A Sixth-Order Hurwitz Interval Polynomial Family Without Degree-Five Weak Strictly Positive Real Numerator

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hexagon:2611.00127v1

Abstract

We construct an explicit sixth-order Hurwitz-stable interval polynomial family with independent closed coefficient intervals that admits a common strictly positive real (SPR) numerator of degree six, yet admits no common weak strictly positive real (WSPR) numerator of degree five. More strongly, no nonzero real polynomial of degree at most five makes the real part of the associated transfer functions nonnegative across the family at all real frequencies. This counterexample shows that, for sixth-order interval polynomial families, robust Hurwitz stability does not guarantee the existence of a strictly proper common SPR or WSPR numerator of degree five, and equal numerator degree six is necessary in general.

Keywords: strictly positive real synthesis; weak strictly positive real; interval polynomial family; Hurwitz stability; numerator degree necessity

1 Introduction

Strictly positive real (SPR) and weak strictly positive real (WSPR) transfer functions play a fundamental role in passivity, absolute stability, adaptive control, and output-error system identification [3, 10]. When the coefficients of the plant denominator vary within uncertainty intervals, the robust SPR synthesis problem seeks a single fixed real numerator polynomial that renders the transfer function strictly positive real for every admissible denominator in the family and at all real frequencies.

For a prescribed numerator and an interval family of denominators, frequency-domain positive realness reduces to checking four extreme vertex combinations [1, 6], just as Kharitonov's theorem [5] reduces robust Hurwitz stability to four vertex polynomials. Synthesis, however, requires finding a common numerator satisfying infinitely many frequency inequalities simultaneously. For Hurwitz-stable interval families of order $n \leq 4$, Anderson et al. [1], Betser and Zeheb [2], and Yu and Wang [7, 6] proved that a common SPR numerator always exists, including strictly proper solutions with numerator degree $n-1$. Related synthesis results for convex combinations (segments) of fifth-order, sixth-order, and arbitrary-order polynomials were established in [8, 9, 11].

A basic question in robust SPR synthesis is whether every robustly Hurwitz-stable nth-order interval polynomial family admits a common SPR or WSPR numerator of degree $n-1$, or whether an equal-degree numerator of degree n is strictly required. In the classical two-stage synthesis framework [10], the existence of a degree-n SPR numerator is often approached by showing that the monic degree-$(n-1)$ WSPR numerator sets of the family members have a nonempty intersection. In this paper, we show that this property fails for sixth-order interval polynomial families ($n = 6$). Specifically, we construct an explicit Hurwitz-stable sixth-order interval family that possesses a common SPR numerator of degree six, but admits no common WSPR numerator—and indeed no nonzero common weak positive-real numerator—of degree at most five.

2 Problem Formulation and the Main Counterexample

Consider an nth-order monic interval polynomial family

$$\mathcal{K} = \left\{ a(s) = s^n + \sum_{k=0}^{n-1} a_k s^k : \ell_k \le a_k \le u_k,\ 0 \le k \le n-1 \right\}, \tag{2.1}$$

where the coefficients $a_0, \dots, a_{n-1}$ vary independently in the closed intervals $[\ell_k, u_k]$. A polynomial $a(s) \in \mathbb{R}[s]$ is Hurwitz stable if all of its complex roots lie in the open left half-plane $\{s \in \mathbb{C} : \operatorname{Re} s < 0\}$, and the family $\mathcal{K}$ is robustly Hurwitz stable if every $a \in \mathcal{K}$ is Hurwitz stable.

Following [10], for a Hurwitz-stable denominator $a(s)$ of degree n and a numerator $b(s) \in \mathbb{R}[s]$, the transfer function $b(s)/a(s)$ is called:

(i) *equal-degree strictly positive real (SPR)* if $\deg b = n$ and

$$\operatorname{Re} \frac{b(i\omega)}{a(i\omega)} > 0 \qquad (\forall\, \omega \in \mathbb{R}); \tag{2.2}$$

(ii) *weak strictly positive real (WSPR)* of degree $n-1$ if $\deg b = n-1$ and (2.2) holds for all $\omega \in \mathbb{R}$ (so that $\operatorname{Re}[b(i\omega)/a(i\omega)] > 0$ at all finite frequencies while $\lim_{|\omega|\to\infty} \operatorname{Re}[b(i\omega)/a(i\omega)] = 0$);

(iii) *weakly positive real* on the imaginary axis if

$$\operatorname{Re} \frac{b(i\omega)}{a(i\omega)} \ge 0 \qquad (\forall\, \omega \in \mathbb{R}). \tag{2.3}$$

Every common WSPR numerator of degree $n-1$ (or at most $n-1$) for $\mathcal{K}$ is, in particular, a nonzero polynomial of degree at most $n-1$ satisfying (2.3) for all $a \in \mathcal{K}$.

For $n = 6$ and $t = \omega^2 \ge 0$, decompose any denominator $a(s) = s^6 + \sum_{k=0}^{5} a_k s^k \in \mathcal{K}$ and any candidate numerator $p(s) = \sum_{k=0}^{6} \beta_k s^k$ into their even and odd parts along the imaginary axis:

$$a(i\omega) = E_a(t) + i\omega O_a(t), \qquad p(i\omega) = P_p(t) + i\omega Q_p(t), \tag{2.4}$$

where

$$\begin{aligned} E_a(t) &= -t^3 + a_4 t^2 - a_2 t + a_0, & O_a(t) &= a_5 t^2 - a_3 t + a_1, \\ P_p(t) &= -\beta_6 t^3 + \beta_4 t^2 - \beta_2 t + \beta_0, & Q_p(t) &= \beta_5 t^2 - \beta_3 t + \beta_1. \end{aligned} \tag{2.5}$$

Define the frequency pairing polynomial

$$\Phi(a, p; t) \coloneqq P_p(t) E_a(t) + t\, Q_p(t) O_a(t). \tag{2.6}$$

Whenever $a(i\omega) \neq 0$, we have

$$\operatorname{Re} \frac{p(i\omega)}{a(i\omega)} = \frac{\Phi(a, p; \omega^2)}{E_a(\omega^2)^2 + \omega^2 O_a(\omega^2)^2} = \frac{\Phi(a, p; \omega^2)}{|a(i\omega)|^2}. \tag{2.7}$$

The extreme even and odd denominator polynomials on $t \ge 0$ are

$$\begin{aligned} \widehat{E}_0(t) &= -t^3 + \ell_4 t^2 - u_2 t + \ell_0, & \widehat{E}_1(t) &= -t^3 + u_4 t^2 - \ell_2 t + u_0, \\ \widehat{O}_0(t) &= \ell_5 t^2 - u_3 t + \ell_1, & \widehat{O}_1(t) &= u_5 t^2 - \ell_3 t + u_1, \end{aligned} \tag{2.8}$$

and the four Kharitonov vertex polynomials $a^{(i,j)} \in \mathcal{K}$ $(i, j \in \{0, 1\})$ are determined by $a^{(i,j)}(i\omega) = \widehat{E}_i(\omega^2) + i\omega \widehat{O}_j(\omega^2)$.

Lemma 2.1. For every $a \in \mathcal{K}$ and $t \geq 0$, $\widehat{E}_0(t) \leq E_a(t) \leq \widehat{E}_1(t)$ and $\widehat{O}_0(t) \leq O_a(t) \leq \widehat{O}_1(t)$. Consequently, for any fixed $p(s) \in \mathbb{R}[s]$ and $t \geq 0$, $\Phi(a, p; t) > 0$ (respectively, ≥ 0) holds for all $a \in \mathcal{K}$ if and only if $\widehat{\Phi}_{ij}(p; t) := \Phi(a^{(i,j)}, p; t) > 0$ (respectively, ≥ 0) holds for all $i, j \in \{0, 1\}$.

Proof. For fixed $t \geq 0$, $E_a(t)$ and $O_a(t)$ depend affinely on disjoint subsets of the independent interval coefficients $a_k \in [\ell_k, u_k]$, so $(E_a(t), O_a(t))$ ranges over the rectangle $[\widehat{E}_0(t), \widehat{E}_1(t)] \times [\widehat{O}_0(t), \widehat{O}_1(t)]$. Since $\Phi(a, p; t)$ is affine in $(E_a(t), O_a(t))$, its extrema over $\mathcal{K}$ are attained at the four corners $(\widehat{E}_i(t), \widehat{O}_j(t))$. □

We now specify the counterexample family $\mathcal{K}$. Let the auxiliary interval endpoints $[L_k, U_k]$ $(k = 0, \ldots, 5)$ be given by Table 1, fix $\rho = 1 - 10^{-9}$, and define the actual interval bounds by

$$\ell_k = L_k + \frac{1-\rho}{2}(U_k - L_k), \qquad u_k = U_k - \frac{1-\rho}{2}(U_k - L_k) \qquad (0 \leq k \leq 5), \tag{2.9}$$

where $(1-\rho)/2 = \frac{1}{2 \cdot 10^9}$. Each interval $[\ell_k, u_k]$ has positive width $\rho(U_k - L_k) > 0$.

Table 1: Auxiliary endpoints L_k, U_k and actual interval endpoints ℓ_k, u_k $(\rho = 1 - 10^{-9})$.

k	0	1	2	3	4	5
L_k	2000	1000	4083	920	10201/10	1
U_k	3060	2400	20102	1001	1024	40
ℓ_k	$\frac{200000000053}{100000000}$	$\frac{10000000007}{10000000}$	$\frac{8166000016019}{2000000000}$	$\frac{1840000000081}{2000000000}$	$\frac{20402000000039}{20000000000}$	$\frac{2000000039}{2000000000}$
u_k	$\frac{305999999947}{100000000}$	$\frac{23999999993}{10000000}$	$\frac{40203999983981}{2000000000}$	$\frac{2001999999919}{2000000000}$	$\frac{20479999999961}{20000000000}$	$\frac{79999999961}{2000000000}$

Theorem 2.2. Let $\mathcal{K}$ be the sixth-order interval polynomial family defined by (2.9) and Table 1. Then:

(i) Every polynomial $a \in \mathcal{K}$ is Hurwitz stable.

(ii) The sixth-degree polynomial

$$\begin{aligned} b(s) = \frac{1}{25}s^6 + s^5 + \frac{41\,536\,780\,744\,817\,023}{1\,020\,896\,695\,363\,150}s^4 + \frac{3\,548\,520\,832\,221\,173\,603}{5\,104\,483\,476\,815\,750}s^3 \\ + \frac{382\,280\,686\,567\,850\,379}{1\,020\,896\,695\,363\,150}s^2 + \frac{6\,832\,048\,626\,988\,833\,273}{5\,104\,483\,476\,815\,750}s + 18 \end{aligned} \tag{2.10}$$

is a common equal-degree SPR numerator for $\mathcal{K}$, satisfying $\mathrm{Re}[b(i\omega)/a(i\omega)] > 0$ for all $a \in \mathcal{K}$ and $\omega \in \mathbb{R}$.

(iii) There exists no nonzero polynomial $v \in \mathbb{R}[s]$ with $\deg v \leq 5$ such that

$$\mathrm{Re}\,\frac{v(i\omega)}{a(i\omega)} \geq 0 \qquad (\forall\, a \in \mathcal{K},\ \forall\, \omega \in \mathbb{R}). \tag{2.11}$$

In particular, $\mathcal{K}$ admits no common WSPR numerator of degree five (or at most five).

3 Hurwitz Stability and the Six-Degree Common SPR Numerator

3.1 Robust Hurwitz Stability of $\mathcal{K}$

The even and odd endpoint polynomials associated with the auxiliary bounds $[L_k, U_k]$ factor over $\mathbb{Q}$ as

$$\begin{aligned} E_0(t) &= -t^3 + L_4 t^2 - U_2 t + L_0 = -\left(t - \frac{1}{10}\right)(t-20)(t-1000), \\ E_1(t) &= -t^3 + U_4 t^2 - L_2 t + U_0 = -(t-1)(t-3)(t-1020), \\ O_0(t) &= L_5 t^2 - U_3 t + L_1 = (t-1)(t-1000), \\ O_1(t) &= U_5 t^2 - L_3 t + U_1 = 40(t-3)(t-20), \end{aligned} \tag{3.1}$$

with $E_1(t) - E_0(t) = 1060 + 16019t + \frac{39}{10}t^2 > 0$ and $O_1(t) - O_0(t) = 1400 + 81t + 39t^2 > 0$ on $t \geq 0$. Writing $E_c = (E_0 + E_1)/2$ and $O_c = (O_0 + O_1)/2$, the actual endpoints in (2.8) take the form

$$\widehat{E}_i(t) = \rho E_i(t) + (1-\rho)E_c(t), \qquad \widehat{O}_j(t) = \rho O_j(t) + (1-\rho)O_c(t) \qquad (i,j \in \{0,1\}). \tag{3.2}$$

Evaluating (3.2) at the six nodes $t \in \{1/10, 1, 3, 20, 1000, 1020\}$ and using Lemma 2.1 gives the strict sign pattern of $E_a(t)$ and $O_a(t)$ for every $a \in \mathcal{K}$:

t	$1/10$	1	3	20	1000	1020
$\operatorname{sgn} E_a(t)$	$+$	$-$	$-$	$+$	$+$	$-$
$\operatorname{sgn} O_a(t)$	$+$	$+$	$-$	$-$	$+$	$+$

By the intermediate value theorem, for each $a \in \mathcal{K}$, the cubic $E_a(t)$ has three simple roots $e_1 \in (1/10, 1)$, $e_2 \in (3, 20)$, $e_3 \in (1000, 1020)$ and the quadratic $O_a(t)$ (with $a_5 \geq \ell_5 > 0$) has two simple roots $o_1 \in (1, 3)$, $o_2 \in (20, 1000)$, satisfying

$$0 < e_1 < o_1 < e_2 < o_2 < e_3. \tag{3.3}$$

By the Hermite–Biehler interlacing criterion [4], every $a \in \mathcal{K}$ is Hurwitz stable. Equivalently, expanding $a(s)/E_a(-s^2)$ in partial fractions gives

$$\frac{a(s)}{E_a(-s^2)} = 1 + \sum_{k=1}^{3} \frac{\gamma_k s}{s^2 + e_k}, \qquad \gamma_k = \frac{O_a(e_k)}{\prod_{j \neq k}(e_k - e_j)} > 0,$$

whose real part exceeds 1 throughout $\operatorname{Re} s > 0$, while (3.3) and $a_0 > 0$ exclude imaginary-axis zeros. This proves Theorem 2.2(i).

3.2 Verification of the Degree-Six Common SPR Numerator

Write the even-odd decomposition of $b(s)$ in (2.10) as $b(i\sqrt{t}) = P(t) + i\sqrt{t}\,Q(t)$, and let $\Phi_{ij}(t) = P(t)E_i(t) + t\,Q(t)O_j(t)$ $(i,j \in \{0,1\})$ denote its pairings with the auxiliary endpoints (3.1). Direct expansion yields the four double-contact factorizations

$$\begin{aligned} \Phi_{00}(t) &= (t-1000)^2 F_0(t), \quad \Phi_{01}(t) = (t-20)^2 F_1(t), \\ \Phi_{10}(t) &= (t-1)^2 F_2(t), \qquad \Phi_{11}(t) = (t-3)^2 F_3(t), \end{aligned} \tag{3.4}$$

where $F_m(t) = \sum_{k=0}^{4} c_{mk} t^k$ $(m = 0, 1, 2, 3)$ are quartic polynomials with leading coefficient $c_{m4} = 1/25$; their exact rational coefficients are listed in Table 3 of the Appendix.

For $m \in \{0, 1, 3\}$, the coefficients satisfy $c_{m0} > 0$, $c_{m1} > 0$, $c_{m2} > 0$, and $4c_{m4}c_{m2} - c_{m3}^2 > 0$:

m	c_{m0}	$c_{m1}>$	$c_{m2}>$	$4c_{m4}c_{m2}-c_{m3}^2>$
0	9/250	1/5	5	3/5
1	90	5200	12000	380
3	6120	225000	13000	400

Hence $c_{m4}t^2+c_{m3}t+c_{m2}>0$ for all $t\in\mathbb{R}$, and $F_m(t)=t^2(c_{m4}t^2+c_{m3}t+c_{m2})+c_{m1}t+c_{m0}>0$ for all $t\geq 0$. For $F_2(t)$, setting $R=1010$, $T=2440/3$, $L=c_{23}+2Rc_{24}>23/100$, $Z_2=c_{22}-R^2c_{24}+2RL>11$, and $Z_1=c_{21}-R^2L$, we have the identity

$$F_2(t)=(t-R)^2(c_{24}t^2+Lt)+Z_2t^2+Z_1t+55080,$$

together with $TZ_2+Z_1>97>0$ and $L(R-T)^2+Z_1>174>0$. Thus $Z_2t+Z_1>0$ for $t\geq T$ and $L(t-R)^2+Z_1>0$ for $0\leq t<T$, implying $F_2(t)>0$ for all $t\geq 0$.

Consequently, $\Phi_{ij}(t)\geq 0$ on $t\geq 0$, and because the four contact points $1000, 20, 1, 3$ are distinct, their sum $\Sigma(t)=\sum_{k,l=0}^{1}\Phi_{kl}(t)$ is strictly positive for all $t\geq 0$. By (3.2), the four vertex pairings of the actual family $\mathcal{K}$ satisfy

$$\widehat{\Phi}_{ij}(b;t)=\rho\,\Phi_{ij}(t)+\frac{1-\rho}{4}\,\Sigma(t)\geq\frac{1-\rho}{4}\,\Sigma(t)>0\qquad(\forall\, t\geq 0,\ i,j\in\{0,1\}).$$

Lemma 2.1 and (2.7) then yield $\mathrm{Re}[b(i\omega)/a(i\omega)]>0$ for all $a\in\mathcal{K}$ and $\omega\in\mathbb{R}$, proving Theorem 2.2(ii).

4 Nonexistence of Degree-Five WSPR Numerators

We now establish the main negative result, Theorem 2.2(iii): no nonzero polynomial $v(s)=\sum_{k=0}^{5}\beta_k s^k$ of degree at most five can satisfy $\mathrm{Re}[v(i\omega)/a(i\omega)]\geq 0$ for all $a\in\mathcal{K}$ and $\omega\in\mathbb{R}$.

4.1 The Four-Contact Tangency Mechanism on the Auxiliary Boundary

To see why degree-five numerators fail while the degree-six numerator (2.10) succeeds, consider first any polynomial $v(s)=\sum_{k=0}^{5}\beta_k s^k$ ($\beta_6=0$) with $P_v(t)=\beta_4t^2-\beta_2t+\beta_0$ and $Q_v(t)=\beta_5t^2-\beta_3t+\beta_1$ on the uncontracted auxiliary endpoints (3.1).

1. *High-frequency t^5 ceiling:* At the corner (E_0,O_0) where $L_5=1$, the pairing polynomial has degree at most five with leading term

$$\Phi_{00}(v;t)=P_v(t)E_0(t)+t\,Q_v(t)O_0(t)=(\beta_5-\beta_4)t^5+O(t^4)\qquad(t\to\infty).$$

   Nonnegativity for large t requires $\beta_5-\beta_4\geq 0$, i.e., $\beta_4\leq\beta_5$. (By contrast, when $\beta_6>0$, $P(t)E_0(t)=\beta_6t^6+O(t^5)$ dominates at high frequencies and removes this upper bound on β_4.)

2. *Four shared-root tangency equations:* Notice from (3.1) that each auxiliary corner (E_i,O_j) has a positive common root t_{ij}^* where $E_i(t_{ij}^*)=O_j(t_{ij}^*)=0$:

$$t_{00}^*=1000,\qquad t_{01}^*=20,\qquad t_{10}^*=1,\qquad t_{11}^*=3.$$

   Hence $\Phi_{ij}(v;t_{ij}^*)=0$ identically for every v. For $\Phi_{ij}(v;t)$ to be nonnegative on $[0,\infty)$, its derivative must vanish at each interior root t_{ij}^*:

$$\Phi_{ij}'(v;t_{ij}^*)=P_v(t_{ij}^*)E_i'(t_{ij}^*)+t_{ij}^*Q_v(t_{ij}^*)O_j'(t_{ij}^*)=0\qquad(i,j\in\{0,1\}).\tag{4.1}$$

Solving the nonsingular 4×4 linear system (4.1) for $(\beta_1, \beta_2, \beta_3, \beta_4)$ in terms of (β_0, β_5) gives

$$\beta_4 = \frac{1\,600\,159}{1\,681\,866\,100}\beta_0 + \left(1 + \frac{51\,219}{1\,681\,866\,100}\right)\beta_5,$$

so that

$$\beta_5 - \beta_4 = -\frac{1\,600\,159}{1\,681\,866\,100}\beta_0 - \frac{51\,219}{1\,681\,866\,100}\beta_5.$$

Since $\beta_0 \geq 0$ and $\beta_5 \geq 0$, this contradicts $\beta_5 - \beta_4 \geq 0$ unless $\beta_0 = \beta_5 = 0$ (and hence $v \equiv 0$). Because the strict gap $\frac{51\,219}{1\,681\,866\,100} \approx 3.045 \times 10^{-5}$ is positive, this infeasibility persists on the contracted box $\mathcal{K}$ ($\rho = 1 - 10^{-9}$) when the four derivative equations are replaced by point evaluations near $1000, 20, 1, 3$. We now give the complete proof on $\mathcal{K}$.

4.2 Necessary Coefficient Inequalities on $\mathcal{K}$

Let $v(s) = \sum_{k=0}^{5} \beta_k s^k \in \mathbb{R}[s]$ be any polynomial of degree at most five satisfying (2.11).

Lemma 4.1. Suppose $v(s) = \sum_{k=0}^{5} \beta_k s^k$ satisfies $\mathrm{Re}[v(i\omega)/a(i\omega)] \geq 0$ for all $a \in \mathcal{K}$ and $\omega \in \mathbb{R}$, and set

$$\alpha := \ell_5 = 1 + \frac{39}{2 \cdot 10^9} = \frac{2\,000\,000\,039}{2\,000\,000\,000}. \tag{4.2}$$

Then:

(i) $\Phi(a, v; t) = P_v(t)E_a(t) + t\,Q_v(t)O_a(t) \geq 0$ for all $a \in \mathcal{K}$ and $t \geq 0$;

(ii) $\beta_0 \geq 0$;

(iii) $h(v) := \alpha\beta_5 - \beta_4 \geq 0$.

Proof. Assertion (i) follows from (2.7) and $|a(i\sqrt{t})|^2 > 0$. Evaluating $\Phi(a^{(0,0)}, v; 0) = \ell_0\beta_0 \geq 0$ with $\ell_0 > 0$ gives (ii). For (iii), expanding $\Phi(a^{(0,0)}, v; t)$ at the corner $(\widehat{E}_0, \widehat{O}_0)$ where $[t^2]\widehat{O}_0 = \ell_5 = \alpha$ gives

$$\begin{aligned}\Phi(a^{(0,0)}, v; t) &= (\beta_4 t^2 - \beta_2 t + \beta_0)(-t^3 + \ell_4 t^2 - u_2 t + \ell_0) + t(\beta_5 t^2 - \beta_3 t + \beta_1)(\alpha t^2 - u_3 t + \ell_1) \\ &= (\alpha\beta_5 - \beta_4)t^5 + O(t^4).\end{aligned}$$

Since $\Phi(a^{(0,0)}, v; t) \geq 0$ for all $t \geq 0$, dividing by t^5 and letting $t \to +\infty$ yields $h(v) = \alpha\beta_5 - \beta_4 \geq 0$. □

Lemma 4.2. If $v(s) = \sum_{k=0}^{5} \beta_k s^k \neq 0$ satisfies $\mathrm{Re}[v(i\omega)/a(i\omega)] \geq 0$ for some Hurwitz-stable sixth-degree monic polynomial $a \in \mathcal{K}$ and all $\omega \in \mathbb{R}$, then $\beta_5 \geq 0$.

Proof. Since $a(s)$ is Hurwitz stable of degree six and $\deg v \leq 5$, the rational function $f(z) = v(z)/a(z)$ is analytic on the open right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$, continuous on $\overline{\mathbb{C}_+}$, and satisfies $\lim_{|z|\to\infty,\, z\in\overline{\mathbb{C}_+}} f(z) = 0$. Its real part $u(z) = \mathrm{Re}\, f(z)$ is harmonic on $\mathbb{C}_+$ with boundary values $u(i\omega) \geq 0$ and limit zero at infinity, so the harmonic minimum principle gives $u(z) \geq 0$ on $\mathbb{C}_+$. If $u(z_0) = 0$ at an interior point $z_0 \in \mathbb{C}_+$, the strong minimum principle implies $u \equiv 0$, so $f(z)$ is a purely imaginary constant, which must be zero by the limit at infinity, contradicting $v \neq 0$. Thus $u(z) > 0$ on $\mathbb{C}_+$. Restricting to the positive real axis $z = x > 0$, where $a(x) > 0$, gives $v(x) = a(x)u(x) > 0$ for all $x > 0$. Letting $x \to +\infty$ implies $\beta_5 \geq 0$. □

4.3 Linear Evaluation System and Dual Separation Identity

Choose four rational squared frequencies $t_0, t_1, t_2, t_3 > 0$ near $1000, 20, 1, 3$ and pair them with the four Kharitonov vertices $a^{(i_j,k_j)} \in \mathcal{K}$:

$$\begin{aligned} (i_0,k_0,t_0) &= (0,0,999), \qquad (i_1,k_1,t_1) = \left(0,1,\frac{1001}{50}\right), \\ (i_2,k_2,t_2) &= \left(1,0,\frac{1001}{1000}\right), \quad (i_3,k_3,t_3) = \left(1,1,\frac{2997}{1000}\right). \end{aligned} \tag{4.3}$$

For $j \in \{0,1,2,3\}$, define the linear functional

$$f_j(v) \coloneqq \widehat{\Phi}_{i_j k_j}(v;t_j) = P_v(t_j)\widehat{E}_{i_j}(t_j) + t_j\, Q_v(t_j)\widehat{O}_{k_j}(t_j) = \sum_{m=0}^{5} r_{jm}\beta_m, \tag{4.4}$$

where the row vector $r_j = (r_{j0}, r_{j1}, r_{j2}, r_{j3}, r_{j4}, r_{j5}) \in \mathbb{Q}^6$ is given by

$$r_j = \Big(\widehat{E}_{i_j}(t_j),\ t_j\widehat{O}_{k_j}(t_j),\ -t_j\widehat{E}_{i_j}(t_j),\ -t_j^2\widehat{O}_{k_j}(t_j),\ t_j^2\widehat{E}_{i_j}(t_j),\ t_j^3\widehat{O}_{k_j}(t_j)\Big). \tag{4.5}$$

The exact values of $\widehat{E}_{i_j}(t_j)$ and $\widehat{O}_{k_j}(t_j)$ at these four nodes are listed in Table 2.

Table 2: Exact values of the contracted even and odd endpoints $\widehat{E}_{i_j}(t_j)$ and $\widehat{O}_{k_j}(t_j)$ at the four evaluation nodes (4.3).

j	(i_j,k_j)	t_j	$\widehat{E}_{i_j}(t_j)$	$\widehat{O}_{k_j}(t_j)$
0	$(0,0)$	999	$\dfrac{1959819999804213839}{20000000000}$	$-\dfrac{1995999960999239}{2000000000}$
1	$(0,1)$	$\dfrac{1001}{50}$	$-\dfrac{15283427271601383101}{50000000000000}$	$\dfrac{6814807999993185231079}{5000000000000}$
2	$(1,0)$	$\dfrac{1001}{1000}$	$\dfrac{407195971760000003999}{200000000000000000}$	$-\dfrac{199799799238920999961}{2000000000000000}$
3	$(1,1)$	$\dfrac{2997}{1000}$	$\dfrac{1220352620415599996001}{200000000000000000}$	$\dfrac{4150275599995849704951}{2000000000000000}$

Let $H = (r_{jm})_{0\le j\le 3,\,1\le m\le 4} \in \mathbb{Q}^{4\times 4}$ be the 4×4 submatrix corresponding to the intermediate coefficients $(\beta_1,\beta_2,\beta_3,\beta_4)$, and set $q = (0,0,0,1)^{\mathsf{T}}$. Direct rational evaluation shows that $\det H \neq 0$. Define the unique solution $w = (w_0,w_1,w_2,w_3)^{\mathsf{T}} \in \mathbb{Q}^4$ of $H^{\mathsf{T}}w = q$ by Cramer's rule, together with the two endpoint constants $A, B \in \mathbb{Q}$:

$$w_j = \frac{\det H_j}{\det H} \quad (0 \le j \le 3), \qquad A = -\sum_{j=0}^{3} w_j r_{j0}, \qquad B = -\alpha - \sum_{j=0}^{3} w_j r_{j5}, \tag{4.6}$$

where H_j is obtained by replacing the jth column of H^{T} with q. Their exact reduced fractions are given in (A.1)–(A.6) of the Appendix.

Proposition 4.3. The rational constants w_0, w_1, w_2, w_3, A, B in (4.6) are strictly positive and satisfy

$$\begin{aligned} 10^{-12} < w_0 < 1.1\times 10^{-12}, &\qquad 7\times 10^{-8} < w_1 < 8\times 10^{-8}, \\ 3\times 10^{-4} < w_2 < 4\times 10^{-4}, &\qquad 5\times 10^{-5} < w_3 < 6\times 10^{-5}, \\ \frac{9}{10\,000} < A < \frac{1}{1\,000}, &\qquad \frac{3}{100\,000} < B < \frac{4}{100\,000}. \end{aligned} \tag{4.7}$$

Moreover, for every polynomial $v(s) = \sum_{k=0}^{5}\beta_k s^k \in \mathbb{R}[s]$ of degree at most five,

$$h(v) + \sum_{j=0}^{3} w_j f_j(v) = -A\beta_0 - B\beta_5. \tag{4.8}$$

Proof. The two-sided bounds (4.7) follow directly from the exact rational expressions in (A.1)–(A.6). Expanding the left-hand side of (4.8) using $h(v) = \alpha\beta_5 - \beta_4$ and $f_j(v) = \sum_{m=0}^{5} r_{jm}\beta_m$ gives

$$h(v) + \sum_{j=0}^{3} w_j f_j(v) = \left(\sum_{j=0}^{3} w_j r_{j0}\right)\beta_0 + \sum_{m=1}^{3}\left(\sum_{j=0}^{3} w_j r_{jm}\right)\beta_m + \left(\sum_{j=0}^{3} w_j r_{j4} - 1\right)\beta_4 + \left(\alpha + \sum_{j=0}^{3} w_j r_{j5}\right)\beta_5.$$

Since $H^{\mathsf{T}} w = (0,0,0,1)^{\mathsf{T}}$, the coefficients of $\beta_1, \beta_2, \beta_3, \beta_4$ vanish identically, while the coefficients of β_0 and β_5 equal $-A$ and $-B$ by (4.6). □

Proof of Theorem 2.2(iii). Suppose for contradiction that a nonzero polynomial $v(s) = \sum_{k=0}^{5} \beta_k s^k$ satisfies (2.11). By Lemmas 4.1 and 4.2, we have $\beta_0 \geq 0$, $\beta_5 \geq 0$, $h(v) \geq 0$, and $f_j(v) = \Phi(a^{(i_j,k_j)}, v; t_j) \geq 0$ for $j = 0,1,2,3$. Since $w_j > 0$, $A > 0$, and $B > 0$ by Proposition 4.3, the identity (4.8) yields

$$0 \leq h(v) + \sum_{j=0}^{3} w_j f_j(v) = -A\beta_0 - B\beta_5 \leq 0.$$

Hence $\beta_0 = \beta_5 = 0$ and $f_0(v) = f_1(v) = f_2(v) = f_3(v) = 0$. Substituting $\beta_0 = \beta_5 = 0$ into (4.4) gives $H(\beta_1, \beta_2, \beta_3, \beta_4)^{\mathsf{T}} = 0$. Because $\det H \neq 0$, we obtain $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, contradicting $v \neq 0$. □

5 Conclusion

We have presented an explicit sixth-order Hurwitz-stable interval polynomial family with independent positive-width coefficient intervals that admits a common SPR numerator of degree six, yet admits no common WSPR numerator and no nonzero common weak positive-real numerator of degree at most five. Unlike low-order interval families ($n \leq 4$) and two-polynomial segments, a six-parameter interval box can enforce four simultaneous corner contacts whose tangency constraints conflict with the high-frequency sign condition of fifth-degree numerators. Consequently, robust SPR synthesis for sixth-order interval polynomial families requires equal-degree synthesis in the full degree-six numerator space.

Acknowledgements

GPT-6 Astra was used for calculations, proof development, and editorial assistance. AI-generated suggestions were checked against the mathematical arguments and the accompanying formal development; the formal claims are supported by Lean kernel checking. Responsibility for the final manuscript rests with the authors. The complete proof of this paper has been formalized in Lean 4, and the source code repository is available at `https://github.com/dgjszy/sextic-spr-degree-necessity`.

A Exact Rational Certificates

Table 3 records the exact rational coefficients of the four quartic polynomials $F_m(t) = \sum_{k=0}^{4} c_{mk} t^k$ $(m = 0,1,2,3)$ in (3.4). Equations (A.1)–(A.6) list the exact reduced fractions for w_0, w_1, w_2, w_3, A, B in Proposition 4.3.

Table 3: Exact rational coefficients c_{mk} of the quartic factors $F_m(t) = \sum_{k=0}^{4} c_{mk} t^k$ ($m = 0, 1, 2, 3$).

k	c_{0k} in $F_0(t)$	c_{1k} in $F_1(t)$	c_{2k} in $F_2(t)$	c_{3k} in $F_3(t)$
0	$\frac{9}{250}$	90	55080	6120
1	$\frac{2325246801607113}{10208966953631500}$	$\frac{53727500481470645241}{10208966953631500}$	$\frac{23406302273232975276}{102089669536315}$	$\frac{4604582188848296296}{20417933907263}$
2	$\frac{57094101348581227}{10208966953631500}$	$\frac{126021414955216817453}{10208966953631500}$	$\frac{102966285353162075756}{2552241738407875}$	$\frac{2700257383808100617}{204179339072630}$
3	$-\frac{1252042944999614}{2552241738407875}$	$-\frac{101810367438269889}{2552241738407875}$	$-\frac{16450028191179889}{204179339072630}$	$-\frac{42271826365478491}{1020896695363150}$
4	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$

$$w_0 = \frac{82560456707358936862845473692880259477433396107599800000000000000}{81268414009215942680302457591130239791618540922907287540125187597409518010 21}, \tag{A.1}$$

$$w_1 = \frac{20737052978367620265590257356173414317038606968000000000000000000000}{27170875387641651296384005121976731591151242880619006820184591615328800 7}, \tag{A.2}$$

$$w_2 = \frac{26733853257328206852213223482085243070454252210100000000000000000000000}{81512626162924845388915201536593019477345372864185702046055374845986402 1}, \tag{A.3}$$

$$w_3 = \frac{12287210228418814088982788512832601510832738713194300000000000000000000000}{24380524202764782804090737277339071937485562276872186262037556279222855403063}, \tag{A.4}$$

$$A = \frac{23046797004185631819643050257180682267915284584118958515075399780915999747000}{24404904726967547586894828014616411009423047839149058448299593835502078258466063}, \tag{A.5}$$

$$B = \frac{49588749713076005177068401199244872601762754911972179527755659986818312218 1}{16286238993591377700082957349968635260208865707130010398812263233686042000000000}. \tag{A.6}$$

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