Finite Field Grothendieck Inequality, Finite Field Johnson-Lindenstrauss Flattening and Finite Field Bourgain-Tzafriri Restricted Invertibility Problems

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Abstract: We ask for finite field versions of following three: (1) Grothendieck Inequality, (2) Johnson-Lindenstrauss Flattening Lemma, (3) Bourgain-Tzafriri Restricted Invertibility Theorem.
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1. Finite Field Grothendieck Inequality Problem

The Grothendieck inequality was derived in the following form by Lindenstrauss and Pelczynski in 1968 [11].

Theorem 1.1. *[1, 2, 5, 8, 9, 11, 13, 14] (**Grothendieck Inequality** or **The Fundamental Theorem in the Metric Theory of Tensor Products**) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. There is a universal constant $K_G^{\mathbb{K}} > 0$ satisfying the following: For every Hilbert space $\mathcal{H}$ over $\mathbb{K}$ and for all $m, n \in \mathbb{N}$, if a scalar matrix $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n}$ satisfies*

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}s_jt_k\right| \leq \left(\max_{1\leq j\leq m}|s_j|\right)\left(\max_{1\leq k\leq n}|t_k|\right), \quad \forall s_j, t_k \in \mathbb{K},$$

then

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}\langle u_j, v_k\rangle\right| \leq K_G^{\mathbb{K}}\left(\max_{1\leq j\leq m}\|u_j\|\right)\left(\max_{1\leq k\leq n}\|v_k\|\right), \quad \forall u_j, v_k \in \mathcal{H}.$$

Further,

$$1.67 \leq K_G^{\mathbb{R}} \leq 1.79, \quad 1.33 \leq K_G^{\mathbb{C}} \leq 1.41.$$

We formulate problem which is finite field analogue of Theorem 1.1. Let p be a prime, $\mathbb{Z}_p := \{0, 1, \ldots, p-1\}$ be the standard field of integers modulo p. Given $n \in \mathbb{Z}_p$, we define

$$|n| := \text{ unique } a \text{ such that } 0 \leq a \leq p-1 \text{ and } n \equiv a (\text{mod } p).$$

Since p is a prime, we have the following:

(i) If $n \in \mathbb{Z}_p$ is such that $|n| = 0$, then $n = 0$.

(ii) $|mn| \leq |m||n|$ for all $m, n \in \mathbb{Z}_p$.
(iii) $|m+n| \leq |m| + |n|$ for all $m, n \in \mathbb{Z}_p$.

For $d \in \mathbb{N}$, let $\mathbb{Z}_p^d$ be the standard vector space over $\mathbb{Z}_p$. We define the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^d a_j b_j \in \mathbb{Z}_p, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Z}_p^d$$

and the norm

$$\|(c_j)_{j=1}^d\|^2 := \sum_{j=1}^d |c_j|^2 \in \{0, 1, \ldots, d(p-1)^2\}, \quad \forall (c_j)_{j=1}^d \in \mathbb{Z}_p^d.$$

Note that

$$\sum_{j=1}^d |c_j^2| \leq \sum_{j=1}^d |c_j|^2 = \|(c_j)_{j=1}^d\|^2, \quad \forall (c_j)_{j=1}^d \in \mathbb{Z}_p^d.$$

Inner product and norm will have following properties:

(i) If $x \in \mathbb{Z}_p^d$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathbb{Z}_p^d$, then $x = 0$.
(ii) $\langle nx, y \rangle = n\langle x, y \rangle$ for all $n \in \mathbb{Z}_p$, for all $x, y \in \mathbb{Z}_p^d$.
(iii) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathbb{Z}_p^d$.
(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in \mathbb{Z}_p^d$.
(v) $|\langle x, y \rangle| \leq \|x\|\|y\|$ for all $x, y \in \mathbb{Z}_p^d$.
(vi) If $x \in \mathbb{Z}_p^d$ is such that $\|x\| = 0$, then $x = 0$.
(vii) $\|x\|^2 \in \mathbb{N} \cup \{0\}$ for all $x \in \mathbb{Z}_p^d$.
(viii) $\|nx\| \leq |n|\|x\|$ for all $n \in \mathbb{Z}_p$, for all $x \in \mathbb{Z}_p^d$.
(ix) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{Z}_p^d$.

Based on Theorem 1.1, we formulate following problem.

Problem 1.2. ***(Finite Field Grothendieck Inequality Problem)*** *Let p be a prime. Whether there is a universal constant $K_p > 0$ (which may depend upon p) satisfying the following property: for every $d \in \mathbb{N}$, for all $m, n \in \mathbb{N}$, if $[a_{j,k}]_{1 \leq j \leq m, 1 \leq k \leq n} \in \mathbb{M}_{m \times n}(\mathbb{Z}_p)$ satisfies*

$$\left| \sum_{j=1}^m \sum_{k=1}^n a_{j,k} s_j t_k \right| \leq \left(\max_{1 \leq j \leq m} |s_j| \right) \left(\max_{1 \leq k \leq n} |t_k| \right), \quad \forall s_j, t_k \in \mathbb{Z}_p,$$

then

$$\left| \sum_{j=1}^m \sum_{k=1}^n a_{j,k} \langle \mathbf{u}_j, \mathbf{v}_k \rangle \right| \leq K_p \left(\max_{1 \leq j \leq m} \|\mathbf{u}_j\| \right) \left(\max_{1 \leq k \leq n} \|\mathbf{v}_k\| \right), \quad \forall \mathbf{u}_j, \mathbf{v}_k \in \mathbb{Z}_p^d.$$

2. Finite Field Johnson-Lindenstrauss Flattening Problem

Johnson and Lindenstrauss showed that points in high dimensional Euclidean space can be mapped to low dimensional Euclidean space with small distortion to distances between every pair of points, using Lipschitz maps [10].

Theorem 2.1. *[10,12]* ***(Johnson-Lindenstrauss Flattening Lemma)*** *Let $M, N \in \mathbb{N}$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M \in \mathbb{R}^N$. For every $0 < \varepsilon < 1$, there exists a Lipschitz map $f : \mathbb{R}^N \to \mathbb{R}^m$ and a real number $r > 0$ such that*

$$r(1-\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\| \leq \|f(\mathbf{x}_j) - f(\mathbf{x}_k)\| \leq r(1+\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|, \quad \forall 1 \leq j, k \leq M,$$

where

$$m = O\left(\frac{\log M}{\varepsilon^2}\right).$$

Over the period, some improvements of Theorem 2.1 have been obtained. We recall them.

Theorem 2.2. *[7, 10]* ***(Johnson-Lindenstrauss Flattening Lemma: Frankl-Maehara form)*** *Let $0 < \varepsilon < \frac{1}{2}$ and $M \in \mathbb{N}$. Define*

$$m(\varepsilon, M) \coloneqq \left\lceil 9\frac{1}{\varepsilon^2 - \frac{2\varepsilon^3}{3}} \log M \right\rceil + 1.$$

If $M > m(\varepsilon, M)$, then for every $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M \in \mathbb{R}^M$, there exists a map $f : \{\mathbf{x}_j\}_{j=1}^M \to \mathbb{R}^m$ such that

$$(1-\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|^2 \leq \|f(\mathbf{x}_j) - f(\mathbf{x}_k)\|^2 \leq (1+\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|^2, \quad \forall 1 \leq j, k \leq M.$$

Theorem 2.3. *[4, 10]* ***(Johnson-Lindenstrauss Flattening Lemma: Dasgupta-Gupta form)*** *Let $M, N \in \mathbb{N}$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M \in \mathbb{R}^N$. Let $0 < \varepsilon < 1$. Choose any natural number m such that*

$$m > 4\frac{1}{\frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3}} \log M.$$

Then there exists a map $f : \mathbb{R}^N \to \mathbb{R}^m$ such that

$$(1-\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|^2 \leq \|f(\mathbf{x}_j) - f(\mathbf{x}_k)\|^2 \leq (1+\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|^2, \quad \forall 1 \leq j, k \leq M.$$

The map f can be determined in randomized polynomial time.

Theorem 2.4. *[6, 10]* ***(Johnson-Lindenstrauss Flattening Lemma: matrix form)*** *There is a universal constant $C > 0$ satisfying the following. Let $0 < \varepsilon < 1$, $M, N \in \mathbb{N}$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M \in \mathbb{R}^N$. For each natural number*

$$m > \frac{C}{\varepsilon^2} \log M,$$

there exists a matrix $M \in \mathbb{M}_{m \times N}(\mathbb{R})$ such that

$$(1-\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|^2 \leq \|M(\mathbf{x}_j - \mathbf{x}_k)\|^2 \leq (1+\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|^2, \quad \forall 1 \leq j, k \leq M.$$

Based on Theorem 2.4, we formulate following problem.

Problem 2.5. ***(Finite Field Johnson-Lindenstrauss Flattening Problem)*** *Let p be a prime. What is the best function $\phi : (0,1) \times \mathbb{N} \to (0, \infty)$ satisfying the following: There is a universal constant $C_p > 0$ (which may depend upon p) satisfying the following. Let $0 < \varepsilon < 1$, $M, N \in \mathbb{N}$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M \in \mathbb{Z}_p^N$. For each natural number*

$$m > C_p \phi(\varepsilon, M),$$

there exists a matrix $M \in \mathbb{M}_{m \times N}(\mathbb{Z}_p)$ such that

$$(1-\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\| \leq \|M(\mathbf{x}_j - \mathbf{x}_k)\| \leq (1+\varepsilon)\|\mathbf{x}_j - \mathbf{x}_k\|, \quad \forall 1 \leq j, k \leq M.$$

3. Finite Field Bourgain-Tzafriri Restricted Invertibility Problem

Let $\mathbb{R}^d$ be the standard d-dimensional Euclidean space with canonical orthonormal basis $\{\mathbf{e}_j\}_{j=1}^d$. In 1987, Bourgain and Tzafriri showed that every matrix can be inverted restrictively with control over the norm [3].

Theorem 3.1. *[3, 17] **(Bourgain-Tzafriri Restricted Invertibility Theorem)** There are universal constants $A > 0$, $c > 0$ satisfying the following property. If $d \in \mathbb{N}$, and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear operator with $\|T\mathbf{e}_j\| = 1$, $\forall 1 \leq j \leq d$, then there exists a subset $\sigma \subseteq \{1, \dots, d\}$ of cardinality*

$$\text{Card}(\sigma) \geq \frac{cd}{\|T\|^2}$$

such that

$$\left\| T\left(\sum_{j\in\sigma} a_j \mathbf{e}_j \right) \right\|^2 \geq A \sum_{j\in\sigma} |a_j|^2, \quad \forall a_j \in \mathbb{R}, \forall j \in \sigma.$$

It is clear that we can state Theorem 3.1 also in following way.

Theorem 3.2. *[3, 17] **(Bourgain-Tzafriri Restricted Invertibility Theorem)** There are universal constants $A > 0$, $c > 0$ satisfying the following property. If $d \in \mathbb{N}$, and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear operator with*

$$\|T\mathbf{e}_1\|^2 = \cdots = \|T\mathbf{e}_d\|^2 =: r \quad \textit{(say)},$$

then there exists a subset $\sigma \subseteq \{1, \dots, d\}$ of cardinality

$$\text{Card}(\sigma) \geq \frac{rcd}{\|T\|^2}$$

such that

$$\left\| T\left(\sum_{j\in\sigma} a_j \mathbf{e}_j \right) \right\|^2 \geq rA \sum_{j\in\sigma} |a_j|^2, \quad \forall a_j \in \mathbb{R}, \forall j \in \sigma.$$

Spielman and Srivastava gave a simple proof of Theorem 3.1 by proving a generalization of Theorem 3.1 [15] due to Vershynin [16]. Based on Theorem 3.1, we formulate finite field problem. Given a linear operator $T : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$, we define

$$\|T\| := \max\left\{ \frac{\|T\mathbf{x}\|}{\|\mathbf{x}\|} : \mathbf{x} \in \mathbb{Z}_p^d, \mathbf{x} \neq 0 \right\}.$$

Problem 3.3. ***(Finite Field Bourgain-Tzafriri Restricted Invertibility Problem)** Let p be a prime. Whether there are universal constants $A_p > 0$, $c_p > 0$ (which may depend upon p) satisfying the following property: If $d \in \mathbb{N}$, and $T : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ is a linear operator with $\|T\mathbf{e}_j\| = 1$, $\forall 1 \leq j \leq d$, then there exists a subset $\sigma \subseteq \{1, \dots, d\}$ of cardinality*

$$\text{Card}(\sigma) \geq \frac{c_p d}{\|T\|^2}$$

such that

$$\left\| T\left(\sum_{j\in\sigma} a_j \mathbf{e}_j \right) \right\|^2 \geq A_p \sum_{j\in\sigma} |a_j|^2, \quad \forall a_j \in \mathbb{Z}_p, \forall j \in \sigma.$$

Problem 3.4. ***(Finite Field Bourgain-Tzafriri Restricted Invertibility Problem)** Let p be a prime. Whether there are universal constants $A_p > 0$, $c_p > 0$ (which may depend upon p) satisfying the following property: If $d \in \mathbb{N}$, and $T : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ is a linear operator with*

$$\|T\mathbf{e}_1\|^2 = \cdots = \|T\mathbf{e}_d\|^2 =: r \quad \textit{(say)},$$

then there exists a subset $\sigma \subseteq \{1, \ldots, d\}$ *of cardinality*

$$\mathrm{Card}(\sigma) \geq \frac{rc_p d}{\|T\|^2}$$

such that

$$\left\| T\left(\sum_{j\in\sigma} a_j \mathbf{e}_j \right) \right\|^2 \geq rA_p \sum_{j\in\sigma} |a_j|^2, \quad \forall a_j \in \mathbb{Z}_p, \forall j \in \sigma.$$

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