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# The four-dimensional Hessian conjecture for quintic potentials with a ternary cone top

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**Abstract**

The Hessian conjecture $\mathrm{HC}_n$ asserts that if $f \in \mathbb{C}[x_1, \dots, x_n]$ has $\det \mathrm{Hess}(f) \in \mathbb{C}^\times$, then $\nabla f$ is a polynomial automorphism. It is known for $n \le 3$, false for $n \ge 5$, and open for $n = 4$, where it implies the plane Jacobian conjecture. Z. Ni proved $\mathrm{HC}_4$ for potentials of degree at most four. We prove $\mathrm{HC}_4$ for every potential of degree five whose top homogeneous form depends on exactly three linear forms. The proof combines a weighted analysis of the Hessian determinant with exact computer-algebra certificates (Gröbner bases and Rabinowitsch tests over $\mathbb{Q}$); one branch reduces to plane Jacobian pairs of degrees $(4, 3)$, which are excluded by Magnus' theorem. A self-contained verifier is supplied as an ancillary file. The binary and unary cone tops in degree five remain open.

## 1 Introduction

For $f \in \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ write $\mathrm{Hess}(f) = (\partial_i \partial_j f)$. The Hessian conjecture $\mathrm{HC}_n$ [9] states: if $\det \mathrm{Hess}(f) \in \mathbb{C}^\times$ then $\nabla f \colon \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial automorphism. It holds for $n = 2$ [4] and $n = 3$ [1]; it fails for every $n \ge 5$ [10]; for $n = 4$ it is open and implies the plane Jacobian conjecture by the doubling construction $\Phi(x, y) = y^{\mathsf{T}} F(x)$, for which $\det \mathrm{Hess}\, \Phi = (\det JF)^2$ [10, 11]. Ni [11] proved $\mathrm{HC}_4$ for $\deg f \le 4$. In degree five the determinant has more homogeneous layers and the arguments of [11] do not transfer directly ([11, §8]); partial structural reductions for a degree-five branch are recorded in [5].

Let $f \in \mathbb{C}[x_1, \dots, x_4]$ have degree 5 and $\det \mathrm{Hess}(f) = c \in \mathbb{C}^\times$. Its top form $f_5$ has $\det \mathrm{Hess}(f_5) = 0$, hence by Gordan–Noether [7, 2] depends on at most three linear forms. We say the top is a *ternary cone* if, after a linear change of coordinates, $f_5 = \varphi(x_1, x_2, x_3)$ with $\det \mathrm{Hess}_3(\varphi) \not\equiv 0$ (equivalently, $f_5$ depends on exactly three linear forms).

**Theorem 1.** *Let $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ have degree five, $\det \mathrm{Hess}(f) \in \mathbb{C}^\times$, and a ternary cone top. Then $\nabla f$ is a polynomial automorphism of $\mathbb{C}^4$.*

*Remark* 2. The class of Theorem 1 contains doubled potentials $F(x_1, x_2) + x_3 a(x_1, x_2) + x_4 b(x_1, x_2)$, whose Hessian determinant is $J(a, b)^2$ (Section 5.3). Hence no proof of the quintic case can avoid plane Jacobian pairs of low degree; here only pairs of degrees $(4, 3)$ occur, and Magnus' theorem excludes them.

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**Status.** The statements labelled *[C]* below are certified by exact computation (identities of polynomials over $\mathbb{Q}$ with symbolic coefficients, Gröbner bases, and the Rabinowitsch criterion $1 \in I + (1 - Tg) \Leftrightarrow g \in \sqrt{I}$). Each such statement corresponds to a check of the ancillary verifier `verify_hc4_ternary.py` (Section 6); all 29 checks pass. The paper has not been refereed.

## 2 Tools

Throughout, $x' = (x_1, x_2, x_3)$, $s = x_4$, and $A = \mathrm{Hess}_3(\varphi)$. Linear changes of coordinates preserve both hypothesis and conclusion, since $\mathrm{Hess}(f \circ T) = T^{\mathsf{T}}(\mathrm{Hess}\, f) \circ T\, T$.

**Imported results.** (GN) A form in $n \le 4$ variables with vanishing Hessian determinant depends on at most $n - 1$ linear forms [7], [2, Thm. 3.1]. ($\mathrm{HC}_2$, $\mathrm{HC}_3$) [4, 1], together with transfer to arbitrary fields of characteristic zero and descent of polynomial inverses [11, Lemmas 2.6, 2.7]. (dBvdE) Over a field of characteristic zero, a bivariate polynomial without terms of degree $\le 1$ and with vanishing Hessian determinant is a polynomial in one linear form [3, Thm. 3.1]. (Ni) [11, Prop. 6.1]: if $f = P(x') + sQ(x') + as^2$ with $\deg Q \le 2$ and $\det \mathrm{Hess}\, f \in \mathbb{C}^\times$, then $\nabla f$ is an automorphism. (Mag) [8]: a plane Jacobian pair whose degrees are coprime is an automorphism. (JvdK) For a plane automorphism $(F, G)$, $\deg F \mid \deg G$ or $\deg G \mid \deg F$ [6, Ch. 5].

**Lemma 3** (Schur branch, any $Q$)**.** *Let $f = P(x') + sQ(x') + as^2$ with $a \neq 0$ and $P, Q$ arbitrary. If $\det \mathrm{Hess}\, f \in \mathbb{C}^\times$, then $\nabla f$ is a polynomial automorphism.*

*Proof.* With $z = s + Q/(2a)$ and $S_z = P - Q^2/(4a) + zQ$ one has the polynomial identity $\det \mathrm{Hess}\, f = 2a \det \mathrm{Hess}_{x'}(S_z)$ at $z = s + Q/(2a)$ *[C, S6a]*, and $\nabla_{x'} f = \nabla_{x'} S_z$, $\partial_s f = 2az$. Thus $\det \mathrm{Hess}_{x'} S_z$ is a nonzero constant; $\mathrm{HC}_3$ over $\mathbb{C}(z)$ and descent to $\mathbb{C}[z]$ give a polynomial inverse $H(z, Y)$ of $\nabla_{x'} S_z$, and $z = Y_4/(2a)$, $x' = H(z, Y_1, Y_2, Y_3)$, $s = z - Q(x')/(2a)$ invert $\nabla f$. This is the argument of [11, Prop. 6.1, branch $a \neq 0$], which does not use $\deg Q \le 2$. □

**Lemma 4** (constant direction)**.** *Let $F \in \mathbb{C}[x_1, x_2, x_3]$ be homogeneous of degree $d$, nonzero, without terms of degree $\le 1$ in $(x_2, x_3)$, and with $\det \mathrm{Hess}_{(x_2, x_3)} F = 0$. Then there is a constant linear form $l = \lambda x_2 + \mu x_3$ with $F \in \mathbb{C}[x_1][l]$.*

*Proof.* Over $K = \mathbb{C}(x_1)$, (dBvdE) gives $F = G(l)$ with $l = \lambda x_2 + \mu x_3$, $\lambda, \mu \in K$, $G \in K[T]$, $\deg G \ge 2$. Then $\mathrm{Hess}_{(x_2, x_3)} F = G''(l)\,(\lambda, \mu)^{\mathsf{T}}(\lambda, \mu)$ has rank exactly one, so its kernel line is determined by $F$. Since $\mathrm{Hess}_{(x_2, x_3)} F(tx) = t^{d-2} \mathrm{Hess}_{(x_2, x_3)} F(x)$, the kernel line at $tx_1$ equals the one at $x_1$; thus the ratio $\lambda/\mu$ (or $\mu/\lambda$), a rational function of $x_1$, is invariant under all dilations and hence constant. After rescaling $\lambda, \mu \in \mathbb{C}$, $l$ is a coordinate and the coefficients of $F$ as a polynomial in $l$ lie in $\mathbb{C}[x_1]$. □

## 3 The weighted top

**Proposition 5.** *Let $f$ be as in Theorem 1, $f = f_5 + \cdots + f_0$, $f_5 = \varphi(x')$. Then:*

1. $[\det \mathrm{Hess}\, f]_{11} = \det A \cdot \partial_s^2 f_4$ [C, S1b]*; hence $f_4 = g_4(x') + s\, h(x')$.*
2. *With this,* $[\det \mathrm{Hess}\, f]_{10} = \det A \cdot \partial_s^2 f_3 - (\nabla h)^{\mathsf{T}} \mathrm{adj}(A) \nabla h$ [C, S1c]*; writing $f_3 = g_3 + sh_2 + s^2 m(x') + es^3$, it follows that $e = 0$.*
3. *For the weights $(1, 1, 1, 2)$, $f$ has no monomial of weight $> 5$, its weight-5 part is $\psi = \varphi + sh + s^2 m$, and the weight-10 part of $\det \mathrm{Hess}\, f$ equals $\det \mathrm{Hess}\, \psi$* [C, S2]*. Hence $\det \mathrm{Hess}\, \psi = 0$.*

*Proof.* (1) and (2) are identities with fully symbolic $f_4, f_3, f_2$; as $\det \mathrm{Hess}\, f$ is constant and $\det A \neq 0$, $\partial_s^2 f_4 = 0$, and in (2) the term $6es \det A$ is the only $s$-dependent one, so $e = 0$. (3) After (1) and (2), $f$ contains no $s^2x'^2$, $s^3x'$, $s^4$ (degree 4) and no $s^3$ (degree 3); every other monomial outside $\varphi$ has weight $\leq 4$, except $s\,x'^3$ (weight 5, in $sh$) and $s^2x'$ (weight 5, in $s^2m$); the identity holds for arbitrary lower terms, and the constant has weight $0 < 10$. □

# 4 Classification of the weighted top

**Proposition 6.** *Let $\psi = \varphi(x') + sh(x') + s^2m(x')$ with $\varphi$ a quintic, $h$ a cubic, $m$ linear,* $\det \mathrm{Hess}_3 \varphi \not\equiv 0$ *and* $\det \mathrm{Hess}\, \psi = 0$. *After a linear change of coordinates preserving the shape of $\psi$, one of the following holds:*

*(N)* $h = m = 0$;

*(D)* $m = 0$ *and* $\psi = Q(x_1, x_2) + x_1^4x_3 + x_1^3s$ *with* $\partial_2^2 Q \neq 0$;

*(F)* $\psi = x_1(s + x_1x_2)^2 + x_1^4x_3$.

*Proof. Case $m = 0$.* The coefficient of $s^2$ in $\det \mathrm{Hess}(\varphi + sh)$ is $-\frac{3}{2}\, h \det \mathrm{Hess}_3 h$, and those of $s^3, s^4$ vanish *[C, S3a, S3b]*. So $h = 0$ (case (N)) or, by (GN), $h$ is a binary cubic in two linear forms; up to $\mathrm{GL}_3$, $h \in \{x_1^3,\ x_1^2x_2,\ x_1^3 + x_2^3\}$. For $h = x_1^2x_2$ and $h = x_1^3 + x_2^3$, every coefficient of $\det A$ lies in the radical of the ideal of coefficients of $\det \mathrm{Hess}\, \psi$ *[C, S3c]*: excluded. For $h = x_1^3$, $\det \mathrm{Hess}\, \psi = -9x_1^4 \det \mathrm{Hess}_{(x_2,x_3)} \varphi$ *[C, S3d]*. Apply Lemma 4 to the part of $\varphi$ of degree $\geq 2$ in $(x_2, x_3)$, which is nonzero (otherwise $A$ has a zero $2 \times 2$ block and $\det A = 0$). After a rotation of $(x_2, x_3)$, $\varphi = Q(x_1, x_2) + x_1^4(\beta_2x_2 + \alpha x_3) + \beta_0 x_1^5$, and $\det A = -16\alpha^2 x_1^6\, \partial_2^2 Q$ *[C, S8c]*, so $\alpha \neq 0 \neq \partial_2^2 Q$. Replacing $x_3$ by $\alpha x_3 + \beta_2 x_2 + \beta_0 x_1$ gives (D).

*Case $m \neq 0$.* Take $m = x_1$. The coefficient of $s^4$ is $-4 \det \mathrm{Hess}_{(x_2,x_3)} h$ *[C, S4a]*; Lemma 4 (applied to the part of $h$ of degree $\geq 2$ in $(x_2, x_3)$, if nonzero) and a rotation of $(x_2, x_3)$ give $h = ax_2^3 + bx_1x_2^2 + x_1^2(cx_2 + c'x_3) + ex_1^3$. If $a \neq 0$, rescale to $a = 1$; if $a = 0 \neq b$, rescale to $b = 1$ and shift $x_2 \mapsto x_2 - \frac{c}{2}x_1$ to get $c = 0$; in both cases $\det A$ vanishes on all solutions *[C, S4 (I), (II)]*. If $a = b = 0$ and $(c, c') \neq 0$, a rotation, rescaling and shift give $h = x_1^2x_2$; otherwise $h = ex_1^3$, which is excluded *[C, S4 (III.2)]*. For $h = x_1^2x_2$ the solutions with $\det A \not\equiv 0$ are exactly $\varphi = \frac{1}{4}x_1^3x_2^2 + p_{18}x_1^4x_3 + p_{19}x_1^4x_2 + p_{20}x_1^5$ *[C, S4 (III.1), S8a]*, with $\det A = -8p_{18}^2x_1^9$ *[C, S8b]*, so $p_{18} \neq 0$; the substitution $x_3 \leftarrow (x_3 - p_{19}x_2 - p_{20}x_1)/p_{18}$, $x_2 \mapsto 2x_2$ gives (F) *[C, S8d]*. All changes used fix $x_1$ and act on $x'$ only, so they preserve $m = x_1$ and the shape of $\psi$. □

# 5 Closing the three cases

## 5.1 Case (N)

Here $f$ has no monomials $s\,x'^3$, $s^2x'$, $s^3$, so $f = P(x') + sQ(x') + as^2$ with $\deg Q \leq 2$, and (Ni) applies.

## 5.2 Case (F)

**Proposition 7.** *No quintic $f$ with weight-5 part $x_1(s + x_1x_2)^2 + x_1^4x_3$ has* $\det \mathrm{Hess}\, f \in \mathbb{C}^\times$.

*Proof.* Write $f = \psi_F + \sum_\mu c_\mu \mu$ over all monomials $\mu$ of weight $\leq 4$ and degree $\geq 2$ (41 unknowns). The constant term of $\det \mathrm{Hess}\, f$ lies in the radical of the ideal generated by the coefficients of all non-constant monomials *[C, S5]*. □

### 5.3 Case (D)

**Proposition 8.** *Let $f$ be quintic with weight-5 part $Q(x_1,x_2)+x_1^4x_3+x_1^3s$, $\partial_2^2 Q\neq 0$, and* $\det \operatorname{Hess} f = c\in\mathbb{C}^\times$. *Then $\nabla f$ is an automorphism.*

*Proof.* By Proposition 5 and $m=0$, $f=P(x')+sR(x')+as^2$ with $R=x_1^3+(\text{terms of degree}\le 2)$. If $a\neq 0$, Lemma 3 applies. Let $a=0$; then $c=-(\nabla R)^{\mathsf T}\operatorname{adj}(\operatorname{Hess}P+s\operatorname{Hess}R)\nabla R$.

(i) The coefficient of $s^2$ forces $(\partial_2 R,\partial_3 R)$ to be proportional *[C, S6b]*, so $R=\tilde R(x_1,w)$ for a linear form $w$ in $(x_2,x_3)$. Complete to linear coordinates $(x_1,w,v)$ with $\partial_v x_1=\partial_v w=0$; thus $v=(0,v_2,v_3)$.

(ii) The coefficient of $s$ equals $-P_{vv}\cdot\operatorname{Br}(\tilde R)$, $\operatorname{Br}(R)=R_1^2R_{ww}-2R_1R_wR_{1w}+R_w^2R_{11}$ *[C, S6c]*. As $\mathbb{C}[x]$ is a domain, $\operatorname{Br}=0$ or $P_{vv}=0$.

(iii) If $\operatorname{Br}=0$, then $\tilde R=g(x_1)$ *[C, S6d]* and $c=-g'(x_1)^2\det\operatorname{Hess}_{(w,v)}P$ *[C, S6e]*, impossible since $\deg g'=2$.

(iv) If $P_{vv}=0$: the degree-5 part of $P_{vv}$ is $\partial_v^2(Q+x_1^4x_3)=v_2^2\,\partial_2^2Q$ *[C, S8e]*, so $v_2=0$, $v=e_3$, $R=R(x_1,x_2)$ and $P=P_0(x_1,x_2)+x_3P_1(x_1,x_2)$ with $P_1=x_1^4+(\deg\le 3)$. Then $c=J(P_1,R)^2$ *[C, S6f]*, so $(P_1,R)$ is a plane Jacobian pair of degrees $(4,3)$. By (Mag) it is an automorphism, contradicting (JvdK). □

*Proof of Theorem 1.* Propositions 5 and 6 leave the cases (N), (D), (F), settled in this section. □

## 6 Certificates

The ancillary file `verify_hc4_ternary.py` (Python with `python-flint`, `sympy`, and Singular through Sage/passagemath) performs 29 checks, labelled S1a–S8e as cited above, and prints `ALL CHECKS PASSED`; it runs in a few minutes. Identities are verified with fully symbolic coefficients (or with undetermined functions where stated). Exclusions use the Rabinowitsch criterion over $\mathbb{Q}$; since a Gröbner basis equal to $\{1\}$ over $\mathbb{Q}$ is also $\{1\}$ over $\mathbb{C}$, the conclusions hold over $\mathbb{C}$. The classification of S4 (III.1) is certified twice, by minimal associated primes and by Rabinowitsch tests (S8a).

**Scope.** Theorem 1 does not treat quintic potentials whose top form depends on two linear forms or on one; these cases, and hence $\mathrm{HC}_4$ in degree five, remain open.

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