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TERMINAL SYMPLECTIC SINGULARITIES FOR PARABOLIC BASIC AFFINE SPACES

1. Introduction

Let G be a simply connected semisimple group over $\mathbb{C}$, let $P \subseteq G$ be a parabolic subgroup, and let U_P be its unipotent radical. We write

$$A_P := \Gamma(T^*(G/U_P), \mathcal{O}), \qquad \overline{T^*(G/U_P)} := \operatorname{Spec} A_P.$$

The algebra A_P is finitely generated by [5, Corollary 2.1.7]. Thus $\overline{T^*(G/U_P)}$ is an affine variety. It is normal: a rational function integral over A_P belongs to every local ring of the smooth variety $T^*(G/U_P)$, and hence belongs to A_P.

Recall that a normal variety has *symplectic singularities* if its smooth locus admits a symplectic form whose pullback extends to a regular two-form on a resolution of singularities. A normal variety is $\mathbb{Q}$-*factorial* if every Weil divisor has a positive integer multiple that is Cartier. If V is normal and K_V is $\mathbb{Q}$-Cartier, the *discrepancies* of a resolution $f\colon W \to V$ are the coefficients a_E in

$$K_W = f^*K_V + \sum_{E \text{ exceptional}} a_E E.$$

The variety V has *terminal singularities* if all these coefficients are positive on every resolution. The goal of this paper is to prove the following theorem.

Theorem 1.1. *The variety $\overline{T^*(G/U_P)}$ has $\mathbb{Q}$-factorial terminal symplectic singularities. Moreover, A_P is a unique factorization domain.*

When P is a Borel subgroup, Theorem 1.1 recovers the simply connected case of [4, Theorems 1.1 and 1.2]. The proof of factoriality in [4, Section 3.1] applies to U_P without change; we recall it in Proposition 2.1. The remaining assertions are proved separately in Theorems 3.1 and 4.1.

1.1. **Outline of the proof.** Choose a Borel subgroup $B \subseteq P$ and put $\mathfrak{g} = \operatorname{Lie}(G)$ and $\mathfrak{p} = \operatorname{Lie}(P)$. The criterion of Fu–Liu [2, Theorem 1.6(1.6.2)] reduces the symplectic-singularity assertion to a positivity statement for the projective bundle associated to $G \times^B \mathfrak{p}$. We prove that statement in Proposition 3.7 by constructing an effective rational boundary. The main step, Lemma 3.5, uses the Frobenius splitting of a flag variety to obtain discrepancy inequalities for finitely many fixed polynomials on a Levi subalgebra. Its proof transfers these inequalities to characteristic zero on a common log resolution. Lemma 3.6 controls the centers at which equality occurs, and Proposition 3.7 uses this control to perturb the boundary.

To prove terminality, we first show in Proposition 4.5 that every first-order Poisson deformation of $\overline{T^*(G/U_P)}$ is trivial, using the left G-action and the right action of a Levi subgroup. Lemma 4.6 deduces from Namikawa's deformation theory that the singular locus then has codimension at least four. Finally, [10, Corollary 1] identifies this condition with terminality for symplectic singularities.

Remark 1.2. A *conical grading* on a finitely generated $\mathbb{C}$-algebra A means a grading $A = \bigoplus_{d \geq 0} A_d$ with $A_0 = \mathbb{C}$. Such a grading need not exist for A_P. Indeed, if $P = G$, then $U_P = \{1\}$ and

$$\overline{T^*(G/U_P)} = T^*G \cong G \times \mathfrak{g}^*.$$

For $G = \mathrm{SL}_2$, scalar multiplication contracts the cotangent fibers. Polar decomposition gives the deformation retraction $g \mapsto g(g^*g)^{-t/2}$, for $0 \le t \le 1$, onto $\mathrm{SU}(2)$. The description

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

therefore gives homotopy equivalences

$$T^*\mathrm{SL}_2(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \simeq \mathrm{SU}(2) \cong S^3.$$

Consequently,

$$H^i(T^*\mathrm{SL}_2(\mathbb{C}), \mathbb{C}) = \begin{cases} \mathbb{C}, & i = 0, 3, \\ 0, & \text{otherwise.} \end{cases}$$

A conical grading would define a morphism $\mathbb{A}^1 \times \operatorname{Spec} A \to \operatorname{Spec} A$ by $a_d \mapsto t^d a_d$. At $t = 0$ this morphism is constant, because $A_0 = \mathbb{C}$, and at $t = 1$ it is the identity. Restricting t to $[0, 1]$ would therefore contract the complex analytic space, contradicting the displayed nonvanishing of H^3. Thus $\Gamma(T^*\mathrm{SL}_2, \mathcal{O})$ admits no conical grading.

1.2. **Notation and conventions.** All varieties are over $\mathbb{C}$, except during the Frobenius-splitting arguments in Lemmas 3.4 and 3.5. Fix a maximal torus $T \subseteq B$ and the Levi subgroup $L \subseteq P$ containing T. Put $\mathfrak{u}_P = \mathrm{Lie}(U_P)$ and set

$$B_L = B \cap L, \qquad \mathfrak{t} = \mathrm{Lie}(T), \qquad \mathfrak{l} = \mathrm{Lie}(L), \qquad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P.$$

Let Φ_G^+ denote the positive roots of G determined by B. For reduction modulo a prime, we choose a compatible split rational form and a Chevalley integral model. For a representation V of a subgroup $H \subseteq G$, the associated bundle $G \times^H V$ is the quotient by $(g, v)h = (gh, h^{-1}v)$. Our convention for projectivization is

$$\mathbb{P}_S(\mathcal{V}) = \operatorname{Proj}_S \operatorname{Sym} \mathcal{V}.$$

Thus $\mathbb{P}_S(\mathcal{V}^*)$ parametrizes lines in the fibers of $\mathcal{V}$. We use additive notation for rational divisor classes and their associated rational line bundles. If G is trivial, its cotangent bundle and its affinization are a point, so Theorem 1.1 is immediate. We therefore assume $\dim G > 0$ from now on.

2. Finite generation and factoriality

The fiber of $T^*(G/U_P)$ at the identity coset is $(\mathfrak{g}/\mathfrak{u}_P)^* = \mathfrak{u}_P^\perp$. The Killing form pairs opposite root spaces and is nondegenerate on $\mathfrak{t}$, so it identifies this annihilator with $\mathfrak{p}$. Equivariance then identifies the cotangent bundle as

$$M := T^*(G/U_P) \cong G \times^{U_P} \mathfrak{p}. \tag{2.1}$$

In particular,

$$A_P = \mathbb{C}[G \times \mathfrak{p}]^{U_P}.$$

We use the finite generation of this algebra from [5, Corollary 2.1.7].

Proposition 2.1. *The algebra A_P is a unique factorization domain. In particular, $\overline{T^*(G/U_P)}$ is $\mathbb{Q}$-factorial.*

Proof. The homogeneous space G/U_P is quasi-affine by [9, Proposition 14.26]. An affine morphism over a quasi-affine variety has quasi-affine source, by [13, Tag 01VG, Lemma 29.38.7(2)]; applying this to the vector bundle shows that M is quasi-affine. The canonical map

$$j \colon M \longrightarrow \overline{T^*(G/U_P)}$$

is therefore an open immersion, and its complement has codimension at least two, by [2, Theorem 2.5]; the codimension statement is also [4, Lemma 3.4]. Removing a subset of codimension at least

two does not change Weil divisors or their principal relations, while smoothness identifies Weil and Cartier divisor classes. Consequently,

$$\mathrm{Cl}(\overline{T^*(G/U_P)}) \cong \mathrm{Cl}(M) = \mathrm{Pic}(M) \cong \mathrm{Pic}(G/U_P), \tag{2.2}$$

where the last isomorphism is invariance of the Picard group under passage to a vector bundle on a smooth variety, as used in [4, proof of Proposition 3.3].

The U_P-torsor $G \to G/U_P$ is Zariski locally trivial by [9, Proposition 16.55]. Its Picard sequence [9, Theorem 18.32] therefore gives

$$\mathrm{Hom}(U_P, \mathbb{G}_m) \longrightarrow \mathrm{Pic}(G/U_P) \longrightarrow \mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(U_P).$$

The first group vanishes by [9, Corollary 14.18], and $\mathrm{Pic}(G) = 0$ by [9, Corollary 18.24]. Exactness therefore gives $\mathrm{Pic}(G/U_P) = 0$, so (2.2) gives $\mathrm{Cl}(\overline{T^*(G/U_P)}) = 0$. A Noetherian normal domain with trivial divisor class group is a unique factorization domain, as used in [4, Corollary 3.5]. In particular, every Weil divisor on $\overline{T^*(G/U_P)}$ is principal, which implies $\mathbb{Q}$-factoriality. □

Lemma 2.2. *The cotangent symplectic form on M extends uniquely to a symplectic form on the smooth locus of $\overline{T^*(G/U_P)}$.*

Proof. By the proof of Proposition 2.1, the complement of M has codimension at least two. A section of a locally free sheaf on a smooth variety extends uniquely across such a subset: after trivializing the sheaf, this is the extension property for regular functions on a normal variety. Apply this to the sheaf of two-forms on the smooth locus to extend the cotangent form to ω. The form $d\omega$ vanishes on the dense open set M, hence vanishes everywhere. If $\dim M = 2d$, the section $\omega^{\wedge d}$ of the canonical line bundle is nonzero on M. A nonzero section of a line bundle on a smooth integral variety has either an empty zero locus or a divisor as its zero locus. The codimension statement therefore forces $\omega^{\wedge d}$ to be nowhere zero, so ω is symplectic. □

3. Symplectic singularities

We first prove the symplectic-singularity assertion independently of terminality.

Theorem 3.1. *The canonical symplectic form on $T^*(G/U_P)$ extends to the smooth locus of $\overline{T^*(G/U_P)}$ and makes $\overline{T^*(G/U_P)}$ a variety with symplectic singularities.*

We recall the discrepancy terminology needed for the proof. If V is smooth and D is an effective rational divisor, the *log discrepancy* of a prime divisor E on a smooth birational model $f\colon \widetilde{V} \to V$ is

$$a_E(V, D) = 1 + \mathrm{ord}_E(K_{\widetilde{V}} - f^*(K_V + D)).$$

The pair (V, D) is *log canonical* if all its log discrepancies are nonnegative, and *Kawamata log terminal*, or *klt*, if they are positive. A *log canonical center* is the image of a divisor with log discrepancy zero. A smooth projective variety V is *of Fano type* if there is an effective rational divisor Δ such that (V, Δ) is klt and $-(K_V + \Delta)$ is ample.

A *log resolution* of (V, D) is a proper birational morphism $f\colon \widetilde{V} \to V$ with $\widetilde{V}$ smooth such that the exceptional divisor together with the support of f^*D has simple normal crossings. We recall three standard facts about log canonical pairs.

Lemma 3.2. *Let V be smooth and let D be an effective rational divisor.*

1. *Log canonicity of (V, D) is preserved and detected by smooth surjective pullback.*
2. *The pair (V, D) has finitely many log canonical centers if it is log canonical.*
3. *If (V, D) is log canonical and $0 < \epsilon < 1$, then $(V, (1-\epsilon)D)$ is klt.*

Proof. Let $f\colon \widetilde{V} \to V$ be a log resolution, and write $K_{\widetilde{V}} + \widetilde{D} = f^*(K_V + D)$. The log resolution criterion [7, Remark 3.2(1)] says that (V, D) is log canonical, respectively klt, if and only if every coefficient of $\widetilde{D}$ is at most one, respectively less than one. Given a smooth surjection $g\colon W \to V$,

the base change $\widetilde{V} \times_V W \to W$ is a log resolution of (W, g^*D). The relative canonical divisor and the boundary pull back under this base change, so the coefficients in the criterion are unchanged. Surjectivity ensures that every component of $\widetilde{D}$ has nonempty inverse image. Thus the criterion holds after pullback if and only if it holds before pullback, proving (1).

Part (2) is [3, Theorem 9.1(1)]. Finally, $(V, 0)$ is klt by the same log resolution criterion, and D is $\mathbb{Q}$-Cartier because V is smooth. Hence (3) follows from [8, Corollary 2.35(5)], applied with $\Delta = 0$, $\Delta' = D$, and $c = 1 - \epsilon$ in the notation of that reference. □

3.1. **The two projective bundles.** Define

$$E := G \times^B \mathfrak{p}, \qquad F := G \times^P \mathfrak{p}, \qquad Y := \mathbb{P}_{G/B}(E^*), \qquad Z := \mathbb{P}_{G/P}(F^*).$$

Since E is the pullback of F, there is a Cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & G/B \\ \pi \downarrow & & \downarrow \\ Z & \longrightarrow & G/P. \end{array}$$

Let A be the class of $\mathcal{O}_Z(1)$, let $\xi = \pi^* A$, and set $R = -K_{Y/Z}$. Write Φ_L^+ for the positive roots of L determined by B_L, and put

$$n = |\Phi_L^+|, \qquad 2\rho_L = \sum_{\alpha \in \Phi_L^+} \alpha, \qquad r = \dim \mathfrak{p}.$$

Lemma 3.3. *The divisor A is globally generated, nef, and big. The divisor R is π-ample, and*

$$-K_Z = rA, \qquad -K_Y = r\xi + R, \qquad r > n. \tag{3.1}$$

Moreover, R is the pullback from G/B of the line bundle associated to the B-character $-2\rho_L$.

Proof. The inclusion $F \subseteq (G/P) \times \mathfrak{g}$, given by $[g, x] \mapsto (gP, \operatorname{Ad}_g x)$, gives a surjection from a trivial bundle to F^*. Thus A is generated by the subspace of sections coming from $\mathfrak{g}^*$, and this subspace defines the morphism

$$q \colon Z \longrightarrow \mathbb{P}(\mathfrak{g}^*), \qquad [g, [x]] \longmapsto [\operatorname{Ad}_g x]. \tag{3.2}$$

Here $\mathbb{P}(\mathfrak{g}^*)$ parametrizes lines in $\mathfrak{g}$, and $A = q^* \mathcal{O}_{\mathbb{P}(\mathfrak{g}^*)}(1)$ by construction. This proves global generation and nefness, since the degree of this pullback on any curve is nonnegative.

Call an element of $\mathfrak{g}$ *regular semisimple* if it is semisimple and its centralizer is a maximal torus. If x is such an element, a parabolic whose Lie algebra contains x contains $C_G(x)$: apply [9, Proposition 17.90] to that parabolic to obtain a maximal torus centralizing x, and compare dimensions inside $C_G(x)$. There are finitely many parabolics of a given type containing a fixed maximal torus, since their Lie algebras are sums of the toral subalgebra and root spaces. Hence the fibers of q over regular semisimple lines are finite. They are nonempty, since a maximal torus is contained in a parabolic of the prescribed type. As q is proper, its restriction over this open set is finite. Thus q is dominant and generically finite, which proves that $A = q^* \mathcal{O}(1)$ is big.

Opposite Levi root spaces cancel in $\det \mathfrak{p}$, so its character is $\sum_{\alpha \in \Phi_G^+ \setminus \Phi_L^+} \alpha$. The same character is the determinant of $(\mathfrak{g}/\mathfrak{p})^*$, the cotangent fiber of G/P. Hence $\det F = K_{G/P}$. If $s \colon Z \to G/P$ is the projection, the relative Euler sequence for the projective bundle gives

$$K_Z = -rA + s^*(K_{G/P} - \det F) = -rA.$$

The identity $K_Y = \pi^* K_Z + K_{Y/Z}$ now gives the second equality in (3.1); compare [2, Proposition 5.5].

The relative tangent bundle of $G/B \to G/P$ is $G \times^B (\mathfrak{p}/\operatorname{Lie}(B))$. Its determinant has character $-2\rho_L$, because the weights of $\mathfrak{p}/\operatorname{Lie}(B)$ are exactly the negative Levi roots. Base change in the displayed Cartesian square proves the asserted global description of R. On every fiber L/B_L, this line bundle is ample: $2\rho_L$ pairs positively with every simple Levi coroot, so the flag-variety ampleness

criterion [1, Exercise 3.1.E.1] applies. Over each translate of the opposite big cell of G/P, the root-subgroup decomposition trivializes $G \to G/P$ and hence $G/B \to G/P$; see [1, §2.1]. Thus π is locally a product with L/B_L, and its relative anticanonical bundle R is π-ample. Finally, the root decomposition gives

$$r = \dim \mathfrak{l} + \dim \mathfrak{u}_P = 2n + \operatorname{rk} L + \dim \mathfrak{u}_P > n. \qquad \square$$

By [2, Theorem 1.6(1.6.2)], Theorem 3.1 follows once we prove that Y is of Fano type. We will construct a log canonical divisor of class $R + n\xi$ whose log canonical centers meet the open set on which q is finite.

3.2. **A boundary on the Levi subalgebra.** We first state the Frobenius splitting fact used below. For a variety $\mathcal{F}$ in characteristic $p > 0$, its absolute Frobenius is denoted by Fr. A *Frobenius splitting* is an $\mathcal{O}_{\mathcal{F}}$-linear retraction of $\mathcal{O}_{\mathcal{F}} \to \operatorname{Fr}_* \mathcal{O}_{\mathcal{F}}$, $a \mapsto a^p$.

Lemma 3.4 (Brion–Kumar)**.** *Let G_0 be a simply connected semisimple group over an algebraically closed field k of characteristic $p > 0$, and choose a maximal torus T_0 in a Borel subgroup B_0. Put $\mathcal{F} = G_0/B_0$, let ρ_0 be the half-sum of positive roots, and set*

$$\mathcal{L}(\rho_0) = G_0 \times^{B_0} k_{-\rho_0}.$$

Choose nonzero sections s_+ and s_- of $\mathcal{L}(\rho_0)$ of highest and lowest weight, respectively. Under the identifications

$$\mathcal{L}(2\rho_0) \cong \omega_{\mathcal{F}}^{-1}, \qquad H^0(\mathcal{F}, \omega_{\mathcal{F}}^{1-p}) \cong \operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(\operatorname{Fr}_* \mathcal{O}_{\mathcal{F}}, \mathcal{O}_{\mathcal{F}}),$$

a nonzero scalar multiple of $(s_+ s_-)^{p-1}$ defines a Frobenius splitting. On the opposite big cell, with its usual trivialization of $\mathcal{L}(\rho_0)$ and suitable normalizations, $s_- = 1$ and

$$s_+(u) = \ell_-(uv_+),$$

where v_+ is a highest-weight vector of the Weyl module of highest weight ρ_0, and ℓ_- extracts its lowest-weight component.

Proof. The canonical-bundle identification is [1, equation (2.1.8)], and the Frobenius-duality identification is [1, Theorem 1.3.8]. Theorem 2.3.1 of [1] says that the product of the highest- and lowest-weight sections of the Steinberg module defines a splitting up to a nonzero scalar. Equations (4)–(5) of its proof identify those sections with the $(p-1)$st powers of s_+ and s_-. The evaluation formula in equations (1)–(2) of that proof gives the asserted big-cell expressions: the functional extracting the highest-weight component is constant on uv_+ for u in the opposite unipotent subgroup, while the other extremal functional extracts the lowest-weight component. These statements occur on pages 72–74 of [1]. $\square$

The next lemma supplies both the divisor class and the required control of its centers. A polynomial f is called a B_L-semi-invariant of character χ if $f(\operatorname{Ad}_b x) = \chi(b) f(x)$ for all $b \in B_L$.

Lemma 3.5. *Assume $n > 0$. Let $\delta \in \mathbb{C}[\mathfrak{l}]^L$ be a homogeneous invariant polynomial defined over $\mathbb{Q}$ with nonzero restriction to $\mathfrak{t}$. There are nonzero homogeneous polynomials $f_d \in \mathbb{Q}[\mathfrak{l}]$, indexed by a finite set of positive degrees, and nonnegative rational numbers c_d such that*

$$f_d(\operatorname{Ad}_b x) = (-2\rho_L)(b) f_d(x), \qquad \sum_d c_d = 1, \qquad \sum_d d c_d = n. \tag{3.3}$$

For some rational $\eta > 0$, the pair

$$\left(\mathfrak{l}, \sum_d c_d \operatorname{div}(f_d) + \eta \operatorname{div}(\delta)\right) \tag{3.4}$$

is log canonical.

Proof. Let J be the set of simple roots of L. Since G is simply connected, its fundamental weights are characters of T; take λ to be the sum of the fundamental weights indexed by J. Its pairing with every simple Levi coroot is one, so its restriction to the derived torus is ρ_L. The highest-weight theorem [9, Theorem 22.2] supplies an irreducible L-representation V_λ, which we choose over the fixed split rational form. Let v_+ be a rational highest-weight vector and let ℓ_- be a rational functional extracting the lowest-weight component. If $w_{0,L}$ is the longest element of the Levi Weyl group, then $w_{0,L}\lambda - \lambda = -2\rho_L$: the difference $\lambda - \rho_L$ is fixed by that Weyl group and $w_{0,L}\rho_L = -\rho_L$.

For $1 \le d \le d_{\max} := \dim V_\lambda - 1$, set

$$f_d(x) = \frac{1}{d!}\ell_-(x^d v_+), \tag{3.5}$$

where x acts through the differential of the representation. Discard any polynomial that is identically zero. For $b \in B_L$, the extremal weight properties give $b^{-1}v_+ = \lambda(b)^{-1}v_+$ and $\ell_- \circ b = (w_{0,L}\lambda)(b)\ell_-$. Substitution in (3.5) gives

$$f_d(\mathrm{Ad}_b\, x) = (w_{0,L}\lambda - \lambda)(b) f_d(x) = (-2\rho_L)(b) f_d(x).$$

This proves the first equality in (3.3), while (3.5) proves homogeneity of degree d.

Let $\mathfrak{u}_L$ be the nilpotent radical of $\mathrm{Lie}(B_L)$, and let $\mathfrak{u}_L^-$ be the opposite nilpotent subalgebra. Choose root coordinates z_α on $\mathfrak{u}_L^-$, for $\alpha \in \Phi_L^+$. On this subalgebra, the polynomial

$$f := \sum_d f_d = \ell_-(\exp(x)v_+) \tag{3.6}$$

is the extremal matrix coefficient on the opposite big cell of L/B_L: nilpotence on V_λ makes the exponential a sum through degree $d_{\max}$, and the constant term is zero because the two extremal weights are distinct. The line bundle involved restricts to the line bundle of weight ρ_L on the flag variety of the simply connected cover of the derived group; that flag variety is isomorphic to L/B_L. Choose the integral Weyl-module form of this representation, so its reductions give the Weyl modules in Lemma 3.4. After inverting finitely many integers, the polynomials and the exponential isomorphism $\mathfrak{u}_L^- \cong U_L^-$ are defined over the resulting localization of $\mathbb{Z}$. Here the characteristic-zero exponential isomorphism is [9, Proposition 14.32], and its inverse and the identities defining it spread out after clearing finitely many denominators.

Lemma 3.4 implies that, for every sufficiently large prime p, the coefficient

$$\left[\prod_{\alpha\in\Phi_L^+} z_\alpha^{p-1}\right] f^{p-1} \tag{3.7}$$

is nonzero modulo p. Indeed, that lemma identifies the splitting on the opposite big cell with a nonzero scalar multiple of f^{p-1}. Under the exponential isomorphism, the canonical-bundle trivialization used for this splitting differs from $\bigwedge_\alpha dz_\alpha$ by a nonzero constant, because both trivializations are nowhere zero on an affine space and its only units are constants. Thus the Frobenius trace of f^{p-1} is a nonzero constant. The trace formula [1, Theorem 1.3.8] identifies its constant term with the coefficient in (3.7), up to taking a pth root. This proves the required nonvanishing in exponential coordinates.

Expanding f^{p-1}, we may choose nonnegative integers m_d for which

$$F_p := \prod_d f_d^{m_d}, \qquad \sum_d m_d = p-1, \tag{3.8}$$

has nonzero coefficient at $\prod_\alpha z_\alpha^{p-1}$ after restriction to $\mathfrak{u}_L^-$. Since F_p is homogeneous, this also gives

$$\sum_d d m_d = n(p-1). \tag{3.9}$$

We next include the invariant polynomial δ. Write

$$\mathfrak{l} = \mathfrak{u}_L^- \oplus \mathfrak{t} \oplus \mathfrak{u}_L,$$

with coordinates z_α, $h_1, \dots, h_s$, and y_α, respectively, where $s = \dim \mathfrak{t}$, and set $\mathfrak{b}_L^- = \mathfrak{t} \oplus \mathfrak{u}_L^-$. Choose a cocharacter pairing negatively with every simple Levi root, by taking a sufficiently divisible negative sum of the fundamental Levi coweights. It fixes $\mathfrak{t}$ and contracts $\mathfrak{u}_L^-$ as its parameter tends to zero. Invariance of δ then gives $\delta(h + z) = \delta(h)$, so $\delta|_{\mathfrak{b}_L^-}$ is the pullback of $\delta|_{\mathfrak{t}}$. Choose a monomial

$$c h_1^{a_1} \cdots h_s^{a_s}$$

with nonzero coefficient in $\delta|_{\mathfrak{t}}$. Take p larger than all the a_i and avoiding the numerator and denominator of c, and set

$$Q_p = \prod_{\alpha \in \Phi_L^+} y_\alpha^{p-1} \prod_{i=1}^{s} h_i^{p-1-a_i}.$$

The coefficient of

$$\prod_\alpha z_\alpha^{p-1} \prod_i h_i^{p-1} \prod_\alpha y_\alpha^{p-1} \tag{3.10}$$

in $Q_p F_p \delta$ is nonzero. Indeed, the y-exponents already supplied by Q_p force us to restrict $F_p\delta$ to $\mathfrak{b}_L^-$. There δ depends only on h. The required z-degree is $n(p-1)$, which is the whole degree of F_p by (3.9). Thus only $F_p|_{\mathfrak{u}_L^-}$ contributes, and the coefficient is the product of c and the nonzero coefficient chosen in (3.8).

The total degree of $Q_p F_p \delta$ is $(2n+s)(p-1)$, by (3.9) and the definition of Q_p. For the polynomial ring $S = \overline{\mathbb{F}}_p[\mathfrak{l}]$, let $\mathrm{Tr}\colon F_*S \to S$ be the Frobenius trace with respect to the chosen coordinate volume form; here F_*S has scalar action $a \cdot u = a^p u$. By [1, Theorem 1.3.8], this trace kills a monomial unless every exponent is congruent to $p-1$ modulo p, and sends (3.10) to 1. The degree equality leaves only that monomial, so $\mathrm{Tr}(Q_p F_p \delta)$ is a nonzero constant. Rescale Q_p to make this constant 1. This gives a splitting for the pair

$$\left(\mathfrak{l}_{\overline{\mathbb{F}}_p}, \frac{1}{p-1} \operatorname{div}(F_p \delta) \right). \tag{3.11}$$

More explicitly, put $H_p = F_p\delta$. The S-linear map

$$F_*(H_p^{-1} S) \longrightarrow S, \qquad u \longmapsto \mathrm{Tr}(Q_p H_p u)$$

sends 1 to 1 and retracts $S \to F_*(H_p^{-1}S)$, $a \mapsto a^p$. Its iterates retract the analogous inclusions for F^e with pole divisor $\frac{p^e-1}{p-1} \operatorname{div}(H_p)$. Since this divisor is exactly (p^e-1) times the boundary in (3.11), the pair is F-pure in the sense of [7, Definition 2.1].

It remains to justify the passage to characteristic zero, since the integers m_d depend on p. Before choosing p, fix a log resolution of

$$\operatorname{div}\left(\delta \prod_d f_d \right)$$

over a number field. Spread the morphism, its inverse over a dense open set, and its finitely many boundary components out over a localization of the ring of integers. After further localization, the morphism remains proper and birational, its source remains smooth, and its total boundary has relative simple normal crossings. Moreover, the divisor equalities for the pullbacks of the finitely many f_d, of δ, and for the relative canonical divisor persist on each fiber. These are precisely the resolution and divisor data spread out in [7, proof of Theorem 3.7]. Choose a sufficiently large good residue characteristic p, and only then choose the integers m_d in (3.8).

Hara–Watanabe's discrepancy theorem in fixed positive characteristic, [7, Theorem 3.3], applies locally to the F-pure pair (3.11); its ambient space is smooth and its boundary is rational Cartier.

It gives log discrepancies at least zero for the exceptional divisors of the reduced resolution, while [7, Proposition 2.2(4)] gives the same inequality for the strict transforms of boundary components. Writing k_E for the relative canonical coefficient and v_E for the component order, each inequality is

$$1 + k_E - \sum_d \frac{m_d}{p-1} v_E(f_d) - \frac{1}{p-1} v_E(\delta) \geq 0. \tag{3.12}$$

The integers k_E, $v_E(f_d)$, and $v_E(\delta)$ in (3.12) were preserved by the preceding spread. Keeping the chosen coefficients $m_d/(p-1)$ and $1/(p-1)$, the same inequality therefore holds on the fixed characteristic-zero resolution. The log resolution criterion [7, Remark 3.2(1)] proves that (3.4) is log canonical for

$$c_d = \frac{m_d}{p-1}, \qquad \eta = \frac{1}{p-1}.$$

Equations (3.8) and (3.9) give the remaining equalities in (3.3). □

3.3. **Descent and perturbation of the boundary.** We first fix an invariant polynomial detecting the regular semisimple locus. Let δ_G be $(-1)^{|\Phi_G^+|}$ times the coefficient of $t^{\operatorname{rk} G}$ in $\det(t - \operatorname{ad} x)$. This definition makes δ_G homogeneous, invariant, and defined over the fixed rational form; the root decomposition gives

$$\delta_G|_{\mathfrak{t}} = \prod_{\alpha \in \Phi_G^+} \alpha^2.$$

For the Jordan decomposition $x = x_s + x_n$, the commuting nilpotent operator $\operatorname{ad} x_n$ does not change the characteristic polynomial of $\operatorname{ad} x_s$. Thus $\delta_G(x) = \delta_G(x_s)$, and the displayed product is nonzero precisely when x_s is regular. In that case x_n belongs to the toral centralizer of x_s and is nilpotent, so $x_n = 0$. This proves that the nonvanishing locus of δ_G is exactly the regular semisimple locus.

Choose a cocharacter $\gamma\colon \mathbb{G}_m \to Z(L)^\circ$ with strictly positive weights on $\mathfrak{u}_P$: a sufficiently divisible sum of the fundamental coweights indexed by the simple roots outside L has these properties. If $P = G$, take $\gamma = 0$. Since γ fixes $\mathfrak{l}$ and contracts $\mathfrak{u}_P$, invariance gives

$$\delta_G(x_l + x_u) = \lim_{t \to 0} \delta_G(\operatorname{Ad}_{\gamma(t)}(x_l + x_u)) = \delta_G(x_l). \tag{3.13}$$

Set $\delta = \delta_G|_{\mathfrak{l}}$, which is nonzero on $\mathfrak{t}$ by the product formula. Let $Z^{\mathrm{rs}} \subseteq Z$ be the inverse image under q of the regular semisimple lines.

Lemma 3.6. *There is an effective rational divisor D on Y such that*

$$D \sim_{\mathbb{Q}} R + n\xi, \tag{3.14}$$

the pair (Y, D) is log canonical, and every log canonical center of (Y, D) meets $\pi^{-1}(Z^{\mathrm{rs}})$.

Proof. Suppose first that $n > 0$, and apply Lemma 3.5 to the polynomial δ just constructed. Pull its polynomials f_d back through $\mathfrak{p} \to \mathfrak{l}$. This projection is B-equivariant for the quotient $B \to B_L$, so the pulled-back polynomials remain semi-invariants of character $-2\rho_L$. Their semi-invariance and homogeneity imply that their zero divisors descend along the smooth quotient

$$G \times (\mathfrak{p} \setminus \{0\}) \longrightarrow Y$$

to divisors D_d with

$$D_d \sim R + d\xi.$$

Indeed, $\mathcal{O}_Y(-1)$ is the tautological line in the bundle with fiber $\mathfrak{p}$. The polynomial f_d gives a degree-d evaluation map to the line of character $-2\rho_L$, which is $\mathcal{O}_Y(R)$ by Lemma 3.3. Thus f_d defines a section of $\mathcal{O}_Y(d) \otimes \mathcal{O}_Y(R)$, proving the displayed class. Put $D = \sum_d c_d D_d$. Equation (3.3) gives (3.14).

Let D_δ be the divisor on Y induced by $\delta_G|_{\mathfrak{p}}$. Invariance of δ_G and the definition of q show that it is the pullback under $q\pi$ of the hypersurface defined by δ_G. The map $G \times \mathfrak{p} \to \mathfrak{l}$ is smooth and surjective, since it is a product projection followed by a vector-space projection. Pull back the pair

in Lemma 3.5 along this map and restrict to $G \times (\mathfrak{p} \setminus \{0\})$; it remains log canonical by Lemma 3.2(1) and the definition of log canonicity on an open set. The quotient to Y is a torsor under the smooth group $B \times \mathbb{G}_m$, so smooth descent in Lemma 3.2(1), together with (3.13), gives

$$(Y, D + \eta D_\delta) \quad \text{log canonical.} \tag{3.15}$$

Dropping the effective summand ηD_δ in (3.15) proves that (Y, D) is log canonical. If a log canonical center of (Y, D) were contained in D_δ, a prime divisor E defining it would have $\operatorname{ord}_E(D_\delta) > 0$. Its log discrepancy for (3.15) would be $0 - \eta \operatorname{ord}_E(D_\delta) < 0$, a contradiction. Since $Y \setminus \operatorname{Supp} D_\delta = \pi^{-1}(Z^{\mathrm{rs}})$, this proves the assertion about centers.

If $n = 0$, then L is a torus, $P = B$, $Y = Z$, and $R = 0$. Take $D = 0$. The smooth pair $(Y, 0)$ is klt and has no log canonical centers, so all the assertions hold. □

Proposition 3.7. *The variety $Y = \mathbb{P}_{G/B}(E^*)$ is of Fano type.*

Proof. Choose D as in Lemma 3.6 and an ample Cartier divisor H on Z. For sufficiently large m, the linear system $|mA - H|$ has no base points on Z^{rs}, as we now verify. The coherent sheaf $q_*\mathcal{O}_Z(-H)(m)$ is globally generated for m sufficiently large by [13, Tag 01Q3, Proposition 28.27.13(7)]. Over the regular semisimple open set, q is finite by Lemma 3.3, so the evaluation $q^*q_*\mathcal{O}_Z(-H) \to \mathcal{O}_Z(-H)$ is surjective there: on affine charts it is the surjection $B \otimes_A N \to N$. The projection formula and $A = q^*\mathcal{O}(1)$ now identify these sections with sections of $\mathcal{O}_Z(mA - H)$, proving the assertion.

There are finitely many log canonical centers of (Y, D) by Lemma 3.2(2), and each meets $\pi^{-1}(Z^{\mathrm{rs}})$ by Lemma 3.6. For each center, sections vanishing identically on its image form a proper linear subspace, by the base-point assertion just proved. Avoiding these finitely many subspaces, choose

$$\Gamma \in |mA - H|$$

so that $\pi^*\Gamma$ contains none of those centers. Fix a log resolution of $D + \pi^*\Gamma$, and let E run over its exceptional divisors and the strict transforms of this boundary. Put $a_E = a_E(Y, D)$ and $b_E = \operatorname{ord}_E(\pi^*\Gamma)$. If $a_E = 0$, the center of E is a log canonical center, so the choice of Γ gives $b_E = 0$. For the finitely many remaining E with $b_E > 0$, choose a positive rational t with $2tb_E \le a_E$, and also require $tm < r - n$, which is possible by Lemma 3.3. The log resolution criterion then gives

$$(Y, D + 2t\pi^*\Gamma) \text{ log canonical}, \qquad tm < r - n. \tag{3.16}$$

For a rational number $0 < \epsilon < 1/2$, set

$$\Delta = (1 - \epsilon)D + t\pi^*\Gamma.$$

By Lemma 3.2(3), the pair obtained from (3.16) by multiplying its boundary by $1 - \epsilon$ is klt. The inequality

$$\Delta \le (1 - \epsilon)(D + 2t\pi^*\Gamma)$$

and (3.16) show that (Y, Δ) is klt. Moreover, (3.1) and (3.14) give

$$-(K_Y + \Delta) \sim_{\mathbb{Q}} \pi^*((r - n - tm)A + tH) + \epsilon D. \tag{3.17}$$

Put $C = (r - n - tm)A + tH$. This divisor is ample because A is globally generated, H is ample, and both coefficients are positive; after clearing denominators this is [13, Tag 0890]. Also, D is π-ample by (3.14) and Lemma 3.3. Thus $D + k\pi^*C$ is ample for a sufficiently large integer k, by [13, Tag 0892(1)], after clearing denominators. For $0 < \epsilon < \min(1/2, 1/k)$ we have

$$\pi^*C + \epsilon D = \epsilon(D + k\pi^*C) + (1 - \epsilon k)\pi^*C.$$

The first summand is ample and the second is semiample, so their sum is ample by the same ample-plus-globally-generated criterion. Equation (3.17) therefore proves that $-(K_Y + \Delta)$ is ample, completing the proof. □

Proof of Theorem 3.1. Lemma 2.2 extends the cotangent symplectic form to the smooth locus of $\overline{T^*(G/U_P)}$. The Killing-form calculation preceding (2.1) identifies the bundle $G \times^B \mathfrak{u}_P^\perp$ in [2, Theorem 1.6] with E. Proposition 3.7 and [2, Theorem 1.6(1.6.2)] therefore imply that $\overline{T^*(G/U_P)}$ has symplectic singularities. This criterion applies to the canonical cotangent form: its proof uses the extension and rational-singularity criterion in [2, Proposition 3.4 and its proof]. Hence the form from Lemma 2.2 extends to a resolution, as asserted. □

4. Terminal singularities

We now use Theorem 3.1 to prove the second singularity assertion.

Theorem 4.1. *The singular locus of $\overline{T^*(G/U_P)}$ has codimension at least four. Consequently, $\overline{T^*(G/U_P)}$ has terminal singularities.*

Only Theorem 3.1 from the preceding section will be needed here. Let $K = G \times L$, and regard the right L-action on G/U_P as a left action by taking inverses. In the model (2.1), the cotangent action is

$$(g_0, l) \cdot [g, x] = [g_0 g l^{-1}, \mathrm{Ad}_l\, x]. \tag{4.1}$$

The corresponding left action on functions is $(k \cdot f)(m) = f(k^{-1}m)$. For $v \in \mathrm{Lie}(K)$, write $v \cdot f$ for the derivative of this representation, and use v_M and v_{G/U_P} for the resulting vector-field derivations on M and G/U_P.

Let θ be the canonical one-form on M, and put $\omega = -d\theta$. The cotangent action preserves θ. Thus Cartan's identity gives

$$\iota_{v_M}\omega = d(\iota_{v_M}\theta) = dh_v, \qquad h_v(q, \alpha) = \alpha(v_{G/U_P}(q)).$$

Use the Poisson-bracket convention $\iota_{\{f,-\}}\omega = df$. The displayed identity then gives

$$v \cdot f = \{h_v, f\}. \tag{4.2}$$

Since the action preserves ω, it preserves this bracket. Its action on global functions therefore defines a Poisson action on $\overline{T^*(G/U_P)}$.

4.1. **An invariant cohomology vanishing.** We begin with the coherent cohomology group that controls first-order Poisson deformations of M.

Lemma 4.2. *The invariant subspace $H^1(M, \mathcal{O}_M)^K$ is zero.*

Proof. The map $G \times \mathfrak{p} \to M$ in (2.1) is an affine faithfully flat U_P-torsor. Every term of its Čech nerve is the affine scheme $G \times \mathfrak{p} \times U_P^a$. The Čech spectral sequence and comparison with flat cohomology therefore show that its cochain complex computes $H^\bullet(M, \mathcal{O}_M)$; see [13, Tags 03OW and 03P2]. If

$$I = \ker(\mathbb{C}[U_P] \longrightarrow \mathbb{C})$$

is the augmentation ideal, removing the degenerate cochains gives the normalized complex with degree-a term

$$\mathbb{C}[G] \otimes \mathbb{C}[\mathfrak{p}] \otimes I^{\otimes a}.$$

The left G-action is on the first factor. Invariants of rational G-modules form an exact functor because G is reductive in characteristic zero; see [9, Theorem 12.56 and Proposition 15.15]. Since the only left-invariant functions on G are constants, $H^1(M, \mathcal{O}_M)^G$ is therefore the first cohomology of the complex with degree-a term

$$\mathbb{C}[\mathfrak{p}] \otimes I^{\otimes a}. \tag{4.3}$$

If $P = G$, then $I = 0$ and the assertion follows directly. Otherwise take a sufficiently divisible sum of the fundamental coweights indexed by the simple roots outside L. This defines a cocharacter $\gamma\colon \mathbb{G}_m \to Z(L)^\circ$: it pairs to zero with every Levi root and positively with every root of $\mathfrak{u}_P$. Its adjoint

weights are therefore zero on $\mathfrak{l}$ and strictly positive on $\mathfrak{u}_P$, so its weights on $\mathbb{C}[\mathfrak{p}]$ are nonpositive. On the nerve, (4.1) acts on the U_P coordinates by conjugation by l. The L-equivariant exponential isomorphism $\mathfrak{u}_P \cong U_P$ from [9, Proposition 14.32] identifies I with polynomials of positive degree on $\mathfrak{u}_P$. Hence every γ-weight of I is strictly negative. The degree-one term of (4.3), and therefore its cohomological subquotient $H^1(M, \mathcal{O}_M)^G$, has only strictly negative γ-weights. It has no L-invariant vector, which proves the lemma. □

4.2. **First-order Poisson deformations.** Let $\mathbb{D} = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. For a Poisson variety V, write $\mathrm{PD}_V(\mathbb{D})$ for the vector space of isomorphism classes of flat Poisson deformations over $\operatorname{Spec}\mathbb{D}$ with an identification of their special fiber with V. An isomorphism is required to respect that identification.

Lemma 4.3. *There is an isomorphism*

$$\Phi\colon H^1(M, \mathcal{O}_M) \xrightarrow{\sim} \mathrm{PD}_M(\mathbb{D})$$

with the following description. Choose an affine open cover $\{W_i\}$ of M, and represent a class by a Čech cocycle (ϕ_{ij}). Then $\Phi([\phi])$ is obtained by gluing the trivial Poisson deformations of the W_i with transition maps on functions

$$g_{ij} = 1 + \varepsilon\{\phi_{ij}, -\}, \qquad g_{ij}\colon \mathcal{O}_{W_i\cap W_j} \otimes \mathbb{D} \longrightarrow \mathcal{O}_{W_i\cap W_j} \otimes \mathbb{D}, \tag{4.4}$$

where g_{ij} maps functions in the jth trivialization to functions in the ith.

Proof. Since M is separated, intersections in an affine open cover are affine; hence this cover computes coherent cohomology. Write $H_\phi = \{\phi, -\}$ and use the Čech differential with $(\check{\delta}\psi)_{ij} = \psi_j - \psi_i$ in degree zero. The Jacobi identity makes H_ϕ a Poisson derivation, so the maps in (4.4) are Poisson automorphisms. The cocycle identity $\phi_{ij} + \phi_{jk} = \phi_{ik}$ and $\varepsilon^2 = 0$ give $g_{ij}g_{jk} = g_{ik}$. Replacing ϕ_{ij} by $\phi_{ij} + \psi_j - \psi_i$ gives an isomorphic deformation via the local coordinate changes $1 - \varepsilon H_{\psi_i}$. Refinement does not change the glued deformation. Thus (4.4) defines Φ on $H^1(M, \mathcal{O}_M)$.

For a smooth symplectic variety, first-order Poisson deformations are classified by the degree-two hypercohomology of the truncated algebraic de Rham complex; see [11, Propositions 8 and 9]. Thus

$$\mathrm{PD}_M(\mathbb{D}) \cong \mathbb{H}^2(M, \Omega_M^{\geq 1}).$$

The relevant part of the exact sequence for

$$0 \longrightarrow \Omega_M^{\geq 1} \longrightarrow \Omega_M^\bullet \longrightarrow \mathcal{O}_M \longrightarrow 0$$

is

$$H^1_{\mathrm{dR}}(M) \longrightarrow H^1(M, \mathcal{O}_M) \longrightarrow \mathbb{H}^2(M, \Omega_M^{\geq 1}) \longrightarrow H^2_{\mathrm{dR}}(M).$$

Under the Čech description in the proof of [11, Proposition 8], the deformation (4.4) has transition derivations $H_{\phi_{ij}}$ and zero local deformation of the bracket. Choose the sign of the symplectic comparison in [11, Proposition 9] so that H_ϕ corresponds to $d\phi$. With the total-complex differential $d + (-1)^q\check{\delta}$ on forms of degree q, the corresponding hypercohomology class is represented by $(d\phi_{ij})$, with zero two-form entries. This is exactly the connecting class of (ϕ_{ij}) in the displayed exact sequence. It therefore suffices to prove that its two outer terms vanish.

The vector bundle $M \to G/U_P$ has contractible fibers. The torsor $G \to G/U_P$ is locally trivial by [9, Proposition 16.55], and its fiber is an affine space by [9, Proposition 14.32]. These two locally trivial bundles with contractible fibers identify the singular cohomology of M^{an} with that of $G(\mathbb{C})$. By algebraic de Rham comparison [6, Theorem 1′], it suffices to show that $H^1_{\mathrm{dR}}(G) = H^2_{\mathrm{dR}}(G) = 0$.

Here is an algebraic verification of this last vanishing. Since G is affine, its algebraic de Rham cohomology is computed by global differential forms. Cartan's formula $\mathcal{L}_v = d\iota_v + \iota_v d$ makes the infinitesimal left G-action on this cohomology zero. The action is rational, and G is connected, so it is trivial: in a finite-dimensional rational submodule, the stabilizer of a vector killed by $\mathrm{Lie}(G)$ has the full Lie algebra and hence is all of G. Exactness of invariants, used in Lemma 4.2, now identifies

$H^\bullet_{\mathrm{dR}}(G)$ with the cohomology of the complex of left-invariant forms, namely the Lie algebra cochain complex $\bigwedge^\bullet \mathfrak{g}^*$.

In degree one this complex has zero cohomology because $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$: root vectors and their coroot brackets span the semisimple Lie algebra $\mathfrak{g}$. For completeness, let c be a degree-two cocycle and write $c(x,y) = \kappa(Dx,y)$, using the nondegenerate Killing form κ. The cocycle identity and invariance of κ give

$$\kappa(D[x,y],z) = \kappa([Dx,y]+[x,Dy],z),$$

so D is a derivation. Choose $a \in \mathfrak{g}$ with $\kappa(a,z) = \operatorname{tr}(D \operatorname{ad} z)$. The derivation identity $[D, \operatorname{ad} x] = \operatorname{ad}(Dx)$ and cyclicity of trace give

$$\kappa(Dx,y) = \operatorname{tr}([D,\operatorname{ad} x]\operatorname{ad} y) = \operatorname{tr}(D \operatorname{ad}[x,y]) = \kappa(a,[x,y]).$$

Thus c is a coboundary, that of $z \mapsto -\kappa(a,z)$, proving the degree-two vanishing. Together with the cohomology comparison above, this proves

$$H^1_{\mathrm{dR}}(M) = H^2_{\mathrm{dR}}(M) = 0.$$

The connecting map is consequently an isomorphism. Its Čech description above proves that Φ is an isomorphism as well. □

Lemma 4.4. *Restriction induces an injective map*

$$\mathrm{PD}_{\overline{T^*(G/U_P)}}(\mathbb{D}) \longrightarrow \mathrm{PD}_M(\mathbb{D}).$$

Proof. Any nilpotent thickening of an affine scheme is affine by [13, Tag 06AD]. Let $\mathcal{A}$ be the flat $\mathbb{D}$-algebra of a deformation of $\overline{T^*(G/U_P)}$, and let $\mathcal{M} \subseteq \operatorname{Spec}\mathcal{A}$ be the open set with underlying topological space M. Its special fiber is M because a nilpotent thickening does not change the underlying topological space. Tensoring $0 \to (\varepsilon) \to \mathbb{D} \to \mathbb{C} \to 0$ with the flat module $\mathcal{A}$ gives

$$0 \longrightarrow A_P \xrightarrow{\varepsilon} \mathcal{A} \longrightarrow A_P \longrightarrow 0.$$

On $\mathcal{M}$, it gives $0 \to \mathcal{O}_M \to \mathcal{O}_\mathcal{M} \to \mathcal{O}_M \to 0$. The map $\Gamma(\mathcal{M},\mathcal{O}_\mathcal{M}) \to A_P$ is surjective, since every element of A_P lifts to $\mathcal{A}$. Taking global sections therefore yields another short exact sequence with outer terms A_P. The restriction map between these two sequences is the identity on the outer terms. Hence

$$\mathcal{A} \xrightarrow{\sim} \Gamma(\mathcal{M},\mathcal{O}_\mathcal{M}).$$

This is an isomorphism of Poisson algebras when the deformation is Poisson. An isomorphism between the restrictions of two deformations therefore induces an isomorphism between their algebras of global functions, with the prescribed special-fiber identification. This proves injectivity. □

Proposition 4.5. *Every first-order Poisson deformation of $\overline{T^*(G/U_P)}$ is trivial.*

Proof. Let $\mathcal{A}$ be the flat Poisson $\mathbb{D}$-algebra of a deformation of $\overline{T^*(G/U_P)}$, and restrict it to the deformation $\mathcal{M}$ of M in Lemma 4.4. By Lemma 4.3, choose an affine cover $\{W_i\}$ and a cocycle (ϕ_{ij}) for which $\mathcal{M}$ is the gluing (4.4).

Fix $v \in \mathrm{Lie}(K)$. The regular Hamiltonian h_v belongs to A_P and lifts to an element $\widetilde{h}_v \in \mathcal{A}$, by the surjection in the proof of Lemma 4.4. In the chosen local trivializations its restrictions have the form $h_v + \varepsilon a_i$, with $a_i \in \mathcal{O}_M(W_i)$. Compatibility with (4.4) gives

$$h_v + \varepsilon a_i = g_{ij}(h_v + \varepsilon a_j) = h_v + \varepsilon\big(a_j + \{\phi_{ij}, h_v\}\big).$$

By (4.2), it follows that

$$v \cdot \phi_{ij} = \{h_v, \phi_{ij}\} = a_j - a_i. \tag{4.5}$$

We check that the left side computes the derivative of the actual K-action on $H^1(M,\mathcal{O}_M)$. The tangent vector v defines a point of K over $\mathbb{C}[\tau]/(\tau^2)$ reducing to the identity. Its action on functions is $1 + \tau(v \cdot -)$ by the convention preceding (4.2). The underlying topological map is the identity, so

it preserves every W_i and acts on the Čech complex by applying this same operator to each entry. Thus the induced derivative on cohomology sends $[\phi]$ to $[(v \cdot \phi_{ij})]$. Equation (4.5) identifies this cocycle with the coboundary of (a_i), so the derivative vanishes on $[\phi]$.

The group $H^1(M, \mathcal{O}_M)$ is a rational K-module, since it is the cohomology of the rational K-equivariant Čech complex in Lemma 4.2. Because K is connected, the stabilizer argument in the proof of Lemma 4.3 identifies its Lie algebra invariants with its group invariants. The preceding paragraph therefore places $[\phi]$ in $H^1(M, \mathcal{O}_M)^K$, which is zero by Lemma 4.2. Lemma 4.3 now makes $\mathcal{M}$ trivial, and injectivity in Lemma 4.4 makes the original deformation trivial. □

4.3. **Codimension of the singular locus.** We isolate the consequence of Namikawa's deformation theory used in the final step.

Lemma 4.6. *Let V be an affine variety with symplectic singularities. If $\mathrm{PD}_V(\mathbb{D}) = 0$, then its singular locus has codimension at least four.*

Proof. We use the numbering of arXiv version 7 of [12]; its Theorem (5.1) is Theorem (4.1) in version 6. By [12, §(1.8)], there is an open subset $V^\circ \subseteq V$, with complement of codimension at least four, whose singularities are analytically products of Du Val surface singularities and smooth spaces. The same section identifies

$$\mathrm{PD}_V(\mathbb{D}) \cong \mathrm{PD}_{V^\circ}(\mathbb{D}).$$

Let $\widetilde{V}^\circ \to V^\circ$ be its minimal resolution, let e be the number of irreducible exceptional divisors, and put

$$b_2 = \dim_{\mathbb{C}} H^2((V^\circ)^{\mathrm{an}}, \mathbb{C}), \qquad t = \dim_{\mathbb{C}} \mathrm{PD}_{V^\circ}(\mathbb{D}), \qquad \widetilde{t} = \dim_{\mathbb{C}} \mathrm{PD}_{\widetilde{V}^\circ}(\mathbb{D}).$$

Part (ii) of the proof of [12, Theorem (5.1)] gives

$$\widetilde{t} = b_2 + e, \qquad t \le \widetilde{t}.$$

To explain why equality holds, let $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be the complete local rings of the deformation hulls for V° and $\widetilde{V}^\circ$, respectively, as in part (i) of that proof. Part (iii) shows that $\operatorname{Spec} \widetilde{\mathcal{R}} \to \operatorname{Spec} \mathcal{R}$ has finite closed fiber. Part (iv) proves unobstructedness for $\widetilde{V}^\circ$, so $\widetilde{\mathcal{R}}$ is regular. These facts give the dimension comparison in part (v):

$$\widetilde{t} = \dim \widetilde{\mathcal{R}} \le \dim \mathcal{R} \le t \le \widetilde{t}.$$

Here the first inequality uses the zero-dimensional closed fiber, and the second is the bound of Krull dimension by tangent-space dimension. All inequalities are therefore equalities. Together with the identification from §(1.8), this proves

$$\dim_{\mathbb{C}} \mathrm{PD}_V(\mathbb{D}) = b_2 + e. \tag{4.6}$$

This is an equality of dimensions; it does not assert that the tangent map induced by the resolution is an isomorphism. No $\mathbb{C}^*$-action occurs in the hypotheses of [12, Theorem (5.1)].

If $\mathrm{PD}_V(\mathbb{D}) = 0$, equation (4.6) forces $e = 0$. Any singular point of V° has a nontrivial Du Val transverse slice, whose minimal resolution contains an exceptional curve. Its product with the smooth factor gives an exceptional divisor locally, and its closure gives an exceptional divisor of $\widetilde{V}^\circ \to V^\circ$. This would contradict $e = 0$. Thus V° is smooth, and

$$\mathrm{Sing}(V) \subseteq V \setminus V^\circ.$$

The codimension bound defining V° proves the lemma. □

Proof of Theorem 4.1. Theorem 3.1 and Proposition 4.5 allow us to apply Lemma 4.6 to $\overline{T^*(G/U_P)}$. Hence

$$\operatorname{codim}_{\overline{T^*(G/U_P)}} \mathrm{Sing}(\overline{T^*(G/U_P)}) \ge 4.$$

Finally, [10, Corollary 1] identifies this condition with terminality for symplectic singularities. □

Proof of Theorem 1.1. Combine Proposition 2.1, Theorem 3.1, and Theorem 4.1. □

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