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Disproof of Xi's conjecture on the representation dimension of idempotent corners

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Abstract

We disprove Xi's conjecture on the representation dimension of idempotent corners. Over every field and for each integer $L \geq 3$, we construct an algebra of representation dimension $L+1$ with a corner of representation dimension $2L-1$.

1 Introduction

Auslander introduced representation dimension in 1971 to measure how far an Artin algebra is from finite representation type [1]. Let C be a finite-dimensional algebra over a field k. Write $D = \mathrm{Hom}_k(-,k)$, let DC be the dual of the right regular module, and let $\operatorname{add} M$ consist of the direct summands of finite direct sums of copies of M, up to isomorphism. Write pd_C for projective dimension over C and $\operatorname{gl.dim} C$ for its supremum over C-modules. The representation dimension is

$$\operatorname{repdim} C = \inf_{C \oplus DC \in \operatorname{add} M} \operatorname{gl.dim} \operatorname{End}_C(M).$$

The infimum ranges over finite-dimensional left C-modules satisfying the displayed condition, called generator-cogenerators. For Artin algebras the definition uses duality over the commutative Artinian base. Auslander's criterion says that $\operatorname{repdim} C \leq 2$ exactly when there are finitely many indecomposable finite-dimensional C-modules up to isomorphism. Xi's early work gave upper bounds for tensor products and triangular matrix algebras and raised the finiteness question [6, 7]. Iyama proved finiteness for Artin algebras [2]. Rouquier proved that the exterior algebra on an n-dimensional vector space has representation dimension $n+1$ for $n \geq 1$ [5, Theorem 4.1].

Xi traced to Auslander the question of how representation dimension changes on passing to the endomorphism algebra of a finitely generated projective module [8, §7]. If A is an Artin algebra and $e^2 = e \in A$, right multiplication identifies eAe with $\operatorname{End}_A(Ae)^{\mathrm{op}}$. Indeed, $f \in \operatorname{End}_A(Ae)$ is determined by $f(e) \in eAe$, and $f(ae) = af(e)$. Duality exchanges left and right generator-cogenerators and reverses their endomorphism rings. Since left and right global dimensions agree for Artin algebras, the comparison for Ae is the comparison between $\operatorname{repdim}(eAe)$ and $\operatorname{repdim} A$. Xi proved the inequality when the two adjoints of $X \mapsto eX$, namely $Ae \otimes_{eAe} -$ and $\mathrm{Hom}_{eAe}(eA,-)$, are isomorphic [8, Theorem 7.2]. He conjectured that for every Artin algebra A and idempotent e,

$$\operatorname{repdim}(eAe) \leq \operatorname{repdim} A.$$

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His problem list records the Artin-algebra formulation in January 2003 and a finite-dimensional formulation in December 2012 [9]. He restated the conjecture in [10, p. 40].

When A is representation-finite, so is eAe. Indeed, every finite eAe-module Y is the image under $e(-)$ of $Ae \otimes_{eAe} Y$. By Krull–Schmidt, every indecomposable Y therefore occurs as a summand of the image of one of the finitely many indecomposable A-modules. For $\operatorname{repdim} A \leq 3$, Zhang and Zhang proved that eAe has finite finitistic dimension, the supremum of the finite projective dimensions of its finitely generated modules [11]. Psaroudakis proved $\operatorname{repdim}(eAe) \leq 3$ in the same range by studying recollements of module categories [4, Corollary 5.2.5]. The first possible increase thus occurs at ambient representation dimension four.

A finite-dimensional k-algebra is Frobenius if it admits a linear functional λ for which the pairing $(x, y) \mapsto \lambda(xy)$ is nondegenerate.

Theorem 1. *For every field k and integer $L \geq 3$, there is a Frobenius k-algebra A_L and an idempotent $e \in A_L$ with commutative local corner $B_L = eA_Le$ such that*

$$\operatorname{repdim} A_L = L + 1, \qquad \operatorname{repdim} B_L = 2L - 1.$$

For $L = 3$, the algebras A_3 and B_3 have dimensions 132 *and* 33*, respectively.*

In the proof k is an arbitrary field. Algebras are unital and finite-dimensional over k, and modules are finite-dimensional left modules, except for families over parameter rings. Opposite endomorphism rings make $\operatorname{Hom}_C(M, -)$ left-module-valued.

The difference $L - 2$ is unbounded. The lower bounds use products of first-order extensions. For the corner, the calculation is made on the projective and injective summands present in the additive closure of every generator-cogenerator. The upper bound for the corner uses radical quotients together with modules defined by coefficient multiplication.

2 The algebras

For the rest of the paper fix $L \geq 3$ and put

$$\Sigma = k[x_0, \dots, x_{2L-3}]/(x_0^2, \dots, x_{2L-3}^2), \qquad B_L = k1 \oplus \bigoplus_{a=1}^{L-1} t^a\Sigma \subseteq \Sigma[t]/(t^L).$$

The indeterminate t is central. Write $J = \bigoplus_{a=1}^{L-1} t^a\Sigma$. This ideal is nilpotent with quotient k, so it is the Jacobson radical of B_L. Since $1 \in \Sigma$, $J^j = \bigoplus_{a=j}^{L-1} t^a\Sigma$ for $1 \leq j \leq L$, with the empty sum zero.

Let $\tau : \Sigma \to k$ take the coefficient of $x_0 \cdots x_{2L-3}$ in the squarefree monomial basis. The pairing $(a, b) \mapsto \tau(ab)$ is nondegenerate because each monomial pairs with its complement.

Give DB_L the bimodule action $(bfc)(x) = f(cxb)$ for $b, c, x \in B_L$ and $f \in DB_L$. On $(B_L \times B_L) \oplus DB_L \oplus DB_L$, define A_L by

$$(b_0, b_1, f, g)(c_0, c_1, h, j) = (b_0c_0, b_1c_1, b_0h + fc_1, b_1j + gc_0).$$

Here $b_i, c_i \in B_L$ for $i = 0, 1$ and $f, g, h, j \in DB_L$. The first dual copy has left action from the first factor and right action from the second factor of $B_L \times B_L$; the second has the actions reversed. The bimodule identities give associativity of the square-zero extension. The unit is

$(1,1,0,0)$, and the corner for $e=(1,0,0,0)$ identifies with the first copy of B_L. The functional $\lambda(b_0,b_1,f,g)=f(1)+g(1)$ defines the pairing

$$\lambda\big((b_0,b_1,f,g)(c_0,c_1,h,j)\big)=h(b_0)+f(c_1)+j(b_1)+g(c_0).$$

Each coordinate pairs with its full dual, so A_L is Frobenius. The pairing identifies its left regular module with the dual of its right regular module by $y\mapsto(x\mapsto\lambda(xy))$. Thus the regular module is injective.

Lemma 2. $\operatorname{repdim} A_L\le L+1$.

Proof. Consider the ideal

$$R=(J\times J)\oplus(DB_L\oplus DB_L).$$

Its quotient is $k\times k$. A product in R with two off-diagonal factors is zero, since the off-diagonal bimodule has square zero. For $n\ge 1$, the diagonal blocks of R^n are J^n, and each off-diagonal block is

$$\sum_{i+j=n-1}J^i(DB_L)J^j.$$

The sum is over nonnegative i,j, with $J^0=B_L$. If $i+j\ge L$, then $(afb)(x)=f(bxa)=0$ for $a\in J^i, b\in J^j$, because $bxa\in J^{i+j}=0$. Hence $R^{L+1}=0$, and the semisimple quotient gives $R=\operatorname{rad}A_L$. Choose $0\ne a\in J^{L-1}$ and $f\in DB_L$ with $f(a)\ne 0$. Then af is a nonzero element of an off-diagonal copy of DB_L in R^L, since $(af)(1)\ne 0$.

The Loewy length, the least exponent killing the radical, is therefore $L+1$. We give the radical-layer argument for the self-injective Loewy-length bound [5, Proposition 3.9].

The idempotents $(1,0,0,0)$ and $(0,1,0,0)$ give two nonisomorphic indecomposable projective left modules P_1,P_2, since $A_L/R\cong k^2$. They are injective. Each has a simple essential socle, so every nonzero R^aP_i is indecomposable. For each i the dimensions of the nonzero radical powers strictly decrease by Nakayama's lemma. For different i their socles differ because the injective hulls P_1,P_2 differ. Put

$$M=\bigoplus_{i=1}^{2}\bigoplus_{a=0}^{L}R^aP_i,\qquad \Lambda=\operatorname{End}_{A_L}(M)^{\mathrm{op}},\qquad H=\operatorname{Hom}_{A_L}(M,-).$$

The projective-injective summands at $a=0$ make M a generator-cogenerator.

A map $R^bP_j\to R^aP_i$, composed with inclusion into P_i, extends to P_j by injectivity. Its image on R^bP_j lies in R^bP_i, and hence in $R^{a+1}P_i$ if $b>a$. If $b=a$ and the original map is noninvertible, its extension is noninvertible, since an isomorphism restricts to an isomorphism of the layers. A map between these indecomposable projectives that is nonzero on the target top is surjective by Nakayama's lemma, hence splits and is an isomorphism. Thus the noninvertible extension has image in RP_i, and its restriction again lands in $R^{a+1}P_i$.

For each nonzero R^aP_i, let T be the simple top of $H(R^aP_i)$, and put $\Delta=H(R^aP_i)/H(R^{a+1}P_i)$. Both Hom modules are projective, so $\operatorname{pd}_\Lambda\Delta\le 1$. By the factorization just proved, the kernel of $\Delta\to T$ is supported only on the idempotents for the summands R^bP_j with $b<a$. Applying these idempotent functors to a composition series shows that the kernel has only simple composition factors indexed by such summands. At $a=0$ the kernel is zero and $T=\Delta$. Inductively the factors have projective dimension at most a, and so does the kernel by the long exact Ext sequences of its composition series. Thus T has projective dimension at most $a+1$. Every simple has projective dimension at most $L+1$, giving $\operatorname{gl.dim}\Lambda\le L+1$. □

3 Parameter extensions

Lemma 3. *Let C be a finite-dimensional k-algebra and let $G \in \operatorname{add}(C \oplus DC)$. Put $\Gamma = \operatorname{End}_C(G)^{op}$. For an integer $d \geq 1$, let $R = k[\lambda_1, \ldots, \lambda_d]$, and let X_R be a $C \otimes_k R$-module which is finite free over R. Write $V_T = V \otimes_k T$ for base change of a k-vector space V to a commutative k-algebra T. Assume that $\operatorname{Hom}_C(G, X_R)$ is free over R and commutes with base change to every commutative R-algebra. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in K^d$ over a field extension K/k, let X_α be the fiber given by $\lambda_j \mapsto \alpha_j$. For $1 \leq i \leq d$, specializing by $\lambda_j \mapsto \alpha_j + \delta_{ij}\epsilon$ to the dual numbers $K[\epsilon]/(\epsilon^2)$ gives an extension of the fiber by itself. Here $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. The submodule is the image of ϵ, and the quotient is the reduction modulo ϵ. Applying $\operatorname{Hom}_{C_K}(G_K, -)$ gives an exact extension by the freeness and base-change hypotheses. Let ξ_i be its class. Assume that the Yoneda product $\xi_d \cdots \xi_1$, with ξ_1 applied first, is nonzero for every K, α. Then*

$$\operatorname{repdim} C \geq d + 2.$$

Proof. The proof uses the parameter-lattice method of [3]. Fix a generator-cogenerator M, adding G as a summand if necessary. This preserves $\operatorname{add} M$ and the Morita class of its endomorphism ring. Write $\Lambda = \operatorname{End}_C(M)^{\mathrm{op}}$ and let $p \in \Lambda$ project onto G. Then $p\Lambda p = \Gamma$. The functor $Y \mapsto pY$ is exact and sends $\operatorname{Hom}_C(M, X)$ to $\operatorname{Hom}_C(G, X)$.

In a finite free presentation $C^a \to C^b \to M \to 0$, application of $\operatorname{Hom}_C(-, X_R)$ identifies $\operatorname{Hom}_C(M, X_R)$ with the kernel of a matrix between finite free R-modules. If its rank over the fraction field is zero, the matrix vanishes. Otherwise, write s for that rank and invert a nonzero determinant of an s-by-s submatrix. After permutations the matrix has blocks $\left(\begin{smallmatrix} U & V \\ W & Z \end{smallmatrix}\right)$, with U invertible. Row and column operations give the blocks U and $Z - WU^{-1}V$. The second vanishes over the fraction field by the rank condition, hence over the localization. The kernel is thus free and commutes with every base change on this principal open.

Choose a point α of this open over $K = k$ if k is infinite, or over a sufficiently large finite extension K/k otherwise. Such a point exists because a nonzero polynomial cannot vanish at all points of a field with more elements than its degree in each variable. Flat base change applied to a finite presentation gives

$$\Lambda_K = \operatorname{End}_{C_K}(M_K)^{\mathrm{op}}.$$

Let $Y = \operatorname{Hom}_{C_K}(M_K, X_\alpha)$. The two base changes to K and to the dual numbers $K[\epsilon]/(\epsilon^2)$ identify the coordinate extensions of Y with those from the localized Hom family. They are exact, since the Hom family is free over the localization and the dual-number extension is obtained from $0 \to K \to K[\epsilon]/(\epsilon^2) \to K \to 0$, with the first map given by multiplication by ϵ. Multiplication by p sends these extensions to the given coordinate extensions for G_K. Exactness preserves Yoneda splicing, so the product in $\operatorname{Ext}^d_{\Lambda_K}(Y, Y)$ is nonzero.

Embed X_α into a finite injective I^0 and its cokernel into a finite injective I^1. Apply $\operatorname{Hom}_{C_K}(M_K, -)$ and call the cokernel of the induced map Q. This gives

$$0 \longrightarrow Y \longrightarrow P_0 \longrightarrow P_1 \longrightarrow Q \longrightarrow 0, \qquad P_j = \operatorname{Hom}_{C_K}(M_K, I^j).$$

Both P_j are projective because $DC_K \cong (DC)_K \in \operatorname{add} M_K$, hence $I^j \in \operatorname{add} M_K$. Splitting the sequence at the image of $P_0 \to P_1$ into two short exact sequences and applying $\operatorname{Hom}_{\Lambda_K}(-, Y)$ gives $\operatorname{Ext}^{d+2}_{\Lambda_K}(Q, Y) \cong \operatorname{Ext}^d_{\Lambda_K}(Y, Y)$. It follows that $\operatorname{gl.dim} \Lambda_K \geq d + 2$.

For finite k, the chosen extension is separable. A separability tensor $\sum_i a_i \otimes b_i \in K \otimes_k K$ satisfies $\sum_i a_i b_i = 1$ and $\sum_i c a_i \otimes b_i = \sum_i a_i \otimes b_i c$ for all $c \in K$. For every Λ_K-module Z these

identities show that $z \mapsto \sum_i a_i \otimes b_i z$ is a Λ_K-linear section of $K \otimes_k Z \to Z$, where Z in the tensor product is restricted to Λ. Inducing a projective resolution from Λ and passing to the summand Z gives $\operatorname{gl.dim} \Lambda_K \leq \operatorname{gl.dim} \Lambda$. Taking the infimum over M proves the lemma. □

4 The lower bound for the corner

Lemma 4. $\operatorname{repdim} B_L \geq 2L - 1$.

Proof. Put $n = 2L - 3$.

Matrices and the parameter family. Write $h = L - 2$, so $n = 2h + 1$. If $\operatorname{char} k \neq 2$, put $r = 2^h$ and use

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Write I_j for the j-by-j identity matrix. A zeroth tensor power means the scalar 1. For $1 \leq j \leq h$, define

$$P_{2j-1} = Z^{\otimes(j-1)} \otimes X \otimes I_2^{\otimes(h-j)}, \quad P_{2j} = Z^{\otimes(j-1)} \otimes Y \otimes I_2^{\otimes(h-j)}, \quad P_{2h+1} = Z^{\otimes h}.$$

These integer matrices anticommute in pairs. For matrices from different tensor positions there is exactly one interchange of X or Y with Z, and the same-position assertion is $XY = -YX$. Moreover,

$$P_{2j} P_{2j-1} = I_2^{\otimes(j-1)} \otimes Z \otimes I_2^{\otimes(h-j)}.$$

The latter matrices commute, so

$$\boxed{P_n P_{n-1} \cdots P_1 = I_r.} \tag{4.1}$$

In characteristic two instead take $r = 1$ and $P_i = (1)$ for all i. The pairwise anticommutator is still zero, (4.1) still holds, and $r \neq 0$ in the field. Thus in both cases r is nonzero in k.

Let $S = k[x_1, \ldots, x_n]/(x_i^2)$, let $R = k[\lambda_1, \ldots, \lambda_n]$, and define

$$C_\lambda = \sum_{i=1}^{n} \lambda_i x_i P_i, \qquad N_R = S_R^r.$$

All cross terms in C_λ^2 vanish by pairwise anticommutation, and the diagonal terms contain x_i^2. Give N_R its ordinary left Σ_R-module structure by letting x_i for $i \geq 1$ act as scalar multiplication and x_0 act as $-C_\lambda$. This is well defined: $C_\lambda^2 = 0$, and C_λ commutes with the scalar actions of the other generators. The family is R-free of rank $r2^n$.

At a parameter point α over a coefficient field K, put

$$C_\alpha = \sum_i \alpha_i x_i P_i, \qquad Q_+ = x_0 I_r + C_\alpha, \qquad Q_- = x_0 I_r - C_\alpha.$$

These are fixed matrices in $\operatorname{Mat}_r(\Sigma_K)$ while the parameters λ vary. They satisfy

$$Q_+ Q_- = Q_- Q_+ = 0. \tag{4.2}$$

The action on the canonical Hom module. Put $I = DB_L$ and let $v \in I$ take the degree-zero coefficient of B_L. The functionals

$$g_i(s)(t^{L-1-i}a) = \tau(as), \qquad 0 \le i \le L-2,$$

are defined for $a, s \in \Sigma$ and are zero on the other summands. For $1 \le a \le L-1$, the action of $t^a b$, $b \in \Sigma$, is

$$(t^a b)g_i(s) = \begin{cases} g_{i+a}(bs) & i+a \le L-2, \\ \tau(bs)v & i+a = L-1, \\ 0 & i+a \ge L. \end{cases}$$

For $1 \le m \le L$ and a Σ-module N, let $F_m(N) = N^m$, with coordinates $0, \dots, m-1$. For $1 \le a < L$ and $b \in \Sigma$, $t^a b$ multiplies by b and advances a coordinates, with indices beyond the last sent to zero. The Hom space has the following description, valid also after extension to any commutative k-algebra, with the modules and Hom spaces taken over the extended algebras.

$$\mathrm{Hom}_{B_L}(I, F_L(N)) = N^{L-2} \oplus J_N, \qquad J_N = \mathrm{Hom}_k(\Sigma, N). \tag{4.3}$$

The tuple $(m_1, \dots, m_{L-2}, f)$ corresponds to

$$g_0(s) \longmapsto (0, sm_1, \dots, sm_{L-2}, f(s)), \qquad v \longmapsto 0,$$

and the image of $g_i(s)$ for $i \ge 1$ is the image of $g_0(s)$ shifted by i layers.

For a homomorphism $h : I \to F_L(N)$, the vector $h(v)$ is killed by t, so $h(v) = (0, \dots, 0, z)$. Applying h to $t^{L-1}g_0(s) = \tau(s)v$ gives $h_0(g_0(s)) = \tau(s)z$, where h_0 is the first coordinate of h. Since $\tau(1) = 0$, the relation $(t^{L-1}x_0 \cdots x_n)g_0(1) = v$ gives $z = (x_0 \cdots x_n)h_0(g_0(1)) = 0$. Comparing $(tb)g_0(s)$ with $tg_0(bs)$ makes the middle components Σ-linear; the final component is arbitrary. These elementary equations suffice: $t^a b$ shifts the first $L-1$ components by a and multiplies them by b, and kills the final component. This gives the prescribed image of $g_{i+a}(bs)$ if $i + a \le L-2$, and zero otherwise. All equations hold over the coefficient ring. In particular the final summand is $D\Sigma \otimes_k N$, compatibly with base change.

For $p, q \in \Sigma$, define B_L-maps

$$C_p : I \to B_L, \quad C_p(g_i(s)) = t^{i+1}sp, \ C_p(v) = 0, \qquad G_q : B_L \to I, \quad G_q(1) = g_0(q).$$

The first is B_L-linear even at the last dual layer: one side then contains $C_p(v) = 0$, and the other has t-degree L. For matrices $P = (p_{ij})$, $Q = (q_{ij})$ over Σ, let C_P and G_Q have these maps as their entries, with row i specifying the target and column j the source, for $1 \le i, j \le r$. In the opposite ring we use the same vector-space elements and reverse the order of multiplication. Set

$$\widehat{G} = B_L^r \oplus I^r, \qquad \widehat{\Gamma} = \mathrm{End}_{B_L}(\widehat{G})^{op}, \qquad A_P = C_P^{op}, \quad B_Q = G_Q^{op}.$$

The two compositions are $C_P G_Q = t(PQ)$ on B_L^r and $G_Q C_P = t(QP)$ on I^r. Indeed $C_p G_q(1) = tpq$, and $G_q C_p(g_i(s)) = (t^{i+1}sp)g_0(q) = (tqp)g_i(s)$, while both sides kill v. Matrix composition sums these identities over the common index. In particular, writing $A_0 = A_{I_r}$, $B_0 = B_{I_r}$, $A_\pm = A_{Q_\pm}$, $B_\pm = B_{Q_\pm}$, we have exact algebra identities

$$A_0 B_- = A_- B_0, \qquad B_0 A_+ = B_+ A_0, \tag{4.4}$$

$$A_- B_+ = B_+ A_- = A_+ B_- = B_- A_+ = 0. \tag{4.5}$$

The Hom module is

$$E_N = \mathrm{Hom}_{B_L}(\widehat{G}, F_L(N)) = U_N \oplus V_N, \quad U_N = (N^r)^L, \quad V_N = (N^r)^{L-2} \oplus J_N^r. \tag{4.6}$$

Write U_a for the copy of N^r in U_N of index $0 \le a \le L-1$, and V_a for the copy of N^r in V_N of index $1 \le a \le L-2$. For $N = N_R$, identify N^r with $\mathrm{Mat}_r(S_R)$ by taking the r vectors as columns. In the family $N = N_R$ and its specializations, write σ_λ for the coefficient action and ρ_λ for the induced action on E_N. For T with columns T_i and $f = (f_i) \in J_N^r$, put

$$\big(\mathcal{R}_P(T)\big)_j = \sum_i \sigma_\lambda(p_{ij}) T_i, \qquad \big(\eta_\lambda(T)\big)_j(s) = \sigma_\lambda(s) T_j,$$

$$\big(\mathrm{ev}_Q(f)\big)_j = \sum_i f_i(q_{ij}).$$

Then A_P takes U_a to V_{a+1} by $\mathcal{R}_P$ for $0 \le a \le L-3$, and takes U_{L-2} to J_N^r by $\eta_\lambda \mathcal{R}_P$; it kills the last U-coordinate and all of V. The map B_Q takes V_a to U_a by $\mathcal{R}_Q$, and takes J_N^r to U_{L-1} by ev_Q; it is zero on U. The A_P action follows by evaluating C_P on $g_0(s)$ and shifting the image of 1 once. The B_Q action evaluates at $G_Q(1)$, with entries $g_0(q_{ij})$. In particular,

$$\mathcal{R}_{Q_\pm}(T) = -C_\lambda T \pm T C_\alpha. \tag{4.7}$$

Endpoint identities. At $\lambda = \alpha$, take u to be I_r in U_0, zero elsewhere, and put $u' = A_0 u$. Let $\theta : \mathrm{Mat}_r(S_K) \to K$ be the coefficient of $x_1 \cdots x_n$ in the ordinary matrix trace. Let ϕ_U apply θ to U_{L-1}, zero on all other coordinates, and define

$$\phi_V = \phi_U B_+.$$

The bottom component of B_+ is evaluation at the fixed matrix Q_+; it is parameter independent. Consequently $\phi_U \rho_\lambda(B_+) = \phi_V$ throughout the parameter family.

In the following identities the algebra elements act at $\lambda = \alpha$. By (4.7), $\mathcal{R}_{Q_+}(I_r) = 0$, giving $A_+ u = 0$ and $B_+ u' = 0$ since u' is I_r in V_1. Cyclicity of trace and $B_+ A_- = 0$ give the other endpoint identities

$$A_+ u = 0, \quad A_0 u = u', \quad B_+ u' = 0, \quad \phi_V A_0 = 0, \quad \phi_V A_- = 0, \quad \phi_U B_+ = \phi_V. \tag{4.8}$$

Indeed, $\phi_V A_0$ is zero outside U_{L-2}, and there sends T to $\theta(-C_\alpha T + T C_\alpha) = 0$.

Write $D_i = \partial_{\lambda_i} \rho_\lambda|_\alpha$ for the derivative of the complete representation on (4.6). We also have

$$\phi_U D_i(B_+) = 0, \qquad \phi_V D_i(A_-) = 0. \tag{4.9}$$

The first holds because the bottom evaluation component is constant. For the second, differentiate the algebra identity $B_+ A_- = 0$, then apply ϕ_U; the first identity removes the other derivative term.

The bar functional. For $1 \le j \le n$, define an n-tuple of algebra elements W_j: its letter types are $A, B, A, \ldots, A$; position j is A_0 or B_0 according to type; before that position the letters are A_-, B_+; after it they are A_+, B_-. Let

$$W_* = (B_+, A_-, B_+, \ldots, A_-, B_+).$$

For a K-algebra T and left T-module E, an ordinary bar cochain of degree j is a K-linear map $T^{\otimes_K j} \to \mathrm{End}_K(E)$. Here $T = \widehat{\Gamma} \otimes_k K$ and $E = E_\alpha$, the specialization of the Hom module at α. For a cochain f of degree j, its differential evaluated at $a_1, \ldots, a_{j+1} \in T$ is

$$a_1 f(a_2, \ldots, a_{j+1}) + \sum_{i=1}^{j} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{j+1}) + (-1)^{j+1} f(a_1, \ldots, a_j) a_{j+1}.$$

The end terms compose with the action on E. These cochains are obtained by applying $\mathrm{Hom}_T(-, E)$ to the free left T-modules $T^{\otimes_K (j+1)} \otimes_K E$. The bar differential multiplies consecutive factors with alternating signs, using the module action for the last multiplication. Augmentation is the action $T \otimes_K E \to E$. Inserting 1 on the left gives a K-linear contraction of the augmented complex: its first face is the identity and all later faces cancel with those obtained by inserting 1 after applying the differential. Thus the cochains compute $\mathrm{Ext}^\bullet_T(E, E)$.

For any ordinary bar n-cochain F, with arbitrary values in $\mathrm{End}_K(E_\alpha)$, set

$$Z_n(F) = \sum_{j=1}^{n} (-1)^{j-1} \phi_V F(W_j) u + \phi_U F(W_*) u'. \tag{4.10}$$

Consider an arbitrary ordinary $(n-1)$-cochain f. All left faces of the W_j vanish by $\phi_V A_0 = \phi_V A_- = 0$. All right faces vanish except the last unit position, by $A_+ u = 0$. Every internal product away from the unit position vanishes by (4.5). The product adjoining the unit between positions j and $j+1$ is identical in W_j and W_{j+1}, by (4.4). Its bar-face sign is the same in the two words and the word coefficients are opposite, so these faces cancel in pairs.

The sole surviving right face from the unit words has total coefficient $(-1)^{n-1}(-1)^n = -1$, and is

$$-\phi_V f(A_-, B_+, \ldots, A_-, B_+) u'.$$

For W_*, the internal faces vanish by (4.5), the right face vanishes by $B_+ u' = 0$, and the left face is precisely the opposite of this remaining term, since $\phi_U B_+ = \phi_V$. Thus

$$\boxed{Z_n(df) = 0 \quad \text{for every ordinary bar cochain } f.} \tag{4.11}$$

The parameter product. On the i-th coordinate extension the action is $\rho_\alpha + \epsilon D_i$. Its off-diagonal part D_i satisfies $D_i(ab) = \rho_\alpha(a) D_i(b) + D_i(a) \rho_\alpha(b)$ for $a, b \in T$. The splitting over K into the constant and ϵ coordinates identifies the extension with the bar 1-cocycle D_i. To compose these classes, a cocycle D_i lifts to maps from bar degree $j+1$ to degree j by

$$a_0 \otimes \cdots \otimes a_{j+1} \otimes w \longmapsto a_0 \otimes \cdots \otimes a_j \otimes D_i(a_{j+1}) w.$$

These maps commute with the differentials in positive degrees: the earlier faces commute term by term, and the last two faces give the derivation identity. At degree zero augmentation recovers the cocycle. Composing the lifts therefore represents the Yoneda product by

$$\Phi(c_n, \ldots, c_1) = D_n(c_n) \cdots D_1(c_1).$$

On the coefficient coordinates, the derivative of either $\mathcal{R}_{Q_+}$ or $\mathcal{R}_{Q_-}$, with the matrices $Q_\pm$ held fixed, is

$$T \longmapsto -x_i P_i T.$$

For W_1, the first $n-1 = 2L-4$ derivative factors, applied from right to left, alternate between $U_a \to V_{a+1}$ and $V_{a+1} \to U_{a+1}$ for $0 \le a \le L-3$. They move I_r to U_{L-2} and give $(-1)^{n-1}(x_{n-1}\cdots x_1)P_{n-1}\cdots P_1$. The last derivative is $D_n(A_0)$. Evaluation at Q_+ after applying A_0 on U_{L-2} gives $\mathrm{ev}_{Q_+}\eta_\lambda = \mathcal{R}_{Q_+}$. Evaluation is constant in the parameters, so the derivative contributes left multiplication by $-x_nP_n$. Therefore (4.1) gives

$$\phi_V\Phi(W_1)u = (-1)^n\theta((x_n\cdots x_1)P_n\cdots P_1) = -\operatorname{tr}(I_r) = -r \neq 0. \tag{4.12}$$

Every W_j with $j>1$ starts with A_-, and its contribution vanishes by $\phi_V D_n(A_-)=0$. The W_* term vanishes by $\phi_U D_n(B_+)=0$. Hence

$$Z_n(\Phi) = -r \neq 0. \tag{4.13}$$

Equations (4.11) and (4.13) prove nonvanishing in $\mathrm{Ext}^n_{\widehat{\Gamma}\otimes_k K}(E_\alpha, E_\alpha)$ at every parameter point over every extension field.

The hypotheses of Lemma 3 hold with $C = B_L$, $G = B_L^r \oplus (DB_L)^r$ and $d = n$. The Hom formula (4.3) gives freeness and commutation with base change, and the bar functional gives nonvanishing at every point. □

5 The lower bound for the Frobenius algebra

Lemma 5. $\operatorname{repdim} A_L \ge L+1$.

Proof. Put $d = L-1$. In characteristic different from two, put $h = \lfloor d/2 \rfloor$ and $r = 2^h$. Use the matrices X, Y, Z from the proof of Lemma 4.

Take

$$P_{2j-1} = Z^{\otimes(j-1)} \otimes X \otimes I_2^{\otimes(h-j)}, \quad P_{2j} = Z^{\otimes(j-1)} \otimes Y \otimes I_2^{\otimes(h-j)}$$

for $1 \le j \le h$; for odd $d = 2h+1$, append $P_d = Z^{\otimes h}$. These matrices anticommute in pairs. Write $\Omega = Z^{\otimes h}$. Then

$$P_d\cdots P_1 = \begin{cases} I_r & d \text{ odd}, \\ \Omega & d \text{ even}, \end{cases} \qquad \Omega^2 = I_r.$$

For even d, Ω anticommutes with every P_i. These assertions follow from $YX = Z$, the tensor-position rule, and the displayed definitions. In each P_i exactly one factor anticommutes with the corresponding Z in Ω.

In characteristic two, instead take $r = 1$, all $P_i = 1$, and $\Omega = 1$. The same anticommutator identities hold and r is nonzero in every case.

Parameter family. Let

$$S = k[x_1, \ldots, x_{2L-3}]/(x_i^2), \quad R = k[\lambda_1, \ldots, \lambda_d], \quad C_\lambda = \sum_{i=1}^{d} \lambda_i x_i P_i.$$

The square of C_λ is zero. Give $N_R = S_R^r$ its ordinary left Σ_R action by scalar multiplication by x_i for $i \ge 1$ and by $-C_\lambda$ for x_0. These actions commute, and the x_0 action squares to zero.

Set $X_R = F_L(N_R) = N_R^L$, where $t^a s$ acts by a shift of a places and the action of s. This family is finite free over R.

Apply the Morita equivalence

$$H = \mathrm{Hom}_{B_L}(B_L^r, -) : B_L\text{-mod} \longrightarrow \Delta\text{-mod}, \qquad \Delta = \mathrm{End}_{B_L}(B_L^r)^{op}.$$

This is an exact equivalence, hence preserves the parameter extensions and their Yoneda products in both directions. In particular, nonvanishing proved in Ext_Δ below is nonvanishing in ordinary Ext_{B_L}.

Identify each layer of $H(X_R)$ with $\mathrm{Mat}_r(S_R)$ by taking the r vectors as columns. At a point α over a field extension K/k, keep

$$Q_+ = x_0 I_r + C_\alpha, \qquad Q_- = x_0 I_r - C_\alpha$$

fixed while differentiating the parameters. Both products Q_+Q_- and Q_-Q_+ are zero over Σ_K.

The elements z_+ and z_- of $\Delta_K = \Delta \otimes_k K$ represented by the matrices tQ_+ and tQ_- act, by precomposition, as a one-layer shift followed by

$$T \longmapsto -C_\lambda T \pm T C_\alpha. \tag{5.1}$$

The opposite-ring order reverses matrix products, but the two needed products are both zero. Thus $z_+z_- = z_-z_+ = 0$ as algebra identities.

The bar functional. Let v be I_r in the first layer, zero elsewhere. Write ρ_λ for the action on $H(X_R)$. At the chosen point, $\rho_\alpha(z_+)v = 0$ by (5.1). Let ϕ be zero except on the last layer and there set

$$\phi(T) = \begin{cases} [x_1 \cdots x_d]\,\mathrm{tr}(T) & d \text{ odd}, \\ [x_1 \cdots x_d]\,\mathrm{tr}(\Omega T) & d \text{ even}. \end{cases}$$

Here $[x_1 \cdots x_d]$ selects the indicated squarefree monomial, with exponent zero on all other generators. For odd d, cyclicity of matrix trace gives $\phi\rho_\alpha(z_+) = 0$ on the whole module. For even d, the relation $\Omega C_\alpha + C_\alpha \Omega = 0$ gives

$$\mathrm{tr}\big(\Omega(-C_\alpha T - T C_\alpha)\big) = 0,$$

so $\phi\rho_\alpha(z_-) = 0$ on the whole module. In characteristic two the displayed trace is zero with $\Omega = 1$, since the trace is cyclic and $2 = 0$.

Let W be the alternating word of length d whose rightmost letter is z_+. Its leftmost letter is z_+ when d is odd and z_- when d is even. For an ordinary bar d-cochain F, put

$$Z(F) = \phi F(W) v.$$

For a cochain f of degree $d-1$, every internal face of $Z(df)$ contains a zero product z_+z_- or z_-z_+. The right face contains $\rho_\alpha(z_+)v = 0$. The left face vanishes by the appropriate whole-module identity for ϕ. Consequently

$$Z(df) = 0$$

for every ordinary bar $(d-1)$-cochain.

Write D_i for the derivative of the Δ-action in the ith parameter direction. The action on the i-th coordinate extension is $\rho_\alpha + \epsilon D_i$. By (5.1), the derivative of either fixed z_+ or z_- advances one layer and left-multiplies by $-x_i P_i$. Thus the ordered cup product takes v to

$$(-1)^d x_d \cdots x_1 P_d \cdots P_1$$

in the last layer. All the x_i commute. The detector evaluates this to $-r$ in odd degree and r in even degree. This is nonzero in every field by our choice of r. Hence the parameter product is nonzero at every point, over every extension field.

Inflate the family X_R along the unital projection $A_L \to B_L$ onto the first diagonal coordinate. The exact functor $e(-)$ recovers the original family and its coordinate extensions. It therefore takes their product over A_L to the nonzero product over B_L. Apply Lemma 3 with $C = G = A_L$, since $\mathrm{Hom}_{A_L}(A_L, -)$ identifies with the identity under evaluation at 1. The resulting lower bound is $d + 2 = L + 1$. □

6 The upper bound for the corner

Lemma 6. $\mathrm{repdim}\, B_L \le 2L - 1$.

Proof. Let $s = \dim_k \Sigma$.

Contraction modules. Write $S = B_L/J$ for the simple B_L-module, and put

$$Q_a = B_L/J^a \quad (1 \le a \le L), \qquad q = s - 1, \qquad P = \Sigma \otimes_k \Sigma.$$

Thus $Q_1 = S$ and $Q_L = B_L$. Set $Q_0 = 0$. The multiplication map

$$\mu : P \longrightarrow \Sigma, \qquad \mu(a \otimes c) = ac$$

is a split epimorphism of left Σ-modules; use the section $c \mapsto c \otimes 1$. Its kernel Z is free of rank q, since it is projective over the local algebra Σ. We use the same fact for subsequent split surjections of finite free Σ-modules.

For the strings $F_r(E)$ defined above, set $F_0(E) = 0$ and write $F_r = F_r(\Sigma)$. In particular $F_1 \cong S^s$.

For $0 \le d \le L - 2$, define C_d by its layers in degrees $0, \dots, L-1$:

$$C_d = \left(\Sigma, \underbrace{P, \dots, P}_{d \text{ layers}}, \underbrace{\Sigma, \dots, \Sigma}_{L-2-d \text{ layers}}, k \right).$$

The parentheses denote the direct sum of the listed k-spaces, ordered by increasing degree. When $d > 0$, the first action is insertion, $(tb)c = b \otimes c$. Between tensor layers, tb multiplies the first factor on the left. At the transition to an ordinary coefficient layer it acts by $a \otimes c \mapsto bac$; thereafter it acts by left multiplication. The last action is the Frobenius pairing $c \mapsto \tau(bc)$, or $a \otimes c \mapsto \tau(bac)$ if the preceding layer is still a tensor layer. For $d = 0$, all coefficient layers are ordinary. For $t^a b$ with $a > 1$, apply tb and then t another $a - 1$ times. The same formulas result if the coefficient b is applied at any of the a steps, since insertion is followed by multiplication on the first factor and both contraction and the final trace use the product in order. Thus the action

respects multiplication in B_L. They identify C_0 with DB_L, using the left action $(b\xi)(x) = \xi(xb)$ on the dual.

Set

$$M = \bigoplus_{a=1}^{L} Q_a \ \oplus \ \bigoplus_{d=0}^{L-2} C_d, \qquad \Gamma = \mathrm{End}_{B_L}(M)^{op}, \qquad H = \mathrm{Hom}_{B_L}(M, -).$$

The functor H takes values in left Γ-modules. Write pd_Γ for projective dimension over Γ. Because B_L and DB_L occur, M is a generator-cogenerator.

All $2L-1$ displayed summands are indecomposable and pairwise nonisomorphic. Indeed, an endomorphism of C_d has scalar top: commutation with the maps $t^{L-1}b$ and nondegeneracy of τ force its top component A to equal $\lambda \,\mathrm{id}_\Sigma$: if the map on the final layer is multiplication by λ, then $\tau(bA(c)) = \lambda\tau(bc)$ for all $b, c \in \Sigma$. Each C_d is generated by its degree-zero layer, since $t^a b$ on a top vector c gives $b \otimes c$, bc, or $\tau(bc)$ in degree a. Endomorphisms with zero top raise degree and form a nilpotent ideal. Thus $\mathrm{End}_{B_L}(C_d)$ is local with residue field k. The dimensions of the C_d strictly increase with d; their tops have dimension s, whereas every Q_a has one-dimensional top. The assertions for the Q_a follow from their cyclic local quotient structure.

We call an epimorphism $X \to Y$ *M-proper* if every map $M \to Y$ lifts through it. An exact sequence is proper if it remains exact after applying H. For a summand X of M, write T_X for the simple top of the projective module $H(X)$. A relative radical approximation of X is a map from $\mathrm{add}\, M$ whose image under H is $\mathrm{rad}\, H(X)$. The latter consists of maps whose component at X lies in the radical of $\mathrm{End}_{B_L}(X)$, with unrestricted components at the other summands. If the approximation has kernel K and domain P', left exactness gives

$$0 \to H(K) \to H(P') \to H(X) \to T_X \to 0.$$

The two middle terms are projective, so $\mathrm{pd}_\Gamma T_X \le \mathrm{pd}_\Gamma H(K) + 2$.

Radical splittings and the two lifting rules. Write $I_n = DQ_n$ for $1 \le n \le L$, with $n-1$ coefficient layers and a final scalar layer as in C_0. In the positive degrees of C_d, split each tensor layer as $Z \oplus (\Sigma \otimes 1)$. The Z-layers have zero action into the first layer after contraction, and the other summands continue through the final trace. This gives

$$\mathrm{rad}\, C_d \cong F_d(Z) \oplus I_{L-1}.$$

For $d \ge 1$, there is a quotient $\pi_d : C_d \to C_{d-1}$ that contracts the tensor layer in degree d and is the identity in every other degree. Its kernel is S^{sq}, and on radicals it is

$$\left[F_d(Z) \longrightarrow F_{d-1}(Z)\right] \oplus \mathrm{id}_{I_{L-1}}.$$

Here the bracketed map truncates the final layer.

All maps below decompose into homogeneous maps, where degree j means that a source layer of index i maps to target layer $i + j$. Each source is generated in degree zero, so only degrees $j \ge 0$ occur. A zero-top map into a module generated in degree zero has image in its positive layers. When the target is a shifted string, the same tests apply to the portion starting in the image degree.

The following calculation will be used for maps out of C_e. For a homogeneous map, let A be its map from the source top into the first layer of the target portion. The value on $(t^a b)c$ must be $t^a bA(c)$. When the source layer at distance a is tensor, this prescribes the value on $b \otimes c$ with

no relation. At the first ordinary source layer, it must depend only on bc. If the target layer at that distance is ordinary, this gives $bA(c) = A(bc)$ by setting $b = 1$ in the required factorization. If the target layer is tensor, it gives $b \otimes A(c) = 1 \otimes A(bc)$. Taking b linearly independent of 1 forces $A = 0$. If the target ends in a scalar layer at this distance, the equation is $\tau(bA(c)) = f(bc)$ for a functional f, which implies $A(c) = cu$ by writing $f(x) = \tau(xu)$. When the target has only ordinary layers after the contraction, the relations propagate by multiplication, followed by τ if it has a final scalar layer. At the source's final scalar layer, a full-length map to C_d satisfies $\tau(bA(c)) = \lambda\tau(bc)$, so $A(c) = \lambda c$. For a degree-zero map to $F_r(E)$, the top map is consequently arbitrary if $e \geq r - 1$, and must be Σ-linear if $e \leq r - 2$.

A map out of Q_a is determined by the image of $1 + J^a$, which must be killed by J^a. In an ordinary string a nonzero vector with b layers remaining is killed by J^a exactly when $a \geq b$, since t shifts by the identity. In a string ending in a trace the same test holds on coefficient layers: the last coefficient-to-scalar map can detect any nonzero coefficient by varying the multiplier. These tests will also be used for lifts through a quotient or projective cover.

Truncation lifting. Suppose the restriction of a map to radicals, after left Σ-linear splittings, has the form

$$\left[F_{r+1}(E) \longrightarrow F_r(E)\right] \oplus \mathrm{id}_R, \qquad r + 1 \leq L - 1,$$

where E is free. Add the cover $Q_r^{\dim_k E} \to F_r(E)$ induced by a k-basis of the first layer E. Then every zero-top map from any Q_a or C_e lifts through the augmented map.

Here R is the complementary target summand, which lifts by the identity. Maps from a Q_a into the first layer of the defect string lift through its Q_r-cover; maps landing farther down lift through a left Σ-linear section of the coefficient multiplication map. For a C_e-source, an arbitrary top occurs only when $e \geq r - 1$, and then factors through the standard Q_r-cover. A coefficient-linear top, when $e \leq r - 2$, extends to $F_{r+1}(E)$. Positive-degree components lift through the positive radical layers of the Q_r-cover, again using a left Σ-linear section. This treats all homogeneous components.

The kernel contributed by this augmented defect map is the module

$$U_r(E) = \left(E, \underbrace{\Sigma \otimes_k E, \ldots, \Sigma \otimes_k E}_{r-1 \text{ layers}}, E \right),$$

with insertion at the first transition, multiplication on the first factor between tensor layers, and coefficient multiplication at the last transition. For $r = 1$, tb sends $x \in E$ in the first layer to $bx \in E$ in the second. To see the kernel, identify its top with the top of the added cover and write $\mu_E(b \otimes x) = bx$ for its multiplication map. In each degree $1, \ldots, r - 1$ the pair of coordinates in the two channels is $(-\mu_E(z), z)$ for $z \in \Sigma \otimes_k E$. In degree r only the first channel remains. Taking its negative as the final coordinate gives the stated last action. For $E = \Sigma^b$, it is U_r^b, where $U_r = U_r(\Sigma)$. Set $U_0(E) = E$, viewed as semisimple. The case $r = 0$ is the semisimple kernel of the contraction itself, without an added quotient cover.

Trace lifting. Every map from a Q_a or C_e to I_n, for $n \leq L - 1$, lifts through

$$F_n \longrightarrow I_n,$$

which is the identity on its first $n - 1$ coefficient layers and τ on its last layer. If a homogeneous map starts in degree j, the two tails beginning there have the same form with length $n - j$. It suffices to lift into the first layer of those tails.

For a C_e-map to the first coefficient layer, if $e \geq n-1$, its top A is arbitrary. A lift uses the same A on top and $b \otimes c \mapsto bA(c)$ on every subsequent layer, including the last. Composing the last map with τ recovers the given map. When $e \leq n-2$, comparison at that contraction yields

$$\tau(xA(c)) = f(xc)$$

for a functional f. Write $f(x) = \tau(xu)$. Nondegeneracy forces $A(c) = cu$, which lifts coefficient-wise. If the tail has length one, its maps lift through any k-linear section of τ. For a Q_a-source, a nonzero map into the first coefficient layer requires $a \geq n$: otherwise its required annihilation by a radical power contradicts nondegeneracy of the final pairing. The chosen image of $1 + J^a$ lifts into the first layer of the ordinary tail, which is killed by J^a.

Resolutions of contracted strings. For $1 \leq d \leq L-2$, the module

$$U_d = (\Sigma, \underbrace{P, \ldots, P}_{d-1 \text{ layers}}, \Sigma)$$

is the initial $d+1$-layer truncation of C_{d-1}, and

$$\operatorname{rad} U_d \cong F_{d-1}(Z) \oplus F_d.$$

Notice that $U_1 = F_2$ and $U_0 = F_1 = S^s$.

Choose a basis $e_1, \ldots, e_s$ of Σ, and let

$$\ell : \Sigma^s \longrightarrow \Sigma, \qquad \ell(c_1, \ldots, c_s) = \sum_j c_j e_j.$$

For the j-th C_{d-1}-summand, truncate to U_d and multiply the top and ordinary coefficients on the right by e_j, acting by $a \otimes c \mapsto a \otimes ce_j$ on tensor layers. Together with the cover $Q^s_{d+1} \to U_d$ determined by the basis e_j of its top, these maps define

$$p_d : C^s_{d-1} \oplus Q^s_{d+1} \longrightarrow U_d.$$

This map is proper. On radicals, the $F_{d-1}(Z)$-summand splits through the C_{d-1}-channels, because a scalar combination of the e_j is 1. The F_d-summand splits through the radical of Q^s_{d+1}. In the coordinates $\Sigma \otimes_k \Sigma$ supplied by the basis e_j, the section is $a \mapsto a \otimes 1$ in each positive layer. Its image in the tensor layers lies in $\Sigma \otimes 1$ and in the final layer maps to a. For a degree-zero C_e-map, comparison of insertion and contraction shows that the top is zero if $e < d-1$, is coefficient-linear if $e = d-1$, and is arbitrary if $e \geq d$. The linear top $c \mapsto cu$ lifts by writing u as a linear combination of the e_j in the C_{d-1}-summands. An arbitrary top lifts to Q^s_{d+1}, since the first d positive layers of C_e are tensor. A Q_a-map with nonzero top requires $a \geq d+1$, because t^d is the identity from the first to the last layer of U_d. Such a map lifts through Q^s_{d+1}, which is killed by J^a. The sections on radicals lift positive-degree maps.

Let $H'_d = \ker p_d$. Projecting the kernel to C^s_{d-1} identifies its top and layers $1, \ldots, d-1$ with Σ^s and P^s, since the maps from the Q^s_{d+1}-layers there are isomorphisms onto the target layers. In degrees below d, the coordinate in the Q^s_{d+1} summand is the negative of the image of the C^s_{d-1} coordinate under the first component of p_d, transported by the isomorphism. In degree d the layer is

$$V = \{(f, z) \in \Sigma^s \oplus P : \ell(f) + \mu(z) = 0\},$$

and its later layers are Σ^s, ending in k^s. There is a map $\alpha : C_d^s \to H'_d$ that is the identity in the common coordinates except in degree d, where

$$\alpha(a \otimes (c_j)_j) = \big((ac_j)_j, -a \otimes \ell((c_j)_j)\big).$$

In degree d, the first coordinate is contraction in C_{d-1}^s, and the second is the negative of the tensor map from the preceding degree. The next layer receives only the first coordinate, by multiplication or, at the last step, by trace. Thus α commutes with the tb-actions, including $d = 1$, when the preceding degree is the top. To see surjectivity, lift z through $-\operatorname{id}_\Sigma \otimes \ell$. The discrepancy in the first coordinate then lies in $\ker \ell$ and can be corrected in $\Sigma \otimes_k \ker \ell$, since coefficient multiplication surjects onto $\ker \ell$. Its coefficient kernel is

$$E_0 = \ker(\mu^s) \cap \ker(\operatorname{id}_\Sigma \otimes \ell) \cong \Sigma^{q^2}.$$

Indeed, $\ker \ell \cong \Sigma^q$, and E_0 is the kernel of the multiplication map $\Sigma \otimes_k \ker \ell \to \ker \ell$. For a left Σ-basis $u_1, \dots, u_q$ of $\ker \ell$, the map $\sum_{i=1}^q a_i u_i \mapsto \sum_{i=1}^q a_i \otimes u_i$, with $a_i \in \Sigma$, is a left Σ-linear section of multiplication for the first-factor action. Hence E_0 is projective over the local algebra Σ, so it is free of rank $sq - q = q^2$.

Using the same left Σ-linear splitting of $P^s \to V$ in degrees $1, \dots, d$ identifies the radical restriction of α with

$$\big[F_d(E_0) \longrightarrow F_{d-1}(E_0)\big] \oplus \operatorname{id}_R .$$

The truncation lifting rule therefore supplies a proper cover of the zero-top part after adjoining $Q_{d-1}^{sq^2}$. For a degree-zero map $C_e \to H'_d$, the final scalar layers force its top to be $c \mapsto (\lambda_j c)$. If $e < d$, its projection to Q_{d+1}^s has zero top, so $\ell((\lambda_j c)_j) = 0$; independence of the e_j forces all $\lambda_j = 0$. If $e \geq d$, its top lifts through the canonical quotient $C_e \to C_d$. All Q_a with $a < L$ have zero top into this full-length kernel, and B_L lifts by projectivity. Thus there is a proper exact sequence

$$0 \longrightarrow U_{d-1}^{q^2} \longrightarrow C_d^s \oplus Q_{d-1}^{sq^2} \longrightarrow C_{d-1}^s \oplus Q_{d+1}^s \longrightarrow U_d \longrightarrow 0.$$

Applying H leaves an exact sequence with two projective middle terms. Since $U_0 \cong S^s$ belongs to $\operatorname{add} M$, induction gives

$$\operatorname{pd}_\Gamma H(U_d) \leq 2d.$$

Ordinary strings and the projective summand. For $2 \leq r \leq L - 1$, consider

$$C_0^s \oplus Q_r^s \longrightarrow F_r,$$

using truncation followed by right multiplication by e_j on the j-th C_0, and the top basis e_j for the quotient cover. Coefficient-linear C_e-tops lift through C_0^s, using the maps $C_e \to C_0$ and scalar combinations of the e_j. Arbitrary tops are allowed when $e \geq r - 1$, and lift through Q_r^s. A Q_a-map with nonzero top requires $a \geq r$, so it lifts through Q_r^s. The radical of the quotient cover splits onto the target radical by $a \mapsto a \otimes 1$ in each positive layer, lifting the positive-degree components.

Its kernel H_r has top Σ^s, with the k^s coordinate fixed by cancellation of the two maps on top. In degrees $1, \dots, r - 1$ the equation is $\ell(f) + \mu(z) = 0$, so these layers are V. At degree r and above, only the C_0^s layers remain. The formula for α on degree d in the preceding calculation, now used at every degree $1, \dots, r - 1$, defines

$$C_{r-1}^s \longrightarrow H_r$$

with kernel $F_{r-1}^{q^2}$. Here the entire radical restriction splits: use the same left Σ-linear section of $P^s \to V$ in each of the first $r-1$ layers. It commutes with left multiplication between repeated layers, and its composite with μ^s is projection to the first coordinate of V, which gives the next transition. Every zero-top map therefore lifts. The full-length final trace again forces scalar-coordinate tops from C_e, by projection to C_0^s. If $e < r-1$, the projection to Q_r^s has zero top by the tensor target test, hence $\ell((\lambda_j c)_j) = 0$ for every $c \in \Sigma$. At $c = 1$, independence of the e_j gives $\lambda_j = 0$. If $e \geq r-1$, the top lifts by contracting $C_e \to C_{r-1}$ in the corresponding channels. A Q_a-map into the kernel with nonzero top projects to a Q_a-map into C_0^s with nonzero top, requiring $a = L$; it lifts by projectivity. Consequently

$$0 \longrightarrow F_{r-1}^{q^2} \longrightarrow C_{r-1}^s \longrightarrow C_0^s \oplus Q_r^s \longrightarrow F_r \longrightarrow 0$$

is proper. After applying H, there are two projective middle terms. The base $F_1 \cong S^s$ gives

$$\operatorname{pd}_\Gamma H(F_r) \leq 2r - 2.$$

A map from an indecomposable summand of M to B_L that is nonzero on the top is surjective by Nakayama's lemma. It splits by projectivity and is therefore an isomorphism by indecomposability. Thus the maps in the radical of $H(B_L)$ are exactly those with image in $J = F_{L-1}$, and $0 \to H(F_{L-1}) \to H(B_L) \to T_{B_L} \to 0$ is exact. Since $H(B_L)$ is projective,

$$\operatorname{pd}_\Gamma T_{B_L} \leq 2L - 3.$$

The simples attached to the contraction modules. For $0 \leq d \leq L-3$, a relative radical approximation is

$$C_{d+1} \oplus Q_d^{sq} \longrightarrow C_d,$$

The first component is π_{d+1}, the contraction at degree $d+1$. The second covers $F_d(Z)$ inside the radical. Its kernel is U_d^q, by the truncation lifting rule. The same rule lifts every zero-top map. A nonzero top map from a different C_e occurs precisely when $e \geq d+1$, by the full-length scalar top condition and the tensor target test. It is the corresponding scalar multiple of the iterated contraction. The radical endomorphisms of C_d have zero top. Among Q_a-sources only Q_L can have nonzero top, and it lifts through the surjection by projectivity. Hence

$$\operatorname{pd}_\Gamma T_{C_d} \leq 2d + 2.$$

For the last contraction module, use

$$B_L^s \oplus Q_{L-2}^{sq} \longrightarrow C_{L-2}.$$

On radicals, the splitting $P = Z \oplus \Sigma$ identifies the projective-cover part with

$$\big[F_{L-1}(Z) \longrightarrow F_{L-2}(Z)\big] \oplus \big[F_{L-1} \longrightarrow I_{L-1}\big].$$

The projective map sends the basis of the top of B_L^s to the basis e_j of the top of C_{L-2}. In its positive layers the action is insertion up to degree $L-2$ and the trace of multiplication in degree $L-1$, giving the displayed splitting. All C_e-maps in the radical have zero top, and the only Q_a allowing nonzero top is B_L. The two lifting rules handle the zero-top components and projectivity handles the B_L-source. The kernel is

$$U_{L-2}^q \oplus S^q,$$

where the semisimple term is the kernel of the last trace. Therefore

$$\operatorname{pd}_\Gamma T_{C_{L-2}} \leq 2L - 2.$$

The simples attached to the nonmaximal quotients. Choose a basis $\phi_1, \dots, \phi_s$ of $D\Sigma$ with $\phi_1(1) = 0$, possible since $s > 1$. Choose $c_0 \in \Sigma$ with $\phi_1(c_0) = 1$. For $1 \le a \le L-2$, the relative radical approximation is

$$Q_{a+1} \oplus C_{a-1}^s \longrightarrow Q_a,$$

using the quotient map and the maps with top components ϕ_j, which on tensor layers send $b \otimes c$ to $b\phi_j(c)$. A C_e-map can have nonzero top only when $e \ge a-1$, by the tensor target test on the positive layers of Q_a. Its top is a linear combination of the ϕ_j and lifts by contraction to C_{a-1}^s. A nonzero top from a different Q_b requires $b > a$, and factors through Q_{a+1}. If $b = a$, the endomorphism is radical exactly when its top is zero. All zero-top maps lift because

$$b \otimes c_0 - bc_0 \otimes 1 \in Z$$

maps to b under the first functional channel. This gives a left Σ-linear section throughout $\operatorname{rad} Q_a$. For $a = 1$ the radical is zero, and the top argument suffices.

Let H_a^Q be the kernel of this approximation. In degrees $0, \dots, a-1$, projection to C_{a-1}^s identifies the kernel with its layers, since the quotient map from Q_{a+1} is an isomorphism in those degrees. In degree a both source layers survive and the target layer is zero, giving $\Sigma^s \oplus \Sigma$. Above degree a, only C_{a-1}^s contributes. The natural map $C_a^s \to H_a^Q$ is the identity on the common coordinates and in degree a is

$$\beta\big(b \otimes (c_j)\big) = \left((bc_j), -b \sum_j \phi_j(c_j) \right).$$

Given the first coordinate, lift it through the componentwise multiplication $P^s \to \Sigma^s$. The difference in the last coordinate is corrected by the displayed element of Z in the first channel. This proves surjectivity. The transition into degree a multiplies in the C_{a-1}^s coordinates and evaluates the negative functional in the quotient coordinate. The transition out of degree a uses only the first coordinate, so β commutes with both. Its coefficient kernel $E_S \subseteq \ker \mu^s$ is free of rank

$$b_S = s^2 - s - 1.$$

This is the difference between the ranks s^2 of P^s and $s+1$ of $\Sigma^s \oplus \Sigma$. Choose a left Σ-linear section of β and use it in degrees $1, \dots, a$. Since the first component of β is μ^s, the complement continues through the remaining coefficient and scalar layers. The E_S-component truncates from $F_a(E_S)$ to $F_{a-1}(E_S)$. Adjoining $Q_{a-1}^{sb_S}$ therefore gives a cover with kernel $U_{a-1}^{b_S}$. It is proper on zero-top maps by the truncation lifting rule. A degree-zero C_e-map to H_a^Q has scalar-coordinate top. If $e < a$, its Q_{a+1}-component has zero top. The kernel relation on the top then gives $\sum_j \lambda_j \phi_j(c) = 0$ for all c, so $\lambda_j = 0$ by independence. If $e \ge a$, the top lifts through $C_e \to C_a$. A nonzero top from Q_b projects to a nonzero top in C_{a-1}^s, so $b = L$; it lifts by projectivity. Applying H to this cover gives $\operatorname{pd}_\Gamma H(H_a^Q) \le 1 + 2(a-1)$. The radical approximation adds two projective terms, giving

$$\operatorname{pd}_\Gamma T_{Q_a} \le 2a + 1 \qquad (1 \le a \le L-2).$$

The maximal proper quotient. For Q_{L-1}, a relative radical approximation is

$$C_{L-2}^s \longrightarrow Q_{L-1},$$

with the same functional channels. For C_e, a nonzero top requires $e = L-2$ and lifts through these channels. The same section on radicals as for Q_a lifts zero-top maps. The projective B_L-source lifts through this surjection. Put

$$\Phi : \Sigma^s \longrightarrow k, \qquad \Phi(c) = \sum_j \phi_j(c_j), \qquad W = \ker \Phi, \qquad v = \dim_k W = s^2 - 1.$$

The kernel is the module

$$G(W) = \left(W, \underbrace{\Sigma \otimes_k W, \ldots, \Sigma \otimes_k W}_{L-2 \text{ layers}}, k^s \right),$$

with $(tb)w = b \otimes w$ from the top, left multiplication on the first factor between tensor layers, and $a \otimes (c_j)_j \mapsto (\tau(bac_j))_j$ at the last transition.

The subspace W satisfies

$$\Sigma W = \Sigma^s, \qquad \{\lambda \in k^s : (\lambda_j c)_j \in W \text{ for all } c \in \Sigma\} = 0.$$

For the first, if ΣW were proper, the codimension-one condition would force $\Sigma W = W$. The one-dimensional quotient would give a character $\chi : \Sigma \to k$ satisfying $\Phi(bc) = \chi(b)\Phi(c)$. Applying this coordinatewise and using that the ϕ_j form a basis gives $bc = \chi(b)c$ for all b, c, hence $b = \chi(b)1$, contrary to $s > 1$. For the second identity, the condition says $\sum_j \lambda_j \phi_j = 0$, so every $\lambda_j = 0$.

Thus the multiplication map

$$\mu_W : \Sigma \otimes_k W \longrightarrow \Sigma^s, \qquad b \otimes w \longmapsto bw$$

is a split left Σ-epimorphism, with free kernel E_W of rank $v - s = s^2 - s - 1$. On radicals,

$$\operatorname{rad} G(W) \cong F_{L-2}(E_W) \oplus I_{L-1}^s.$$

There is a mixed cover

$$B_L^v \oplus Q_{L-2}^{s(v-s)} \longrightarrow G(W).$$

The projective part sends the standard generators to a k-basis of W, and the second part covers $F_{L-2}(E_W)$ inside the radical. On positive layers the projective part is the identity on $\Sigma \otimes_k W$ up to degree $L-2$, followed by μ_W and the coordinatewise trace. After splitting μ_W this gives

$$\big[F_{L-1}(E_W) \longrightarrow F_{L-2}(E_W)\big] \oplus \big[F_{L-1}^s \longrightarrow I_{L-1}^s\big].$$

The added quotient cover and the truncation lifting rule handle the defect string, while the trace lifting rule handles the second summand. The composite of a C_e-map into $G(W)$ with its inclusion in C_{L-2}^s has degree-zero top $c \mapsto (\lambda_j c)_j$, by the trace relation. Its image lies in W, so the second identity forces $\lambda = 0$. If $a < L$, a Q_a-map has zero top because a nonzero $w \in W$ is detected in at least one coordinate by $t^{L-1}b$ for a suitable b; the B_L-source lifts by projectivity. Hence the mixed cover is proper. Its kernel is

$$U_{L-2}^{v-s} \oplus S^{s(s-1)}.$$

The trace part contributes $(\ker \tau)^s$ in the final layer, on which J acts by zero. The truncation part contributes the other summand by the truncation rule. The projective dimension of H

applied to this kernel is at most $2(L-2)$. One term for its proper cover and two for the radical approximation give

$$\operatorname{pd}_\Gamma T_{Q_{L-1}} \le 2L-1.$$

The bounds for the simple left Γ-modules are

$$\begin{aligned} \operatorname{pd}_\Gamma T_{C_d} &\le 2d+2 \quad (0 \le d \le L-2),\\ \operatorname{pd}_\Gamma T_{Q_a} &\le 2a+1 \quad (1 \le a \le L-2),\\ \operatorname{pd}_\Gamma T_{Q_{L-1}} &\le 2L-1,\\ \operatorname{pd}_\Gamma T_{B_L} &\le 2L-3. \end{aligned}$$

Therefore $\operatorname{gl.dim}\Gamma \le 2L-1$. Since M is a generator-cogenerator, $\operatorname{repdim} B_L \le 2L-1$. □

Proof of Theorem 1. Lemmas 2 and 5 give the value for A_L, and Lemmas 4 and 6 give the value for B_L. At $L=3$ the squarefree monomial basis has 16 elements, so $\dim_k B_3 = 1+2\cdot 16 = 33$ and $\dim_k A_3 = 4\cdot 33 = 132$. □

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