Three-Relator Universal Group, Four-Relator Adian–Rabin Family, and Unrecognizability of $\#^2(S^2 \times S^2)$

Abstract

We construct a universal group with 2 generators and 3 relators, and an Adian–Rabin family with 2 generators and 4 relators whose nontrivial groups are universal. The resulting presentations show that the standard smooth manifold $\#^n(S^2 \times S^2)$ is unrecognizable up to homeomorphism or diffeomorphism for every fixed $n \geq 2$.

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1 Introduction

A group is *universal* if it contains an isomorphic copy of every finitely presented group. An *Adian–Rabin family* is a computable sequence of finite group presentations $(P_i)_{i\in\mathbb{N}}$ for which there is no algorithm deciding, given i, whether the group presented by P_i is trivial. We ask how few relators are needed to present a universal group or to form an Adian–Rabin family. Families with few relators also give unrecognizable four-manifolds, with the number of relators governing the complexity of the manifold construction.

Higman's embedding theorem gives a finitely presented universal group [12]. Boone and Collins constructed one with 26 relators [5], and Valiev reduced this to 21 [22]. The previously known presentation of a universal group with the fewest relators has 2 generators and 13 relators [20]. Starting from that group, we construct a universal group with 2 generators and 3 relators.

Novikov [17] and Boone [4] proved that finitely presented groups can have unsolvable word problem. Borisov's 12-relator example [7] was improved by Kegel–Li–Ren to a group with 3 generators and 9 relators [13, Theorem 1.1], the smallest previous relator count for a group with unsolvable word problem. Our universal group reduces this count to 3. Indeed, every finitely generated universal group has unsolvable word problem: it contains a finitely presented group with unsolvable word problem, and solvability of the word problem passes to finitely generated subgroups.

The triviality problem concerns a family of presentations rather than words in one fixed group. Its undecidability goes back to Adian [1] and Rabin [18]. Here we track two quantities: the number of relators and the *relator excess*, defined as the number of relators minus the number of generators. Kegel–Li–Ren constructed a 2-generator 10-relator Adian–Rabin family, giving the smallest previous relator count, and a 4-generator 11-relator family, giving the smallest previous relator excess, $11 - 4 = 7$ [13, Theorems 1.3 and 1.4]. Applying Gordon's construction [9, Lemma 2.1] to our universal group gives a family with 2 generators and 4 relators, reducing these bounds to 4 relators and excess 2. Every nontrivial group in our family is universal.

These algebraic bounds have geometric consequences through Markov's construction, which turns undecidable triviality into unrecognizability of four-manifolds [15, 6]. Here topological (respectively smooth) unrecognizability means that no algorithm can decide whether a given closed connected smooth four-manifold is homeomorphic (respectively diffeomorphic) to the specified manifold.

For connected sums of $S^2 \times S^2$, Gordon obtained a bound of 12 summands [9], and Tancer reduced the topological bound to 9 using relator excess [21]. Kegel–Li–Ren then obtained topological

unrecognizability for 7 summands and smooth unrecognizability for 9 [13]. Gordon's recent preprint reduces the smooth bound to 7 as well [10, Theorem 1.1]. Our 2-generator 4-relator Adian–Rabin family yields topological and smooth unrecognizability of $\#^2(S^2 \times S^2)$.

Theorem 1.1. *The following statements hold.*

1. *There exists a universal group with a presentation having 2 generators and 3 relators.*
2. *There exists an Adian–Rabin family $(P_i)_{i\in\mathbb{N}}$ in which every presentation has 2 generators and 4 relators, and every nontrivial group presented by a member of the family is universal.*
3. *For every fixed integer $n \geq 2$, the standard smooth four-manifold*

$$M_n = \#^n(S^2 \times S^2)$$

is topologically and smoothly unrecognizable.

Universal groups necessarily contain torsion. We use a separate construction to obtain torsion-free groups with unsolvable word problem. The 9-relator group of Kegel–Li–Ren is torsion-free; see also [11, proof of Theorem 2.2]. Their 2-generator 10-relator Adian–Rabin family is torsion-free as well: in Gordon's construction its nontrivial members are amalgamated products of torsion-free groups [9, Lemma 2.1]. The following theorem reduces these torsion-free bounds to 4 relators for the word problem and 5 for an Adian–Rabin family, in both cases with 2 generators.

Theorem 1.2. *The following statements hold.*

1. *There exists a torsion-free group with unsolvable word problem and a presentation having 2 generators and 4 relators.*
2. *There exists an Adian–Rabin family in which every presentation has 2 generators and 5 relators and every presented group is torsion-free.*

The four group-theoretic results in Theorem 1.1(1)–(2) and Theorem 1.2 have been formalized in Lean 4.

Group presentations also give applications to embedded surfaces. One can encode a presentation in the fundamental group of a surface complement and use that group to obstruct unknotting. Gordon used this approach to prove that, for every surface type, topological and smooth unknotting are undecidable in $\#^7(S^2 \times S^2)$ [10, Theorem 3.7]. Applying his construction to our presentations gives the same result in $\#^2(S^2 \times S^2)$. He also constructed surfaces of every type with universal complement group in $\#^{12}(S^2 \times S^2)$ [10, Theorem 3.9]; our presentations give such surfaces in $\#^2(S^2 \times S^2)$ as well. Both conclusions are stated in Corollary 10.1. We state unknotting results in the topological and smooth categories; for the locally flat PL surfaces considered here, the PL and smooth formulations are equivalent [11, Section 1].

In S^4, Gordon's companion paper constructs surfaces of orientable genus 40 and nonorientable genus 121 with universal complement groups [11, Theorems 1.4 and 5.3]. The observation that central extensions of universal groups are universal (Lemma 10.2) allows his constructions to be applied directly to the earlier 13-relator universal group, giving genera 26 and 79. Our 3-relator presentation gives genera 6 and 19 in Corollary 10.3.

Proof strategy and organization

The main construction combines many relators into a single relation while preserving an embedded copy of the original group. We first embed that group into a *superperfect* group Q, meaning that $H_1(Q) = H_2(Q) = 0$. Every central extension of Q splits, so an embedded copy of Q in a quotient lifts to the extension group. Sections 2–3 carry out this preparation for a universal group and for a torsion-free group with unsolvable word problem. The resulting groups have presentations on 2 generators with 84 and 216 relators, respectively; these intermediate counts will determine the parameters in the construction.

To combine the relators, we substitute words in distinct factors of a direct product of free groups into the relators, then multiply the resulting words. Two auxiliary generators act by conjugation, permuting the free generators by translation and multiplication by 2 on a finite set of indices. These permutations let us move any one relator to a prescribed pair of generators. Taking the commutator of the product with a generator in the selected direct factor isolates the chosen relator, since the other terms in the product commute with that generator. Thus one product relation makes all the substituted relators central in the image of this direct product. The quotient obtained by setting them individually equal to the identity contains Q, and the splitting property lifts this copy of Q to the group defined by the product relation. Section 4 provides the words used to embed Q on each ordered pair; Sections 5–6 construct the group realizing the permutation action and carry out the reduction to 4 relators. We also compute the central kernel as a free abelian group, which proves that this construction preserves torsion-freeness.

The reduction from 4 relators to 3 in Section 7 uses an additional relation between the chosen indices and the conjugation action. We support each substituted relator on a pair of generators whose indices are powers of 2. Conjugates of a single commutation relation then make every letter in their product commute with a distinguished generator. Equating the product to a power of the translation generator then forces the periodicity relation used to make the index set finite. This combines 2 relations into one. The embedding argument still applies, although the resulting group has torsion.

The constructed groups have infinite cyclic abelianization, with one displayed generator mapping to zero. This supplies the hypothesis for Gordon's attachment, which Section 8 uses to obtain the two Adian–Rabin families. The presentations are chosen so that, when the word used in the attachment is trivial in the source group, generators can be successively eliminated using their defining relations. Section 9 realizes these eliminations by smooth handle cancellations, producing the standard manifold required for the recognition result. Finally, Section 10 applies Gordon's surface constructions. For the applications in $\#^2(S^2 \times S^2)$, we use a displayed generator that normally generates the universal group: adjoining this generator as a relator permits the explicit Andrews–Curtis reduction of Lemma 7.2. For the universal complement groups in S^4, we use the fact that universality passes from the quotient of a central extension to the extension group.

Conventions

We write

$$w^g = g^{-1}wg, \qquad [w, g] = w^{-1}g^{-1}wg.$$

The normal closure of a subset is denoted by $\langle\langle \cdot \rangle\rangle$, with a subscript when the ambient group needs to be specified. Homology has integral coefficients. All presentations are finite; an abbreviation for a word does not introduce a generator or a relator.

Provenance and formalization

This manuscript and its Lean 4 formalization were developed with assistance from OpenAI language models under the direction of Qiuyu Ren. The Lean code [19] formalizes the four group-theoretic results above; the four-manifold and surface consequences are outside its scope.

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2 Two groups to be embedded

We start with a universal group and a torsion-free group with unsolvable word problem. Both have free abelian first homology, which will allow their embeddings to lift through a central extension in Section 3.

Fix a universal instance of the presentation in [20, Section 7.1], as supplied by its Theorem 1.1, and rename its generators c, r as z_1, z_2:

$$Q_0 = \langle z_1, z_2 \mid R_1, \dots, R_{13} \rangle = F(z_1, z_2)/N_0. \tag{2.1}$$

We regard each defining equation as the relator obtained by multiplying its left side by the inverse of its right side. All subsequent presentations are given by finite substitutions in these 13 words, so their construction does not require expanding the original coding words.

The abelianization of Q_0 is

$$H_1(Q_0) \cong \mathbb{Z}. \tag{2.2}$$

Indeed every auxiliary word in the cited Section 7.1 has r-exponent zero, so the same is true of all 13 relators. The seventh equation is $\bar{g}_1 \ell = \ell \bar{g}_1 f u_1 f^{-1}$, where u_1 is a conjugate of c; its relator has c-exponent -1. The exponent-sum vectors of the relators therefore span exactly $\mathbb{Z}(1, 0)$, proving (2.2).

To obtain a torsion-free group, we first express the word problem of Q_0 as a subgroup membership problem in a product of free groups. We use Mihailova's fibre-product construction [16]. Take two commuting free groups of rank 2, $F_x = F(x_1, x_2)$ and $F_y = F(y_1, y_2)$, and the subgroup

$$D = \langle x_1 y_1, x_2 y_2, R_1(x_1, x_2), \dots, R_{13}(x_1, x_2) \rangle \leq F_x \times F_y.$$

Identify the bases of the two free groups and transport N_0 to F_x. Then D has the fibre-product description

$$D = \{(v, w) : vN_0 = wN_0\} = (N_0 \times 1) \rtimes \mathrm{diag}(F_x), \qquad D \cap (F_x \times 1) = N_0 \times 1. \tag{2.3}$$

To see this, the two diagonal elements generate the diagonal free group, and diagonal conjugation of the 13 relators generates $N_0 \times 1$. Conversely every pair in the fibre product is $(vw^{-1}, 1)(w, w)$.

Adjoining a stable letter that centralizes D turns this membership problem into a word problem. Define the group with 5 generators and 19 relators

$$\begin{aligned} G = \langle x_1, x_2, y_1, y_2, k \mid & [x_i, y_j] \ (1 \leq i, j \leq 2), \\ & [k, x_1 y_1], \ [k, x_2 y_2], \ [k, R_j(x_1, x_2)] \ (1 \leq j \leq 13) \rangle. \end{aligned} \tag{2.4}$$

This is the proper HNN extension of $F_x \times F_y$ centralizing D; for the normal-form and torsion theorems, see [14, Chapter IV]. The base embeds and the HNN extension introduces no new torsion

orders, so G is torsion-free. All its relators are commutators, giving $H_1(G) = \mathbb{Z}^5$. Britton's lemma and (2.3) imply, for every word $w \in F_x$,

$$[k, w] = 1 \text{ in } G \quad \Longleftrightarrow \quad w \in N_0 \quad \Longleftrightarrow \quad w = 1 \text{ in } Q_0. \tag{2.5}$$

Thus G has undecidable word problem. No finite presentation of the subgroup D is needed.

Both Q_0 and G therefore have unsolvable word problem and free abelian first homology. To transfer unsolvability to a group containing either of them, we only need a fixed embedding: the images of their finitely many generators are fixed words, and substitution of those words transfers any solution of the ambient word problem to the subgroup.

3 Embedding in a superperfect group

The next construction embeds each group from Section 2 in a finitely presented superperfect group. We will also arrange that the presentation has 2 generators and that its first 2 relators have exponent vectors $(1, 0)$ and $(0, 1)$, with all remaining relators in the commutator subgroup of the free group on the two generators. This normalization will make the central kernels in Section 4 explicit.

Let J be a finitely presented group with $H_1(J)$ free abelian. We first embed J in a group H so that the induced map on H_2 vanishes. We then embed H in a perfect group Q' and take the universal central extension of Q'. The vanishing map on H_2, together with the hypothesis on $H_1(J)$, makes this extension split over J. The construction preserves torsion-freeness. For related perfectification and lifting arguments in the surface setting, see [11, Section 5].

3.1 Two finite mitoses and their integral homology

We apply the finite mitosis construction twice to make the inclusion of J induce zero on H_1 and H_2. The construction comes from [2, Section 5, Lemma 5.1]. It adjoins stable letters that conjugate the first copy of J in $J \times J$ to the diagonal and to the second copy. More precisely, form a graph of groups with vertex group $J \times J$ and two loop edges with group J, with pairs of injections

$$(i_1, \Delta), \qquad (i_1, i_2), \qquad i_1(a) = (a, 1), \quad i_2(a) = (1, a), \quad \Delta(a) = (a, a).$$

Write its fundamental group as

$$m(J) = \langle J \times J, d, s \mid (a, 1)^d = (a, a),\ (a, 1)^s = (1, a)\ (a \in J)\rangle. \tag{3.1}$$

The first-coordinate map $i_J : J \to m(J)$ is injective by the HNN normal-form theorem. If J is torsion-free, so is $m(J)$.

If $J = \langle a_1, \dots, a_n \mid T_1, \dots, T_r\rangle$, the second copy can be eliminated to give the finite mitosis presentation

$$\begin{aligned} m(J) = \langle a_1, \dots, a_n, d, s \mid & T_1, \dots, T_r, \\ & a_i^d = a_i a_i^s\ (1 \le i \le n),\ [a_i, a_j^s] = 1\ (1 \le i, j \le n)\rangle. \end{aligned} \tag{3.2}$$

This presentation has $n + 2$ generators and $r + n + n^2$ relators. The relations for the eliminated copy of J follow by conjugating those for the first copy. Since the two copies commute, the displayed equations on generators imply the equations for every $a \in J$ in (3.1).

Lemma 3.1. *For every group J,*

$$H_1(m(J)) \cong \mathbb{Z}^2, \qquad H_2(m(J)) \cong H_1(J) \otimes H_1(J), \qquad H_1(i_J) = 0.$$

Moreover the composite $J \longrightarrow m(J) \longrightarrow m^2(J)$ induces zero on H_2.

Proof. Put $A = H_1(J)$, $B = H_2(J)$, and $C = A \otimes A$. The integral Künneth decomposition is

$$H_1(J \times J) = A \oplus A, \qquad H_2(J \times J) = B \oplus B \oplus C.$$

There are no degree-two Tor terms, because $H_0(J) = \mathbb{Z}$. Let $\delta_J : B \to C$ be the reduced diagonal component. The degree-two maps induced by i_1, i_2, Δ are respectively

$$b \longmapsto (b, 0, 0), \qquad b \longmapsto (0, b, 0), \qquad b \longmapsto (b, b, \delta_J(b)).$$

The graph-of-groups Mayer–Vietoris edge maps in degrees two and one are therefore

$$\begin{aligned} f_2 &: B \oplus B \longrightarrow B \oplus B \oplus C, & f_2(b, c) &= (c, -b - c, -\delta_J(b)), \\ f_1 &: A \oplus A \longrightarrow A \oplus A, & f_1(b, c) &= (c, -b - c). \end{aligned}$$

The map f_1 is an isomorphism, and the degree-zero edge map $\mathbb{Z}^2 \to \mathbb{Z}$ is zero. The Mayer–Vietoris sequence therefore gives $H_1(m(J)) = \mathbb{Z}^2$, with $H_1(J \times J)$ mapping to zero, and $H_2(m(J)) = \operatorname{coker} f_2$.

An explicit identification of that cokernel with C is

$$\psi : B \oplus B \oplus C \longrightarrow C, \qquad \psi(u, v, w) = w - \delta_J(u + v).$$

It is onto and annihilates f_2. If $\psi(u, v, w) = 0$, choose $c = u$ and $b = -u - v$; then $(u, v, w) = f_2(b, c)$. Thus $\ker \psi = \operatorname{im} f_2$. Under this identification $H_2(i_J)$ is $-\delta_J$.

Naturality of the reduced diagonal and $H_1(i_J) = 0$ give

$$\delta_{m(J)} H_2(i_J) = (H_1(i_J) \otimes H_1(i_J))\delta_J = 0.$$

Applying the inclusion formula once more proves $H_2(i_{m(J)})H_2(i_J) = -\delta_{m(J)}H_2(i_J) = 0$. □

Thus two mitoses give the required embedding $J \hookrightarrow H = m^2(J)$ with zero map on H_2. If J has n generators and r relators, (3.2) gives a presentation

$$H = \langle h_1, \ldots, h_k \mid T_1, \ldots, T_\mu \rangle, \qquad k = n + 4, \quad \mu = r + 2n^2 + 6n + 6. \tag{3.3}$$

The old generators come first and the 4 new stable letters last. Lemma 3.1 gives

$$H_1(H) = \mathbb{Z}^2, \qquad H_2(H) = \mathbb{Z}^4, \qquad H_2(J \longrightarrow H) = 0. \tag{3.4}$$

The last two stable letters give a basis of $H_1(H)$; all earlier generators vanish there.

3.2 Explicit perfectification

We next embed H in a perfect group on 2 generators. The words E_i below replace its generators, while two further relators D_1, D_2 kill the abelianization. Their long, distinct powers of y ensure the small-cancellation condition needed to preserve the embedding of H and compute the second homology of the resulting group.

For $1 \le i \le k$ use the words

$$E_i(x, y) = \prod_{\ell=1}^{32} x y^{2000+32i+\ell}.$$

Let $\mathcal{E}_1$ be the ordered list $1000-10j, 1000+10j$ for $1 \le j \le 16$, with its final entry 1160 replaced by 1159. Let $\mathcal{E}_2$ be 999 followed by the ordered pairs $1000-(10j+3), 1000+(10j+3)$ for $1 \le j \le 16$. Define

$$D_1 = \prod_{e \in \mathcal{E}_1} xy^e, \qquad D_2 = \prod_{e \in \mathcal{E}_2} xy^e. \tag{3.5}$$

For a word w in x, y, write $\exp(w)$ for its vector of exponent sums. The words D_1, D_2 satisfy

$$\exp(D_1) = (32, 31999), \qquad \exp(D_2) = (33, 32999), \qquad 32 \cdot 32999 - 33 \cdot 31999 = 1. \tag{3.6}$$

The exponents of the y syllables in all the words E_i, D_1, D_2 are positive, pairwise distinct, and at least two.

Consider

$$Q' = (H * \langle x \rangle * \langle y \rangle)/\langle\langle h_i^{-1} E_i(x, y) \ (1 \le i \le k), D_1, D_2 \rangle\rangle. \tag{3.7}$$

We check the free-product small-cancellation condition, including the classifying-space conclusion needed below. Each cyclically reduced relator has syllable length 64, 65, or 66, whether or not its coefficient h_i is trivial. A common y syllable determines the relator and its position, because all the y exponents are distinct. A segment without a y syllable has length at most 2, and inversion reverses the signs of the x and y syllables. Thus distinct cyclic conjugates of the relators and their inverses have common prefixes of length at most 2. The estimate also holds when some coefficients coincide or are trivial.

For the generalized free-product presentation, a piece is semi-initial in another cyclic relator: only its last syllable may differ within its factor. The prefix bound of 2 therefore gives piece length at most 3. Every relator has length at least 64 and hence needs at least 22 pieces. Thus generalized $C(6)$ holds. Together with the automatic $T(3)$ condition, this implies that the augmented presentation is aspherical by Bogley–Pride's implications (3.10)–(3.11) and Theorem 2.2.

The distinct y exponents also make every relator a nonpower and rule out cyclic or inverse-cyclic identifications between different relators. The augmented relative asphericity criterion and its factor-injection, torsion, and classifying-space consequences apply; see [3, Section 3.4, Theorem 2.2, Lemma 3.2, and Theorems 3.8 and 4.3]. Hence H embeds in Q', a classifying space for Q' is obtained from a tree of spaces for $H * \langle x \rangle * \langle y \rangle$ by attaching exactly $k+2$ disks, and Q' is torsion-free if H is torsion-free. Factor classifying spaces may be infinite. The nonpower condition makes the relator-root groups trivial. In the classifying-space construction of [3, Theorem 4.3], the spaces attached for these trivial root groups may therefore be chosen as disks, with no additional relative higher cells.

Using (3.4), the relative homology sequence becomes

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow H_2(Q') \longrightarrow \mathbb{Z}^{k+2} \xrightarrow{\partial} \mathbb{Z}^2 \oplus \mathbb{Z}^2 \longrightarrow H_1(Q') \longrightarrow 0.$$

The two summands in the target of ∂ are $H_1(H)$ and the free abelian group on x, y. The last two columns are the vectors in (3.6) in the second summand; the other columns have first components $-[h_i]$, which generate $H_1(H)$. Thus ∂ is onto. Its kernel is free of rank $k-2$, so

$$H_1(Q') = 0, \qquad H_2(Q') \cong \mathbb{Z}^{k+2}. \tag{3.8}$$

Eliminating $h_1, \dots, h_k$ gives the finite presentation on 2 generators

$$\begin{aligned} &Q' = \langle x, y \mid S_1, \dots, S_{\mu+2} \rangle, \\ &S_j = T_j(E_1, \dots, E_k) \ (j \le \mu), \qquad S_{\mu+1} = D_1, \quad S_{\mu+2} = D_2. \end{aligned} \tag{3.9}$$

3.3 Lifting the original group to a superperfect group

Let $\widehat{Q}$ be the universal central extension of the perfect group Q':

$$1 \longrightarrow H_2(Q') \longrightarrow \widehat{Q} \longrightarrow Q' \longrightarrow 1. \tag{3.10}$$

It is superperfect; see [8, Section 2.2]. If J is torsion-free, then H, Q' and the free abelian kernel in (3.10) are torsion-free, so $\widehat{Q}$ is torsion-free too.

To obtain an embedding of J in $\widehat{Q}$, we must split (3.10) over the subgroup $j(J)$, where $j : J \to H \to Q'$ is the embedding constructed above. Put $Z = H_2(Q')$. Central extensions of J by the trivial J-module Z are classified by $H^2(J; Z)$, which fits into the natural universal coefficient sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_1(J), Z) \longrightarrow H^2(J; Z) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_2(J), Z) \longrightarrow 0.$$

The Ext term vanishes because $H_1(J)$ is free abelian. By naturality, the image of the pulled-back extension class in the Hom term factors through $H_2(j)$, which is zero by (3.4). The extension class therefore vanishes, and the pullback splits. This embeds J in $\widehat{Q}$. The argument requires neither a finiteness assumption on $H_2(J)$ nor a computation of the splitting; a fixed embedding suffices to transfer the unsolvability of the word problem.

It remains to give $\widehat{Q}$ a presentation with the exponent vectors specified at the start of the section. We do this by first making the relators of Q' central, then removing the free abelian part. Let $F = F(x, y)$, let R be the normal closure of the S_j, and set

$$M = \langle x, y \mid [x, S_j], [y, S_j] \ (1 \le j \le \mu + 2) \rangle = F/[F, R].$$

The derived subgroup $[M, M] = [F, F]/[F, R]$ is $\widehat{Q}$. In M the words

$$u = D_1^{32999} D_2^{-31999}, \qquad v = D_1^{-33} D_2^{32} \tag{3.11}$$

are central and have exponent vectors $(1, 0)$ and $(0, 1)$. Since $M_{\mathrm{ab}} = \mathbb{Z}^2$, they give a central section of abelianization. Therefore

$$M = [M, M] \times \langle u, v \rangle, \qquad \langle u, v \rangle \cong \mathbb{Z}^2.$$

Quotienting M by this central copy of $\mathbb{Z}^2$ gives the required presentation

$$Q = \langle x, y \mid \rho_1, \ldots, \rho_L \rangle \cong \widehat{Q}, \qquad L = 2\mu + 6, \tag{3.12}$$

where the relators are

$$\begin{aligned} \rho_1 = u, \quad \rho_2 = v, \\ \rho_{2j+1} = [x, S_j], \quad \rho_{2j+2} = [y, S_j] \quad (1 \le j \le \mu + 2). \end{aligned} \tag{3.13}$$

In particular

$$\exp(\rho_1) = (1, 0), \quad \exp(\rho_2) = (0, 1), \quad \exp(\rho_i) = (0, 0) \ (i \ge 3). \tag{3.14}$$

Proposition 3.2. *The construction gives a superperfect universal group Q_{u} with 2 generators and* 84 *relators, and a torsion-free superperfect group Q_{t} with 2 generators,* 216 *relators, and unsolvable word problem. Both presentations have the form* (3.13).

Proof. Use $J = Q_0$ and $J = G$, respectively. Their first homology groups are free abelian by Section 2, so the lifting argument embeds each group. The presentation counts are

	J	$m(J)$	$m^2(J)$	Q'	Q
universal	$(2, 13)$	$(4, 19)$	$(6, 39)$	$(2, 41)$	$(2, 84)$
torsion-free	$(5, 19)$	$(7, 49)$	$(9, 105)$	$(2, 107)$	$(2, 216)$.

Each entry records the numbers of generators and relators, in that order. The group Q_{u} contains Q_0 and is universal. The group Q_{t} is torsion-free and contains G, so it has unsolvable word problem. Both groups are noncyclic. □

4 Copies of a superperfect group and their central extensions

The construction below embeds a copy of a given group into the subgroup generated by each ordered pair of distinct generators of a larger group. Permuting those generators will then permute the embedded copies. We also describe what happens when the defining relators of these copies are made central instead of trivial. This central extension is the one that arises when we combine the relators in Section 6.

Let $Q = \langle x, y \mid \rho_1, \dots, \rho_L \rangle$ be a noncyclic superperfect group with the normalization (3.14). To specify the images of its generators using an ordered pair, we use the words

$$V_1(x, y) = \left(\prod_{q=2}^{16} xy^q \right) xy^{-150}, \qquad V_2(x, y) = \left(\prod_{q=17}^{31} xy^q \right) xy^{-376}. \tag{4.1}$$

Products are in increasing order. Each word has 16 x syllables and 16 y syllables. The 32 absolute y exponents are distinct and at least 2; this bounds overlaps between the words in the small-cancellation argument below. Their exponent vectors and their values when both generators are identified are

$$\exp(V_1) = (16, -15), \qquad \exp(V_2) = (16, -16), \qquad V_1(z, z) = z, \quad V_2(z, z) = 1. \tag{4.2}$$

For $d \geq 2$ put $F_d = F(v_1, \dots, v_d)$ and

$$B_d = F_d / N_d, \quad N_d = \langle\langle \rho_i(V_1(v_r, v_u), V_2(v_r, v_u)) : 1 \leq i \leq L,\ r \neq u \rangle\rangle. \tag{4.3}$$

The defining family is preserved by every permutation of the generators $v_1, \dots, v_d$, which we call a vertex permutation. For each ordered pair (r, u) with $r \neq u$, let $Q_{r,u}$ denote a copy of Q.

Lemma 4.1. *Every ordered-pair map $x \mapsto V_1(v_r, v_u)$, $y \mapsto V_2(v_r, v_u)$ embeds Q in B_d. A classifying space for B_d is obtained from a tree of spaces for*

$$\mathcal{T}_d = (*_{r \neq u} Q_{r,u}) * \left(*_{r=1}^{d} \langle v_r \rangle \right)$$

by attaching exactly $2d(d-1)$ disks. If Q is torsion-free, then B_d is torsion-free.

Proof. For each ordered pair, take a copy $Q_{r,u}$ with generators $z_1^{r,u}, z_2^{r,u}$. We will show that these factors embed after adjoining to $\mathcal{T}_d$ the relators

$$(z_j^{r,u})^{-1} V_j(v_r, v_u), \qquad j = 1, 2.$$

Each relator is cyclically reduced and has syllable length 33. Since Q is noncyclic, its two displayed generators are nontrivial, distinct, and not mutual inverses.

Call a syllable ordinary if it belongs to one of the cyclic factors generated by the v_r. A matching segment of 2 ordinary syllables in cyclic rotations of the relators or their inverses contains one x-type and one y-type syllable. The absolute exponent 1 identifies the x type, while the other absolute exponent is at least 2. Thus the factor labels determine the ordered pair, the sign of the x syllable determines whether the relator was inverted, and the distinct absolute y exponents determine the word and the position. The two rotations must therefore coincide. This also covers negative y exponents, since orientation is determined by the x syllable.

A matching syllable in a Q factor identifies the factor and its displayed generator. If the orientations agree, it determines the relator. Opposite orientations can match only when the generator is an involution. In that case no adjacent ordinary syllable matches: either the signs differ or one absolute exponent is 1 and the other is at least 2. Any segment of 3 syllables contains 2 consecutive ordinary syllables or a Q syllable adjacent to an ordinary one. Distinct rotations of the relators and their inverses therefore have common prefixes of length at most 2.

For the generalized free-product presentation, a piece is semi-initial in another cyclic relator: its last syllable may differ within its factor, while the preceding syllables agree. The prefix bound therefore gives piece length at most 3. Each relator requires at least 11 pieces, so generalized $C(6)$ holds. Together with the automatic $T(3)$ condition, this gives asphericity of the augmented presentation. Each relator is a nonpower because it has exactly one nontrivial syllable in a specified Q factor, and distinct relators have no cyclic or inverse-cyclic identification. The augmented relative asphericity criterion [3, Section 3.4, Theorem 2.2], factor-injection result [3, Lemma 3.2], and classifying-space theorem [3, Theorem 4.3] apply. Since the root groups are trivial, the classifying-space construction uses disk attachments and has no relative higher cells. The torsion theorem [3, Theorem 3.8] places every finite subgroup in a conjugate of a factor or a relator-root group. Thus B_d is torsion-free when Q is torsion-free. Eliminating the displayed generators of the Q factors gives (4.3). □

Superperfectness allows an embedded copy of Q to lift through a central extension. More precisely, every central extension $1 \to Z \to E \to Q \to 1$ splits uniquely. Write $E' = [E, E]$. The homology five-term sequence makes $Z \to E_{\mathrm{ab}}$ injective, so $Z \cap E' = 1$, while perfectness of Q makes $E' \to Q$ onto. Hence the restriction to E' is an isomorphism onto Q. Two sections differ by a homomorphism $Q \to Z$, which vanishes because Q is perfect.

We now replace the relations defining B_d by the weaker requirement that those relators be central. Set

$$C_d = F_d/[N_d, F_d], \qquad A_d = N_d/[N_d, F_d].$$

The subgroup A_d is the central kernel of $C_d \to B_d$. Its calculation will let us determine whether the further quotient in Section 6 introduces torsion.

Proposition 4.2. *The group A_d is free abelian of rank $2d(d-1)$, with basis*

$$c_{\alpha,j} = \rho_j(V_1(v_r, v_u), V_2(v_r, v_u)), \qquad \alpha = (r, u), \quad r \neq u, \quad j = 1, 2. \tag{4.4}$$

Every relator obtained from ρ_i with $i \geq 3$ by the substitutions in (4.3) is trivial in C_d. A vertex permutation σ takes $c_{\alpha,j}$ to $c_{\sigma(\alpha),j}$, preserving the index j and the sign.

Proof. Pull back $C_d \to B_d$ over an embedded Q_α. The unique splitting gives lifts $q_{\alpha,1}, q_{\alpha,2}$ of its generators. The words defining the embedding differ from these lifts by central elements, so write

$$V_j(v_r, v_u) = q_{\alpha,j} a_{\alpha,j}, \qquad a_{\alpha,j} \in A_d.$$

Since the lifts satisfy the defining relators of Q and the elements $a_{\alpha,j}$ are central, we have

$$\rho_i(V_1, V_2) = a_{\alpha,1}^{\exp_1(\rho_i)} a_{\alpha,2}^{\exp_2(\rho_i)}.$$

By (3.14), these values are $a_{\alpha,1}$, $a_{\alpha,2}$, and 1 for $i = 1$, $i = 2$, and $i \geq 3$, respectively. As α and i vary, these substituted relators normally generate N_d and have central images in A_d. Thus the $2d(d-1)$ elements in (4.4) generate A_d.

To prove independence, put $b = d(d-1)$. Since every Q_α is superperfect, the tree of factor classifying spaces in Lemma 4.1 has $H_2 = 0$ and $H_1 = \mathbb{Z}^d$. The relative homology sequence for its disk attachments is

$$0 \longrightarrow H_2(B_d) \longrightarrow \mathbb{Z}^{2b} \longrightarrow \mathbb{Z}^d \longrightarrow H_1(B_d) \longrightarrow 0.$$

The Hopf five-term sequence for $F_d \to B_d$ is

$$0 \longrightarrow H_2(B_d) \longrightarrow A_d \longrightarrow \mathbb{Z}^d \longrightarrow H_1(B_d) \longrightarrow 0.$$

Comparing ranks in these sequences gives $\operatorname{rank} A_d = 2b$. The resulting surjection $\mathbb{Z}^{2b} \to A_d$ has a kernel of rank 0. This kernel is a subgroup of a finite-rank free abelian group and hence is free abelian, so it vanishes. Thus (4.4) is an integral basis. Its defining words also give the stated permutation action; reversing an ordered pair leaves the index j unchanged. □

5 An affine action on a product of free groups

To combine relators supported in different direct factors, we need conjugations that move ordered pairs of generators between those factors. Translation and multiplication by powers of 2 provide these conjugations on indices modulo p^2. For an odd prime p such that 2 is primitive modulo p^2, define

$$K(p) = \langle a, t, s \mid a^s = a,\ t^s = t^2,\ a^{t^{p^2}} = a,\ [a, a^t] = 1 \rangle. \tag{5.1}$$

The next proposition identifies $K(p)$ with a semidirect product whose base is a product of free groups. Each free group in the product has a specified basis, and the action permutes these basis elements without replacing them by nontrivial powers.

Proposition 5.1. *Writing $a_j = a^{t^j}$, with subscripts modulo p^2, we have*

$$K(p) = \mathcal{D}_p \rtimes \mathrm{BS}(1,2), \qquad \mathcal{D}_p = \prod_{r \in \mathbb{Z}/p\mathbb{Z}} F(a_r, a_{r+p}, \ldots, a_{r+(p-1)p}), \tag{5.2}$$

where $\mathrm{BS}(1,2) = \langle t, s \mid t^s = t^2 \rangle$ acts on the right by

$$a_j^t = a_{j+1}, \qquad a_j^s = a_{2j}.$$

In particular $K(p)$ is torsion-free.

Proof. Translation $T_0 : j \mapsto j+1$ and doubling $S_0 : j \mapsto 2j$ permute the residue classes modulo p. As permutations of the vertex set, they satisfy the right-action identity $S_0^{-1} T_0 S_0 = T_0^2$: the successive labels are $2^{-1}j$, $2^{-1}j + 1$, and $j + 2$. They therefore define the stated action of $\mathrm{BS}(1,2)$. The resulting semidirect product satisfies the 4 defining relations of $K(p)$, giving a homomorphism from $K(p)$ to this semidirect product.

Conversely, the periodicity relation in $K(p)$ defines a_j modulo p^2, and the two conjugation relations give $a_j^s = a_{2j}$. Conjugating $[a, a^t] = 1$ by $s^k t^i$ gives

$$[a_i, a_{i+2^k}] = 1.$$

Since 2 generates the units modulo p^2, these relations make every pair of generators from distinct residue classes commute. They define a homomorphism from $\mathcal{D}_p$ to $K(p)$ compatible with the actions of t and s. Doubling is a permutation, so compatibility also holds for s^{-1}. The homomorphism therefore extends to the proposed semidirect product. The two maps are inverse on a, t, s and every displayed a_j, proving (5.2). In particular, the generators in each residue factor satisfy no further relations. Finally, $\mathcal{D}_p$ is a product of free groups and $\mathrm{BS}(1,2) \cong \mathbb{Z}[1/2] \rtimes \mathbb{Z}$ is torsion-free, so their semidirect product is torsion-free. □

Lemma 5.2. *In a presentation containing the 2 relations $a^s = a$ and $t^s = t^2$, the substitution*

$$b = at, \qquad T = [b, s], \qquad A = bT^{-1} \tag{5.3}$$

eliminates a, t in favor of b and replaces the 2 relations by $T^s = T^2$. In every other relator, substitute $a \mapsto A$ and $t \mapsto T$, with s and any additional generators fixed. Both the number of generators and the number of relators decrease by 1.

Proof. In the old group,

$$b^s = a^s t^s = at^2 = bt, \qquad t = b^{-1}b^s = [b, s] = T, \qquad a = bT^{-1} = A.$$

Conversely, in the free group on b, s the definitions give $b^s = bT$ and $AT = b$ identically. Subject only to $T^s = T^2$,

$$A^s = b^s (T^s)^{-1} = bTT^{-2} = bT^{-1} = A.$$

The map from the new presentation sends b to at, and the reverse map sends a to A and t to T. Both compositions fix all generators by the displayed identities, so these maps give the required isomorphism. □

6 A 4-relator construction

We now combine the defining relators of Q into a single relation by placing them in distinct commuting factors of $\mathcal{D}_p$. Conjugation by the affine action then makes each of these relators central in the image of the base. This is enough to retain a copy of Q, because the resulting central extension splits over Q. To preserve torsion-freeness as well, we compute the central kernel and show that it is free abelian.

Let $Q = \langle x, y \mid \rho_1, \dots, \rho_L \rangle$ be as in Section 4, and let $p \geq L$ be an odd prime with 2 primitive modulo p^2. In $K(p)$, use the words (4.1) to define

$$X = V_1(a_0, a_p), \qquad Y = V_2(a_0, a_p), \qquad W(a, t) = \prod_{i=1}^{L} \rho_i(X, Y)^{t^{i-1}}. \tag{6.1}$$

The ith term in this product belongs to the direct factor indexed by the residue class $i - 1$. Thus the terms lie in L distinct direct factors, indexed by $0, \dots, L-1$. Set

$$P_5(Q, p) = K(p) / \langle\langle W(a, t) \rangle\rangle. \tag{6.2}$$

The subscript records the number of relators in this presentation on a, t, s.

6.1 The central kernel

Write $\mathcal{A} = \mathcal{D}_p$ for the base in (5.2). In the direct factor F_r indexed by a residue class, let N_r be the normal closure of the defining relators of Q evaluated on every ordered pair of distinct basis letters by means of V_1, V_2. Then

$$N_{\mathcal{A}} = \prod_r N_r \ \triangleleft\ K(p), \qquad \mathcal{A}/N_{\mathcal{A}} \cong \prod_{r \in \mathbb{Z}/p\mathbb{Z}} B_p.$$

Translation and doubling preserve the defining family on all ordered pairs, so the action of $\mathrm{BS}(1,2)$ descends to this quotient.

Let $J = \langle\langle W \rangle\rangle_{K(p)}$. To describe the quotient by J, we first compare it with the normal subgroup generated by all the substituted relators:

$$[N_{\mathcal{A}}, \mathcal{A}] \le J \le N_{\mathcal{A}}. \tag{6.3}$$

The upper inclusion follows from $W \in N_{\mathcal{A}}$ and the invariance of $N_{\mathcal{A}}$ under $K(p)$. For the lower inclusion, the image of $\langle t, s \rangle$ on the indices consists of all affine maps $j \mapsto uj + v$ modulo p^2, with u a unit. These maps act transitively on ordered pairs of distinct indices lying in a common residue class: every such pair is $(v, v + pk)$ with $k \not\equiv 0 \pmod p$, and some power of 2 is congruent to k modulo p.

Fix an index i and an ordered pair (u, v) in any residue factor. Choose $h \in \langle t, s \rangle$ whose affine action takes $(i - 1, p + i - 1)$ to (u, v). The other factors of W^h lie in different residue factors: the difference of their residue indices is a unit multiple of $j - i$, with $0 < |j - i| < p$. In $K(p)/J$, take the commutator of $W^h = 1$ with any basis letter a_w in the residue factor containing a_u, a_v. Every factor of W^h except the ith commutes with a_w, so

$$[\rho_i(V_1(a_u, a_v), V_2(a_u, a_v)), a_w] = 1.$$

The same relator commutes with every other residue factor by (5.2). Thus every substituted relator, and hence the image of its normal closure, is central in the image of $\mathcal{A}$. This proves the lower inclusion in (6.3). To obtain centrality in the whole image of $\mathcal{A}$, we used every basis letter in the selected factor, including those outside the chosen ordered pair.

The inclusions (6.3) allow us to compute the remaining relations in the central extension

$$\mathcal{A}/[N_{\mathcal{A}}, \mathcal{A}] \cong \prod_r C_p.$$

By Proposition 4.2, the kernel over $\prod_r B_p$ is free abelian, with 2 basis elements for each ordered pair in each residue factor. The substituted relators of type $i \ge 3$ vanish. In additive notation, the image of W is therefore the sum of the type-1 basis element from its first factor and the type-2 basis element from its second factor. The images of its conjugates under the affine action generate the image of J: conjugation by the base is trivial on the central kernel, and $[N_{\mathcal{A}}, \mathcal{A}]$ is already contained in J by (6.3).

Label an ordered pair by (v, k), meaning $(v, v + pk)$, where $v \in \mathbb{Z}/p^2\mathbb{Z}$ and $k \in (\mathbb{Z}/p\mathbb{Z})^\times$, and write its two central basis elements as $c_{v,k,1}, c_{v,k,2}$. The affine map $j \mapsto uj + v$ takes the ordered pairs supporting the first two factors of W to $(v, v + up)$ and $(v + u, v + u + up)$. Thus the defining relations for the remaining central quotient are

$$c_{v,k,1} + c_{v+u,k,2} = 0, \qquad u \in (\mathbb{Z}/p^2\mathbb{Z})^\times, \quad u \equiv k \pmod p. \tag{6.4}$$

Proposition 4.2 shows that the affine action preserves the type and sign of each basis element, so both coefficients in these relations are 1.

These relations can be read from a bipartite graph: its vertices are the central basis elements, separated by type, and its edges are (6.4). For fixed $r \in \mathbb{Z}/p\mathbb{Z}$ and $k \neq 0 \pmod p$, every type-1 vertex with $v \equiv r$ is adjacent to every type-2 vertex with $w \equiv r + k$, since $w - v$ runs over the units congruent to k. This gives a component isomorphic to $K_{p,p}$. Every vertex belongs to one such component, so there are $p(p-1)$ components.

Negating all type-2 basis elements turns each relation into the difference of its endpoints. Quotienting the free abelian group on the vertices by these differences gives one copy of $\mathbb{Z}$ for each connected component. Indeed, a spanning tree identifies all vertices in a component, and the sum-of-coordinates map shows that the remaining generator has infinite order. Consequently,

$$N_{\mathcal{A}}/J \cong \mathbb{Z}^{p(p-1)}. \tag{6.5}$$

Thus the quotient of the base fits into the central extension

$$1 \longrightarrow \mathbb{Z}^{p(p-1)} \longrightarrow \mathcal{A}/J \longrightarrow \prod_{r \in \mathbb{Z}/p\mathbb{Z}} B_p \longrightarrow 1, \tag{6.6}$$

and the whole group is

$$P_5(Q,p) = (\mathcal{A}/J) \rtimes \mathrm{BS}(1,2). \tag{6.7}$$

The kernel in (6.6) is central in $\mathcal{A}/J$; the action of $\mathrm{BS}(1,2)$ on it may be nontrivial.

Theorem 6.1. *The group $P_5(Q,p)$ contains a copy of Q. If Q is torsion-free, then $P_5(Q,p)$ is torsion-free. The abelianization of $P_5(Q,p)$ is $\mathbb{Z}$ generated by the image of s, and s normally generates the group. It has an isomorphic presentation with 2 generators and 4 relators.*

Proof. The quotient group $\prod_r B_p$ in (6.6) contains a copy of Q by Lemma 4.1. Pulling back the central extension to this copy and using superperfectness embeds Q in $\mathcal{A}/J$. The embedded generators can be written as

$$X\rho_1(X,Y)^{-1}, \qquad Y\rho_2(X,Y)^{-1}. \tag{6.8}$$

Indeed, if q_1, q_2 are the lifts given by the splitting, write $X = q_1c_1$, $Y = q_2c_2$ with c_1, c_2 central. The normalization gives $\rho_1(X,Y) = c_1$ and $\rho_2(X,Y) = c_2$, so (6.8) gives q_1, q_2. Projection to the copy of Q in the quotient proves injectivity. Thus universality or an undecidable word problem in Q passes to $P_5(Q,p)$.

If Q is torsion-free, Lemma 4.1 makes every B_p torsion-free. Both kernel and quotient in (6.6) are then torsion-free, hence so is $\mathcal{A}/J$. Formula (6.7) proves torsion-freeness of the full group.

To prove normal generation, impose $s = 1$ in P_5. The relation $t^s = t^2$ gives $t = t^2$, so $t = 1$. Then $X = V_1(a,a) = a$ and $Y = V_2(a,a) = 1$, and (3.14) makes W freely reduce to a. Hence $a = 1$, proving that s normally generates the group. All defining relators have total s-exponent zero, so sending s to $1 \in \mathbb{Z}$ and a, t to 0 gives an epimorphism onto $\mathbb{Z}$. Since normal generation makes the abelianization cyclic, this epimorphism identifies it with $\mathbb{Z}$.

Finally Lemma 5.2 gives, with $T = [b,s]$ and $A = bT^{-1}$,

$$P_4(Q,p) = \langle b, s \mid s^{-1}TsT^{-2},\ T^{-p^2}AT^{p^2}A^{-1},\ [A, T^{-1}AT],\ W(A,T)\rangle \cong P_5(Q,p). \tag{6.9}$$

The isomorphism fixes s, so all conclusions persist. □

6.2 An explicit modulus and the torsion-free group

Take

$$p = 227, \qquad p^2 = 51529, \qquad p(p-1) = 51302 = 2 \cdot 113 \cdot 227. \tag{6.10}$$

Trial division by $2, 3, 5, 7, 11, 13$ proves 227 prime; trial division by $2, 3, 5, 7$ proves 113 prime. Repeated squaring gives, modulo 51529,

$$2^{51302} \equiv 1, \qquad 2^{25651} \equiv 51528, \qquad 2^{454} \equiv 4317, \qquad 2^{226} \equiv 45628.$$

The last 3 exponents are 51302 divided by its three prime divisors, and none of the corresponding powers of 2 has residue 1. Hence 2 has order 51302. Also $227 \geq 216 \geq 84$.

Apply Theorem 6.1 to Q_{u} and Q_{t} from Proposition 3.2, and denote the resulting groups by P_{u} and P_{t}, respectively. Both have presentations

$$\langle b, s \mid r_1, r_2, r_3, r_4 \rangle, \qquad T = [b, s], \quad A = bT^{-1}, \tag{6.11}$$

where

$$\begin{aligned} r_1 &= s^{-1}TsT^{-2}, & r_2 &= T^{-51529}AT^{51529}A^{-1}, \\ r_3 &= [A, T^{-1}AT], & r_4 &= W(A, T), \end{aligned} \tag{6.12}$$

where $W(A, T)$ is given explicitly by

$$\begin{aligned} X_* &= V_1(A, T^{-227}AT^{227}), \qquad Y_* = V_2(A, T^{-227}AT^{227}), \\ W(A, T) &= \prod_{i=1}^{L} \rho_i(X_*, Y_*)^{T^{i-1}}. \end{aligned} \tag{6.13}$$

Use the $L = 84$ defining relators of Q_{u} for P_{u}, and the $L = 216$ defining relators of Q_{t} for P_{t}. The group P_{u} is universal, while P_{t} is torsion-free and has undecidable word problem. Both have abelianization $\mathbb{Z}_s$ and normal generator s. This proves Theorem 1.2(1). In the next section we combine the periodicity relation with the relation containing W to obtain the 3-relator presentation of Theorem 1.1(1).

7 A universal group with 3 relators

We reduce the presentation by one further relator by combining the periodicity relation with the product of the substituted relators of Q. We equate this product to a power of t, choosing the letters in the product so that they all commute with a. The new relation then forces the same power of t to centralize a, recovering the required periodicity. Conjugation by powers of s provides the required letters, at indices that are powers of 2.

Let

$$Q = \langle x, y \mid \rho_1, \ldots, \rho_L \rangle$$

be a noncyclic superperfect group with the exponent vectors (3.14), and use the words V_1, V_2 of (4.1). Choose an odd prime p such that 2 is primitive modulo p^2 and $L \leq p - 1$, and put $N = p^2$. Define the words

$$\begin{aligned} u_i(a, t, s) &= (a^t)^{s^{i-1}}, & v_i(a, t, s) &= (a^t)^{s^{i+p-2}}, \\ X_i &= V_1(u_i, v_i), & Y_i &= V_2(u_i, v_i), \\ W_{\mathrm{d}}(a, t, s) &= \prod_{i=1}^{L} \rho_i(X_i, Y_i). \end{aligned} \tag{7.1}$$

Products are in increasing order, and $u_i, v_i, X_i, Y_i, W_{\mathrm{d}}$ abbreviate words in a, t, s. Define

$$G(Q,p) = \langle a,t,s \mid a^s = a,\ t^s = t^2,\ [a,a^t] = 1,\ t^N = W_{\mathrm{d}}(a,t,s)\rangle. \tag{7.2}$$

Theorem 7.1. *The group $G(Q,p)$ contains a copy of Q, has abelianization $\mathbb{Z}$ generated by s, and is normally generated by s. It has the isomorphic 2-generator 3-relator presentation*

$$\langle b,s \mid s^{-1}TsT^{-2},\ [A,T^{-1}AT],\ T^N W_{\mathrm{d}}(A,T,s)^{-1}\rangle, \qquad T=[b,s], \quad A = bT^{-1}. \tag{7.3}$$

In particular a 2-generator 3-relator universal group exists.

Proof. Step 1: Recover the periodicity relation. Conjugating $[a,a^t]=1$ by s^e for $e \geq 0$ and using $a^{s^e} = a$ gives

$$[a,(a^t)^{s^e}] = 1.$$

Every u_i and v_i, and consequently W_{d}, commutes with a. The relation $t^N = W_{\mathrm{d}}$ therefore implies

$$[a,t^N] = 1. \tag{7.4}$$

Thus adjoining (7.4) is redundant, and

$$G(Q,p) = K(p)/J, \qquad J = \langle\langle t^N W_{\mathrm{d}}^{-1}\rangle\rangle_{K(p)},$$

where $K(p)$ is the group defined in (5.1). By Proposition 5.1,

$$K(p) = \mathcal{A} \rtimes \mathrm{BS}(1,2), \qquad \mathcal{A} = \prod_{r\in\mathbb{Z}/p\mathbb{Z}} F(a_r, a_{r+p}, \ldots, a_{r+(p-1)p}), \qquad a_j = a^{t^j},$$

with indices modulo N and with t, s acting by translation and multiplication by 2.

Step 2: Identify the quotient of the base group. The conjugation relations identify the words in (7.1) as

$$u_i = a_{2^{i-1}}, \qquad v_i = a_{2^{i+p-2}}. \tag{7.5}$$

Their two indices agree modulo p, because $2^{p-1} \equiv 1 \pmod p$, but are distinct modulo p^2, because $2^{p-1} \not\equiv 1 \pmod{p^2}$. The first indices for $1 \leq i \leq L$ have distinct residues modulo p: their exponents lie between 0 and $L-1 \leq p-2$, whereas the order of 2 modulo p is $p-1$. Thus the L relators occupy distinct residue factors, each on an ordered pair of distinct basis letters.

In each free group factor F_r, let N_r be the normal closure of all substituted relators of Q on every ordered pair of distinct basis letters, and set $N_{\mathcal{A}} = \prod_r N_r$. This subgroup is invariant under the group $K(p)$, and Lemma 4.1 gives

$$\mathcal{A}/N_{\mathcal{A}} \cong \prod_{r\in\mathbb{Z}/p\mathbb{Z}} B_p,$$

with a copy of Q embedded on each ordered pair in every factor. Write $\overline{\mathcal{A}}$ for the image of $\mathcal{A}$ in $G(Q,p)$. We will prove the central exact sequence

$$1 \longrightarrow \mathrm{im}(N_{\mathcal{A}}) \longrightarrow \overline{\mathcal{A}} \longrightarrow \prod_{r\in\mathbb{Z}/p\mathbb{Z}} B_p \longrightarrow 1. \tag{7.6}$$

To construct the quotient map in (7.6), let

$$\Gamma_N = (\mathbb{Z}/N\mathbb{Z}) \rtimes \mathbb{Z} = \langle t,s \mid t^N = 1,\ t^s = t^2\rangle.$$

Translation and doubling define an action of this group on the N indices, since N is odd. Hence there is a homomorphism

$$K(p) \longrightarrow \left(\prod_{r \in \mathbb{Z}/p\mathbb{Z}} B_p \right) \rtimes \Gamma_N$$

sending each a_j to its image in the corresponding factor B_p and t, s to the displayed affine generators. It kills W_{d} and t^N, and therefore kills J. The resulting map from $G(Q,p)$ restricts to a surjection from $\overline{\mathcal{A}}$ onto $\prod_r B_p$. Its kernel on $\overline{\mathcal{A}}$ is exactly $\mathrm{im}(N_{\mathcal{A}})$: represent an element of $\overline{\mathcal{A}}$ by $z \in \mathcal{A}$ and observe that its projection is trivial precisely when $z \in N_{\mathcal{A}}$.

To prove centrality of the kernel, observe that t^N centralizes $\mathcal{A}$ in $K(p)$. Thus its image W_{d} centralizes $\overline{\mathcal{A}}$. Normality of $\mathcal{A}$ in $K(p)$ implies that every conjugate W_{d}^h with $h \in K(p)$ centralizes $\overline{\mathcal{A}}$ as well. The affine image of $\langle t, s \rangle$ contains every map $j \mapsto uj + v$ modulo N, with u a unit, and can send any ordered pair of distinct vertices in a residue factor to any such pair in any residue factor. Indeed, the differences between the indices in the source and target pairs are p times nonzero residues modulo p. A unit multiplier matches these residues, and a translation then matches the first vertices.

Fix an index i and an ordered pair (a_u, a_v) in any residue factor. Choose $h \in \langle t, s \rangle$ carrying the pair (u_i, v_i) from (7.5) to (a_u, a_v). Every other relator factor remains in a different residue factor. For any basis letter a_w of the selected factor, the equation $[W_{\mathrm{d}}^h, a_w] = 1$ therefore gives

$$[\rho_i(V_1(a_u, a_v), V_2(a_u, a_v)), a_w] = 1 :$$

all the other relator factors commute with a_w in $K(p)$. The selected relator also commutes with every other residue factor already in $K(p)$. Hence every substituted relator is central in $\overline{\mathcal{A}}$, and so is its normal closure $\mathrm{im}(N_{\mathcal{A}})$. This proves (7.6). The two parts of the argument also give

$$[N_{\mathcal{A}}, \mathcal{A}] \;\leq\; J \cap \mathcal{A} \;\leq\; N_{\mathcal{A}}.$$

For the embedding below, centrality of this kernel suffices.

Step 3: Embed Q and obtain the presentation. Pull back (7.6) over one of the copies of Q in the quotient. Because Q is superperfect, this central extension splits, so Q embeds in $G(Q,p)$. Explicitly, for $X = V_1(a_0, a_p)$ and $Y = V_2(a_0, a_p)$ the embedded generators are

$$X\rho_1(X,Y)^{-1}, \qquad Y\rho_2(X,Y)^{-1}.$$

As in Theorem 6.1, the split lifts differ from X, Y by central elements. The exponent normalization makes the first 2 relators evaluate to those elements, so the displayed corrections give the split lifts. Projection to that copy of Q proves injectivity.

Lemma 5.2 now replaces a, t by $b = at$ and replaces the relations $a^s = a$ and $t^s = t^2$ by $T^s = T^2$, giving (7.3). The occurrences of s in the words (7.1) are unchanged by this Tietze transformation.

Finally impose $s = 1$ in (7.2). The relation $t^s = t^2$ forces $t = 1$. Every u_i, v_i becomes a, so $X_i = a$, $Y_i = 1$, and normalization gives $W_{\mathrm{d}}(a, 1, 1) = a$. The relation $t^N = W_{\mathrm{d}}$ gives $a = 1$. Thus s normally generates the group. All relators have total s-exponent zero, so the map $s \mapsto 1$, $a, t \mapsto 0$ is an epimorphism onto $\mathbb{Z}$. Normal generation makes the abelianization cyclic, proving it is exactly $\mathbb{Z}$ generated by s. Taking $Q = Q_{\mathrm{u}}$ proves universality. Since Q_{u} has undecidable word problem and its generators have the explicit images above, substitution of words gives a many-one reduction to the word problem of $G(Q_{\mathrm{u}}, p)$. Thus the 3-relator group itself has undecidable word problem. □

7.1 The universal group and its cyclic specialization

For $Q = Q_{\mathrm{u}}$, we have $L = 84$ and take

$$p = 101, \qquad N = 10201, \qquad h = p(p-1) = 10100. \tag{7.7}$$

Trial division by $2, 3, 5, 7$ proves 101 prime. Repeated squaring gives, modulo 10201,

$$2^{10100} \equiv 1, \qquad 2^{5050} \equiv 10200, \qquad 2^{2020} \equiv 4943, \qquad 2^{100} \equiv 9293.$$

The last 3 exponents are 10100 divided by its distinct prime divisors $2, 5, 101$, and none of the corresponding powers of 2 has residue 1. Thus 2 has exact order 10100, and $84 \leq 100 = p - 1$. The prime $p = 227$ also satisfies the hypotheses for both superperfect groups of Proposition 3.2, since $216 \leq 226$. For the universal construction, fix

$$P_{\mathrm{u}}^{(3)} = G(Q_{\mathrm{u}}, 101), \qquad L = 84, \qquad N = 10201.$$

This is the 3-relator source used below. Its displayed generators satisfy $b \mapsto 0$, $s \mapsto 1$ under the abelianization map to $\mathbb{Z}$.

We will need the following identity when the Gordon attachment identifies b and s with powers of the same element:

$$W_{\mathrm{d}}(z, 1, z^d) = z \qquad (d \in \mathbb{Z}). \tag{7.8}$$

After this substitution, each u_i, v_i freely reduces to a conjugate of z by a power of z, hence to z. The identities $V_1(z, z) = z$, $V_2(z, z) = 1$ and the exponent normalization of the relators now give the claim. The case $d = 3$ applies when that attachment gives $b = z$ and $s = z^3$.

We use Andrews–Curtis equivalence allowing the addition or deletion of a generator together with a defining relator, as in [10, Section 2].

Lemma 7.2. *The presentation obtained from $P_{\mathrm{u}}^{(3)}$ by adjoining the relator s is Andrews–Curtis equivalent to the presentation with no generators and 2 empty relators.*

Proof. Use the added relator to clear s from the three source relators, then delete s and its defining relator. Now $T = [b, 1] = 1$ and $A = b$, so the relators freely reduce to $1, 1, b^{-1}$ by (7.8) with $d = 0$. Invert the last relator and cancel b with its defining relator. These are Andrews–Curtis moves, including defining-generator elimination [10, Lemma 2.3]. □

Remark. The group constructed in this section has nontrivial torsion. Put $h = p(p-1)$. Multiplication by 2^h is the identity on the N indices, so s^h fixes every element of the base group $\mathcal{A}$. Since $W_{\mathrm{d}} = t^N$ in the quotient,

$$W_{\mathrm{d}}^{s^h} = W_{\mathrm{d}}, \qquad (t^N)^{s^h} = t^{N2^h}, \qquad t^{N(2^h-1)} = 1.$$

The quotient map to the semidirect product with Γ_N in Step 2 sends t to an element of exact order N in Γ_N, so t is a nontrivial finite-order element. The torsion-free conclusion therefore uses the 4-relator construction of Section 6.

8 Two small Adian–Rabin families

Gordon's attachment converts the word problem in a fixed group into the triviality problem for a family of presentations. We apply it to each of the groups constructed above, then eliminate the

auxiliary generators to obtain the desired Adian–Rabin families. Write m for the number of relators in the source presentation, so that

$$P = P_{\mathrm{u}}^{(3)}, \quad m = 3, \qquad \text{or} \qquad P = P_{\mathrm{t}}, \quad m = 4.$$

The first is the group of (7.3) with $Q = Q_{\mathrm{u}}$ and $p = 101$; the second is the torsion-free group of (6.11)–(6.13) with $L = 216$ and $p = 227$. Both have undecidable word problem and abelianization $\mathbb{Z}$ generated by the image of s, while b maps to zero. In particular, the image of b in the abelianization has order 1, as required for Gordon's attachment. To analyze the case when the chosen word is trivial, we will use the specializations

$$W(z, 1) = z, \qquad W_{\mathrm{d}}(z, 1, z^d) = z \quad (d \in \mathbb{Z}). \tag{8.1}$$

The first identity follows from (4.2) and (3.14); the second is (7.8).

For each word in b, s, the construction below gives a presentation with 2 generators and $m + 1$ relators whose group is trivial precisely when that word is trivial in P.

8.1 The common Gordon attachment

Given a word $w = w(b, s)$, adjoin generators $\mathsf{A}, \alpha, \mathsf{B}, \beta$ to the presentation of P, together with the following 5 relations. We use A to distinguish the new generator from the word $A = bT^{-1}$ in the source presentation.

$$\mathsf{A}\alpha\mathsf{A}^{-1} = \mathsf{B}^2, \tag{8.2}$$
$$\alpha\mathsf{A}\alpha^{-1} = \mathsf{B}\beta\mathsf{B}^{-1}, \tag{8.3}$$
$$\mathsf{A}^{-1}b\alpha = \beta^{-1}\mathsf{B}\beta, \tag{8.4}$$
$$\mathsf{A}^{-3}s\alpha^3 = \beta^{-2}\mathsf{B}\beta^2, \tag{8.5}$$
$$w\alpha^2 w^{-1}\alpha^{-2} = \beta^{-3}\mathsf{B}\beta^3. \tag{8.6}$$

Denote the resulting presentation by $\mathcal{Q}_w$ and its group by G_w. Thus $\mathcal{Q}_w$ has 6 generators and $m + 5$ relators. The commutator on the left of (8.6) is written out because Gordon uses the opposite commutator convention.

Apply [9, Lemma 2.1] with ordered source generators $(x_1, x_2) = (b, s)$ and parameters $p = 1$ and $q_1 = q = 1$. This gives the exponent $q + 2 = 3$ in (8.5) and the exponent $n + 1 = 3$ on the right of (8.6). If $w \neq 1$ in P, the left sides of these 5 relations freely generate a rank-5 subgroup H_L of $P * F(\mathsf{A}, \alpha)$, and the right sides freely generate a rank-5 subgroup H_R of $F(\mathsf{B}, \beta)$. Matching the two bases identifies G_w as the amalgamated product

$$G_w \cong \big(P * F(\mathsf{A}, \alpha)\big) *_{H_L \cong H_R} F(\mathsf{B}, \beta). \tag{8.7}$$

Each amalgamated subgroup is proper in its vertex group. On the right, H_R and $F(\mathsf{B}, \beta)$ have free ranks 5 and 2, respectively. On the left, H_L has abelianization $\mathbb{Z}^5$, whereas $P * F(\mathsf{A}, \alpha)$ has abelianization $\mathbb{Z}^3$. The normal-form theorem embeds both vertex groups, and hence P, in G_w. Thus $G_w \neq 1$, and it is universal when $P = P_{\mathrm{u}}^{(3)}$. When $P = P_{\mathrm{t}}$, both vertex groups are torsion-free, so the torsion theorem for amalgams shows that G_w is torsion-free as well.

If $w = 1$ in P, then (8.6) gives $\mathsf{B} = 1$ in G_w. Relations (8.2)–(8.5) then give, successively,

$$\alpha = 1, \qquad \beta = \mathsf{A}, \qquad b = \mathsf{A}, \qquad s = \mathsf{A}^3. \tag{8.8}$$

Consequently $T = [\mathsf{A}, \mathsf{A}^3] = 1$ and $A = \mathsf{A}$. By (8.1), the 3 source relators become $1, 1, \mathsf{A}^{-1}$ when $P = P_{\mathrm{u}}^{(3)}$, and the 4 source relators become $1, 1, 1, \mathsf{A}$ when $P = P_{\mathrm{t}}$. In either case the last relator gives $\mathsf{A} = 1$, so all 6 generators are trivial. We have proved

$$G_w = 1 \quad \Longleftrightarrow \quad w = 1 \text{ in } P. \tag{8.9}$$

8.2 Presentations with 4 and 5 relators

The first 4 attachment relations express α, β, b, s successively in terms of A, B. To carry out these eliminations, define words in $F(\mathsf{A}, \mathsf{B})$ by

$$\begin{aligned} \alpha_* &= \mathsf{A}^{-1}\mathsf{B}^2\mathsf{A}, \\ \beta_* &= \mathsf{B}^{-1}\alpha_*\mathsf{A}\alpha_*^{-1}\mathsf{B}, \\ b_* &= \mathsf{A}\beta_*^{-1}\mathsf{B}\beta_*\alpha_*^{-1}, \\ s_* &= \mathsf{A}^3\beta_*^{-2}\mathsf{B}\beta_*^2\alpha_*^{-3}, \\ T_* &= [b_*, s_*], \qquad A_* = b_*T_*^{-1}, \qquad w_* = w(b_*, s_*). \end{aligned} \tag{8.10}$$

For $P = P_{\mathrm{u}}^{(3)}$, substitution in the remaining relators gives the presentation $\mathcal{P}_w$ on generators A, B with relators

$$\begin{aligned} &s_*^{-1}T_*s_*T_*^{-2}, \\ &[A_*, T_*^{-1}A_*T_*], \\ &T_*^{10201}W_{\mathrm{d}}(A_*, T_*, s_*)^{-1}, \\ &w_*\alpha_*^2w_*^{-1}\alpha_*^{-2}\beta_*^{-3}\mathsf{B}^{-1}\beta_*^3. \end{aligned} \tag{8.11}$$

Here W_{d} is the word in (7.1), with $p = 101$, $L = 84$, and the defining relators of Q_{u}. For $P = P_{\mathrm{t}}$, the same substitutions give the 5 relators

$$\begin{aligned} &s_*^{-1}T_*s_*T_*^{-2}, \\ &T_*^{-51529}A_*T_*^{51529}A_*^{-1}, \\ &[A_*, T_*^{-1}A_*T_*], \\ &W(A_*, T_*), \\ &w_*\alpha_*^2w_*^{-1}\alpha_*^{-2}\beta_*^{-3}\mathsf{B}^{-1}\beta_*^3. \end{aligned} \tag{8.12}$$

In this list, W is the word in (6.13), with $L = 216$ and the defining relators of Q_{t}. The expressions in (8.10) are abbreviations for words, so they add no generators or relations. Each elimination deletes a generator together with its defining relation, and is therefore a Tietze transformation. Thus $\mathcal{P}_w$ presents G_w with 2 generators and $m + 1$ relators.

Theorem 8.1. *The presentations (8.11) form an effective 2-generator 4-relator Adian–Rabin family. The group presented by a member is trivial exactly when $w = 1$ in $P_{\mathrm{u}}^{(3)}$; otherwise it contains $P_{\mathrm{u}}^{(3)}$ and is universal. The presentations (8.12) form an effective 2-generator 5-relator Adian–Rabin family whose presented groups are all torsion-free. The group presented by a member is trivial exactly when $w = 1$ in P_{t}; otherwise it contains P_{t}.*

Proof. Enumerate the freely reduced words on b, s in length-lexicographic order. The enumeration and its inverse on reduced words are computable, as are the substitutions defining $\mathcal{P}_w$. Hence (8.9) reduces the word problem in P to the triviality problem for the resulting natural-number-indexed family. Both choices of P have undecidable word problem. The embedding and torsion assertions follow from (8.7), together with the fact that the trivial group is torsion-free. This proves Theorem 1.1(2) and Theorem 1.2(2). □

9 Unrecognizable four-manifolds

We realize the presentations of Section 8 by smooth 5-dimensional handlebodies. The boundary corresponding to a word w has fundamental group G_w. When w is nontrivial in the source group, this fundamental group distinguishes the boundary from the simply connected target manifold. When w is trivial, an explicit Andrews–Curtis reduction of the presentation gives handle cancellations that identify the boundary with the standard $\#^2(S^2 \times S^2)$.

The smooth recognition problem is expressed using finite PL triangulations and their compatible smooth structures in dimension 4; see [13, Section 1.2].

Use $P = P_{\mathrm{u}}^{(3)}$ and the presentation $\mathcal{Q}_w$ before generator elimination. For each word w, construct an orientable smooth 5-dimensional handlebody V_w with 1 zero-handle, 6 one-handles, and 8 two-handles whose attaching words are the relators of $\mathcal{Q}_w$. Fix a spin structure on the one-handlebody and choose each attaching framing to extend it. This construction is effective: realize the finite words by disjoint polygonal circles in the boundary of a triangulated one-handlebody, choose tubular neighborhoods and the prescribed framings, and attach the handles after finite subdivisions. It produces a finite PL triangulation with its compatible smooth structure; see [9, Section 3] and [6]. Viewed relative to the boundary, the dual handle decomposition has handles of index at least 3, so boundary inclusion induces

$$\pi_1(\partial V_w) \cong \pi_1(V_w) \cong G_w. \tag{9.1}$$

The cancellations use the geometric realizations of Andrews–Curtis moves: conjugating or inverting an attaching word, multiplying a relator by a conjugate of another relator through a two-handle slide, and cancelling a one–two handle pair when a defining relator contains its generator once. Before a cancellation, we clear that generator from the other relators; see [10, Theorem 2.1, Corollary 2.2, and Lemma 2.3]. In the 4-dimensional boundary, a homotopy of a finite collection of attaching circles can be chosen to be an isotopy, since parametric general position removes simultaneous intersections by $1 + 1 + 1 < 4$. Isotopy extension gives an ambient isotopy; compare [9, Section 3]. These operations preserve the smooth handlebody and its spin property.

We write $(x_1, \dots, x_g : r_1, \dots, r_k)$ for a presentation, with 1 denoting an empty relator. We write $\mathcal{P} \sim_{\mathrm{AC}} \mathcal{Q}$ for Andrews–Curtis equivalence, allowing the addition or deletion of a generator together with a defining relator, as in Lemma 7.2.

Lemma 9.1. *If $w = 1$ in P, then*

$$\mathcal{Q}_w \ \sim_{\mathrm{AC}} \ (\varnothing : 1, 1). \tag{9.2}$$

Consequently, V_w is diffeomorphic to the standard boundary connected sum $\natural_2(S^2 \times D^3)$.

Proof. We first prove the AC reduction. Let $D_w = w\alpha^2 w^{-1}\alpha^{-2}$. Since w lies in the normal closure of the 3 source relators in $F(b, s)$, the word D_w is a finite product of conjugates of these relators and their inverses in the enlarged free group. Retaining the source relators, Gordon's relator-clearing lemma [10, Lemma 2.5] removes D_w from the relator (8.6). The resulting relator is $\beta^{-3}\mathsf{B}^{-1}\beta^3$, which becomes B after conjugation and inversion.

Cancel B using this relator, then cancel α using (8.2), β using (8.3), and b, s using (8.4)–(8.5), clearing the other occurrences before each cancellation. These are the substitutions in (8.8). The 5 cancellations leave the generator A and the 3 source relators. Since $T = [\mathsf{A}, \mathsf{A}^3]$ freely reduces to the empty word and $A = bT^{-1}$ becomes A, these relators reduce to

$$1, \qquad 1, \qquad \mathsf{A}^{-1}.$$

The third relator is A^{-1} because $W_{\mathrm{d}}(\mathsf{A}, 1, \mathsf{A}^3) = \mathsf{A}$ by (7.8). Invert this relator and cancel A, leaving 2 empty relators. This proves (9.2).

Realizing the AC reduction by handle slides and cancellations leaves 2 two-handles attached to a 5-ball along an unlink. The spin-compatible framings give the standard handlebody $\natural_2(S^2 \times D^3)$, whose boundary is $\#^2(S^2 \times S^2)$; see [10, Theorem 2.1 and Corollary 2.2]. The construction of V_w is effective for every w, while the slides used to identify it with the standard handlebody need only exist when $w = 1$. □

Proof of Theorem 1.1(3). First take $n = 2$. If $w = 1$ in P, Lemma 9.1 identifies ∂V_w smoothly with $M_2 = \#^2(S^2 \times S^2)$. If $w \neq 1$, the amalgam description (8.7) and (9.1) give a nontrivial fundamental group, excluding even a homeomorphism to the simply connected M_2. A recognition algorithm in either category would therefore decide the word problem in P, which is impossible.

For fixed $n > 2$, add $n - 2$ empty spin-framed relators before constructing the handlebody. They leave the fundamental group unchanged. When $w = 1$, the same cancellations leave n unlinked spin-framed two-handles on D^5, whose boundary is the standard $M_n = \#^n(S^2 \times S^2)$. This proves both undecidability assertions for every fixed $n \geq 2$.

Every manifold produced above is closed and connected, so the reduction uses only inputs allowed in the statement. □

10 Applications to embedded surfaces

We apply Gordon's surface constructions [10, 11] to our universal group. The applications in $\#^2(S^2 \times S^2)$ use the explicit Andrews–Curtis reduction obtained by adjoining the relator s. For the applications in S^4, we use the fact that a central extension of a universal group is universal.

A surface will mean a closed connected locally flat PL embedded surface. We use compatible smoothings of these pairs; for locally flat PL surfaces in four-manifolds, the PL and smooth isotopy classifications agree [11, Section 1]. We call F topologically or smoothly unknotted if it is ambiently isotopic in the indicated category to a standard unknotted surface contained in a 4-ball in X. The standard orientable model bounds a handlebody; a standard nonorientable model is a connected sum of standard projective planes. Nonorientable genus counts projective-plane summands. Algorithms take finite triangulations of pairs, with the surface a subcomplex after subdivision.

Throughout this section, let

$$\mathcal{P} = \langle b, s \mid r_1, r_2, r_3 \rangle$$

be the presentation (7.3) of $U = P_{\mathrm{u}}^{(3)}$. The group U has unsolvable word problem, s normally generates U, and s has infinite order. Lemma 7.2 gives an explicit Andrews–Curtis equivalence

$$\mathcal{P} + (s) \ \sim_{\mathrm{AC}} \ (\varnothing : 1, 1). \tag{10.1}$$

10.1 Surfaces in $\#^2(S^2 \times S^2)$

We apply Gordon's surface constructions to our presentation, using the AC reduction (10.1) to verify their hypothesis. As in Section 9, AC reductions identify the standard geometric objects. Here adjoining the meridian as a relator identifies the ambient manifold for every word w; retaining the meridian generator proves unknotting when $w = 1$.

Corollary 10.1. *Let $M_2 = \#^2(S^2 \times S^2)$. For each homeomorphism type of closed connected surface, the following hold.*

1. *There is a surface of that type in M_2 whose complement group is universal.*
2. *The topological and smooth unknotting problems for surfaces of that type in M_2 are undecidable.*

Proof. The presentation $\mathcal{P}$ has relator excess $3-2=1$, so Gordon's Proposition 3.1 applies in M_2 by (10.1). Its parts (1) and (3) give a sphere with complement group U and a projective plane whose complement group contains U, respectively. For part (3), take the auxiliary nontrivial element to be s. Connected sums with standard surfaces give all remaining surface types, as in [10, Theorem 3.9].

For the unknotting assertion, apply Gordon's four-relation attachment [10, Section 3, before Lemma 3.2] with source group U, distinguished element s, and word w. Denote the resulting presentation by $\mathcal{R}_w$, its group by H_w, and its meridian generator by A. The attachment adds four generators and four relators, so $\mathcal{R}_w$ has 6 generators and 7 relators, again with relator excess 1.

The AC reduction in Gordon's Lemma 3.2 gives

$$\mathcal{R}_w + (\mathsf{A}) \ \sim_{\mathrm{AC}} \ \mathcal{P} + (s) \ \sim_{\mathrm{AC}} \ (\varnothing : 1, 1) \qquad \text{for every } w. \tag{10.2}$$

His Proposition 3.1(1) therefore produces an effective family of spheres $S_w \subset M_2$ with complement group H_w and meridian A. The reduction is uniform in w, so identifying the ambient manifold does not require deciding whether $w = 1$ in U.

If $w = 1$ in U, Gordon's relator-clearing argument [10, Lemma 2.5 and Proposition 3.5(1)] cancels the auxiliary generators other than A and gives

$$\mathcal{R}_w \ \sim_{\mathrm{AC}} \ (b, s, \mathsf{A} : r_1, r_2, r_3, s) \ \sim_{\mathrm{AC}} \ (\mathsf{A} : 1, 1).$$

The last reduction uses (10.1) while retaining the meridian generator. Gordon's Proposition 3.1(2) then shows that S_w is smoothly unknotted.

If $w \neq 1$ in U, the amalgam argument in the proof of Gordon's Proposition 3.1(3) embeds U in $H_w/\langle\langle \mathsf{A}^2 \rangle\rangle$. This argument does not require the source group to be torsion-free. Thus both H_w and $H_w/\langle\langle \mathsf{A}^2 \rangle\rangle$ are noncyclic, giving the obstruction to topological unknotting for orientable and nonorientable surfaces, respectively.

For each fixed surface type, Gordon's connected-sum argument [10, Theorem 3.7] now gives an effective family $F_w \subset M_2$ that is smoothly unknotted when $w = 1$ and topologically knotted otherwise. Hence either unknotting algorithm would decide the word problem of U. □

10.2 Universal complement groups of surfaces in the four-sphere

Gordon's constructions in S^4 place the source group in a quotient of the complement group by a central subgroup. The following observation shows that universality of the quotient suffices. All coefficient groups carry the trivial module action.

Lemma 10.2. *Let $1 \to A \to E \xrightarrow{q} B \to 1$ be a central extension. If B is universal, then E is universal.*

Proof. There exists a finitely presented universal group V such that $H^2(V; C) = 0$ for every abelian group C; Gordon constructs one in [11, Section 5, proof of Theorem 1.4]. (One may also take $V = Q_{\mathrm{u}}$ from Proposition 3.2, since superperfectness gives the required vanishing.) Since B is universal, it contains a copy of V. Pulling back the extension to this copy gives a central extension of V by A, which splits because $H^2(V; A) = 0$. Thus V embeds in E, proving universality. Only the existence of the embedding and splitting is needed. □

Corollary 10.3. *There is an orientable ribbon surface of genus 6 in S^4 with universal complement group, and a nonorientable surface of genus 19 in S^4 with universal complement group. Both complement groups have unsolvable word problem.*

Proof. Apply [11, Proposition 2.7] to the 3-relator presentation $\mathcal{P}$ of U. It constructs an orientable ribbon surface $F \subset S^4$ of genus $2 \cdot 3 = 6$ and an embedding

$$U \hookrightarrow \pi_1(S^4 \setminus F)/Z(\pi_1(S^4 \setminus F)).$$

The quotient is universal because it contains U. Lemma 10.2 therefore shows that $\pi_1(S^4 \setminus F)$ is universal.

For the nonorientable example, let G_s denote the group G_w defined in [11, Section 3, immediately before Lemma 3.2], with source group $G = U$ and nontrivial word $w = s$. The construction in the proof of Proposition 3.1 there gives an orientable ribbon surface F_s of genus $3 \cdot 3 = 9$ and an epimorphism $\pi_1(S^4 \setminus F_s) \to G_s$ with central kernel. If $a \in G_s$ is the image of the meridian, his Lemma 3.2(4) embeds U in $G_s/\langle\langle a^2 \rangle\rangle$.

As in [11, Theorem 5.3], take the connected sum $K = F_s \# P_+$ with a standard projective plane. This has nonorientable genus $2 \cdot 9 + 1 = 19$. The induced epimorphism

$$\pi_1(S^4 \setminus K) \longrightarrow G_s/\langle\langle a^2 \rangle\rangle$$

has central kernel, since its kernel is the image of the original central kernel. Its target contains U and is therefore universal, so Lemma 10.2 proves universality of $\pi_1(S^4 \setminus K)$.

Both complement groups have unsolvable word problem because they are finitely presented and universal. □

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