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Exact Intervals and Arithmetic Structure in Fuchs' Extremal Problem

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Abstract

We determine Fuchs' weighted unit-circle extremum for monic integer polynomials for $0 < \lambda \leq 3$. The proof combines a fractional difference energy with a sharp rearrangement inequality and an explicit gap for nonnegative integer sequences that are not interval indicators. We obtain the finite-degree minimum on $(1, 2)$ at every degree and on $(2, 3)$ at every degree at least five. Every minimizing sequence on these intervals eventually consists, up to monomial factors, of Dirichlet polynomials and products of two adjacent Dirichlet polynomials, respectively. At integer parameters, Parseval's identity reduces the problem to integer coefficient norms. The lower bound is attained exactly when a multiplicity-free ideal Prouhet–Tarry–Escott pair exists. Classical PTE constructions give exact values for the integers 1 through 10 and for 12, and a two-value alternative at 11. Subset counting gives a quadratic upper bound in general. The interval formulas and explicit constructions show that 2, 3 and 4 are strict local maxima; the formula on $(2, 3)$ has a unique minimum. A Pell construction gives an upper bound on $(4, 5)$, and coefficient-norm gaps exclude Dirichlet products uniformly near certain integers. For each fixed degree bound, the extremum, divided by a common gamma factor, is locally the lower envelope of finitely many rational functions. Exact lattice enumeration identifies a nonproduct minimizer on an open interval containing $17/4$ for every degree bound from 61 to 80.

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1 Introduction and main results

For a polynomial P and a real parameter $\lambda > -1/2$, put

$$\mathcal{E}_\lambda(P) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - e^{i\theta}|^{2\lambda} |P(e^{i\theta})|^2 \, d\theta. \tag{1.1}$$

Let $\mathcal{P}$ be the set of all monic polynomials in $\mathbb{Z}[z]$, including the constant polynomial 1. For integers $N \geq 0$, define

$$F(\lambda) = 2\pi \inf_{P \in \mathcal{P}} \mathcal{E}_\lambda(P), \qquad F_N(\lambda) = 2\pi \min_{\substack{P \in \mathcal{P} \\ \deg P \leq N}} \mathcal{E}_\lambda(P), \qquad \lambda > 0. \tag{1.2}$$

The determination of F is Problem 4.25 in Hayman and Lingham's *Research Problems in Function Theory*, where it is attributed to W. H. J. Fuchs [8]. The monic integer condition prevents arbitrary rescaling, while the degree remains unrestricted. The problem therefore asks both how small the weighted norm can be and whether a polynomial attains its infimum.

We write

$$\mathcal{C}(\alpha) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)^2}, \qquad D_L(z) = 1 + z + \cdots + z^L \quad (L \in \mathbb{Z},\ L \geq 0). \tag{1.3}$$

The following formulas determine the extremal function through $\lambda = 3$.

Theorem 1.1 (The first three intervals)**.** *One has*

$$F(\lambda) = 2\pi\mathcal{C}(\lambda), \qquad 0 < \lambda \le 1, \tag{1.4}$$

$$F(1+\alpha) = 4\pi\mathcal{C}(\alpha), \qquad 0 < \alpha \le 1, \tag{1.5}$$

$$F(2+\alpha) = 4\pi\frac{\alpha+2}{\alpha+1}\mathcal{C}(\alpha), \qquad 0 < \alpha \le 1. \tag{1.6}$$

The infimum is attained for $0 < \lambda \le 1$ *and at* $\lambda = 2, 3$*. It is not attained on* $(1,2) \cup (2,3)$*. For* $0 < \lambda < 1$*, the minimizing polynomials are exactly the monomials* z^s*. At* $\lambda = 1$*, they are exactly* $z^s D_L$*, with* $s, L \ge 0$*.*

The proof on $(2,3)$ uses a quantitative rearrangement inequality. Among nonnegative integer sequences of fixed mass, a block of consecutive ones minimizes the fractional energy, and every other sequence has an explicit positive energy gap. Applied to the positive and negative parts of $(1-z)^2 P$, this gap forces both masses to equal two in every sufficiently advanced member of a minimizing sequence. The second difference then factors, up to a monomial, into two binomials. The same estimates also determine the finite-degree minimum.

Theorem 1.2 (Finite degree and minimizing sequences)**.** *Let* $0 < \alpha < 1$ *and let* $c_j(\alpha)$ *be the Fourier coefficients of* $|1 - e^{i\theta}|^{2\alpha}$*, as in equation (2.1). For every* $N \ge 0$*,*

$$F_N(1+\alpha) = 4\pi\big(\mathcal{C}(\alpha) - c_{N+1}(\alpha)\big), \tag{1.7}$$

with unique minimizer D_N*. For* $N \ge 5$*, put* $M = \lfloor (N+1)/2 \rfloor$*. Then*

$$\frac{F_N(2+\alpha)}{2\pi} = 4\mathcal{C}(\alpha) + 2c_1(\alpha) + 2c_{2M+1}(\alpha) - 4c_M(\alpha) - 4c_{M+1}(\alpha). \tag{1.8}$$

Its minimizers are precisely

$$z^s D_{M-1} D_M, \qquad 0 \le s \le N - (2M-1).$$

Furthermore, every minimizing sequence for $F(1+\alpha)$ *eventually has the form* $z^{s_j} D_{L_j}$ *with* $L_j \to \infty$*. Every minimizing sequence for* $F(2+\alpha)$ *eventually has the form* $z^{s_j} D_{L_j - 1} D_{L_j}$ *with* $L_j \to \infty$*.*

For each fixed degree bound, the minimizers are independent of the parameter within $(0,1)$ and within $(1,2)$. The same holds on $(2,3)$ for every $N \ge 5$. The sequence statements also classify all asymptotically optimal polynomials on $(1,2)$ and $(2,3)$.

At an integer m, set

$$\mathcal{M}_m = \min\left\{ \|Q\|_2^2 : Q \in \mathbb{Z}[z],\ (1-z)^m \mid Q,\ \mathrm{lc}(Q) = (-1)^m \right\}, \tag{1.9}$$

where $\|Q\|_2^2$ is the sum of the squares of the coefficients. The moment interpretation of divisibility by $(1-z)^m$ connects the problem with equal sums of like powers.

Theorem 1.3 (Integer parameters)**.** *For* $m \ge 1$*, one has* $F(m) = 2\pi\mathcal{M}_m$ *and*

$$4\pi m \le F(m) \le 2\pi \min\{2^m, m(m-1)+2\}. \tag{1.10}$$

Equality in the lower bound holds exactly when there are disjoint m*-element sets* $A, B \subset \mathbb{Z}$ *with*

$$\sum_{a \in A} a^r = \sum_{b \in B} b^r \qquad (1 \le r < m).$$

In particular,

$$F(m) = 4\pi m \quad (1 \le m \le 10 \text{ and } m = 12), \qquad F(11) \in \{44\pi, 48\pi\}.$$

The sets in the equality criterion are multiplicity-free ideal PTE pairs. The classical PTE problem allows multisets; the distinction matters because equality in a coefficient-square lower bound forces all nonzero coefficients to have absolute value one.

1.1 Relation to earlier work

The equivalence between equal power sums and high-order vanishing of a signed incidence polynomial is classical; see Borwein and Ingalls [1], the survey [11], and the formulation in [6]. Borwein and Mossinghoff [3] study the least degree of a height-one polynomial with prescribed vanishing at one. Maltby [9, 10] and Cipu [5] study coefficient norms of pure products

$$\prod_{i=1}^{m}(1-z^{L_i}), \qquad L_i \in \mathbb{N}.$$

In particular, Euclidean coefficient norms already occur in that literature. The admissible class in equation (1.9) contains all integer polynomials with the required zero and leading coefficient. The equality criterion identifies exactly which classical PTE constructions attain the Fuchs bound.

Our principal analytic results are the interval formulas, their finite-degree versions, and the classification of minimizing sequences. Their proofs combine coefficient integrality with the sign and complete monotonicity of the fractional Fourier kernel. We also show that known pure-product norm obstructions persist on open parameter intervals around certain integers. These intervals separate the full extremal problem from the Dirichlet-product restriction even when the degree is unrestricted.

The integer pairs used below come from the PTE literature; their moment identities are verified by integer arithmetic. The quadratic bound in equation (1.10) is a direct application of the classical subset-counting argument, also recorded in [7]. A Pell family of symmetric four-element pairs gives a further bound on $(4, 5)$, and complete lattice enumeration exhibits a nonproduct minimizer near $17/4$ at degree bounds 61 through 80. The finite-degree algorithm uses exact rational arithmetic after removing $\mathcal{C}(\lambda)$ from the objective. Its mathematical basis and the reported search ranges are given in section 6.

2 Discrete energies and exact interval formulas

2.1 The fractional Fourier kernel

All coefficient sequences are extended by zero to $\mathbb{Z}$. For real finite sequences a, b, the notation $\langle a, b\rangle_\alpha$ denotes the bilinear form associated with equation (1.1). A sequence and its coefficient polynomial will be used interchangeably.

Lemma 2.1 (Fourier coefficients)**.** *For $\alpha > -1/2$ and $j \in \mathbb{Z}$,*

$$c_j(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |1-e^{i\theta}|^{2\alpha} e^{-ij\theta}\, d\theta = (-1)^j \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+j+1)\Gamma(\alpha-j+1)}. \tag{2.1}$$

Reciprocal gamma factors at poles are interpreted as zero. For $0 < \alpha < 1$, put $d_j(\alpha) = -c_j(\alpha)$ for $j \geq 1$. Then

$$d_j(\alpha) = \frac{\sin(\pi\alpha)}{\pi}\int_0^1 t^{j-\alpha-1}(1-t)^{2\alpha}\, dt, \tag{2.2}$$

$$\frac{d_{j+1}(\alpha)}{d_j(\alpha)} = \frac{j-\alpha}{j+\alpha+1}, \tag{2.3}$$

$$\mathcal{C}(\alpha) = 2\sum_{j\geq 1} d_j(\alpha). \tag{2.4}$$

The sequences d_j and $\delta_j = d_j - d_{j+1}$ are strictly positive, strictly decreasing and strictly convex.

Proof. The substitution $u = \theta/2$ and the beta integral give $c_0 = \mathcal{C}(\alpha)$. Put $w(\theta) = (2-2\cos\theta)^\alpha$. Away from zero,

$$(w \sin\theta)' = \alpha w + (\alpha+1) w \cos\theta.$$

Since $w(\theta)\sin\theta \to 0$ as $\theta \to 0$ when $\alpha > -1/2$, integration by parts against $e^{-ij\theta}$ gives

$$(\alpha+1-j)c_{j-1} + 2\alpha c_j + (\alpha+1+j)c_{j+1} = 0.$$

At $j = 0$ this yields $c_1 = -\alpha c_0/(\alpha+1)$. Induction then gives the multiplicative recurrence $(j+\alpha+1)c_{j+1} = (j-\alpha)c_j$ for $j \geq 0$. Together with $c_{-j} = c_j$, it proves equation (2.1), including the vanishing coefficients at integer parameters. For $0 < \alpha < 1$, the reflection formula gives

$$d_j = \frac{\Gamma(2\alpha+1)\sin(\pi\alpha)}{\pi} \frac{\Gamma(j-\alpha)}{\Gamma(j+\alpha+1)},$$

and the beta integral gives equation (2.2). Successive differences insert a factor $1-t$ in that integral, proving all the strict inequalities. The gamma ratio is $O(j^{-2\alpha-1})$, so the Fourier series converges absolutely. Its value at $\theta = 0$ is zero, which proves equation (2.4). □

Lemma 2.2 (Difference energy)**.** *For $0 < \alpha < 1$ and every real finite sequence a,*

$$\mathcal{E}_\alpha(a) = \sum_{h\geq 1} d_h(\alpha) \sum_{j\in\mathbb{Z}} (a_{j+h} - a_j)^2. \tag{2.5}$$

At $\alpha = 1$, the identity becomes $\mathcal{E}_1(a) = \sum_j (a_{j+1} - a_j)^2$.

Proof. Expanding the squares in equation (2.5) and using equation (2.4) gives the Fourier quadratic form $\sum_{i,j} a_i a_j c_{i-j}(\alpha)$. All exchanges of sums are justified by finite support and $\sum d_h < \infty$. At $\alpha = 1$ the only nonzero Fourier coefficients are $c_0 = 2$ and $c_{\pm 1} = -1$. □

Proposition 2.3 (The base interval)**.** *For every nonzero integer polynomial A and $0 < \alpha \leq 1$, $\mathcal{E}_\alpha(A) \geq \mathcal{C}(\alpha)$. For $0 < \alpha < 1$, equality holds exactly for $A = \pm z^s$. At $\alpha = 1$, equality holds exactly for $A = \pm z^s D_L$. Consequently equation (1.4) and its equality assertions hold.*

Proof. For each $h \geq 1$, a nonzero residue class of the integer sequence modulo h starts at zero, reaches a nonzero integer and returns to zero. Thus $\sum_j (a_{j+h} - a_j)^2 \geq 2$. Summing with the weights d_h gives the result for $0 < \alpha < 1$. Equality forces each such squared variation to equal two. Taking h larger than the support diameter gives $2\sum a_j^2 = 2$, hence a single coefficient ± 1. At $\alpha = 1$, equality in the unit-step variation means that exactly two jumps have magnitude one and every other jump is zero. The sequence is therefore a single block of ones or minus ones. The leading coefficient condition selects the positive sign. □

2.2 Rearrangement and a strict gap for integer masses

For an integer $s \geq 1$, define

$$S_s = \mathcal{E}_\alpha(D_{s-1}) = s\mathcal{C}(\alpha) - 2\sum_{h=1}^{s-1} (s-h) d_h, \qquad S_0 = 0. \tag{2.6}$$

Throughout this subsection $0 < \alpha < 1$ is fixed. The increments are

$$S_{s+1} - S_s = 2\sum_{h>s} d_h > 0. \tag{2.7}$$

They strictly decrease, so S_s is strictly increasing and strictly concave as a sequence. In particular, $S_{a+b} \leq S_a + S_b$.

Lemma 2.4 (Integer rearrangement with a gap)**.** *Let p be a nonnegative integer sequence of finite support and mass $\sum_j p_j = s \geq 1$. Then*

$$\mathcal{E}_\alpha(p) \geq S_s, \tag{2.8}$$

with equality exactly when p is the indicator of an interval of s consecutive integers. If $s \geq 2$ and p is not such an indicator, then

$$\mathcal{E}_\alpha(p) \geq S_s + 2(d_1 - d_s). \tag{2.9}$$

The constant in equation (2.9) is attained by the indicator of $\{0, 1, \ldots, s-2, s\}$.

Proof. First let $p = \mathbf{1}_A$, where $A = \{x_1 < \cdots < x_s\}$. Since $x_j - x_i \geq j - i$ and d_h strictly decreases,

$$\mathcal{E}_\alpha(\mathbf{1}_A) = s\mathcal{C}(\alpha) - 2\sum_{i<j} d_{x_j - x_i} \geq s\mathcal{C}(\alpha) - 2\sum_{i<j} d_{j-i} = S_s.$$

Equality requires consecutive points. If $x_{q+1} - x_q \geq 2$, compress all other gaps to one and this gap to two. Such compression increases each pair interaction $d_{x_j - x_i}$ and hence decreases the energy. Relative to a consecutive interval, the remaining gap increases the energy by

$$2\sum_{i \leq q < j} (d_{j-i} - d_{j-i+1}).$$

For every $r = 1, \ldots, s-1$, at least one pair with $j - i = r$ crosses the cut q. Thus this sum is at least $2\sum_{r=1}^{s-1} \delta_r = 2(d_1 - d_s)$. A gap next to an endpoint attains this bound.

For general p, use its nested integer level sets $A_r = \{j : p_j \geq r\}$, so $p = \sum_r \mathbf{1}_{A_r}$. If $B \subseteq A$, the difference representation gives

$$\langle \mathbf{1}_A, \mathbf{1}_B \rangle_\alpha = \sum_{h \geq 1} d_h \sum_j (\mathbf{1}_A(j+h) - \mathbf{1}_A(j))(\mathbf{1}_B(j+h) - \mathbf{1}_B(j)) \geq 0.$$

Every summand is nonnegative because the sets are nested. Therefore

$$\mathcal{E}_\alpha(p) \geq \sum_r \mathcal{E}_\alpha(\mathbf{1}_{A_r}) \geq \sum_r S_{|A_r|} \geq S_s.$$

If there is more than one nonempty level, strict concavity and $S_0 = 0$ imply

$$\sum_r S_{|A_r|} \geq S_{s-1} + S_1.$$

For completeness, the minimum of $S_a + S_{s-a}$ over $1 \leq a \leq s-1$ occurs at an endpoint by concavity; combining levels reduces the assertion to this two-part inequality. Moreover,

$$S_{s-1} + S_1 - S_s = 2\sum_{h=1}^{s-1} d_h > 2(d_1 - d_s).$$

This proves equation (2.9) for repeated coefficients and completes the equality classification. □

2.3 The second interval and finite-degree attainment

Proposition 2.5 (A zero-sum energy)**.** *Let $0 < \alpha < 1$, let $b \neq 0$ be an integer sequence with $\sum b_j = 0$, and let its support lie in an interval of integer length $H \geq 1$, meaning that its support diameter is at most H. Then*

$$\mathcal{E}_\alpha(b) \geq 2(\mathcal{C}(\alpha) + d_H). \tag{2.10}$$

Equality holds exactly for two coefficients $+1, -1$ at distance H. If the positive mass $s = \sum_j \max(b_j, 0)$ is at least two, then

$$\mathcal{E}_\alpha(b) > 2S_2 = 2\mathcal{C}(\alpha) + \frac{2(1-\alpha)}{1+\alpha}\mathcal{C}(\alpha). \tag{2.11}$$

Proof. Write $b = p - n$ for its nonnegative positive and negative parts. They have disjoint supports and the same mass $s \geq 1$. All their cross interactions have nonzero index, so

$$\mathcal{E}_\alpha(b) = \mathcal{E}_\alpha(p) + \mathcal{E}_\alpha(n) + 2\sum_{i,j} d_{|i-j|} p_i n_j. \tag{2.12}$$

By proposition 2.3, the first two terms are at least $\mathcal{C}(\alpha)$. The cross term is at least $2d_H s^2$. This proves equation (2.10), and its equality case follows from proposition 2.3. If $s \geq 2$, lemma 2.4 and strict positivity of the cross term give $\mathcal{E}_\alpha(b) > 2S_s \geq 2S_2$. □

Proof of equations (1.5) and (1.7). For a monic P of degree at most N, put $B = (1-z)P$. Its coefficient sum is zero and its support diameter is at most $N+1$. By proposition 2.5, $\mathcal{E}_{1+\alpha}(P) = \mathcal{E}_\alpha(B) \geq 2(c_0 + d_{N+1})$. The polynomial D_N attains the bound. Its uniqueness follows because equality requires the two extreme indices $0, N+1$ and the leading coefficient of B is -1. Letting $N \to \infty$ proves equation (1.5) for $0 < \alpha < 1$ and shows that the infimum is not attained. At $\alpha = 1$, a nonzero integer zero-sum sequence takes both positive and negative values and is zero outside a finite interval. Its total unit-step variation is therefore at least four. Each jump is an integer, so its squared variation is also at least four. The polynomial $1 + z$ attains four. Thus $F(2) = 8\pi$. □

Corollary 2.6 (Exact error and sequence rigidity)**.** *For $0 < \alpha < 1$,*

$$F_N(1+\alpha) - F(1+\alpha) = 4\pi d_{N+1}, \tag{2.13}$$

$$4\pi d_{N+1} \sim 4\Gamma(2\alpha+1)\sin(\pi\alpha)(N+1)^{-2\alpha-1}. \tag{2.14}$$

Every minimizing sequence for $F(1+\alpha)$ eventually has the form $z^{s_j} D_{L_j}$, where $L_j \to \infty$.

Proof. The error formula follows from the preceding result, and its asymptotic form follows from the gamma ratio in lemma 2.1. The positive gap in equation (2.11) forces the positive and negative masses of $(1-z)P_j$ to equal one for all sufficiently large j. The leading coefficient then gives $(1-z)P_j = z^{s_j}(1 - z^{L_j+1})$. Formula (2.13) implies $L_j \to \infty$. □

2.4 The third interval

Fix $0 < \alpha < 1$. The second difference $(1-z)^2 P$ has vanishing zeroth and first moments. The rearrangement gap restricts its positive mass to two whenever its energy is sufficiently close to the infimum. Define

$$T_\alpha = 4\mathcal{C}(\alpha) - 2d_1 = \frac{2(\alpha+2)}{\alpha+1}\mathcal{C}(\alpha), \qquad g_\alpha = \frac{6(1-\alpha)(2-\alpha)}{(\alpha+1)(\alpha+2)(\alpha+3)}\mathcal{C}(\alpha) > 0. \tag{2.15}$$

Lemma 2.7 (Two vanishing moments)**.** *Let $0 < \alpha < 1$, and let $B(z) = \sum_{j \geq 0} b_j z^j$ be a nonzero integer polynomial with $\sum_j b_j = \sum_j j b_j = 0$, and let s be its positive mass. Then $s \geq 2$. If $s \geq 3$, then*

$$\mathcal{E}_\alpha(b) > T_\alpha + g_\alpha. \tag{2.16}$$

If $s = 2$ and the leading coefficient is $+1$, its polynomial has the form

$$B(z) = z^u(1-z^a)(1-z^b), \qquad u \geq 0, \quad a, b \geq 1. \tag{2.17}$$

The possibility $a = b$ is included.

Proof. Mass one would give $B = z^u - z^v$ with $u = v$ by the first-moment condition, contrary to $B \neq 0$. Write $B = p - n$ as in equation (2.12). If both parts were interval indicators of mass s, equality of their first moments would force the same starting index. This contradicts

their disjoint supports. At least one part therefore satisfies equation (2.9). The cross term is positive, and hence

$$\mathcal{E}_\alpha(B) > 2S_s + 2(d_1 - d_s) \geq 2S_3 + 2(d_1 - d_3)$$

when $s \geq 3$. The second inequality uses the increase of S_s and the decrease of d_s. Substituting the first three Fourier coefficients gives

$$2S_3 + 2(d_1 - d_3) - T_\alpha = 2c_0 - 4d_1 - 4d_2 - 2d_3 = g_\alpha.$$

For $s = 2$, the positive and negative parts each consist of two points counted with multiplicity, with equal sums. Their supports are disjoint. The pair containing the largest point also contains the smallest: otherwise its sum would be larger than the other sum. The prescribed leading coefficient makes this the positive pair. Subtract its smallest point u from all four indices. The negative indices are $a, b > 0$, and the remaining positive index is $a + b$. This proves equation (2.17). □

For later use write

$$I_\alpha(a, b) = \mathcal{E}_\alpha((1 - z^a)(1 - z^b)) = 4c_0 + 2c_{a+b} + 2c_{|a-b|} - 4c_a - 4c_b. \tag{2.18}$$

For $a < b$,

$$I_\alpha(a, b) - T_\alpha = 2(d_1 - d_{b-a}) + 4d_a + 4d_b - 2d_{a+b} > 0. \tag{2.19}$$

For $a = b$, one has $I_\alpha(a, a) = 6c_0 - 2d_{2a} + 8d_a > 6c_0 > T_\alpha$. On the other hand, $I_\alpha(a, a+1) \to T_\alpha$ as $a \to \infty$.

Proof of equation (1.6). Put $B = (1-z)^2 P$. Its first two moments vanish. The cases $s \geq 3$ and $s = 2$ are covered by lemma 2.7 and equation (2.19), respectively, and both give $\mathcal{E}_\alpha(B) > T_\alpha$. The sequence $P = D_{a-1} D_a$ gives energies tending to T_α. This proves the formula and nonattainment for $0 < \alpha < 1$. At $\alpha = 1$, the integer moment lower bound in theorem 3.2 gives $F(3) \geq 12\pi$; equality is attained by $D_1 D_2 = 1 + 2z + 2z^2 + z^3$, whose third difference has six coefficients of absolute value one. This completes theorem 1.1. □

Corollary 2.8 (Rigidity on the third interval)**.** *Let $0 < \alpha < 1$. If P_j is a minimizing sequence for $F(2 + \alpha)$, then, for all sufficiently large j,*

$$P_j = z^{s_j} D_{L_j - 1} D_{L_j}, \qquad L_j \to \infty.$$

Proof. The gap g_α forces positive mass two eventually. In equation (2.19), a difference $b - a \geq 2$ incurs the fixed gap $2(d_1 - d_2) > 0$, while diagonal pairs have a fixed positive gap as well. Thus $b = a + 1$ eventually. The remaining energy excess is $4d_a + 4d_{a+1} - 2d_{2a+1} > 4d_a$. It tends to zero as $a \to \infty$, and this lower bound shows that the converse also holds. □

2.5 The shape of the extremal function

Proposition 2.9 (A local maximum and a unique interior minimum)**.** *The function F is continuous on $(0, 3]$, with*

$$F'(2-) = 8\pi, \qquad F'(2+) = -4\pi.$$

Consequently 2 is a strict local maximum. On $(2, 3)$ the function F has exactly one critical point, a strict minimum. Writing it as $2 + \alpha_*$, the number $\alpha_* \in (0, 1)$ is the unique solution of

$$2\psi(2\alpha + 1) - 2\psi(\alpha + 1) - \frac{1}{(\alpha + 1)(\alpha + 2)} = 0, \tag{2.20}$$

where $\psi = \Gamma'/\Gamma$.

Proof. The endpoint values agree in theorem 1.1. Since $\mathcal{C}'(0) = 0$ and $\mathcal{C}'(1) = 2$, the stated one-sided derivatives follow directly. The logarithmic derivative of $F(2+\alpha)$ equals the left side of equation (2.20). By the duplication formula it is

$$h(\alpha) = \psi(\alpha + 1/2) - \psi(\alpha + 1) + 2\log 2 - \frac{1}{(\alpha+1)(\alpha+2)}.$$

The trigamma function ψ' strictly decreases on the positive real axis, as follows from $\psi'(x) = \sum_{n\geq 0}(x+n)^{-2}$. Thus $h'(\alpha) > 0$. Finally $h(0) = -1/2$ and $h(1) = 5/6$, so h has exactly one zero. □

The critical point has the numerical approximations

$$\alpha_* = 0.1646512225\ldots, \qquad F(2+\alpha_*) = 24.2137162258\ldots.$$

Figure 1 plots the exact formulas and enlarges the minimum on the third interval.

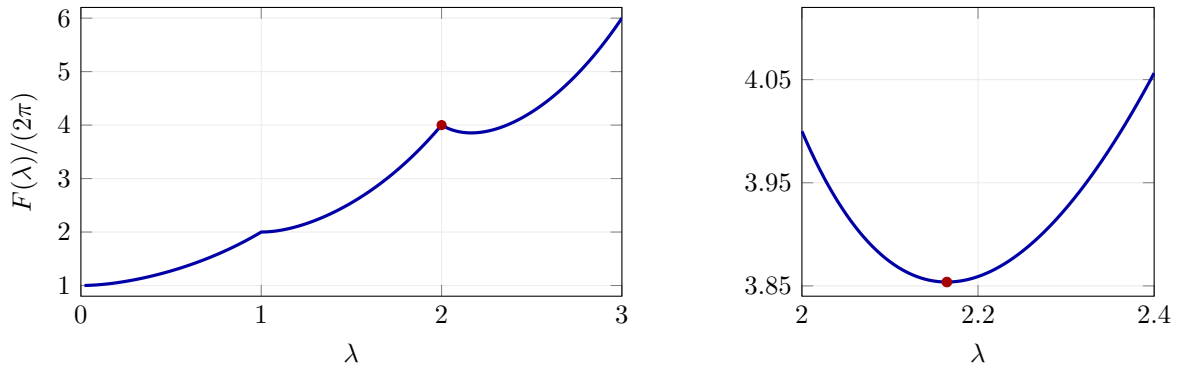


Figure 1: The exact extremal function on $(0, 3]$ (left), and the unique minimum on $(2, 3)$ (right). Both curves use the analytic formulas of theorem 1.1.

3 Integer parameters and ideal PTE solutions

3.1 Parseval reduction and the equality case

At integer parameters the normalized weighted integral is a sum of coefficient squares. Divisibility by a power of $1 - z$ then turns the minimization into a problem about equal power sums.

For a coefficient polynomial $Q = \sum_j b_j z^j$, write $\|Q\|_1 = \sum_j |b_j|$ and $\|Q\|_2^2 = \sum_j b_j^2$. Divisibility by $(1-z)^m$ is equivalent to

$$\sum_j j^r b_j = 0 \qquad (0 \leq r < m). \tag{3.1}$$

Indeed, derivatives at one give the corresponding falling-factorial moments, and the falling factorials and ordinary powers span the same polynomial space. This is the standard incidence-polynomial formulation of the PTE problem [1, 5].

Lemma 3.1 (The integer moment lower bound)**.** *If $Q \neq 0$ is an integer polynomial divisible by $(1-z)^m$, then its positive and negative coefficient masses are equal to an integer $s \geq m$. Consequently*

$$\|Q\|_2^2 \geq \|Q\|_1 = 2s \geq 2m. \tag{3.2}$$

Proof. Replace each positive coefficient b_j by b_j copies of the integer j, and each negative coefficient by $-b_j$ copies in a second multiset. The identities in equation (3.1) give equal cardinalities s and equal power sums through degree $m-1$. If $s < m$, their first s power sums agree. Newton's identities then imply that their elementary symmetric functions agree, so the two monic polynomials having these multisets as roots coincide. The multisets are identical. Their supports were disjoint, which contradicts $Q \neq 0$. Finally, $b_j^2 \geq |b_j|$ for every integer coefficient. □

Theorem 3.2 (Arithmetic reduction)**.** *The minimum in equation (1.9) exists, and* $F(m) = 2\pi\mathcal{M}_m \geq 4\pi m$*. Equality holds exactly when a multiplicity-free ideal PTE pair of size* m *exists. In that event, all minimizing polynomials are obtained from such pairs by*

$$P(z) = \frac{\sum_{a\in A} z^a - \sum_{b\in B} z^b}{(1-z)^m}, \tag{3.3}$$

after a common translation to nonnegative exponents and an interchange of A, B*, if necessary, to impose the leading coefficient condition.*

Proof. The admissible class for $\mathcal{M}_m$ is nonempty, since it contains $(1-z)^m$. Its squared norms are positive integers, so a least value is attained. The map $P \mapsto Q = (1-z)^m P$ is a bijection from monic integer polynomials to this class. Conversely, division by the monic polynomial $(z-1)^m$ shows that $Q/(1-z)^m \in \mathbb{Z}[z]$, and the specified leading coefficient makes the quotient monic. Parseval gives

$$\mathcal{E}_m(P) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Q(e^{i\theta})|^2\, d\theta = \|Q\|_2^2\,.$$

The lower bound follows from lemma 3.1. If $\|Q\|_2^2 = 2m$, equality throughout equation (3.2) forces positive mass m and coefficients in $\{-1, 0, 1\}$. Their positive and negative supports are the desired disjoint sets. Conversely, such sets give a polynomial of squared norm $2m$ and satisfy equation (3.1). Common translation preserves the moment identities by the binomial theorem, and interchange adjusts the sign. This also proves the classification in equation (3.3). □

Corollary 3.3 (Discreteness and elementary operations)**.** *For* $m, n \geq 1$*,*

$$\mathcal{M}_m \in 2\mathbb{N}, \qquad \mathcal{M}_m \leq \mathcal{M}_{m+1}, \qquad \mathcal{M}_{m+n} \leq \mathcal{M}_m\mathcal{M}_n. \tag{3.4}$$

If $\mathcal{M}_m = 2m$*, every minimizing* Q *has a zero of exactly order* m *at one. If equality is impossible, then* $F(m) \geq 4\pi(m+1)$*.*

Proof. Modulo two, $\sum b_j^2 \equiv \sum b_j = 0$. Changing the sign of an order-$(m+1)$ admissible polynomial makes it admissible at order m, proving monotonicity. Choose minimizers Q_m, Q_n and an integer $L > \deg Q_m$. The coefficients of $Q_m(z)Q_n(z^L)$ are products without overlap, so their squared sum is $\mathcal{M}_m\mathcal{M}_n$. The zero at one has order at least $m+n$ and the leading sign is correct. The exact-order assertion follows by applying lemma 3.1 at order $m+1$. □

3.2 Explicit integer values

The pairs in table 1 have equal power sums through degree $m-1$. Their incidence polynomials have the required leading signs. Thus they give $F(m) = 4\pi m$ for $1 \leq m \leq 6$.

Table 1: Multiplicity-free ideal PTE pairs for the first six parameters. The degree is that of the monic quotient in equation (3.3).

m	degree	A_m	B_m
1	0	$\{0\}$	$\{1\}$
2	1	$\{0,3\}$	$\{1,2\}$
3	3	$\{0,4,5\}$	$\{1,2,6\}$
4	7	$\{0,4,7,11\}$	$\{1,2,9,10\}$
5	13	$\{0,4,8,16,17\}$	$\{1,2,10,14,18\}$
6	16	$\{0,5,6,16,17,22\}$	$\{1,2,10,12,20,21\}$

For $m = 7, 8, 9, 10, 12$, the following pairs give the remaining values in theorem 1.3:

$$\begin{aligned}
A_7 &= \{0,18,27,58,64,89,101\},\\
B_7 &= \{1,13,38,44,75,84,102\},\\
A_8 &= \{0,4,9,23,27,41,46,50\},\\
B_8 &= \{1,2,11,20,30,39,48,49\},\\
A_9 &= \{0,24,30,83,86,133,157,181,197\},\\
B_9 &= \{1,17,41,65,112,115,168,174,198\},\\
A_{10} &= \{0,12,125,213,214,412,413,501,614,626\},\\
B_{10} &= \{5,6,133,182,242,384,444,493,620,621\},\\
A_{12} &= \{0,11,24,65,90,129,173,212,237,278,291,302\},\\
B_{12} &= \{3,5,30,57,104,116,186,198,245,272,297,299\}.
\end{aligned} \tag{3.5}$$

The size-seven pair is recorded in [4, Section 5.4.2, p. 48]. The size-eight pair is the expansion of the pure product with lengths $(1,2,3,5,7,8,11,13)$ listed in [5, Table 1]. The size-nine, size-ten and size-twelve pairs are obtained by translation and, where necessary, interchange from the symmetric examples in [6, equations (3)–(5)]; see also [2]. The accompanying program verifies the moment identities for every displayed pair using integer arithmetic. The quotient degrees for $m = 7, 8, 9, 10, 12$ are $95, 42, 189, 616, 290$, respectively.

The size-twelve pair also gives an admissible polynomial at order eleven after changing its sign. Hence $22 \le \mathcal{M}_{11} \le 24$. By parity, $\mathcal{M}_{11} \in \{22, 24\}$, proving the stated two-value alternative. The value 22 is equivalent to the existence of a multiplicity-free ideal pair of size eleven. The symmetric search in [6, Section 4.1 and Table 2] excludes such integer solutions up to height 3500 in that symmetric class. Here height means $\max_{a\in A} |a|$ in the centered normalization $B = -A$. The general ideal existence question remains open; see also the discussion in [7].

Remark 3.4 (Extremizers need not be palindromic). The pair $\{0,9,10\}, \{1,6,12\}$ has equal first and second moments. It gives the nonpalindromic minimizer at $m = 3$

$$1 + 2z + 3z^2 + 4z^3 + 5z^4 + 6z^5 + 6z^6 + 5z^7 + 3z^8 + z^9.$$

Indeed its third difference is $1 - z - z^6 + z^9 + z^{10} - z^{12}$. Consequently palindromicity cannot be imposed as a general search restriction. Palindromicity is therefore not imposed in the coefficient searches reported below.

4 General bounds and lifting operations

4.1 A quadratic upper bound

Proposition 4.1 (Subset counting). *For every $m \ge 1$, one has $\mathcal{M}_m \le m(m-1) + 2$.*

Proof. Put $s = m(m-1)/2 + 1$ and consider all s-element subsets of $\{1, \dots, H\}$. There are $\binom{H}{s} \asymp_m H^s$ such subsets. The vector of power sums of orders $1, \dots, m-1$ takes at most

$$\prod_{r=1}^{m-1} (sH^r + 1) = O_m(H^{s-1})$$

values. For sufficiently large H, two distinct subsets have the same vector. Deleting their common elements gives two nonempty disjoint sets of equal cardinality at most s, with the same moments through order $m-1$, including order zero. Their signed incidence polynomial has squared norm at most $2s$ and is divisible by $(1-z)^m$. Its sign can be chosen to meet the leading coefficient condition. □

This is the classical pigeonhole construction for equal power sums. Starting with subsets makes the resulting coefficients have absolute value one; a bound on the cardinality of a multiset alone would not give the same squared-norm estimate. For comparison, Prouhet's binary construction

$$\prod_{r=0}^{m-1} (1 - z^{2^r})$$

has 2^m coefficients of absolute value one. Combining these two constructions proves equation (1.10) and completes theorem 1.3.

4.2 Dilation and adjacent parameter values

Multiplication by a long Dirichlet polynomial raises the parameter by one and asymptotically doubles the energy. Dilating the exponents of an integer extremizer gives a second comparison, which retains the factor $\mathcal{C}(\alpha)$.

Proposition 4.2 (Dirichlet lifting)**.** *For every $\lambda > 0$ and integer $r \geq 0$,*

$$F(\lambda + 1) \leq 2F(\lambda), \qquad F(\lambda + r) \leq 2^r F(\lambda). \tag{4.1}$$

Proof. For a fixed monic integer P, the polynomial PD_N is admissible and

$$\mathcal{E}_{\lambda+1}(PD_N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - e^{i\theta}|^{2\lambda} |P(e^{i\theta})|^2 (2 - 2\cos((N+1)\theta))\, d\theta.$$

The Riemann–Lebesgue lemma gives the limit $2\mathcal{E}_\lambda(P)$. Take the infimum over P and iterate. □

Theorem 4.3 (Interpolation from integer extrema)**.** *For every integer $k \geq 1$ and $0 < \alpha < 1$,*

$$\frac{\mathcal{C}(\alpha)}{2} F(k+1) \leq F(k+\alpha) \leq \mathcal{C}(\alpha) F(k). \tag{4.2}$$

Consequently,

$$2\pi(k+1)\mathcal{C}(\alpha) \leq F(k+\alpha) \leq 2\pi\mathcal{C}(\alpha) \min\{2^k, k(k-1)+2\}. \tag{4.3}$$

In particular $F(\lambda) = \Omega(\lambda)$ and $F(\lambda) = O(\lambda^2)$ as $\lambda \to \infty$.

Proof. For $Q = (1-z)^k P$, equation (2.5) gives

$$\mathcal{E}_{k+\alpha}(P) = \sum_{h \geq 1} d_h \left\| (1 - z^h) Q \right\|_2^2.$$

Each polynomial in the sum is admissible at order $k+1$, so its squared norm is at least $\mathcal{M}_{k+1}$. Summing and using equation (2.4) proves the lower bound. For the upper bound, choose a minimizer Q_k for $\mathcal{M}_k$ and, for positive integers L, put

$$P_L(z) = Q_k(z^L)/(1-z)^k.$$

This is a monic integer polynomial. Its energy equals $\mathcal{E}_\alpha(Q_k(z^L))$, whose nonconstant Fourier interactions tend to zero as $L \to \infty$. The limit is $\mathcal{C}(\alpha)\,\|Q_k\|_2^2 = \mathcal{C}(\alpha)\mathcal{M}_k$. The remaining assertions follow from the integer bounds and $1 \le \mathcal{C}(\alpha) \le 2$ on $[0,1]$. The latter follows either by the gamma formula or by differentiating it: ψ is increasing and $\mathcal{C}(0) = 1$, $\mathcal{C}(1) = 2$. □

5 Dirichlet products and their limitations

5.1 An exact two-factor optimization

The first three interval formulas lead to Dirichlet products as natural candidates at higher parameters. We first optimize the two-factor case, then construct higher-interval bounds and identify intervals where these products are suboptimal.

For $\lambda = k+\alpha$, $k \ge 1$ and $0 < \alpha < 1$, a Dirichlet product with positive integer lengths $L_1, \dots, L_k$ is

$$P_{L_1,\dots,L_k}(z) = \prod_{i=1}^{k} D_{L_i-1}(z). \tag{5.1}$$

It has degree $\sum_i (L_i - 1)$, and

$$\mathcal{E}_{k+\alpha}(P_{L_1,\dots,L_k}) = \mathcal{E}_\alpha\left(\prod_{i=1}^{k}(1-z^{L_i})\right). \tag{5.2}$$

Theorem 5.1 (Two-factor minimum at a degree bound)**.** *Let $0 < \alpha < 1$ and $N \ge 1$. Put $M = \lfloor (N+1)/2 \rfloor$. Among the positive integer pairs a, b with $a + b - 2 \le N$, the unique unordered minimizing pair for $I_\alpha(a,b)$ is $\{M, M+1\}$.*

Proof. For $a < b$, write $s = a + b$ and $h = b - a$. Formula (2.18) becomes

$$I_\alpha(a,b) = 4c_0 - 2d_s - 2d_h + 4d_{(s-h)/2} + 4d_{(s+h)/2}.$$

For fixed s, increasing the positive difference h increases $-2d_h$; the sum of the last two terms also increases by convexity. Thus the best non-diagonal pair for total length $2m+1$ is $(m, m+1)$, and that for total length $2m+2$ is $(m, m+2)$. With $\delta_j = d_j - d_{j+1}$,

$$I_\alpha(m, m+2) - I_\alpha(m, m+1) = 2(\delta_1 + \delta_{2m+1} - 2\delta_{m+1}) > 0 \tag{5.3}$$

by strict convexity of δ.

If $f_m = I_\alpha(m, m+1)$, direct subtraction gives

$$f_{m+1} - f_m = 2(\delta_{2m+1} + \delta_{2m+2}) - 4(\delta_m + \delta_{m+1}) < 0.$$

It remains to exclude diagonal pairs. For $m \ge 2$,

$$I_\alpha(m,m) - I_\alpha(m-1,m) = 2(c_0 + \delta_{2m-1} + d_1 - 2\delta_{m-1}) > 0,$$

because $\delta_{m-1} < d_1 < c_0$. Finally,

$$I_\alpha(1,1) - I_\alpha(1,2) = 2c_0 + 6(d_1 - d_2) + 2d_3 > 0.$$

The latter comparison is feasible since $N \ge 1$. Thus the largest admissible adjacent pair is the unique unordered minimizer. □

Theorem 5.2 (The finite-degree third interval)**.** *For $0 < \alpha < 1$ and $N \geq 5$, the formula and complete minimizer classification in equation (1.8) hold. With $M = \lfloor (N+1)/2 \rfloor$,*

$$F_N(2+\alpha) - F(2+\alpha) = 2\pi(4d_M + 4d_{M+1} - 2d_{2M+1}). \tag{5.4}$$

Writing $p = 2\alpha + 1$ and $A_\alpha = \Gamma(2\alpha+1)\sin(\pi\alpha)/\pi$, the error is asymptotic to

$$2\pi A_\alpha (8 - 2^{-2\alpha}) M^{-2\alpha-1}. \tag{5.5}$$

Proof. By theorem 5.1, the best mass-two candidate is $D_{M-1}D_M$. Since $M \geq 3$ and the adjacent-pair energy decreases,

$$I_\alpha(M, M+1) - T_\alpha \leq 4d_3 + 4d_4 - 2d_7 < 4d_3 + 4d_4 = \frac{28}{\alpha+4} d_3.$$

The gap from equation (2.15) satisfies $g_\alpha = 6d_3/\alpha$. Moreover

$$\frac{28d_3/(\alpha+4)}{g_\alpha} = \frac{14\alpha}{3(\alpha+4)} < 1 \quad (0 < \alpha < 1).$$

Thus every polynomial with positive mass at least three in its second difference has strictly larger energy, by lemma 2.7. All minimizers have mass two and are classified by theorem 5.1. Their monomial shifts are exactly those allowed by the degree bound. Formula (5.4) is immediate; the gamma-ratio asymptotic $d_n \sim A_\alpha n^{-p}$ gives equation (5.5). □

5.2 Separated clusters of exponents

Proposition 5.3 (A product limit)**.** *Fix $0 < \alpha < 1$, positive integers $u_1, \ldots, u_k$ and integers $v_1, \ldots, v_k$. Put $L_i(t) = u_i t + v_i$ and, for $S \subseteq \{1, \ldots, k\}$, write*

$$U_S = \sum_{i \in S} u_i, \quad V_S = \sum_{i \in S} v_i, \quad \varepsilon_S = (-1)^{|S|}.$$

As $t \to \infty$ through integers for which all lengths are positive,

$$\mathcal{E}_{k+\alpha}(P_{L_1(t),\ldots,L_k(t)}) \longrightarrow \sum_{\substack{S,T \subseteq \{1,\ldots,k\} \\ U_S = U_T}} \varepsilon_S \varepsilon_T c_{V_S - V_T}(\alpha). \tag{5.6}$$

Proof. Expand the binomial product as $\sum_S \varepsilon_S z^{U_S t + V_S}$ and insert it into the Fourier quadratic form. Terms with $U_S \neq U_T$ tend to zero, while those with $U_S = U_T$ remain constant. □

For example, the lengths $(t, t+1, 2t+1)$ give

$$(1-z^t)(1-z^{t+1})(1-z^{2t+1}) = 1 - z^t - z^{t+1} + z^{3t+1} + z^{3t+2} - z^{4t+2}.$$

There are two isolated coefficients and two adjacent two-term clusters. Consequently

$$F(3+\alpha) \leq 2\pi(6c_0 + 4c_1) = 4\pi \frac{\alpha+3}{\alpha+1} \mathcal{C}(\alpha). \tag{5.7}$$

In particular $F(7/2) \leq 112/3 < 12\pi = F(3)$.

5.3 A Pell construction on the fifth interval

Theorem 5.4 (An eight-term Pell construction)**.** *For every $0 < \alpha < 1$,*

$$F(4+\alpha) \le 2\pi\big(8c_0(\alpha) + 6c_1(\alpha)\big) = 4\pi\frac{\alpha+4}{\alpha+1}\mathcal{C}(\alpha). \tag{5.8}$$

More precisely, let

$$u_n + w_n\sqrt{2} = (3+2\sqrt{2})^n, \qquad n \ge 2,$$

and put

$$A_n = \left\{0, \frac{u_n-1}{2}, \frac{u_n+1}{2}, u_n\right\},$$
$$B_n = \left\{\frac{u_n-w_n-1}{2}, \frac{u_n-w_n+1}{2}, \frac{u_n+w_n-1}{2}, \frac{u_n+w_n+1}{2}\right\}.$$

The polynomials

$$Q_n(z) = \sum_{a\in A_n} z^a - \sum_{b\in B_n} z^b, \qquad P_n(z) = \frac{Q_n(z)}{(1-z)^4}$$

are well defined, P_n is monic with integer coefficients and degree $u_n - 4$, and

$$\lim_{n\to\infty} \mathcal{E}_{4+\alpha}(P_n) = 8c_0(\alpha) + 6c_1(\alpha). \tag{5.9}$$

Proof. The Pell recurrence

$$u_{n+1} = 3u_n + 4w_n, \qquad w_{n+1} = 2u_n + 3w_n$$

shows that u_n, w_n are positive integers, u_n is odd, w_n is even, and $u_n^2 - 2w_n^2 = 1$. For $n \ge 2$ the displayed sets are nonnegative, disjoint, and each has four elements. Under the affine change $x \mapsto 2x - u_n$, they become

$$\{-u_n, -1, 1, u_n\}, \qquad \{-w_n - 1, -w_n + 1, w_n - 1, w_n + 1\}.$$

Their odd power sums agree by symmetry, and their second power sums agree because

$$u_n^2 + 1 = (w_n+1)^2 + (w_n-1)^2.$$

Thus the original sets have equal moments of orders zero through three. Their fourth-moment difference is

$$\sum_{a\in A_n} a^4 - \sum_{b\in B_n} b^4 = \frac{w_n^2(w_n^2-4)}{4} > 0.$$

Consequently Q_n has a zero of exactly order four at one. Division by $(1-z)^4$ gives an integer polynomial, and the leading coefficient of Q_n is one, so P_n is monic of degree $u_n - 4$.

Write $a_n = (u_n - 1)/2$ and $b_n = (u_n - w_n - 1)/2$. The eight terms of Q_n form the five clusters

$$\{0\}, \quad \{b_n, b_n+1\}, \quad \{a_n, a_n+1\}, \quad \{u_n - b_n - 1, u_n - b_n\}, \quad \{u_n\}.$$

Each of the three two-point clusters has coefficients of the same sign. Since $u_n/w_n \to \sqrt{2}$ and $w_n \to \infty$, the distances between distinct clusters tend to infinity. In the Fourier expression for $\mathcal{E}_\alpha(Q_n)$, all interactions between different clusters therefore vanish in the limit. The eight diagonal terms contribute $8c_0(\alpha)$, and the three adjacent same-sign pairs contribute $6c_1(\alpha)$. This proves equation (5.9). Finally, $\mathcal{E}_{4+\alpha}(P_n) = \mathcal{E}_\alpha(Q_n)$ and $c_1(\alpha) = -\alpha c_0(\alpha)/(\alpha+1)$, which gives equation (5.8). □

The first two pairs are

$$(A_2, B_2) = (\{0, 8, 9, 17\}, \{2, 3, 14, 15\}), \qquad (A_3, B_3) = (\{0, 49, 50, 99\}, \{14, 15, 84, 85\}).$$

Their quotient degrees are 13 and 95. The first quotient is $D_1 D_2 D_4 D_6$; none of the subsequent quotients is a product of four Dirichlet polynomials.

Corollary 5.5 (Nonproduct members of the Pell family)**.** *For $n \geq 3$, the polynomial P_n in theorem 5.4 is not a product of four Dirichlet polynomials.*

Proof. Put $u = u_n$, $w = w_n$ and $a = (u - w - 1)/2$. The first negative terms of Q_n occur at $a, a + 1$. If Q_n were a product of four binomials, these would force its first lengths to be $a, a + 1$. The vanishing coefficient at $2a + 1$ then forces the third length to be $2a + 1$. The fourth length is determined by the total degree as $d = 2w - u$. For $u > 17$, the Pell identity gives

$$2a + 1 < d < 3a + 1 < (u - 1)/2.$$

Indeed $3w > 2u$ follows from $9w^2 - 4u^2 = (u^2 - 9)/2 > 0$, and the middle inequality follows from

$$(5u - 1)^2 - 49w^2 = (u - 17)(u - 3)/2 > 0.$$

The first three factors have their next nonzero term at $3a+1$. Multiplication by $1-z^d$ therefore produces coefficient -1 at exponent d, where Q_n has coefficient zero. This is impossible. □

The Pell construction gives the explicit bounds

$$F(9/2) \leq 48, \qquad \frac{F(17/4)}{2\pi} \leq \frac{34}{5}\mathcal{C}(1/4). \tag{5.10}$$

It also yields additional information about the shape of F.

Corollary 5.6 (Three strict local maxima)**.** *The parameters 2, 3 and 4 are strict local maxima of F.*

Proof. For $1 \leq l \leq 4$ put

$$H_l(\alpha) = 4\pi \frac{\alpha + l}{\alpha + 1}\mathcal{C}(\alpha).$$

On $0 < \alpha < 1$, the exact interval formulas and equations (5.7) and (5.8) give $F(l + \alpha) \leq H_l(\alpha)$. For $k = 2, 3, 4$, the integer values imply

$$H_k(0) = H_{k-1}(1) = 4\pi k = F(k).$$

Since $\mathcal{C}'(0) = 0$ and $\mathcal{C}'(1) = 2$,

$$H_k'(0) = 4\pi(1 - k) < 0, \qquad H_{k-1}'(1) = 2\pi(k + 2) > 0.$$

The respective upper bounds are strictly below $F(k)$ in right and left punctured neighborhoods of k, which proves the assertion. □

5.4 Persistence of arithmetic obstructions

Let

$$\mathcal{B}_m = \left\{ \prod_{i=1}^{m} (1 - z^{L_i}) : L_1, \ldots, L_m \geq 1 \right\}.$$

Repeated lengths are allowed and the degrees are unrestricted. The coefficient-length results of Maltby [9], recorded explicitly in [5, p. 39], give

$$\min_{Q\in\mathcal{B}_7}\|Q\|_1=16,\qquad \min_{Q\in\mathcal{B}_9}\|Q\|_1=20,\qquad \min_{Q\in\mathcal{B}_{10}}\|Q\|_1=24. \tag{5.11}$$

These are also the minima of $\|Q\|_2^2$. Integrality gives the lower bounds, and the respective length vectors

$$(1,2,3,4,5,7,11),\quad (1,2,3,4,5,7,9,11,13),\quad (1,2,3,4,5,7,9,11,13,17)$$

in [5, Table 1] attain them by direct expansion. Thus pure products have strict squared-norm gaps above $\mathcal{M}_7=14$, $\mathcal{M}_9=18$ and $\mathcal{M}_{10}=20$.

Lemma 5.7 (Comparison with the coefficient norm)**.** *For $-1/2<\beta<1$ and every real-coefficient polynomial Q,*

$$\left|\frac{\mathcal{E}_\beta(Q)}{\mathcal{C}(\beta)}-\|Q\|_2^2\right|\le\frac{|\beta|}{1+\beta}(\|Q\|_1^2-\|Q\|_2^2). \tag{5.12}$$

Proof. The Fourier recurrence in lemma 2.1 shows that

$$\frac{|c_j(\beta)|}{\mathcal{C}(\beta)}\le\frac{|\beta|}{1+\beta}\qquad(j\ge1),$$

since the successive absolute ratios are $(j-\beta)/(j+\beta+1)<1$ when $\beta\neq0$. At zero all off-diagonal coefficients vanish. Sum the absolute off-diagonal terms of the quadratic form, using $\sum_{i\neq j}|b_ib_j|=\|Q\|_1^2-\|Q\|_2^2$. □

Theorem 5.8 (Persistence of a pure-product gap)**.** *Suppose $Q_*=(1-z)^mP_*$ with $P_*\in\mathcal{P}$, and put $h=\|Q_*\|_2^2$, $L=\|Q_*\|_1$. Assume $\|Q\|_2^2\ge G>h$ for every $Q\in\mathcal{B}_m$. If*

$$0<\delta<\min\left\{\frac12,\frac{G-h}{4^m+L^2-2h}\right\}, \tag{5.13}$$

then, for $|\lambda-m|\le\delta$ and $k=\lfloor\lambda\rfloor$,

$$\mathcal{E}_\lambda(P_*)<\inf_{s\ge0,\ L_1,\dots,L_k\ge1}\mathcal{E}_\lambda\left(z^s\prod_{i=1}^{k}D_{L_i-1}\right). \tag{5.14}$$

Proof. Write $\beta=\lambda-m$ and $r=|\beta|/(1+\beta)$. Every $Q\in\mathcal{B}_m$ has $\|Q\|_1\le2^m$. Hence lemma 5.7 gives

$$\frac{\mathcal{E}_\beta(Q)}{\mathcal{C}(\beta)}\ge G-r(4^m-G),\qquad\frac{\mathcal{E}_\beta(Q_*)}{\mathcal{C}(\beta)}\le h+r(L^2-h).$$

The difference is at least $(G-h)-r(4^m+L^2-G-h)$, uniformly in Q. It is strictly positive by equation (5.13) and $r\le\delta/(1-\delta)$. If $\lambda\ge m$, then $k=m$ and

$$(1-z)^m\prod_{i=1}^{k}D_{L_i-1}=\prod_{i=1}^{m}(1-z^{L_i})\in\mathcal{B}_m.$$

If $\lambda<m$, then $k=m-1$ and an additional factor $1-z$ gives an element of the same class. Finally, $\mathcal{E}_{m+\beta}(P)=\mathcal{E}_\beta((1-z)^mP)$, and monomial factors do not affect either energy. □

5.5 Explicit intervals of uniform product suboptimality

Bounding the interactions separately on the positive and negative supports gives larger exclusion intervals. Fix $0 < \delta < 1/2$ and define

$$\rho_\delta(1) = \frac{\delta}{1-\delta}, \qquad \rho_\delta(j+1) = \rho_\delta(j)\frac{j+\delta}{j+1-\delta}, \tag{5.15}$$

$$H_\delta(p) = \sum_{j=1}^{p} \rho_\delta(\lceil j/2 \rceil), \qquad H_\delta(0) = 0. \tag{5.16}$$

The Fourier formula implies

$$\frac{|c_j(\beta)|}{\mathcal{C}(\beta)} \le \rho_\delta(j) \qquad (|\beta| \le \delta,\ j \ge 1). \tag{5.17}$$

For negative β, each factor is bounded by its value at $\beta = -\delta$. For nonnegative β the left side is at most $\beta/j \le \delta/j \le \rho_\delta(j)$.

Lemma 5.9 (A sparse-support bound)**.** *Let $Q \in \mathcal{B}_m$, write $\|Q\|_2^2 = 2s$, and put $M = 2^{m-1}$. For $|\beta| \le \delta$,*

$$\frac{\mathcal{E}_\beta(Q)}{\mathcal{C}(\beta)} \ge 2s(1 - H_\delta(\min\{s, M\})). \tag{5.18}$$

Proof. The coefficient sum is zero, so the positive and negative masses agree. Each is at most M, by the expansion into 2^m signed subset terms, and at most s, since $\|Q\|_1 \le \|Q\|_2^2 = 2s$. Each sign support therefore contains at most $p = \min\{s, M\}$ points. The integer s is well-defined because the squared norm is even.

At a fixed integer exponent there are at most two other exponents at each positive distance. Since $\rho_\delta(j)$ decreases, the sum of any p off-diagonal entries bounded by equation (5.17) is at most $H_\delta(p)$. If $\beta > 0$, the off-diagonal Fourier coefficients are negative. Discard the nonnegative interactions between opposite signs and apply the row-sum bound to each same-sign block. If $\beta < 0$, the off-diagonal Fourier coefficients are positive. Discard the nonnegative same-sign interactions. The remaining rectangular matrix has row sums and column sums at most $H_\delta(p)$, and therefore has operator norm at most $H_\delta(p)$. Apply $2\|u\|_2\|v\|_2 \le \|u\|_2^2 + \|v\|_2^2$ to the two coefficient vectors. At $\beta = 0$ the assertion is immediate. □

Suppose $G \le 2^m$ is an even lower bound for the squared norm on $\mathcal{B}_m$, and $H_\delta(M) < 1$. Define the finite minimum

$$B_{m,G}(\delta) = \min_{\substack{G/2 \le s \le M \\ s \in \mathbb{Z}}} 2s(1 - H_\delta(s)). \tag{5.19}$$

For $s \ge M$, the right side of equation (5.18) increases linearly in s. Thus equation (5.19) is a uniform lower bound for $\mathcal{E}_\beta(Q)/\mathcal{C}(\beta)$ over all $Q \in \mathcal{B}_m$ and all $|\beta| \le \delta$.

For a height-one polynomial $Q_* = \sum_{a\in A} z^a - \sum_{b\in B} z^b$ with disjoint sets A, B satisfying $|A| = |B| = m$, put

$$U_\delta(A,B) = 2\max\left\{ \sum_{a\in A,\, b\in B} \rho_\delta(|a-b|),\ \sum_{\{a,a'\}\subset A} \rho_\delta(|a-a'|) + \sum_{\{b,b'\}\subset B} \rho_\delta(|b-b'|) \right\}. \tag{5.20}$$

The within-set sums are over unordered two-element subsets. Keeping only the interactions that can increase the energy gives

$$\frac{\mathcal{E}_\beta(Q_*)}{\mathcal{C}(\beta)} \le 2m + U_\delta(A,B) \qquad (|\beta| \le \delta). \tag{5.21}$$

Theorem 5.10 (Explicit product exclusion intervals)**.** *For* (m, δ, η) *equal to*

$$(7, 1/40, 1/50), \qquad (9, 1/60, 1/4), \qquad (10, 1/35, 1/10),$$

let P_m *be the monic quotient of the corresponding pair in equation (3.5). For* $|\lambda - m| \le \delta$ *and* $k = \lfloor \lambda \rfloor$*,*

$$\inf_{s \ge 0,\ L_1,\dots,L_k \ge 1} \mathcal{E}_\lambda \left(z^s \prod_{i=1}^{k} D_{L_i - 1} \right) \ge \mathcal{E}_\lambda(P_m) + \eta \mathcal{C}(\lambda - m). \tag{5.22}$$

In particular, this product family cannot attain or approach the full infimum anywhere in the indicated closed intervals.

Proof. Use the squared-norm bounds in equation (5.11). At the indicated rational values of δ, every term in equations (5.15), (5.19) and (5.20) is rational. Direct evaluation gives the strict bounds in table 2; each row also satisfies $H_\delta(2^{m-1}) < 1$.

Table 2: Rational bounds used in the product exclusion intervals.

m	G	δ	$B_{m,G}(\delta) >$	$2m + U_\delta(A_m, B_m) <$
7	16	1/40	14243/1000	14222/1000
9	20	1/60	18418/1000	18146/1000
10	24	1/35	20392/1000	20281/1000

The differences between the last two columns exceed 1/50, 1/4 and 1/10, respectively. The finite minima in equation (5.19) require 57, 247 and 501 rational comparisons; the program `anc/verify_pte.py` performs them using integers and fractions. Combining equations (5.18) and (5.21) gives the uniform gaps for pure m-factor products. The transformation in the proof of theorem 5.8 covers both sides of each integer and yields equation (5.22). □

6 Finite-degree structure and exact computation

6.1 Rational energy functions and algebraic transitions

Dividing the energy by $\mathcal{C}(\lambda)$ makes every Fourier coefficient rational in λ. At a fixed degree, positive definiteness bounds the coefficients of all possible minimizers locally uniformly in the parameter. These two facts reduce changes of minimizer to crossings of finitely many rational functions.

Define

$$r_0(\lambda) = 1, \qquad r_j(\lambda) = \frac{c_j(\lambda)}{\mathcal{C}(\lambda)} = (-1)^j \prod_{h=0}^{j-1} \frac{\lambda - h}{\lambda + h + 1} \quad (j \ge 1). \tag{6.1}$$

For $P(z) = \sum_{i=0}^{N} a_i z^i$, write

$$R_P(\lambda) = \frac{\mathcal{E}_\lambda(P)}{\mathcal{C}(\lambda)} = \sum_{i=0}^{N} a_i^2 + 2 \sum_{j=1}^{N} r_j(\lambda) \sum_{i=0}^{N-j} a_i a_{i+j}. \tag{6.2}$$

This is a rational function with no pole on $(0, \infty)$. Multiplication by a monomial leaves R_P unchanged. We regard two polynomials as equivalent if removing their maximal monomial factors gives the same polynomial. Each such class with a representative of degree at most N has a unique representative of exact degree N.

Theorem 6.1 (Local finiteness and algebraic transitions)**.** *Fix $N \geq 0$ and a compact interval $K \subset (0, \infty)$. There is a finite collection of monic integer polynomials of exact degree N such that*

$$\frac{F_N(\lambda)}{2\pi \mathcal{C}(\lambda)} = \min_P R_P(\lambda) \qquad (\lambda \in K). \tag{6.3}$$

Two energy functions in this collection are identical if and only if the corresponding coefficient autocorrelations are identical. After identical functions are merged, every crossing of two branches is an algebraic number of degree at most N, and there are only finitely many such crossings in K. Between them, F_N is real analytic and the set of minimizing polynomials, modulo monomial factors, is constant.

Proof. Let $T_N(\lambda) = (r_{|i-j|}(\lambda))_{0 \leq i,j \leq N}$. Its quadratic form at a nonzero real vector is the positive integral in equation (1.1), divided by $\mathcal{C}(\lambda)$. The weight is positive almost everywhere, so $T_N(\lambda)$ is positive definite. Continuity and compactness give a uniform lower bound $\nu_K > 0$ for its smallest eigenvalue. The monomial z^N has normalized energy one. Thus every minimizing integer vector satisfies

$$\nu_K \sum_{i=0}^{N} a_i^2 \leq 1.$$

There are only finitely many such vectors with $a_N = 1$. This proves existence of the finite-degree minimum and equation (6.3).

The common denominator

$$\Delta_N(\lambda) = \prod_{h=1}^{N} (\lambda + h)$$

is positive on K, and $\Delta_N R_P$ is an integer polynomial of degree at most N. Differences between distinct branches therefore have at most N zeros, each algebraic of degree at most N. For q distinct branches, their total number of crossings is at most $N\binom{q}{2}$. The endpoints of K need not be algebraic.

Finally, $r_0, \ldots, r_N$ are linearly independent as rational functions. For $N = 0$ this is immediate. For $N \geq 1$, only r_N has a pole at $-N$, with nonzero residue. Successively examining $-N, -N+1, \ldots, -1$ proves independence. The formula in equation (6.2) gives the autocorrelation assertion. On an interval without crossings, the ordering of all distinct branches is constant, proving the final statements. □

6.2 Exact enumeration of the full coefficient lattice

For rational $\lambda > 0$ and $N \geq 1$, the normalized energy is a positive definite rational quadratic form in the free integer coefficients. Completing the square and using a triangular decomposition gives finite bounds at each stage of the enumeration. Write $P = z^N + \sum_{i=0}^{N-1} a_i z^i$, and set

$$T = (r_{|i-j|})_{0 \leq i,j < N}, \qquad v = (r_N, r_{N-1}, \ldots, r_1)^{\mathsf{T}}.$$

Then

$$R_P = a^{\mathsf{T}} T a + 2v^{\mathsf{T}} a + 1 = \mu + (a - c)^{\mathsf{T}} T (a - c), \tag{6.4}$$

where $c = -T^{-1}v$ and $\mu = 1 - v^{\mathsf{T}} T^{-1} v$. Take the exact rational decomposition $T = LDL^{\mathsf{T}}$, with L unit lower triangular and D positive diagonal. Assigning $a_{N-1}, a_{N-2}, \ldots, a_0$ in this order gives

$$R_P = \mu + \sum_{i=0}^{N-1} D_{ii} \left(a_i - c_i + \sum_{j>i} L_{ji}(a_j - c_j) \right)^2. \tag{6.5}$$

Proposition 6.2 (Completeness of the enumeration)**.** *Start with any admissible polynomial and its exact rational energy B. At each step of equation (6.5), enumerate all integers whose contribution does not exceed the remaining bound, and replace B whenever a smaller value is reached. If all branches are exhausted, the final value equals $F_N(\lambda)/(2\pi\mathcal{C}(\lambda))$. Retaining equality branches also determines all minimizers modulo monomial factors.*

Proof. Every diagonal contribution is nonnegative. A branch whose partial sum exceeds the current bound cannot contain an improving point or a tie. Conversely, every vector with value at most the current bound satisfies all partial inequalities and is visited. Positive definiteness makes the initial ellipsoid bounded, so it contains finitely many integer vectors. Bound updates only remove points whose values exceed an already attained value. Complete exhaustion therefore leaves the minimum and all ties. The monomial representation at exact degree N covers every polynomial of degree at most N. □

All branch endpoints are computed without rounding. If a conditional center is p/q in lowest terms and the squared radius is $R \geq 0$, then admissible integers satisfy

$$|qa_i - p| \leq \operatorname{isqrt}(\lfloor q^2 R \rfloor),$$

where isqrt denotes the integer square root. The implementation clears denominators in equation (6.5), so each pruning decision uses only integer arithmetic. At every leaf it independently reevaluates equation (6.2). The search imposes no symmetry or sign restriction on the coefficients.

The program `anc/exact_toeplitz.py` uses the Python standard library. Its output records the rational objective, all minimizing representatives, the recursive node count, the completion status and a hash of the program. A node limit yields only an admissible upper bound. The separate product enumeration visits all nondecreasing tuples $L_1 \leq \cdots \leq L_k$ with $\sum_i (L_i - 1) \leq N$ and evaluates their sparse binomial products exactly.

6.3 Finite-degree values

Table 3 gives the quarter-step grid through six at degree bound 34. The first three intervals follow from the analytic theorems, and the integer rows follow from the displayed ideal pairs. All nine noninteger rows above three were obtained by complete enumeration of the full integer lattice. In those rows a product representative attains the finite-degree minimum. The listed lengths mean $\prod_i D_{L_i-1}$; factors of length one are omitted. The displayed decimals approximate exact values of the form $r\mathcal{C}(\lambda)$, with $r \in \mathbb{Q}$ recorded in the accompanying data.

The larger searches are summarized in table 4. At $7/2$ and $N = 80$, complete enumeration gives a unique minimizer modulo monomial factors, represented by $D_{19}D_{20}D_{40}$ of degree 79, and

$$\frac{F_{80}(7/2)}{2\pi\mathcal{C}(7/2)} = \frac{397421143002128}{2488650531906825}. \tag{6.6}$$

At $9/2$ and $11/2$, the degree-80 runs stopped at 10^7 nodes; their values in the table are upper bounds. These computations do not determine the unrestricted infimum for $\lambda > 3$.

Table 3: Exact minima $F_{34}(\lambda)/(2\pi)$ on the quarter-step grid.

λ	$F_{34}(\lambda)/(2\pi)$	degree	lengths	basis
1/4	1.0787052024	0	$\varnothing$	Interval theorem
1/2	1.2732395447	0	$\varnothing$	Interval theorem
3/4	1.5737874654	0	$\varnothing$	Interval theorem
1	2.0000000000	0	$\varnothing$	Integer theorem
5/4	2.1593372006	34	(35)	Interval theorem
3/2	2.5469988852	34	(35)	Interval theorem
7/4	3.1476575393	34	(35)	Interval theorem
2	4.0000000000	1	(2)	Integer theorem
9/4	3.9032487457	33	(17, 18)	Interval theorem
5/2	4.2519542939	33	(17, 18)	Interval theorem
11/4	4.9479850131	33	(17, 18)	Interval theorem
3	6.0000000000	3	(2, 3)	Integer theorem
13/4	5.6866081883	31	(8, 9, 17)	Enumeration
7/2	5.9830183496	31	(8, 9, 17)	Enumeration
15/4	6.7578167415	31	(8, 9, 17)	Enumeration
4	8.0000000000	7	(2, 3, 5)	Integer theorem
17/4	7.8857045648	34	(5, 6, 11, 16)	Enumeration
9/2	8.2295955039	13	(2, 3, 5, 7)	Enumeration
19/4	8.9152179618	13	(2, 3, 5, 7)	Enumeration
5	10.0000000000	13	(2, 3, 5, 7)	Integer theorem
21/4	10.5879471968	30	(3, 4, 7, 10, 11)	Enumeration
11/2	10.2373568706	30	(3, 4, 7, 10, 11)	Enumeration
23/4	10.7270972270	30	(3, 4, 7, 10, 11)	Enumeration
6	12.0000000000	16	(2, 3, 4, 5, 7)	Integer theorem

Table 4: Larger exact lattice searches. Completed searches determine the minimum; searches stopped at the node limit give upper bounds.

λ	N	energy $\mathcal{E}_\lambda$	nodes	status
17/4	61	7.7625934099	3 894 122	Complete
3/2	80	2.5465761241	1 398	Complete
5/2	80	4.2455882227	149 987	Complete
7/2	80	5.9487961058	2 226 136	Complete
17/4	80	7.7625934099	34 239 539	Complete
9/2	80	8.2295955039	10 000 000	Stopped: upper bound
11/2	80	9.9081780706	10 000 000	Stopped: upper bound

6.4 A nonproduct interval at degree bounds 61 through 80

Set

$$\begin{aligned} Q_*(z) &= 1 - z^9 - z^{10} + z^{31} + z^{34} - z^{55} - z^{56} + z^{65}, \\ P_*(z) &= Q_*(z)/(1-z)^4. \end{aligned} \tag{6.7}$$

Its positive and negative supports $\{0, 31, 34, 65\}$ and $\{9, 10, 55, 56\}$ are symmetric about $65/2$. The identity $65^2 + 3^2 = 47^2 + 45^2$ gives equal second centered moments; symmetry gives the odd moments. Thus Q_* is divisible by $(1-z)^4$ and $P_* \in \mathcal{P}$ has degree 61. The fourth power-sum difference is 1 108 800, so the order of vanishing is exactly four.

Theorem 6.3 (A nonproduct interval from exact enumeration)**.** *For every* $61 \le N \le 80$*,*

$$\frac{F_N(17/4)}{2\pi\mathcal{C}(17/4)} = \frac{32788246355412478151251141810474849551841554248764 16}{40639891549624156371693126611609552341028229338209875}. \tag{6.8}$$

All minimizing polynomials are $z^s P_*$ *with* $0 \le s \le N - 61$*. There is an* $\varepsilon > 0$ *such that the same complete minimizer classification holds for every* $|\lambda - 17/4| < \varepsilon$ *and every* $61 \le N \le 80$*. None of these minimizers is a monomial times a product of four Dirichlet polynomials.*

Proof. Complete enumeration using proposition 6.2 at $N = 61$ visits 3 894 122 nodes and returns P_* as its sole representative. At $N = 80$ it visits 34 239 539 nodes and returns the same sole representative. The rational value is equation (6.8); the records for these runs include the full coefficient vector and report no other minimizer modulo monomial factors. The degree-80 computation also proves the assertion at every intermediate degree, since P_* is admissible from degree 61 onwards.

For the open interval assertion, apply theorem 6.1 on a compact neighborhood of $17/4$. Only finitely many coefficient vectors can minimize. At $17/4$ the exact-degree-80 representative of P_* has strictly smaller energy than every other vector in this finite collection. Continuity preserves these inequalities in a sufficiently small neighborhood. The argument again covers all intermediate degree bounds.

Finally, if Q_* were a product of four binomials, its lowest nonzero coefficients would force the first two lengths to be $9, 10$. The zero coefficient at exponent 19 forces the third length to be 19. The total degree then forces the fourth length to be 27. That product has coefficient -1 at exponent 27, contrary to Q_*. The nonzero constant term also excludes a nontrivial monomial factor in Q_* itself. □

Numerically, $\mathcal{E}_{17/4}(P_*) = 7.7625934099\ldots$. Enumeration of all 92 582 four-factor length tuples with degree bound 80 gives its minimum at $(11, 12, 23, 34)$, with energy $7.8094477516\ldots$. The difference is $0.0468543417\ldots$; its exact value after division by $\mathcal{C}(17/4)$ is recorded in `anc/results/nonproduct_17_4.json`. The Pell construction in theorem 5.4 gives the smaller unrestricted upper bound $F(17/4)/(2\pi) \le 7.3351953762\ldots$. Exact rational comparison gives $\mathcal{E}_{17/4}(P_*)/\mathcal{C}(1/4) > 34/5$. Thus the finite-degree minimum in theorem 6.3 lies strictly above the unrestricted infimum.

7 Further questions

The next analytic problem is to determine whether the explicit constructions on $(3, 4)$ and $(4, 5)$ are sharp. Combining theorem 4.3 with their upper bounds gives

$$8\pi\mathcal{C}(\alpha) \le F(3+\alpha) \le 4\pi\frac{\alpha+3}{\alpha+1}\mathcal{C}(\alpha), \tag{7.1}$$

$$10\pi\mathcal{C}(\alpha) \le F(4+\alpha) \le 4\pi\frac{\alpha+4}{\alpha+1}\mathcal{C}(\alpha), \qquad 0 < \alpha < 1. \tag{7.2}$$

The third-interval proof suggests seeking energy gaps under three or four vanishing moment constraints. Such gaps would have to distinguish the possible equal-moment supports, whose arithmetic is richer than the two-binomial factorization in lemma 2.7.

Question 7.1. Does either upper bound in equations (7.1) and (7.2) hold with equality throughout the corresponding parameter interval? What are the possible shapes of minimizing sequences there?

The Pell family shows that separated clusters of exponents can occur along a sparse arithmetic sequence of degrees. A classification of such limiting configurations may therefore require Diophantine parametrizations in addition to inequalities for the Fourier kernel. At 17/4, the degree-61 example establishes nonproduct minimizers for every degree bound from 61 to 80, whereas the Pell family gives a strictly smaller unrestricted upper bound.

Question 7.2. For which $\lambda > 0$ is $F(\lambda)$ attained, and is F continuous on all of $(0, \infty)$?

The function F is upper semicontinuous, being an infimum of continuous energy functions. The interval formulas settle continuity and characterize attainment on $(0, 3]$, while the integer reduction gives attainment at every positive integer. Lower semicontinuity outside the determined region requires control as the polynomial degree grows.

Question 7.3. Does a multiplicity-free ideal PTE pair of size eleven exist? Equivalently, is $F(11) = 44\pi$ or 48π?

More generally, the arithmetic sequence $\mathcal{M}_m$ is even, nondecreasing and submultiplicative, with $2m \le \mathcal{M}_m \le m(m-1)+2$. Determining its growth would sharpen the interpolation bounds for the full real-parameter problem.

For each fixed degree, theorem 6.1 gives finitely many algebraic transitions on compact parameter intervals. Determining their exact locations is a separate finite problem: once all relevant energy functions are known, their crossings reduce to polynomial root isolation. Quantitative bounds on the size of the relevant coefficient set would make this description effective over whole parameter intervals, and would clarify how the possible minimizers change as the degree bound grows.

Computational material

The accompanying `anc` directory contains the complete programs and exact data used in this paper. Its `README.yaml` specifies the normalization, commands and completion states. The PTE and product-exclusion identities are checked by `verify_pte.py`; the Pell identities by `verify_pell.py`; the full coefficient-lattice and restricted-product enumerations by `exact_toeplitz.py` and `exact_products.py`; and the table consistency by `verify_results.py`. The critical-point decimals are calculated separately by `critical_point.py`. The existence and uniqueness of that critical point follow from proposition 2.9 independently of its numerical approximation.

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