \le B$, $v$ odd and $0 < v < 2d$. There are $\varphi(d)$ such numerators for odd $d$, and $2\varphi(d)$ for even $d$. Summing gives (3.4). □

At the seed, $-(t+2)^2 = y^2$ on the midpoint line. The multiplier $(t+2)h_b(-(t+2)^2)$ therefore produces the desired factor $h_b(y^2)$ there, while its odd factor $t+2$ makes the product with $R_1$ reflection invariant. To lift this choice to every odd index, we make it homogeneous in $n, t$:

$$\begin{aligned} W_{n,b}(t) &= n^{2J}(t+2n)h_b\left(-\frac{(t+2n)^2}{n^2}\right) \\ &= \sum_{k=0}^{J} h_{b,k}(-1)^k n^{2J-2k}(t+2n)^{2k+1}, \qquad h_b(x) = \sum_{k=0}^{J} h_{b,k}x^k. \end{aligned} \tag{3.5}$$

The expanded expression shows that the coefficients are integers. Each term has the same total degree in $n, t$:

$$E = 2J + 1 = 2N - 1 \le 4D - 1 < s. \tag{3.6}$$

The last inequality follows from $S > DL$ and $L \ge 10$. These polynomials are odd about $-2n$ and satisfy the useful scaling identity

$$W_{mp,b}(pt) = p^E W_{m,b}(t). \tag{3.7}$$

The degree bound ensures decay at infinity; the scaling identity will control prime dilation.

### 3.3 The common positive weight

Let $M_{b,j}$ be the $b$th rational constant associated with $W_{1,j}R_1$. We now prove that the matrix $M = (M_{b,j})$ is nonsingular.

Write $z = t + 2$. The monic numerator $P_1$ has simple real roots and is odd in $z$, so for a set $\Lambda$ of $DL/2$ positive rational numbers,

$$P_1(t) = z \prod_{\alpha \in \Lambda} (z^2 - \alpha^2).$$

On the line $t = -2 + iy$, the denominator $t^S(t+4)^S$ is $-(y^2+4)^S$, since $S$ is odd. Substitution gives

$$W_{1,j}(-2+iy)R_1(-2+iy) = (-1)^{DL/2}C_1 \frac{y^2 \prod_{\alpha \in \Lambda}(y^2+\alpha^2)}{(y^2+4)^S} h_j(y^2). \tag{3.8}$$

Every pole of $H_b$ is a point $-k/b \in (-4, 0)$ where $P_1$ vanishes. These poles cancel in $W_{1,j}R_1H_b$, leaving only 0 and $-4$. Close the upward midpoint line by a large semicircle to its right. The contour encloses 0 clockwise, so the residue theorem and (3.1) give

$$M_{b,j} = \frac{\sigma C_1}{2\pi} \int_{-\infty}^{\infty} \omega(y) h_b(y^2) h_j(y^2)\, dy, \qquad \sigma = (-1)^{DL/2+1}, \tag{3.9}$$

where

$$\omega(y) = \frac{y^2 \prod_{\alpha\in\Lambda}(y^2+\alpha^2)}{(y^2+4)^S Q(y^2)}.$$

For completeness, $H_b(t) = O(t^{-2})$ because its character weights sum to zero. The contour integrand therefore has degree at most $DL + 2N - 2S - 2 < -2$ at infinity. The arc integral vanishes, and the integral in (3.9) converges absolutely.

The weight $\omega(y)$ is positive for every real $y \neq 0$. For a nonzero real vector $v$, Lemma 3.2 shows that $\sum_b v_b h_b$ is a nonzero polynomial. Consequently

$$\sigma v^{\mathsf{T}} M v = \frac{C_1}{2\pi} \int_{-\infty}^{\infty} \omega(y) \left( \sum_b v_b h_b(y^2) \right)^2 dy > 0.$$

Thus $\sigma M$ is positive definite and $M$ is nonsingular.

The entries of $M$ are rational, being constants of rational partial fractions. We can therefore choose fixed integers $c_1, \dots, c_B$ such that

$$M(c_1, \dots, c_B)^{\mathsf{T}} = \eta e_B, \qquad \eta \in \mathbb{Q} \setminus \{0\}.$$

For all positive odd $n$, set

$$W_n = \sum_{j=1}^{B} c_j W_{n,j}, \qquad T_n = W_n R_n. \tag{3.10}$$

This gives the prescribed seed vector (1.4). The coefficients $c_j$ choose one multiplier $W_n$ shared by all $B$ forms. The weights introduced in Section 7 will combine those forms to cancel specified beta terms. The $c_j$ remain fixed as $n$ varies. Their size can affect fixed constants in estimates, but not the exponential decay rate.

# 4 The linear forms and their denominators

## 4.1 Extracting the beta coefficients

The function $T_n$ in (3.10) is reflection invariant and has pole order at most $s$ at each of $0, -4, \dots, -4n$. Its degree at infinity is at most $-(S - DL)n - s + E$. Thus it has a partial-fraction expansion

$$T_n(t) = \sum_{h=0}^{n} \sum_{i=1}^{s} \frac{a_{i,h}(n)}{(t+4h)^i}, \qquad \rho_i(n) = \sum_{h=0}^{n} a_{i,h}(n). \tag{4.1}$$

Reflection gives $a_{i,n-h}(n) = (-1)^i a_{i,h}(n)$, so $\rho_i(n) = 0$ for odd $i$.

For $1 \le b \le B$, define the absolutely convergent alternating sum

$$F_{n,b} = \sum_{k=0}^{\infty} (-1)^k T_n\left(\frac{2k+1}{b}\right) = \sum_{j\ge 1} \chi_4(j) T_n(j/b). \tag{4.2}$$

In the term belonging to the pole $-4h$, replace $j$ by $j + 4bh$. The character is unchanged, since $4bh$ is a multiple of its period. Therefore

$$F_{n,b} = q_b(n) + \sum_{\substack{2\le i\le s \\ i \text{ even}}} b^i \rho_i(n) \beta(i), \tag{4.3}$$

$$q_b(n) = -\sum_{h=0}^{n} \sum_{i=1}^{s} a_{i,h}(n) b^i \sum_{j=1}^{4bh} \frac{\chi_4(j)}{j^i}. \tag{4.4}$$

For $i = 1$, each individual shifted character series converges conditionally. There are only finitely many partial-fraction sums, so the decomposition remains valid, and its total coefficient at $\beta(1)$ vanishes by reflection. At $n = 1$, (4.4) is precisely the seed constant used in Section 3.

### 4.2 A progression estimate for numerator blocks

We need one common multiplier that makes the coefficients and constants integral. Put

$$d_n = \operatorname{lcm}(1, \dots, n), \qquad \delta_n = 4d_n.$$

The key point is that taking a Taylor coefficient of order $k$ loses at most $k$ prime-power contributions at each level. The power $d_n^k$ restores them.

**Lemma 4.1.** *For $\theta = a/d \in \Theta$, define*

$$B_\theta(t) = \frac{A_d^n}{(n!)^L} \prod_{j=0}^{Ln-1} (t - rn + j + a/d).$$

*For every integer $v$ and every $k \geq 0$,*

$$d_n^k [t^k] B_\theta(t + v) \in \mathbb{Z}. \tag{4.5}$$

*Proof.* First clear the factor $d$ in every denominator. The factor $d^{Ln}$ in $A_d^n$ does this exactly. Fix a prime $q$ not dividing $d$. In the resulting progression of integer constants, divide the $Ln$ terms into $L$ blocks of length $n$. At each level $q^e \leq n$, each block contains at least $\lfloor n/q^e \rfloor$ divisible constants. A term in the coefficient of $t^k$ replaces $k$ of these constants by variable coefficients. It therefore removes at most $k$ contributions at each level. Summing over $e$ gives valuation at least

$$Lv_q(n!) - kv_q(d_n).$$

The factors $(n!)^{-L}$ and $d_n^k$ compensate for this bound. A term retaining a zero constant is zero and needs no estimate.

If $q \mid d$, the cleared constants are units at $q$. The extra factor in $A_d^n$ has valuation $n\lceil L/(q-1) \rceil \geq Ln/(q-1) \geq Lv_q(n!)$. It covers the factorial denominator. These arguments apply to each coefficient term, proving (4.5). □

**Proposition 4.2.** *For every positive odd $n$,*

$$\delta_n^{S-i} a_{i,h}(n) \in \mathbb{Z}, \qquad \delta_n^{S-i} \rho_i(n) \in \mathbb{Z}, \qquad \delta_n^S q_b(n) \in \mathbb{Z}. \tag{4.6}$$

*Proof.* Write $B_n(t) = K^n P_n(t)/(n!)^{DL}$. Suppress the index $n$ on $a_{i,h}$. Expanding at the pole $-4h$ gives

$$\begin{aligned} a_{i,h} = (n+1)^S (-1)^{Sh} \binom{n}{h}^S \\ \times [t^{S-i}] \left\{ W_n(t-4h) B_n(t-4h) \prod_{k \neq h} \left(1 + \frac{t}{4(k-h)}\right)^{-S} \right\}. \end{aligned} \tag{4.7}$$

To see the normalization, the constants from the other poles have product $4^{Sn}(-1)^{Sh}(h!(n-h)!)^S$. Their power of 4 cancels the factor $4^{Sn}$ in $C_n$, leaving the integer prefactor in (4.7).

Every $4(k-h)$ divides $\delta_n$, and $W_n$ is an integer polynomial. Lemma 4.1, followed by the binomial series for the remaining factors, proves the first bound in (4.6). Summing in $h$ proves the second.

The last bound needs the numerator zeros as well as the coefficient bounds. Fix an odd $j \leq 4bh$ and put $t_0 = j/b - 4h$. This is a numerator zero in $(-4n, 0)$ and is not a pole. Hence

$$\sum_i \frac{a_{i,h}}{(j/b)^i} = -\sum_{k \neq h} \sum_i \frac{a_{i,k}}{(j/b + 4(k-h))^i}. \tag{4.8}$$

At a prime $q$, let $v = v_q(\delta_n)$ and write $x_k = j/b + 4(k-h)$. At most one $x_k$ has valuation greater than $v$: otherwise the difference of two such terms, $4(k-\ell)$, would also have valuation greater than $v$, although it divides $\delta_n$. The coefficient bound gives

$$v_q\left(\frac{\delta_n^S a_{i,k}}{x_k^i}\right) \geq i(v - v_q(x_k)). \tag{4.9}$$

If $v_q(x_h) \leq v$, the scaled left side of (4.8) is $q$-integral. Otherwise all groups on the right are $q$-integral, so the left side is again $q$-integral. This holds at every prime. Summing these integral groups with the signs in (4.4) proves $\delta_n^S q_b(n) \in \mathbb{Z}$. □

The grouping is important: the summands with a fixed $i$ need not be integral separately. For example, at $n = 1$, $b = 3$, $h = 1$, $j = 5$, the zero $T_1(-7/3) = 0$ gives

$$\sum_i a_{i,1}(1)\left(\frac{3}{5}\right)^i = -\sum_i a_{i,0}(1)\left(-\frac{3}{7}\right)^i.$$

The right side is integral at 5, by the coefficient bounds, so the powers of 5 in the left-side denominators cancel after the terms are grouped. Equation (4.8) makes this cancellation available at every index and every prime.

# 5 Decay after clearing denominators

Clearing denominators multiplies each form by $\delta_n^S$. The prime number theorem gives $\delta_n^{1/n} \to \mathrm{e}$, so it suffices to bound the $n$th-root limsup of $|F_{n,b}|$ strictly below $\mathrm{e}^{-S}$. We estimate the alternating sums by their absolute values. The initial grid points are zeros, and the remaining terms lie to the right of $rn$, where $R_n$ is positive. The polynomial $W_n$ has fixed degree and will not change the exponential rate.

## 5.1 The exponential profile

We measure distance beyond the last zero in units of $n$, writing $t = nx$. The function $\Phi(x)$ below records the exponential size of $R_n(rn + t)$ at that scaled position. For $t > 0$, the numerator and denominator products have the gamma representations

$$P_n(rn+t) = t\prod_{\theta\in\Theta}\frac{\Gamma(t+Ln+\theta)}{\Gamma(t+\theta)},$$
$$\prod_{h=0}^{n}(rn+t+4h) = 4^{n+1}\frac{\Gamma(((r+4)n+t)/4+1)}{\Gamma((rn+t)/4)}.$$

Together with the factorials in $C_n$, Stirling's formula gives

$$\begin{aligned}\Phi(x) &= \lim_{\substack{n\to\infty\\ n\text{ odd}}}\frac{1}{n}\log R_n(n(r+x))\\ &= \log K + D\big((x+L)\log(x+L) - x\log x\big)\\ &\quad - \frac{S}{4}\big((x+r+4)\log(x+r+4) - (x+r)\log(x+r)\big) + S\log 4.\end{aligned} \tag{5.1}$$

The coefficient of $n\log n$ from the factorials is $S - DL$; the product ratio contributes its negative. The linear Stirling terms also cancel. The remaining expression is (5.1).

Let $f$ be the function in (2.4). Direct differentiation gives $4\Phi'(x) = \log f(x)$. The derivative of $\log f$ has positive denominator and numerator

$$4\big((S-DL)x^2 + L(S-DL)x - DLr(r+4)\big).$$

This quadratic has exactly one positive zero. Thus $\log f$ first decreases and then increases. It starts at $+\infty$ and approaches 0 from below, with leading term $-4(S-DL)/x$. Consequently it has exactly one zero $x_0 > 0$, where $\Phi$ attains its global maximum. The identity $\Phi'(x_0) = 0$ simplifies that maximum to

$$G = \exp(\Phi(x_0)) = K(x_0+L)^{DL}4^S\left(\frac{(x_0+r)^r}{(x_0+r+4)^{r+4}}\right)^{S/4}. \tag{5.2}$$

## 5.2 Summing the profile and controlling the tail

Fix $A > 0$. We must allow $t$ to stay as small as $1/B$ while $n$ grows. Only the gamma factors $\Gamma(t+\theta)$ can have bounded arguments. Stirling's formula for $t \geq 1$ and continuity on the compact interval $[1/B, 1]$ give, uniformly for $\theta \in \Theta$,

$$\log\Gamma(t+\theta) = t\log t - t + O_B(\log(2+t)), \qquad t \geq 1/B.$$

All other gamma arguments grow proportionally to $n$ on $1/B \leq t \leq An$. Their logarithmic errors are $O_A(\log n)$, as are the error above and the extra term $\log t$ from $P_n$. Substitution in the gamma formulas therefore gives the uniform estimate

$$\log R_n(rn+t) = n\Phi(t/n) + O_A(\log n), \qquad 1/B \leq t \leq An.$$

The lower endpoint $1/B$ is sufficient: the first nonzero point on any of our grids lies at least that distance beyond $rn$. Homogeneity gives $|W_n(rn+t)| = O_A(n^E)$ on this range, which contains $O_A(n)$ grid points. Their absolute sum therefore has $n$th-root limsup at most $G$.

We also need a bound uniform over the infinite tail. Set

$$\tau = s - E > 0, \tag{5.3}$$

and take $A \geq \max(L, r+2)$. For $t \geq An$, each numerator factor is at most $2t$, each denominator factor is at least $t$, and $|W_n(rn+t)| \leq Ct^E$ for a fixed constant $C$. Thus

$$|T_n(rn+t)| \leq CC_n 2^{DLn+1} t^{-(S-DL)n-\tau}.$$

For $q_n = (S-DL)n + \tau > 1$, the tail on a grid of spacing $1/b$ is bounded by a first term and an integral:

$$\sum_{\substack{t \text{ on the grid} \\ t \geq An}} t^{-q_n} \leq (An)^{-q_n} + b\frac{(An)^{1-q_n}}{q_n - 1}.$$

Stirling's formula for $C_n$ then gives the tail $n$th-root bound

$$4^S K 2^{DL} \mathrm{e}^{-(S-DL)} A^{-(S-DL)}. \tag{5.4}$$

The powers of $n^{S-DL}$ from the factorials cancel the corresponding powers from $(An)^{-(S-DL)n}$. Since $S > DL$, the bound in (5.4) tends to zero as $A$ grows. Combining the bounded range with the tail proves

$$\limsup_{\substack{n\to\infty \\ n \text{ odd}}} |F_{n,b}|^{1/n} \leq G. \tag{5.5}$$

### 5.3 The two rational inequalities imply decay

The estimate (5.5) reduces the required decay to $\mathrm{e}^S G < 1$. If $f(u) < 1$, the description of $f$ above implies $x_0 < u$. Also

$$\frac{(x+r)^r}{(x+r+4)^{r+4}}$$

decreases for $x \geq 0$: its logarithmic derivative is $-4x/((x+r)(x+r+4))$. Finally, the exponential series gives

$$\mathrm{e} < \frac{163}{60} + \frac{7}{4320} < \frac{68}{25}.$$

Using these bounds in (5.2) yields $(\mathrm{e}^S G)^4 < U$, where $U$ is the rational expression in (2.5). We have proved the following consequence of the two conditions in Proposition 2.1:

$$\limsup_{\substack{n\to\infty \\ n \text{ odd}}} |\delta_n^S F_{n,b}|^{1/n} < 1. \tag{5.6}$$

The same conclusion holds for every fixed integer combination of the $B$ forms.

## 6 Prime dilation and nonvanishing

The preceding estimates will make the rational combinations small. We now show how the nonzero seed constant prevents them from being zero at every large index. The coefficient of 1 and the beta coefficients scale differently at indices divisible by a large prime. After one common scaling, the seed constant survives modulo that prime, while every beta coefficient vanishes.

For a prime $p$, write $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}$, and let $v_p$ be the usual prime valuation. Recall $\tau = s - E > 0$ from (5.3).

**Proposition 6.1.** *Fix a positive odd integer $m$. For all sufficiently large primes $p$,*

$$p^\tau \rho_i(mp) \in p^i \mathbb{Z}_{(p)}, \tag{6.1}$$

$$p^\tau q_b(mp) \equiv \chi_4(p)(m+1)^{-S} q_b(m) \pmod{p}. \tag{6.2}$$

*In particular, the second congruence at $m = 1$ transfers the seed constant with the nonzero factor $\chi_4(p)2^{-S}$.*

### 6.1 Dilation of the rational function

We first compare the rational functions before extracting their coefficients. In the following calculations $p$ is large enough that it divides none of the fixed denominators, and $p > m+1$. Both $(mp)!$ and $(mp+1)!$ have valuation $m$. After removing $p^m$, their units reduce to $(-1)^m m!$, by Wilson's theorem. Fermat's theorem treats the fixed powers in $C_{mp}$. Hence $C_{mp}$ has valuation $(S - DL)m$, and its unit reduces to

$$(-1)^{m(S-DL)} \frac{C_m}{(m+1)^S}. \tag{6.3}$$

In a numerator block with fraction class $a/d$, the factors of $P_{mp}(pz)$ divisible by $p$ are selected by $dj + a \equiv 0 \pmod{p}$. After division by $p$, their fraction class is $a'/d$, where $pa' \equiv a \pmod{d}$ and $1 \leq a' \leq d$. This permutes the reduced classes of each denominator and hence permutes $\Theta$.

For example, division by 5 interchanges the classes $1/3$ and $2/3$: the first selected factors have $(3+1/3)/5 = 2/3$ and $(1+2/3)/5 = 1/3$. Keeping all reduced classes ensures that such a permutation leaves the numerator construction unchanged.

Each block has $Lm$ selected factors. Together with the extra factor $p(z-rm)$, their product is $p^{DLm+1}P_m(z)$. The unselected numerator units contribute $(-1)^{DLm}$, since each complete block of nonzero residues has product $-1$.

In the denominator, the selected indices are $h = py$, $0 \le y \le m$. They give $p^{S(m+1)}\prod_{y=0}^{m}(z+4y)^S$. The unselected denominator unit is $(-1)^{Sm}$. The signs cancel:

$$m(S-DL) + DLm - Sm = 0.$$

The net valuation is $(S-DL)m + (DLm+1) - S(m+1) = -s$. Thus, after removing these powers, the resulting rational functions have the congruence

$$p^s R_{mp}(pz) \equiv (m+1)^{-S} R_m(z) \pmod{p}. \tag{6.4}$$

Homogeneity, (3.7), changes the exponent by $E$:

$$p^\tau T_{mp}(pz) \equiv (m+1)^{-S} T_m(z) \pmod{p}. \tag{6.5}$$

This is why the multiplier affects the prime-dilation exponent, even though it leaves the denominator exponent and exponential decay estimate unchanged.

## 6.2 Extracting coefficients at the poles

To deduce coefficient congruences from (6.5), we must know that reduction modulo $p$ commutes with the local series expansion. Fix a selected pole $h = py$ and put $z = -4y + u$. The denominator factorization above shows that

$$u^S p^\tau T_{mp}(p(u-4y)) \in \mathbb{Z}_{(p)}[[u]]. \tag{6.6}$$

After the selected powers of $p$ and the pole factor $u^S$ are removed, all other denominator factors have unit constant term. Their inverses therefore have coefficients in $\mathbb{Z}_{(p)}$. The same factorization reduces to the local expansion of the right side of (6.5).

The coefficient of $u^{S-i}$ in (6.6) is $p^{\tau-i}a_{i,py}(mp)$. We may therefore extract it before or after reduction, obtaining

$$p^{\tau-i}a_{i,py}(mp) \equiv (m+1)^{-S} a_{i,y}(m) \pmod{p}. \tag{6.7}$$

Now consider a pole $h = x + py$ with $1 \le x < p$. Substitute $t = pz - 4x$. The extra numerator factor is a unit, the numerator blocks have $DLm$ selected factors, and the denominator has $m$ selected factors, each of multiplicity $S$. Including $C_{mp}$, the net valuation is zero. The same local expansion at $z = -4y$ therefore gives, for the base function, $p^{-i}a_{i,h}^{\mathrm{base}}(mp) \in \mathbb{Z}_{(p)}$.

Multiplication by $W_{mp}$ preserves the bound: its $k$th Taylor coefficient is an integer and multiplies a base coefficient of order $i+k$, with valuation at least $i+k$. Consequently

$$v_p(a_{i,h}(mp)) \ge i \qquad (p \nmid h). \tag{6.8}$$

Summing (6.7) and (6.8) over all poles proves (6.1).

### 6.3 The surviving constant term

Apply the constant formula (4.4), taking $p > 4Bm$ as well. Every denominator $j \le 4bmp$ in that formula is then smaller than $p^2$, so $v_p(j) \le 1$. Multiply the formula by $p^\tau$. At a pole with $p \nmid h$, (6.8) leaves valuation at least $\tau > 0$, even when $p \mid j$. At a selected pole $h = py$, the terms with $p \nmid j$ have valuation at least $i > 0$, by (6.7).

Only the terms $h = py$, $j = pk$ survive. For these terms the upper limit becomes $k \le 4by$, and $\chi_4(pk) = \chi_4(p)\chi_4(k)$. The surviving sum is therefore $\chi_4(p)(m+1)^{-S}$ times the seed constant formula at index $m$. This proves (6.2) and completes the proof of Proposition 6.1.

## 7 Completing the irrationality contradiction

We now prove Proposition 2.1. Assume its two inequalities hold and suppose that at most $B-1$ of $\beta(2), \dots, \beta(s)$ are irrational. Choose $B-1$ distinct even indices $i_1 < \cdots < i_{B-1}$ containing all possible irrational indices, adding further indices if necessary. There are enough choices, since $s \ge DL \ge 10D \ge 10B$.

The coefficients $c_j$ have already fixed the common multiplier $W_n$. We now choose integer weights $w = (w_1, \dots, w_B)$ on the $B$ forms to annihilate the columns

$$(1^{i_j}, 2^{i_j}, \dots, B^{i_j})^{\mathsf{T}}, \qquad j = 1, \dots, B-1,$$

by using their signed maximal minors. Its last entry $w_B$ is nonzero. Indeed, the corresponding minor is a generalized Vandermonde determinant at the distinct positive points $1^2, \dots, (B-1)^2$, with exponents $i_j/2$. A singular minor would give a nonzero polynomial with at most $B-1$ monomials vanishing at $B-1$ positive points. A polynomial with $k$ monomials has at most $k-1$ distinct positive zeros: factor out its smallest power and apply Rolle's theorem inductively to its derivative. This proves the required nonsingularity. For $B = 1$, use the empty determinant 1 and $w = (1)$.

Form the single sequence

$$H_n = \sum_{b=1}^{B} w_b F_{n,b}, \qquad q_w(n) = \sum_{b=1}^{B} w_b q_b(n).$$

The weights are fixed, so the chosen beta coefficients cancel at every $n$. All remaining beta values are rational under our assumption. Let $D_0$ be a common denominator for those rational values. Proposition 4.2 and (5.6) show that

$$D_0 \delta_n^S H_n \in \mathbb{Z}, \qquad D_0 \delta_n^S H_n \longrightarrow 0 \quad (n \to \infty,\ n \text{ odd}). \tag{7.1}$$

The seed vector was chosen to make this sequence nonzero infinitely often. By (1.4), $q_w(1) = w_B \eta \neq 0$. Exclude the finitely many primes dividing $D_0$ or the numerator or denominator of $q_w(1)$, and take $p$ large enough for Proposition 6.1. Its constant congruence at $m = 1$ gives

$$p^\tau q_w(p) \equiv \chi_4(p) 2^{-S} q_w(1) \not\equiv 0 \pmod{p}.$$

Every remaining beta term in $p^\tau H_p$ is divisible by $p$ in $\mathbb{Z}_{(p)}$, by (6.1) and the rationality assumption. Thus $H_p \neq 0$ for infinitely many primes $p$.

Equation (7.1) now gives infinitely many nonzero integers tending to zero. Since a nonzero integer has absolute value at least 1, this is impossible. Therefore $I(s) \ge B$, as claimed.

# 8 Choosing the endpoints

## 8.1 One choice that works for every count

We now prove Theorem 1.2 using one convenient choice of $r$ and $u$ for every $B$. Set

$$r = 7, \quad L = 18, \quad u = \frac{1}{50}, \quad e_0 = \frac{68}{25}, \qquad A = \frac{11\log 11 - 7\log 7}{4} - \log 4 - \log e_0.$$

In the second criterion inequality,

$$\frac{1}{4}\log U = \log K + 18D\log(18+u) - AS.$$

The first two terms are the costs from the normalization and the numerator factors; $A$ measures the decay gained per unit of $S$. We will bound those costs by $36D\log B + (261/5)D$ and show $A > 4/5$. Thus $S \geq D(66 + 45\log B)$ will suffice: the coefficient $45 = 36/(4/5)$ covers the logarithmic cost, and $66 > 261/4$ covers the remaining cost. We choose $s_B$ by rounding $D(66 + 45\log B)$ up to an even integer, so $S = s_B + 1$ exceeds this threshold.

We now verify these bounds and the first criterion inequality. The following three comparisons are over the rationals:

$$901^4 \left(\frac{351}{551}\right)^{66} < 1, \tag{8.1}$$

$$(901/50)^{10} < (163/60)^{29}, \tag{8.2}$$

$$\left((4e_0)^4 \frac{7^7}{11^{11}}\right)^5 e_0^{16} < 1. \tag{8.3}$$

Since $163/60 < \mathrm{e} < e_0$, the last two give

$$\log(18+u) < \frac{29}{10}, \qquad A > \frac{4}{5}. \tag{8.4}$$

For example, the logarithm of the left side of (8.3) is $-20A + 16\log e_0$, so it implies $A > (4/5)\log e_0 > 4/5$.

The arithmetic cost also has a simple bound. From (2.1), $\lceil 18/(p-1)\rceil \leq 18$ gives

$$A_d \leq d^{36}, \qquad K \leq B^{36D}, \qquad \log K \leq 36D\log B.$$

Take $s = s_B$ from (1.2). Then

$$S = s + 1 \geq 66D + 45D\log B + 1 > 18D.$$

Raising (8.1) to the $D$th power proves $f(u) < 1$. For the second inequality, (8.4) gives

$$\frac{1}{4}\log U < 36D\log B + \frac{261}{5}D - \frac{4}{5}S \leq -\frac{3}{5}D - \frac{4}{5} < 0.$$

Thus $U < 1$, and Proposition 2.1 proves $I(s_B) \geq B$ for every $B \geq 1$.

## 8.2 Sharper small endpoints

For particular counts we can use $K$ itself in (2.5), instead of the upper bound $B^{36D}$. The following rational choices satisfy $S > DL$, $f(u) < 1$ and $U < 1$; the two inequalities are checked exactly in the supplied construction checker.

| $B$ | $s$ | $r$ | $u$ |
|---|---|---|---|
| 1 | 64 | 7 | $3/250$ |
| 2 | 160 | 7 | $21/10\,000$ |
| 3 | 364 | 7 | $3/5\,000$ |
| 4 | 588 | 7 | $7/25\,000$ |
| 5 | 1030 | 8 | $29/50\,000$ |
| 6 | 1296 | 8 | $7/20\,000$ |
| 7 | 1992 | 7 | $67/1\,000\,000$ |
| 8 | 2502 | 7 | $47/1\,000\,000$ |
| 9 | 3260 | 7 | $7/200\,000$ |
| 10 | 3828 | 8 | $11/100\,000$ |

This proves the finite refinements in Theorem 1.2. The same rank argument applies in every row; these calculations only check the decay inequalities.

# 9 A bound in terms of the endpoint

Theorem 1.2 takes a desired count $B$ and produces an endpoint. We now reverse that description. For every even $s$, it gives the computable bound

$$I(s) \geq \max\bigl(\{0\} \cup \{B \geq 1 : D_B(66 + 45 \log B) \leq s\}\bigr). \tag{9.1}$$

The ceiling disappears here because $s$ is even. To obtain the simpler formula in Theorem 1.1, we need an explicit estimate for $D_B$ and control of the intervals between endpoints.

## 9.1 The totient estimate and the square-root constant

**Lemma 9.1.** *For every integer $k \geq 1$, with $a = 3/\pi^2$,*

$$D_k \leq ak^2 + k(\log k + 2). \tag{9.2}$$

*Proof.* Counting coprime ordered pairs in $\{1, \dots, k\}^2$ gives

$$2D_k - 1 = \sum_{d=1}^{k} \mu(d) \lfloor k/d \rfloor^2,$$

where $\mu$ is the Möbius function. The subtraction of 1 accounts for the only coprime diagonal pair, $(1, 1)$. Replacing each squared floor by $(k/d)^2$ costs at most $2k \sum_{d=1}^{k} 1/d$. Extending the resulting Möbius series to infinity costs at most $k^2 \sum_{d>k} d^{-2} \leq k$. Since $\sum_{d \geq 1} \mu(d)/d^2 = 6/\pi^2$, we obtain

$$D_k \leq ak^2 + k \sum_{d=1}^{k} \frac{1}{d} + \frac{k}{2} + \frac{1}{2} \leq ak^2 + k(\log k + 2).$$

□

The same absolute-error estimate gives $D_B = aB^2 + O(B \log B)$. The coefficient $a = 3/\pi^2$ is half the density $6/\pi^2$ of coprime pairs, since $D_B$ counts fractions with numerator no greater than denominator. Thus the leading endpoint in (1.2) is

$$s_B \sim 45aB^2 \log B.$$

The coefficient 45 records the balance used in Section 8: the denominator cost is at most $36D_B \log B$, while the decay gain per unit of $S = s + 1$ is greater than $4/5$. Their ratio gives $45 = 36/(4/5)$.

Since $\log B \sim \frac{1}{2}\log s_B$, solving for $B$ gives the constant

$$\sqrt{\frac{2}{45a}} = \pi\sqrt{\frac{2}{135}} = c$$

in front of $\sqrt{s/\log s}$. The factor 2 comes from the logarithm of the quadratic term $B^2$. We next show that this leading constant already satisfies the required finite inequalities.

## 9.2 The unrounded bound for large endpoints

**Proposition 9.2.** *For every even integer $s \geq 42\,170$,*

$$I(s) \geq c\sqrt{\frac{s}{\log s}}.$$

*Proof.* The endpoints $s_k$ increase strictly. Between $s_{k-1}$ and $s_k$, the family guarantees at least $k-1$ irrational values. Since $x/\log x$ increases for $x > \mathrm{e}$, it suffices to check the largest even point in each interval, namely $s_k - 2$:

$$c^2(s_k - 2) < (k-1)^2 \log(s_k - 2), \qquad k \geq 26. \tag{9.3}$$

The first interval starts at $s_{25} = 42\,170$.

We first prove (9.3) for every $k \geq 256$. We record the error in the totient estimate as a relative factor $q_k$ and use it to form an upper bound $\mathcal{U}(k)$ for $s_k - 2$. Put

$$\ell = \log k, \qquad b_0 = \frac{66}{45} = \frac{22}{15}, \qquad q_k = 1 + \frac{\ell + 2}{ak}, \qquad \mathcal{U}(k) = 45ak^2(\ell + b_0)q_k.$$

Lemma 9.1 gives $D_k \leq ak^2 q_k$. Together with the ceiling inequality $2\lceil x/2\rceil - 2 < x$, this yields

$$s_k - 2 < D_k(66 + 45\ell) \leq \mathcal{U}(k). \tag{9.4}$$

It is therefore enough to prove $c^2\mathcal{U}(k) < (k-1)^2 \log \mathcal{U}(k)$. Since $c^2 = 2/(45a)$, this is

$$2(\ell + b_0)\frac{q_k}{(1 - 1/k)^2} < \log \mathcal{U}(k). \tag{9.5}$$

Separate the common term $2(\ell + b_0)$ on both sides. The excess on the left comes from the totient error and from having only $k-1$ guaranteed values before the next endpoint. We bound this cost by

$$\mathcal{T}(k) = \frac{2(\ell + b_0)((\ell + 2)/a + 2)}{k(1 - 1/k)^2},$$

because

$$\frac{q_k}{(1 - 1/k)^2} - 1 = \frac{((\ell + 2)/a + 2)/k - 1/k^2}{(1 - 1/k)^2}.$$

The excess on the right is the margin available to absorb this cost:

$$\mathcal{G}(k) + \log q_k, \qquad \mathcal{G}(k) = \log(45a(\ell + b_0)) - 2b_0.$$

As $q_k > 1$, it suffices to show $\mathcal{T}(k) < \mathcal{G}(k)$. The margin $\mathcal{G}$ increases. We will show that the error bound $\mathcal{T}$ decreases for $k \geq 256$, so a comparison at 256 will settle every $k \geq 256$. Its logarithmic derivative with respect to $\ell$ is

$$\frac{1}{\ell + b_0} + \frac{1}{\ell + 2 + 2a} - 1 - \frac{2}{k - 1} < 0 \qquad (k \geq 256).$$

Indeed, $\ell > 11/2$, and the sum of the first two terms is less than $2/11 + 2/15 < 1$. It remains to compare the functions at $k = 256$.

Use $\pi < 22/7$ and $11/2 < \log 256 < 111/20$. Direct rational comparison gives

$$\mathcal{T}(256) < 2\left(\frac{111}{20} + \frac{22}{15}\right)\left(\left(\frac{111}{20} + 2\right)\frac{484}{147} + 2\right)\frac{256}{255^2} < \frac{3}{2}, \tag{9.6}$$

$$45a(\log 256 + b_0) > 45\frac{147}{484}\left(\frac{11}{2} + \frac{22}{15}\right) > 95. \tag{9.7}$$

Moreover, $\mathrm{e} < 11/4$ and $(11/4)^9 < 95^2$, so $\log 95 > 9/2$. Thus

$$\mathcal{G}(256) > \frac{9}{2} - \frac{44}{15} = \frac{47}{30} > \frac{3}{2}.$$

Monotonicity proves (9.5) throughout the infinite range $k \geq 256$.

For the remaining 230 indices $26 \leq k \leq 255$, use the exact totient sums in (1.2). The rational logarithm bounds specified in Appendix A.1 verify

$$\frac{(k-1)^2\,\underline{\log}(s_k - 2)}{s_k - 2} - \frac{968}{6615} > \frac{24529}{10^8}.$$

Here $\underline{\log}(x) \leq \log x$ is the explicit rational lower bound, and $968/6615 = 2(22/7)^2/135 > c^2$. This proves (9.3) for all remaining indices, completing the proposition. □

### 9.3 Keeping the floor down to the two-value count

It remains to cover $160 \leq s < 42\,170$. Let $t_k$ be the sharper endpoint in Section 8 for $2 \leq k \leq 10$, and let $t_k = s_k$ for $11 \leq k \leq 25$. The resulting increasing list is

| $k$ | $t_k$ | $k$ | $t_k$ |
|---|---|---|---|
| 2 | 160 | 14 | 11 826 |
| 3 | 364 | 15 | 13 528 |
| 4 | 588 | 16 | 15 262 |
| 5 | 1030 | 17 | 18 576 |
| 6 | 1296 | 18 | 20 000 |
| 7 | 1992 | 19 | 23 820 |
| 8 | 2502 | 20 | 25 704 |
| 9 | 3260 | 21 | 28 422 |
| 10 | 3828 | 22 | 30 766 |
| 11 | 7306 | 23 | 35 622 |
| 12 | 8180 | 24 | 37 624 |
| 13 | 10 524 | 25 | 42 170 |

On $t_{k-1} \leq s < t_k$, we have $I(s) \geq k - 1$. To prove the rounded bound, it is enough for the expression inside the floor to stay strictly below $k$. Once again, its largest value on the even endpoints occurs at $s = t_k - 2$. The required test is now

$$c^2(t_k - 2) < k^2 \log(t_k - 2), \qquad 3 \leq k \leq 25. \tag{9.8}$$

The 23 exact comparisons in Appendix A.1 give the stronger bound

$$\frac{k^2\,\underline{\log}(t_k - 2)}{t_k - 2} - \frac{968}{6615} > \frac{14325}{10^8}.$$

It follows that $\lfloor c\sqrt{s/\log s}\rfloor \le k-1 \le I(s)$ on every one of these intervals. They cover all even integers from 160 through 42 168, and Proposition 9.2 covers the rest. This proves Theorem 1.1.

The floor also gives a completely explicit error form:

$$I(s) > \left(c - \sqrt{\frac{\log s}{s}}\right)\sqrt{\frac{s}{\log s}}, \qquad s \ge 160 \text{ even}.$$

The coefficient is positive throughout this range. In fact, $c > 19/50$ and $\sqrt{\log 160/160} < 9/50$, while $\log s/s$ decreases for $s > \mathrm{e}$. Hence the coefficient is greater than $1/5$.

# A Exact arithmetic checks

## A.1 Logarithm bounds and endpoint comparisons

Here is the finite rational calculation used in Section 9. For a rational $x \ge 1$, write $x = 2^j y$ with $j \ge 0$ and $1 \le y < 2$, and put $z = (y-1)/(y+1)$. With $M = 18$, define

$$L_M(y) = 2\sum_{h=0}^{M-1} \frac{z^{2h+1}}{2h+1}, \qquad R_M(y) = \frac{2z^{2M+1}}{(2M+1)(1-z^2)}.$$

The positive series $\log y = 2\,\operatorname{atanh} z$ gives

$$L_M(y) \le \log y \le L_M(y) + R_M(y).$$

Applying the same formula to 2 gives bounds for $\log 2$. Adding $j$ times those bounds gives rational numbers $\underline{\log}(x)$ and $\overline{\log}(x)$ enclosing $\log x$. The remainder follows by replacing all denominators in the tail by $2M+1$ and summing the resulting geometric series.

The checker computes $\varphi(d)$ by counting coprime integers and then forms $D_k$. For each closed endpoint $s_k$, it computes

$$2\left\lceil D_k\left(33 + \frac{45}{2}\underline{\log}(k)\right)\right\rceil, \qquad 2\left\lceil D_k\left(33 + \frac{45}{2}\overline{\log}(k)\right)\right\rceil.$$

These integers agree in every required case, thereby isolating the endpoint exactly. It then performs the following rational comparisons:

| Comparison | Number of cases | Index of smallest margin |
|---|---|---|
| Unrounded, $26 \le k \le 255$ | 230 | 29 |
| Rounded, $3 \le k \le 25$ | 23 | 3 |

After subtraction of $968/6615$, the margins are strictly greater than $24529/10^8$ and $14325/10^8$, respectively, as used in the proof. All calculations use integer or rational arithmetic. The checker also verifies the constants in (9.6)–(9.7), the two small floors at $s = 160, 364$, and the positive coefficient bound at the end of Section 9.

The elementary bounds on $\pi$ can be verified independently of decimal approximations. For the upper bound,

$$0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

For the lower bound $\pi > 157/50$, the checker uses Machin's identity $\pi = 16\arctan(1/5) - 4\arctan(1/239)$ with eight terms of the alternating arctangent series and its next-term error bound.

### A.2 Checks of the construction identities

The positive-definiteness proof in Section 3 works for every $B$. We also check its normalization and the arithmetic identities by separate finite calculations. At each of the ten parameter choices in Section 8, two partial-fraction algorithms over the field with 1 000 003 elements agree with an exact rational calculation of the Gram matrix.

The rational calculation uses explicit moments. Every element of $\mathcal{A}$ occurs in $\Lambda$, because the kernel poles are numerator zeros. With all fractions reduced, put

$$\Delta = \prod_{\alpha=v/d\in\Lambda} d^2, \qquad V(x) = \prod_{\alpha=v/d\in\Lambda\setminus\mathcal{A}} (d^2x + v^2).$$

Then $\prod_{\alpha\in\Lambda}(x+\alpha^2)/Q(x) = V(x)/\Delta$. For $0 \le k < S$, the elementary beta integral gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{y^{2k}}{(y^2+4)^S}\,dy = 2^{2k-4S+2}\,\frac{(2k)!(2S-2k-2)!}{k!(S-k-1)!(S-1)!}. \tag{A.1}$$

Thus $M_{b,j}/C_1$ is $\sigma/\Delta$ times the moment functional in (A.1) applied to $xV(x)h_b(x)h_j(x)$. The largest moment index is $DL/2 + N - 1 < S$, so every integral used is convergent.

At the ten endpoints, the computations check 385 rational Gram entries, 55 positive leading principal minors and ten integer adjugate identities. Full rational computations at $(B, s, r) = (1, 20, 3), (2, 24, 3)$ and $n = 1, 17, 19$ also check the coefficient and constant congruences and grouped integrality. These smaller parameter sets test identities; decay is checked at the stated theorem endpoints.

Run `checks/verify_construction.py` for the construction identities and `checks/verify_endpoints.py` for the endpoint comparisons. Both use only the Python standard library and should be run without `-O`.

### A.3 Independent audit

An independent mathematical audit reported on 5 November 2026 found no substantive error or gap in the 19-page revision identified in the accompanying audit record. The reviewer checked the general arguments and independently reproduced the finite calculations, without running or importing the supplied checkers. The audit recommended the explanation of uniformity now included in Section 5.2; no mathematical correction was requested. It was not formal verification or a fresh audit of all cited literature.

## B The counting consequence of the earlier bound

We justify the bound $I(s) \ge 0.21\sqrt{s/\log s}$ used in the introduction to compare the present result with [4]. The application of Fischler's criterion in [4, Section 7] gives a lower bound for the span including 1. With the parameter estimates from that section, it yields

$$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{1, \beta(2), \dots, \beta(s)\} \ge 1 + (c_* + o(1))\sqrt{\frac{s}{\log s}}, \qquad c_* = 0.2174562348\ldots,$$

as $s$ tends to infinity through multiples of 34. All rational beta values lie in $\mathbb{Q}\cdot 1$, and each irrational value adds at most one dimension. The dimension on the left is therefore at most $1 + I(s)$, so

$$I(s) \ge (c_* + o(1))\sqrt{\frac{s}{\log s}} \qquad (34 \mid s).$$

For an arbitrary large even $s$, put $t = 34\lfloor s/34\rfloor$. Then $I(s) \ge I(t)$ and

$$\frac{\sqrt{t/\log t}}{\sqrt{s/\log s}} \longrightarrow 1.$$

Since $c_* > 0.21$, the preceding estimate gives $I(s) \geq 0.21\sqrt{s/\log s}$ for every sufficiently large even $s$.

# References


[1] V. Kumar and R. Tosi, *Irrationality of values of L-functions for real Dirichlet characters*, preprint, 30 October 2026. arXiv:2610.41458v1.

[2] L. Lai and J. Sprang, *Many p-adic odd zeta values are irrational*, preprint, version of 17 February 2025. arXiv:2306.10393v2.

[3] L. Lai and P. Yu, *A note on the number of irrational odd zeta values*, Compos. Math. **156** (2020), no. 8, 1699–1717. arXiv:1911.08458v2.

[4] twinheterodox, *Linear independence of even Dirichlet beta values*, 3 November 2026. Hexagon 2611.00044v2.