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UNIVERSAL MODULES, NATURAL PAIRINGS, AND WEIGHTED ISOMETRIES FOR NONCOMMUTATIVE L^P-SPACES

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ABSTRACT. For $1 \le p < \infty$, we consider the functor L^p on finite tracial von Neumann algebras with trace-preserving embeddings, including nonunital ones. For a functor of unital contractive left modules with compatible distinguished vectors ξ_M, finite direct sums of scalars determine the best constant in $\|x\xi_M\| \le K\|x\|_p$. The pointed L^p functor is initial among the families satisfying this estimate with $K = 1$. Retaining only the bound on scalar points gives an atomic norm. On $\mathbb{C}^n$ with unit weights the norm of the identity from ℓ_n^p to this atomic space is asymptotic to $p^{-1}(\log n)^{1-1/p}$ for $p > 1$. For $1 \le p, q < \infty$, every objectwise bounded natural bilinear family on $L^p \times L^q$ has the form $c\,\tau(xy) + d\,\tau(x)\tau(y)$ for $x, y \in M$, with c, d independent of the object and $c = 0$ when $1/p + 1/q > 1$. A uniformly bounded natural inner product on every $L^p(M, \tau)$ exists exactly at $p = 2$. For finite-dimensional unital embeddings preserving a faithful positive functional, we determine the nearest modularly compatible target density both in Hilbert–Schmidt norm and in the Hilbert–Schmidt distance between positive square roots. The minimum squared distance between square roots lies between a weighted sum of the losses in squared L^2 norm on matrix units and twice that sum. Both constants are sharp.

1. INTRODUCTION

For a finite von Neumann algebra M with faithful normal finite trace τ, each nonzero projection e defines a trace-preserving embedding of the scalars into M by $z \mapsto ze$, provided the scalar trace has mass $\tau(e)$. Such embeddings relate the value of a functor on a scalar algebra to its behavior on individual projections in M. We use them to characterize the pointed L^p-module functor and to classify the scalar pairings that are natural for trace-preserving embeddings. In finite dimensions we also ask how much an embedding can distort the L^2 norm when it preserves a positive functional with a noncentral density.

Throughout the tracial part of the paper, morphisms are normal injective $*$-homomorphisms preserving the given traces. We allow nonunital morphisms and all positive finite trace masses, and take $1 \le p, q < \infty$. For background on tracial integration, general noncommutative L^p-spaces, and their interpolation theory, see [Seg53, Nel74, Haa79, Kos84, Ter82, PX03].

Scalar tests and universal modules. Leinster characterized the classical L^p functor on finite measure spaces by a universal property and obtained integration from the case $p = 1$ [Lei23]. Liu, Liu, Huang, and Zhou studied categorical integration for normed modules over finite-dimensional algebras [LLHZ26]. For the theory of noncommutative L^p modules, see Junge and Sherman [JS05].

Let F be a functor to Banach spaces and linear contractions. Give each $F(M)$ a unital contractive left M-action and a distinguished vector ξ_M. Assume that the maps $F(\pi)$ intertwine the actions and satisfy

$$F(\pi)\xi_M = \pi(1_M)\xi_N \qquad (\pi : (M, \tau) \to (N, \sigma)).$$

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Write $\mathbb{C}_s$ for the scalars with trace $z \mapsto sz$. On a finite direct sum $D_{\mathbf{s}} = \bigoplus_j \mathbb{C}_{s_j}$ with coordinate projections e_j, consider the inequalities

$$\left\| \sum_j z_j e_j \xi_{D_{\mathbf{s}}} \right\| \le K \left(\sum_j s_j |z_j|^p \right)^{1/p}. \tag{1.1}$$

The optimal constant equals the supremum of $\|x\xi_M\|/\|x\|_{p,\tau}$ over all (M,τ) and $0 \neq x \in M$. When finite, it is the uniform norm of the unique natural family of bounded module maps $T_M : L^p(M,\tau) \to F(M)$ sending 1_M to ξ_M. The proof uses spectral approximation for $|x|$ and the module action of the partial isometry in the polar decomposition of x. With $K = 1$, this gives the initial property of the pointed L^p functor (Theorem 3.2 and Corollary 3.3).

Suppose we retain only $\|\xi_{\mathbb{C}_s}\| \le s^{1/p}$ from (1.1). The initial module is then the completion of M in the norm

$$\alpha_{p,\tau}(x) = \inf_{x=\sum_j a_j e_j} \sum_j \|a_j\|_\infty \, \tau(e_j)^{1/p},$$

where $a_j \in M$ and the e_j are arbitrary projections. For $p = 1$ this is the trace norm. On finite scalar algebras the formula gives a weighted Lorentz norm. For $p > 1$ and unit weights, the norm of the identity $\ell_n^p \to (\mathbb{C}^n, \alpha_p)$ is asymptotic to $p^{-1}(\log n)^{1/p'}$, where $p' = p/(p-1)$ (Theorem 3.6 and Proposition 3.7). The role of the module action is already visible on $M_2(\mathbb{C})$. Theorem 3.9 constructs a pointed functor satisfying a direct-sum estimate for which the prescribed values on rank-one projections admit no linear extension.

Natural linear maps and scalar pairings. A family of maps is objectwise bounded if each component is bounded and uniformly bounded if one bound holds for all components. The objectwise bounded natural linear functionals on L^p are the multiples of the trace. The objectwise bounded natural linear maps $L^p \to L^q$ are the multiples of the finite-trace inclusion for $p \ge q$, and zero for $p < q$ (Lemma 4.1 and Theorem 4.3). For $p > 1$, a uniformly bounded family of linear functionals ℓ_M satisfies the sharp defect estimate of Proposition 4.4,

$$\sup_M \|\ell_M\| \le \sup_{\pi : M \to N} \|\ell_N \circ L^p(\pi) - \ell_M\|.$$

Izumi studied bilinear and sesquilinear forms associated with interpolation realizations of noncommutative L^p-spaces [Izu98]. Here naturality asks that pullback along every tracial embedding preserve the pairing on its source. If $B_M : L^p(M,\tau) \times L^q(M,\tau) \to \mathbb{C}$ is such a family of bounded bilinear forms, then on $M \times M$,

$$B_M(x,y) = c\,\tau(xy) + d\,\tau(x)\tau(y) \tag{1.2}$$

for scalars c, d independent of the object. The product of traces extends continuously for every pair (p,q). The single-trace term extends for $1/p + 1/q \le 1$, and its coefficient vanishes otherwise. To obtain the two terms, we use inner automorphisms of $M_2(M)$ to isolate a balanced bilinear form. The remaining form is natural in two independently varying algebras (Theorem 5.2).

Varying the scalar trace mass in (1.2) gives the uniform classification. Only $c\,\tau(xy)$ survives when $1/p + 1/q = 1$, only $d\,\tau(x)\tau(y)$ when $p = q = 1$, and all other uniformly bounded families vanish. Thus a uniformly bounded natural family of inner products exists exactly when $p = 2$. Normalizing the form to take value one at $(1,1)$ on $\mathbb{C}_1$ gives $(x,y) \mapsto \tau(x^*y)$ (Corollaries 5.3 and 5.5).

Weighted embeddings. Tracial embeddings induce complete isometries with completely contractive left inverses (Section 6). We use the predual of the opposite algebra at the L^1 endpoint [JRS05, Section 2], [Pis14, Section 9]. Complete isometries of noncommutative L^p-spaces and their relation to conditional expectations are studied in [JRS05, She06]. For extensions of maps in the nontracial setting, see [HJX10].

Consider now finite-dimensional algebras with faithful densities h, k and an injective $*$-homomorphism π satisfying $\mathrm{Tr}_N(k\pi(x)) = \mathrm{Tr}_M(hx)$. The symmetric weighted norm is

$\|x\|_{p,h} = \|h^{1/(2p)} x h^{1/(2p)}\|_p$, with unnormalized matrix traces. The embedding preserves the weighted L^2 norm precisely when its support commutes with k and it intertwines the modular groups. This criterion is related to the self-polar forms of Haagerup and Størmer [HS95] and to Takesaki's theorem on conditional expectations [Tak72]. Theorem 7.2 gives a finite-dimensional proof. For a unital embedding, in multiplicity coordinates, the criterion reads

$$\pi(x)_j = \bigoplus_i x_i \otimes I_{r_{ji}}, \qquad k_j = \bigoplus_i h_i \otimes \rho_{ji}, \qquad \rho_{ji} > 0, \quad \sum_j \operatorname{Tr} \rho_{ji} = 1 \quad \text{for each } i.$$

Such an embedding preserves the weighted norms for all $1 \le p < \infty$.

Fix a unital π and h. Among compatible densities ℓ with this source density, Theorem 7.5 gives the unique minimizer of $\|k - \ell\|_2$ and the unique minimizer of $\|\sqrt{k} - \sqrt{\ell}\|_2$. Let V_h be the subspace of block matrices $\bigoplus_{j,i} \sqrt{h_i} \otimes z_{ji}$, where $z_{ji} \in M_{r_{ji}}$, and put $\delta(k) = \operatorname{dist}_2(\sqrt{k}, V_h)^2$. The minimum of $\|\sqrt{k} - \sqrt{\ell}\|_2^2$ lies between $\delta(k)$ and $2\delta(k)$, with both constants sharp over finite dimensions. In an eigenbasis of h_i with eigenvalues λ_{ia} and matrix units $e_{ab}^{(i)}$, we have

$$\delta(k) = \sum_i \frac{1}{\operatorname{Tr} h_i} \sum_{a,b} \sqrt{\lambda_{ia}\lambda_{ib}}\, (\|e_{ab}^{(i)}\|_{2,h}^2 - \|\pi(e_{ab}^{(i)})\|_{2,k}^2). \tag{1.3}$$

Each summand is nonnegative. Thus equality of the weighted norms on these matrix units characterizes modular compatibility. The same identity bounds the distance to the compatible densities in terms of the largest relative loss of squared norm. For a nonunital embedding, the squared norm loss at 1_M is exactly $\frac{1}{2}\|[\sqrt{k}, \pi(1_M)]\|_2^2$ (Propositions 7.7 and 7.9 and Theorem 7.8).

2. Tracial embeddings and dense cores

Let $\mathbf{Nrm}_1$ and $\mathbf{Ban}_1$ denote the categories of complex normed spaces and complex Banach spaces, respectively, with linear contractions as morphisms. Write $U : \mathbf{Ban}_1 \to \mathbf{Nrm}_1$ for the forgetful functor. We fix a universe of objects and form functor categories in a larger universe. An object of $\mathbf{TrEmb}_{\mathrm{fin}}$ is a nonzero von Neumann algebra M with a faithful normal finite trace τ. Write $\mathcal{P}(M)$ for its projections. An arrow $\pi : (M, \tau) \to (N, \sigma)$ is a normal injective $*$-homomorphism satisfying $\sigma \circ \pi = \tau$. We allow nonunital embeddings. One may also adjoin the zero algebra, with $L^p(0) = 0$. In that convention formulas involving the total trace mass refer to nonzero objects.

For $1 \le p < \infty$ put

$$\|x\|_{p,\tau} = \tau(|x|^p)^{1/p} \quad (x \in M), \qquad L^p(M, \tau) = \overline{(M, \|\cdot\|_{p,\tau})}.$$

We use the standard tracial integration, Hölder, and duality theorems [Seg53, Nel74, PX03]. In particular,

$$\|axb\|_{p,\tau} \le \|a\|_\infty \|x\|_{p,\tau} \|b\|_\infty, \qquad \|a\|_{p,\tau} \le \tau(1)^{1/p} \|a\|_\infty. \tag{2.1}$$

The first inequality extends left and right multiplication by continuity to operators $L_M(a)$ and $R_M(b)$ on $L^p(M, \tau)$, with norms at most $\|a\|_\infty$ and $\|b\|_\infty$, respectively. We identify M with its dense image in $L^p(M, \tau)$ and use τ also for the continuous extension of the trace. Its norm is

$$\|\tau : L^p(M, \tau) \to \mathbb{C}\| = \tau(1)^{1-1/p}. \tag{2.2}$$

Hölder's inequality gives the upper bound, and evaluation at $1_M / \|1_M\|_{p,\tau}$ gives equality.

Lemma 2.1. *Every arrow $\pi : (M, \tau) \to (N, \sigma)$ induces an isometry $L^p(\pi) : L^p(M, \tau) \to L^p(N, \sigma)$ extending π. These extensions define a functor $L^p : \mathbf{TrEmb}_{\mathrm{fin}} \to \mathbf{Ban}_1$.*

Proof. With $e = \pi(1)$, functional calculus in the corner eNe gives $|\pi(x)|^p = \pi(|x|^p)$. Hence $\|\pi(x)\|_{p,\sigma}^p = \sigma(\pi(|x|^p)) = \tau(|x|^p)$. Thus π extends uniquely to an isometry. The identity and composition laws pass from the dense algebras to their completions. □

For $s > 0$ we write $\mathbb{C}_s = (\mathbb{C}, \tau_s)$, where $\tau_s(z) = sz$. Both summand embeddings into $(M, \tau) \oplus (N, \sigma) = (M \oplus N, \tau \oplus \sigma)$ preserve traces. If $e \neq 0$ is a projection of M, the map

$$\mathbb{C}_{\tau(e)} \longrightarrow (M, \tau), \qquad z \longmapsto ze \tag{2.3}$$

is an arrow. We give $M_n(M)$ the trace $\mathrm{Tr}_n \otimes \tau$, with Tr_n the unnormalized matrix trace. Each matrix-corner map $x \mapsto e_{jj} \otimes x$ is then an arrow of $\mathbf{TrEmb}_{\mathrm{fin}}$.

Proposition 2.2. *The set of finite sums of pairwise orthogonal projections with complex coefficients is precisely the set of finite-spectrum normal elements. It is a complex vector space if and only if M is abelian. The linear span of all projections,*

$$\mathcal{P}_{\mathrm{lin}}(M) = \mathrm{span}_{\mathbb{C}}\{e \in M : e = e^* = e^2\},$$

is preserved by embeddings in $\mathbf{TrEmb}_{\mathrm{fin}}$. It is dense in M for the operator norm and in $L^p(M, \tau)$ for the L^p norm.

Proof. The first assertion is the finite-spectrum spectral theorem. If projections e, f do not commute, then $(e + if)^*(e + if) - (e + if)(e + if)^* = 2i(ef - fe) \neq 0$. Thus closure under addition forces all projections to commute and hence, by spectral approximation, forces M to be abelian. Conversely, in an abelian algebra a common refinement of two finite spectral partitions gives a finite spectral partition for the sum. For arbitrary M, approximate the real and imaginary parts of any $x \in M$ in operator norm by finite-spectrum selfadjoint elements. Their linear combinations belong to $\mathcal{P}_{\mathrm{lin}}(M)$. Equation (2.1) then gives L^p-density. Functoriality follows because $*$-homomorphisms preserve projections. □

Lemma 2.3 (Functorial completion)**.** *Let $\mathcal{C}$ be a small category and let $A : \mathcal{C} \to \mathbf{Nrm}_1$. Objectwise completion defines a functor $\widehat{A} : \mathcal{C} \to \mathbf{Ban}_1$. For every $F : \mathcal{C} \to \mathbf{Ban}_1$, restriction along the dense inclusions gives a bijection*

$$\mathrm{Nat}_{\mathbf{Ban}_1}(\widehat{A}, F) \simeq \mathrm{Nat}_{\mathbf{Nrm}_1}(A, UF).$$

The same assertion holds for matrix-normed spaces and complete contractions.

Proof. Each component of a transformation on the right extends uniquely to a contraction on the completion. The two sides of every naturality square agree on a dense subspace, so the extensions are natural. Conversely, restriction recovers the original transformation. For operator spaces the same extension is contractive at every matrix level, since finite matrices over a dense subspace are dense. □

Leinster uses this completion adjunction in the proof of the classical L^p universal property [Lei23, Theorem 3.7].

3. Scalar tests and universal pointed modules

Fix $1 \leq p < \infty$. Given a distinguished vector ξ_M in a Banach left M-module, consider the map $x \mapsto x\xi_M$ on M. Under the compatibility conditions below, its optimal uniform bound with respect to the L^p norm can be computed on finite direct sums of scalar algebras.

3.1. **Supported points and scalar testing.** We abbreviate $F(M, \tau)$ to $F(M)$ when the trace is understood. A *supported pointed module family* consists of a functor $F : \mathbf{TrEmb}_{\mathrm{fin}} \to \mathbf{Ban}_1$, a unital contractive Banach left M-module structure on every $F(M)$, and a vector $\xi_M \in F(M)$ for every object. The compatibility conditions are

$$F(\pi)(a \cdot u) = \pi(a) \cdot F(\pi)(u), \qquad a \in N,\ u \in F(N), \tag{3.1}$$

$$F(\pi)(\xi_N) = \pi(1_N) \cdot \xi_M, \qquad \pi : (N, \sigma) \longrightarrow (M, \tau). \tag{3.2}$$

Here contractivity of the action means $\|a \cdot u\| \leq \|a\|_\infty \|u\|$. A morphism of such families is a natural transformation whose components are contractive module maps preserving the distinguished vectors.

For a finite tuple $\mathbf{s} = (s_1, \dots, s_n)$ of positive numbers, put

$$D_{\mathbf{s}} = \bigoplus_{j=1}^{n} \mathbb{C}_{s_j}, \qquad \iota_j : \mathbb{C}_{s_j} \longrightarrow D_{\mathbf{s}}.$$

Define the scalar test map

$$V_{\mathbf{s}}(z) = \sum_{j=1}^{n} z_j F(\iota_j)\xi_{\mathbb{C}_{s_j}}, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n. \tag{3.3}$$

If e_j is the jth coordinate projection of $D_{\mathbf{s}}$, then (3.2) gives $F(\iota_j)\xi_{\mathbb{C}_{s_j}} = e_j \xi_{D_{\mathbf{s}}}$. Thus $V_{\mathbf{s}}(z) = z \cdot \xi_{D_{\mathbf{s}}}$.

Lemma 3.1. *For every supported pointed module family, the following statements hold.*

(i) *For $x \in M$,*

$$\|x \cdot \xi_M\| = \||x| \cdot \xi_M\|.$$

(ii) *For any $K \geq 0$, the scalar bounds*

$$\|\xi_{\mathbb{C}_s}\| \leq K s^{1/p} \qquad (s > 0)$$

are equivalent to

$$\|\xi_M\| \leq K \tau(1_M)^{1/p} \qquad ((M, \tau) \in \mathbf{TrEmb}_{\mathrm{fin}}).$$

Under these bounds, $\|e \cdot \xi_M\| \leq K\tau(e)^{1/p}$ for every projection $e \in M$.

Proof. Write $x = v|x|$ for the polar decomposition in M. The identities $x = v|x|$ and $|x| = v^*x$, together with contractivity of the action, prove (i). For (ii), use the unital trace-preserving embedding $\mathbb{C}_{\tau(1_M)} \to M$ in (3.2) and take norms. The converse is the case $M = \mathbb{C}_s$. The estimate for $e \neq 0$ follows from the scalar corner embedding (2.3), and the case $e = 0$ is immediate. □

Theorem 3.2 (Scalar testing with the optimal constant)**.** *For a supported pointed module family, define*

$$C_{\mathrm{sc},p} = \sup_{n,\, \mathbf{s},\, z \neq 0} \frac{\|V_{\mathbf{s}}(z)\|}{(\sum_{j=1}^{n} s_j |z_j|^p)^{1/p}},$$

$$C_{\mathrm{alg},p} = \sup_{(M,\tau),\, 0 \neq x \in M} \frac{\|x \cdot \xi_M\|}{\tau(|x|^p)^{1/p}}.$$

These quantities agree, with $+\infty$ allowed. In the first supremum it suffices to take $z_j \geq 0$ for every j.

If their common value is finite, there is a unique natural family of bounded module maps

$$T_M : L^p(M, \tau) \longrightarrow F(M), \qquad T_M(1_M) = \xi_M,$$

and its uniform norm is

$$\sup_{(M,\tau)} \|T_M\| = C_{\mathrm{sc},p}. \tag{3.4}$$

Conversely, a uniformly bounded natural family with these properties can exist only if $C_{\mathrm{sc},p} < \infty$.

Proof. Since $V_{\mathbf{s}}(z) = z\xi_{D_{\mathbf{s}}}$, each scalar test occurs in the supremum defining $C_{\mathrm{alg},p}$. Moreover, replacing z by $|z|$ preserves the test ratio by Lemma 3.1(i). It remains to prove $C_{\mathrm{alg},p} \leq C_{\mathrm{sc},p}$ when $C_{\mathrm{sc},p} < \infty$.

Let $a = \sum_{j=1}^{n} \lambda_j e_j$ be a positive finite-spectrum element of M, with nonzero pairwise orthogonal projections e_j and $\lambda_j \geq 0$. Set $s_j = \tau(e_j)$. The map

$$\pi : D_{\mathbf{s}} \longrightarrow M, \qquad (z_j) \longmapsto \sum_j z_j e_j$$

is a morphism of $\mathbf{TrEmb}_{\mathrm{fin}}$, including when $\sum_j e_j < 1_M$. Equivariance and the supported-point condition give

$$F(\pi)V_{\mathbf{s}}(\lambda) = \sum_j \lambda_j e_j \xi_M = a\xi_M.$$

Since $F(\pi)$ is contractive,

$$\|a\xi_M\| \le C_{\mathrm{sc},p} \left(\sum_j \lambda_j^p \tau(e_j) \right)^{1/p} = C_{\mathrm{sc},p} \|a\|_{p,\tau}.$$

Every positive element of M is an operator-norm limit of positive finite-spectrum elements. The action is operator-norm continuous, and $\|b\|_{p,\tau} \le \tau(1_M)^{1/p}\|b\|_\infty$. The preceding inequality therefore holds for all positive $a \in M$. For arbitrary x, apply it to $|x|$ and use Lemma 3.1(i). This proves $C_{\mathrm{alg},p} \le C_{\mathrm{sc},p}$.

The map $x \mapsto x\xi_M$ now has a unique bounded extension T_M to $L^p(M,\tau)$. On $N \subseteq L^p(N,\sigma)$,

$$F(\pi)T_N(x) = \pi(x)\pi(1_N)\xi_M = T_M(\pi(x)),$$

so the extensions are natural. The identity $T_M(ax) = aT_M(x)$ also extends by density. A module map sending 1_M to ξ_M must agree with T_M on M, which proves uniqueness and bounds each scalar test by the uniform norm of any such family. Since extension preserves operator norms, (3.4) follows. □

Let PM_p^0 be the category of supported pointed module families satisfying $\|\xi_{\mathbb{C}_s}\| \le s^{1/p}$. Let PM_p be its full subcategory defined by the scalar partition inequalities

$$\left\| \sum_j z_j F(\iota_j)\xi_{\mathbb{C}_{s_j}} \right\| \le \left(\sum_j s_j |z_j|^p \right)^{1/p}. \tag{3.5}$$

Corollary 3.3 (Reconstruction of the pointed L^p-module functor)**.** *The left regular module functor $(L^p, L, \mathbf{1})$ is initial in PM_p. If $p = 1$, the scalar bounds alone imply (3.5), so $\mathsf{PM}_1^0 = \mathsf{PM}_1$. More generally,*

$$C_{\mathrm{sc},1} = \sup_{s>0} \frac{\|\xi_{\mathbb{C}_s}\|}{s}.$$

Proof. For the left regular family, the norm in (3.5) is exactly the displayed weighted ℓ^p norm. The initial morphism to any other object is supplied by Theorem 3.2. For $p = 1$, the triangle inequality and contractivity of the $F(\iota_j)$ give

$$\|V_{\mathbf{s}}(z)\| \le \sum_j |z_j| \|\xi_{\mathbb{C}_{s_j}}\|.$$

Taking $n = 1$ gives the reverse inequality between the two suprema. □

Proposition 3.4 (Exact cyclic representation)**.** *Let (M,τ) be fixed, let X be a unital contractive Banach left M-module, and let $\xi \in X$. The following are equivalent.*

(i) *For every finite orthogonal family of projections e_j and every $\lambda_j \ge 0$,*

$$\left\| \sum_j \lambda_j e_j \xi \right\|^p = \sum_j \lambda_j^p \tau(e_j).$$

(ii) *The map $x \mapsto x\xi$ extends to an isometric M-module isomorphism from $L^p(M,\tau)$ onto $\overline{M\xi}$.*

In particular, if $\overline{M\xi} = X$ and (i) *holds, the isomorphism onto X is uniquely determined by $1_M \mapsto \xi$.*

Proof. Positive spectral approximation and the polar-decomposition argument of Lemma 3.1 turn (i) into $\|x\xi\| = \|x\|_{p,\tau}$ for every $x \in M$. The isometric extension has range $\overline{M\xi}$ and is module-linear by density. The converse follows by applying the isometry to positive finite-spectrum elements. A module map with the prescribed distinguished value is determined on M. □

Remark 3.5. For a family in PM_p^0, a sufficient condition for (3.5) is the direct-sum estimate

$$\|F(i)u + F(j)v\| \le (\|u\|^p + \|v\|^p)^{1/p}, \tag{3.6}$$

for arbitrary vectors in $F(M)$ and $F(N)$, where $i : M \to M \oplus N$ and $j : N \to M \oplus N$ are the summand maps. Iterating this estimate on scalar summands and applying $\|\xi_{\mathbb{C}_s}\| \le s^{1/p}$ gives (3.5). For $p > 1$, (3.6) is strictly stronger than the defining conditions of PM_p. Indeed,

$$F(M) = L^p(M,\tau) \oplus_\infty L^1(M,\tau), \qquad \xi_M = (1_M, 0),$$

with the diagonal action and induced functorial maps, belongs to PM_p. Taking $M = N = \mathbb{C}_1$ and $u = v = (0,1)$ makes the left side of (3.6) equal to 2, while its right side is $2^{1/p}$. The scalar tests constrain precisely the cyclic submodules generated by the distinguished vectors.

3.2. **The initial object without the partition estimate.** For $x \in M$, define

$$\alpha_{p,\tau}(x) = \inf\left\{ \sum_{j=1}^n \|a_j\|_\infty \tau(e_j)^{1/p} : x = \sum_{j=1}^n a_j e_j, \quad a_j \in M, \quad e_j \in \mathcal{P}(M) \right\}. \tag{3.7}$$

The projections e_j need not be orthogonal. The one-term decomposition $x = x1_M$ is always admissible. Write $\mathcal{A}_p(M,\tau)$ for the completion in this norm.

Theorem 3.6 (The scalar-bounded initial module)**.** *The function $\alpha_{p,\tau}$ is a norm, and*

$$\|x\|_{p,\tau} \le \alpha_{p,\tau}(x) \le \tau(1_M)^{1/p}\|x\|_\infty. \tag{3.8}$$

Left multiplication extends to a unital contractive action of M on $\mathcal{A}_p(M,\tau)$. Trace-preserving embeddings extend to contractions and make $\mathcal{A}_p$ a functor on $\mathbf{TrEmb}_{\mathrm{fin}}$. With distinguished vector 1_M, this functor is initial in PM_p^0. For every projection e,

$$\alpha_{p,\tau}(e) = \tau(e)^{1/p}. \tag{3.9}$$

For $p = 1$, the equality $\alpha_{1,\tau}(x) = \|x\|_{1,\tau}$ holds for all $x \in M$.

Proof. Homogeneity and the triangle inequality follow by scaling and concatenating decompositions. For every summand in (3.7),

$$\|a_j e_j\|_{p,\tau} \le \|a_j\|_\infty \tau(e_j)^{1/p}.$$

The triangle inequality in L^p gives the lower bound in (3.8), including definiteness of the norm. The decomposition $x = x1_M$ gives the upper bound. Replacing a_j by ba_j in a decomposition proves

$$\alpha_{p,\tau}(bx) \le \|b\|_\infty \alpha_{p,\tau}(x).$$

The module action therefore extends to the completion. If $\pi : (N,\sigma) \to (M,\tau)$ is an embedding, then $\|\pi(a_j)\|_\infty = \|a_j\|_\infty$ and $\tau(\pi(e_j)) = \sigma(e_j)$. Applying π to a decomposition of x therefore gives a decomposition of $\pi(x)$ with the same sum in (3.7). Taking the infimum gives $\alpha_{p,\tau}(\pi(x)) \le \alpha_{p,\sigma}(x)$. Uniqueness of continuous extensions proves functoriality and equivariance. The identity $\mathcal{A}_p(\pi)(1_N) = \pi(1_N)\cdot 1_M$ proves the supported-point condition. The lower bound in (3.8) and the single-atom decomposition $e = 1_M e$ give (3.9), including the required scalar bounds.

Let (F,ξ) belong to PM_p^0. By Lemma 3.1, $\|e\xi_M\| \le \tau(e)^{1/p}$. Hence every atomic decomposition gives

$$\|x\xi_M\| \le \sum_j \|a_j\|_\infty \|e_j\xi_M\| \le \sum_j \|a_j\|_\infty \tau(e_j)^{1/p}.$$

Taking the infimum shows that $x \mapsto x\xi_M$ extends to a contraction $\mathcal{A}_p(M,\tau) \to F(M)$. As in Theorem 3.2, equivariance and the supported-point identity give naturality on the dense algebras, hence on their completions. Each component is a module map and sends 1_M to ξ_M. These two properties determine it on M, proving uniqueness.

For $p = 1$, a positive finite-spectrum element $a = \sum_j \lambda_j e_j$ satisfies $\alpha_{1,\tau}(a) \le \sum_j \lambda_j \tau(e_j) = \|a\|_{1,\tau}$. Positive spectral approximation is valid in the atomic norm by the upper bound in (3.8). Finally, the contractive module action and polar decomposition give $\alpha_{1,\tau}(x) = \alpha_{1,\tau}(|x|)$. This proves equality with the L^1 norm for all x. □

The initial morphism $\mathcal{A}_p \to L^p$ extends the identity on each algebra by continuity. On a finite scalar algebra its inverse is bounded, and its norm can be computed explicitly for unit weights.

Proposition 3.7 (Finite scalar formula and the sharp distortion)**.** *Let $D_{\mathbf{s}} = \bigoplus_{j=1}^n \mathbb{C}_{s_j}$. Relabel coordinates so that $a_1 = |z_1| \ge \cdots \ge a_n = |z_n|$, and set $a_{n+1} = 0$, $S_0 = 0$, and $S_k = \sum_{j=1}^k s_j$. Then*

$$\alpha_{p,\mathbf{s}}(z) = \sum_{k=1}^{n} (a_k - a_{k+1}) S_k^{1/p} \tag{3.10}$$

$$= \int_0^\infty \left(\sum_{\{j : |z_j| > t\}} s_j \right)^{1/p} dt. \tag{3.11}$$

Thus $\alpha_{p,\mathbf{s}}$ is the weighted $L^{p,1}$ norm, normalized so that a characteristic function of mass s has norm $s^{1/p}$. In Lorentz's notation this is the $\Lambda(\varphi, 1)$ norm with $\varphi(u) = p^{-1}u^{1/p-1}$ [Lor51].

For $p > 1$, put $p' = p/(p-1)$. On $\mathbb{C}^n$ with unit coordinate weights, the norm of the identity from ℓ_n^p to the atomic norm is exactly

$$D_{p,n} = \left(\sum_{k=1}^{n} \left(k^{1/p} - (k-1)^{1/p}\right)^{p'} \right)^{1/p'}. \tag{3.12}$$

In particular,

$$D_{p,n} \sim \frac{1}{p} (\log n)^{1/p'} \qquad (n \to \infty). \tag{3.13}$$

Proof. Multiplication by a diagonal unitary is an atomic-norm isometry, so we may assume $z_j = a_j \ge 0$ in the stated order. Let E_k be the projection onto the first k coordinates. The decomposition $z = \sum_k (a_k - a_{k+1}) E_k$ gives the upper bound in (3.10).

For the reverse bound, put $d_k = S_k^{1/p} - S_{k-1}^{1/p}$ and define $\ell(w) = \sum_k d_k w_k$. For every subset $J \subseteq \{1, \dots, n\}$, concavity of $g(t) = t^{1/p}$ gives

$$\sum_{k \in J} d_k \le \left(\sum_{k \in J} s_k \right)^{1/p}. \tag{3.14}$$

Indeed, for $k \in J$ put $t_k = \sum_{j \in J,\, j<k} s_j$. Since $t_k \le S_{k-1}$, concavity gives $d_k \le g(t_k + s_k) - g(t_k)$. Summing over J in increasing order gives (3.14) by telescoping. If e_J is the projection corresponding to J, then

$$|\ell(be_J)| \le \|b\|_\infty \sum_{k \in J} d_k \le \|b\|_\infty \tau(e_J)^{1/p}.$$

It follows from the definition of α that $|\ell(w)| \le \alpha_{p,\mathbf{s}}(w)$. Summation by parts gives $\ell(z) = \sum_k (a_k - a_{k+1}) S_k^{1/p}$, proving equality. On each nonempty interval (a_{k+1}, a_k) the distribution function has value S_k, which gives (3.11).

For unit weights, write $\delta_k = k^{1/p} - (k-1)^{1/p}$. Formula (3.10) becomes $\alpha_p(z) = \sum_k \delta_k a_k$. Hölder's inequality gives the upper bound $D_{p,n}$ in (3.12). The sequence δ_k is decreasing, and

equality is attained by the decreasing sequence $a_k = \delta_k^{p'-1}$, up to an overall scalar. Finally, $\delta_k^{p'} \sim p^{-p'}k^{-1}$, so summation gives (3.13). □

Corollary 3.8. *For $p > 1$, the pointed L^p-module functor is not initial in PM_p^0. There is no uniformly bounded natural family of module maps $L^p \to \mathcal{A}_p$ preserving the distinguished vectors.*

Proof. Every such component would be the identity on the dense algebra. Already on $\mathbb{C}_1 \oplus \mathbb{C}_1$,

$$\alpha_p(2,1) = 1 + 2^{1/p} > (2^p + 1)^{1/p} = \|(2,1)\|_p,$$

where strictness follows from the strict triangle inequality in ℓ_2^p for the nonproportional vectors $(1,0)$ and $(1,1)$. Thus the required initial morphism cannot be contractive. The constants in (3.12) tend to infinity, which excludes every uniform bound. □

3.3. **A functorial obstruction without the module action.** We now drop the module action and retain the conditions on embeddings, direct sums, and distinguished vectors specified below. On $M_2(\mathbb{C})$ these conditions allow maps that are linear on each commutative subalgebra but have no common linear extension.

Let PF_p denote the category of pairs (F, ξ), where $F : \mathbf{TrEmb}_{\mathrm{fin}} \to \mathbf{Ban}_1$ is a functor and the distinguished vectors $\xi_M \in F(M)$ satisfy

(i) $F(\pi)\xi_N = \xi_M$ for every *unital* embedding $\pi : N \to M$;
(ii) $\xi_{M\oplus N} = F(i)\xi_M + F(j)\xi_N$ for the summand maps;
(iii) $\|\xi_M\| \le \tau(1_M)^{1/p}$;
(iv) the direct-sum estimate (3.6) holds.

Morphisms are point-preserving natural contractions. The pointed functor $(L^p, \mathbf{1})$ belongs to PF_p.

Theorem 3.9. *For every $1 \le p < \infty$, there is an object (F, ξ) of PF_p for which no point-preserving natural transformation from L^p to F has linear components. In particular, $(L^p, \mathbf{1})$ is not initial in PF_p.*

Proof. Let $\mathcal{D}$ be the full subcategory consisting of the zero algebra, if present, and the algebras isomorphic to $\mathbb{C}$, $\mathbb{C}^2$, or $M_2(\mathbb{C})$, with arbitrary faithful finite traces. If an algebra embeds into an object of $\mathcal{D}$, then it also belongs to $\mathcal{D}$. Indeed, a proper nonzero $*$-subalgebra of $M_2(\mathbb{C})$ is either one-dimensional or unitarily conjugate to the diagonal algebra. Outside $\mathcal{D}$ set $F(M) = 0$ and $\xi_M = 0$. Send every arrow with target outside $\mathcal{D}$ to zero. It remains to define F on $\mathcal{D}$.

On the commutative objects of $\mathcal{D}$, set $F(M) = L^p(M, \tau)$ and $\xi_M = 1_M$. To define F on a two-by-two matrix algebra, write $\tau = c\,\mathrm{Tr}_2$, $c > 0$, and let

$$\mathbb{S}_M = \{a \in M : a = a^*,\ \mathrm{Tr}_2(a) = 0,\ a^2 = 1_M\}.$$

This is the unit sphere in the three-dimensional real space of traceless selfadjoint matrices, for the inner product $\frac{1}{2}\mathrm{Tr}_2(ab)$. For $n \in \mathbb{S}_M$, define

$$f_n(a) = \left(\tfrac{1}{2}\mathrm{Tr}_2(na)\right)^3, \qquad a \in \mathbb{S}_M,$$

and let H_M be the complex span of the functions f_n in $C(\mathbb{S}_M)$. The space H_M is finite dimensional because its elements are restrictions of homogeneous cubics in three variables. Each f_n has supremum norm one, and $f_{-n} = -f_n$. Set

$$F(M) = \mathbb{C} \oplus H_M, \qquad \|(b,h)\| = (2c)^{1/p}\max\{|b|, \|h\|_\infty\}, \qquad \xi_M = (1,0).$$

Every normal $x \in M$ can be written as $x = \lambda e + \mu(1_M - e)$ with a rank-one projection e. Put $n = 2e - 1_M$ and define

$$Q_M(x) = \left(\frac{\lambda+\mu}{2}, \frac{\lambda-\mu}{2} f_n\right). \tag{3.15}$$

Swapping the two spectral projections leaves this expression unchanged, and it is independent of e when x is scalar. Thus Q_M is well defined on normal elements and is complex-linear on every commutative $*$-subalgebra. Hölder's inequality gives

$$\begin{aligned}\|Q_M(x)\| &= \frac{(2c)^{1/p}}{2}\max\{|\lambda+\mu|,|\lambda-\mu|\}\\ &\le c^{1/p}(|\lambda|^p+|\mu|^p)^{1/p} = \|x\|_{p,\tau}.\end{aligned}$$

For $p=1$ this is simply the triangle inequality.

Between commutative objects of $\mathcal{D}$, use the usual L^p maps. For an embedding $\pi : N \to M$ with N commutative and $M \cong M_2(\mathbb{C})$, set $F(\pi) = Q_M \circ \pi$. The range of π is commutative, so this composition is linear, and the preceding estimate makes it contractive. An embedding $\pi : N \to M$ between two-by-two matrix algebras is a unital $*$-isomorphism and preserves the trace coefficient c. It induces an isometry $\mathbb{S}_N \to \mathbb{S}_M$. Set $F(\pi)(b,h) = (b, h\circ\pi^{-1})$. Since $f_n \circ \pi^{-1} = f_{\pi(n)}$, this action intertwines Q_N and Q_M. The resulting maps respect composition in $\mathcal{D}$. No arrow enters $\mathcal{D}$ from outside, so extension by zero gives a functor on $\mathbf{TrEmb}_{\mathrm{fin}}$.

The distinguished points are preserved by unital embeddings, since $Q_M(1_M) = (1,0)$. They have norm at most $\tau(1_M)^{1/p}$. If a direct sum of two nonzero algebras lies in $\mathcal{D}$, both summands are scalar. In this case additivity of the points is the identity $(1,1) = (1,0) + (0,1)$, and the weighted ℓ^p norm gives the direct-sum estimate. For a direct sum outside $\mathcal{D}$ its target space is zero, so both conditions hold there as well. The zero algebra, if included, contributes only the zero space. Thus (F,ξ) is an object of PF_p.

Suppose that a point-preserving natural transformation $T : L^p \to F$ had linear components. On $\mathbb{C}_1$, point preservation forces $T_{\mathbb{C}_1} = \mathrm{id}$. For a rank-one projection $e \in M_2(\mathbb{C})$ with the usual trace, naturality along $\mathbb{C}_1 \to M_2(\mathbb{C})$ gives $T_{M_2}(e) = Q_{M_2}(e)$. Let

$$z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad n = \frac{x+z}{\sqrt{2}}.$$

For $a \in \{x,z,n\}$, the projection $e = (1+a)/2$ has rank one. Thus $T_{M_2}(a) = 2Q_{M_2}(e) - \xi_{M_2} = (0, f_a)$. Linearity would give $f_x + f_z = \sqrt{2} f_n$. Evaluating this identity at $z \in \mathbb{S}_{M_2}$ gives $1 = 1/2$. This contradiction proves the assertion. □

4. Scalar and linear natural transformations

A natural family is *objectwise bounded* if each component is bounded and *uniformly bounded* if the component norms have a common finite bound. In $\mathbf{TrEmb}_{\mathrm{fin}}$ the total trace mass can be any $s > 0$.

4.1. **Scalar naturality.**

Lemma 4.1. *Fix $1 \le u < \infty$. Let*

$$\ell_{M,\tau} : L^u(M,\tau) \longrightarrow \mathbb{C}$$

be a bounded linear functional at each object, and suppose that

$$\ell_{M,\tau} \circ L^u(\pi) = \ell_{N,\sigma}$$

for every $\pi : (N,\sigma) \to (M,\tau)$ in $\mathbf{TrEmb}_{\mathrm{fin}}$. There is a unique $c \in \mathbb{C}$ such that

$$\ell_{M,\tau}(x) = c\,\tau(x) \qquad (x \in L^u(M,\tau)),$$

where τ denotes its continuous extension to $L^u(M,\tau)$. The component norms are

$$\|\ell_{M,\tau}\| = |c|\,\tau(1_M)^{1-1/u}.$$

Proof. For $s > 0$, put $a(s) = \ell_{\mathbb{C}_s}(1)$. Naturality for the diagonal map $\mathbb{C}_{s+t} \to \mathbb{C}_s \oplus \mathbb{C}_t$ and the two summand embeddings gives

$$a(s+t) = a(s) + a(t).$$

Let $A = L^\infty([0,1])$ with its integral trace, and for $0 < s \le 1$ let e_s be the characteristic function of $[0,s]$. The map $\mathbb{C}_s \to A$, $\lambda \mapsto \lambda e_s$, is a morphism. Hence

$$|a(s)| = |\ell_A(e_s)| \le \|\ell_A\| s^{1/u}.$$

Thus a extends to a continuous additive function on $\mathbb{R}$, so $a(s) = cs$ with $c = a(1)$.

For each nonzero projection $e \in M$, naturality for $\mathbb{C}_{\tau(e)} \to M$, $\lambda \mapsto \lambda e$, gives $\ell_{M,\tau}(e) = c\tau(e)$. Since the linear span of the projections is dense in $L^u(M,\tau)$ by Proposition 2.2, the formula holds on all of $L^u(M,\tau)$. The norm now follows from (2.2). □

Corollary 4.2. *For $p = 1$, the uniformly bounded scalar natural families are precisely the multiples $c\tau$, with uniform norm $|c|$. They are contractive exactly when $|c| \le 1$. For $1 < p < \infty$, the zero family is the only uniformly bounded scalar natural family.*

Proof. By Lemma 4.1, the component norm at total mass s is $|c|s^{1-1/p}$. The assertion follows by allowing s to range over all positive real numbers. □

4.2. **Natural maps between different exponents.**

Theorem 4.3. *Fix $1 \le p, q < \infty$. Let*

$$T_{M,\tau} : L^p(M,\tau) \longrightarrow L^q(M,\tau)$$

be an objectwise bounded natural family of complex linear maps. If $p \ge q$, there is a unique $c \in \mathbb{C}$ such that every component is c times the canonical inclusion $L^p(M,\tau) \subseteq L^q(M,\tau)$, and

$$\|T_{M,\tau}\| = |c|\tau(1_M)^{1/q-1/p}. \tag{4.1}$$

If $p < q$, the family is zero.

Consequently, the uniformly bounded natural families $L^p \to L^q$ are $c\,\mathrm{id}$ for $p = q$, with uniform norm $|c|$, and zero for $p \ne q$.

Proof. Write $T_{\mathbb{C}_s}(z) = a(s)z$. In the object $\mathbb{C}_s \oplus \mathbb{C}_t$, naturality for the two summand embeddings gives

$$T_{\mathbb{C}_s\oplus\mathbb{C}_t}(1,0) = (a(s),0), \qquad T_{\mathbb{C}_s\oplus\mathbb{C}_t}(0,1) = (0,a(t)).$$

Naturality for the unital diagonal embedding $\mathbb{C}_{s+t} \to \mathbb{C}_s \oplus \mathbb{C}_t$ now gives

$$(a(s),a(t)) = a(s+t)(1,1).$$

Thus $a(s) = c$ is independent of $s > 0$. The projection embeddings (2.3) imply $T_{M,\tau}(e) = ce$ for every projection $e \in M$. For $x \in M$, take a sequence in $\mathcal{P}_{\mathrm{lin}}(M)$ converging to x in operator norm (Proposition 2.2). This convergence holds in both L^p and L^q, so linearity and continuity give $T_{M,\tau}(x) = cx$.

If $p \ge q$, Hölder's inequality gives

$$\|x\|_{q,\tau} \le \tau(1_M)^{1/q-1/p}\|x\|_{p,\tau}.$$

Equality holds at 1_M. The inclusion therefore extends from M to L^p, and density gives (4.1). If $p < q$, work in $A = L^\infty([0,1])$ with the integral trace and projections e_s of mass s. Boundedness of T_A yields

$$|c|s^{1/q} \le \|T_A\| s^{1/p} \qquad (0 < s \le 1).$$

Letting $s \downarrow 0$ forces $c = 0$. Finally, when $p > q$, the component norms in (4.1) are unbounded over arbitrary trace masses unless $c = 0$. □

On the subcategory of unital embeddings, $x \mapsto \tau(x)1_M/\tau(1_M)$ is also a natural endomorphism. For a proper corner embedding π, applying naturality to the unit would equate a nonzero multiple of the target unit with $\pi(1)$.

4.3. **A sharp estimate for approximate scalar naturality.**

Proposition 4.4. *Fix $1 < p < \infty$, and let $\ell_{M,\tau} : L^p(M,\tau) \to \mathbb{C}$ be an arbitrary family of bounded linear functionals with*

$$K := \sup_{(M,\tau)} \|\ell_{M,\tau}\| < \infty.$$

Define its naturality defect by

$$\delta := \sup_{\pi:(N,\sigma)\to(M,\tau)} \|\ell_{M,\tau} \circ L^p(\pi) - \ell_{N,\sigma}\|.$$

Then

$$K \le \delta. \tag{4.2}$$

The coefficient 1 *in this inequality is optimal.*

Proof. Fix (N,σ), $x \in L^p(N,\sigma)$, and a positive integer n. Equip $N^{\oplus n}$ with the sum trace, and let $i_j : N \to N^{\oplus n}$ be its jth summand embedding. By the definition of δ,

$$\left| \sum_{j=1}^{n} \ell_{N^{\oplus n}}(L^p(i_j)x) - n\ell_N(x) \right| \le n\delta\|x\|_p.$$

The sum of the embedded vectors is $(x,\dots,x)$ and has norm $n^{1/p}\|x\|_p$. Consequently,

$$|\ell_N(x)| \le (Kn^{1/p-1} + \delta)\|x\|_p.$$

Letting $n \to \infty$ gives $|\ell_N(x)| \le \delta\|x\|_p$ at every object, which proves (4.2).

To see that equality can occur, fix $\varepsilon > 0$, put $p' = p/(p-1)$, and set, at each nonzero object,

$$\ell_{M,\tau} = \varepsilon\, \tau(1_M)^{-1/p'}\tau.$$

Each such component has norm ε. For an embedding $\pi : (N,\sigma) \to (M,\tau)$ with nonzero source, let $S = \sigma(1_N)$ and $T = \tau(1_M)$. Then $0 < S \le T$, and

$$\|\ell_{M,\tau} \circ L^p(\pi) - \ell_{N,\sigma}\| = \varepsilon\left(1 - (S/T)^{1/p'}\right) \le \varepsilon.$$

Taking summand embeddings with $T/S \to \infty$ makes this norm tend to ε. Thus $K = \delta = \varepsilon$ for this family. □

5. Natural bilinear forms and the Hilbert exponent

5.1. **Classification of bilinear forms.**

Definition 5.1. Let $1 \le p, q < \infty$. A *natural bilinear family* of exponents (p,q) consists of bounded complex bilinear forms

$$B_{M,\tau} : L^p(M,\tau) \times L^q(M,\tau) \longrightarrow \mathbb{C}$$

such that

$$B_{M,\tau}(L^p(\pi)x, L^q(\pi)y) = B_{N,\sigma}(x,y) \tag{5.1}$$

for every morphism $\pi : (N,\sigma) \to (M,\tau)$ and every $x \in L^p(N,\sigma)$, $y \in L^q(N,\sigma)$. Its uniform norm, possibly infinite, is

$$\|B\|_{\mathrm{unif}} := \sup_{(M,\tau)} \|B_{M,\tau}\|.$$

For $\alpha = p^{-1} + q^{-1} \le 1$, the form $(x,y) \mapsto \tau(xy)$ on $M \times M$ has a unique bounded extension to $L^p(M,\tau) \times L^q(M,\tau)$. Indeed, writing $T = \tau(1_M)$,

$$|\tau(xy)| \le T^{1-\alpha}\|x\|_{p,\tau}\|y\|_{q,\tau}. \tag{5.2}$$

We use the same notation for this extension. The separate trace product $(x,y) \mapsto \tau(x)\tau(y)$ is bounded for every $1 \le p, q < \infty$, by (2.2).

Theorem 5.2. *Let B be a natural bilinear family of exponents (p, q), and set $\alpha = p^{-1} + q^{-1}$.*

(i) *If $\alpha \le 1$, there are unique $c, d \in \mathbb{C}$ such that*

$$B_{M,\tau}(x, y) = c\,\tau(xy) + d\,\tau(x)\tau(y) \tag{5.3}$$

at every object and for all $x \in L^p(M, \tau)$, $y \in L^q(M, \tau)$.

(ii) *If $\alpha > 1$, there is a unique $d \in \mathbb{C}$ such that*

$$B_{M,\tau}(x, y) = d\,\tau(x)\tau(y).$$

Conversely, each of these formulas defines an objectwise bounded natural bilinear family.

Proof. Work first with $x, y \in M$ and suppress τ from the notation for B. Equip $M_2(M)$ with the trace $\mathrm{Tr}_2 \otimes \tau$, where Tr_2 is unnormalized. Write $E_{ij}(x) = e_{ij} \otimes x$ and set

$$C_M(x, y) := B_{M_2(M)}(E_{12}(x), E_{21}(y)), \tag{5.4}$$

$$D_M(x, y) := B_{M_2(M)}(E_{11}(x), E_{22}(y)). \tag{5.5}$$

Since $|E_{ij}(x)|^u = E_{jj}(|x|^u)$, the map $x \mapsto E_{ij}(x)$ preserves the L^u-norm. Hence C_M and D_M extend to bounded bilinear forms on $L^p(M, \tau) \times L^q(M, \tau)$.

The matrix identities. For selfadjoint $H \in M_2(M)$, conjugation by $\exp(itH)$ is a trace-preserving automorphism. Differentiating (5.1) at $t = 0$ gives

$$B_{M_2(M)}([H, X], Y) + B_{M_2(M)}(X, [H, Y]) = 0. \tag{5.6}$$

The derivatives of the two arguments exist in operator norm, hence also in the respective L^p- and L^q-norms. Linearity in H extends the identity to every $H \in M_2(M)$.

The flip matrix interchanges the two coordinates. Naturality under conjugation by this unitary gives

$$B_{M_2(M)}(E_{22}(x), E_{11}(y)) = D_M(x, y), \tag{5.7}$$

$$B_{M_2(M)}(E_{21}(x), E_{12}(y)) = C_M(x, y). \tag{5.8}$$

In (5.6), choose

$$H = E_{12}(a), \qquad X = E_{21}(x), \qquad Y = E_{11}(y).$$

The commutators are

$$[H, X] = E_{11}(ax) - E_{22}(xa), \qquad [H, Y] = -E_{12}(ya).$$

Using naturality for the diagonal corner embeddings and (5.7)–(5.8), we obtain

$$B_M(ax, y) - D_M(xa, y) - C_M(x, ya) = 0. \tag{5.9}$$

Taking $a = 1_M$ gives

$$B_M(x, y) = C_M(x, y) + D_M(x, y). \tag{5.10}$$

The second choice

$$H = E_{12}(a), \qquad X = E_{11}(x), \qquad Y = E_{21}(y)$$

gives

$$[H, X] = -E_{12}(xa), \qquad [H, Y] = E_{11}(ay) - E_{22}(ya),$$

and hence

$$-C_M(xa, y) + B_M(x, ay) - D_M(x, ya) = 0. \tag{5.11}$$

Conjugation by $\mathrm{diag}(u, 1_M)$, with u unitary in M, shows that $D_M(uxu^*, y) = D_M(x, y)$. Conjugation by $\mathrm{diag}(1_M, u)$ gives the corresponding invariance in the second variable. Differentiating and complexifying as above yields

$$D_M(ax, y) = D_M(xa, y), \qquad D_M(x, ay) = D_M(x, ya). \tag{5.12}$$

Substituting (5.10) in (5.9) and (5.11), the D_M terms cancel by (5.12), leaving

$$C_M(ax, y) = C_M(x, ya), \qquad C_M(xa, y) = C_M(x, ay). \tag{5.13}$$

The single-trace part. Set $f_M(z) = C_M(1_M, z)$. Setting $x = 1_M$ in the two identities of (5.13) gives

$$C_M(x, y) = f_M(xy), \qquad f_M(xy) = f_M(yx). \tag{5.14}$$

Moreover,

$$|f_M(z)| \le \|B_{M_2(M)}\|\tau(1_M)^{1/p}\|z\|_{q,\tau}. \tag{5.15}$$

Thus f_M extends to a bounded linear functional on $L^q(M, \tau)$.

If $\pi : (N, \sigma) \to (M, \tau)$ is a morphism, so is its entrywise amplification $\pi_2 : M_2(N) \to M_2(M)$. Equations (5.4) and (5.1) give

$$C_M(\pi(x), \pi(y)) = C_N(x, y).$$

Writing $e = \pi(1_N)$ and using (5.14), we obtain

$$f_M(\pi(z)) = f_M(e\pi(z)) = C_M(e, \pi(z)) = C_N(1_N, z) = f_N(z).$$

The extensions of f_M therefore form an objectwise bounded scalar natural family. By Lemma 4.1, there is $c \in \mathbb{C}$, independent of M, such that

$$C_M(x, y) = c\tau(xy) \qquad (x, y \in M). \tag{5.16}$$

The separate-trace part. For (A, ρ) and (B, σ), equip $A \oplus B$ with the sum trace and set

$$\Delta_{A,B}(x, y) := B_{A\oplus B}(x \oplus 0, 0 \oplus y).$$

This is a bounded bilinear form on $L^p(A, \rho) \times L^q(B, \sigma)$. An embedding $\pi : A \to A'$ induces $\pi \oplus \mathrm{id}_B : A \oplus B \to A' \oplus B$, so Δ is natural in the first variable. The same argument applies to the second variable. Fix B and $y \in L^q(B, \sigma)$. By Lemma 4.1,

$$\Delta_{A,B}(x, y) = \rho(x)\, g_B(y), \qquad g_B(y) = \Delta_{\mathbb{C}_1,B}(1, y).$$

Since $g_B(y) = \Delta_{\mathbb{C}_1,B}(1, y)$, the functionals g_B are bounded and natural in B. Applying Lemma 4.1 once more gives $g_B(y) = d\sigma(y)$ for a common $d \in \mathbb{C}$. Thus

$$\Delta_{A,B}(x, y) = d\rho(x)\sigma(y). \tag{5.17}$$

The diagonal block embedding $M \oplus M \to M_2(M)$ is trace-preserving. Naturality for this embedding identifies D_M with $\Delta_{M,M}$, so

$$D_M(x, y) = d\tau(x)\tau(y). \tag{5.18}$$

Combining (5.10), (5.16), and (5.18) establishes (5.3) on $M \times M$.

The exponent restriction and extension. Let $A = L^\infty([0, 1])$ carry the integral trace and let $e_s = \mathbf{1}_{[0,s]}$ for $0 < s \le 1$. If $\alpha > 1$, then

$$|cs + ds^2| = |B_A(e_s, e_s)| \le \|B_A\| s^\alpha.$$

Dividing by s and letting $s \downarrow 0$ gives $c = 0$. When $\alpha \le 1$, (5.2) bounds the single-trace term. For all p, q, the separate-trace term is bounded in modulus by

$$|d|\, \tau(1_M)^{2-\alpha}\|x\|_{p,\tau}\|y\|_{q,\tau}.$$

Density now extends the formula to the stated product of L^p-spaces.

For uniqueness, evaluate on the coordinate projections of $\mathbb{C}_1 \oplus \mathbb{C}_1$. The ordered pair of distinct projections gives d, and taking both arguments equal to the first projection gives $c + d$. Conversely, trace preservation and multiplicativity give naturality of the indicated forms on $M \times M$, hence on the completions. □

Corollary 5.3. *A uniformly bounded natural bilinear family of exponents (p, q) has one of the following forms.*

(i) *If $p^{-1} + q^{-1} = 1$, then*

$$B_{M,\tau}(x, y) = c\tau(xy)$$

for some $c \in \mathbb{C}$.

(ii) *If* $p = q = 1$, *then*

$$B_{M,\tau}(x, y) = d\,\tau(x)\tau(y)$$

for some $d \in \mathbb{C}$.

(iii) *For all other pairs of exponents,* $B = 0$.

The uniform norms in (i) *and* (ii) *are* $|c|$ *and* $|d|$, *respectively. In every case* $B_{M,\tau}$ *is symmetric on* $M \times M$.

Proof. Write $\alpha = p^{-1} + q^{-1}$ and $K = \|B\|_{\mathrm{unif}}$. Evaluating Theorem 5.2 on $(1, 1)$ in $\mathbb{C}_s$ gives

$$|cs + ds^2| \le Ks^\alpha \qquad (s > 0). \tag{5.19}$$

If $\alpha < 2$, divide by s^2 and let $s \to \infty$ to obtain $d = 0$. For $\alpha < 1$, the same limit after division by s gives $c = 0$. For $\alpha > 1$, the objectwise classification already gives $c = 0$. Thus a single-trace term can survive only for $\alpha = 1$, and a separate-trace term only for $\alpha = 2$, that is, $p = q = 1$. Hölder's inequality gives the stated norm bounds, with equality at $(1, 1)$ in $\mathbb{C}_1$. □

Remark 5.4. Conjugation in $M_2(M)$ separates the two trace terms for every finite tracial algebra, including those that are not hyperfinite. Uniform boundedness then restricts the exponents by varying the trace mass on $\mathbb{C}$.

5.2. **Natural sesquilinear forms.**

Corollary 5.5. *Fix* $1 \le p < \infty$. *Suppose that each* $L^p(M, \tau)$ *is equipped with a bounded sesquilinear form* $H_{M,\tau}$, *conjugate-linear in the first variable, and that*

$$H_{M,\tau}(L^p(\pi)x, L^p(\pi)y) = H_{N,\sigma}(x, y)$$

for every morphism $\pi : (N, \sigma) \to (M, \tau)$. *Assume* $\sup_{(M,\tau)} \|H_{M,\tau}\| < \infty$. *Then*

$$H_{M,\tau}(x, y) = \begin{cases} c\,\tau(x^*y), & p = 2, \\ d\,\overline{\tau(x)}\tau(y), & p = 1, \\ 0, & p \notin \{1, 2\}, \end{cases}$$

with a single coefficient independent of the object. If every form is positive and nondegenerate, then $p = 2$ *and* $c > 0$. *If, in addition,* $H_{\mathbb{C}_1}(1, 1) = 1$, *then* $c = 1$.

Proof. The adjoint map on M extends to a conjugate-linear isometry on every $L^p(M, \tau)$ and commutes with the maps induced by $*$-monomorphisms. Hence

$$B_{M,\tau}(x, y) := H_{M,\tau}(x^*, y)$$

is a uniformly bounded natural complex bilinear family. Apply Corollary 5.3 with $q = p$. Under positivity, evaluation at $(1, 1)$ in $\mathbb{C}_1$ makes the coefficient nonnegative. When $p = 1$, the nonzero vector $(1, -1) \in \mathbb{C}_1 \oplus \mathbb{C}_1$ has trace zero and pairs to zero with every vector. Nondegeneracy therefore forces $p = 2$ and $c > 0$. In that case $H_{\mathbb{C}_1}(1, 1) = c$. □

6. Complete isometries and conditional expectations

For an operator space E, the opposite space E^{op} has matrix norms $\|[x_{ij}]\|_{M_n(E^{\mathrm{op}})} = \|[x_{ji}]\|_{M_n(E)}$. Under the trace pairing $x \mapsto (a \mapsto \tau(ax))$, the standard operator-space structures are given by

$$L^1_{\mathrm{os}}(M, \tau) = (M^{\mathrm{op}})_*, \qquad L^p_{\mathrm{os}}(M, \tau) = [M, L^1_{\mathrm{os}}(M, \tau)]_{1/p} \quad (1 < p < \infty). \tag{6.1}$$

At the Banach space level the pairing identifies $L^1(M, \tau)$ isometrically with M_*. The L^1 endpoint carries the matrix norms of the opposite predual, and the interpolation formula holds at every matrix level [JRS05, Section 2, (2.7)], [Pis14, Section 9].

Remark 6.1. For $M = M_n$ with the usual trace, consider the subspace of first-row matrices. Its endpoint spaces in (6.1) are the row space R_n and the column space C_n. At $p = 2$ it is the operator Hilbert space OH_n. Transporting the matrix norms of M_* through the same pairing gives R_n at both endpoints [Pis14, Section 9].

For the embeddings in $\mathbf{TrEmb}_{\mathrm{fin}}$, the conditional expectation on the image corner gives an explicit left inverse. For the general theory of noncommutative L^p-isometries, see [JRS05, She06].

Proposition 6.2. *For every $\pi : (M,\tau) \to (N,\sigma)$ in $\mathbf{TrEmb}_{\mathrm{fin}}$ and every $1 \le p < \infty$, $L^p(\pi)$ is a complete isometry for (6.1). It has a canonical completely contractive left inverse $R_{\pi,p}$. If $\rho : (N,\sigma) \to (P,\omega)$ is another arrow, then*

$$R_{\rho\pi,p} = R_{\pi,p} R_{\rho,p}.$$

Proof. Put $e = \pi(1)$. The finite tracial inclusion $\pi(M) \subseteq eNe$ admits a unique normal conditional expectation E_π preserving $\sigma|_{eNe}$ [Tak72]. Set

$$R_\pi(a) = \pi^{-1} E_\pi(eae), \qquad a \in N.$$

Compression and conditional expectation are normal completely positive maps, so R_π is normal, unital and completely positive. It satisfies $R_\pi \pi = \mathrm{id}$. Trace preservation and the bimodule property of E_π [Tom57] give

$$\tau(x R_\pi(a)) = \sigma(\pi(x)a) \qquad (x \in M,\ a \in N). \tag{6.2}$$

At $p = \infty$ the maps are π and R_π. At $p = 1$, under (6.1), the embedding is $(R_\pi^{\mathrm{op}})_*$ and its left inverse is $(\pi^{\mathrm{op}})_*$. Both preadjoints are completely contractive. By (6.2) they agree with π and R_π, respectively, on M and N. Interpolation gives complete contractions $L^p(\pi)$ and $R_{\pi,p}$ with $R_{\pi,p} L^p(\pi) = \mathrm{id}$ by density. At each matrix level this left inverse gives the reverse norm inequality, so $L^p(\pi)$ is a complete isometry.

Applying (6.2) twice gives

$$\tau(x R_\pi R_\rho(a)) = \omega(\rho\pi(x)a).$$

The nondegeneracy of the trace pairing identifies $R_\pi R_\rho$ with $R_{\rho\pi}$ on P. Density proves the asserted identity on L^p. □

Corollary 6.3. *If $M \subseteq N$ is a unital finite tracial inclusion, its trace-preserving conditional expectation extends to a completely contractive M-bimodule projection*

$$E_p : L^p(N,\sigma) \longrightarrow L^p(M, \sigma|_M).$$

For a commutative square of unital inclusions $M \subseteq N$ and $M' \subseteq N'$ with vertical arrows $\alpha : M \to M'$ and $\beta : N \to N'$, if $E'\beta = \alpha E$, then $E'_p L^p(\beta) = L^p(\alpha) E_p$.

Proof. Apply Proposition 6.2 to the inclusion. The bimodule identity extends by (2.1), and the identity for the square extends by continuity. □

A commutative square of inclusions need not satisfy $E'\beta = \alpha E$. For example, take the inclusions $\mathbb{C}I \subseteq M_2$ and $D_2 \subseteq M_2$ with normalized traces, where D_2 is the diagonal algebra. If α is the scalar inclusion and $\beta = \mathrm{id}_{M_2}$, then $\alpha E(e_{11}) = \frac{1}{2} I$, whereas $E'\beta(e_{11}) = e_{11}$.

Let $\mathcal{M}_p^{\mathrm{os}}$ send (M,τ) to the dense subspace $M \subseteq L^p_{\mathrm{os}}(M,\tau)$ with its induced matrix norms. By Lemma 2.3 and Proposition 6.2, its objectwise operator-space completion is L^p_{os}. Equivalently, every natural complete contraction from these dense cores to a functor of complete operator spaces extends uniquely to L^p_{os}.

7. Symmetric weighted embeddings and modular compatibility

For a unital inclusion $A \subseteq B$ and a faithful normal state φ on B, the self-polar forms of φ and $\varphi|_A$ agree on A precisely when there is a φ-preserving conditional expectation onto A [HS95, Theorem 4.2]. Junge, Ruan, and Sherman use this criterion in their classification of 2-isometries [JRS05, proof of Theorem 4.9]. Takesaki characterizes the existence of the expectation by invariance under the modular group [Tak72]. For finite-dimensional embeddings we give a direct proof that also identifies the obstruction on the support of a nonunital embedding. The resulting block decomposition lets us compute the distance to the nearest compatible density.

7.1. **The qualitative criterion.** Let

$$M = \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}), \qquad N = \bigoplus_{j=1}^{s} M_{m_j}(\mathbb{C})$$

carry the sums of the unnormalized matrix traces on their summands. Fix positive invertible densities $h \in M$ and $k \in N$. For $1 \le p < \infty$, write

$$\varphi_h(x) = \mathrm{Tr}_M(hx), \qquad \varphi_k(a) = \mathrm{Tr}_N(ka), \qquad \|x\|_{p,h} = \|h^{1/(2p)} x h^{1/(2p)}\|_p.$$

Define $\|\cdot\|_{p,k}$ on N in the same way. All unweighted L^p norms in this section use the unnormalized traces. Let $\pi : M \to N$ be an injective $*$-homomorphism satisfying

$$\varphi_k \pi = \varphi_h. \tag{7.1}$$

We allow $\mathrm{Tr}_N \pi \neq \mathrm{Tr}_M$. Put $e = \pi(1_M)$. The trace adjoint $\pi^\dagger : N \to M$ is characterized by

$$\mathrm{Tr}_M(x\pi^\dagger(a)) = \mathrm{Tr}_N(\pi(x)a).$$

It is completely positive, and (7.1) is equivalent to $\pi^\dagger(k) = h$.

Lemma 7.1. *The map $\pi : (M, \|\cdot\|_{2,h}) \to (N, \|\cdot\|_{2,k})$ is a contraction. Define*

$$R_{h,k}(a) = h^{-1/2}\pi^\dagger(k^{1/2} a k^{1/2}) h^{-1/2}, \qquad S_{h,k} = R_{h,k}\pi. \tag{7.2}$$

Then $R_{h,k}$ is unital and completely positive, and $S_{h,k}$ is a positive self-adjoint contraction for the inner product

$$\langle x, y\rangle_h = \mathrm{Tr}_M(x^* h^{1/2} y h^{1/2}).$$

In particular,

$$\mathcal{D}_\pi(x) := \|x\|_{2,h}^2 - \|\pi(x)\|_{2,k}^2 = \langle x, (I - S_{h,k})x\rangle_h \ge 0, \tag{7.3}$$

and π preserves the weighted L^2 norm if and only if $S_{h,k} = I$.

Proof. The formula for $R_{h,k}$ gives complete positivity, and $R_{h,k}(1_N) = h^{-1/2}\pi^\dagger(k) h^{-1/2} = 1_M$. The trace adjoint identity gives, for $x, y \in M$,

$$\langle x, S_{h,k} y\rangle_h = \mathrm{Tr}_N(\pi(x)^* k^{1/2} \pi(y) k^{1/2}) = \langle \pi(x), \pi(y)\rangle_k.$$

Thus $S_{h,k}$ is positive and self-adjoint on the weighted Hilbert space. As a map on the C^*-algebra M, it is completely positive and subunital since $S_{h,k}(1_M) = R_{h,k}(e) \le 1_M$. Its operator norm on the C^*-algebra M is therefore at most one, so its eigenvalues lie in $[0, 1]$. Hence $0 \le S_{h,k} \le I$ on the weighted Hilbert space. Taking $y = x$ gives (7.3) and the criterion for equality. □

After unitary changes of coordinates in the summands of N, every such embedding has the form

$$\mathbb{C}^{m_j} = \left(\bigoplus_{i : r_{ji} > 0} \mathbb{C}^{n_i} \otimes \mathbb{C}^{r_{ji}} \right) \oplus H_{j,0}, \qquad \pi(x)_j = \left(\bigoplus_i x_i \otimes I_{r_{ji}} \right) \oplus 0. \tag{7.4}$$

Terms with $r_{ji} = 0$ are omitted throughout. For $a \in N$, let $a_{j,ii}$ denote its compression to $\mathbb{C}^{n_i} \otimes \mathbb{C}^{r_{ji}}$. The ordinary partial trace over the second tensor factor gives

$$\pi^\dagger(a)_i = \sum_j \mathrm{Tr}_{r_{ji}}(a_{j,ii}).$$

Theorem 7.2. *Under (7.1), the following conditions are equivalent.*

1. *$\|\pi(x)\|_{2,k} = \|x\|_{2,h}$ for all $x \in M$.*
2. *$[k, e] = 0$ and there is a $\varphi_k|_{eNe}$-preserving conditional expectation from eNe onto $\pi(M)$.*
3. *$[k, e] = 0$ and*

$$k^{it}\pi(x)k^{-it} = \pi(h^{it} x h^{-it}) \qquad (x \in M,\ t \in \mathbb{R}).$$

(4) In (7.4), the density has the form

$$k_j = \left(\bigoplus_i h_i \otimes \rho_{ji}\right) \oplus k_{j,0}, \qquad \rho_{ji} > 0, \quad k_{j,0} > 0, \quad \sum_j \mathrm{Tr}(\rho_{ji}) = 1 \quad (1 \le i \le r). \tag{7.5}$$

The complement term is omitted when $H_{j,0} = 0$.

When these conditions hold, $\|\pi(x)\|_{p,k} = \|x\|_{p,h}$ for every $x \in M$ and $1 \le p < \infty$. The conditional expectation in (2) is unique and equals πR, where $\omega_\rho(b) = \mathrm{Tr}(\rho b)$ and

$$R(a)_i = \sum_j (\mathrm{id}_{M_{n_i}} \otimes \omega_{\rho_{ji}})(a_{j,ii}), \qquad a \in eNe. \tag{7.6}$$

Proof. Assume (1). Since $\mathrm{Tr}_M(h) = \mathrm{Tr}_N(ke)$, evaluating (7.3) at 1_M gives

$$0 = \mathrm{Tr}_N(ke) - \mathrm{Tr}_N(ek^{1/2}ek^{1/2}) = \|(1-e)k^{1/2}e\|_2^2. \tag{7.7}$$

Hence $[k, e] = 0$. By Lemma 7.1, the restriction $R_{h,k} : eNe \to M$ is a unital completely positive left inverse of π. Equality in the completely positive Schwarz inequality on $\pi(M)$ implies

$$R_{h,k}(\pi(x)a\pi(y)) = xR_{h,k}(a)y \qquad (x, y \in M,\ a \in eNe).$$

Consequently $\pi R_{h,k}$ is a conditional expectation. For $a \in eNe$,

$$\varphi_h(R_{h,k}(a)) = \mathrm{Tr}_N(ek^{1/2}ak^{1/2}) = \varphi_k(a),$$

which proves (2).

Assume (2), let E be the stated expectation, and put $R = \pi^{-1}E$. Bimodularity over the central projections of M implies that R vanishes on blocks joining distinct source summands. On the (j, i) diagonal block, bimodularity over M_{n_i} gives

$$R_i(x \otimes b) = x\,\omega_{ji}(b)$$

for a positive functional ω_{ji} on $M_{r_{ji}}$. Indeed, $R_i(I \otimes b)$ commutes with every matrix in M_{n_i} and is therefore scalar. Unitality gives $\sum_j \omega_{ji}(I) = 1$. Write $\omega_{ji}(b) = \mathrm{Tr}(\rho_{ji}b)$. Evaluating $\varphi_h R = \varphi_k|_{eNe}$ on each block gives $k_{j,ii} = h_i \otimes \rho_{ji}$ and zero off-diagonal blocks in eNe. Since k is invertible, each ρ_{ji} is positive definite. Together with $[k, e] = 0$, this proves (4). The blocks of k determine the functionals ω_{ji} uniquely, so (7.6) also proves uniqueness of E.

Taking imaginary powers in (7.5) gives (3). Conversely, apply (3) to the minimal central projections of M. Their images commute with every k^{it}, hence with k. On a (j, i) block, differentiation at $t = 0$ now gives

$$[\log k_{j,ii} - \log h_i \otimes I,\ x \otimes I] = 0 \qquad (x \in M_{n_i}).$$

The commutant is $I \otimes M_{r_{ji}}$. Exponentiation therefore yields $k_{j,ii} = h_i \otimes \rho_{ji}$ with $\rho_{ji} > 0$. Weight preservation gives $\sum_j \mathrm{Tr}(\rho_{ji}) = 1$, proving (4).

Finally, (4) implies

$$k_j^{1/(2p)}\pi(x)_j k_j^{1/(2p)} = \left(\bigoplus_i h_i^{1/(2p)} x_i h_i^{1/(2p)} \otimes \rho_{ji}^{1/p}\right) \oplus 0.$$

Taking pth powers of Schatten norms and summing over j, i gives $\|\pi(x)\|_{p,k}^p = \|x\|_{p,h}^p$. This includes (1) and completes the proof. □

Corollary 7.3. *Under the equivalent conditions of Theorem 7.2, equip M and N with the operator-space structures transported from their standard tracial L^p-spaces through*

$$J_{p,h}(x) = h^{1/(2p)}xh^{1/(2p)}, \qquad J_{p,k}(a) = k^{1/(2p)}ak^{1/(2p)}.$$

Then π is a complete isometry for every $1 \le p < \infty$.

Proof. In the coordinates of (7.5), the map

$$T_p = J_{p,k}\pi J_{p,h}^{-1} : L^p(M, \mathrm{Tr}_M) \longrightarrow L^p(N, \mathrm{Tr}_N)$$

has the form

$$T_p(x)_j = \left(\bigoplus_i x_i \otimes \rho_{ji}^{1/p}\right) \oplus 0.$$

Consequently, for every positive integer n and $X \in L^p(M_n \otimes M)$,

$$\|(\mathrm{id}_{M_n} \otimes T_p)(X)\|_p^p = \sum_i \|X_i\|_p^p \sum_j \mathrm{Tr}(\rho_{ji}) = \|X\|_p^p.$$

By [JRS05, (2.8)–(2.9)], the corresponding operator-space matrix norm is

$$\|X\|_{M_n(L^p(M))} = \sup_{\substack{\|a\|_{S^n_{2p}} \le 1 \\ \|b\|_{S^n_{2p}} \le 1}} \|(a \otimes 1_M)X(b \otimes 1_M)\|_{L^p(M_n \otimes M)},$$

where $S_r^n = L^r(M_n, \mathrm{Tr}_n)$. Entrywise application of T_p commutes with multiplication by the scalar matrices a and b. The preceding norm equality therefore gives equality of the operator-space matrix norms at every level. □

Example 7.4. For any invertible density matrix $\rho \in M_d$ with $\mathrm{Tr}(\rho) = 1$, the embedding $\pi : M_n \to M_n \otimes M_d$, $\pi(x) = x \otimes I_d$, with target density $k = h \otimes \rho$, satisfies (7.5). Hence

$$\|h^{1/(2p)} x h^{1/(2p)} \otimes \rho^{1/p}\|_p = \|x\|_{p,h}.$$

These embeddings scale the ordinary trace by d.

7.2. **Nearest compatible densities.** In this subsection and the next, assume π is unital, so that the complement spaces in (7.4) vanish. Fix h, and let $\mathcal{K}_{\pi,h}$ be the set of all invertible positive $\ell \in N$ with $\pi^\dagger(\ell) = h$ that satisfy the equivalent conditions of Theorem 7.2. In these coordinates,

$$\mathcal{K}_{\pi,h} = \left\{ \bigoplus_{j,i} h_i \otimes \rho_{ji} : \rho_{ji} > 0, \quad \sum_j \mathrm{Tr}(\rho_{ji}) = 1 \text{ for every } i \right\}. \tag{7.8}$$

Define

$$d_{\mathrm{mat}}(k) = \inf_{\ell \in \mathcal{K}_{\pi,h}} \|k - \ell\|_2, \qquad d_{\mathrm{sq}}(k) = \inf_{\ell \in \mathcal{K}_{\pi,h}} \|k^{1/2} - \ell^{1/2}\|_2.$$

The second distance compares positive square roots in the given matrix algebra.

Set $q = k^{1/2}$, $t_i = \mathrm{Tr}(h_i)$, $u_i = \mathrm{Tr}(h_i^2)$, and define

$$\begin{aligned} c_{ji} &= u_i^{-1} \mathrm{Tr}_{n_i}[(h_i \otimes I)k_{j,ii}], \\ b_{ji} &= t_i^{-1} \mathrm{Tr}_{n_i}[(h_i^{1/2} \otimes I)q_{j,ii}], \qquad r_i = \sum_j \mathrm{Tr}(b_{ji}^2). \end{aligned} \tag{7.9}$$

Here Tr_{n_i} traces over the source factor. The partial trace in $\pi^\dagger$ traces over the multiplicity factor.

Theorem 7.5. *The matrices c_{ji} and b_{ji} are positive definite and satisfy*

$$\sum_j \mathrm{Tr}(c_{ji}) = 1, \qquad 0 < r_i \le 1.$$

The unique minimizers of $d_{\mathrm{mat}}(k)$ and $d_{\mathrm{sq}}(k)$ are, respectively,

$$\bar{k} = \bigoplus_{j,i} h_i \otimes c_{ji}, \qquad \widehat{k} = \bigoplus_{j,i} h_i \otimes \frac{b_{ji}^2}{r_i}. \tag{7.10}$$

The minimum squared distances are

$$d_{\mathrm{mat}}(k)^2 = \|k\|_2^2 - \sum_i u_i \sum_j \mathrm{Tr}(c_{ji}^2), \tag{7.11}$$

$$d_{\mathrm{sq}}(k)^2 = 2\sum_i t_i(1-\sqrt{r_i}). \tag{7.12}$$

Let

$$V_h = \left\{ \bigoplus_{j,i} h_i^{1/2} \otimes z_{ji} : z_{ji} \in M_{r_{ji}} \right\} \subseteq N$$

and put

$$\delta(k) = \mathrm{dist}_2(q, V_h)^2 = \sum_i t_i(1-r_i). \tag{7.13}$$

Then

$$\delta(k) \le d_{\mathrm{sq}}(k)^2 \le 2\delta(k), \tag{7.14}$$

and $\delta(k) = 0$ if and only if $k \in \mathcal{K}_{\pi,h}$.

Proof. The Hilbert–Schmidt orthogonal projection of k onto

$$W_h = \left\{ \bigoplus_{j,i} h_i \otimes z_{ji} : z_{ji} \in M_{r_{ji}} \right\}$$

is $\bar{k}$, by the defining property of the partial trace and $\|h_i\|_2^2 = u_i$. Each coefficient is positive definite because

$$\mathrm{Tr}_{n_i}[(h_i \otimes I)k_{j,ii}] = \mathrm{Tr}_{n_i}[(h_i^{1/2} \otimes I)k_{j,ii}(h_i^{1/2} \otimes I)] > 0.$$

Moreover, $\pi^\dagger(k) = h$ gives

$$\sum_j \mathrm{Tr}(c_{ji}) = u_i^{-1}\,\mathrm{Tr}\left(h_i \sum_j \mathrm{Tr}_{r_{ji}}\, k_{j,ii} \right) = 1.$$

Thus $\bar{k} \in \mathcal{K}_{\pi,h}$. Since this set is contained in W_h, $\bar{k}$ is its unique point nearest to k. The Pythagorean identity gives (7.11).

Similarly, the orthogonal projection of q onto V_h is $q_0 = \bigoplus_{j,i} h_i^{1/2} \otimes b_{ji}$. Applying the same partial trace argument to $q_{j,ii}$ and $h_i^{1/4}$ gives $b_{ji} > 0$. Let P_{ji} be the projection onto the (j,i) block. Orthogonal projection of each diagonal block gives

$$t_i r_i \le \sum_j \|q_{j,ii}\|_2^2 \le \sum_j \|q_j P_{ji}\|_2^2 = \mathrm{Tr}_N(k\pi(1_{M_{n_i}})) = t_i.$$

Hence $0 < r_i \le 1$. Since $\mathrm{Tr}_N(k) = \mathrm{Tr}_M(h) = \sum_i t_i$, subtracting $\|q_0\|_2^2 = \sum_i t_i r_i$ from $\|q\|_2^2$ gives (7.13).

Every $\ell \in \mathcal{K}_{\pi,h}$ has $\ell^{1/2} = \bigoplus_{j,i} h_i^{1/2} \otimes a_{ji}$ with $a_{ji} > 0$ and $\sum_j \mathrm{Tr}(a_{ji}^2) = 1$. For each i, the Hilbert–Schmidt Cauchy–Schwarz inequality gives

$$\sum_j \mathrm{Tr}(b_{ji}a_{ji}) \le \sqrt{r_i},$$

with equality precisely when $a_{ji} = b_{ji}/\sqrt{r_i}$ for every j. Since $\mathrm{Tr}(\ell) = \mathrm{Tr}(k) = \sum_i t_i$, minimizing

$$\|q - \ell^{1/2}\|_2^2 = 2\sum_i t_i - 2\sum_i t_i \sum_j \mathrm{Tr}(b_{ji}a_{ji})$$

gives the stated $\widehat{k}$ and (7.12). Finally, for $0 < r \le 1$,

$$1 - r \le 2(1-\sqrt{r}) \le 2(1-r).$$

Summing with weights t_i gives (7.14). Since the minimum defining d_{sq} is attained, $\delta(k) = 0$ exactly when $k \in \mathcal{K}_{\pi,h}$. □

Example 7.6. Both constants in (7.14) are optimal uniformly over finite dimensions. For $n \geq 2$, take $M = M_n$, $h = I_n/n$, $N = M_n \otimes M_n$, and $\pi(x) = x \otimes I_n$. For $\Omega = n^{-1/2} \sum_{a=1}^{n} e_a \otimes e_a$ and $0 < \eta < 1$, set

$$k_\eta = (1-\eta)|\Omega\rangle\langle\Omega| + \eta I_{n^2}/n^2.$$

These densities are faithful and have source marginal h. Let $P = |\Omega\rangle\langle\Omega|$ and $q = k_\eta^{1/2}$. Since q is a linear combination of P and I, and $\mathrm{Tr}_n(P) = I_n/n$, its partial trace over the source is $(\mathrm{Tr}(q)/n)I_n$. Thus the single coefficient r in (7.9) is $(\mathrm{Tr}(q)/n)^2$. It is less than one for $0 < \eta < 1$ by strict Cauchy–Schwarz, since $\mathrm{Tr}(q^2) = 1$ and q is nonscalar. As $\eta \downarrow 0$, q tends to P and r tends to $1/n^2$. Hence

$$\frac{d_{\mathrm{sq}}(k_\eta)^2}{\delta(k_\eta)} = \frac{2}{1+\sqrt{r}} \longrightarrow \frac{2n}{n+1}.$$

Letting $n \to \infty$ proves optimality of the upper constant. For fixed n, letting $\eta \uparrow 1$ gives $q \to I_{n^2}/n$ and $r \to 1$. The ratio then tends to one.

7.3. **Finite tests and quantitative stability.** In each summand choose matrix units $e_{ab}^{(i)}$ with respect to an eigenbasis of h_i, with eigenvalues $\lambda_{i1}, \ldots, \lambda_{in_i}$. Define the maximal relative loss of squared L^2 norm by

$$\varepsilon(k) = \sup_{x \neq 0} \frac{\mathcal{D}_\pi(x)}{\|x\|_{2,h}^2}, \qquad t = \mathrm{Tr}(h), \qquad \mu_h = \min_i \frac{\lambda_{\min}(h_i)^2}{t_i}. \tag{7.15}$$

Here $0 \leq \varepsilon(k) < 1$ and $\mu_h > 0$. Indeed, injectivity of π makes the Gram operator $S_{h,k}$ positive definite.

Proposition 7.7. *The losses of squared norm on the matrix units determine $\delta(k)$ by*

$$\delta(k) = \sum_i \frac{1}{t_i} \sum_{a,b=1}^{n_i} \sqrt{\lambda_{ia}\lambda_{ib}}\, \mathcal{D}_\pi(e_{ab}^{(i)}). \tag{7.16}$$

Consequently,

$$\frac{\delta(k)}{t} \leq \varepsilon(k) \leq \frac{\delta(k)}{\mu_h}, \qquad \frac{d_{\mathrm{sq}}(k)^2}{2t} \leq \varepsilon(k) \leq \frac{d_{\mathrm{sq}}(k)^2}{\mu_h}. \tag{7.17}$$

In particular, if π preserves the weighted L^2 norm of every $e_{ab}^{(i)}$, it preserves that norm on all of M.

Proof. Let $Q_{ji,a}$ denote the (a,a) multiplicity block of $q_{j,ii}$ in the chosen basis. Matrix multiplication gives

$$\|\pi(e_{ab}^{(i)})\|_{2,k}^2 = \sum_j \mathrm{Tr}(Q_{ji,a} Q_{ji,b}), \qquad \|e_{ab}^{(i)}\|_{2,h}^2 = \sqrt{\lambda_{ia}\lambda_{ib}}.$$

Weight the first formula by $\sqrt{\lambda_{ia}\lambda_{ib}}$ and sum over a, b. By (7.9) this gives

$$\sum_j \mathrm{Tr}\left[\left(\sum_a \sqrt{\lambda_{ia}} Q_{ji,a}\right)^2\right] = t_i^2 r_i.$$

The same weighted sum for the second formula is t_i^2. Subtract, divide by t_i, and sum over i to obtain (7.16).

The vectors $f_{ab}^{(i)} = (\lambda_{ia}\lambda_{ib})^{-1/4} e_{ab}^{(i)}$ form an orthonormal basis of $(M, \langle\cdot,\cdot\rangle_h)$. By Lemma 7.1, $A = I - S_{h,k}$ is positive, with $\|A\| = \varepsilon(k)$, and

$$\delta(k) = \sum_{i,a,b} \frac{\lambda_{ia}\lambda_{ib}}{t_i} \langle f_{ab}^{(i)}, A f_{ab}^{(i)} \rangle_h.$$

The coefficients have sum t and are at least μ_h. It follows that $\mu_h \|A\| \leq \mu_h \,\mathrm{tr}(A) \leq \delta(k) \leq t\|A\|$, where tr is the trace of a linear operator on this finite-dimensional Hilbert space. The remaining inequalities follow from (7.14). □

Theorem 7.8. *Let $m = \lambda_{\min}(k)$, $M_0 = \|k\|_\infty$, and $H_0 = \max_i \|h_i\|_\infty$. Then*

$$\sqrt{m}\, d_{\mathrm{sq}}(k) \le d_{\mathrm{mat}}(k) \le (\sqrt{M_0} + \sqrt{H_0}) d_{\mathrm{sq}}(k). \tag{7.18}$$

In particular,

$$\sqrt{m\delta(k)} \le d_{\mathrm{mat}}(k) \le \sqrt{2}(\sqrt{M_0} + \sqrt{H_0})\sqrt{\delta(k)}, \tag{7.19}$$

and

$$\frac{d_{\mathrm{mat}}(k)^2}{2t(\sqrt{M_0} + \sqrt{H_0})^2} \le \varepsilon(k) \le \frac{d_{\mathrm{mat}}(k)^2}{m\mu_h}. \tag{7.20}$$

Proof. For any positive ℓ, write $X = k^{1/2} - \ell^{1/2}$. The identity

$$k - \ell = k^{1/2}X + X\ell^{1/2}$$

and the Hilbert–Schmidt inner product give

$$\sqrt{m}\|X\|_2^2 \le \mathrm{Tr}(X^* k^{1/2} X) + \mathrm{Tr}(X^* X \ell^{1/2}) \le \|X\|_2 \|k - \ell\|_2.$$

Taking the infimum over $\mathcal{K}_{\pi,h}$ proves the lower bound. For the upper bound, take $\ell = \widehat{k}$ from Theorem 7.5. Its multiplicity densities have total trace one for each i, so $\|\widehat{k}\|_\infty \le H_0$. The same algebraic identity yields

$$d_{\mathrm{mat}}(k) \le \|k - \widehat{k}\|_2 \le (\sqrt{M_0} + \sqrt{H_0})\|k^{1/2} - \widehat{k}^{1/2}\|_2.$$

Equations (7.14) and (7.17) give the remaining estimates. □

For $M = M_n$ and $h = I_n/n$, the bounds in (7.17) reduce to $\delta(k) \le \varepsilon(k) \le n^2\delta(k)$. Here the factor n^2 is the reciprocal of the least coefficient in the weighted sum for $\delta(k)$ over the orthonormal matrix units.

7.4. **The support defect for nonunital embeddings.** For a possibly nonunital π, testing the unit measures the part of $k^{1/2}$ that couples $e = \pi(1_M)$ to its complement.

Proposition 7.9. *Put $\Delta_e = \mathcal{D}_\pi(1_M)$. Then*

$$\Delta_e = \|(1 - e)k^{1/2}e\|_2^2 = \tfrac{1}{2}\|[k^{1/2}, e]\|_2^2. \tag{7.21}$$

The pinched density

$$k_e = eke + (1 - e)k(1 - e)$$

is positive definite, has the same trace as k, and satisfies $\pi^\dagger(k_e) = h$. It is the unique Hilbert–Schmidt nearest density commuting with e, and

$$\|k - k_e\|_2^2 = \|[k, e]\|_2^2. \tag{7.22}$$

If $mI \le k \le M_0 I$, then

$$8m\Delta_e \le \|k - k_e\|_2^2 \le 8M_0\Delta_e. \tag{7.23}$$

Proof. Subtracting $\mathrm{Tr}_N(ek^{1/2}ek^{1/2})$ from $\mathrm{Tr}_N(ke)$, as in (7.7), gives the first equality in (7.21). The two off-diagonal blocks of $[k^{1/2}, e]$ are adjoints up to sign, giving the second equality. Pinching is the Hilbert–Schmidt orthogonal projection onto the algebra commuting with e. It preserves lower spectral bounds and the trace. Since $\pi(x) = e\pi(x)e$, replacing k by k_e also preserves the functional on $\pi(M)$. The two off-diagonal blocks of $k - k_e$ agree up to sign with those of $[k, e]$, proving (7.22).

In an eigenbasis of k, with eigenvalues κ_a, the entries of the two commutators satisfy

$$[k, e]_{ab} = (\sqrt{\kappa_a} + \sqrt{\kappa_b})[k^{1/2}, e]_{ab}.$$

Squaring, summing, and using $2\sqrt{m} \le \sqrt{\kappa_a} + \sqrt{\kappa_b} \le 2\sqrt{M_0}$ proves (7.23). □

The embedding $M \to eNe$ is unital, with target density eke. Let ℓ_e be the nearest compatible density to eke given by (7.10) for the matrix distance in eNe. By (7.5), a compatible density on N may have any positive definite complementary block. The off-diagonal part $k - k_e$ is Hilbert–Schmidt orthogonal to both diagonal corners. Thus the unique nearest compatible density on N is $\ell_e + (1-e)k(1-e)$, and its squared distance from k is $\|k - k_e\|_2^2 + \|eke - \ell_e\|_2^2$.

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