Acyclic Networks: Causal Comparisons, Convex Minors, and Gauge Descent

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Abstract

We study natural transformations, minors, and symmetric monoidal decorations of finite directed acyclic networks. For any weakly connected small source category, a natural transformation into a free acyclic path category is determined by one component. If more than one transformation exists, common prefixes and suffixes give a canonical factorization through two constant functors and identify the transformation set with a single set of directed paths. Consequences include boundary rigidity, a decision algorithm, and a fundamental-group obstruction to non-uniqueness. For fusion–embedding minors, right fillings exist exactly when the graph pushout is acyclic. A support invariant detects a nonexact minor under the full gauge congruence, while path-convex embeddings admit two-sided fillings. Inverse-image pullbacks give canonical partial-map normal forms, a scanning algorithm, and a coherent presentation of the cartesian quotient. For open networks, wire splicing gives units, and diagram groupoids satisfy descent by homotopy pullback over the interface groupoid, including through-wires. This descent is invariant under strong symmetric monoidal equivalence and compatible with admissible event contractions within each piece. Finite-field examples yield orbit classifications, stabilizer weights, and boundary convolution formulas. Roof-refinement spaces realize spheres whose higher homotopy is lost under groupoid completion.

Keywords. Acyclic category; causal-net; natural transformation; convex embedding; minor calculus; gauge groupoid; homotopy pullback; tensor network.

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1 Introduction and main results

A finite directed acyclic network has a free path category recording its directed compositions. Natural transformations compare functors between such categories, while quotients and embeddings describe changes to the network itself. Decorating wires by objects and events by morphisms in a symmetric monoidal category adds a further question: how do diagram groupoids behave when networks are glued along an interface? We classify non-unique comparisons, identify obstructions to two-sided minor normal forms, and prove a descent law for these diagram groupoids.

Write $\mathrm{P}(G)$ for the free path category of a finite directed acyclic multigraph G. The category Cau has these graphs as objects and path functors as morphisms. A fusion is a graph homomorphism surjective on vertices and edges; an embedding identifies a graph with a subgraph. An embedding is path-convex when every ambient directed path between its vertices lies in that subgraph. Formal minors have forward embeddings and backward quotients. Sections 2–6 give the definitions and proofs of the following results.

Theorem A (Classification of causal comparisons). Let C be a nonempty weakly connected small category, let H be a finite acyclic graph, and let $F, F' : C \to \mathrm{P}(H)$. Every component evaluation is injective on $\mathrm{Nat}(F, F')$. If this set contains at least two elements, there are canonical natural transformations

$$p : F \Rightarrow \Delta d, \qquad q : \Delta c \Rightarrow F'$$

for vertices d, c of H, inducing a bijection

$$\mathrm{P}(H)(d, c) \xrightarrow{\cong} \mathrm{Nat}(F, F'), \qquad w \longmapsto q \circ \Delta(w) \circ p.$$

Consequently both classifying-space maps BF and BF' are null-homotopic whenever comparisons are non-unique. If either induced fundamental-group map is nontrivial, there is at most one comparison.

The injectivity part is a special case of a categorical characterization: evaluation on every weakly connected source is faithful exactly when every target arrow is both monic and epic (Theorem 2.4). The constant-factorization assertion uses the free path structure (Theorem 2.9); a product of two free acyclic categories shows that cancellation alone does not imply it. For a finite causal-net source, if each component contains a marked vertex and comparisons are identities there, the comparison category is a finite poset. A spanning forest algorithm decides the existence of a comparison between two given functors by prefix and suffix cancellation.

Theorem B (Minor obstruction and convex repair). Let $A \xleftarrow{q} H \xhookrightarrow{i} C$ be a fusion–embedding corner. It admits a fusion–embedding right filling if and only if the pushout $A \sqcup_H C$ in finite directed multigraphs is acyclic. There is a fusion minor from the directed path of length two to the directed path of length three which has no sub–quotient representative, even under the full seven-operation gauge congruence.

For path-convex embeddings, both left and right fillings always exist. Thus $(\mathsf{FU}, \mathsf{Emb}_{\mathrm{cvx}})$ is balanced and every minor for this pair has both a quotient–sub and a sub–quotient normal form.

The right-filling criterion is Theorem 3.4. The nonexact minor is detected by a two-edge support invariant (Theorem 3.15), which controls arbitrary sequences of gauge operations. Convexity excludes the ambient path excursions that could create a cycle in the pushout (Theorem 3.8).

Theorem C (Canonical normalization and presentations). Pullbacks of arbitrary causal-net functors along graph embeddings are inverse-image subgraphs. For coarse-grainings or fusions, using these pullbacks gives a canonical roof $A \xleftarrow{\nu_T} D_T \hookrightarrow B$ for every formal minor T, unique up to its unique apex isomorphism among cartesian normal forms with fixed endpoint maps. The cartesian word quotient is the corresponding category of partial maps in the reverse direction. Allowing all commutative fillings, while retaining nontrivial automorphism factors and identity endpoint maps in word isomorphisms, gives the category of connected components of roof-refinement posets. The full gauge congruence is a further quotient by automorphism-erasure relations at every intermediate object.

For a word of $n \geq 1$ coarse-graining–embedding factors with intermediate graphs of size at most $M \geq 1$, a right-to-left scan computes the canonical roof in $O(nM)$ elementary table operations. Contextual moves with Peiffer interchange define a free strict $(2,1)$-category; explicit normalization relations give a coherent presentation of the cartesian quotient.

The cartesian normal form, complexity bound, and quotient comparisons are proved in Theorems 4.6, 4.9, and 4.10. The two-dimensional construction and its normalization relations are given in Theorem 4.14 and Proposition 4.16. Canonical-roof uniqueness belongs to the cartesian quotient; general fillings and automorphism erasure impose the additional identifications stated in Theorem C.

Theorem D (Open composition and gauge descent). Let $\mathcal{S}$ be a small symmetric monoidal category. Acyclic port networks compose by splicing interface wires, with event-free identity networks. For $G : A \to B$ and $H : B \to C$, let $\mathcal{D}_{\mathcal{S}}(G)$ be the groupoid of diagrams and all edge isomorphisms satisfying the vertex equations, and put $\mathcal{B}_{\mathcal{S}}(B) = (\mathcal{S}^{\simeq})^{B}$. Cutting the interface induces an equivalence

$$\mathcal{D}_{\mathcal{S}}(H \circ_B G) \;\simeq\; \mathcal{D}_{\mathcal{S}}(G) \times^{h}_{\mathcal{B}_{\mathcal{S}}(B)} \mathcal{D}_{\mathcal{S}}(H).$$

These equivalences respect external boundaries, iterated composition, and disjoint union. They are invariant under strong symmetric monoidal equivalences of coefficient categories. Evaluation of admissible event blocks defines contraction functors; contractions performed within the two pieces are compatible with this descent law.

The port model and its units are constructed in Theorem 5.7; the descent and contraction statements are Theorems 6.7 and 6.15. The boundary analysis also identifies the cartesian morphisms of the leaf-open category and gives an acyclicity criterion for gluing along arbitrary injective vertex interfaces.

1.1 Context and scope

Lu's causal-net category and its quotient–embedding calculus [Lu23] supply the combinatorial setting. The path-functor comparison theory developed here uses pointwise category theory [Mac98], word cancellation, and the graph interpretation of free-category classifying spaces [Cit01]. The factorization theorem identifies each transformation set containing at least two elements with a single target path set. The support invariant and the pushout criterion locate an obstruction in the unrestricted fusion calculus, while convex embeddings give a stable class with two-sided normalization.

Partial-map categories are studied in [CL02]. Here the inverse-image model separates canonical normalization from refinement and from automorphism erasure. The higher presentation gives interchange equations and a possibly infinite family of normalization relations. Critical-branching methods in higher-dimensional rewriting provide a way to seek finite homotopy bases [GM12].

The open-network model is compatible with the graphical calculus of symmetric monoidal categories [JS91] and with the study of open graphs [DK13]. Structured and decorated cospans provide general approaches to compositional systems [Fon15, BC20, BCV22]. The edge-isomorphism gauge relation is already present in causal-network condensation [LT22]. We retain the gauge transformations as groupoid morphisms and prove the interface equivalence, its boundary transport, and its compatibility with event evaluation. These results produce symmetric monoidal span semantics and explicit finite-groupoid convolution weights [BHW10].

Section 7 computes the finite-field gauge orbits and stabilizers of a state–costate network, including the homotopy-fiber weights of scalar contraction. It also realizes the standard $\mathrm{Hom}(K_2, K_n)$ spheres [BK06] as roof-refinement spaces. These examples distinguish gauge isotropy from the higher topology of refinement. Section 8 formulates the remaining higher-homotopy and semantic-extension questions.

1.2 Conventions

All graphs are finite directed multigraphs unless stated otherwise; parallel edges, isolated vertices, and empty graphs are permitted. Acyclicity excludes every nonempty directed closed walk. Categories and groupoids are small in a fixed universe, with a larger universe available for categories of such objects. Categorical composition is written right to left. A category is weakly connected only if it is nonempty. Tensor expressions carry specified orders and bracketings, and their canonical transports use symmetric monoidal coherence. Quotients by relations, groupoid equivalences, and homotopy fibers are distinguished throughout by their definitions.

2 Acyclic categories and causal comparisons

The comparison theory depends on three properties of the target category. Acyclicity excludes nonidentity endomorphisms and comparisons in both directions between distinct functors. Over a weakly connected source, cancellation determines a comparison from any one of its components. When comparisons into a free acyclic target are non-unique, common prefixes and suffixes identify the transformation set with a single set of paths.

2.1 Acyclic and cancellative targets

Definition 2.1. A small category D is *acyclic* if every endomorphism is an identity and the existence of arrows $a \to b$ and $b \to a$ implies $a = b$. It is *cancellative* if every arrow is both a monomorphism and an epimorphism. A category is *weakly connected* if it is nonempty and any two objects are joined by a finite zigzag of arrows. A finite category has finitely many objects and morphisms. □

A *causal-net* is a finite acyclic directed multigraph; parallel edges and isolated vertices are allowed. We write $\mathrm{P}(G)$ for its free path category. Its morphisms are directed edge words, including the empty word at each vertex. A directed path cannot repeat a vertex, so $\mathrm{P}(G)$ is finite and acyclic. Cancellation of edge words makes it cancellative. We use the usual order for categorical composition. When manipulating edge words, we write $p \cdot q$ for the path that first traverses p and then q; thus $p \cdot q = q \circ p$.

The category Cau has causal-nets as objects and functors between their path categories as morphisms [Lu23]. The functor $G \mapsto \mathrm{P}(G)$ identifies it, up to equivalence, with the full subcategory of finite free acyclic categories: the generating graph of a free category is recovered from its indecomposable nonidentity arrows. Admitting natural transformations gives the full strict 2-subcategory of Cat on these path categories, with

$$\mathsf{Cau}_2(G,H) = [\mathrm{P}(G), \mathrm{P}(H)].$$

The functor-category operations and their interchange law are inherited from Cat [Mac98].

Proposition 2.2. For every small category C and every acyclic category D, the functor category $[C,D]$ is acyclic. If C and D are finite, then $[C,D]$ is finite. If D is cancellative, then $[C,D]$ is cancellative.

Proof. An endomorphism $\eta : F \Rightarrow F$ has components $\eta_x : F(x) \to F(x)$. Acyclicity of D makes every component an identity, so η is the identity transformation. If there are transformations $\eta : F \Rightarrow F'$ and $\theta : F' \Rightarrow F$, then acyclicity gives $F(x) = F'(x)$ and $\eta_x = \theta_x = 1_{F(x)}$ for every x. Naturality at an arbitrary arrow $f : x \to y$ consequently gives $F(f) = F'(f)$, and hence $F = F'$. This proves acyclicity.

When C and D are finite, there are only finitely many assignments of objects and arrows defining functors, and finitely many families of components defining natural transformations. The functor category is therefore finite. Finally, an equality between composites of natural transformations is an equality at every object. If every arrow of D is monic and epic, cancellation at each component makes every arrow of $[C,D]$ monic and epic. □

Corollary 2.3. The full subcategory of Cat on finite acyclic categories is cartesian closed. Its full subcategory on finite cancellative acyclic categories is also cartesian closed. In both cases, products and exponentials are the ordinary categorical products and functor categories.

Proof. The terminal category belongs to both classes. A finite product is finite, and acyclicity and cancellation hold componentwise. Proposition 2.2 proves closure under functor categories. The usual currying bijection

$$\mathsf{Cat}(A \times C, D) \cong \mathsf{Cat}(A, [C,D])$$

therefore restricts to each full subcategory. □

The finite free objects are not closed under these products. For example, $[1] \times [1]$ is the four-object square poset, whose two length-two routes agree. Its four indecomposable arrows freely generate a category with two distinct such routes. The distinction between free and cancellative targets will reappear in Example 2.11.

2.2 Evaluation and fixed boundary data

Theorem 2.4 (Evaluation and cancellation). For a small category D, the following conditions are equivalent.

1. Every arrow of D is both monic and epic.
2. For every weakly connected small category C and every object $r \in C$, the evaluation functor
$$\mathrm{ev}_r : [C, D] \longrightarrow D$$
is faithful.
3. Both evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 : [[1], D] \to D$ are faithful.

In particular, if D is cancellative and C is weakly connected, then

$$\mathrm{Nat}(F, F') \longrightarrow D(F(r), F'(r)), \qquad \eta \longmapsto \eta_r, \tag{2.1}$$

is injective for every $F, F' : C \to D$.

Proof. Suppose every arrow of D is monic and epic, and let $\eta, \theta : F \Rightarrow F'$ have equal components at r. For an arrow $f : x \to y$ in C, naturality gives

$$F'(f) \circ \eta_x = \eta_y \circ F(f), \qquad F'(f) \circ \theta_x = \theta_y \circ F(f).$$

If $\eta_x = \theta_x$, epimorphy of $F(f)$ gives $\eta_y = \theta_y$. If $\eta_y = \theta_y$, monomorphy of $F'(f)$ gives $\eta_x = \theta_x$. Equality propagates along any zigzag from r, proving faithfulness. The implication from the second condition to the third is immediate.

Assume the third condition. Let $f : a \to b$ and let $u, v : b \to c$ satisfy $u \circ f = v \circ f$. Regard f as a functor $[1] \to D$. The pairs $(u \circ f, u)$ and $(v \circ f, v)$ are natural transformations from this functor to the constant functor at c. They agree at 0, so faithfulness of ev_0 gives $u = v$. Thus f is epic. For monomorphy, let $u, v : z \to a$ satisfy $f \circ u = f \circ v$. The pairs $(u, f \circ u)$ and $(v, f \circ v)$ are transformations from the constant functor at z to the functor f. They agree at 1, so faithfulness of ev_1 gives $u = v$. Thus f is monic as well. □

Let G be a causal-net, let $B \subseteq V(G)$ be a set of marked vertices, and fix an object assignment $b : B \to \mathrm{Ob}(D)$. We write $\mathrm{Fun}_b(\mathrm{P}(G), D)$ for the category whose objects are functors F with $F(x) = b(x)$ for $x \in B$, and whose morphisms are natural transformations satisfying $\eta_x = 1_{b(x)}$ for $x \in B$. Marked vertices need not be leaves for the following result.

Corollary 2.5 (Boundary rigidity). Suppose D is a finite cancellative acyclic category and B meets every weakly connected component of G. Then $\mathrm{Fun}_b(\mathrm{P}(G), D)$ is a finite poset, regarded as a category.

Proof. Two parallel transformations in this category agree at a marked vertex of each component. Theorem 2.4, applied componentwise, makes them equal. Thus the category is thin. It is finite and acyclic by Proposition 2.2 and by restriction of its morphisms. A thin acyclic category is a poset. □

The marking hypothesis cannot be omitted uniformly. For a single unmarked source vertex and a target with two parallel edges $a \rightrightarrows b$, the two constant functors at a and b have two distinct comparisons.

Proposition 2.6 (Deciding a boundary comparison). Let G, H be causal-nets, let $B \subseteq V(G)$ meet every weakly connected component, and let $F, F' : \mathrm{P}(G) \to \mathrm{P}(H)$ agree on B. Suppose that the image of each generating edge is given as an explicit list of edges of H. There is an algorithm that decides whether a transformation $F \Rightarrow F'$ with identity components on B exists and returns it when it does. If $h(H)$ is the maximum length of a directed path in H, with $h(\varnothing) = 0$, its running time in edge-word operations is

$$O(|V(H)| + |E(H)| + (|V(G)| + |E(G)|)(h(H) + 1)). \tag{2.2}$$

In particular, the time is polynomial in the explicit input size.

Proof. The empty source has one transformation and is dealt with immediately. For each nonempty component, choose a marked root r, choose an undirected spanning tree, and set $\eta_r = 1_{F(r)}$. Traverse the tree. For an edge $e : x \to y$, the required equation in edge-word notation is

$$F(e) \cdot \eta_y = \eta_x \cdot F'(e). \tag{2.3}$$

If η_x is known, test whether $F(e)$ is a prefix of $\eta_x \cdot F'(e)$. If it is, delete that prefix and assign the remaining path to η_y; otherwise reject. If the tree traversal reaches the edge from y, test whether $F'(e)$ is a suffix of $F(e) \cdot \eta_y$, delete it, and assign the remaining path to η_x; reject if the suffix test fails. Empty paths are included in these operations, with their specified endpoints.

Every assignment is forced by (2.3). Consequently, a failure excludes any transformation. After assigning all vertices, check (2.3) on each non-tree edge and check that the component at every marked vertex is an identity. If these tests succeed, the equations hold on all generators and therefore on all paths, so the constructed family is natural. The procedure decides existence. Uniqueness follows from Corollary 2.5.

Spanning forests and a topological ordering of H can be constructed in linear time. Each component path, when it exists, has length at most $h(H)$, and every concatenation or prefix or suffix test above uses at most $2h(H)$ edge symbols. There are linearly many tree steps, remaining-edge tests, and boundary tests. This proves (2.2); the term $h(H) + 1$ includes graphs having no edges. Since $h(H) \le |V(H)| - 1$ for a nonempty target, the stated polynomial bound follows. The algorithm concerns two given functors and does not enumerate the functor category. □

2.3 Positive transport in free groupoids

For a directed graph G, let $\Pi(G)$ be its free groupoid. Its arrows are reduced words in edges and formal inverse edges; reduction deletes adjacent inverse pairs. The path category embeds faithfully in $\Pi(G)$, because a positive edge word is already reduced. A functor $F : \mathrm{P}(G) \to \mathrm{P}(H)$ extends uniquely to $\overline{F} : \Pi(G) \to \Pi(H)$.

Proposition 2.7 (Transport and monodromy). Let G, H be causal-nets with G weakly connected. Fix a root r and an undirected spanning tree of G, and let $t_x : r \to x$ denote the corresponding tree word in $\Pi(G)$. For every non-tree edge $e : x \to y$, put

$$\ell_e = t_y^{-1} \circ e \circ t_x \in \Pi(G)(r, r).$$

For $F, F' : \mathrm{P}(G) \to \mathrm{P}(H)$, evaluation at r identifies $\mathrm{Nat}(F, F')$ with the paths $w \in \mathrm{P}(H)(F(r), F'(r))$ satisfying the following two conditions:

1. for every vertex x, the groupoid arrow

$$\eta_x(w) = \overline{F}'(t_x) \circ w \circ \overline{F}(t_x)^{-1} \tag{2.4}$$

   belongs to the positive path subcategory $\mathrm{P}(H)$;

2. for every non-tree edge e,

$$\overline{F}'(\ell_e) \circ w = w \circ \overline{F}(\ell_e). \tag{2.5}$$

There are $|E(G)| - |V(G)| + 1$ equations in the second condition.

Proof. If $\eta : F \Rightarrow F'$ is natural, its naturality equation can be solved for an inverse source edge after passing to $\Pi(H)$. It therefore holds for every signed edge word. In particular, naturality along t_x gives (2.4) with $w = \eta_r$, and naturality along ℓ_e gives (2.5). The components are positive by assumption.

Conversely, suppose w satisfies the two conditions and define the positive components by (2.4). For a tree edge $e : x \to y$, the identity $t_y = e \circ t_x$ holds in $\Pi(G)$, irrespective of the direction in which the tree was traversed. Formula (2.4) then gives $F'(e) \circ \eta_x = \eta_y \circ F(e)$ in $\Pi(H)$. For a non-tree edge, substitute $\ell_e = t_y^{-1} \circ e \circ t_x$ into (2.5). Multiplying on the left by $\overline{F}'(t_y)$ and on the right by $\overline{F}(t_x)^{-1}$ yields the same naturality equation. All arrows in that equation are positive, and the embedding $\mathrm{P}(H) \to \Pi(H)$ is faithful. Thus the equation holds in $\mathrm{P}(H)$. Naturality on generators proves naturality on all paths. Evaluation recovers w because t_r is empty, proving the claimed bijection. A spanning tree has $|V(G)| - 1$ edges, which gives the final count. □

2.4 The factorization of non-unique comparisons

The factorization rests on a property of parallel paths in an acyclic graph: their common prefix and suffix cannot overlap unless all the paths are equal. The length of an edge word p is written $|p|$.

Lemma 2.8. Let $\mathcal{R}$ be a finite set containing at least two distinct directed paths with the same endpoints in an acyclic graph H. Let p be their longest common prefix and q their longest common suffix. Then

1. no two distinct members of $\mathcal{R}$ are comparable by the prefix relation or by the suffix relation;
2. for every $u \in \mathcal{R}$, one has $|p| + |q| < |u|$, so $u = p \cdot w_u \cdot q$ for a uniquely determined nonempty path w_u;
3. adjoining a common suffix does not change the longest common prefix, and adjoining a common prefix does not change the longest common suffix.

Adjoining a prefix to every member adjoins that prefix to the longest common prefix; the corresponding assertion holds for suffixes.

Proof. If $v = u \cdot a$ and u, v have the same endpoints, then a is a directed loop. Acyclicity makes a empty. The suffix assertion is proved in the same way, giving the first claim.

For the second claim, suppose that a member u of length L has $|p| + |q| \geq L$. Put $k = |p|$ and $l = |q|$. The initial vertex c of q occurs at position $L - l \leq k$ in u, and hence occurs at that position in the common prefix p. In any member v of length M, the same vertex c consequently occurs at position $L - l$ in its prefix p and at position $M - l$ as the initial vertex of its suffix q. A directed path in an acyclic graph cannot visit a vertex twice, so $M - l = L - l$ and $M = L$. The prefix and suffix cover all L edge positions; since both are shared, $v = u$. This holds for every $v \in \mathcal{R}$, contradicting the existence of two distinct members. Thus $k + l < L$, and the intermediate word exists uniquely.

By the first claim, the longest common prefix ends before the end of every member. At least two members have different next edges there. Adjoining a suffix cannot alter that first disagreement. The argument for a common suffix is the reverse one. Finally, prepending the same word shifts the first disagreement by its length, and appending the same word shifts the last disagreement in the analogous way. These observations prove the remaining claims. □

Theorem 2.9 (Factorization of non-unique comparisons). Let C be a weakly connected small category, let H be a causal-net, and let $F, F' : C \to \mathrm{P}(H)$. If $\mathrm{Nat}(F, F')$ has at least two elements, then there are vertices $d, c \in H$ and natural transformations

$$p : F \Rightarrow \Delta d, \qquad q : \Delta c \Rightarrow F'$$

such that

$$\mathrm{P}(H)(d, c) \xrightarrow{\cong} \mathrm{Nat}(F, F'), \qquad w \longmapsto q \circ \Delta(w) \circ p \tag{2.6}$$

is a bijection. Here Δ denotes the constant-functor embedding. The components p_x and q_x can be chosen canonically as the longest common prefix and longest common suffix, respectively, of all components η_x with $\eta \in \mathrm{Nat}(F, F')$.

Proof. We extract the common prefix and suffix at each object, use naturality to identify their endpoints across the source, and then cancel them to obtain a path independent of the source object.

Put $T = \mathrm{Nat}(F, F')$. Evaluation at any object embeds T into a finite hom-set of $\mathrm{P}(H)$ by Theorem 2.4; hence T is finite even when C is infinite. The same theorem shows that distinct transformations have distinct components at every object. Thus, for each x, the set $\{\eta_x : \eta \in T\}$ contains at least two distinct parallel paths. Let p_x, q_x be its longest common prefix and suffix.

For an arrow $f : x \to y$ in C, all transformations satisfy

$$F(f) \cdot \eta_y = \eta_x \cdot F'(f). \tag{2.7}$$

Take the longest common prefix of the two families in this equality. Lemma 2.8 gives

$$F(f) \cdot p_y = p_x. \tag{2.8}$$

Taking the longest common suffix similarly gives

$$q_y = q_x \cdot F'(f). \tag{2.9}$$

The endpoints of p_x, p_y therefore agree, as do the initial vertices of q_x, q_y. Weak connectedness supplies common vertices d, c such that

$$p_x : F(x) \longrightarrow d, \qquad q_x : c \longrightarrow F'(x)$$

for every x. Equations (2.8) and (2.9) are precisely the naturality conditions for $p : F \Rightarrow \Delta d$ and $q : \Delta c \Rightarrow F'$.

Lemma 2.8 also shows that these prefix and suffix words are disjoint inside every component, with a nonempty intermediate path. Write, uniquely,

$$\eta_x = p_x \cdot w_{\eta,x} \cdot q_x, \qquad w_{\eta,x} : d \longrightarrow c.$$

Substituting this expression in (2.7) and using (2.8) and (2.9) yields

$$p_x \cdot w_{\eta,y} \cdot q_y = p_x \cdot w_{\eta,x} \cdot q_y.$$

Cancellation gives $w_{\eta,x} = w_{\eta,y}$. Weak connectedness makes this path independent of x; denote it by w_η. Thus every η has the asserted form, with a unique intermediate path.

Conversely, for any directed path $w : d \to c$, the composites $p_x \cdot w \cdot q_x$ are defined. The two transport equations give

$$F(f) \cdot p_y \cdot w \cdot q_y = p_x \cdot w \cdot q_x \cdot F'(f),$$

so these composites define a natural transformation. Distinct intermediate paths give distinct components by cancellation. This proves the bijection and its stated canonical construction. □

Corollary 2.10 (Topological rigidity). Under the hypotheses of Theorem 2.9, the existence of two distinct transformations $F \Rightarrow F'$ implies that both maps

$$BF, BF' : BC \longrightarrow B\mathrm{P}(H)$$

are null-homotopic. Consequently, if either induced homomorphism on fundamental groups is nontrivial, then $|\mathrm{Nat}(F, F')| \leq 1$.

Proof. A natural transformation determines a functor from $C \times [1]$ to its target and hence a homotopy between the induced classifying-space maps. Apply this construction to $p : F \Rightarrow \Delta d$ and $q : \Delta c \Rightarrow F'$. The classifying-space map of a constant functor factors through a point, so both maps are null-homotopic. A null-homotopic map induces the trivial homomorphism on the fundamental group; this property is unchanged by the basepoint conjugation associated with a homotopy. □

For $C = \mathrm{P}(G)$, the homotopy equivalences $B\mathrm{P}(G) \simeq |G|$ and $B\mathrm{P}(H) \simeq |H|$ identify the obstruction with the induced map on the fundamental groups of the underlying graphs [Cit01, Example 4.3]. Proposition 2.7 expresses these maps using finitely many reduced loop words.

Example 2.11 (A cancellative target with relations). Let A be the graph with two parallel edges $a_0 \rightrightarrows a_1$, and let B have two parallel edges $u, v : b_0 \rightrightarrows b_1$. The category

$$D = \mathrm{P}(A) \times \mathrm{P}(B)$$

is finite, cancellative, and acyclic. Define $F, F' : \mathrm{P}(A) \to D$ by

$$F(x) = (x, b_0), \quad F(f) = (f, 1_{b_0}), \qquad F'(x) = (x, b_1), \quad F'(f) = (f, 1_{b_1}).$$

The families $(1_x, u)_x$ and $(1_x, v)_x$ are distinct natural transformations $F \Rightarrow F'$. There are exactly two, by evaluation at either source vertex.

Projection onto the first factor composes with either functor to the identity of $\mathrm{P}(A)$. Since $B\mathrm{P}(A) \simeq S^1$, both induced fundamental-group homomorphisms are nontrivial. Thus the factorization and topological rigidity conclusions require more than acyclicity and cancellation. The commuting squares in D supply the additional relations; indeed $BD \simeq S^1 \times S^1$. ♡

3 Minor obstructions and convex embeddings

Inverse images give quotient–sub factorizations of formal minors. The reverse ordering requires extending a fusion across an embedding, which can create directed cycles. A graph pushout characterizes when such an extension exists, and path-convexity guarantees its existence. We also construct a minor for which no sequence of gauge transformations produces the reverse normal form.

3.1 Classes of causal-net morphisms

Write $V(G)$ and $E(G)$ for the vertex and edge sets of a finite directed multigraph G. We distinguish the following classes of morphisms in Cau.

Definition 3.1. A morphism $q : G \to H$ is a *coarse-graining* if it is surjective on vertices, maps every generating edge to either a generating edge or an identity, and every generating edge of H is the image of a generating edge of G. It is a *fusion* if, in addition, no generating edge is mapped to an identity. These classes are denoted by CG and FU, respectively.

A morphism $i : G \to H$ is an *embedding* if it maps generating edges to generating edges and is injective on both vertices and edges. Its image is then a directed subgraph of H. We write Emb for this class.

An embedding $i : G \hookrightarrow H$ is *path-convex* if every directed path in H whose endpoints belong to $i(V(G))$ is a path in the image subgraph $i(G)$. The class of path-convex embeddings is denoted by $\mathsf{Emb}_{\mathrm{cvx}}$. □

These definitions of coarse-graining, fusion, and embedding agree with [Lu23, Sections 2.1, 2.6, and 3.4]. Path-convexity includes paths of length one. In particular, a path-convex subgraph contains every ambient edge between its vertices, as well as every intermediate vertex and edge on an ambient path between them. Equivalently, an embedding i is path-convex precisely when $\mathrm{P}(i)$ is a full functor.

Lemma 3.2. The classes CG, FU, Emb, and $\mathsf{Emb}_{\mathrm{cvx}}$ contain all isomorphisms and are closed under composition. A fusion preserves the length of every directed path. If $q : G \to H$ is a coarse-graining, then

$$|V(H)| \le |V(G)|, \qquad |E(H)| \le |E(G)|.$$

If $i : G \hookrightarrow H$ is an embedding, the opposite inequalities hold.

Proof. Surjectivity on vertices and the prescribed images of generating edges are preserved by composition. For the edge-surjectivity of a composite rq, choose first a generating edge mapping under r to the desired target edge, and then a generating edge mapping to it under q. A composite of fusions contracts no edge. Injectivity on vertices and edges is also preserved by composition.

Suppose that $G \hookrightarrow H \hookrightarrow K$ are path-convex embeddings. A path in K with endpoints in G first belongs to H by convexity of $H \hookrightarrow K$, and then belongs to G by convexity of $G \hookrightarrow H$. Thus their composite is path-convex. A fusion replaces each generating edge by exactly one generating edge, so it preserves path length. The cardinality inequalities follow from the surjectivity or injectivity in Definition 3.1. □

3.2 Inverse images and the obstruction to a right filling

Let $\mathcal{Q}$ and $\mathcal{E}$ be composition-closed classes in a category, containing all isomorphisms. A *left filling* of a cospan $A \xrightarrow{i} B \xleftarrow{q} C$, with $i \in \mathcal{E}$ and $q \in \mathcal{Q}$, is a commutative square

$$\begin{array}{ccc} D & \xrightarrow{j} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[i]{} & B \end{array} \tag{3.1}$$

with $p \in \mathcal{Q}$ and $j \in \mathcal{E}$. A *right filling* of $A \xleftarrow{q} H \xrightarrow{i} C$ is a commutative square

$$\begin{array}{ccc} H & \xrightarrow{i} & C \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ A & \xrightarrow[j]{} & P \end{array} \tag{3.2}$$

with $r \in \mathcal{Q}$ and $j \in \mathcal{E}$. The pair is *left-dominating* or *right-dominating* when the corresponding filling exists for every corner, and is *balanced* when both conditions hold.

Proposition 3.3. Both $(\mathsf{CG}, \mathsf{Emb})$ and $(\mathsf{FU}, \mathsf{Emb})$ are left-dominating. Their left fillings can be chosen to be pullbacks in Cau. If the given embedding is path-convex, the embedding in the filling is path-convex as well.

Proof. After identifying A with its image in B, define a subgraph $D \subseteq C$ by

$$V(D) = q^{-1}(V(A)), \qquad E(D) = \{e \in E(C) : q(e) \text{ is a morphism of } \mathrm{P}(A)\}. \tag{3.3}$$

For a coarse-graining, the second condition retains every edge mapping to an edge of A, together with every contracted edge mapping to an identity at a vertex of A. The restriction $p : D \to A$ is surjective on vertices and generating edges, because q is. It is therefore a coarse-graining, and it is a fusion when q is a fusion. The inclusion $D \hookrightarrow C$ provides the other side of (3.1).

If A is path-convex in B and γ is a path in C with endpoints in D, then $q(\gamma)$ is a path in A. Concatenation in a free path category has no cancellation. Hence the image of every edge of γ belongs to $\mathrm{P}(A)$, so γ belongs to D. Thus D is path-convex in C. The pullback universal property, valid for arbitrary causal-net functors in place of q, is proved in Proposition 4.1. □

Let $\mathsf{Digraph}_{\mathrm{fin}}$ be the category of finite directed multigraphs and graph homomorphisms. Its objects may have loops and directed cycles; a graph homomorphism sends an edge to an edge. Pushouts are obtained by taking the corresponding pushouts of vertex sets and edge sets and using the induced source and target maps.

Theorem 3.4 (Right-filling criterion). Let $q : H \to A$ be a fusion of causal-nets and $i : H \hookrightarrow C$ an embedding. Form the pushout

$$P = A \sqcup_H C$$

in $\mathsf{Digraph}_{\mathrm{fin}}$. The following are equivalent:

1. The corner $A \xleftarrow{q} H \xrightarrow{i} C$ admits a right filling with a fusion $C \to D$ and an embedding $A \hookrightarrow D$.
2. The graph P has no directed cycle.

When these conditions hold, the pushout square itself is such a filling, and it is also a pushout in Cau.

Proof. Identify H with its image in C. Since q is surjective on vertices and edges, every vertex and edge of P has a representative in C. Explicitly, its vertex set is obtained from $V(C)$ by identifying vertices of H having the same q-image, and its edge set is obtained from $E(C)$ by identifying edges of H having the same q-image. All vertices and edges outside H remain distinct. The induced map $j : A \to P$ is injective on vertices and edges, while $r : C \to P$ is surjective on both. Both are graph homomorphisms.

If P is acyclic, these statements say exactly that j is an embedding and r is a fusion. Conversely, suppose that a filling with target D exists. Its two maps are graph homomorphisms, so the pushout property gives a graph homomorphism $P \to D$. Every nonempty directed closed walk in P would map to a nonempty directed closed walk in D, since graph homomorphisms do not contract edges. As D is acyclic, P is acyclic.

Finally, suppose P is acyclic and let $A \to X$ and $C \to X$ be causal-net morphisms agreeing on H. Regard their edge assignments as graph homomorphisms to the underlying directed graph of $\mathrm{P}(X)$, whose edges are all morphisms, including identities. The graph pushout gives a unique such assignment on P. Freeness then extends it uniquely to a functor $\mathrm{P}(P) \to \mathrm{P}(X)$. This is the pushout property in Cau. □

Example 3.5. Let H consist of two isolated vertices x, z, let $q : H \to A$ merge them to a single vertex, and let C be the single-edge graph $x \to z$. The map q is a fusion and $H \hookrightarrow C$ is an embedding. Their graph pushout has one vertex and one loop. Consequently this corner has no fusion–embedding right filling. In particular, $(\mathsf{FU}, \mathsf{Emb})$ is not balanced. ♡

Remark 3.6. The assertion that $(\mathsf{FU}, \mathsf{Emb})$ is balanced appears in [Lu23, Section 4.4], using its Lemma 4.2.7. Example 3.5 exhibits the obstruction to that lemma: merging vertices after an ambient edge has been added can produce a directed cycle. Theorem 3.4 specifies exactly when that construction remains in the causal-net category. ♢

3.3 Convex embeddings and two-sided fillings

Lemma 3.7. Under the hypotheses of Theorem 3.4, suppose that $H \hookrightarrow C$ is path-convex. Then $P = A \sqcup_H C$ is acyclic, and $A \hookrightarrow P$ is path-convex.

Proof. Use the explicit description of P in the proof of Theorem 3.4. Vertices outside the image of A correspond bijectively to vertices of $C \setminus H$. An edge of P outside A has a unique representative in $C \setminus H$.

Consider a directed path segment in P whose endpoints belong to A and whose internal vertices lie outside A. If the segment has any edge outside A, its consecutive outside vertices and edges lift uniquely to C, and its endpoints lift to vertices of H. It therefore gives a path in C between vertices of H. Path-convexity forces all its edges to belong to H, a contradiction. This also covers a single outside edge between two vertices of A. Splitting a general path with endpoints in A at its successive visits to A shows that every such path lies wholly in A. Thus $A \hookrightarrow P$ is path-convex.

A directed closed walk in P avoiding A would lift to a directed closed walk in C. One meeting A lies entirely in A by the preceding argument. Both possibilities contradict the acyclicity of C and A. Hence P is acyclic. □

Theorem 3.8. The pair $(\mathsf{FU}, \mathsf{Emb}_{\mathrm{cvx}})$ is balanced. Its left fillings can be chosen as inverse-image pullbacks, and its right fillings can be chosen as graph pushouts. In both cases the resulting embedding is path-convex.

Proof. Left fillings and their convexity are given by Proposition 3.3. Lemma 3.7 and Theorem 3.4 give the right fillings and their convexity. □

3.4 Formal minors and their two normal forms

Definition 3.9. Let $\mathcal{Q}$ and $\mathcal{E}$ be composition-closed classes in a category, containing all isomorphisms. A *formal minor* from A to B is a finite word of oriented factors

$$T = (A = X_0 \overset{\alpha_1}{\longleftrightarrow} X_1 \overset{\alpha_2}{\longleftrightarrow} \cdots \overset{\alpha_n}{\longleftrightarrow} X_n = B),$$

where a forward factor $X_{k-1} \to X_k$ belongs to $\mathcal{E}$ and a backward factor $X_k \to X_{k-1}$ belongs to $\mathcal{Q}$. The orientation is part of the data, including for isomorphisms. Words compose by concatenation, and the empty word is the identity. □

We use the following elementary gauge operations, in the terminology of [Lu23, Section 4.1].

1. A *dual transformation* interchanges the two boundary routes around a commutative square of the form (3.1).
2. A *flip* replaces an isomorphism by its inverse with the opposite orientation.
3. A *composable reduction* replaces adjacent factors of the same orientation by their composite.
4. A *factor decomposition* is the inverse of a composable reduction.
5. An *automorphism deletion* removes a factor which is an automorphism of its object.
6. An *automorphism insertion* is the inverse operation.
7. An *isomorphic transformation* replaces a word by a word of the same orientation pattern through a family of vertex isomorphisms making all factor squares commute.

For fixed endpoints, the last operation allows endpoint automorphisms. The *fixed-endpoint gauge congruence* is the congruence generated by these operations in arbitrary composable contexts. Thus replacing a subword by a gauge-equivalent subword is allowed. The congruence is closed under contexts by definition, even when a contextual instance requires several elementary operations. Section 4.6 records these instances as derivations.

Definition 3.10. A *quotient–sub normal form*, or *roof*, is a minor

$$A \overset{q}{\leftarrow} D \overset{i}{\rightarrow} B, \qquad q \in \mathcal{Q}, \quad i \in \mathcal{E}.$$

A *sub–quotient normal form* is a minor $A \overset{i}{\rightarrow} N \overset{q}{\leftarrow} B$ with the same class requirements. Identity factors may be omitted. A minor is *exact* if its gauge class contains normal forms of both kinds. □

Theorem 3.11. If $(\mathcal{Q}, \mathcal{E})$ is left-dominating, every formal minor has a quotient–sub normal form. If it is right-dominating, every formal minor has a sub–quotient normal form. These transformations use only dual transformations, composable reductions, and the insertion or deletion of identity factors. In particular, every minor for a balanced pair is exact.

Proof. For a word T of length n, let

$$\operatorname{inv}(T) = \#\{(r,s) : r < s,\ \alpha_r \text{ is forward and } \alpha_s \text{ is backward}\}.$$

If this number is positive, the orientation word contains an adjacent forward–backward pair. A left filling replaces this pair by a backward–forward pair. Its replacement removes exactly one inversion: all other factors retain their order relative to both replaced positions. After $\operatorname{inv}(T)$ such replacements, all backward factors precede all forward factors. Compose each of these two blocks. An empty block is represented by an omitted identity, giving a roof.

For the right-dominating statement, use the number of backward factors occurring before forward factors and replace adjacent backward–forward pairs by forward–backward pairs. The same argument gives a sub–quotient normal form. The permitted operations are all gauge operations, so the final assertion follows.□

Corollary 3.12. Every $\mathsf{CG/Emb}$ minor and every $\mathsf{FU/Emb}$ minor has a quotient–sub normal form. Every $\mathsf{FU/Emb}_{\mathrm{cvx}}$ minor has both normal forms, and hence is exact.

Proof. Apply Theorem 3.11 to Proposition 3.3 and Theorem 3.8. □

3.5 A nonexact fusion minor

Let P_m be the causal-net consisting of a single directed path of length m, with vertices $0, \dots, m$ and one edge from $k-1$ to k for each k. Its only automorphism is the identity.

Lemma 3.13. Every fusion with source P_m is an isomorphism.

Proof. Every two distinct vertices of P_m are joined in one direction by a nonempty directed path. If a fusion identified them, this path would map to a nonempty directed closed walk, because a fusion preserves path length. Thus the vertex map is injective and, by surjectivity, bijective. Distinct edges then have distinct ordered endpoint pairs, so the edge map is also injective and bijective. The map is an isomorphism of directed graphs. □

Fix $A = P_2$ and $B = P_3$, and denote the three edges of B in order by e_1, e_2, e_3.

Lemma 3.14. Every formal $\mathsf{FU/Emb}$ minor $T : A \rightsquigarrow B$ determines a two-element subset

$$I(T) \subseteq \{e_1, e_2, e_3\}.$$

This subset is invariant under the fixed-endpoint gauge congruence.

Proof. We distinguish the two-edge and three-edge stages of the word. Every three-edge object is uniquely isomorphic to the target, so the transition between these stages selects two target edges. We then check that each gauge operation preserves this selection.

The number of edges is nondecreasing along a formal minor, by Lemma 3.2. Thus every intermediate object has two or three edges. If an intermediate object X has three edges, normalize the suffix from X to B by Corollary 3.12:

$$X \xleftarrow{q} D \hookrightarrow B, \qquad q \in \mathsf{FU}.$$

The graph D must contain all three edges of B, and hence all its vertices, so $D = B$. Lemma 3.13 implies that $X \cong B$. The isomorphism is unique, because B is rigid.

Consider the first factor at which the edge count increases from two to three. It cannot be a backward fusion: that would be a fusion from a graph isomorphic to P_3 onto a graph with two edges. It is therefore an embedding. Identify its three-edge target with B by the unique isomorphism and define $I(T)$ to be the image of its two edges. All subsequent objects are isomorphic to B, and every subsequent factor is an isomorphism. This makes the definition independent of the chosen description of the terminal part of the word.

We verify invariance under the generators of the congruence. Within the two-edge part of a word, every fusion or embedding is a bijection on edges. Within the three-edge part, every factor is an isomorphism. Consequently composable reductions, factor decompositions, flips, automorphism insertions and deletions, and isomorphic transformations preserve the two-element image. This reasoning also applies to these operations in a larger context: endpoint automorphisms in a two-edge stage permute its whole two-element edge set, and endpoint automorphisms in a three-edge stage are identities after the unique identification with B.

It remains to check a dual transformation. If its two boundary endpoints both have two edges, or both have three edges, the entire square lies within the corresponding stage and cannot change I. Suppose its left endpoint X has two edges and its right endpoint Z has three. Write the two routes as

$$X \xrightarrow{e} Y \xleftarrow{q} Z, \qquad X \xleftarrow{p} W \xrightarrow{j} Z, \qquad qj = ep.$$

The graph Y has three edges: otherwise the fusion q would contradict Lemma 3.13. Hence q is an isomorphism. The graph W has two edges: if it had three, the fusion p would give the same contradiction. The map p is a bijection on edges, and the equation $qj = ep$ identifies the two-edge images of j and e under the unique identifications of Y and Z with B. Thus the dual transformation preserves I. All generating operations, in every context, preserve it. □

Theorem 3.15 (A nonexact fusion minor). Let $H \subseteq B = P_3$ be the subgraph obtained by deleting the middle edge e_2 and retaining all four vertices. Let $q : H \to A = P_2$ map e_1 and e_3 to the first and second edges of A, respectively, and identify the two middle vertices of B. Then q is a fusion, and

$$T = \left(A \xleftarrow{q} H \hookrightarrow B\right)$$

is not exact, even when all seven gauge operations are allowed.

Proof. The specified map is surjective on vertices and edges and contracts no edge, so it is a fusion. Its support invariant is $I(T) = \{e_1, e_3\}$.

Suppose T were gauge equivalent to a sub–quotient normal form

$$A \xrightarrow{j} N \xleftarrow{r} B, \qquad j \in \mathsf{Emb}, \quad r \in \mathsf{FU}.$$

Lemma 3.13 makes r an isomorphism. Under the resulting identification $N = B$, an embedding of P_2 into P_3 uses two consecutive edges. Thus this normal form has support either $\{e_1, e_2\}$ or $\{e_2, e_3\}$. Both contradict Lemma 3.14. □

The support invariant rules out every gauge-equivalent sub–quotient form, extending the local obstruction of Example 3.5. It therefore disproves the claim in [Lu23, Section 4.4] that all fusion minors are exact for unrestricted embeddings. Indeed, if endpoint-changing isomorphic transformations are allowed along a gauge sequence, every source remains isomorphic to P_2 and every target remains isomorphic to P_3. The rigidity of these two paths gives unique identifications of all endpoints with the fixed A and B. Transporting the sequence along these identifications produces fixed-endpoint operations, to which the support invariant applies. The path-convex hypothesis in Corollary 3.12 supplies a class for which the two normal forms coexist.

4 Canonical normalization and coherent presentations

Inverse-image fillings interpret formal minors as partial maps in the opposite direction. This interpretation determines a canonical roof and a normalization algorithm. General commutative fillings relate partial maps by domain refinement. We describe the resulting quotient categories and present their relations in strict $(2,1)$-categories.

4.1 Pullbacks along graph embeddings

An embedding identifies its source with a subgraph of its target. We will use this identification when taking inverse images. For a functor $F : \mathrm{P}(C) \to \mathrm{P}(B)$ and a subgraph $A \subseteq B$, define

$$\begin{aligned} V(F^{-1}A) &= \{x \in V(C) : F(x) \in V(A)\}, \\ E(F^{-1}A) &= \{e \in E(C) : F(e) \in \mathrm{Mor}(\mathrm{P}(A))\}. \end{aligned} \tag{4.1}$$

The edge condition includes edges mapped to identities at vertices of A. Every retained edge has both endpoints in the specified vertex set, so (4.1) defines a subgraph.

Proposition 4.1 (Inverse-image pullback). For every causal-net morphism $F : C \to B$ and embedding $i : A \hookrightarrow B$, the inverse-image graph $D = F^{-1}(i(A))$ is a causal-net. Restriction and inclusion give a pullback square

$$\begin{array}{ccc} D & \xhookrightarrow{j} & C \\ {\scriptstyle F_A}\downarrow & & \downarrow{\scriptstyle F} \\ A & \xhookrightarrow[i]{} & B \end{array} \tag{4.2}$$

in Cau. In fact,

$$\mathrm{P}(D) \cong \mathrm{P}(A) \times_{\mathrm{P}(B)} \mathrm{P}(C)$$

in Cat. If F is a coarse-graining or a fusion, then F_A belongs to the same class. If i is path-convex, then j is path-convex.

Proof. It suffices to treat i as the inclusion of A in B. The graph D is a subgraph of the finite acyclic graph C, so it is finite and acyclic. The restriction F_A is defined on vertices and edges by (4.1) and extends to paths by concatenation.

A path $\gamma = e_m \cdots e_1$ in C has $F(\gamma)$ in $\mathrm{P}(A)$ if and only if γ is a path in D. For an identity path this is the vertex condition in (4.1). For a nonempty path, the reverse implication follows by composition. For the forward implication, write each $F(e_k)$ as a possibly empty word of edges of B. Their concatenation belongs to $\mathrm{P}(A)$. Since no letters can cancel, every nonempty word $F(e_k)$ uses only edges of A. Its intermediate objects, and the objects supporting any empty words, are vertices of that same path in A. Thus every edge and vertex of γ belongs to D.

The objects of the categorical pullback are pairs (a, c) with $i(a) = F(c)$. Since i is injective on vertices, these pairs are in bijection with $V(D)$. A morphism in the pullback is a pair of paths (α, γ) with $i(\alpha) = F(\gamma)$. The path α is unique when it exists, and the preceding characterization says exactly that γ belongs to $\mathrm{P}(D)$. These bijections preserve identities and concatenation, proving the displayed isomorphism and the pullback property in Cau.

For a coarse-graining, every vertex of A has a preimage vertex in D, and every generating edge of A has a preimage generating edge in D. Its restriction maps edges to edges or identities. It is therefore a coarse-graining. The restriction contracts no edge when F is a fusion. Finally, a path in C between vertices of D has an image path whose endpoints are in A. If A is path-convex, this image lies in $\mathrm{P}(A)$; the same characterization makes the original path a path in D. This proves path-convexity of j. □

Lemma 4.2. For causal-net morphisms $G : D \to C$ and $F : C \to B$ and a subgraph $A \subseteq B$, one has an equality of subgraphs

$$(FG)^{-1}(A) = G^{-1}(F^{-1}(A)).$$

Proof. The equality on vertices is immediate. For an edge e of D, the condition $F(G(e)) \in \mathrm{P}(A)$ is equivalent, by the path characterization in the preceding proof, to $G(e) \in \mathrm{P}(F^{-1}(A))$. These are precisely the two edge membership conditions. □

4.2 Partial maps and refinement

Let $\mathcal{Q}$ be a composition-closed class of causal-net morphisms containing all isomorphisms and stable under the pullbacks of Proposition 4.1. Both CG and FU satisfy these conditions. The class of all causal-net morphisms is another example.

Definition 4.3. For causal-nets A, B, let $\mathcal{R}_{\mathcal{Q}}(A, B)$ have as objects pairs

$$(D, f), \qquad D \subseteq B, \qquad f : D \to A \text{ in } \mathcal{Q}.$$

Set $(D, f) \leq (E, g)$ if $D \subseteq E$ and $g|_D = f$. The pair represents the roof $A \xleftarrow{f} D \hookrightarrow B$. Its partial-map direction is from B to A, opposite to the direction $A \rightsquigarrow B$ of the formal minor. □

The relation in Definition 4.3 is a partial order: reflexivity and transitivity follow from restriction, and mutual extension forces equality of both domains and maps.

Proposition 4.4. The posets $\mathcal{R}_{\mathcal{Q}}(A,B)$ are the hom-categories of a strict 2-category $\mathsf{Ref}_{\mathcal{Q}}$. Its objects are causal-nets. If $T=(D,f):A\rightsquigarrow B$ and $S=(E,g):B\rightsquigarrow C$, their composite is

$$S\circ T=\big(g^{-1}(D),\ f\circ g|_{g^{-1}(D)}\big), \tag{4.3}$$

where $g^{-1}(D)$ is regarded as a subgraph of C. The identity at A is (A,id_A).

Proof. Pullback stability gives $g|_{g^{-1}(D)}:g^{-1}(D)\to D$ in $\mathcal{Q}$. Closure under composition then makes (4.3) a valid object of $\mathcal{R}_{\mathcal{Q}}(A,C)$.

For three composable pairs, the two iterated domains agree by Lemma 4.2, applied also to the relevant restrictions. On that common domain, both maps are the same composite of the three functors. Thus composition is strictly associative. Inverse image under an identity, and inverse image of the entire target, give the two strict unit laws.

Suppose $(D,f)\le(D',f')$ and $(E,g)\le(E',g')$. The defining restrictions imply

$$g^{-1}(D)\subseteq(g')^{-1}(D'),$$

and the two composite maps agree on the smaller domain. Hence composition is monotone in the two variables jointly. It therefore defines a functor between the product hom-poset and the target hom-poset. The strict associativity and unit equalities already proved are equalities of these functors. This gives a strict 2-category; its interchange law also follows from uniqueness of morphisms in a poset. □

Write $\mathsf{PM}_{\mathcal{Q}}$ for the underlying ordinary category obtained by retaining all the pairs (D,f) as distinct morphisms and forgetting the order relations. This is the opposite-direction partial-map construction for graph embeddings, restricted to maps in $\mathcal{Q}$; partial-map categories and their restriction structure are studied in [CL02].

Proposition 4.5. Every roof $A\xleftarrow{q}M\xrightarrow{i}B$, with $q\in\mathcal{Q}$ and $i\in\mathsf{Emb}$, determines the unique pair

$$\big(i(M),\ q\circ(i:M\xrightarrow{\sim}i(M))^{-1}\big). \tag{4.4}$$

Two roofs determine the same pair if and only if there is an apex isomorphism commuting with both legs, and that isomorphism is unique. More generally, maps of apices commuting with both legs correspond to the order relations of $\mathcal{R}_{\mathcal{Q}}(A,B)$.

Proof. An embedding is an isomorphism onto its image subgraph, so (4.4) is defined and its map belongs to $\mathcal{Q}$. An apex isomorphism commuting with both legs preserves the image subgraph and its map to A. Conversely, equality of the two image pairs gives the unique apex isomorphism obtained by passing through their common image in B.

If $u:M\to N$ is any causal-net morphism satisfying $ju=i$ for two embeddings $i:M\hookrightarrow B$ and $j:N\hookrightarrow B$, then u is itself an embedding. Indeed, j preserves path length, while i sends each edge to an edge, so u sends each edge to an edge; injectivity on vertices and edges follows from the injectivity of i. Such a map exists precisely when $i(M)\subseteq j(N)$. The remaining equality of the left legs is precisely agreement of the two partial maps on that subgraph. Uniqueness follows from the injectivity of j. □

In particular, an apex automorphism preserving the right leg of a roof is the identity. The topology of the nerve of the refinement poset records additional information from its noninvertible morphisms; an explicit family appears in Proposition 7.4.

4.3 Cartesian normalization and its uniqueness

For the following constructions, one may work in a small concrete model of Cau closed under subgraphs, for example with finite vertex and edge sets contained in fixed countable sets.

Let $\mathsf{W}_{\mathcal{Q},\mathsf{Emb}}$ denote the category of formal minor words. Interpret a forward embedding $i:A\hookrightarrow B$ and a backward factor $q:B\to A$ by

$$\begin{aligned}\mathcal{N}(i^{+})&=(i(A),i^{-1}),\\ \mathcal{N}(q^{-})&=(B,q).\end{aligned} \tag{4.5}$$

The superscripts record the word orientation; q^- is a formal symbol, not an inverse functor. Concatenation and Proposition 4.4 extend these assignments uniquely to a functor

$$\mathcal{N} : \mathsf{W}_{\mathcal{Q},\mathsf{Emb}} \longrightarrow \mathsf{PM}_{\mathcal{Q}}.$$

Write $\mathcal{N}(T) = (D_T, \nu_T)$.

A *cartesian dual transformation* is a dual transformation whose square is a pullback. In the *cartesian congruence*, we allow cartesian dual transformations, composable reductions and their inverses, insertion and deletion of identity factors, flips, and isomorphic transformations whose two endpoint isomorphisms are identities. Nontrivial automorphism factors are retained in this congruence.

Theorem 4.6 (Canonical cartesian normal form). Every formal $\mathcal{Q}/\mathsf{Emb}$ minor $T : A \rightsquigarrow B$ is cartesian equivalent to

$$A \xleftarrow{\nu_T} D_T \hookrightarrow B. \tag{4.6}$$

The following additional statements hold.

1. Every cartesian gauge operation preserves $\mathcal{N}(T)$.
2. Orient cartesian dual transformations from a forward–backward pair to a backward–forward pair, and orient composable reductions and identity deletions toward shorter words. Every reduction sequence terminates. Any two resulting roofs admit a unique apex isomorphism commuting with their two legs, after omitted identity legs are restored.
3. Two formal minors are cartesian equivalent if and only if they have the same image under $\mathcal{N}$.

Proof. For a cartesian square (4.2), the partial map interpreting the route $A \hookrightarrow B \xleftarrow{F} C$ has domain $F^{-1}(A)$ and map F_A. This is exactly the pair interpreting the other route $A \xleftarrow{F_A} D \hookrightarrow C$. The two words therefore have the same interpretation. Every other pullback filling is uniquely isomorphic to this one over both legs and determines the same image pair. Composing two forward factors amounts to composing their partial inverses, and composing two backward factors amounts to composing their total maps. Units and flips also preserve (4.5).

For an isomorphism of words with vertex isomorphisms $u_0, \dots, u_n$, each factor interpreted in the opposite partial-map direction is conjugated by u_{k-1} on its target and u_k^{-1} on its source. These intermediate conjugations cancel in the composite. When $u_0 = u_n = \mathrm{id}$, the resulting partial map has the same domain and the same values. All these equalities are preserved under concatenation, because $\mathcal{N}$ is a functor. This proves the first statement.

For termination, use the lexicographically ordered pair $(\mathrm{inv}(T), \ell(T))$, where $\ell(T)$ is word length and inv is defined in the proof of Theorem 3.11. A leftward cartesian swap decreases the first entry by one. A composable reduction or identity deletion never increases the first entry and strictly decreases the second. Thus no infinite reduction sequence exists.

Every irreducible orientation word consists of a backward block followed by a forward block, each containing at most one factor. Pullback stability supplies each required swap, so the argument of Theorem 3.11 gives a roof $A \xleftarrow{q} M \xrightarrow{i} B$. Its interpretation is (4.4); since interpretation is unchanged, that pair is (D_T, ν_T). Transporting the apex along its embedding onto its image identifies this roof with (4.6). This transport is an isomorphic transformation with identity endpoint maps. Proposition 4.5 gives the asserted uniqueness of comparison isomorphisms between all resulting roofs.

Finally, cartesian equivalence implies equality of interpretations by the first statement. If two interpretations are equal, normalize both words to (4.6). The resulting roofs are the same after the indicated apex transports, so the original words are cartesian equivalent. □

The unique comparison isomorphisms in Theorem 4.6 commute in every diagram of normalized roofs: two composites with the same source and target are maps of apices preserving both legs, hence equal. This supplies the associativity and unit coherence of the roof description. Formal derivation paths are retained separately by the presentations in Section 4.6.

Proposition 4.7 (General fillings as refinements). Let $A \xrightarrow{i} B \xleftarrow{q} C$ be a corner, and let $A \xleftarrow{p} D \xrightarrow{j} C$ be any commuting filling with $p \in \mathcal{Q}$ and $j \in \mathsf{Emb}$. There is a unique map from this roof to the cartesian filling, commuting with both legs. It is an embedding. Consequently the pair interpreted by the given roof is a refinement of the pair interpreted by the original corner.

Proof. The pullback property gives a unique map $u : D \to q^{-1}(i(A))$. Its composite with the inclusion into C is j, so Proposition 4.5 makes u an embedding. The equality of the left legs gives the required restriction equality. □

Example 4.8. For discrete causal-nets, let A be a singleton, let C have two vertices, and let $q : C \to A$ be the unique map. The corner $A \xrightarrow{\mathrm{id}} A \xleftarrow{q} C$ has the cartesian filling $A \xleftarrow{q} C \xrightarrow{\mathrm{id}} C$. Choosing either vertex of C gives a filling whose apex has one vertex instead of two. Allowing all commutative fillings therefore identifies partial maps with different domains; cartesian normalization retains the full inverse-image domain. ♡

4.4 An explicit normalization algorithm

The partial-map description computes the normal roof by scanning the word from right to left. Assume that $\mathcal{Q}$ is CG or FU. For a word $T = (X_0, \alpha_1, X_1, \ldots, \alpha_n, X_n)$, put $B = X_n$ and initialize $D = B$, $f = \mathrm{id}_B$. Process the factors for $k = n, n-1, \ldots, 1$. At the beginning of step k, the maintained map is $f : D \to X_k$.

1. If α_k is a backward factor $q : X_k \to X_{k-1}$, retain D and replace f by qf.
2. If α_k is a forward embedding $i : X_{k-1} \hookrightarrow X_k$, replace

$$D \longleftarrow f^{-1}(i(X_{k-1})), \qquad f \longleftarrow i^{-1} f|_D. \tag{4.7}$$

In (4.7), the inverse is the isomorphism from the image subgraph to X_{k-1}, and the restriction uses the new domain. At termination output $X_0 \xleftarrow{f} D \hookrightarrow B$.

Theorem 4.9 (Complexity of normalization). The algorithm computes (D_T, ν_T) of Theorem 4.6. Suppose the word has $n \geq 1$ factors and

$$M = \max\Big\{1, \max_{0 \leq k \leq n} \big(|V(X_k)| + |E(X_k)|\big)\Big\}.$$

Index vertices and edges by consecutive integers and represent every factor by its vertex-image array and edge-image array, with an edge image tagged as an edge or an identity. Then the algorithm uses $O(nM)$ deterministic array operations and $O(M)$ auxiliary storage beyond the input. Its maintained domain is always a subgraph of B and only decreases. For the empty word, initialization directly gives the identity roof in $O(M)$ operations.

Proof. After initialization, (D, f) interprets the empty suffix. If a backward factor is prepended, formula (4.3) composes f with the total map q, leaving its domain unchanged. If a forward factor is prepended, its interpretation is the partial inverse of i, and the same formula is exactly (4.7). Induction on the processed suffix proves correctness. Pullback stability and closure under composition ensure at every step that f remains in $\mathcal{Q}$.

For a backward factor, each maintained vertex image requires one array lookup. An edge already mapped to an identity remains an identity at the new vertex image. An edge mapped to a generating edge is updated by one lookup in the edge-image array of q. Thus this step costs $O(|V(D)| + |E(D)|)$.

For a forward factor, construct inverse arrays for the images of its vertices and edges in X_k, marking indices outside the images. Their construction costs $O(M)$. Retain a vertex of D precisely when its current image is marked as a vertex in the image of i. Retain an edge precisely when its current image is either an edge in the image of i or an identity at a retained image vertex. The source and target conditions then hold automatically. The inverse arrays give the new images of all retained vertices and edges. This also costs $O(M)$.

There are n steps, and the domain is always a subgraph of B, so the time bound follows. The maintained membership and image arrays, and the inverse arrays for the current factor, occupy $O(M)$ storage and can be reused at each step. All assertions use the stated indexed representation; they count scalar operations on its indices. □

4.5 The two fixed-endpoint quotient categories

Let $\equiv_{\mathrm{cart}}$ be the cartesian congruence used in Theorem 4.6. Let $\equiv_{\square}$ be generated by the same operations with arbitrary commutative dual squares in place of pullback squares. In both congruences, word isomorphisms have identity maps at their two endpoints, and nontrivial automorphisms are retained as factors.

For the strict 2-category $\mathsf{Ref}_{\mathcal{Q}}$, define $\pi_0\mathsf{Ref}_{\mathcal{Q}}$ to have the same objects and hom-sets $\pi_0\mathcal{R}_{\mathcal{Q}}(A,B)$. Here two objects of a poset are in the same component when they can be joined by a finite zigzag of order relations. Joint monotonicity in Proposition 4.4 induces its composition.

Theorem 4.10 (Fixed-endpoint comparison). There are isomorphisms of ordinary categories, identity on objects,

$$\mathsf{W}_{\mathcal{Q},\mathsf{Emb}}/{\equiv_{\mathrm{cart}}} \cong \mathsf{PM}_{\mathcal{Q}}, \qquad \mathsf{W}_{\mathcal{Q},\mathsf{Emb}}/{\equiv_{\square}} \cong \pi_0\mathsf{Ref}_{\mathcal{Q}}. \tag{4.8}$$

Proof. The first isomorphism is induced by $\mathcal{N}$. Every partial pair is represented by its roof word. Equality of interpretations is exactly cartesian equivalence by Theorem 4.6. This proves bijectivity on hom-sets and compatibility with composition.

For the second comparison, a general dual square changes the interpretation by a refinement, by Proposition 4.7. In a context, joint monotonicity preserves that refinement. All other generators preserve the partial map itself. Hence interpretation followed by taking the component induces a functor to $\pi_0\mathsf{Ref}_{\mathcal{Q}}$.

Conversely, let $(D,f) \le (E,g)$ in $\mathcal{R}_{\mathcal{Q}}(A,B)$. Their defining equality $g|_D = f$ gives a commutative square

$$\begin{array}{ccc} D & \hookrightarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\mathrm{id}_A]{} & A. \end{array} \tag{4.9}$$

Its dual transformation relates $A \xleftarrow{f} D \hookrightarrow E$ to $A \xrightarrow{\mathrm{id}} A \xleftarrow{g} E$. Append $E \hookrightarrow B$, compose the adjacent embeddings, and delete the identity factor. The two resulting words are exactly the roofs for (D,f) and (E,g). Thus every refinement, and consequently every zigzag of refinements, is realized by $\equiv_{\square}$.

Normalize arbitrary words to their roofs by cartesian moves. If their interpretations lie in the same component, the preceding construction connects those roofs by $\equiv_{\square}$. This proves injectivity on hom-sets; surjectivity again follows because every partial pair has a roof word. All constructions respect concatenation, completing the proof. □

Proposition 4.11. The fixed-endpoint gauge congruence of Section 3.4 is obtained from $\equiv_{\square}$ by imposing

$$\alpha^{+} \equiv 1_X \qquad \text{for every automorphism } \alpha : X \to X, \tag{4.10}$$

and taking the congruence generated by these relations. Here 1_X denotes the empty word. These relations act at every intermediate object of a composite.

Proof. Forward automorphism deletions are precisely (4.10); backward deletions follow by a flip. Insertion is the inverse relation. The remaining difference is that an isomorphic transformation in the full gauge congruence may use nonidentity endpoint automorphisms. A word isomorphism with components $u_0,\dots,u_n$ replaces each factor by its conjugate. Factor decomposition and flips express the transformed word as the original word preceded by u_0^{-1} and followed by u_n, with internal isomorphisms cancelling in pairs. After applying (4.10) to its two boundary factors, the words are equivalent. The same calculation in an arbitrary context proves that all the operations of Section 3.4 are generated. The reverse containment follows because every operation and every relation (4.10) is among those gauge operations. □

Example 4.12. The additional quotient in Proposition 4.11 can identify distinct components even when both endpoints have trivial automorphism groups. Take $\mathcal{Q}$ to be the isomorphisms, let A be one vertex, and let B consist of an edge $a \to c$ and an isolated vertex b. The poset $\mathcal{R}_{\mathcal{Q}}(A,B)$ has three isolated objects, corresponding to the three choices of a one-vertex domain.

Let K be the two-vertex subgraph with vertices a, b and no edges, and let $\sigma : K \to K$ exchange them. The two embeddings $A \to K$ selecting a and b become gauge equivalent after inserting σ and composing it with the first embedding. Appending the fixed inclusion $K \hookrightarrow B$ identifies the corresponding two roof classes. Both A and B are rigid, so their endpoint automorphism groups cannot identify these two roof classes. ♡

4.6 Contextual derivations and Peiffer interchange

Contextual derivations and interchange relations give strict $(2,1)$-categorical presentations of the preceding congruences. The construction follows the higher-dimensional rewriting approach of [GM12].

Let Σ_0 be the object set of a small category, and let Σ_1 be the directed graph with a tagged generator e^+ for each $\mathcal{E}$-morphism and a reverse-direction tagged generator q^- for each $\mathcal{Q}$-morphism. Its free path category is the word category W. Let $\mathcal{R}$ be a set of generating relations

$$\rho : s(\rho) \Longrightarrow t(\rho)$$

between parallel words. Each generator will be invertible as a 2-cell. Thus a reduction and its inverse decomposition are represented by one generator and its formal inverse. Identity relations need not be included as separate generators.

For the cartesian presentation, take composition and unit relations, flips, cartesian dual relations, and endpoint-identity word isomorphisms. For the square presentation, allow every commutative dual square. For the full fixed-endpoint gauge presentation, add the automorphism-erasure generators (4.10). By Proposition 4.11, their generated congruence is the one defined by the seven gauge operations.

A *contextual step* is a triple (U, ρ, V) replacing the subword $s(\rho)$ by $t(\rho)$ in a word $U\,s(\rho)\,V$. In this notation words are written in their traversal order, with U a prefix and V a suffix. Empty subwords and empty contexts are permitted, and an occurrence is specified by its context. Let $\Gamma_{\mathcal{R}}(A, B)$ be the graph of words from A to B and contextual steps between them, and let $\Pi\Gamma_{\mathcal{R}}(A, B)$ be its free path groupoid.

Definition 4.13. For two relation occurrences in successive subwords, the *Peiffer square* is

$$\begin{array}{ccc} U\,s(\rho)\,V\,s(\sigma)\,W & \xrightarrow{\ \rho\ } & U\,t(\rho)\,V\,s(\sigma)\,W \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ U\,s(\rho)\,V\,t(\sigma)\,W & \xrightarrow[\ \rho\ ]{} & U\,t(\rho)\,V\,t(\sigma)\,W. \end{array} \tag{4.11}$$

The edge labels abbreviate the indicated contextual occurrences. Impose equality of its two length-two derivations, including the versions using inverse generators and empty subwords. Let $\mathcal{G}_{\mathcal{R}}(A, B)$ be the quotient of $\Pi\Gamma_{\mathcal{R}}(A, B)$ by the smallest family of groupoid congruences containing these equalities and preserved by adding composable prefixes and suffixes. □

Theorem 4.14 (Free strict gauge presentation). There is a strict $(2, 1)$-category $\mathbb{M}_{\mathcal{R}}$ whose objects are Σ_0, whose 1-cells are the words of W, and whose hom-groupoids are $\mathcal{G}_{\mathcal{R}}(A, B)$. It has the following universal property. For a strict 2-category $\mathbb{D}$, assigning objects to Σ_0, 1-cells to Σ_1, and an invertible 2-cell to each $\rho \in \mathcal{R}$ with the prescribed source and target extends uniquely to a strict 2-functor

$$\mathbb{M}_{\mathcal{R}} \longrightarrow \mathbb{D}.$$

Proof. Adding a fixed prefix or suffix sends contextual steps to contextual steps. It therefore induces a functor of free path groupoids, and this functor descends to their quotients because the defining congruences are stable under contexts. These are the two whiskering operations.

Let $\alpha : T \Rightarrow T'$ and $\beta : S \Rightarrow S'$ be composable 2-cells, where $T, T' : A \to B$ and $S, S' : B \to C$ are words. Define their horizontal composite by first applying α in the T-part and then applying β in the S-part:

$$\beta * \alpha = (\beta \triangleright T') \circ (S \triangleleft \alpha). \tag{4.12}$$

Here $S \triangleleft \alpha$ denotes left whiskering and $\beta \triangleright T'$ right whiskering; the symbol $\circ$ in (4.12) is vertical composition. Peiffer squares imply

$$(\beta \triangleright T') \circ (S \triangleleft \alpha) = (S' \triangleleft \alpha) \circ (\beta \triangleright T). \tag{4.13}$$

For elementary steps this is one of (4.11); for finite derivations it follows by successively exchanging their elementary steps. Formal inverses obey the same equation because the hom-categories are groupoids.

Equation (4.13), together with functoriality of whiskering, shows that (4.12) is a functor of the pair (β, α). Explicitly, in a composite of two such horizontal pairs, the middle steps acting in the two different subwords can be exchanged by (4.13); the steps in each subword then compose vertically. This is the interchange law. Expanding the horizontal composite of three derivations performs the first, second, and third derivations in that order in either parenthesization. Strict associativity of word concatenation makes their contexts identical. Empty words and empty derivations give the strict unit laws. Thus the claimed strict 2-category exists, and every 2-cell is invertible.

For the universal property, extend the assignments on Σ_1 to words by strict composition in $\mathbb{D}$. Send a contextual generator to the corresponding whiskering of its assigned 2-cell, and send a formal inverse to the inverse 2-cell. This first defines functors from the free path groupoids. Interchange in $\mathbb{D}$ makes every Peiffer square commute, so they factor through $\mathcal{G}_{\mathcal{R}}(A,B)$. Their construction respects both compositions and all units, giving the required strict 2-functor. Every 1-cell and 2-cell is generated by the specified data, which proves uniqueness. □

Proposition 4.15 (Connected components of the presentation). Let $\equiv_{\mathcal{R}}$ be the smallest congruence on W containing the pairs $(s(\rho), t(\rho))$. Then

$$\pi_0 \mathbb{M}_{\mathcal{R}} \cong \mathsf{W}/\equiv_{\mathcal{R}}$$

by an isomorphism identity on objects. In particular, the cartesian and square presentations have connected-component categories $\mathsf{PM}_{\mathcal{Q}}$ and $\pi_0\mathsf{Ref}_{\mathcal{Q}}$, respectively. The full gauge presentation has the further quotient described in Proposition 4.11.

Proof. Two words are in the same component of $\Pi\Gamma_{\mathcal{R}}(A,B)$ exactly when they are connected by a finite sequence of generating relations, their inverses, and their instances in contexts. This is precisely $\equiv_{\mathcal{R}}$. Taking the Peiffer quotient identifies parallel derivations and does not change which pairs of objects admit a derivation. Therefore the components are unchanged. Composition is concatenation on both sides. The two stated identifications now follow from Theorem 4.10. □

Proposition 4.16 (Coherence from the canonical normal form). Let $\mathbb{M}_{\mathrm{cart}}$ be the cartesian presentation of Theorem 4.14. Write $n(T)$ for the fixed roof word (4.6), including its identity legs when necessary, and choose a normalizing 2-cell

$$r_T : T \Longrightarrow n(T), \qquad r_{n(T)} = \mathrm{id}_{n(T)}.$$

Such choices exist by Theorem 4.6. For each elementary contextual step $\alpha : T \Rightarrow T'$, impose the relation between 2-cells

$$r_T = r_{T'} \circ \alpha. \tag{4.14}$$

This is an unrestricted, possibly infinite family indexed by all elementary contextual steps. It forms a homotopy basis for the cartesian presentation: in the resulting strict $(2,1)$-category, any two parallel 2-cells are equal. The induced strict 2-functor to the locally discrete 2-category $\mathsf{PM}_{\mathcal{Q}}$ is a biequivalence.

Proof. For a cartesian step, $\mathcal{N}(T) = \mathcal{N}(T')$ by Theorem 4.6; hence $n(T) = n(T')$ as the chosen image-roof words. Thus (4.14) compares parallel 2-cells. Equivalently, one may adjoin a 3-cell between its two sides and then identify 2-cells connected by these 3-cells in arbitrary contexts.

In the quotient, every elementary contextual step satisfies

$$\alpha = r_{T'}^{-1} \circ r_T.$$

The same formula holds for its inverse. For a composite derivation $\theta : T \Rightarrow T'$, the consecutive factors r_U and r_U^{-1} cancel, giving

$$\theta = r_{T'}^{-1} \circ r_T.$$

Thus all parallel derivations are equal. A derivation between two words exists precisely when their partial-map interpretations agree, by Theorem 4.6. It follows that the quotient hom-groupoid has exactly one morphism between any two words with the same interpretation and none between words with different interpretations. Its functor to the discrete set of partial maps is therefore fully faithful and essentially surjective. Interpretation is identity on objects and compatible with both compositions, so these local equivalences give the claimed biequivalence. □

Example 4.17. The Peiffer relations are already necessary for two independent composable reductions. In the chain category $[4]$, take all arrows as $\mathcal{E}$ and only identities as $\mathcal{Q}$. Starting with the word $0 \to 1 \to 2 \to 3 \to 4$, one may compose the first two arrows and then the last two, or do these operations in the reverse order. The intermediate words are respectively $0 \to 2 \to 3 \to 4$ and $0 \to 1 \to 2 \to 4$, and both derivations end at $0 \to 2 \to 4$. They are different reduced words in the free path groupoid. Their equality in $\mathcal{G}_{\mathcal{R}}(0,4)$ is exactly the corresponding Peiffer square. ♡

The cartesian and square presentations in Theorem 4.14 retain formal derivations and have the connected-component categories described in Theorem 4.10. Proposition 4.16 identifies all parallel cartesian derivations. The full gauge quotient also erases automorphisms at intermediate objects.

5 Open interfaces and composition

We describe boundaries by leaves or by ports. The leaf description admits boundary-compatible path functors and determines the cartesian changes of boundary. The port description composes networks by joining interface wires and removing the interface; it is used for the decorated networks below.

5.1 Boundary leaves and cartesian change of boundary

Definition 5.1. A *leaf-open causal-net* from a finite set A to a finite set B is a causal-net G with injections

$$i^- : A \hookrightarrow V(G), \qquad i^+ : B \hookrightarrow V(G)$$

having disjoint images. Each $i^-(a)$ has indegree zero and outdegree one, and each $i^+(b)$ has indegree one and outdegree zero. A morphism $(f, F, g) : (G, i^-, i^+) \to (H, j^-, j^+)$ consists of functions $f : A \to A'$, $g : B \to B'$ and a functor $F : \mathrm{P}(G) \to \mathrm{P}(H)$ such that

$$F_0 i^- = j^- f, \qquad F_0 i^+ = j^+ g.$$

Write OpCau for the resulting category and

$$\partial : \mathsf{OpCau} \longrightarrow \mathsf{FinSet} \times \mathsf{FinSet}$$

for its boundary functor. □

Composition in OpCau is componentwise. The boundary equations are preserved by composition, so this defines a category. The assignment $j(G) = (G, \varnothing, \varnothing)$ is fully faithful: a map between two such objects is specified by its path functor alone.

Recall that $u : X \to Y$ over $a : \partial X \to \partial Y$ is cartesian if every map $v : Z \to Y$ over ab has a unique factorization $v = uw$ with $\partial w = b$.

Theorem 5.2 (Cartesian boundary maps). A morphism (f, F, g) of OpCau is ∂-cartesian if and only if F is an isomorphism in Cau. Its two boundary maps are then injective. Consequently ∂ is not a Grothendieck fibration over FinSet^2, whereas its restriction to boundary injections is a split fibration over FinInj^2.

Proof. Suppose $u = (f, F, g) : G \to H$ is cartesian. For every causal-net K, the boundary of $j(K)$ is $(\varnothing, \varnothing)$. The unique map from this boundary to ∂H factors uniquely through ∂G. Cartesian lifting therefore gives a bijection

$$\mathrm{Hom}_{\mathsf{Cau}}(K, G) \xrightarrow{F \circ -} \mathrm{Hom}_{\mathsf{Cau}}(K, H).$$

Taking $K = H$ supplies $R : H \to G$ with $FR = \mathrm{id}_H$; injectivity for $K = G$ gives $RF = \mathrm{id}_G$. Thus F is an isomorphism. Since $F_0 i^-$ and $F_0 i^+$ are injective, the boundary equations force f and g to be injective.

Conversely, suppose F is invertible. Given $v = (a, V, b) : L \to H$ and a factorization of its boundary map through (f, g), the only possible lift has underlying functor $F^{-1}V$. If the factorization is $(a, b) = (fh, gk)$, then

$$F_0^{-1} V_0 i_L^- = F_0^{-1} j^- f h = i_G^- h, \qquad F_0^{-1} V_0 i_L^+ = F_0^{-1} j^+ g k = i_G^+ k.$$

This lift is an open morphism and is unique.

There are open nets with a one-element input boundary. The map from a two-element input set onto that boundary has no cartesian lift, by the first assertion. Hence ∂ is not a fibration over all finite-set maps. For injections $f : A \hookrightarrow A'$ and $g : B \hookrightarrow B'$ and an open net (H, j^-, j^+) over (A', B'), choose the lift with the same underlying net and boundary maps $j^- f, j^+ g$. Its underlying functor is the identity. These choices preserve identities and composition literally, which gives the asserted splitting over FinInj^2. □

5.2 Acyclicity of a glued interface

For interfaces with arbitrary vertices, gluing can create directed cycles. The following criterion characterizes when the directed-graph pushout is acyclic.

Theorem 5.3 (Interface criterion). Let G, H be finite acyclic directed multigraphs and let $i : B \hookrightarrow V(G)$ and $j : B \hookrightarrow V(H)$ be injections. For $b, b' \in B$, set

$$bR_G b' \iff \text{a nonempty directed path in } G \text{ joins } i(b) \text{ to } i(b'),$$

and define R_H similarly. The graph $G \cup_B H$, obtained by identifying $i(b)$ with $j(b)$ for every b, is acyclic if and only if the relation $R_G \cup R_H$ has no directed cycle. Equivalently, there is a strict total order on B extending both interface reachability relations.

Proof. A directed cycle in the graph of $R_G \cup R_H$ determines nonempty paths in G or H whose endpoints are successive interface vertices. Their concatenation is a nonempty closed directed walk in $G \cup_B H$, and a finite graph containing such a walk contains a directed cycle.

Conversely, a directed cycle in $G \cup_B H$ cannot be contained entirely in one of the original graphs. Record its successive visits to the interface and subdivide it further whenever the graph supplying its edges changes. After segments of length zero are omitted, each segment is a nonempty path in G or H between interface vertices. Thus the cycle gives a nonempty closed walk in $R_G \cup R_H$, which contains a cycle. Finally, a finite relation is acyclic precisely when its transitive closure admits a linear extension. □

Corollary 5.4. Identifying output leaves of a leaf-open causal-net with input leaves of another preserves acyclicity.

Proof. There is no nonempty path between two output leaves in the first graph, or between two input leaves in the second. Both interface relations in Theorem 5.3 are empty. □

The graph pushout retains a bivalent vertex at each joined pair of leaves. For example, gluing two single-edge nets produces a two-edge path. To obtain identity wires, we use the following port description and remove the interface during composition. Such descriptions of open graphs and their composition are part of the graphical treatment of monoidal categories [JS91, DK13]; structured and decorated cospans provide complementary general constructions [BC20, BCV22].

5.3 Port networks and wire splicing

Definition 5.5. An *acyclic port network* $G : A \to B$ consists of a finite set $V(G)$ of events, finite disjoint sets $I_G(v), O_G(v)$ of input and output flags for each event, and a bijection

$$w_G : A \sqcup O_G \xrightarrow{\cong} B \sqcup I_G, \qquad I_G = \coprod_{v \in V(G)} I_G(v), \quad O_G = \coprod_{v \in V(G)} O_G(v). \tag{5.1}$$

All coproducts are tagged. A *wire* is a pair $(s, w_G(s))$; their set is denoted $E(G)$. The directed multigraph on $V(G)$ with an edge $v \to u$ for each wire from $O_G(v)$ to $I_G(u)$ is required to be acyclic. A wire from A to B is called a *through-wire*.

An isomorphism of port networks is a bijection of events and of their flags preserving input and output flags and the matching. For networks with the same boundary, it is required to be the identity on A and B.□

Attaching a separate source vertex to each input and a separate sink vertex to each output converts a port network into a leaf-open causal-net. Removing those boundary vertices gives the inverse construction, up to boundary-preserving isomorphism. An event of valence $(1, 1)$ remains an event; only the designated interface is removed by the following operation.

Definition 5.6. For $G : A \to B$ and $H : B \to C$, define $H \circ_B G : A \to C$ to have events $V(G) \sqcup V(H)$, retaining all their flags. Its matching is

$$w_{H \circ_B G}(s) = \begin{cases} w_G(s), & s \in A \sqcup O_G,\ w_G(s) \in I_G, \\ w_H(b), & s \in A \sqcup O_G,\ w_G(s) = b \in B, \\ w_H(s), & s \in O_H. \end{cases} \tag{5.2}$$

In particular, the two wires incident with each $b \in B$ become one wire. The *identity network* $1_A : A \to A$ has no events and matching id_A. □

Theorem 5.7 (Composition of port networks). Finite sets and boundary-preserving isomorphism classes of acyclic port networks form a symmetric monoidal category $\mathsf{Net}_{\mathrm{ac}}$. Composition is wire splicing, identities are 1_A, and tensor product is disjoint union. The groupoids of actual port networks and their boundary-preserving isomorphisms form the hom-groupoids of a bicategory with the same composition and canonical associators and unitors.

Proof. First, (5.2) is a bijection from $A \sqcup O_G \sqcup O_H$ to $C \sqcup I_G \sqcup I_H$. A target in I_G has its unique preimage under w_G. A target in $C \sqcup I_H$ has a unique preimage under w_H; if that preimage is $b \in B$, replace it by the unique preimage of b under w_G. This describes the inverse matching. The new event graph contains edges within G, within H, and from an event of G to an event of H. There are no edges in the reverse direction. Concatenating topological orders of the two event graphs therefore gives a topological order of the composite.

For three composable networks, a composite wire is obtained by following the unique matching at every interface until an event input or an external output is reached. This description is independent of the order in which the two interfaces are removed. Both parenthesizations have the same events and flags, up to the canonical reassociation of tagged unions, and the same wire matching under this reassociation. Thus composition is associative on isomorphism classes. Following a matching through 1_A changes no wire, so 1_A is a two-sided identity. Isomorphisms preserve this construction and splice to isomorphisms of composites.

Disjoint union preserves acyclicity. Splicing a disjoint union performs the two splicings separately, which proves the interchange equation for tensor product and composition. The empty set and the empty network give the tensor units. Associativity, unit, and symmetry maps are the event-free networks corresponding to the canonical bijections of finite sets. Their coherence equations hold because they are the corresponding equations for these bijections.

At the level of actual networks, the reassociations just described give the bicategorical associators and unitors. Every composite in the pentagon identifies an event or flag with the same tagged original event or flag and follows the same wire through the interfaces. Hence the pentagon commutes. The same argument with an event-free identity network proves the triangle. □

The category OpCau allows boundary-compatible path functors. The hom-groupoids in Theorem 5.7 use boundary-preserving network isomorphisms. Wire splicing and the descent theorem below are defined on the latter groupoids.

6 Gauge groupoids and descent

Let $\mathcal{S}$ be a small symmetric monoidal category and let $\mathcal{S}^{\simeq}$ denote its subgroupoid of objects and isomorphisms. Diagrams on a fixed network form a groupoid whose morphisms are compatible families of edge isomorphisms. We use the gauge relation of Lu and Tang [LT22, Definition 4] and retain these isomorphisms as morphisms. Proposition 6.4 distinguishes this relation from the symmetry-generated relation in [Lu23, Section 1.2].

6.1 Diagram groupoids and monoidal equivalence

Choose linear orders on the input and output flags at each event of a port network $G : A \to B$. We call this choice a polarization. Fix a bracketing of each ordered tensor product, and use the monoidal unit for an empty product. For a flag h, the notation X_h means the label of its unique incident wire.

Definition 6.1. An $\mathcal{S}$-diagram on G consists of an object X_e of $\mathcal{S}$ for every wire e and a morphism

$$d_v : \bigotimes_{h \in I_G(v)} X_h \longrightarrow \bigotimes_{k \in O_G(v)} X_k \qquad (v \in V(G)).$$

A gauge transformation $(X, d) \to (X', d')$ is a family of isomorphisms $\gamma_e : X_e \to X'_e$ satisfying

$$\left(\bigotimes_{k \in O_G(v)} \gamma_k \right) d_v = d'_v \left(\bigotimes_{h \in I_G(v)} \gamma_h \right) \qquad (v \in V(G)). \tag{6.1}$$

These transformations compose componentwise; their inverses are the componentwise inverses. The resulting groupoid is denoted $\mathcal{D}_{\mathcal{S}}(G)$. □

Boundary wires are included in the family γ. The resulting boundary isomorphisms allow the output labels of one diagram to be matched with the input labels of another.

Lemma 6.2 (Polarization independence). For two polarizations P, P' of G, canonical reordering isomorphisms define an isomorphism of the corresponding diagram groupoids. These isomorphisms preserve all wire labels and satisfy $R_{P',P''}R_{P,P'} = R_{P,P''}$ and $R_{P,P} = \mathrm{id}$.

Proof. Let r_v^{in} and r_v^{out} be the symmetry and associativity isomorphisms from the tensors ordered by P to those ordered by P'. Replace d_v by

$$r_v^{\mathrm{out}} d_v (r_v^{\mathrm{in}})^{-1}$$

and retain each X_e and γ_e. Naturality of the symmetry and associativity maps carries (6.1) to the corresponding equation for P'. The coherence theorem for symmetric monoidal categories identifies successive reorderings with the reordering of the composite permutation [JS91]. It gives the two asserted identities and makes $R_{P',P}$ the inverse. □

Consequently a network isomorphism induces an isomorphism of diagram groupoids by relabelling flags, followed, when needed, by the canonical change of polarization. These maps respect composition. In statements about splicing, the polarization at an event is the one inherited from the factor containing that event.

Theorem 6.3 (Invariance under monoidal equivalence). A strong symmetric monoidal functor $F : \mathcal{S} \to \mathcal{T}$ induces a functor

$$F_G : \mathcal{D}_{\mathcal{S}}(G) \longrightarrow \mathcal{D}_{\mathcal{T}}(G).$$

If the underlying functor F is an equivalence, then F_G is an equivalence for every finite port network G. It is compatible with the boundary functors induced by $F : \mathcal{S}^{\simeq} \to \mathcal{T}^{\simeq}$.

Proof. Write

$$\mu_{(X_i)} : \bigotimes_i F(X_i) \xrightarrow{\cong} F\left(\bigotimes_i X_i\right)$$

for the iterated tensorator, including the unit comparison when the index set is empty. The induced diagram has labels $F(X_e)$ and vertex maps

$$\mu^{-1}_{O_G(v)} F(d_v) \mu_{I_G(v)}.$$

On gauges it applies F to every edge isomorphism. Naturality of the tensorator proves (6.1) for the images and establishes functoriality.

Suppose F is fully faithful. A gauge between two image diagrams has a unique preimage on each wire, since the map on hom-sets is bijective. These preimages are isomorphisms: their inverses are the preimages of the inverse gauges. Faithfulness, together with naturality of μ, shows that they satisfy the vertex equations. Hence F_G is fully faithful.

Suppose also that F is essentially surjective, and let (Y, a) be a $\mathcal{T}$-diagram. Choose $X_e \in \mathcal{S}$ and isomorphisms $z_e : F(X_e) \to Y_e$. Transport each a_v along the tensors of the z_e and along μ to obtain a morphism

$$F\left(\bigotimes_{I_G(v)} X_h\right) \longrightarrow F\left(\bigotimes_{O_G(v)} X_k\right).$$

Fullness gives a preimage d_v. The family (z_e) is a gauge from $F_G(X, d)$ to (Y, a). This proves essential surjectivity. The assertion about boundaries follows directly from the labels on their incident wires. □

The theorem justifies computations in a strict symmetric monoidal model of $\mathcal{S}$. The boundary and descent constructions remain equivalent under that replacement.

Proposition 6.4 (Symmetry-generated gauges). The quotient of diagrams by gauges generated only by symmetry isomorphisms need not be invariant under strict symmetric monoidal equivalence.

Proof. Let $\mathbf{1}$ be the terminal strict symmetric monoidal category. Let $\mathcal{T}$ have objects $\mathbb{N}$, exactly one morphism between any two objects, tensor product given by addition, and unit 0. It is a strict symmetric monoidal groupoid, and the projection $\mathcal{T} \to \mathbf{1}$ is a strict symmetric monoidal equivalence. Every symmetry in $\mathcal{T}$ is the identity of the object $m+n = n+m$. Thus any composite of symmetries preserves each edge label.

Take the closed network with one wire from a source event to a sink event. For each $n \in \mathbb{N}$, there is a unique diagram with that wire label: its event maps are the unique arrows $0 \to n$ and $n \to 0$. Symmetry-generated gauge identifies none of the distinct labels. Its quotient is therefore $\mathbb{N}$, whereas the corresponding quotient for $\mathbf{1}$ is a singleton. With all edge isomorphisms admitted, the diagram groupoids in this example are equivalent, as Theorem 6.3 asserts. □

Remark 6.5. A graph of generating gauge steps determines a free path groupoid, but its loops also record the chosen presentation. For example, let $\mathcal{S} = \mathbf{1}$ and let three source events feed one sink event. The six polarizations at the sink are connected by unique reordering maps, so their reordering groupoid is thin. In the free path groupoid generated by these maps, a triangle through three distinct polarizations is a nontrivial reduced loop. Symmetric monoidal coherence identifies this composite with the identity. Definition 6.1 imposes these relations by composing edge isomorphisms in $\mathcal{S}$. ♢

6.2 A description by group actions

Choose one object X_c in every isomorphism class $c \in \pi_0(\mathcal{S}^{\simeq})$. For a tuple $\boldsymbol{c} = (c_e)_{e \in E(G)}$, put

$$T_G(\boldsymbol{c}) = \prod_{v \in V(G)} \mathrm{Hom}_{\mathcal{S}}\left(\bigotimes_{h \in I_G(v)} X_{c_h}, \bigotimes_{k \in O_G(v)} X_{c_k} \right), \tag{6.2}$$

$$K_G(\boldsymbol{c}) = \prod_{e \in E(G)} \mathrm{Aut}_{\mathcal{S}}(X_{c_e}). \tag{6.3}$$

The group in (6.3) acts on (6.2) by

$$(\gamma \cdot d)_v = \left(\bigotimes_{k \in O_G(v)} \gamma_k \right) d_v \left(\bigotimes_{h \in I_G(v)} \gamma_h \right)^{-1}. \tag{6.4}$$

For a group K acting on a set T, the notation $T//K$ denotes the action groupoid, whose arrows $t \to t'$ are the elements k with $kt = t'$.

Proposition 6.6 (Gauge decomposition). There is an equivalence

$$\mathcal{D}_{\mathcal{S}}(G) \simeq \coprod_{\boldsymbol{c} \in \pi_0(\mathcal{S}^{\simeq})^{E(G)}} T_G(\boldsymbol{c})//K_G(\boldsymbol{c}). \tag{6.5}$$

In particular, the automorphism group of a diagram is the stabilizer specified by (6.1) with $d' = d$.

Proof. The full subgroupoid consisting of diagrams whose wire labels are the chosen representatives is essentially surjective: choose an isomorphism on every wire and transport the event maps by (6.4). Two objects of this subgroupoid have isomorphic edge labels only when their tuples $\boldsymbol{c}$ coincide. For a fixed tuple, the objects are $T_G(\boldsymbol{c})$ and the arrows are precisely the elements of $K_G(\boldsymbol{c})$ satisfying the action equation. This proves (6.5) and the stabilizer assertion. □

6.3 Boundary groupoids and cutting

For a finite set A, define

$$\mathcal{B}_{\mathcal{S}}(A) = (\mathcal{S}^{\simeq})^A.$$

Restriction to incident wire labels gives functors

$$\mathcal{B}_{\mathcal{S}}(A) \xleftarrow{s_G} \mathcal{D}_{\mathcal{S}}(G) \xrightarrow{t_G} \mathcal{B}_{\mathcal{S}}(B). \tag{6.6}$$

On arrows they restrict the family of edge isomorphisms.

For functors $p : \mathcal{X} \to \mathcal{Z}$ and $q : \mathcal{Y} \to \mathcal{Z}$ between groupoids, their *homotopy pullback* $\mathcal{X} \times^h_{\mathcal{Z}} \mathcal{Y}$ is the groupoid of triples (x, y, α) with an isomorphism $\alpha : p(x) \to q(y)$. A morphism $(x, y, \alpha) \to (x', y', \alpha')$ is a pair (u, v) satisfying

$$\alpha' p(u) = q(v)\alpha. \tag{6.7}$$

This is the weak pullback used to compose spans of groupoids [BHW10].

Theorem 6.7 (Descent across an interface). For acyclic port networks $G : A \to B$ and $H : B \to C$, cutting every wire at the removed interface defines an equivalence

$$\mathrm{Cut}_{G,H} : \mathcal{D}_{\mathcal{S}}(H \circ_B G) \xrightarrow{\simeq} \mathcal{D}_{\mathcal{S}}(G) \times^h_{\mathcal{B}_{\mathcal{S}}(B)} \mathcal{D}_{\mathcal{S}}(H), \tag{6.8}$$

where the two functors into the base are t_G and s_H. The cutting functor preserves the two external boundary functors strictly. It has a quasi-inverse which preserves the input boundary strictly and the output boundary by a specified natural isomorphism. The statement includes networks with through-wires.

Proof. Write $K = H \circ_B G$ and $\mathcal{P}$ for the groupoid on the right of (6.8). Each original wire of G or H lies in a unique wire of K. Given a diagram on K, give every original wire that wire's label and retain all event maps. At an interface b, the two labels agree, so choose the identity as the comparison isomorphism. This defines $\mathrm{Cut}_{G,H}$ on objects. On gauges it copies the isomorphism of a spliced wire to both of its pieces. The resulting pair satisfies (6.7).

For two diagrams on K, a morphism between their cuts is a pair of gauges whose two components agree at every interface wire, because both comparison isomorphisms are identities. The pair therefore gives one isomorphism on each wire of K. Its vertex equations are exactly the vertex equations in the two factors. This establishes a bijection on every hom-set, so cutting is fully faithful.

For essential surjectivity, take $(D, E, \alpha) \in \mathcal{P}$. Write

$$\alpha_b : X_b \xrightarrow{\cong} Y_b$$

for the comparison from the output label of D at b to the input label of E at b. Form a diagram E^α on H by replacing the label Y_b of every wire beginning at b with X_b. Let $r_e : E^\alpha_e \to E_e$ be α_b on such a wire and the identity on every other wire. Each wire has only one source, so these assignments are unambiguous. Define its event maps by transport,

$$d^\alpha_v = \left(\bigotimes_{k \in O_H(v)} r_k \right)^{-1} d^E_v \left(\bigotimes_{h \in I_H(v)} r_h \right). \tag{6.9}$$

Thus $r : E^\alpha \to E$ is a gauge. Every altered wire starts at a boundary input and hence is not an output flag of any event. The left factor in (6.9) is therefore the identity.

The diagrams D and E^α now have equal labels at the interface. Join their incident wires and retain their event maps to obtain a diagram $L(D, E, \alpha)$ on K. Its cut is $(D, E^\alpha, \mathrm{id})$, and

$$(\mathrm{id}_D, r) : \mathrm{Cut}_{G,H} L(D, E, \alpha) \longrightarrow (D, E, \alpha)$$

is an isomorphism in $\mathcal{P}$. Indeed, its interface equation is $\alpha_b = r_b$.

For a morphism (γ, δ) in $\mathcal{P}$, define $L(\gamma, \delta)$ using γ on every wire contributed by G, including spliced wires, and δ on the remaining wires of H. The equation $\alpha'_b \gamma_b = \delta_b \alpha_b$ ensures the vertex equations after the transport (6.9). It also proves that the displayed isomorphisms form a natural transformation $\mathrm{Cut}_{G,H} L \Rightarrow \mathrm{id}_{\mathcal{P}}$. The definitions give $L \,\mathrm{Cut}_{G,H} = \mathrm{id}$ under the canonical identification of wires. Thus L is a quasi-inverse.

The input boundary of $L(D, E, \alpha)$ is that of D. At an output $c \in C$, the label remains that of E unless the incident wire of H is a through-wire beginning at some b. In the latter case the new label is X_b, and $\alpha_b : X_b \to Y_b$ supplies the comparison with the original output. These comparisons are the image of r under t_H and are natural in (D, E, α). Hence they give the specified natural isomorphism of output boundary functors. Cutting itself copies labels at the external endpoints and preserves both boundary functors strictly, including on through-wires. □

6.4 Spans and boundary framings

We write $\mathsf{HSpan}(\mathsf{Grpd})$ for the category whose objects are small groupoids and whose arrows are equivalence classes of spans of groupoids. Here an equivalence of spans is an equivalence of their middle groupoids together with specified natural isomorphisms commuting with the two legs; the equivalence relation is generated by such maps. Composition is homotopy pullback, and tensor product is cartesian product. These operations are well defined on equivalence classes. In fact, equivalences induce equivalences on the explicit groupoids (6.7); iterated homotopy pullbacks have the same description by tuples and successive comparison isomorphisms. This also gives associativity. The identity span of $\mathcal{X}$ has middle groupoid $\mathcal{X}$ and both legs the identity, and the corresponding unit laws follow by forgetting an isomorphic copy of an object.

Corollary 6.8 (Compositional gauge semantics). The assignments

$$A \longmapsto \mathcal{B}_{\mathcal{S}}(A), \qquad G \longmapsto [\mathcal{B}_{\mathcal{S}}(A) \xleftarrow{s_G} \mathcal{D}_{\mathcal{S}}(G) \xrightarrow{t_G} \mathcal{B}_{\mathcal{S}}(B)]$$

define a symmetric monoidal functor

$$\mathsf{Net}_{\mathrm{ac}} \longrightarrow \mathsf{HSpan}(\mathsf{Grpd}).$$

The cutting equivalences are natural under boundary-preserving network isomorphisms and satisfy the associativity and unit coherence equations before passage to equivalence classes of spans.

Proof. Relabelling by a network isomorphism gives an isomorphism of diagram groupoids preserving boundary labels, so the span depends only on the network's isomorphism class. Theorem 6.7 identifies its composition with homotopy-pullback composition. There is a canonical isomorphism

$$\mathcal{D}_{\mathcal{S}}(1_A) \cong \mathcal{B}_{\mathcal{S}}(A)$$

under which both boundary maps are the identity: the identity network has precisely one through-wire for each a and no event maps. This proves preservation of identities.

A diagram, or a gauge, on a disjoint union is exactly a pair of diagrams, or gauges, on its two components. Therefore

$$\mathcal{D}_{\mathcal{S}}(G \sqcup H) \cong \mathcal{D}_{\mathcal{S}}(G) \times \mathcal{D}_{\mathcal{S}}(H), \qquad \mathcal{B}_{\mathcal{S}}(A \sqcup C) \cong \mathcal{B}_{\mathcal{S}}(A) \times \mathcal{B}_{\mathcal{S}}(C).$$

The empty network gives the terminal groupoid. These isomorphisms commute with the two boundary maps and with permutations of components, proving the symmetric monoidal assertion.

For a chain of three networks, either iterated cutting functor assigns the same labels to every wire piece, the same event maps, and identity comparisons at both interfaces. After the canonical reassociation of the homotopy pullback, the two functors agree. The argument with four networks proves the pentagon, and cutting an identity network proves the triangle. Relabelling events and flags commutes with restriction and duplication of labels, giving naturality under network isomorphisms. □

For prescribed boundary objects $\lambda^- \in \mathcal{B}_{\mathcal{S}}(A)$ and $\lambda^+ \in \mathcal{B}_{\mathcal{S}}(B)$, define the groupoid of *framed diagrams* by the homotopy fiber

$$\mathcal{D}_{\mathcal{S}}(G; \lambda^-, \lambda^+) := \mathrm{hofib}_{(\lambda^-, \lambda^+)}(s_G, t_G). \tag{6.10}$$

An object consists of D and isomorphisms $\theta^- : s_G D \to \lambda^-$, $\theta^+ : t_G D \to \lambda^+$. An arrow $\gamma : D \to D'$ preserves these framings when $\theta'^- s_G \gamma = \theta^-$ and $\theta'^+ t_G \gamma = \theta^+$. These are explicit boundary data; the isomorphisms allow a through-wire to carry a transport between differently presented boundary objects.

Proposition 6.9. The equivalence of Theorem 6.7 induces an equivalence on homotopy fibers over any prescribed external boundary objects. For the identity network one has an equivalence with a discrete groupoid,

$$\mathcal{D}_{\mathcal{S}}(1_A; \lambda, \mu) \simeq \mathrm{Iso}_{\mathcal{B}_{\mathcal{S}}(A)}(\lambda, \mu)_{\mathrm{disc}}. \tag{6.11}$$

Proof. An equivalence of groupoids commuting with maps to a base induces an equivalence of their homotopy fibers. This follows either from (6.7), or by transporting the fiber's specified isomorphism along the unit and counit of a quasi-inverse. Cutting commutes strictly with both external boundary functors, so this observation applies to (6.8).

For 1_A, an object of the fiber is a triple (X, θ^-, θ^+) with $\theta^- : X \to \lambda$ and $\theta^+ : X \to \mu$. Send it to $\theta^+(\theta^-)^{-1}$. Between two such triples there is a unique framing-preserving arrow precisely when these isomorphisms agree. Every isomorphism $u : \lambda \to \mu$ is represented by $(\lambda, \mathrm{id}_\lambda, u)$. This proves (6.11). □

Example 6.10. Let K be a nontrivial group and let BK be its one-object groupoid. The homotopy pullback of the two unique functors $* \to BK$ is the discrete groupoid with object set K. Consequently

$$\pi_0(* \times^h_{BK} *) \cong K, \qquad \pi_0(*) \times_{\pi_0(BK)} \pi_0(*) \cong \{*\}.$$

Thus connected components alone do not preserve interface composition. The comparison isomorphisms in Theorem 6.7 retain the information lost by replacing every boundary groupoid with its set of components.♡

6.5 Event contraction and descent

Contracting a block of events replaces its diagram by the corresponding composite morphism. The contractions considered here preserve all external ports and retain each wire between distinct blocks. We show that they commute with interface descent when each block lies within one piece.

Definition 6.11. An *admissible event partition* of a port network $G : A \to B$ is a partition $\mathcal{W}$ of $V(G)$ into nonempty blocks such that the following quotient is acyclic. For each $W \in \mathcal{W}$, replace its events by one event, delete the wires whose two incident events belong to W, and retain all other wires. The input and output flags of the new event are the original flags incident with the retained wires. The matching and boundary ports are inherited from G. Denote the resulting network by $G/\mathcal{W}$ and the contraction by $q : G \to G/\mathcal{W}$. The empty event set has its unique empty partition. □

Admissibility is determined by acyclicity of the quotient event graph and can be tested by a topological sort. Distinct retained wires remain distinct. On the associated leaf-open causal-nets, q is a coarse-graining: internal edges are contracted and all other edges are preserved individually.

Lemma 6.12. Each block of an admissible event partition is path-convex in the event graph. A partition into path-convex blocks need not be admissible.

Proof. If a directed path leaves a block and subsequently returns to it, its image in the quotient contains a nonempty closed directed walk. This contradicts admissibility. All edges of the induced subgraph on a block are included, so this is exactly path-convexity.

For the converse assertion, take four events x_0, x_1, y_0, y_1 and the four edges

$$x_0 \longrightarrow x_1, \qquad y_0 \longrightarrow y_1, \qquad x_0 \longrightarrow y_1, \qquad y_0 \longrightarrow x_1.$$

There are no paths of length two. The two blocks $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are connected and path-convex, whereas their quotient has an edge in each direction between its two events. □

Fix a symmetric monoidal category $\mathcal{S}$. We write tensor products in a fixed parenthesization and order the incident flags at every event. Symmetry and associativity isomorphisms give the required reordering maps. Different choices are related by the polarization isomorphisms of Lemma 6.2.

Lemma 6.13 (Evaluation on an acyclic block). An $\mathcal{S}$-decoration D of an acyclic port network with ordered input and output ports determines a morphism

$$\mathrm{ev}(D) : \bigotimes_{a \in A} X_a \longrightarrow \bigotimes_{b \in B} X_b.$$

It is independent of the order in which available events are evaluated. If $\gamma : D \to D'$ is an edge gauge, then

$$\left(\bigotimes_{b \in B} \gamma_b \right) \mathrm{ev}(D) = \mathrm{ev}(D') \left(\bigotimes_{a \in A} \gamma_a \right). \tag{6.12}$$

Replacing disjoint admissible blocks by their evaluations preserves the evaluation of the whole network.

Proof. Choose a topological order of the event graph. Initially the active wires are the input wires. At each event, move its incoming active wires to the front in their flag order, preserving the relative order of all other active wires. Apply its vertex morphism tensored with identities on those other wires, and replace the input prefix by its ordered output wires. At the end, reorder the active wires into the prescribed output order. A

through-wire remains active throughout and carries an identity. Events with no inputs or no outputs are treated using the monoidal unit.

Two topological orders are related by interchanges of adjacent incomparable events: move the first event of one order to the beginning of the other, passing only events incomparable to it, and continue inductively. For each such interchange, the two event maps act on disjoint active factors. Bifunctoriality of tensor product and naturality of the symmetry identify the two resulting maps. The associativity and symmetry coherence equations identify their reorderings. Thus all topological orders give the same morphism, as in the graphical calculus of [JS91].

To prove (6.12), insert the gauge equation at each event in a topological evaluation. On an internal wire, the outgoing gauge from one event is the incoming gauge at the next event and cancels after solving the adjacent vertex equations. Equivalently, induction on the active frontier carries its tensor product of gauge maps from the initial frontier to the final one. This gives precisely (6.12), including through-wires.

For an admissible partition, first choose a topological order of the quotient and then a topological order within each block. Their concatenation is a topological order of the original event graph. Evaluating in this order evaluates the blocks consecutively. The preceding independence therefore identifies blockwise and global evaluation. □

Theorem 6.14 (Contraction functor). An admissible event contraction $q : G \to G/\mathcal{W}$ induces a functor

$$q_* : \mathcal{D}_{\mathcal{S}}(G) \longrightarrow \mathcal{D}_{\mathcal{S}}(G/\mathcal{W}).$$

It preserves boundary labels and their gauge maps. On objects it retains the labels on surviving wires and decorates each quotient event by the evaluation of its block. On morphisms it retains the gauge isomorphisms on surviving wires. Nested admissible contractions q and r give an admissible composite partition. Under the canonical network isomorphism between the iterated and direct quotient, there is a natural isomorphism

$$r_* q_* \;\cong\; (rq)_*. \tag{6.13}$$

These comparisons satisfy the associativity and unit coherence equations. The construction also commutes with disjoint unions and with strong symmetric monoidal equivalences of coefficient categories, up to their canonical natural isomorphisms.

Proof. Apply Lemma 6.13 to the network induced by each block, with every wire crossing its boundary treated as an external wire. Equation (6.12) is exactly the vertex gauge equation in the quotient. Thus the proposed maps define gauge morphisms. Identities and composites are preserved because their surviving edge components are preserved. Boundary wires survive individually, proving the assertion about the boundary functors.

For nested partitions, the evaluation at an event of the final quotient is the evaluation of the same original diagram on both sides of (6.13). The final assertion of Lemma 6.13 identifies the vertex maps. Their surviving wire labels agree. After the canonical polarization transports, identity maps on these labels give (6.13). For three nested partitions, all composites of these comparisons identify the same original evaluation and the same surviving wire components. Hence the associativity diagram commutes. Singleton partitions give the unit equations.

Disjoint union evaluates by tensor product. A strong symmetric monoidal functor carries every local tensor, composition, unit, and symmetry in the evaluation to the corresponding expression through its structural isomorphisms. Induction over the topological evaluation gives the asserted natural comparison. The structural coherence equations make these comparisons compatible with nested partitions. □

An admissible contraction usually changes the decoration groupoid. For example, contracting the two events of a closed state–costate network replaces (u, φ) by the scalar $\varphi(u)$. The distinct gauge orbits with scalar zero in Proposition 7.3 all map to the same scalar decoration.

Theorem 6.15 (Contraction and interface descent). Let $G : A \to B$ and $H : B \to C$ be port networks with admissible event contractions q_G and q_H. The union of their event partitions is an admissible contraction q of $H \circ_B G$, and there is a canonical network isomorphism

$$(H \circ_B G)/q \;\cong\; (H/q_H) \circ_B (G/q_G).$$

Under this isomorphism, the descent equivalences fit into a naturally commutative square. Write $\mathcal{P}(G,H) = \mathcal{D}_{\mathcal{S}}(G) \times^{h}_{\mathcal{B}_{\mathcal{S}}(B)} \mathcal{D}_{\mathcal{S}}(H)$; then

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{S}}(H \circ_B G) & \xrightarrow[\simeq]{\text{Cut}} & \mathcal{P}(G,H) \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle (q_G)_* \times^h (q_H)_*} \\ \mathcal{D}_{\mathcal{S}}((H/q_H) \circ_B (G/q_G)) & \xrightarrow[\simeq]{\text{Cut}} & \mathcal{P}(G/q_G, H/q_H). \end{array} \tag{6.14}$$

The comparison is coherent under iterated splicing and nested contractions.

Proof. Every new event edge created by splicing points from a G-event to an H-event. Consequently, concatenating topological orders of G/q_G and H/q_H proves admissibility of the union partition. Deleting wires internal to a block and splicing wires at B concern disjoint incidences; each original surviving wire follows the same route under either order of operations. This gives the stated network isomorphism.

The right-hand functor in (6.14) exists because the contractions preserve both boundary functors. Cutting a decoration of the composite gives the same label on the two halves of each interface wire, with identity interface isomorphism. Evaluation of a block uses exactly the same vertex maps and internal wires before and after this cut, since the block lies in one piece. Thus the square commutes after the canonical polarization identifications. For a general object of the homotopy pullback, interface isomorphisms transport the incident block input labels. Equation (6.12) identifies this transport with transport after evaluating the block. This proves naturality for the explicit gluing quasi-inverse as well. Through-wires are retained unchanged by these contractions, so the output boundary comparisons carried by their interface isomorphisms are retained as well.

For several interfaces and nested contractions, every comparison retains the same external edge gauges and evaluates the same original diagram. The coherence of Theorem 6.14 and of successive cuts in Theorem 6.7 therefore proves the final assertion. □

7 Computations and examples

For essentially finite groupoids, descent gives a weighted sum over interface labels. Finite-field vector spaces yield explicit formulas for these weights, and a state–costate network illustrates how contraction changes gauge orbits and stabilizers. A separate family of discrete networks gives refinement spaces with nontrivial higher homotopy.

7.1 Groupoid cardinality and the interface measure

A groupoid $\mathcal{X}$ is *essentially finite* if it has finitely many isomorphism classes of objects and every automorphism group is finite. Its cardinality is

$$|\mathcal{X}| = \sum_{[x] \in \pi_0 \mathcal{X}} \frac{1}{|\operatorname{Aut}_{\mathcal{X}}(x)|}. \tag{7.1}$$

It is invariant under equivalence. Products multiply cardinalities and disjoint unions add them. For a finite group K acting on a finite set T, the orbit–stabilizer formula gives

$$|T \,//\, K| = \frac{|T|}{|K|}. \tag{7.2}$$

The resulting calculus of spans is a form of groupoidification [BHW10].

Proposition 7.1 (Finite homotopy integration). Let $\mathcal{X} \xrightarrow{f} \mathcal{B} \xleftarrow{g} \mathcal{Y}$ be functors between essentially finite groupoids. Write $\mathcal{X}_b$ and $\mathcal{Y}_b$ for their homotopy fibers at b. Then

$$|\mathcal{X}| = \sum_{[b] \in \pi_0 \mathcal{B}} \frac{|\mathcal{X}_b|}{|\operatorname{Aut}_{\mathcal{B}}(b)|}, \tag{7.3}$$

$$|\mathcal{X} \times^h_{\mathcal{B}} \mathcal{Y}| = \sum_{[b] \in \pi_0 \mathcal{B}} \frac{|\mathcal{X}_b|\,|\mathcal{Y}_b|}{|\operatorname{Aut}_{\mathcal{B}}(b)|}. \tag{7.4}$$

Proof. Choose one representative x for every isomorphism class in $\mathcal{X}$. For a fixed b, its contribution to the homotopy fiber is the action groupoid of $\mathrm{Aut}_{\mathcal{X}}(x)$ on $\mathrm{Iso}_{\mathcal{B}}(f(x), b)$, with action induced by f. This set is empty unless $f(x) \cong b$; otherwise its size is $|\mathrm{Aut}_{\mathcal{B}}(b)|$. Equation (7.2) therefore gives

$$|\mathcal{X}_b| = |\mathrm{Aut}_{\mathcal{B}}(b)| \sum_{[x]:\, f(x)\cong b} \frac{1}{|\mathrm{Aut}_{\mathcal{X}}(x)|}.$$

Summing proves (7.3). The homotopy fiber over b of the projection $\mathcal{X} \times^h_{\mathcal{B}} \mathcal{Y} \to \mathcal{B}$ is equivalent to $\mathcal{X}_b \times \mathcal{Y}_b$. Indeed, isomorphisms $f(x) \to b$ and $g(y) \to b$ determine the interface isomorphism $f(x) \to g(y)$ by composition, and conversely. Apply the first formula and multiplicativity to obtain (7.4). □

For boundary labels λ on A and μ on B, put

$$K_G(\lambda, \mu) = |\mathcal{D}_{\mathcal{S}}(G; \lambda, \mu)|.$$

Assume that these homotopy fibers and the interface sum below are finite. In the vector-space examples, one may restrict to finitely many wire-dimension assignments, provided the restrictions on the pieces and their composite correspond under cutting and gluing. The framed descent theorem and Proposition 7.1 give

$$K_{H\circ_B G}(\lambda, \mu) = \sum_{[\nu]\in\pi_0\mathcal{B}_{\mathcal{S}}(B)} \frac{K_G(\lambda, \nu)K_H(\nu, \mu)}{|\mathrm{Aut}_{\mathcal{B}_{\mathcal{S}}(B)}(\nu)|}. \tag{7.5}$$

The unit has kernel

$$K_{1_A}(\lambda, \mu) = \begin{cases} |\mathrm{Aut}(\lambda)|, & \lambda \cong \mu, \\ 0, & \lambda \ncong \mu. \end{cases} \tag{7.6}$$

This follows from the discrete isomorphism groupoid in (6.11) and is the identity for the weighted composition (7.5).

For a nontrivial finite group K, the homotopy pullback in Example 6.10 has cardinality $|K|$. Its set-theoretic counterpart after passage to components has one element. Passing to connected components therefore changes the interface weight.

7.2 Finite-field tensor networks

Let q be a prime power and take $\mathcal{S}$ to be a small skeleton of the symmetric monoidal category of finite-dimensional $\mathbb{F}_q$-vector spaces. Fix a nonnegative dimension d_e for every wire e of G. For an event v, define

$$d_{\mathrm{in}}(v) = \prod_{e\in I_G(v)} d_e, \qquad d_{\mathrm{out}}(v) = \prod_{e\in O_G(v)} d_e, \qquad N(G, \mathbf{d}) = \sum_{v\in V(G)} d_{\mathrm{in}}(v)d_{\mathrm{out}}(v),$$

where a flag indexes its incident wire and an empty product is 1. Write $\mathcal{D}_{\mathbf{d}}(G)$ for the corresponding dimension component of the decoration groupoid.

Proposition 7.2. For fixed $\mathbf{d}$, the groupoid $\mathcal{D}_{\mathbf{d}}(G)$ is essentially finite and

$$|\mathcal{D}_{\mathbf{d}}(G)| = \frac{q^{N(G,\mathbf{d})}}{\prod_{e\in E(G)} |\mathrm{GL}_{d_e}(\mathbb{F}_q)|}, \qquad |\mathrm{GL}_d(\mathbb{F}_q)| = \prod_{j=0}^{d-1} (q^d - q^j). \tag{7.7}$$

The value for $d = 0$ is 1. Let E_{int} consist of wires joining two events and let E_{th} consist of through-wires. If boundary labels λ, μ have the prescribed incident dimensions, then the framed homotopy fiber has cardinality

$$|\mathcal{D}_{\mathbf{d}}(G; \lambda, \mu)| = q^{N(G,\mathbf{d})} \frac{\prod_{e\in E_{\mathrm{th}}} |\mathrm{GL}_{d_e}(\mathbb{F}_q)|}{\prod_{e\in E_{\mathrm{int}}} |\mathrm{GL}_{d_e}(\mathbb{F}_q)|}. \tag{7.8}$$

It is empty if a boundary dimension does not match its incident wire.

Proof. Choose $V_e = \mathbb{F}_q^{d_e}$. A vertex decoration is an arbitrary linear map from a space of dimension $d_{\mathrm{in}}(v)$ to one of dimension $d_{\mathrm{out}}(v)$. The product T of all these spaces of maps has $q^{N(G,\mathbf{d})}$ elements. The gauge group is $K = \prod_e \mathrm{GL}(V_e)$, acting by output changes of basis and inverse input changes of basis. Proposition 6.6 identifies the decoration component with $T /\!/ K$, so (7.2) proves (7.7). The displayed order of GL_d counts its ordered bases: the next column must avoid the span of the preceding columns.

The prescribed boundary component is a one-object groupoid with automorphism group

$$K_\partial = \prod_{a\in A} \mathrm{GL}(V_a) \times \prod_{b\in B} \mathrm{GL}(V_b).$$

By (7.3), its homotopy fiber has cardinality $|T /\!/ K|\,|K_\partial|$. A wire with exactly one boundary endpoint contributes one group factor to both numerator and denominator, so these factors cancel. A through-wire contributes two factors to K_∂ and one to K, leaving one numerator factor. An internal wire contributes only its denominator factor. This gives (7.8), without requiring a free gauge action. □

For a network without through-wires, (7.8) divides only by the internal changes of basis. For an identity network it gives the number of boundary isomorphisms, as required by (7.6). These are two cases of the same homotopy-fiber calculation.

7.3 A state–costate network and its stabilizers

Let $G : \varnothing \to \{b\}$ consist of one event with one outgoing wire, and let $H : \{b\} \to \varnothing$ consist of one event with one incoming wire. At a fixed space $V = \mathbb{F}_q^d$, their decorations are a vector $u \in V$ and a covector $\varphi \in V^*$. Descent becomes the equivalence

$$[V /\!/ \mathrm{GL}(V)] \times^h_{B\mathrm{GL}(V)} [V^* /\!/ \mathrm{GL}(V)] \simeq [(V \times V^*) /\!/ \mathrm{GL}(V)], \tag{7.9}$$

where g acts on a pair by $(gu, \varphi g^{-1})$.

Proposition 7.3. For $d \geq 2$, the groupoid on the right of (7.9) has $q+3$ isomorphism classes. Their stabilizers are as follows, with one class for each listed value of $a \in \mathbb{F}_q^\times$:

Condition	Representative	Stabilizer order
$u = 0,\ \varphi = 0$	$(0,0)$	$\lvert\mathrm{GL}_d(\mathbb{F}_q)\rvert$
$u \neq 0,\ \varphi = 0$	$(e_1,0)$	$q^{d-1}\lvert\mathrm{GL}_{d-1}(\mathbb{F}_q)\rvert$
$u = 0,\ \varphi \neq 0$	$(0,e_1^*)$	$q^{d-1}\lvert\mathrm{GL}_{d-1}(\mathbb{F}_q)\rvert$
$u,\varphi \neq 0,\ \varphi(u) = 0$	(e_1,e_2^*)	$q^{2d-3}\lvert\mathrm{GL}_{d-2}(\mathbb{F}_q)\rvert$
$\varphi(u) = a \neq 0$	(e_1,ae_1^*)	$\lvert\mathrm{GL}_{d-1}(\mathbb{F}_q)\rvert$

For $d = 1$, the fourth row is absent and there are $q+2$ classes. For $d = 0$ there is one class. In every dimension,

$$|[(V \times V^*) /\!/ \mathrm{GL}(V)]| = \frac{q^{2d}}{|\mathrm{GL}_d(\mathbb{F}_q)|}. \tag{7.10}$$

Proof. The vanishing of u, the vanishing of φ, and the scalar $a = \varphi(u)$ are gauge invariants. Each displayed case is a single orbit. For the two one-sided cases, extend the nonzero vector to a basis or use the dual argument. If $a \neq 0$, decompose $V = \langle u\rangle \oplus \ker\varphi$ to obtain the stated normal form. If both entries are nonzero and $a = 0$, extend u to a basis of $\ker\varphi$ and choose a vector sent to 1 by φ, giving (e_1, e_2^*) after reordering. These cases exhaust all pairs.

The stabilizer of $(e_1, 0)$ fixes the first column and has order $q^{d-1}|\mathrm{GL}_{d-1}|$; the covector case is dual. For (e_1, ae_1^*) with $a \neq 0$, both the first column and first row are fixed, so the remaining block is an arbitrary element of GL_{d-1}. For (e_1, e_2^*), the first column is e_1 and the second row is e_2^*. In the ordered decomposition with first two coordinates distinguished, the stabilizer matrices have the form

$$\begin{pmatrix} 1 & \alpha & r \\ 0 & 1 & 0 \\ 0 & c & M \end{pmatrix}, \qquad M \in \mathrm{GL}_{d-2}(\mathbb{F}_q).$$

There are $1+(d-2)+(d-2)=2d-3$ free scalar entries outside M, giving the stabilizer order in the fourth row. The lower-dimensional cases follow directly. Finally, the whole decoration set has q^{2d} elements, so (7.2) proves (7.10). □

For $d \geq 2$, the stabilizer calculation gives the explicit identity

$$\frac{1}{|\mathrm{GL}_d|} + \frac{2}{q^{d-1}|\mathrm{GL}_{d-1}|} + \frac{1}{q^{2d-3}|\mathrm{GL}_{d-2}|} + \frac{q-1}{|\mathrm{GL}_{d-1}|} = \frac{q^{2d}}{|\mathrm{GL}_d|}. \tag{7.11}$$

Here every general linear group is over $\mathbb{F}_q$. The two framed pieces in (7.9) have cardinality q^d, and the interface has automorphism group GL_d. Thus the same value follows directly from the descent sum.
Some exact values are

q	d	Number of gauge classes	Groupoid cardinality
2	1	4	4
2	2	5	8/3
2	3	5	8/21
3	1	5	9/2
3	2	6	27/16

Contracting the two events evaluates the scalar $a = \varphi(u)$. For $d \geq 1$, the cardinality of the fiber at a is

$$W_d(a) = \begin{cases} \dfrac{q^d + (q^d-1)q^{d-1}}{|\mathrm{GL}_d(\mathbb{F}_q)|}, & a = 0, \\ \dfrac{1}{|\mathrm{GL}_{d-1}(\mathbb{F}_q)|}, & a \neq 0. \end{cases} \tag{7.12}$$

Indeed, $u = 0$ permits every covector when $a = 0$, and each $u \neq 0$ admits q^{d-1} covectors with any specified value on u. The target scalar groupoid is discrete, so these are also the homotopy-fiber weights of the contraction functor. Their sum is (7.10).

7.4 Spherical refinement spaces

Consider discrete causal-nets $A = \{0,1\}$ and $B = [n] = \{1,\dots,n\}$ with $n \geq 2$. A roof $A \leftarrow D \hookrightarrow B$ with a surjective left map is specified, after identifying D with its image, by two disjoint nonempty subsets $S_0, S_1 \subseteq [n]$. Its left map sends S_i to i. Refinement is inclusion of both subsets. Let $\mathcal{R}_n$ be this finite poset. It is the roof-refinement poset for these endpoints in either the coarse-graining or the fusion calculus.

Proposition 7.4. The nerve of $\mathcal{R}_n$ has geometric realization homeomorphic to S^{n-2}. More precisely, $\mathcal{R}_n$ is the poset of nonempty proper faces of the polytope

$$P_n = \mathrm{conv}\{e_i - e_j : i,j \in [n],\ i \neq j\} \ \subset \ \left\{ z \in \mathbb{R}^n : \sum_i z_i = 0 \right\}. \tag{7.13}$$

Proof. For disjoint nonempty S, T, define

$$F_{S,T} = \mathrm{conv}\{e_i - e_j : i \in S,\ j \in T\} = \Delta_S - \Delta_T.$$

It is exposed by the linear functional taking coefficient 1 on S, -1 on T, and 0 elsewhere. Since the supports are disjoint, $F_{S,T}$ is affinely a product of simplices of dimension $|S|+|T|-2$.

Conversely, any nonzero linear functional on the hyperplane containing P_n is represented by coefficients c_i that are not all equal. Its maximum on the roots is $\max_i c_i - \min_j c_j$. The maximizing roots are exactly $e_i - e_j$ with i in the nonempty maximum set S and j in the disjoint nonempty minimum set T. Thus every proper nonempty face is $F_{S,T}$. The vertices of this face determine S, T, and $F_{S,T} \subseteq F_{S',T'}$ holds precisely when $S \subseteq S'$ and $T \subseteq T'$. This proves the poset assertion.

The polytope has dimension $n-1$ and contains the origin in its relative interior: its roots span the displayed hyperplane and occur in opposite pairs. Radial projection therefore identifies its boundary with S^{n-2}. The nerve of its nonempty proper face poset is the barycentric subdivision of that boundary, proving the result. □

This polyhedral complex is the familiar graph-homomorphism complex $\mathrm{Hom}(K_2, K_n)$, expressed here through roof refinements [BK06, Proposition 4.3]. For $n = 3$, the nerve has twelve vertices and twelve edges and is a circle. For $n = 4$, it has face numbers $(50, 144, 96)$ and is a triangulated 2-sphere. In particular, its fundamental groupoid is equivalent to a point while its second homotopy group is $\mathbb{Z}$. Passing from this refinement category to its groupoid completion therefore loses higher homotopy. The space belongs to the explicit refinement calculus; additional gauge relations must be analyzed before comparing it with a larger minor quotient.

7.5 Finite verification

We also checked the examples by exact finite enumeration. Take the source with edges $0 \to 1$, $1 \to 2$, and $0 \to 2$, and the target with two parallel edges $i \to j$ for each $0 \le i < j \le 2$. Enumeration gives 115 functors and all 13,225 ordered functor pairs. All 11 pairs with more than one natural transformation satisfy the complete factorization of Theorem 2.9. Direct enumeration also agrees with 39,675 rooted boundary decisions and 13,645 root-path transport tests. An independent minor calculation examines 10,482 fusion–embedding corners in the 76 simple acyclic graphs on at most four vertices with fixed topological vertex order, allowing multigraph fusion targets. The pushout criterion agrees with exhaustive enumeration of fusion extensions in every case; all 1,885 path-convex corners admit a right filling. Further calculations enumerate invertible matrices, gauge orbits, and stabilizers for the cases in the preceding table, and compute the nerve homology of $\mathcal{R}_n$ for $2 \le n \le 4$ over $\mathbb{F}_2$. These finite calculations check the examples independently of the proofs, which apply to the full parameter ranges stated above.

8 Further questions

The results leave questions about comparison targets, classes of embeddings, refinement topology, and the extension of diagram semantics to more general causal maps.

8.1 Comparison classification beyond free targets

Theorem 2.4 characterizes the targets for which one component determines a comparison. The stronger factorization of Theorem 2.9 uses freeness, and Example 2.11 shows how commuting relations can support ambiguity along a nontrivial loop. It remains to characterize finite cancellative acyclic categories D such that every set $\mathrm{Nat}(F, F')$ containing at least two elements, with weakly connected source, factors through two constant functors. Thin targets satisfy this property vacuously. Such a classification would identify which relations between parallel composites introduce new kinds of ambiguity and whether finitely many source categories suffice to test the property.

8.2 Embedding classes and effective gauge invariants

Theorem 3.4 decides each fusion corner by a finite acyclicity test. Path-convexity supplies a composition-closed class for which every such test succeeds. A further problem is to characterize larger classes of embeddings whose left and right fillings remain in the class. The dependence on both ambient reachability and the identifications made by a fusion suggests a relative formulation for families of allowed fusions.

The support invariant of Theorem 3.15 is finite and survives the full gauge congruence. For larger endpoints, one can seek invariants that combine terminal edge supports with the identifications at successive edge-count levels. A complete invariant, or an effective decision procedure under fixed size bounds, would sharpen the comparison between cartesian normalization, refinement, and automorphism erasure.

8.3 Higher refinement topology and finite presentations

The hom-posets of Ref_Q have monotone composition. Taking their nerves therefore gives a simplicial enrichment, since the nerve preserves products. Proposition 7.4 shows that its mapping spaces can carry nonzero homotopy groups in arbitrarily high degrees. A localization comparison should describe how these spaces change when the additional gauge relations are imposed, including how deletion of intermediate automorphisms acts on refinement data.

The normalization relations of Proposition 4.16 give an explicit, possibly infinite family. One can seek a finite or locally finite coherent presentation by finding a smaller family that generates the critical-branching coherences. Which bounded graph classes admit such a presentation, and can the associated rewriting procedure compute the higher homotopy of their refinement spaces?

8.4 Semantic extension along more general causal maps

Admissible event contractions evaluate an existing diagram and preserve each external wire separately. More general fusions identify wires whose labels may be unrelated objects of the coefficient category. Extending the semantic assignment to such maps requires additional operations on labels and compatible rules for event morphisms. The problem is to identify the algebraic data that make these operations functorial and satisfy a base-change law extending Theorem 6.15.

The present construction uses diagram groupoids and homotopy pullbacks. A semantic theory retaining noninvertible transformations or higher refinement data should replace them with suitable categories or spaces and specify the corresponding descent condition. The boundary transport and unit calculations in Sections 6 and 7 provide the boundary and unit conditions that such an extension must preserve.

References

[BK06] Eric Babson and Dmitry N. Kozlov. *Complexes of graph homomorphisms*. Israel Journal of Mathematics, 152:285–312, 2006. URL https://arxiv.org/abs/math/0310056.

[BC20] John C. Baez and Kenny Courser. *Structured cospans*. Theory and Applications of Categories, 35 (48):1771–1822, 2020. URL https://tac.mta.ca/tac/volumes/35/48/35-48abs.html.

[BCV22] John C. Baez, Kenny Courser, and Christina Vasilakopoulou. *Structured versus decorated cospans*. Compositionality, 4(3), 2022. doi: 10.32408/compositionality-4-3.

[BHW10] John C. Baez, Alexander E. Hoffnung, and Christopher D. Walker. *Higher-dimensional algebra VII: Groupoidification*. Theory and Applications of Categories, 24(18):489–553, 2010. URL https://tac.mta.ca/tac/volumes/24/18/24-18abs.html.

[Cit01] Maurizio G. Citterio. *Classifying spaces of categories and term rewriting*. Theory and Applications of Categories, 9(5):92–105, 2001. URL https://tac.mta.ca/tac/volumes/9/n5/9-05abs.html.

[CL02] J. Robin B. Cockett and Stephen Lack. *Restriction categories I: categories of partial maps*. Theoretical Computer Science, 270(1–2):223–259, 2002. doi: 10.1016/S0304-3975(00)00382-0.

[DK13] Lucas Dixon and Aleks Kissinger. *Open-graphs and monoidal theories*. Mathematical Structures in Computer Science, 23(2):308–359, 2013. doi: 10.1017/S0960129512000138.

[Fon15] Brendan Fong. *Decorated cospans*. Theory and Applications of Categories, 30(33):1096–1120, 2015. URL https://tac.mta.ca/tac/volumes/30/33/30-33abs.html.

[GM12] Yves Guiraud and Philippe Malbos. *Higher-dimensional normalisation strategies for acyclicity*. Advances in Mathematics, 231(3–4):2294–2351, 2012. doi: 10.1016/j.aim.2012.05.010.

[JS91] André Joyal and Ross Street. *The geometry of tensor calculus, I*. Advances in Mathematics, 88(1): 55–112, 1991. doi: 10.1016/0001-8708(91)90003-P.

[Lu23] Xuexing Lu. *Causal-net category*, 2023. URL https://arxiv.org/abs/2201.08963v5. arXiv:2201.08963, version 5.

[LT22] Xuexing Lu and Ziqian Tang. *Causal network condensation*, 2022. URL https://arxiv.org/abs/2112.15515v2. arXiv:2112.15515, version 2.

[Mac98] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, New York, 2 edition, 1998. doi: 10.1007/978-1-4757-4721-8.