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SUPREMUM ESTIMATES FOR $\bar{\partial}$ ON BOUNDED CONVEX DOMAINS IN $\mathbb{C}^2$

Abstract. For every bounded convex domain $\Omega \subset \mathbb{C}^2$, we give a bounded distributional solution of $\bar{\partial} u = f$ for every distributionally $\bar{\partial}$-closed form $f \in L^\infty_{(0,1)}(\Omega)$. The solution satisfies $\|u\|_\infty \le C_\Omega \|f\|_\infty$, where $C_\Omega > 0$ depends only on Ω. The estimate follows from holomorphic localization of Leray rows and boundary cancellation.

1. Introduction

On smoothly bounded strictly pseudoconvex domains, Henkin [13] and, independently, Grauert–Lieb [11] obtained sup-norm estimates by integral representation. Pseudoconvexity alone gives a different picture even in dimension two. Berndtsson [2] constructed a smoothly bounded pseudoconvex Hartogs domain and a closed $(0,1)$-form, continuous on its closure, for which every solution is unbounded.

The Levi form of a convex domain may be degenerate. Range [17] established Hölder estimates on bounded convex domains in $\mathbb{C}^2$ with real-analytic boundary. Fefferman–Kohn [5] obtained local Hölder estimates for the canonical solution near finite-type boundary points of smoothly bounded pseudoconvex domains in $\mathbb{C}^2$. On smoothly bounded convex domains of finite type m in $\mathbb{C}^n$, Diederich–Fischer–Fornæss [4] and Cumenge [3] obtained solutions with Hölder exponent $1/m$ for bounded closed data.

Range [19] obtained an estimate in bounded mean oscillation, measured with respect to volume, for an integral solution operator on smooth convex domains in $\mathbb{C}^2$. His subsequent survey [20] records the L^p estimates for $1 < p < \infty$ alongside the volume-BMO bound.

Supremum estimates are also available on several classes of convex domains of infinite type. Fornæss–Lee–Zhang [8] proved sup-norm estimates on particular convex domains of infinite type, including a bidisc rounded with an exponential profile of exponent $0 < \alpha < 1$. Khanh [15] extended this approach to convex models whose profiles satisfy logarithmic integrability conditions; under those conditions, he also obtained estimates with an associated generalized Hölder modulus. On the bidisc, Henkin [14] obtained the supremum estimate for data of class C^1 up to the boundary. For products of bounded planar domains with C^2 boundary, Yuan [21] proved an L^p estimate for the canonical solution for every $1 \le p \le \infty$.

The difficulty in the general case already arises at smooth boundary points where a complex tangent line has contact of infinite order. At such a point no positive power of the tangential distance gives a uniform lower bound for the separation from the supporting plane along that line.

Range [20, Problem 4] asked in his 1995 survey whether L^1 and L^∞ estimates for $\bar{\partial}$ hold on convex domains in $\mathbb{C}^2$. Fornæss–Kim [7, Problem 3] later asked whether C^1 regularity of the boundary of a bounded convex domain $\Omega \subset \mathbb{C}^2$ suffices for a supremum estimate. Fornæss–Lee–Zhang [8, p. 496] posed it for smooth convex domains and $C^1(\overline{\Omega})$ data; their preprint [9, p. 1] uses arbitrary bounded convex domains and distributionally closed L^∞ data. The theorem below establishes the supremum estimate in this distributional setting.

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For $f = f_1\,\mathrm{d}\bar{z}_1 + f_2\,\mathrm{d}\bar{z}_2$, write $\|f\|_\infty$ for the essential supremum of the Euclidean norm of its coefficient vector. Closedness means that $\partial_{\bar{z}_1} f_2 = \partial_{\bar{z}_2} f_1$ in distributions.

Theorem 1.1. *For every bounded convex domain $\Omega \subset \mathbb{C}^2$ there is a constant $C_\Omega > 0$ such that every $\bar{\partial}$-closed form $f \in L^\infty_{(0,1)}(\Omega)$ admits a solution $u \in L^\infty(\Omega)$ with*

$$\bar{\partial} u = f \quad \text{in } \mathcal{D}'(\Omega), \qquad \|u\|_\infty \le C_\Omega \|f\|_\infty.$$

For each $\kappa \ge 1$ there is a constant $C_\kappa > 0$, depending only on κ, such that one can take $C_\Omega = R C_\kappa$ whenever $B(p,a) \subset \Omega \subset B(p,R)$ and $R/a \le \kappa$.

Every bounded convex domain satisfies the ball conditions of the theorem for some center and radii. In particular, Theorem 1.1 answers Problem 3 of Fornæss–Kim [7] and the L^∞ part of Range's Problem 4 [20]. The quantitative estimate is uniform after fixing the outer radius and an upper bound for R/a.

On a smooth strongly convex approximation, the proof uses a Cauchy–Fantappiè representation in the form of the Leray–Koppelman formula for differential forms [16, 18]. A mixed boundary form couples a generating form adapted to the domain to the Bochner–Martinelli form. Gong [10] records both the mixed form and the resulting boundary and volume integrals.

At a source point ζ on the smooth boundary, a normalized Leray row $\mathcal{L}(\zeta, Z)$ is a covector, holomorphic in the target Z, whose contraction with $Z - \zeta$ equals one. In two complex dimensions the boundary term in the solution formula is linear in this row. When the boundary is nearly flat, the usual row obtained by dividing the complex normal by the supporting linear form can be large even at sources some distance from the target. At each horizontal scale, the mean height of a disk above its complex tangent plane determines the vertical size of the plates. The associated plates control the variation of the normal, while a packing estimate bounds the number of separated plates that can meet a narrow range of the supporting linear form. Two supporting slopes separated by an amount comparable to the disk height divided by its radius yield entire factors that suppress distant sources while preserving holomorphicity in the target. A radial holomorphic difference decomposition then corrects the localized row to restore its contraction.

Closedness of the data enters through the cancellation in the boundary integral. For a support function G with $A = \Re G > 0$, the identity

$$G^{-1} + \bar{G}^{-1} + 2A\bar{G}^{-2} = \frac{4A^2}{G\bar{G}^2}$$

separates an integrable boundary term from a term to which Stokes' formula applies. A partial Legendre transform yields a positive divergence identity that controls the required weighted integral of $4A^2/|G|^3$. The frequency localization used in the holomorphic row is retained in the extension to the interior. Differentiation brings in first derivatives of the normal, whose weighted integrals are controlled by convexity. The resulting solution bound is controlled by the chart sizes, buffers, slopes, and cutoff bounds. These quantities remain uniform along the approximating domains, which permits passage to the original domain.

Section 2 derives the phase pushforward estimates and the curvature bounds associated with the plate decomposition. The localized Leray row is constructed in Section 3, where boundary cancellation and the uniform approximation argument complete the proof of Theorem 1.1.

2. Phase pushforward and curvature integrals

Lebesgue measure on $\mathbb{C}^2$ is denoted by $\mathrm{d}V$ and planar area by $\mathrm{d}A$. All scalar products between complex rows and displacements below are bilinear. For real horizontal vectors the

dot denotes the Euclidean scalar product. Write $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar z} = (\partial_x + i\partial_y)/2$ for $z = x + iy$.

A smooth domain is called strongly convex here when the real second fundamental form of its boundary is positive definite. Suppose for now that $D \Subset \mathbb{C}^2$ has smooth boundary and that f is smooth and closed on a neighborhood of $\overline{D}$. Assume that the row $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ has coefficients in $C^{1,\infty}(\partial D \times D)$, with one source derivative and arbitrary target derivatives, is holomorphic in Z and satisfies

$$\mathcal{L}(\zeta, Z) \cdot (Z - \zeta) = 1. \tag{2.1}$$

For $E = (e, b) = Z - \zeta$, write

$$\mathcal{J}(\zeta, Z) = \frac{\mathcal{L}_2 \bar e - \mathcal{L}_1 \bar b}{|E|^2}.$$

The row $-\mathcal{L}$ is normalized against $\zeta - Z$. The Leray–Koppelman formula [18, Ch. IV, Theorem 3.6], in the mixed-form notation of Gong [10, eqs. (2.4)–(2.6)], gives

$$\begin{aligned} u(Z) &= u_0(Z) + \frac{1}{4\pi^2} \int_{\partial D} \mathcal{J}(\zeta, Z) f \wedge \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2, \\ u_0(Z) &= \frac{1}{\pi^2} \int_D \frac{\bar e f_1(\zeta) + \bar b f_2(\zeta)}{|E|^4} \, \mathrm{d}V(\zeta), \end{aligned} \tag{2.2}$$

and $\bar\partial u = f$ in D. Holomorphicity in Z and $\bar\partial f = 0$ remove the remaining terms of the homotopy formula. In complex dimension two the source component of the mixed form is $-\mathcal{J}\, \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2/(4\pi^2)$; the boundary minus sign in that formula gives the positive coefficient in (2.2), with the outward boundary orientation. Integration of $|E|^{-3}$ in four dimensions gives $\|u_0\|_\infty \le C \operatorname{diam}(D) \|f\|_\infty$.

Local normalized rows may be added using a smooth partition of unity depending only on the source. Under a unitary coordinate change $\zeta' = U\zeta$, the row in the new coordinates is $\mathcal{L}U^*$, and $\mathcal{J}$ gains the factor $\overline{\det U}$, since both rows in the determinant transform by right multiplication by U^*. The form $\mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2$ gains the factor $\det U$, so their product is invariant. Thus each chart integral can be estimated in its own unitary coordinates.

For the geometric estimates, assume in addition that D is convex and its boundary has positive definite real second fundamental form. To estimate the boundary kernel, use a unitary chart in which the source is (s, γ), with $\gamma = P(s, t) + it$, $s = x + iy$ and $t \in \mathbb{R}$. The domain locally lies on the side $\Re\xi > P(s, \Im\xi)$. We work on compact subboxes of a larger convex parameter box on which P is smooth with positive definite real Hessian. These conditions hold for the smooth strongly convex domains used in the approximation. The covectors of the tangent planes support the whole convex domain. The estimates are uniform over targets $Z = (z, w)$ anywhere in D. Write $X = (x, y, t)$, $\mathrm{d}X = \mathrm{d}A_s \, \mathrm{d}t$ and

$$\begin{gathered} e = z - s, \quad b = w - \gamma, \quad r = |e|, \quad c = P_x - iP_y, \quad q = P_t, \quad \beta = 1 + iq, \quad V = c/\beta, \\ l = (-c, \beta), \qquad F = b - Ve, \qquad G = l \cdot (e, b) = \beta F = A + iY. \end{gathered} \tag{2.3}$$

Thus $A > 0$ is the unnormalized real supporting-plane gap. If a formula involves $\lesssim$ or $\asymp$, its constants may depend on a bound for $|\nabla P|$, the sizes of the buffered chart and the bounding ball, and any fixed exponents displayed in it. The estimates hold uniformly in the target, the scales and all smooth models with these chart bounds. For annular estimates, $r \sim R$ means $R/2 < r < 2R$.

The component of the boundary correction in these coordinates has the exact decomposition, for $e \neq 0$,

$$\mathcal{J} = \frac{\mathcal{L}_2}{e} - \frac{\bar b}{e(r^2 + |b|^2)}. \tag{2.4}$$

Multiplication by $e(r^2+|b|^2)$ and (2.1) give the identity. The second term has uniformly bounded absolute integral on the graph. Indeed, with $t_* = \Im w$, its absolute value is at most $C/[r(r+|t-t_*|)]$. On bounded parameter intervals the integral in t is at most $C(1+\log^+(1/r))/r$. The remaining planar integral is finite uniformly in z, and the graph area element is bounded by a constant times $\mathrm{d}X$.

Lemma 2.1. *On a buffered graph as above, the following estimates hold.*

(i) *The supporting-plane gap satisfies*

$$\int_{r\sim R} \frac{A^2}{|G|^3}\frac{\mathrm{d}X}{r} \le CR. \tag{2.5}$$

(ii) *For $m>1$ and $N>0$,*

$$\int_{r\sim R} (1+N|F|)^{-m}\,\mathrm{d}X \le C_m R^2/N. \tag{2.6}$$

Both integrals are restricted to the source subbox.

Proof. Write $w = u_* + it_*$. We first perform the calculation for a globally smooth, superlinear convex P with positive definite Hessian and $P(z,t_*) < u_*$. The reduction of the local case to this calculation is given at the end of the proof.

For each fixed s, strict monotonicity and superlinear growth make $t \mapsto q = P_t(s,t)$ a bijection of $\mathbb{R}$ onto $\mathbb{R}$. The inverse function $t = t(s,q)$ gives the partial Legendre transform

$$Q(s,q) = P(s,t) - qt, \qquad Q_q = -t, \qquad g = \nabla_s Q = \nabla_s P,$$

where the gradients in s are real two-dimensional gradients. Differentiating $P_t(s,t(s,q)) = q$ yields

$$L_0 = -Q_{qq} = 1/P_{tt} > 0, \qquad H = Q_{ss} = P_{ss} - P_{st}P_{ts}/P_{tt} \ \ge 0.$$

Evaluating D^2P on $(v, -P_{ts}v/P_{tt})$ gives the last inequality. With J_0 the positive quarter-turn in the horizontal plane, the real vector $h = s - z$ has a unique representation $h = (I + qJ_0)\sigma$. Write $\rho = |\sigma|$, $a = J_0\sigma$. In these coordinates

$$\tau = t - t_* + a\cdot g, \quad A = u_* - qt_* + h\cdot g - Q, \qquad Y = qA - (1+q^2)\tau.$$

Expansion of $\beta(w-\gamma)+ch$, together with $(J_0+qI)h = (1+q^2)J_0\sigma$, gives the last identity. In particular $\Im F = -\tau$, and $|G| \asymp (A^2+\tau^2)^{1/2}$ on bounded q-intervals. The two changes of variables give

$$\mathrm{d}X = (1+q^2)L_0\ \mathrm{d}A_\sigma\ \mathrm{d}q. \tag{2.7}$$

In fact the first change of variables is in t at fixed s, with derivative L_0. At fixed q the horizontal change has determinant $1+q^2$.

(i) In the following identities, a q-derivative holds σ fixed. A radial derivative holds q and the direction of σ fixed. Since $s_q = a$ and $\rho s_\rho = h$, differentiation gives

$$\begin{aligned}
\tau_q &= -Q_{qq} - Q_{qs}\cdot a + a\cdot Q_{sq} + H[a,a] = L_0 + H[a,a],\\
\rho\tau_\rho &= -Q_{qs}\cdot h + a\cdot g + H[a,h],\\
A_q &= t - t_* + h\cdot Q_{sq} + H[h,a] = \tau - \rho\tau_\rho + 2H[h,a],\\
\rho A_\rho &= H[h,h].
\end{aligned}$$

Writing $S = A^2+\tau^2$, these relations imply

$$\partial_q\frac{\tau}{\sqrt{S}} + \partial_\rho\frac{\rho A}{\sqrt{S}} = \frac{A^3 + A^2L_0 + H[Aa-\tau h, Aa-\tau h]}{S^{3/2}}. \tag{2.8}$$

Indeed, multiplication of the left side by $S^{3/2}$ gives

$$A^2\tau_q - A\tau A_q + AS + \rho\tau^2 A_\rho - \rho A\tau\tau_\rho.$$

The two terms involving τ_ρ cancel on substitution. All terms on the right of (2.8) are nonnegative. Integration at a fixed angle on $[\rho_1, \rho_2] \times [-M_0, M_0]$ consequently gives

$$\int_{\rho_1}^{\rho_2} \int_{-M_0}^{M_0} \frac{A^2 L_0}{S^{3/2}} \, \mathrm{d}q \, \mathrm{d}\rho \le 2(\rho_2 - \rho_1) + 2M_0\rho_2.$$

Here $|\tau|/\sqrt{S} \le 1$ and $0 \le A/\sqrt{S} \le 1$. Choose M_0 greater than the local bound for $|q|$. The image of the source annulus lies in such a rectangle with $\rho_1 = R/(2\sqrt{1+M_0^2})$ and $\rho_2 = 2R$. Since $r = \sqrt{1+q^2}\rho$, polar coordinates in (2.7) give

$$\frac{\mathrm{d}X}{r} = \sqrt{1+q^2}\, L_0 \, \mathrm{d}\rho \, \mathrm{d}\vartheta \, \mathrm{d}q.$$

Integration in angle proves (2.5).

(ii) The image of the physical annulus is contained in $\{|\sigma| < 2R\} \times [-M_0, M_0]$. Since $1 + q^2$ is bounded there and $|F| \ge |\tau|$, (2.7) gives

$$\begin{aligned}\int_{r\sim R} (1 + N|F|)^{-m} \, \mathrm{d}X &\le C \int_{|\sigma|<2R} \int_{-M_0}^{M_0} (1 + N|\tau|)^{-m} L_0 \, \mathrm{d}q \, \mathrm{d}A_\sigma \\ &\le C_m R^2/N.\end{aligned}$$

At fixed σ, the map $q \mapsto \tau$ has derivative at least L_0. The inner integral is therefore at most $2/((m-1)N)$, which gives the last inequality.

To pass to a graph restricted to a box, take the supremum of its affine tangent functions over a compact convex box whose interior contains the source subbox. This gives a globally Lipschitz convex extension equal to P on that box. Each of those affine functions is a global support for D. At (z, t_*) the extension is at most $u_* - \operatorname{dist}(Z, \partial D)$, by testing the vertical radius of the open ball about Z in D. Add a nonnegative convex function which vanishes on a ball containing the larger box and (z, t_*) and grows quadratically at infinity, for example the squared distance to that ball. Smooth by positive convolution and add $\epsilon|X|^2$, with the convolution radius and ϵ tending to zero. On the source subbox the result converges in C^1 to P. At the fixed target its value stays below u_* for all sufficiently small parameters. Thus all tangent-plane gaps of these global smooth functions are positive. Their q-ranges on the source box are bounded by a common M_0, and the bounds just proved depend only on M_0 and the annulus. Fatou's lemma on the original physical annulus gives both assertions. □

The plate covering uses a common upper bound M_P for $|DP|$ on the smooth strongly convex graph. All chart cutoffs have compact support in a smaller coordinate box. We use a horizontal scale $0 < \ell \le \ell_0 \le 1$, with ℓ_0 small compared with the distance between the boxes. For the sole purpose of evaluating a height outside the larger box, extend P to the supremum of its supporting affine functions there. This gives a convex M_P-Lipschitz function on $\mathbb{R}^3$ which agrees with P in the box. Every tangent plane used below is taken in the original smooth graph.

At a horizontal center x_Q and vertical coordinate T, the disk average is

$$\langle f \rangle_\ell = (\pi\ell^2)^{-1} \int_{|y|<\ell} f(y) \, \mathrm{d}A_y.$$

With $a_+ = \max(a, 0)$, the disk height is

$$B_T(\lambda, y) = P(x_Q + y, T + \Im(\lambda y)) - P(x_Q, T) - \Re(\lambda y),$$
$$h(T) = \inf_{\lambda \in \mathbb{C}} \langle B_T(\lambda, \cdot)_+ \rangle_\ell.$$

The dependence on Q and ℓ is understood. At a smooth center write $\lambda_T = V(x_Q, T)$ and $\beta_T = 1 + iP_t(x_Q, T)$. Its supporting plane gives $B_T(\lambda, y) \geq \Re[\beta_T(\lambda_T - \lambda)y]$. The functional being minimized is continuous and is at least $2\ell|\beta_T||\lambda - \lambda_T|/(3\pi)$, so it tends to infinity with $|\lambda|$ and attains its infimum. Every minimizing slope $\lambda_{\min}$ satisfies

$$\ell|\lambda_{\min} - \lambda_T| \leq Ch(T).$$

The competitor $\lambda = 0$ gives $h(T) \leq M_P\ell$. Since the functions B_T are $(1 + M_P)\ell$-Lipschitz in λ, comparison with $\lambda_{\min}$ gives

$$h(T) \leq \langle B_T(\lambda_T, \cdot) \rangle_\ell \leq Ch(T), \qquad 0 \leq B_T(\lambda_T, y) \leq Ch(T) \quad (|y| \leq \ell/2). \tag{2.9}$$

Indeed the last integrand is nonnegative and convex. Its value at y is at most its mean on the disk of radius $\ell/2$ about y, which is at most four times its mean on the full disk. Strong convexity gives $h(T) > 0$ at every center in use. At every T, including those outside the box, h is $2M_P$-Lipschitz by comparison of the same competitors.

For two smooth centers T, U on the same vertical line, the support at U tested at $(x_Q + y, T + \Im(\lambda_T y))$ gives

$$\tfrac{1}{2}\ell|\beta_U|\,|\lambda_T - \lambda_U| \leq Ch(T) + P(x_Q, T) - P(x_Q, U) - q_U(T - U), \qquad q_U = P_t(x_Q, U).$$

Here y ranges over the half disk in (2.9). Interchanging the two centers and adding proves

$$\ell|\lambda_T - \lambda_U| \leq C\{h(T) + h(U) + (q_T - q_U)(T - U)\}. \tag{2.10}$$

In particular the last term may be replaced by $2M_P|T - U|$.

Lemma 2.2. *There are fixed constants $\alpha_0, \kappa_0 > 0$, depending only on the common chart bounds, such that for every $0 < \ell \leq \ell_0$ there is a smooth nonnegative partition $\chi_{Q,i} = \chi_Q\varphi_{Q,i}$ on a neighborhood of the chart cutoff, with bounded overlap. The horizontal supports have radius $\alpha_0\ell$ about x_Q and bounded multiplicity. Writing $h_i = h(t_i)$, $\lambda_i = \lambda_{t_i}$, the support of $\chi_{Q,i}$ is contained in the plate*

$$E_i = \{(s, t) : |s - x_Q| < \alpha_0\ell,\ |t - t_i - \Im(\lambda_i(s - x_Q))| < \kappa_0 h_i\}.$$

Throughout E_i,

$$|V(s, t) - \lambda_i| \leq Ch_i/\ell, \qquad |\gamma(s, t) - \gamma(x_Q, t_i) - \lambda_i(s - x_Q)| \leq Ch_i. \tag{2.11}$$

The partition satisfies $|D\chi_{Q,i}| \leq C/h_i$. At successive scales ℓ and ℓ/K, with K sufficiently large depending only on M_P and the covering constants, intersecting fine and coarse plates satisfy $h_{\text{fine}} \leq h_{\text{coarse}}/4$.

Proof. Use a square lattice of mesh $\alpha_0\ell/4$ for the centers x_Q. Translate a fixed nonnegative radial bump of $(s - x_Q)/(\alpha_0\ell)$, equal to one on the half disk and supported in the disk of radius $3/4$. Divide by the sum of all translates. At every point of the working patch the denominator is at least one and at most a fixed number of translates are nonzero. This gives the weights χ_Q with $|D\chi_Q| \leq C/\ell$ and support compactly contained in the indicated disk. For each Q take a closed vertical interval J containing the vertical projection of the cutoff neighborhood, with a buffer larger than $2M_P\alpha_0\ell$ at either end. The larger box contains these intervals and all disks of radius ℓ about their centers with slopes λ_T. For estimates involving vertical integration it is convenient to reserve a further fixed buffer in the larger box. These choices are possible uniformly by reducing ℓ_0.

Choose a maximal disjoint family of intervals $I_i^0 = (t_i - \delta h_i, t_i + \delta h_i)$ with centers in J. It is finite because h has positive minimum on J in each fixed model at a fixed scale. Maximality gives, for every $T \in J$, a center satisfying

$$|T - t_i| \le \delta(h(T) + h_i).$$

If $2M_P\delta$ is small, Lipschitz continuity gives $h(T) \asymp h_i$. For $y = s - x_Q$ the continuous map $T \mapsto T + \Im(\lambda_T y)$ moves each endpoint of J by at most $M_P\alpha_0\ell$. Its image therefore covers the vertical projection under consideration. For a preimage T of t, (2.10) gives

$$|t - t_i - \Im(\lambda_i y)| \le C(\delta + \alpha_0)h_i.$$

First choose $\kappa_0 > 0$ small in terms of M_P, then choose α_0, δ small enough to make the last bound at most $\kappa_0 h_i/2$. This construction applies also to cells meeting the edge of the cutoff.

If two plates for the same Q intersect, (2.10) yields

$$|t_i - t_j| \le \kappa_0(h_i + h_j) + C\alpha_0(h_i + h_j + |t_i - t_j|).$$

With the preceding choices, $|t_i - t_j| \le C(\kappa_0 + \alpha_0)(h_i + h_j)$ and the Lipschitz bound on h gives $h_i \asymp h_j$. Their disjoint cores lie in an interval of length $O(h_i)$ whenever the plates contain a common point. This bounds the multiplicity. Choose a smooth bump of the normalized sheared vertical coordinate which equals one for modulus at most $\kappa_0/2$ and vanishes for modulus at least $3\kappa_0/4$. Division by the sum of these bumps within the same Q gives $\varphi_{Q,i}$ wherever χ_Q is needed. The sum is at least one, its active terms have comparable heights, and $|\lambda_i| \le M_P$. Differentiation gives the claimed derivative bound since $h_i \le M_P\ell$.

In the coordinates $y = s - x_Q$, $u = t - t_i - \Im(\lambda_i y)$ the excess over the support at (x_Q, t_i) is

$$U(y,u) = P(x_Q + y, t_i + \Im(\lambda_i y) + u) - P(x_Q, t_i) - \Re(\lambda_i y) - q_i u.$$

It is convex and nonnegative. Equation (2.9) and vertical Lipschitz continuity give

$$U(y,u) \le C(h_i + |u|) \qquad (|y| \le \ell/2).$$

Differences over horizontal segments of length $\ell/4$ in the inner disk therefore yield

$$|D_y U| \le C(h_i + |u|)/\ell \qquad (|y| \le \ell/4).$$

In complex notation,

$$U_{y_1} - iU_{y_2} = \beta(s,t)(V(s,t) - \lambda_i). \tag{2.12}$$

Also $\gamma(s,t) - \gamma(x_Q, t_i) - \lambda_i y = U(y,u) + (q_i + i)u$. These facts give (2.11).

Use subscripts f, p for intersecting fine and coarse plates and take a common point (s,t). Equation (2.11) gives

$$|\lambda_f - \lambda_p| \le C(h_f/\ell_f + h_p/\ell_p).$$

The displacement $d = t_f - t_p - \Im(\lambda_p(x_f - x_p))$ consequently satisfies

$$|d| \le \kappa_0(h_f + h_p) + |\lambda_f - \lambda_p|\alpha_0\ell_f \le C(h_f + h_p).$$

Here x_f, x_p are the horizontal centers. The disk about x_f of radius ℓ_f lies in $|s - x_p| < \ell_p/4$, since $|x_f - x_p| \le \alpha_0(\ell_p + \ell_f)$ and K can be fixed large. Use λ_p as a competitor at (x_f, t_f). The resulting residual is the horizontal increment of the coarse $U(\cdot, d)$ from $x_f - x_p$ at fixed d. The preceding horizontal gradient bound gives

$$h_f \le C(\ell_f/\ell_p)(h_p + h_f).$$

Taking $K \ge 5C$ gives the asserted contraction. □

Lemma 2.3. *Associated with each plate E_i are two fixed boundary points $X_j = (s_j, \gamma(s_j, t_j'))$, $j = 0, 1$. The first is its center, and the second lies on the tangent disk parametrized by $|s - x_Q| \le \ell$ and $t = t_i + \Im(\lambda_i(s - x_Q))$. Their slopes $V_j = V(s_j, t_j')$ and support rows $l_j = (-c_j, \beta_j)$ satisfy*

$$|V_1 - V_0| \asymp h_i/\ell, \qquad |V(s,t) - V_j| \le Ch_i/\ell,$$
$$0 \le \Re[l_j \cdot (\zeta - X_j)] \le Ch_i, \qquad \Re[l_j \cdot (Z - \zeta)] \ge -Ch_i$$

for $(s,t) \in E_i$, $\zeta = (s, \gamma(s,t))$, and every interior target Z.

Proof. The mean of the central tangent excess $U(y, 0)$ is at least h_i. On a radius reaching a point where $U(y, 0) \ge h_i$, convexity gives a radial derivative at that point of at least h_i/ℓ. The gradient vanishes at zero. Along this radius its norm first reaches $h_i/(2\ell)$ at some y_1. The second point lies over $(x_Q + y_1, t_i + \Im(\lambda_i y_1))$. Equation (2.12) and $1 \le |\beta_1| \le 1 + M_P$ prove the separation. The other slope estimate follows from (2.11). The excess at the central point over the second support is

$$\Re[l_1 \cdot (X_0 - X_1)] = D_y U(y_1, 0) \cdot y_1 - U(y_1, 0) \le h_i/2.$$

From X_0 to ζ the second coordinate increment is $\lambda_i(s - x_Q) + O(h_i)$ by (2.11). Applying either row costs at most $C(\ell|V_j - \lambda_i| + h_i)$. Both lower support inequalities follow from convexity, first at ζ and then at Z. □

Fix Q and a target $Z = (z, w)$. For the estimates over this cell, write $n_Q = 1 + |x_Q - z|/\ell$ and $S_i = h_i + \inf_{E_i} |F|$. The sums run over plates for this one horizontal cell. The reference points below are selected for each pair (Q, Z). The plate partitions and the nodes in Lemma 2.3 are fixed before choosing Z.

Lemma 2.4. *The plate family over Q satisfies the following packing estimates.*

(i) *There are at most Cn_Q points $T_* \in J$ and an assignment of each plate to one of them such that*

$$h_* + |t - T_* - \Im(\lambda_*(s - x_Q))| \le Cn_Q S_i \qquad ((s,t) \in E_i), \tag{2.13}$$

where $h_* = h(T_*)$ and $\lambda_* = \lambda_{T_*}$.

(ii) *The plate heights satisfy*

$$\sum_i (h_i/S_i)^2 \le Cn_Q^3. \tag{2.14}$$

Proof. (i) Equation (2.11) and $|z - s| \le n_Q \ell$ on E_i give

$$|w - \gamma(x_Q, t_i) - (z - x_Q)\lambda_i| \le C_0 n_Q S_i.$$

Indeed the difference between the quantity inside the modulus and F is $O(n_Q h_i)$ at every point of the plate. Consider the intervals

$$J_i = [t_i - C_1 n_Q S_i, t_i + C_1 n_Q S_i] \cap J.$$

Since their centers belong to J, clipping them to J preserves pairwise intersection. Two disjoint members satisfy

$$d = |t_i - t_j| > C_1 n_Q (S_i + S_j).$$

Subtracting the imaginary parts of the preceding bounds then gives

$$|z - x_Q||\lambda_i - \lambda_j| \ge d/2 \qquad (C_1 \ge 2C_0).$$

In particular disjointness is impossible for $z = x_Q$. Otherwise (2.10) and $h_i + h_j \le d/(C_1 n_Q)$ give

$$(q_i - q_j)\operatorname{sgn}(t_i - t_j) \ge c/n_Q$$

after increasing C_1, because $\ell/|z-x_Q| \ge 1/n_Q$. Partition $[-M_P, M_P]$ into at most $2+4M_P n_Q/c$ intervals of length at most $c/(2n_Q)$ and group the plates by their q_i values. Within each group the J_i intersect pairwise. The one-dimensional Helly property gives a common point $T_* \in J$: the largest left endpoint belongs to every interval in the group. Assign each plate to the point of its group. The assignment satisfies

$$|t_i - T_*| \le C_1 n_Q S_i.$$

The Lipschitz bound for h and (2.10) therefore yield

$$h_* \le C n_Q S_i, \qquad \ell|\lambda_i - \lambda_*| \le C n_Q S_i.$$

The plate inequality now gives (2.13).

(ii) We first establish the height integral

$$\int_{\mathbb{R}} \frac{h(T_*+u)}{(h_*+|u|)^2}\,\mathrm{d}u \le C. \tag{2.15}$$

Use the same slope λ_* as a competitor at every T_*+u. For $|y| < \ell$ write $r_y = |y|/\ell$, $a = (1-r_y)/2$, and $y' = y/(1-a)$. The map $y \mapsto y'$ takes the disk onto itself and has Jacobian $4/(1+r_y)^3 \in [1/2, 4]$. The point tested by this competitor is the convex combination, with weights $1-a, a$, of $(x_Q+y', T_*+\Im(\lambda_* y'))$ and $(x_Q, T_*+u/a)$. Thus, with $p(u) = P(x_Q, T_*+u)$,

$$B_{T_*+u}(\lambda_*, y)_+ \le (1-a)B_{T_*}(\lambda_*, y') + D_a(u),$$
$$D_a(u) = (1-a)p(0) + ap(u/a) - p(u) \ge 0.$$

The first term has mean at most Ch_* by the Jacobian bound and (2.9). Its integral against $(h_*+|u|)^{-2}$ is bounded. For the second term write $v(u) = p(u) - p(0)$. For $0 < b < B$,

$$\int_b^B \frac{D_a(u)}{u^2}\,\mathrm{d}u = \int_B^{B/a} \frac{v(u)}{u^2}\,\mathrm{d}u - \int_b^{b/a} \frac{v(u)}{u^2}\,\mathrm{d}u.$$

The bounded increasing quotients $v(u)/u$ have limits $p'(0+)$ and $p'(+\infty)$ at the two ends of the positive half-line. Applying the same identity to $p(-u)$ yields

$$\int_{\mathbb{R}} \frac{D_a(u)}{u^2}\,\mathrm{d}u \le 2M_P \log(1/a).$$

The disk mean of $\log(1/a)$ is $2\int_0^1 r\log(2/(1-r))\,\mathrm{d}r < \infty$. Tonelli's theorem proves (2.15).

On I_i^0, $h(T) \asymp h_i$ and, for its assigned point, $h_* + |T - T_*| \le C n_Q S_i$. Since $|I_i^0| = 2\delta h_i$, restricting (2.15) to the disjoint cores assigned to one point gives $\sum (h_i/S_i)^2 \le C n_Q^2$. Summing over the points proves (2.14). □

Lemma 2.5. *The plates over each Q satisfy*

$$\sum_i \int_{E_i} \frac{|DV(s,t)|}{h_i + |F(s,t)|}\,\mathrm{d}A_s\,\mathrm{d}t \le C n_Q^2 \sqrt{\ell}. \tag{2.16}$$

Here D denotes the full real derivative in (s,t). The constant is uniform over the smooth models with the common chart bounds.

Proof. At an assigned point T_* take the coordinates $y = s - x_Q$, $u = t - T_* - \Im(\lambda_* y)$ and let U be the excess over the supporting plane as in the proof of Lemma 2.2. In these coordinates U is nonnegative and convex, $|DU| \le C$, and $U(y,0) \le Ch_*$ for $|y| \le \ell/2$. We integrate in a fixed u-interval about zero containing the assigned plates for $|y| \le \alpha_0 \ell$. Such an interval is obtained by translating J by $-T_*$ and enlarging either end by $C(M_P)\ell$ to cover both the tilt and the plate widths. The chart buffer allows this interval for all $|y| \le \ell/2$ and its length is uniformly bounded.

For $|y| \le \ell/4$ and any horizontal unit vector v, the support at (y,u), tested at $(y \pm \ell v/4, 0)$, gives

$$\tfrac{1}{4}\ell |D_y U(y,u)| \le C_0 h_* + u U_u(y,u) - U(y,u). \tag{2.17}$$

In particular $|D_y U| \le C(h_* + |u|)/\ell$. With y fixed, suppress it in the notation. For $b > 0$ in the working interval, integration by parts gives

$$\begin{aligned}\int_0^b \frac{C_0 h_* + u U'(u) - U(u)}{(h_* + u)^2}\,\mathrm{d}u &= \frac{C_0 b}{h_* + b} + \frac{bU(b)}{(h_* + b)^2} - 2h_* \int_0^b \frac{U(u)}{(h_* + u)^3}\,\mathrm{d}u \\ &\le C,\end{aligned}$$

because $0 \le U(b) \le C(h_* + b)$. Applying (2.17) and reflecting the negative half interval therefore yields

$$\int \frac{|D_y U(y,u)|}{(h_* + |u|)^2}\,\mathrm{d}u \le C/\ell$$

on the entire vertical line segment.

Take a nonnegative horizontal cutoff supported in $|y| < \ell/4$ and equal to one on $|y| \le \alpha_0 \ell$, with gradient bounded by C/ℓ. Write $\mu = \operatorname{tr} U_{yy} \ge 0$. Integration by parts against this cutoff, followed by the line estimate, gives

$$\int_{|y| < \alpha_0 \ell} \frac{\mu}{(h_* + |u|)^2}\,\mathrm{d}A_y\,\mathrm{d}u \le C.$$

All subsequent double integrals are over this inner cylinder, with measure $\mathrm{d}A_y\,\mathrm{d}u$. The same cutoff and $|D_y U| \le C$ give $\int \mu \le C\ell$: the horizontal gradient of the cutoff has L^1 norm $O(\ell)$, and the vertical interval has bounded length. Consequently

$$\int \frac{\mu}{h_* + |u|} \le \Big(\int \frac{\mu}{(h_* + |u|)^2} \Big)^{1/2} \Big(\int \mu \Big)^{1/2} \le C\sqrt{\ell}.$$

Convexity on each vertical line and boundedness of U_u give $\int U_{uu} \le C\ell^2$. As $|U_{yu}|^2 \le \mu U_{uu}$, Cauchy–Schwarz and the pointwise consequence of (2.17) yield, respectively,

$$\int \frac{|U_{yu}|}{h_* + |u|} \le C\ell, \qquad \int \frac{|D_y U| U_{uu}}{h_* + |u|} \le C\ell.$$

Throughout the reference cylinder, (2.12) with the fixed slope λ_* reads

$$q = q_* + U_u, \qquad V = \lambda_* + \frac{U_{y_1} - i U_{y_2}}{\beta}.$$

Differentiation throughout this cylinder gives

$$|DV| \le C\big(\operatorname{tr} U_{yy} + |U_{yu}| + |D_y U| U_{uu} \big)$$

because $|D_y U|$ is bounded, $|\beta| \ge 1$, and both the shear and its inverse have bounded norms. The change of variables has determinant one. Consequently, on the reference cylinder,

$$\int \frac{|DV|}{h_* + |u|}\,\mathrm{d}A_y\,\mathrm{d}u \le C\sqrt{\ell}.$$

On an assigned plate, (2.13) gives

$$(h_i + |F|)^{-1} \le C n_Q/(h_* + |u|).$$

Bounded plate multiplicity and the $O(n_Q)$ reference points now give (2.16). □

The packing estimate also controls derivatives of the partition. For $R/2 < r < 2R$, $m > 3$, and dyadic $N \geq c/h_i$, the factor $(1 + N|F|)^{-2}$ is at most $C(h_i/S_i)^2$ throughout E_i. For the remaining factor, (2.6) with exponent $m - 2 > 1$ bounds its integral on the shell by $C_m R^2/N$, even after the plate restriction is dropped. Summation over these frequencies yields

$$\sum_{N \geq c/h_i} \int_{E_i \cap \{R/2 < r < 2R\}} (1 + N|F|)^{-m} \, \mathrm{d}A_s \, \mathrm{d}t \leq C_m R^2 h_i (h_i/S_i)^2. \tag{2.18}$$

3. Localized Leray rows and boundary cancellation

Retain the notation of (2.3) on a smooth strongly convex domain D, with $X = (s, t)$, $Z = (z, w)$ and $E = (e, b) = (z - s, w - \gamma)$. The local construction takes place on a fixed source parameter patch $\mathcal{U} \subset \mathbb{R}^3$ with compact closure in the larger graph box, covered by the partitions of Lemma 2.2. All unqualified boundary integrals in this section are over $\mathcal{U}$. The estimates for the local construction depend only on the slope and chart bounds and on the fixed parameters of the plate construction.

The two supporting rows associated with a plate i of nominal radius ℓ_i and height h_i are $l_{ij} = (-c_{ij}, \beta_{ij})$, $j = 0, 1$, from Lemma 2.3. Their slopes are $V_{ij} = c_{ij}/\beta_{ij}$, $\beta_{ij} = 1 + iq_{ij}$. For $d \geq 0$ let

$$E_d = (e, b - d), \quad Y_{ij}(d) = (l - l_{ij}) \cdot E_d, \quad M = \bar{G} + \beta d, \qquad n_i = 1 + r/\ell_i.$$

On a plate over Q, the quantities n_i and $n_Q = 1 + |x_Q - z|/\ell_i$ are comparable, uniformly in the source and target. At successive radii $\ell_p = K\ell_i$, the pointwise quantities satisfy $n_p \leq n_i \leq K n_p$.

Choose a nonnegative $\varphi \in C_c^\infty((1/4, 1/2))$ of integral one. The functions

$$\psi(y) = \int \varphi(u) e^{uy} \, du, \qquad \Phi_i(E_d) = \prod_{j=0}^{1} \psi(a Y_{ij}(d)/h_i)$$

are entire in their displayed complex arguments, with $\Phi_i(0) = 1$. Here $a > 0$ is fixed. We choose a fixed C_0 sufficiently large below. Let η be smooth, nondecreasing, zero on $[0, 1]$, and one on $[2, \infty)$. Write $\eta_i(v) = \eta(v h_i/C_0)$.

Lemma 3.1. *Fix a plate i. For every source whose boundary projection $X = (s, t)$ belongs to E_i, every target $Z \in D$, and every depth $d \geq 0$, the packet has the following bounds.*

(i) *For every $m \geq 0$, if $|G| \leq h_i$ and $d = 0$, then $|\Phi_i(E)| \leq C_m n_i^{-m}$.*

(ii) *For $m, k \geq 0$ and $N > 0$ with $N h_i \geq C_0/2$, suppose g is smooth with compact support in $[N/2, 2N]$ and $|g^{(j)}| \leq B_j N^{-j}$ for $0 \leq j \leq J$, with J sufficiently large in terms of m, k. For any fixed integer $p \geq 0$,*

$$\left| \Phi_i(E_d)^2 \int g(v) (vA)^p e^{-vM} \, dv \right| \leq C N e^{-c_1 N(A+d)} n_i^{-m} (1 + N|F|)^{-k}. \tag{3.1}$$

The constant depends also on p and the B_j. The same estimate holds with a single packet, or with any one of its factors ψ replaced by ψ'. At $d = 0$ it holds for the exponential with G in place of M.

(iii) *Under the conditions of (ii), for $\xi = \gamma + d$ and $T = \partial_{\bar{s}}$ or $\partial_{\bar{\xi}}$ in the independent physical coordinates (s, ξ), the left side of (3.1) with $T(\Phi_i(E_d)^2)$ in place of $\Phi_i(E_d)^2$ is bounded by its right side times*

$$C h_i^{-1} \{ r|\nabla_X V| + (|F| + d)|\nabla_X q| \}.$$

Proof. Since $X \in E_i$, the supporting plane bounds in Lemma 2.3 give $\Re Y_{ij}(0) \le A + Ch_i$. Since $Y_{ij}(d) - Y_{ij}(0) = -i(q - q_{ij})d$, the same bound holds at depth d. For each fixed $b_1, j_1 \ge 0$, integration by parts j_1 times against e^{uy} in the integral for $\psi^{(b_1)}$ gives

$$|\psi^{(b_1)}(y)| \le C_{b_1,j_1} e^{\max(0,\Re y)/2}(1 + |y|)^{-j_1}.$$

We use the integral directly for $|y| \le 1$. A product of at most four factors grows at most as $C\exp(2aA/h_i)$. The choice $C_0 \ge 64a$ ensures $2aA/h_i \le NA/16$ whenever $Nh_i \ge C_0/2$; thus $e^{-N(A+d)/2}$ absorbs this growth. We also require $C_0 \ge 2\sup_i h_i$, so all true cutoffs vanish for $v \le 2$. These requirements involve only the common chart bounds and the fixed parameter a.

At $d = 0$ the identity

$$Y_{ij}(0) = i(q - q_{ij})F + \beta_{ij}(V_{ij} - V)e$$

and $|V_{i0} - V_{i1}| \gtrsim h_i/\ell_i$ show that

$$n_i \le C\left(1 + |F|/h_i + d/h_i + \max_j |Y_{ij}(d)|/h_i\right).$$

Indeed at least one node satisfies $|V_{ij} - V| \gtrsim h_i/\ell_i$. This proves (i). For (ii), $\Re M = A + d$ and $|M| \asymp |F| + d$, since $d \le \Re M$ and $|G| \le |M| + |\beta|d$. Direct integration for $N|M| \le 1$ and J integrations by parts against e^{-vM} otherwise give

$$\left|\int g(v)(vA)^p e^{-vM}\,dv\right| \le CN(1 + NA)^p e^{-N(A+d)/2}(1 + N|M|)^{-J}.$$

The compact support eliminates every endpoint term. Since $(|F| + d)/h_i \lesssim N|M|$, the last two inequalities supply n_i^{-m} with k powers of phase decay to spare if $J \ge k + m$. The remaining real exponential absorbs the polynomial in NA. The same calculation applies to G on the boundary. Finally $TE_d = 0$ and the nodes are constant on their plate. Hence

$$TY_{ij}(d) = i(Tq)(F - d) - \beta(TV)e.$$

The boundary projection $(s, \xi) \mapsto (s, \Im\xi)$ has bounded first derivatives. The chain rule and the estimate for ψ' prove (iii). □

Let $\mathcal{I}_k$ denote the plates at level $k \ge 0$ and χ_i their nonnegative partition weights. A formal level $\mathcal{I}_{-1} = \{*\}$ has $\Phi_* = \eta_* = \chi_* = 1$. Adjacent indices in a sum below range over all pairs, with disjoint supports contributing zero. The true radii are $\ell_k = \ell_0 K^{-k}$. Each partition weight and its derivative are supported in the corresponding plate:

$$\operatorname{supp}\chi_i \subset E_i, \qquad \operatorname{supp} D\chi_i \subset \operatorname{supp}\chi_i \subset E_i.$$

The scalar multiplier is

$$H_v = \sum_{k\ge -1} \sum_{i\in\mathcal{I}_k,\ j\in\mathcal{I}_{k+1}} \chi_i\chi_j(\eta_i - \eta_j)\Phi_i^2. \tag{3.2}$$

All packets in any one expression are evaluated at the same E_d. By Lemma 2.2, $h_j \le h_i/4$ on intersecting adjacent true plates, and $h_i \lesssim \ell_i \to 0$ uniformly. Thus $\eta_i - \eta_j \ge 0$ and every fixed frequency involves finitely many levels. Summing once over each adjacent partition gives

$$H_v = 1 + \sum_{k\ge 0} \sum_{i\in\mathcal{I}_k,\ p\in\mathcal{I}_{k-1}} \chi_i\chi_p\eta_i(\Phi_i^2 - \Phi_p^2). \tag{3.3}$$

In particular the weights $\chi_i\chi_j(\eta_i - \eta_j)$ in (3.2) sum to one, by the same identity with all packets equal to one. For fixed X, N, the corresponding support indicators satisfy

$$\sum_{k\ge -1} \sum_{i\in\mathcal{I}_k,\, j\in\mathcal{I}_{k+1}} \chi_i(X)\chi_j(X)\mathbf{1}_{\{[N/2,2N]\cap\mathcal{V}_{ij}\ne\varnothing\}} \le C, \tag{3.4}$$

where $\mathcal{V}_{ij} = [C_0/h_i, 2C_0/h_j]$ for true parents, and $\mathcal{V}_{*j} = [0, 2C_0/h_j]$. At fixed X, N, an active true pair (i, j) with parent level k satisfies

$$h_i \geq C_0/(2N), \qquad h_j \leq 4C_0/N.$$

All plates with positive weights at X intersect there. If another active pair had parent p at level $l \geq k + 3$, repeated height contraction would give

$$h_p \leq 4^{-(l-k-1)} h_j \leq C_0/(4N),$$

contradicting $h_p \geq C_0/(2N)$. Thus at most three true parent levels contribute. Each level has total pair weight one, and the formal initial level contributes at most one, so the left side of (3.4) is at most 4. A band difference may therefore be replaced by its support indicator when estimating derivatives or a dyadic frequency integral.

The packets depend on both components of E. To preserve (2.1) we use the radial form of Hefer's holomorphic difference decomposition [12]. For a child i and its parent p, the radial derivative integral is

$$R_{ip}(E) = \int_0^1 \partial_E(\Phi_i - \Phi_p)(\theta E)\, d\theta, \qquad \partial_E = (\partial_e, \partial_b).$$

With this notation,

$$B_{ip}(E) = (\Phi_i(E) + \Phi_p(E))R_{ip}(E).$$

Since $\Phi_i(0) = \Phi_p(0) = 1$, the fundamental theorem of calculus gives $B_{ip}(E) \cdot E = \Phi_i(E)^2 - \Phi_p(E)^2$. The row on the boundary is

$$L = (L_1, L_2) = \int_0^\infty e^{-vG} \left[lH_v - \sum_{k \geq 0} \sum_{i \in \mathcal{I}_k,\ p \in \mathcal{I}_{k-1}} \chi_i \chi_p \eta_i'(v) B_{ip}(E) \right] dv. \tag{3.5}$$

Its contraction with E is

$$L \cdot E = \int_0^\infty -\partial_v(e^{-vG} H_v)\, dv = 1.$$

In fact $H_0 = 1$. On compact interior target sets $A \geq \delta > 0$ uniformly over the source support. The exponential bounds in Lemma 3.1 give $|e^{-vG} H_v| \leq Ce^{-v\delta/2}$. The estimate for ψ' in Lemma 3.1, applied to θE with the source X fixed, and $h_p \geq 4h_i$ give

$$|R_{ip}(E)| \leq Ch_i^{-1} \exp(CaA/h_i).$$

For $X \in E_i \cap E_p$, the integral for R_{ip} varies θE at the fixed source X. The constant C_0 is chosen once, large enough to absorb this growth together with that of $\Phi_i + \Phi_p$ when $v \asymp C_0/h_i$. The choices are made in the following order: chart bounds, covering constants and K, then a, and finally C_0. Since $|\eta_i'| \leq Ch_i$, the correction in (3.5) has the same integrable bound, using (3.4) at the cutoffs. The integral converges normally and L is holomorphic in Z. For a fixed smooth strongly convex model, Taylor's formula on the disk with the central tangent slope gives

$$c_D|y|^2 \leq B_{t_i}(\lambda_{t_i}, y) \leq C_D|y|^2 \qquad (|y| \leq \ell_i).$$

Indeed the real tangent displacement is $(y, \Im(\lambda_{t_i} y))$, whose length is between $|y|$ and $(1+M_P)|y|$, and the full segment lies in the larger smooth chart. The Hessian on this compact set is bounded above and bounded away from zero. Averaging and using (2.9) gives

$$c_D'\ell_i^2 \leq h_i \leq C_D'\ell_i^2.$$

Thus only $O_D(1 + \log(2 + v))$ levels enter at frequency v, and their partition derivatives and the differentiated packets cost at most fixed powers of $1 + v$ after summation. The same exponential bound with a smaller δ permits one source derivative and arbitrary target

derivatives on compact interior target sets. The norm estimates below use the integrated bounds of Lemmas 2.1, 2.4, and 2.5.

In the remaining norm estimates, fix $m > 6$ and $k > 3$. The packet estimates are used with enough additional decay powers to retain these exponents.

Proposition 3.2. *With $r > 0$ and $S = r^2 + |b|^2$, write*

$$J = \frac{L_2\bar{e} - L_1\bar{b}}{S}, \qquad \mathcal{M}_0 = \frac{\beta}{e}\int_0^\infty H_v(E)(1 + 2vA)e^{-v\bar{G}}\,dv.$$

Uniformly for all interior targets and all smooth strongly convex models with the stated chart bounds,

$$\int_{\mathcal{U}} |J + \mathcal{M}_0|\,dA_s\,dt \le C.$$

Proof. By (2.4) the difference $J - L_2/e$ already has uniformly bounded absolute integral. It remains to estimate $L_2/e + \mathcal{M}_0$.

The factor η_i' restricts the corresponding summand in the correction term of (3.5) to $v \in [C_0/h_i, 2C_0/h_i]$. The substitution $v = C_0x/h_i$ gives

$$I_i(G) := \int_0^\infty \eta_i'(v)e^{-vG}\,dv = \int_1^2 \eta'(x)e^{-C_0xG/h_i}\,dx.$$

Integration by parts against this compactly supported density yields, for every $J \ge 0$,

$$|I_i(G)| \le C_J e^{-C_0A/(2h_i)}(1 + |F|/h_i)^{-J}.$$

The contribution of this pair to L_2/e is therefore

$$\frac{(\Phi_i(E) + \Phi_p(E))R_{ip,2}(E)}{e}\,I_i(G),$$

before multiplication by $-\chi_i\chi_p$. The bound for R_{ip} above supplies h_i^{-1}. The two nodes of Φ_i and the inequality controlling n_i in Lemma 3.1 supply arbitrary spatial decay for its term. The Φ_p term is estimated using $n_i \le Kn_p$ and $|F|/h_p \le |F|/h_i$. Choosing J sufficiently large leaves both the desired phase decay and the spatial decay; the real exponential absorbs the growth of R_{ip} and the outer factor. When the parent is formal, $\Phi_* = 1$ and n_i is bounded at the fixed coarsest radius. Thus each pair contributes at most

$$\frac{C_{m,k}}{rh_i}\,n_i^{-m}(1 + |F|/h_i)^{-k}. \tag{3.6}$$

The factors from η_i' and its integration interval cancel, leaving the single factor h_i^{-1} from R_{ip}.

On a shell $R/2 < r \le R$, Lemma 2.1 and $S_i = h_i + \inf_{E_i}|F|$ give

$$\int_{E_i\cap\{R/2<r\le R\}} h_i^{-1}(1 + |F|/h_i)^{-k}\,dA_s\,dt \le CR^2(h_i/S_i)^2. \tag{3.7}$$

Indeed two powers are bounded by $(h_i/S_i)^2$, and the shell integral for the remaining exponent $k-2 > 1$ is $O(R^2h_i)$. At a true level of radius ℓ, the shell meets at most $C(1+R/\ell)^2$ horizontal cells. In each such cell Lemma 2.4 bounds the sum of $(h_i/S_i)^2$ by $C(1 + R/\ell)^3$. In (3.6) the bound is uniform over the parents at each X and $\sum_p \chi_p(X) = 1$. After summing those weights, (3.7) bounds the cost of the shell and level by

$$CR(1 + R/\ell)^{5-m}. \tag{3.8}$$

The sum over dyadic $R \le \ell$ is $O(\ell)$. The remaining shells contribute at most

$$C\ell\sum_{j\ge0} 2^{j(6-m)} = O(\ell) \qquad (m > 6).$$

The geometric sum over levels is finite.

Since $A > 0$, the unweighted sum of the two exponential integrals is

$$\mathcal{K}(G) = \int_0^\infty [e^{-vG} + (1 + 2vA)e^{-v\bar{G}}]\, dv = \frac{1}{G} + \frac{1}{\bar{G}} + \frac{2A}{\bar{G}^2} = \frac{4A^2}{G\bar{G}^2}.$$

In particular Lemma 2.1 gives $\int_{r\asymp R} |\mathcal{K}(G)|\, dA_s\, dt/r \le CR$. Let $\mathcal{K}_h$ denote the same integral with the scalar factor $\eta(vh/C_0)$. If $|G| \le h$, the omitted interval has length $O(1/h)$ and vA is bounded there. Thus $\mathcal{K}_h = \mathcal{K} + O(1/h)$. If $|G| > h$, then for every k

$$|\mathcal{K}_h(G)| \le C_k h^{-1} e^{-C_0 A/(2h)} (1 + |G|/h)^{-k}. \tag{3.9}$$

To see this, write

$$I_h(g) = \int_0^\infty \eta(vh/C_0) e^{-vg} dv \qquad (\Re g > 0).$$

Integration by parts gives

$$I_h(g) = g^{-1} \int_1^2 \eta'(x) e^{-xC_0 g/h} dx,$$

and hence

$$\mathcal{K}_h = I_h(G) + I_h(\bar{G}) - 2AI_h'(\bar{G}).$$

Differentiation introduces factors $1/g$ or $C_0 x/h$. Integrating the compactly supported densities against the complex exponential by parts gives arbitrary inverse powers of $|g|/h$. The bounds $A \le |g|$ and $|g| > h$ then give (3.9).

In (3.2) the contribution from a true adjacent pair i, j to $L_2/e + \mathcal{M}_0$, apart from the row correction, is

$$\frac{\beta}{e} \chi_i \chi_j \Phi_i(E)^2 \, [\mathcal{K}_{h_i}(G) - \mathcal{K}_{h_j}(G)].$$

Fix either endpoint height $h \in \{h_i, h_j\}$. In the region $|G| \le h$, the inequality $h \le h_i$ and Lemma 3.1(i) give $|\Phi_i(E)|^2 \lesssim n_i^{-m}$. The $\mathcal{K}$ term is bounded, on the entire shell at this level, by $CR(1 + R/\ell_i)^{-m}$ using $\sum_{i,j} \chi_i \chi_j = 1$ before the shell integral. The error $O(1/h)$ on $|G| \le h$ has the bound (3.6) with height h and spatial radius that of the endpoint, since adjacent radii have a fixed ratio and $|F| \lesssim h$. For $|G| > h$, the real exponential in (3.9) absorbs the growth of $\Phi_i(E)^2$, since $h \le h_i$. Its node decay and extra inverse powers of $1 + |G|/h$ supply n_i^{-m} by the inequality in the proof of Lemma 3.1. The resulting bound is again (3.6) with height h.

At the parent endpoint h_i, sum χ_j first and retain χ_i before applying (3.7) on E_i.

The child endpoint h_j is handled using $n_j \le K n_i$: sum χ_i first and apply (3.7) on E_j. In the latter case $|G| \le h_j$ implies $|G| \le h_i$, while for $|G| > h_j$ the exponential with denominator $h_j \le h_i$ absorbs the growth of the parent packet. Thus (3.8) and its summation apply to both endpoints.

In the initial pair $*, j$ the difference is $\mathcal{K} - \mathcal{K}_{h_j}$ and the packet is one. The full $\mathcal{K}$ occurs on this one level and costs $O(R)$ per shell. In the term with $\mathcal{K}_{h_j}$, the initial true radius is fixed, so n_j is bounded on the whole chart. The same small- and large-$|G|$ bounds therefore apply to the true endpoint. Summing all shells and levels proves the estimate. At each fixed interior target the band integrals can be grouped in this order by the positive lower bound on A and (3.4). The shell bounds hold uniformly for $Z \in D$. □

To estimate $\mathcal{M}_0$ against closed data, extend it into the same chart. The interior source coordinates are (s, ξ), where $t = \Im\xi$, $d = \Re\xi - P(s,t)$ and $\xi = \gamma + d$. In the inward column $0 \le d < d_0$ the volume element is $\mathrm{d}X\, \mathrm{d}d$, with $\mathrm{d}X = \mathrm{d}A_s\, \mathrm{d}t$. By reducing the support of the chart cutoff we may take the whole column over this support inside $\overline{D}$. Let θ denote an extension of the boundary cutoff to this column, compactly supported in the chart and in

$d < d_0$. Multiplication of its boundary value at (s,t) by a fixed smooth function of d, equal to one near zero, gives such an extension with controlled first derivatives. The displacement and phase are therefore $E_d = (e, w - \xi)$ and $M = \bar{G} + \beta d$.

Proposition 3.3. *For every interior target and every smooth closed $(0,1)$-form f on a neighborhood of $\overline{D}$, the scalar $\mathcal{M}_0$ of Proposition 3.2 satisfies*

$$\left| \int_{\partial D} \theta \mathcal{M}_0 f \wedge \mathrm{d}s \wedge \mathrm{d}\xi \right| \leq C \|f\|_{L^\infty(D)}.$$

The constant depends only on the chart bounds, the cutoff bounds, and the parameters of the plate and packet constructions.

Proof. The extension of the boundary scalar is $\mathcal{M} = B/e$, where

$$B = \beta \int_0^\infty H_v(E_d)(1 + 2vA) e^{-vM} \, \mathrm{d}v. \tag{3.10}$$

All geometric coefficients are evaluated at $X = (s,t)$. In particular $A = \Re G$ is the boundary value even when $d > 0$. At $d = 0$ the extension agrees with $\mathcal{M}_0$. For a fixed smooth model and interior target, the convergence and differentiated convergence established after (3.5) apply to (3.10), by (3.1) and the unchanged real parts of $Y_{ij}(d)$. Thus B is continuously differentiable up to $d = 0$ there. Extend the product θB by zero to the rest of D. We shall prove the uniform bounds

$$\int_D \frac{|\bar{\partial}(\theta B)|}{r} \, \mathrm{d}V + \int_{\{\xi : (z,\xi) \in D\}} |(\theta B)(z,\xi)| \, \mathrm{d}A_\xi \leq C. \tag{3.11}$$

Let T denote $\partial_{\bar{s}}$ or $\partial_{\bar{\xi}}$ in the independent complex source coordinates. Write $G_d = l \cdot E_d = G - \beta d$. Since $TE_d = 0$ and $M + G_d = 2A$,

$$\begin{aligned} TG_d &= i(Tq)(F - d) - \beta e \, TV, \\ TM &= -i(Tq)(\bar{F} - d) - \bar{\beta}\bar{e} \, T\bar{V}. \end{aligned} \tag{3.12}$$

Indeed the remaining terms in TM are

$$-\bar{\beta} T\bar{\gamma} - \bar{c} \, T\bar{e} + \beta T d.$$

Their sum vanishes for $T = \partial_{\bar{s}}$ because

$$-(\bar{\beta} + \beta)\bar{c}/2 + \bar{c} = 0.$$

When $T = \partial_{\bar{\xi}}$, the cancellation follows from

$$Tt = i/2, \qquad T\bar{\gamma} = \beta/2, \qquad Td = \bar{\beta}/2.$$

In particular

$$\begin{aligned} 2TA &= i(Tq)(F - \bar{F}) - \beta e \, TV - \bar{\beta}\bar{e} \, T\bar{V}, \\ MT\beta - \beta TM &= 2i\bar{F} \, Tq + |\beta|^2 \bar{e} \, T\bar{V}. \end{aligned}$$

Both moduli are bounded by $C\mathcal{E}$, where $\mathcal{E} = |F||\nabla_X q| + r|\nabla_X V|$.

For the derivative of the source-dependent normal factor use

$$T(\beta e^{-vM}) = (T\beta)\partial_v(ve^{-vM}) + ve^{-vM}(MT\beta - \beta TM).$$

Integration by parts on the whole positive v-axis gives no boundary term, since $v = 0$ at the lower endpoint and the exponential dominates the packets at infinity. Writing $H = H_v(E_d)$,

one obtains

$$TB = \int_0^\infty e^{-vM}\Big[\beta(1+2vA)TH - (T\beta)(1+2vA)v\partial_v H \\ + vH\{2\beta TA - 2AT\beta + (1+2vA)(MT\beta - \beta TM)\}\Big]\,\mathrm{d}v. \tag{3.13}$$

The coefficients in braces are bounded by $C(1+vA)\mathcal{E}$ because $A \le C|F|$. This integration by parts precedes the subdivision into frequencies.

Use a smooth dyadic partition on $v \ge 1$ with pieces supported in $[N/2, 2N]$, N a power of two, and a remaining low-frequency cutoff. With the fixed exponents $m > 6$ and $k > 3$, write $W_N = (1+N|F|)^{-k}$ and $W_\ell = (1+r/\ell)^{-m}$. The v-integrals in (3.13) are estimated by (3.1). The depth integrals satisfy

$$\int_0^{d_0} Ne^{-cNd}\,\mathrm{d}d \le C, \qquad \int_0^{d_0} Nde^{-cNd}\,\mathrm{d}d \le C/N.$$

An additional v on a band contributes at most CN, by incorporating v/N in its smooth cutoff. Thus the normal derivative term with vH leaves a factor NW_NW_ℓ after depth integration, whereas a partition derivative bounded by C/h_j leaves $h_j^{-1}W_NW_\ell$. A factor d in a packet derivative gains $1/N$ relative to its boundary counterpart. Adjacent radii differ by the fixed factor K, so either spatial weight can be used with a change in its constant. For the formal parent, the exponential integral has the same phase bound without a packet, and the spatial weight at the coarsest true radius is bounded below throughout the bounded chart and target set. The choice of C_0 makes $H_v = 1$ in the low-frequency range.

To estimate derivatives of the source-dependent normal in (3.13), use (3.2) for the term containing vH. For a parent i and a child j the frequency satisfies $N \lesssim 1/h_j$. The contribution of $r|\nabla_X V|$ loses its factor r against the denominator in (3.11). For $k > 1$,

$$\sum_{N \lesssim 1/h_j} NW_N \le C/(h_j + |F|).$$

Indeed the sum is bounded both by C/h_j and by $C/|F|$ upon splitting at $N = 1/|F|$. Sum the parent weights χ_i first, retaining the spatial weight at the child level. On each child cell Q' of radius ℓ, $1 + r/\ell \asymp n_{Q'}$. Lemma 2.5 therefore bounds the cost for this cell by $Cn_{Q'}^{2-m}\sqrt{\ell}$. There are $O(L^2)$ cell centers with $L \le n_{Q'} < 2L$ for each dyadic $L \ge 1$. Both sums converge:

$$\sum_{L\ \text{dyadic}} L^{4-m} < \infty \quad (m > 4), \qquad \sum_{k \ge 0} \sqrt{\ell_0 K^{-k}} < \infty.$$

Thus the sum over cells and levels is uniformly bounded.

The term $|F||\nabla_X q|$ is controlled by the bound

$$\sum_N N|F|W_N \le C$$

at every source. At any fixed pair of adjacent levels the weights $\chi_i\chi_j$ sum to one. For the term containing $v\partial_v H$, use (3.3), in which only η_i depends on v. Thus $v\partial_v H$ contains the factor $v\eta_i'(v)$. Only a bounded number of dyadic intervals with $N \asymp 1/h_i$ meet its support. Both packets in $\Phi_i^2 - \Phi_p^2$ have the required spatial bound there. Thus the two terms involving $\nabla_X q$ considered so far cost at most

$$C\int W_\ell \frac{|\nabla_X q|}{r}\,\mathrm{d}X \le C\sqrt{\ell} \tag{3.14}$$

per level. To verify the inequality, integrate over a horizontal shell $R/2 < r < R$ in the working box. The variation of P_t on vertical lines gives

$$\int P_{tt}\, \mathrm{d}X \le CR^2.$$

On each horizontal slice, the shell portion of the compact source patch satisfies

$$\int \operatorname{tr} P_{ss}\, \mathrm{d}A_s \le CR.$$

Indeed one may integrate by parts with a cutoff in $|s - z| < 2R$ times a fixed cutoff in the larger chart. The integral of its horizontal gradient is $O(R + R^2) = O(R)$ for the bounded radii under consideration. Positive semidefiniteness gives

$$|P_{st}|^2 \le P_{tt} \operatorname{tr} P_{ss}.$$

Cauchy–Schwarz over the shell and the bounded vertical interval therefore yields

$$\int |P_{st}|\, \mathrm{d}X \le CR^{3/2}.$$

Dividing by R and summing $(R + \sqrt{R})(1 + R/\ell)^{-m}$ over dyadic shells proves (3.14). The sum of $\sqrt{\ell}$ over the levels is finite.

In the part of TH differentiating Φ_i^2, use (3.2) with true parent i and child j. The bounds of Lemma 3.1(iii) apply because $N \gtrsim 1/h_i$ on the support of the band difference. The factor $r|\nabla_X V|/h_i$ is then at most $CNr|\nabla_X V|$, and the upper endpoint still gives $N \lesssim 1/h_j$. After summing χ_i, the resulting sum is bounded by $C|\nabla_X V|/(h_j + |F|)$ with the child spatial weight, so Lemma 2.5 applies as in the normal derivative term. For the other term, integration of the additional factor d gains $1/N$, so it suffices to check

$$\sum_{N \gtrsim 1/h_i} \frac{|F| + N^{-1}}{h_i} (1 + N|F|)^{-k} \le C.$$

The terms containing $1/(Nh_i)$ form a geometric majorant. For $x = |F|/h_i < 1$ the terms containing $|F|/h_i$ with $N \le 1/|F|$ sum to $O(x(1 + |\log x|))$, and the higher frequencies cost $O(x)$. For $x \ge 1$ the bound is $O(x^{1-k})$. The weighted integral of $|\nabla_X q|/r$ at each level is thus covered by (3.14).

For the part of TH differentiating only the partition weights, use (3.3). For every true level,

$$\sum_{i \in \mathcal{I}_k} \chi_i = 1, \qquad \sum_{i \in \mathcal{I}_k} T\chi_i = 0.$$

Consequently (3.2) and (3.3) remain identical when the packets and the cutoffs η_i are held fixed and only the partition weights are differentiated. Each term in the derivative is therefore of the form

$$T(\chi_i \chi_p)\, \eta_i (\Phi_i^2 - \Phi_p^2).$$

If the derivative hits the factor with index $j \in \{i, p\}$, its modulus is at most C/h_j and it vanishes outside E_j. The other partition factor is retained. Its support, together with that of the differentiated factor, places the source in both relevant plates, so Lemma 3.1 applies also to these terms. The active frequencies satisfy $N \gtrsim 1/h_i$, hence $N \gtrsim 1/h_j$ for either differentiated true plate. The formal parent has constant weight. Both packets admit (3.1) with the spatial weight at level j, including the coarsest level as above. At a fixed frequency, the sum of the remaining partition factors at the opposite level is at most one, before the frequency range

is enlarged to $N \gtrsim 1/h_j$. On a shell $R/2 < r < R$, the contribution for j after integration in depth is consequently at most

$$\frac{C(1+R/\ell_j)^{-m}}{Rh_j} \sum_{N \gtrsim 1/h_j} \int_{E_j \cap \{R/2<r<R\}} W_N \, \mathrm{d}X \le CR(1+R/\ell_j)^{-m}(h_j/S_j)^2$$

by (2.18). Lemma 2.4 sums the square factors within one horizontal cell at a cost $C(1+R/\ell_j)^3$. There are at most $C(1+R/\ell_j)^2$ such cells. As in (3.8), the cost of one shell at a level of radius ℓ is $CR(1+R/\ell)^{5-m}$. The sum in R is $O(\ell)$ for $m > 6$, and the sum in ℓ is finite.

The term $(T\theta)B$ in (3.11) has no height derivative. At each X, N use (3.2) and the bounded sum of active pair weights in (3.4). Discarding the spatial weight, (3.1) then gives CW_N for the integral in d of the modulus of the Nth band of B. Lemma 2.1 gives $\int_{R/2<r<R} W_N \, \mathrm{d}X \le CR^2/N$. The shell with its additional divisor r costs CR/N, summable over bounded dyadic radii and dyadic $N \ge 1$. At low frequencies B is bounded directly. In TB, formula (3.13) and $H = 1$ at those frequencies leave the bound $C(|F||\nabla_X q|/r + |\nabla_X V|)$ after division by r. Here $|F|$ is bounded and (3.14) applies with the fixed coarsest radius. The integral of $|\nabla_X V|$ is bounded by Lemma 2.5 at that same radius, since $h_i + |F|$ is bounded above and the plates cover the working patch. The low-frequency term with $T\theta$ contributes only C/r, integrable in s. This proves the first term of (3.11). The second term controls the distribution supported on the complex fiber $s = z$ that arises from differentiating $1/e$.

On the fiber $s = z$ one has $F = w - \gamma$ and $|F| \ge |t - \Im w|$. The same estimate for B after the depth integral, now at $s = z$, costs at most

$$C \int_{\mathbb{R}} (1 + N|t - \Im w|)^{-k} \, \mathrm{d}t \le C/N$$

on each high band. The area element on this fiber is $\mathrm{d}t \, \mathrm{d}d$. Summing in N and including the bounded low-frequency part proves the second term of (3.11).

It remains to identify this fiber term in Stokes' formula. For $\varepsilon > 0$ use $k_\varepsilon(e) = \bar{e}/(|e|^2 + \varepsilon^2)$. With $e = z - s$,

$$\partial_{\bar{s}} k_\varepsilon(e) = -\frac{\varepsilon^2}{(|e|^2+\varepsilon^2)^2} \;\longrightarrow\; -\pi\delta_z(s), \qquad \partial_{\bar{\xi}} k_\varepsilon(e) = 0.$$

By the ordinary Stokes formula with outward orientation and $\bar{\partial} f = 0$,

$$\int_{\partial D} k_\varepsilon \theta B f \wedge \mathrm{d}s \wedge \mathrm{d}\xi = \int_D \left[k_\varepsilon \bar{\partial}(\theta B) + (\theta B)(\partial_{\bar{s}} k_\varepsilon)\, \mathrm{d}\bar{s}\right] \wedge f \wedge \mathrm{d}s \wedge \mathrm{d}\xi.$$

For the fixed interior target the left side converges to the asserted boundary pairing by $|k_\varepsilon| \le 1/r$ and boundedness of B and of the graph slopes. The first volume term converges by the first bound in (3.11). In the second term use the column coordinates (s, t, d) with volume Jacobian one. The coefficient of $\mathrm{d}V$ after removing $\partial_{\bar{s}} k_\varepsilon$ is continuous up to $d = 0$ and compactly supported in the chart, with a bounded range of t, d. Integration in t, d therefore yields a continuous compactly supported function of s. The planar approximate identity gives its value at z multiplied by $-\pi$. Its modulus is bounded by a numerical constant times $\|f\|_\infty$ times the fiber integral in (3.11). Both volume terms thus have the asserted uniform estimate. □

To assemble a global solution on D, cover its boundary by finitely many of the graphs used above. Here and below a smaller source box may be subdivided into a bounded number of boxes so that all enlarged boxes lie in their respective graphs. A smooth partition of unity on ∂D subordinate to the smaller boxes gives weights θ_α. On each graph extend the local row (3.5) back to the original coordinates as a complex covector. Multiplication by θ_α and

summation give a row of class $C^{1,\infty}(\partial D \times D)$, holomorphic in the target and satisfying (2.1). The solution formula (2.2) therefore applies.

On the graph the pullback of $f \wedge \mathrm{d}s \wedge \mathrm{d}\xi$ has density bounded by $C\|f\|_\infty$ with respect to $\mathrm{d}X$, by the bound for the derivative of $(s,t) \mapsto (s,\gamma)$. For the contribution of $\theta_\alpha \mathcal{J}$ in the boundary integral of (2.2), add and subtract the boundary function $\mathcal{M}_0$ of Proposition 3.2. That proposition bounds the integral of $\theta_\alpha(\mathcal{J} + \mathcal{M}_0)$ against the form by $C\|f\|_\infty$. Proposition 3.3 bounds the integral of $\theta_\alpha \mathcal{M}_0$ against the same form by $C\|f\|_\infty$. Combining the finitely many estimates with the bound for u_0 gives

$$\sup_D |u| \leq C\|f\|_{L^\infty(D)}. \tag{3.15}$$

For a fixed bounding ball, the numerical constant here depends on the number and sizes of the graph boxes, their buffers, an upper bound for their slopes, an inward depth on which the cutoff extensions are supported, and a uniform first derivative bound for the chart cutoffs.

Proof of Theorem 1.1. Choose concentric balls $B(p_0, a) \subset \Omega \subset B(p_0, R)$ and fix $\kappa \geq R/a$. After translation and dilation we may take $B(0, 1/\kappa) \subset \Omega \subset B(0,1)$. The reduction to $R = 1$ will be reversed at the end. With $a_0 = 1/(2\kappa)$, we first construct uniform charts for every smooth strongly convex D with $B(0, a_0) \subset D \subset B(0,1)$.

For a real unit vector $\nu \in S^3$ and $y \in \nu^\perp$ with $|y| < a_0$, let $g_\nu(y)$ be the upper endpoint coordinate of the segment $\overline{D} \cap (y + \mathbb{R}\nu)$. Convexity makes g_ν concave, and

$$\sqrt{a_0^2 - |y|^2} \leq g_\nu(y) \leq 1.$$

The lower endpoint coordinate is at most $-\sqrt{a_0^2 - |y|^2}$. Every supporting plane at the upper endpoint has outward normal with strictly positive ν component: a negative component is excluded by the segment below the endpoint, and a zero component would support the orthogonal projection at its interior point y. Hence the implicit function theorem makes g_ν smooth there. Testing the supporting slope at $y \pm (a_0/4)v$ gives

$$|\nabla g_\nu(y) \cdot v| \leq 4/a_0 \qquad (|y| \leq a_0/2,\ v \in \nu^\perp,\ |v| = 1).$$

Hence $|\nabla g_\nu| \leq 4/a_0$ on this smaller ball. The two endpoint bounds show that an inward segment of length $a_0/2$ from the upper graph, except for its initial point, lies in D. A unitary change of coordinates takes $-\nu$ to the increasing real direction of ξ. In this chart $P = -g_\nu$ is convex and its real Hessian is positive definite, since the second fundamental form of ∂D is positive definite.

Choose a finite $a_0/16$-net of S^3 and a smooth partition of unity subordinate to the caps $|\omega - \nu| < a_0/8$ about its members. For example, start with identical smooth bumps in the squared chordal distance, equal to one on the smaller caps and zero outside the larger caps, and divide by their sum. The number of bumps and their first derivative bounds depend only on a_0. On ∂D use the argument $\omega = x/|x|$, whose derivative is bounded by $1/a_0$. A point in the support projects to $|y| \leq a_0/8$ and has $x \cdot \nu > 0$, so it is the upper endpoint. The smoothed cutoffs in the interior can be taken to be their boundary values, evaluated at the graph point with the same y, multiplied by a smooth function of the inward depth equal to one at zero and zero for depth exceeding $a_0/4$. This construction uses only first derivatives of g_ν and uniform cutoff bounds. The inner ranges $|y| \leq a_0/8$ have fixed positive buffers in $|y| < a_0/2$. Finite covers by concentric boxes inside these ranges and their subordinate cutoffs can be chosen in the parameter space once for all D. The first horizontal mesh radius is chosen sufficiently small in terms of these buffers and $4/a_0$, so that every translated and tilted disk

required in Lemma 2.2 remains in the larger graph. Thus (3.15) holds on this whole family with a constant depending only on κ.

For the original possibly nonsmooth domain consider its gauge

$$p(x) = \inf\{t > 0 : x \in t\Omega\}.$$

It is convex and satisfies

$$\Omega = \{p < 1\}, \qquad |x| \le p(x) \le \kappa|x|.$$

Subadditivity applied in both orders gives

$$|p(x) - p(y)| \le \kappa|x - y|.$$

With a nonnegative symmetric smooth probability mollifier ρ_ϵ of radius ϵ, take

$$D_\epsilon = \{x : (p * \rho_\epsilon)(x) + \epsilon|x|^2 < 1 - \epsilon\}.$$

The defining function is smooth with Hessian at least $2\epsilon I$. It is at least p by convexity and symmetry and converges to p uniformly on compact sets. Therefore $\overline{D_\epsilon} \Subset \Omega$ and every compact subset of Ω is contained in D_ϵ for all sufficiently small ϵ. In particular these sets contain $B(0, a_0)$ eventually. Their level boundaries are regular: a point with zero gradient of a convex function is a global minimizer, whereas its value at the origin is strictly below $1 - \epsilon$. The level boundaries have strictly positive second fundamental form by the Hessian bound. They consequently satisfy every hypothesis used for (3.15) with the common constant C_κ.

Take $\epsilon_j \downarrow 0$ small enough for these conclusions. On a neighborhood of $\overline{D_{\epsilon_j}}$, convolve f coefficientwise with a smooth nonnegative probability kernel of radius $\delta_j < \min(1/j, \operatorname{dist}(\overline{D_{\epsilon_j}}, \partial\Omega)/3)$. The convolution $f^{(j)}$ is closed there because distributional differentiation commutes with convolution and the translated test functions are supported in Ω. These approximations satisfy

$$\|f^{(j)}\|_\infty \le \|f\|_\infty, \qquad f^{(j)} \longrightarrow f \quad \text{in } L^1_{\mathrm{loc}}(\Omega).$$

Let u_j be the solution on D_{ϵ_j} constructed above. For weak-star extraction, extend u_j by zero to Ω. The extensions belong to the ball of radius $C_\kappa\|f\|_\infty$ in $L^\infty(\Omega)$. This ball is weak-star compact by the Banach–Alaoglu theorem [1, Theorem 1.3]; see also Folland [6, Theorem 5.18]. Since Lebesgue measure on Ω is finite, Folland [6, Theorem 6.15] gives $L^\infty(\Omega) = (L^1(\Omega))^*$. The predual $L^1(\Omega)$ is separable, so evaluation on a countable dense subset metrizes the weak-star topology on the ball. A subsequence converges weak-star to some u in the same ball:

$$u_j \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(\Omega), \qquad \|u\|_\infty \le C_\kappa\|f\|_\infty.$$

Every compactly supported smooth test form in Ω is eventually supported in D_{ϵ_j}. On this support $\bar\partial u_j = f^{(j)}$ and the two sides converge in distributions to $\bar\partial u$ and f, respectively.

Finally, under $Z = p_0 + RZ'$ a coefficient vector f pulls back to $Rf(p_0 + RZ')$. Apply the unit-scale result to this pullback and compose its scalar primitive with the inverse dilation. This gives precisely the asserted factor R in the original coordinates. □

AI Use Disclosure

GPT-6-Astra was used to explore the proof and draft the manuscript, and GPT-6.1-Sol was used in the revision of the manuscript.

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