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THE CRITICAL CAPACITY OF THE HOPFIELD MODEL

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Abstract. We prove that the critical capacity of the Hopfield model at zero temperature lies in [0.138198129, 0.13820106]. The best rigorous lower bound was 0.113, and our lower end exceeds the published one-step and two-step replica symmetry breaking values. The capacity is the supremum of the loads at which the free energy at fixed overlap with one pattern has a well. At every positive temperature, we give a variational formula for this free energy and show that binary patterns and Gaussian rows have the same wells, which characterizes the capacity. The proof builds on the Hamilton–Jacobi solution of the model by Ali, Boursier, Chen, Ho, Issa and Mourrat, and on a strict form of their Jensen inequality, uniform in the number of patterns, which we prove.

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1. Introduction

The Hopfield model [12] stores M random patterns $\xi^1, \dots, \xi^M \in \{-1,1\}^N$ in the energy

$$H_N(\sigma) = -\frac{1}{2N} \sum_{\mu \le M} (\xi^\mu \cdot \sigma)^2, \qquad \sigma \in \Sigma_N := \{-1,1\}^N. \tag{1.1}$$

Each pattern is meant to sit at the bottom of a well of H_N, and one asks how many patterns the landscape can hold. We let $M/N \to \alpha$ and call this limit the load, the number of patterns per neuron. In this regime, Amit, Gutfreund and Sompolinsky answered the question with the replica method [2, 3]: these wells exist up to a critical load $\alpha_c(T)$, with $\alpha_c(0) \approx 0.138$. Steffan and Kühn later computed the corrections due to replica symmetry breaking, and found 0.138186 at one step and 0.138187733 at two steps [19]. Rigorous results have remained well below these values. Newman proved an energy barrier around each pattern up to load 0.056

[16], and this was improved to 0.071 by Loukianova [15] and to 0.113 by Feng, Shcherbina and Tirozzi [11].

We study the largest load at which the free energy at fixed overlap with one pattern has a well. Our starting point is a formula for this free energy. For one binary pattern in a bulk of Gaussian rows, it is the minimum of an explicit functional over the law of the overlap path, and this functional is strictly convex, while for binary patterns it is a supremum of such formulas over finitely many retained patterns. This gives a variational characterization of the critical load at every temperature, the same for binary patterns and for Gaussian rows. At zero temperature, we then prove $\underline{\alpha} \le \alpha_{\mathrm{c}}(0) \le \overline{\alpha}$ in ball arithmetic, where $\underline{\alpha} := 0.138198129$ and $\overline{\alpha} := 0.13820106$. The lower end is above both published replica symmetry breaking values (Subsection 1.6). Following [11], we call $\alpha_{\mathrm{c}}(T)$ the critical capacity. It is the threshold for wells in the free energy landscape at fixed overlap: the supremum of the loads at which, for some $0 < m_1 < m_2 < 1$, the limiting free energy at overlap m_2 exceeds the one at overlap m_1. The definition is (1.11), and at $T = 0$ the energy takes the place of the free energy, with $m_2 = 1$ allowed.

Two obstacles stand in the way of a direct argument. The first is that the wells cannot be seen at equilibrium. Above load about 0.05, the replica symmetric computation of [2] places the global minimum of H_N at a spin glass state, and the wells around the patterns become metastable. At load 0.1, for instance, the well around a pattern lies above the global minimum. The capacity therefore cannot be read off the equilibrium free energy, although this free energy is now known [1], and this is why we fix the overlap: a well then appears as a local maximum, visible at every load.

The second obstacle is that at fixed overlap one cannot simply replace all the patterns but one by Gaussian rows, because a state can be correlated with several patterns. Such mixtures do exist, and they raise the free energy at low overlap. Informally, a state of the binary model with overlap m with one pattern and overlaps $\boldsymbol{\omega}$ with others is never better than a state of the Gaussian-row model with the single overlap $V = (m^2 + |\boldsymbol{\omega}|^2)^{1/2}$. This is Jensen's inequality on the sphere of overlap vectors, and its field form is due to [1]. The inequality identifies the maximum of the free energy over all overlaps, which is all that the equilibrium problem sees, but it still allows the free energy at a given overlap to rise up to the maximum over larger overlaps and fill a well. The capacity is therefore decided by a strict form of the inequality, with a gap that does not depend on the number of retained patterns, which we prove. Together with the inequality itself, this strict form shows that binary patterns and Gaussian rows have the same wells.

1.1. **Free energy at fixed overlap.** Let $\xi^1, \dots, \xi^M$ be i.i.d. uniform on Σ_N, with $M/N \to \alpha > 0$, and let $\beta = 1/T > 0$. We write $m_\mu(\sigma) = \xi^\mu \cdot \sigma / N$ for the overlap, which is the fraction of sites where the two configurations agree minus the fraction where they differ. Its possible values are $m_k = 2k/N - 1$, and $S_N(k)$ is the set of σ with exactly k plus signs. The free energy at fixed overlap is

$$p_N(k) := \frac{1}{N} \mathbb{E} \log 2^{-N} \sum_{\sigma \colon m_1(\sigma) = m_k} e^{-\beta H_N(\sigma)}. \tag{1.2}$$

The comparison model keeps one binary pattern and replaces the others by Gaussian rows. With the pattern gauged to the all-ones vector and G an $(M-1) \times N$ matrix of i.i.d. standard Gaussians, its free energy at fixed overlap is

$$p_N^G(k) := \frac{\beta m_k^2}{2} + \frac{1}{N} \mathbb{E} \log Y_{N,k}, \qquad Y_{N,k} := 2^{-N} \sum_{\sigma \in S_N(k)} \exp\Big(\frac{\beta}{2N} |G\sigma|^2 \Big). \tag{1.3}$$

Only the limit of the ratio of rows to columns enters the results, so in later sections we will also use $M \times N$ matrices.

Our formula involves the Parisi functional on a slice of Boursier and Mourrat [6]. Let $\mathcal{Q}_1$ be the set of nondecreasing right-continuous paths $x : [0,1) \to [0,1]$, and $\mathcal{Q}_\infty$ the set of bounded nondecreasing right-continuous paths with values in $\mathbb{R}_+$. For a law ν on $\mathbb{R}_+$ with bounded support, we let ϕ_ν solve

$$-\partial_s \phi_\nu = \partial_x^2 \phi_\nu + \nu([0,s])(\partial_x \phi_\nu)^2, \qquad \phi_\nu(S,\cdot) = \log\cosh, \tag{1.4}$$

with $S \geq \sup \operatorname{supp} \nu$. We set $\mathcal{M}_\nu := \partial_x \phi_\nu(0,\cdot)$, and $\lambda(\nu, m)$ denotes the solution of $\mathcal{M}_\nu(\lambda) = m$. We also put $g_m(\nu) := S + \sup_u\{um - \phi_\nu(0,u)\}$, which does not depend on S. For $a \in \mathcal{Q}_\infty$, we write $g_m(a)$ for g_m at the law of $a(U)$, with U uniform on $[0,1)$. The conjugate is $J_m(x) := \sup_{a\in\mathcal{Q}_\infty}\{\int_0^1 xa - g_m(a)\}$. For $x \in \mathcal{Q}_1$, let

$$\delta_x(w) := \int_0^1 (1 - x(u) \vee w)\,\mathrm{d}u, \qquad D_x(w) := 1 - \beta\delta_x(w), \qquad \mathcal{S}_\beta(x) := \frac{\beta}{2}\int_0^1 \frac{\mathrm{d}w}{D_x(w)} \tag{1.5}$$

if $D_x(0) > 0$, and $\mathcal{S}_\beta(x) := +\infty$ otherwise. For $|m| < 1$, we set

$$\Phi(m) = \min_{x\in\mathcal{Q}_1}\{J_m(x) + \alpha\,\mathcal{S}_\beta(x)\}, \qquad \Lambda(m) = \frac{\beta m^2}{2} + \Phi(m), \tag{1.6}$$

extended continuously to $m = \pm 1$. Since the objective is strictly convex in the law of x, the minimizer x^* is unique. Let $\nu^*(m) = \nu^*(m;\alpha,\beta)$ be the law of $a^* = \frac{\alpha\beta^2}{2}\int_0^{x^*} \mathrm{d}s/D_{x^*}(s)^2$. We then put $h(m) = h(m;\alpha,\beta) := \lambda(\nu^*(m), m)$.

To describe states correlated with several patterns, let $r \geq 1$ and $d_r = 2^{r-1}$, and let η run over $\{-1,1\}^{r-1}$, with $\mathbb{E}_\eta$ the uniform average. For $\boldsymbol{t} = (t_\eta) \in [-1,1]^{d_r}$, we put

$$m = \mathbb{E}_\eta t_\eta, \qquad \boldsymbol{\omega}(\boldsymbol{t}) = \mathbb{E}_\eta[\eta\, t_\eta] \in \mathbb{R}^{r-1}, \qquad V(\boldsymbol{t})^2 = m^2 + |\boldsymbol{\omega}(\boldsymbol{t})|^2, \tag{1.7}$$

and $P_r(m) = \{\boldsymbol{t} \in [-1,1]^{d_r} : \mathbb{E}_\eta t_\eta = m\}$. Given $r-1$ further patterns, the sites split into d_r types according to their values, and t_η is the magnetization on type η. We denote by $\Phi^{(r)}(\boldsymbol{t})$ the limit free energy of the Gaussian rows on the corresponding type slice. Its formula is given in Theorem 2.3, and $\Phi^{(1)} = \Phi$.

Theorem 1.1 (Free energy at fixed overlap)**.** *Let $\alpha, \beta > 0$.*

(a) (Gaussian rows.) $\sup_k |p_N^G(k) - \Lambda(m_k)| \to 0$*, with Λ given by* (1.6)*. Λ is C^1 on $(-1,1)$ with $\Lambda'(m) = \beta m - h(m)$.*

(b) (Binary patterns.) $\sup_k |p_N(k) - \Lambda^{\mathrm{R}}(m_k)| \to 0$*, where*

$$\Lambda^{\mathrm{R}}(m) = \sup_{r\geq 1}\ \sup_{\boldsymbol{t}\in P_r(m)} \mathcal{F}_r(\boldsymbol{t}), \qquad \mathcal{F}_r(\boldsymbol{t}) := \frac{\beta}{2}V(\boldsymbol{t})^2 + \Phi^{(r)}(\boldsymbol{t}). \tag{1.8}$$

(c) (Merging.) *For all r and all $\boldsymbol{t} \in [-1,1]^{d_r}$,*

$$\mathcal{F}_r(\boldsymbol{t}) \leq \Lambda(V(\boldsymbol{t})). \tag{1.9}$$

Hence for $m \in [0,1]$

$$\Lambda(m) \leq \Lambda^{\mathrm{R}}(m) \leq \max_{[m,1]}\Lambda, \qquad \Lambda^{\mathrm{R}}(0) = \max_{[0,1]}\Lambda, \tag{1.10}$$

and $\Lambda^{\mathrm{R}}(m) = \Lambda(m)$ wherever $\Lambda(m) = \max_{[m,1]}\Lambda$. If $0 < m < 1$ and $\Lambda(m) < \max_{[m,1]}\Lambda$, then $\Lambda^{\mathrm{R}}(m) < \max_{[m,1]}\Lambda$.

The free energy at fixed overlap does not reduce to $r = 1$ everywhere: by (1.10), $\Lambda^{\mathrm{R}}(0) > \Lambda(0)$ as soon as a state at positive overlap beats $m = 0$. The last statement of part (c) is what keeps this from changing the wells.

Since Λ is the logarithm of a partition function, a well of the free energy is a local maximum of Λ, and it requires Λ to rise at positive overlap. For $m_s \in (0, 1)$, we define

$$\mathcal{W}_T(m_s) := \big\{\alpha > 0 : \ \Lambda(m_2) > \Lambda(m_1) \text{ for some } m_s \le m_1 < m_2 < 1\big\}, \tag{1.11}$$

and $\mathcal{W}_T^{\mathrm{R}}(m_s)$ is the same set with Λ^{R}. We set $\mathcal{W}_T = \bigcup_{m_s} \mathcal{W}_T(m_s)$, $\alpha_{\mathrm{c}}(T; m_s) = \sup \mathcal{W}_T(m_s)$ and $\alpha_{\mathrm{c}}(T) = \sup \mathcal{W}_T$, with $\sup \emptyset = 0$, and similarly with the superscript R. The point $m = 0$ is a strict local maximum of Λ at every $T > 0$ (Theorem 4.9) and $\Lambda' \to -\infty$ at $m = 1$, so a point where $\Lambda' > 0$ always lies between a local minimum and a local maximum at positive overlap.

1.2. **Energy at fixed overlap.** At zero temperature, the role of the free energy is played by the energy at fixed overlap,

$$e_N(k) := \frac{1}{N}\mathbb{E} \max_{m_1(\sigma) = m_k} (-H_N(\sigma)), \tag{1.12}$$

whose limit is e^{R} for binary patterns and e for Gaussian rows (Theorem 1.2). With this sign, larger values mean lower energy. The set $\mathcal{W}_0(m_s)$ consists of the $\alpha > 0$ with $e(m_2) > e(m_1)$ for some $m_s \le m_1 < m_2 \le 1$, and $\mathcal{W}_0^{\mathrm{R}}(m_s)$ is the same set with e^{R}. The numbers $\alpha_{\mathrm{c}}(0; m_s)$, $\alpha_{\mathrm{c}}(0)$, $\alpha_{\mathrm{c}}^{\mathrm{R}}(0; m_s)$, $\alpha_{\mathrm{c}}^{\mathrm{R}}(0)$ are defined as before, so that $\alpha_{\mathrm{c}}(0) = \sup_{m_s > 0} \alpha_{\mathrm{c}}(0; m_s)$.

The zero-temperature formula involves the following recursion. Let $\mathcal{U}$ be the set of nondecreasing right-continuous $\kappa : [0, 1) \to [0, \infty)$ with $\int_0^1 \kappa < \infty$, and for $\kappa \in \mathcal{U}$ and $C \ge 0$, put $\delta(w) = C + \int_w^1 \kappa$. Let Ψ_κ solve $-\partial_w \Psi_\kappa = \frac{1}{2}(\partial_x^2 \Psi_\kappa + \kappa(w)(\partial_x \Psi_\kappa)^2)$ with $\Psi_\kappa(1, x) = |x|$. This is the recursion at unit scale. In Section 5 we will also use a version scaled by α, and there this one carries a tilde.

Theorem 1.2 (Zero temperature)**.** (a) *The energy at fixed overlap converges uniformly in k: to $e^{\mathrm{R}}(m_k)$ for binary patterns, with $e^{\mathrm{R}} = \lim_{\beta\to\infty} \Lambda^{\mathrm{R}}/\beta$, and to $e(m_k)$ for Gaussian rows, where $e = m^2/2 + L^2/2$ and*

$$L(\alpha, m) = \inf_{\kappa, C, \lambda} \Big\{ \frac{C}{2} + \frac{\alpha}{2} \int_0^1 \frac{\mathrm{d}w}{\delta(w)} + \Psi_\kappa(0, \lambda) - \lambda m \Big\}. \tag{1.13}$$

For $|m| < 1$ the infimum is attained at a unique point. L is jointly concave and C^1 on $(0, \infty) \times (-1, 1)$, with $\partial_m L = -\lambda^*$, and $\partial_m e = m - L\lambda^*$.

(b) *$\mathcal{W}_0^{\mathrm{R}}(m_s) = \mathcal{W}_0(m_s)$ for every $m_s \in (0, 1)$, and $\alpha \in \mathcal{W}_0(m_s)$ if and only if $m > L(\alpha, m)\lambda^*(\alpha, m)$ for some $m \in (m_s, 1)$.*

(c) *$\alpha_{\mathrm{c}}(0; m_s) = \alpha_{\mathrm{c}}^{\mathrm{R}}(0; m_s)$ is at least $\underline{\alpha}$ for every $m_s \in (0, 0.9666236]$ and at most $\overline{\alpha}$ for every $m_s \in (0, 1)$.*

(d) *For $\alpha \ge \overline{\alpha}$, $e^{\mathrm{R}} = e$ on $[0, 1]$ and e^{R} is strictly decreasing on $(0, 1)$. Every $\alpha \in (0, \underline{\alpha}]$ lies in $\mathcal{W}_0^{\mathrm{R}}(m_s)$ for all $m_s \le 0.9616$, so there are no holes below $\underline{\alpha}$.*

(e) *If $e^{\mathrm{R}}(m_2) - e^{\mathrm{R}}(m_1) = \Delta > 0$ with $m_1 < m_2$, and $m_{k_i} \to m_i$, then for large N, with probability at least $1 - e^{-cN}$, for all patterns simultaneously,*

$$\min\{H_N(\sigma) : m_\mu(\sigma) = m_{k_1}\} \ge \min\{H_N(\sigma) : m_\mu(\sigma) = m_{k_2}\} + \frac{\Delta}{3} N. \tag{1.14}$$

(f) *Every limit point of $\alpha_{\mathrm{c}}(T)$ as $T \to 0$ lies in $[\underline{\alpha}, \overline{\alpha}]$, and so does every limit point of $\alpha_{\mathrm{c}}(T; m_s)$ for $m_s \in (0, 0.9666236]$.*

Parts (c), (d) and (f) have computer-assisted proofs, while parts (a), (b) and (e) do not use a computer.

1.3. **The capacity at positive temperature.** The next theorem characterizes the wells at positive temperature and gives them their meaning at finite N.

Theorem 1.3 (Capacity at positive temperature)**.** *Let $T > 0$.*

(a) *$\alpha \in \mathcal{W}_T$ if and only if $\beta m > h(m;\alpha,\beta)$ for some $m \in (0,1)$, if and only if $m < \mathcal{M}_{\nu^*(m)}(\beta m)$ for some $m \in (0,1)$. The stationary points $m > 0$ of Λ are exactly the solutions of the generalized Mattis equation $m = \mathcal{M}_{\nu^*(m)}(\beta m)$.*

(b) *$\mathcal{W}_T$ is open. It is empty for $T \geq 1$. For $T < 1$ it contains an interval $(0, \alpha_0(T))$ with $\alpha_0(T) > 0$. For every T, $\mathcal{W}_T \subset (0,2)$.*

(c) *At a positive boundary point of $\mathcal{W}_T$ there is $m_c \in (0,1)$ with $h(m_c) = \beta m_c$, and $\Lambda' \leq 0$ on $[0,1)$. So the well disappears at positive overlap, with no collapse to $m = 0$ or $m = 1$.*

(d) *If $\alpha \in \mathcal{W}_T(m_s)$, there are $m_s \leq m_b < m_* < 1$ and $\Delta > 0$ such that for large N, with probability at least $1 - e^{-cN}$, for all patterns μ simultaneously,*

$$Z^\mu(k_b) \leq e^{-N\Delta/3}\, Z^\mu(k_*), \qquad Z^\mu(k) := 2^{-N} \sum_{m_\mu(\sigma) = m_k} e^{-\beta H_N(\sigma)}, \tag{1.15}$$

where $m_{k_b} \to m_b$, $m_{k_*} \to m_*$.

(e) *If $\alpha \notin \mathcal{W}_T$, then $\Lambda^{\mathrm{R}} = \Lambda$ on $[0,1]$ and Λ^{R} is nonincreasing there. So above $\alpha_{\mathrm{c}}(T)$ there is no well for binary patterns at any overlap.*

By part (d), a well implies that, around every pattern at once, the slice at overlap m_{k_b} carries exponentially less weight than the slice at the larger overlap m_{k_*}, and (1.14) is the analogue for the minimum energy at zero temperature.

1.4. **Main theorem.** Our main result collects the identity of the well sets with the bounds on the capacity from Theorems 1.2 and 1.3.

Theorem 1.4 (Critical capacity)**.** *Let $\underline{\alpha} = 0.138198129$ and $\overline{\alpha} = 0.13820106$.*

(a) *For every $T > 0$ and $m_s \in (0,1)$, $\mathcal{W}_T^{\mathrm{R}}(m_s) = \mathcal{W}_T(m_s)$: binary patterns and Gaussian rows have the same wells. So $\alpha_{\mathrm{c}}^{\mathrm{R}}(T; m_s) = \alpha_{\mathrm{c}}(T; m_s)$, $\alpha_{\mathrm{c}}^{\mathrm{R}}(T) = \alpha_{\mathrm{c}}(T)$, and*

$$\alpha_{\mathrm{c}}(T) = \sup\bigl\{\alpha > 0 :\ \exists\, m \in (0,1),\ \beta m > h(m;\alpha,\beta)\bigr\}. \tag{1.16}$$

(b) *$\alpha_{\mathrm{c}}(T) = 0$ for $T \geq 1$, and $0 < \alpha_{\mathrm{c}}(T) \leq 2$ for $T < 1$.*

(c) *$\underline{\alpha} \leq \alpha_{\mathrm{c}}(0) = \alpha_{\mathrm{c}}^{\mathrm{R}}(0) \leq \overline{\alpha}$. So the capacity at $T = 0$ is 0.13820 to five digits.*

(d) *At $T = 0$, in both models, there is no well at any overlap for $\alpha \geq \overline{\alpha}$, and every $\alpha \in (0, \underline{\alpha}]$ has a well.*

1.5. **Relation to prior work.** The equilibrium free energy is now known. The Hamilton–Jacobi method of Boursier–Mourrat, Chen–Mourrat and Chen–Ho–Issa [6, 9, 8] handles Ising and spherical spins with a non-convex coupling. Camilli, Contucci and Mingione announced the formula for Gaussian patterns [7], Barra and Durante reduced binary patterns to finitely many retained patterns in a Gaussian bulk [5], and Ali, Boursier, Chen, Ho, Issa and Mourrat proved the limit for all $\alpha, \beta > 0$, with one retained pattern [1].

The fixed-overlap formula, the capacity characterization, the zero-temperature results and the certified bounds are not in [1]. The proofs given here do, however, rest on that preprint, which has not been refereed, mainly in two places. The limit of the mixed model at fixed radius in Section 2 is their Theorem 3.3 with their Proposition 4.1, and we also take from them the facts on the Ising and spherical terms in their Lemmas 2.1 and 2.3 and (4.5). The comparison of Rademacher and Gaussian disorder in Section 3 uses their Lemmas 6.1 and 6.2, their estimate (6.20), whose proof also uses their Lemma 6.3, and their concentration bounds (6.3)–(6.5). We first obtained these two steps without [1], by rerunning the Hamilton–Jacobi argument of [6]

on the mixed slice and by extending the sector comparison of Barra and Durante [5]. That longer version is in the repository given under Code and data, as `paper/long_version.pdf`. The Jensen step behind Lemma 6.1 is due to [1, Prop. 5.2], although the lemma has its own short proof. That lemma, the merging theorems built on it, and the extension of the capacity statements to all overlaps came after we had read [1].

The upper bound (1.9) is not stated in [1], but it follows from their Proposition 5.2 in a few lines (Remark 6.3). They also state that a refinement of their analysis gives a strict inequality at a fixed number of fields, without giving the refinement. Theorem 6.6 proves the strict inequality at a fixed slice with a gap uniform in r, and the identity of the well sets needs this uniformity.

Stojnic [21, 22] studies the capacity through lifted random duality and computes finite-level values of the same kind as those in [19]. Talagrand proved the replica symmetric formula in a region of parameters [23]. Compared with [11], (1.14) implies their event for each pattern, at every load up to $\underline{\alpha}$, with an inner point that is not specified.

1.6. **Comparison with physics.** At $T = 0$, replica symmetry gives 0.137906 [2, 19], and [19] gives 0.1381864895 at one step and 0.138187733 at two steps. The one-step value 0.144 of [10] is still widely quoted, although [19] regards it as an error. Finite-size simulations started from a stored pattern report 0.145 ± 0.009 (Amit, Gutfreund and Sompolinsky, as quoted in [19]), about 0.143 ± 0.001 [14] and 0.141 ± 0.0015 [20]. All of these lie above our interval, but they measure retrieval under a dynamics at finite N, whereas α_{c} measures wells of the limiting profile at fixed overlap.

1.7. **Outline of the proof.** We first treat Gaussian rows at fixed overlap. Writing $\exp(\frac{\beta}{2N}|G\sigma|^2)$ as a Gaussian integral over $z \in \mathbb{R}^{M-1}$ and separating the radius of z from its direction, we obtain an Ising slice coupled bilinearly to a sphere. At fixed radius, this mixed model is solved by the Hamilton–Jacobi theorem of [1], uniformly in the slice, and a Laplace principle in the radius gives the slice formula (Section 2).

To pass from binary patterns to retained patterns, we note that after a gauge a slice is a deterministic weight on the cube, and that the comparison lemmas of [1] between Rademacher and Gaussian disorder hold for any weight. Large hidden coordinates go into the weight, and the others are compared under a cutoff. Convex concentration on the cube then gives a union over all patterns and slices (Section 3), and hence Theorem 1.1(b).

After the sphere and the radius are eliminated in closed form, the slice problem becomes (1.6), which is strictly convex in the law of x. Uniqueness gives differentiability and the multiplier h, and the capacity follows from the shape of Λ (Section 4). Section 5 does the same at zero temperature.

The merging step rests on the fact that $q \mapsto \phi_\nu(0, \sqrt{q})$ is concave for every ν. Applied to the effective fields of a state with several overlaps, Jensen's inequality gives (1.9), and a fourth-moment bound for Rademacher sums gives a strict form that is uniform in r. Together they give Theorem 1.4(a) (Section 6).

Finally, the zero-temperature recursion can be evaluated in ball arithmetic. A trial bounds L from above for all α and a tangent of the sphere problem bounds it from below, and together they certify a well at $\underline{\alpha}$ and $e' < 0$ on $[0.89, 1)$ at $\overline{\alpha}$. A separate bound on the slope of the response, $h' \geq (1+m^2/\mathsf{w}(m)^2)^{1/2}$ almost everywhere, excludes $(0, 0.89)$ (Section 7). Appendix A gives the error terms of the ball arithmetic and describes what was computed.

2. Gaussian rows at fixed magnetization

In this section, we identify the limit of the free energy of Gaussian rows on a magnetization slice, uniformly in the slice. We will use it in Section 3, where the free energy at fixed overlap

for binary patterns is reduced to it, and in Section 4, where the capacity is read off its formula. A radial identity turns the rows into an Ising slice coupled to a sphere. At fixed radius, this mixed model is solved in [1, Theorem 3.3, Proposition 4.1], where the Hamilton–Jacobi method of [6] is extended to it, and a Laplace principle in the radius then yields the formula for Φ.

2.1. **The formula.** Let $\mathcal{Q}$ be the set of right-continuous nondecreasing maps $[0,1) \to \mathbb{R}_+$, which we call paths. For a path p, we set $p(1) := \lim_{r\to 1} p(r)$. As before, $\mathcal{Q}_1$ denotes the set of paths bounded by 1, and $\mathcal{Q}_\infty$ the set of bounded paths. We write $\mathcal{L}_y$ for the law of a path y under Lebesgue measure on $[0,1)$.

We recall that, for a law ν on $\mathbb{R}_+$ with bounded support, ϕ_ν solves (1.4) and $g_m(\nu) = S + \sup_u\{um - \phi_\nu(0,u)\}$ [6, (1.7)–(1.8)]; for $a \in \mathcal{Q}_\infty$, $g_m(a)$ is g_m at the law of a. By [1, Lemma 2.1], g_m is convex and 1-Lipschitz for L^1 on paths, $g_m(0) = I(m) := \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m)$, and

$$\big|g_m(a) - g_{m'}(a) - I(m) + I(m')\big| \le 2|a|_{L^1}|m - m'|. \tag{2.1}$$

At a constant path $c \ge 0$ and at $m = \pm 1$, we have

$$g_m(c) = c + \sup_u\big\{um - \mathbb{E}\log\cosh(u + \sqrt{2c}\,z)\big\}, \qquad g_{\pm 1}(a) = \log 2 + \int_0^1 a, \tag{2.2}$$

with z standard normal; the last value is [6, (6.22)]. The conjugate is $J_m(x) := \sup_{a\in\mathcal{Q}_\infty}\{\int_0^1 xa - g_m(a)\}$ for $x \in \mathcal{Q}_1$ [6, (9.1)], and by [1, (4.5)], for every $a \in \mathcal{Q}_\infty$,

$$g_m(a) = \max_{x\in\mathcal{Q}_1}\Big\{\int_0^1 xa - J_m(x)\Big\}. \tag{2.3}$$

For the sphere, we put, as in [1, (2.10)–(2.11)], $\mathsf{k}(\vartheta) := \vartheta(1) - \int_0^1 \vartheta$ and $\mathsf{k}_\vartheta(r) := \int_0^1(\vartheta(1) - \vartheta(u)\vee r)\,\mathrm{d}u$ for $\vartheta \in \mathcal{Q}_\infty$, and

$$\mathsf{T}(\vartheta, b) := \int_0^{\vartheta(1)} \frac{\mathrm{d}r}{b - 2\mathsf{k}_\vartheta(r)} + \frac{1}{2}(b - 1 - \log b) - \vartheta(1), \qquad \psi_{\mathrm{S}}(\vartheta) := -\inf_{b > 2\mathsf{k}(\vartheta)} \mathsf{T}(\vartheta, b). \tag{2.4}$$

Its conjugate is $\psi_{\mathrm{S}}^*(y) := \sup_{\vartheta\in\mathcal{Q}_\infty}\{\int_0^1 y\vartheta - \psi_{\mathrm{S}}(\vartheta)\}$ for $y \in \mathcal{Q}_1$. Recall from (1.5) that $\delta_y(w) = \int_0^1(1 - y(u)\vee w)\,\mathrm{d}u$.

Lemma 2.1 (Sphere conjugate, [1, Lemma 2.3, (2.19)])**.** *Let $y \in \mathcal{Q}_1$ with $\hat{y} := y(1) < 1$. Then*

$$\psi_{\mathrm{S}}^*(y) = \frac{1}{2}\Big[\int_0^{\hat{y}} \frac{\mathrm{d}w}{\delta_y(w)} + \log(1 - \hat{y})\Big] \ge 0, \tag{2.5}$$

with equality only at $y = 0$. In particular $\psi_{\mathrm{S}}^*(q_0) = \frac{1}{2}\big(\frac{q_0}{1-q_0} + \log(1-q_0)\big)$ for a constant $q_0 \in [0,1)$.

Lemma 2.2 (Ising conjugate at constant paths)**.** *Let $m \in [-1,1]$ and $q_0 \in [0,1)$. Then $J_m(q_0) = \sup_{c\ge 0}\{cq_0 - g_m(c)\} < \infty$ if $|m| < 1$. If $q_0 \le m^2$, then $J_m(q_0) = -I(m)$. If $|m| = 1$, then $J_m = -\log 2$ on all of $\mathcal{Q}_1$. Always $J_m \ge -I(m)$.*

Proof. Let $|m| < 1$ and $c \ge 0$. By [6, Proposition 6.10], the supporting inequality holds at the constant path c, with a constant supporting path $\mathsf{q}(c,m) := \mathbb{E}\tanh^2(u_c + \sqrt{2c}\,z)$, where $\mathbb{E}\tanh(u_c + \sqrt{2c}\,z) = m$, and hence $J_m(\mathsf{q}(c,m)) = c\,\mathsf{q}(c,m) - g_m(c)$ by [6, Lemma 9.1]. The map $c \mapsto g_m(c)$ is convex with derivative $\mathsf{q}(c,m)$ [6, (6.44)]. This derivative is continuous, with $\mathsf{q}(0,m) = m^2$ and $\mathsf{q}(c,m) \to 1$ as $c \to \infty$, since $\mathbb{E}\,\mathrm{sech}^2(u + \sqrt{2c}\,z) \le 2(2\pi)^{-1/2}(2c)^{-1/2}$ for every u. Hence, for $q_0 \in [m^2, 1)$, there is c with $\mathsf{q}(c,m) = q_0$, and $cq_0 - g_m(c) = \sup_{c'}\{c'q_0 - g_m(c')\}$. For $q_0 < m^2$, [6, (6.46)] at the path 0, whose slope is m^2, gives $\int_0^1 q_0 a - g_m(a) \le \int_0^1(q_0 - m^2)a - I(m) \le -I(m)$, with equality at $a = 0$. If $|m| = 1$, we obtain from (2.2) that

$J_{\pm1}(x) = -\log 2 + \sup_a \int_0^1 (x-1)a = -\log 2$. Finally, taking $a = 0$ in the definition gives $J_m \geq -g_m(0) = -I(m)$. □

Theorem 2.3 (Gaussian rows on slices)**.** *Let $r \geq 1$, $d_r = 2^{r-1}$, let $(I_\eta)_{\eta\in\{-1,1\}^{r-1}}$ be a partition of $[N]$ with $|I_\eta|/N \to 1/d_r$, let $M/N \to \alpha > 0$, let G be an $M \times N$ matrix of i.i.d. $N(0,1)$ entries, and let $\beta > 0$. For a slice $k = (k_\eta)_\eta$, $0 \leq k_\eta \leq |I_\eta|$, let $\boldsymbol{t}_k := (2k_\eta/|I_\eta| - 1)_\eta$ and*

$$Y_{N,k} := 2^{-N} \sum_{\sigma:k} \exp\Big(\frac{\beta}{2N}|G\sigma|^2\Big),$$

the sum over $\sigma \in \Sigma_N$ with k_η plus signs on I_η for every η. For $\boldsymbol{t} \in [-1,1]^{d_r}$ and $\gamma \geq 0$ let

$$\Phi_r^{\mathrm{mix}}(\gamma;\boldsymbol{t}) := \inf_{(x_\eta)_\eta,\, y\in\mathcal{Q}_1} \Big\{ \frac{\gamma\alpha}{2}\Big(1 - \mathbb{E}_\eta \int_0^1 x_\eta y\Big) + \mathbb{E}_\eta J_{t_\eta}(x_\eta) + \alpha\psi_{\mathrm{S}}^*(y)\Big\}, \tag{2.6}$$

and, with $\mathrm{rad}(v) = \frac{\alpha}{2}(1 + \log v - v)$,

$$\Phi^{(r)}(\boldsymbol{t}) := \sup_{v>0}\big\{\mathrm{rad}(v) + \Phi_r^{\mathrm{mix}}(\beta v;\boldsymbol{t})\big\}. \tag{2.7}$$

Then:

(a) $\sup_k \big|\frac{1}{N}\mathbb{E}\log Y_{N,k} - \Phi^{(r)}(\boldsymbol{t}_k)\big| \to 0$.
(b) *The infimum in* (2.6) *is attained, at some $y \leq \gamma/(1+\gamma)$.*
(c) *The supremum in* (2.7) *is attained at a unique v^*, and v^* lies in a compact interval $[v_0, v_1] \subset (0,\infty)$ depending only on (α,β).*
(d) *With $\tau = \gamma\alpha/2$, for all $\boldsymbol{t}, \boldsymbol{t}'$,*

$$\Big|\big[\Phi_r^{\mathrm{mix}}(\gamma;\boldsymbol{t}) + \mathbb{E}_\eta I(t_\eta)\big] - \big[\Phi_r^{\mathrm{mix}}(\gamma;\boldsymbol{t}') + \mathbb{E}_\eta I(t'_\eta)\big]\Big| \leq 2\tau\,\mathbb{E}_\eta|t_\eta - t'_\eta|, \tag{2.8}$$

and $\Phi^{(r)} + \mathbb{E}_\eta I(t_\eta)$ is Lipschitz on $[-1,1]^{d_r}$, with a constant depending only on (α,β) for the norm $\mathbb{E}_\eta|\cdot|$.

With one type, we drop the index:

$$\Phi^{\mathrm{mix}}(\gamma;m) = \inf_{x,y\in\mathcal{Q}_1} \Big\{ \frac{\gamma\alpha}{2}\Big(1 - \int_0^1 xy\Big) + J_m(x) + \alpha\psi_{\mathrm{S}}^*(y)\Big\},$$
$$\Phi(m) = \sup_{v>0}\big\{\mathrm{rad}(v) + \Phi^{\mathrm{mix}}(\beta v; m)\big\}.$$

2.2. **The mixed model at fixed radius.** If $\mathbf{z} \sim N(0, I_M)$, then $\mathbf{z} = s\tau$ with $s := |\mathbf{z}|/\sqrt{M}$ and $\tau := \sqrt{M}\mathbf{z}/|\mathbf{z}|$ independent. Here τ is uniform on the sphere of radius $\sqrt{M}$, with law u_M, and $s \sim \mathsf{n}_M(\,\mathrm{d}s) := c_M s^{M-1} e^{-Ms^2/2}\,\mathrm{d}s$, where $c_M := 2(M/2)^{M/2}/\Gamma(M/2)$ and $\frac{1}{M}\log c_M \to \frac{1}{2}$. Since $\mathbb{E}_{\mathbf{z}} \exp(\sqrt{\beta/N}\,\mathbf{z}\cdot G\sigma) = \exp(\frac{\beta}{2N}|G\sigma|^2)$, we obtain

$$Y_{N,k} = \int_0^\infty \mathsf{n}_M(\,\mathrm{d}s) Z_{N,k}(s), \qquad Z_{N,k}(s) := 2^{-N} \sum_{\sigma:k} \int \mathsf{u}_M(\,\mathrm{d}\tau) \exp\big(s\sqrt{\beta/N}\,\tau\cdot G\sigma\big). \tag{2.9}$$

We set $\mathsf{A}_N(s) := \frac{1}{N}\log Z_{N,k}(s)$ and $\alpha_N := M/N$.

At fixed radius, we use [1] with $J = \{-1,1\}^{r-1}$, so that $k = r-1$ in their (4.1), with $n = N + M$ sites, species I_η and a spherical species of size M, proportions $\lambda_\eta = d_r^{-1}/(1+\alpha)$, $\lambda_{\mathrm{S}} = \alpha/(1+\alpha)$, and covariance $\xi_{\alpha,r-1}(a) = (1+\alpha)a_{\mathrm{S}}\sum_\eta a_\eta$. A Gaussian field with this covariance is $H_n(\sigma,\tau) = \sqrt{(1+\alpha)/n}\,\tau\cdot G\sigma$, and their measure [1, (3.2)] is $2^{-|I_\eta|}$ times counting measure on each Ising slice, and u_M on the sphere. We let $\gamma = \beta s^2$ and $t_N := \gamma n/(2N(1+\alpha))$,

so that $\sqrt{2t_N}H_n = s\sqrt{\beta/N}\,\tau\cdot G\sigma$ and $nt_N\xi_{\alpha,r-1}(\lambda_n) = \gamma M/2$. Their free energy [1, (3.6)] at $q=0$ is thus

$$\mathbb{E}\mathsf{A}_N(s) = \frac{\gamma\alpha_N}{2} - \frac{n}{N}\bar{F}_{n,k}(t_N,0). \tag{2.10}$$

Since $t_N\to\gamma/2$ and $n/N\to 1+\alpha$, [1, Theorem 3.3 and (3.7)] gives $\bar{F}_{n,k}(t_N,0)-f_{\alpha,r-1}(\gamma/2,\boldsymbol{t}_k)\to 0$ uniformly in k and in γ on bounded sets. Their theorem is stated along a sequence indexed by all n, while our $n=N+M$ may repeat or skip integers; this is harmless, since every sequence of N has a subsequence along which n increases strictly, and the missing n can be filled with any partition satisfying their (3.1).

Finally, [1, (4.7)], with $p_\varepsilon = x_\eta$, $p_{\mathrm{S}} = y$, $\alpha t = \gamma\alpha/2$, $g^*_m = J_m$ and $\psi^{\mathsf{S},*} = \psi^*_{\mathrm{S}}$, reads

$$\Phi^{\mathrm{mix}}_r(\gamma;\boldsymbol{t}) = \frac{\gamma\alpha}{2} - (1+\alpha)f_{\alpha,r-1}\Big(\frac{\gamma}{2},\boldsymbol{t}\Big). \tag{2.11}$$

Hence, for each $\bar{s}$,

$$\sup_{s\le\bar{s}}\sup_k\big|\mathbb{E}\mathsf{A}_N(s) - \Phi^{\mathrm{mix}}_r(\beta s^2;\boldsymbol{t}_k)\big|\to 0. \tag{2.12}$$

With $\tau=\gamma\alpha/2$, minimizing over each x_η in (2.6) by (2.3) gives the form in y,

$$\Phi^{\mathrm{mix}}_r(\gamma;\boldsymbol{t}) = \tau + \inf_{y\in\mathcal{Q}_1}\Big\{\alpha\psi^*_{\mathrm{S}}(y) - \mathbb{E}_\eta g_{t_\eta}(\tau y)\Big\}. \tag{2.13}$$

By [1, Proposition 4.1], the infimum is attained at a path bounded by $a_{\gamma/2} = \gamma/(1+\gamma) < 1$, which is Theorem 2.3(b). From (2.13) and the properties of g_m, we read off three facts. At $\gamma=0$, we have $\Phi^{\mathrm{mix}}_r(0;\boldsymbol{t}) = -\mathbb{E}_\eta I(t_\eta)$, by Lemma 2.1. Since g_m is 1-Lipschitz and $y\le 1$, Φ^{mix}_r is Lipschitz in γ, with constant α, and using (2.1) with $|\tau y|_{L^1}\le\tau$, we obtain (2.8). Moreover, by (2.6), $\gamma\mapsto\Phi^{\mathrm{mix}}_r(\gamma;\boldsymbol{t})$ is an infimum of affine functions, and hence concave.

2.3. **The radial Laplace principle.** We first show that the mixed free energy is monotone in the radius, which allows us to work on a grid in the radius. Since $\log Z_{N,k}(s)$ is a log-Laplace transform, it is convex in s, and it is even because $\tau\mapsto-\tau$ preserves u_M. It follows that A_N is nondecreasing on $[0,\infty)$, and that $\mathsf{A}_N(s)\ge\mathsf{A}_N(0) = \frac{1}{N}\log(2^{-N}\#\mathrm{slice})\ge -\log 2$. On the other hand, since $|\tau\cdot G\sigma|\le\|G\|_{\mathrm{op}}\sqrt{MN}$, we have, on the whole probability space,

$$\mathsf{A}_N(s)\le\mathsf{A}_N(0) + s\mathsf{B}_G,\qquad \mathsf{B}_G := \sqrt{\beta\alpha_N}\,\|G\|_{\mathrm{op}}/\sqrt{N}. \tag{2.14}$$

Proof of Theorem 2.3(c), (d). For (c), taking expectations and limits in (2.14), with (2.12) and $\mathbb{E}\|G\|_{\mathrm{op}}\le\sqrt{N}+\sqrt{M}$, we obtain $\Phi^{\mathrm{mix}}_r(\beta s^2;\boldsymbol{t}) + \mathbb{E}_\eta I(t_\eta)\le s\sqrt{\beta\alpha}(1+\sqrt{\alpha})$, while monotonicity gives $\Phi^{\mathrm{mix}}_r(\beta;\boldsymbol{t}) + \mathbb{E}_\eta I(t_\eta)\ge 0$, and $\mathrm{rad}(1)=0$. Hence any maximizer in (2.7) lies in a compact interval $[v_0,v_1]\subset(0,\infty)$ depending only on (α,β), and the supremum is attained. Uniqueness holds because $v\mapsto\Phi^{\mathrm{mix}}_r(\beta v;\boldsymbol{t})$ is concave and rad is strictly concave.

For (d), (2.8) with $v\le v_1$ gives the Lipschitz bound, with constant $\alpha\beta v_1$. □

Proof of Theorem 2.3(a). Let $\mathsf{E}_N := \{\|G\|_{\mathrm{op}}\le C_0\sqrt{N}\}$ with $C_0 := 2(1+\sqrt{\alpha})+1$, so that $\mathbb{P}(\mathsf{E}_N^{\mathrm{c}})\le e^{-cN}$, and $\mathsf{B}_G\le\mathsf{B}_0$ on E_N for a constant B_0. We fix $\bar{s}>1$ with $\bar{s}^2\ge v_1$ such that, for N large and $s\ge\bar{s}$,

$$\log c_M + (M-1)\log s - Ms^2/2 + Ns\mathsf{B}_0\le -N(1+s), \tag{2.15}$$

which is possible as $M/N\to\alpha>0$. Consider a grid $s_j = j\varepsilon$, $j=0,\dots,\mathsf{j}$, with $s_{\mathsf{j}} = \bar{s}$, and let s^*_j be a maximizer of $s\mapsto\mathrm{rad}(s^2)$ on $[s_j,s_{j+1}]$. In what follows, j runs over $0,\dots,\mathsf{j}-1$.

For the lower bound, monotonicity gives, for each j, $Y_{N,k} \geq \mathsf{n}_M([s_j, s_{j+1}]) \exp(N\mathsf{A}_N(s_j))$, and hence $\frac{1}{N}\mathbb{E}\log Y_{N,k} \geq \frac{1}{N}\log \mathsf{n}_M([s_j, s_{j+1}]) + \mathbb{E}\mathsf{A}_N(s_j)$. Since $\frac{1}{N}\log \mathsf{n}_M([s_j, s_{j+1}]) \to \mathrm{rad}((s_j^*)^2)$, it follows from (2.12) that

$$\liminf_N \inf_k \Big[\frac{1}{N}\mathbb{E}\log Y_{N,k} - \max_j \{\Phi_r^{\mathrm{mix}}(\beta s_j^2; \boldsymbol{t}_k) + \mathrm{rad}((s_j^*)^2)\}\Big] \geq 0.$$

For the upper bound, (2.14) and (2.15) give $\int_{\bar{s}}^{\infty} \mathsf{n}_M(\,\mathrm{d}s) Z_{N,k}(s) \leq e^{N\mathsf{A}_N(0)} e^{-N}$ on E_N. Since $\mathsf{n}_M([0, \bar{s}]) \geq \frac{1}{2}$, the bulk is at least $\frac{1}{2}e^{N\mathsf{A}_N(0)}$, so the tail is smaller than the bulk. By monotonicity on $[0, \bar{s}]$, we obtain, on E_N,

$$\frac{1}{N}\log Y_{N,k} \leq \max_j \Big\{\frac{1}{N}\log \mathsf{n}_M([s_j, s_{j+1}]) + \mathsf{A}_N(s_{j+1})\Big\} + \frac{\log(2\mathsf{j}+2)}{N}.$$

The quantity $\mathsf{A}_N(s)$ is a function of G with Lipschitz constant $s\sqrt{\beta M}/N$, since $\nabla_G \log Z_{N,k}(s) = s\sqrt{\beta/N}\langle \tau\sigma^T\rangle$ has norm at most $s\sqrt{\beta/N}\sqrt{MN}$. Gaussian concentration gives $\mathbb{E}\max_j |\mathsf{A}_N(s_j) - \mathbb{E}\mathsf{A}_N(s_j)| \leq C\bar{s}\sqrt{\beta\alpha_N \log(2\mathsf{j}+2)/N} \to 0$, uniformly in k. On E_N^c, $-\log 2 \leq \frac{1}{N}\log Y_{N,k} \leq \beta\|G\|_{\mathrm{op}}^2/(2N)$, and the expectation of the upper bound on E_N^c is $o(1)$. Hence, uniformly in k,

$$\limsup_N \sup_k \Big[\frac{1}{N}\mathbb{E}\log Y_{N,k} - \max_j \{\Phi_r^{\mathrm{mix}}(\beta s_{j+1}^2; \boldsymbol{t}_k) + \mathrm{rad}((s_j^*)^2)\}\Big] \leq 0.$$

Finally, since $\bar{s}^2 \geq v_1$, we have $\sup_{s\leq\bar{s}}\{\Phi_r^{\mathrm{mix}}(\beta s^2; \boldsymbol{t}_k) + \mathrm{rad}(s^2)\} = \Phi^{(r)}(\boldsymbol{t}_k)$. Let s^* be the maximizer, and $[s_j, s_{j+1}]$ its cell. Then $\mathrm{rad}((s_j^*)^2) \geq \mathrm{rad}((s^*)^2)$, and by the Lipschitz bound in γ the first maximum above is at least $\Phi^{(r)}(\boldsymbol{t}_k) - C\varepsilon$. In the second maximum, s_j^* and s_{j+1} lie in one cell, so it is at most $\Phi^{(r)}(\boldsymbol{t}_k) + C\varepsilon$. As C does not depend on k, letting $\varepsilon \to 0$ completes the proof of (a). □

3. From binary patterns to retained patterns at fixed overlap

The reduction of binary patterns to finitely many retained patterns in a Gaussian bulk was first carried out in [5], for the unconstrained free energy, and in [1] with cutoffs on the auxiliary Gaussian variables. As the capacity depends on the free energy at fixed overlap, we prove in this section the reduction on every slice $m_1 = m_k$, in both directions, with errors independent of the slice, and with concentration strong enough for a union over all patterns and all slices; this is Theorem 3.1. Combined with Theorem 2.3, it gives the limit of the free energy at fixed overlap, $\Lambda^{\mathrm{R}}(m) = \sup_r \Theta_r(m)$.

The reduction extends to slices because a gauge transformation turns the pattern that defines the slice into the constant vector, so that the slice becomes a deterministic measure on the cube, and the comparison lemmas of [1, Section 6] treat an arbitrary weight on the cube, independent of the disorder.

3.1. **Notation.** We fix $\beta > 0$, $\alpha > 0$ and $M = M(N)$ with $M/N \to \alpha$, and put $K := M - 1$. We also fix $0 < \alpha_- < \alpha < \alpha_+$ and N_0 such that $(M - r)/N \in [\alpha_-, \alpha_+]$ for the finitely many r in use and all $N \geq N_0$; all statements are for $N \geq N_0$.

Here a matrix X is $N \times K'$, with columns X^μ, and acts through $X^T\sigma \in \mathbb{R}^{K'}$, so that the rows of Theorem 2.3 are the columns here. We write ε for the $N \times K$ matrix with i.i.d. symmetric ± 1 entries and G for a matrix with i.i.d. standard Gaussian entries, of a size given by the context. We denote by $\gamma_{K'}$ the standard Gaussian measure on $\mathbb{R}^{K'}$ and by P_N the uniform law on Σ_N.

Throughout, ρ denotes a nonzero finite positive deterministic measure on Σ_N, and we set

$$f^\rho(X) := \frac{1}{N}\log \sum_{\sigma\in\Sigma_N} \rho(\sigma)\exp\Big(\frac{\beta}{2N}|X^T\sigma|^2\Big).$$

The exponent lies in $[0, \frac{\beta}{2}\|X\|_{\mathrm{op}}^2]$, so

$$f^\rho(0) \le f^\rho(X) \le f^\rho(0) + \frac{\beta}{2N}\|X\|_{\mathrm{op}}^2, \qquad f^\rho(0) = \frac{1}{N}\log \rho(\Sigma_N). \tag{3.1}$$

As before, $S_N(k)$ denotes the set of configurations with exactly k plus signs and $m_k := 2k/N - 1$. We put $\rho_k := 2^{-N}\mathbf{1}_{S_N(k)}$, so that $f^{\rho_k}(0) \in [-\log 2, 0]$.

Recall that the overlap is $m_\mu(\sigma) = \xi^\mu \cdot \sigma / N$. The gauge $\sigma \mapsto \mathrm{diag}(\xi^1)\sigma$ sends $\{m_1(\sigma) = m_k\}$ to $S_N(k)$, and the vectors $\mathrm{diag}(\xi^1)\xi^\mu$, $\mu \ge 2$, are i.i.d. Rademacher vectors independent of ξ^1. On $S_N(k)$ we have $(\mathbf{1}\cdot\sigma)^2 = N^2 m_k^2$. With ε the matrix of the gauged patterns,

$$p_N(k) := \frac{1}{N}\mathbb{E}\log 2^{-N} \sum_{m_1(\sigma)=m_k} \exp\Big(\frac{\beta}{2N}\sum_{\mu\le M}(\xi^\mu\cdot\sigma)^2\Big) = \frac{\beta m_k^2}{2} + \mathbb{E} f^{\rho_k}(\varepsilon). \tag{3.2}$$

For $1 \le r \le M$, with $\zeta^2, \dots, \zeta^r$ i.i.d. Rademacher vectors, G' a Gaussian $N \times (M-r)$ matrix and $X_r := (\zeta^2, \dots, \zeta^r, G')$, we set

$$p^{(r)}(\rho) := \mathbb{E} f^\rho(X_r), \qquad p_N^{(r)}(k) := \frac{\beta m_k^2}{2} + p^{(r)}(\rho_k).$$

Thus r counts pattern 1 and $r-1$ further Rademacher patterns, and $p_N^{(M)}(k) = p_N(k)$. For $r = 1$, with $Y_{N,k} := 2^{-N}\sum_{S_N(k)} \exp(\frac{\beta}{2N}|G^T\sigma|^2)$ and G Gaussian $N \times K$, we have $p_N^{(1)}(k) = \beta m_k^2/2 + \frac{1}{N}\mathbb{E}\log Y_{N,k}$.

Recall that $d_r := 2^{r-1}$ for $r \ge 1$, that the index η ranges over $\{-1,1\}^{r-1}$, with coordinates η_j, $j = 2, \dots, r$, and that $\mathbb{E}_\eta$ is the uniform average. For $\boldsymbol{t} = (t_\eta) \in [-1,1]^{d_r}$, we set, as before,

$$\boldsymbol{\omega}_j(\boldsymbol{t}) := \mathbb{E}_\eta[\eta_j t_\eta], \qquad V(\boldsymbol{t})^2 := (\mathbb{E}_\eta t_\eta)^2 + |\boldsymbol{\omega}(\boldsymbol{t})|^2, \qquad P_r(m) := \{\boldsymbol{t} \in [-1,1]^{d_r} : \mathbb{E}_\eta t_\eta = m\}.$$

With $\Phi^{(r)}$ from Theorem 2.3, so that $\Phi^{(1)} = \Phi$, let

$$\mathcal{F}_r(\boldsymbol{t}) := \frac{\beta}{2}V(\boldsymbol{t})^2 + \Phi^{(r)}(\boldsymbol{t}), \qquad \Theta_r(m) := \max_{\boldsymbol{t}\in P_r(m)} \mathcal{F}_r(\boldsymbol{t}), \qquad \Lambda^{\mathrm{R}}(m) := \sup_{r\ge 1}\Theta_r(m),$$

and $\Lambda(m) := \Theta_1(m) = \beta m^2/2 + \Phi(m)$.

At zero temperature, we put $L_A(X) := \frac{1}{N}\max_{\sigma\in A}|X^T\sigma|$ for nonempty $A \subset \Sigma_N$, and

$$e_N(k) := \frac{1}{2N^2}\mathbb{E}\max_{m_1(\sigma)=m_k}\sum_{\mu\le M}(\xi^\mu\cdot\sigma)^2 = \frac{m_k^2}{2} + \frac{1}{2}\mathbb{E}L_{S_N(k)}(\varepsilon)^2.$$

Keeping the largest term gives, pointwise on matrices,

$$\frac{\beta}{2}L_A(X)^2 - \log 2 \le f^{2^{-N}\mathbf{1}_A}(X) \le \frac{\beta}{2}L_A(X)^2. \tag{3.3}$$

3.2. **The reduction.** The main result of this section is the following.

Theorem 3.1. *There are constants $C_1, c_1, L, \Lambda > 0$, depending only on $\beta, \alpha_\pm$, such that the following hold.*

(i) (Lower bound.) For fixed r, N large and every ρ,

$$\mathbb{E} f^\rho(\varepsilon) \ge p^{(r)}(\rho) - C_1 N^{-1/2}.$$

So $p_N(k) \ge p_N^{(r)}(k) - C_1 N^{-1/2}$ for all k.

(ii) (Upper bound.) For $\delta \in (0,1]$ put $K_\delta := \lfloor L^2/\delta^2 \rfloor$. For N large and every ρ,

$$\mathbb{E} f^\rho(\varepsilon) \le \max_{1 \le r \le 1+K_\delta} p^{(r)}(\rho) + C_1\delta^2 + \mathrm{err}_N(\delta),$$

where $\mathrm{err}_N(\delta) \to 0$ as $N \to \infty$ for fixed δ, and it does not depend on ρ. In particular $p_N(k) \le \max_{r \le 1+K_\delta} p_N^{(r)}(k) + C_1\delta^2 + \mathrm{err}_N(\delta)$ for all k.

(iii) (Limit form.)

(a) For fixed r, $\sup_k |p_N^{(r)}(k) - \Theta_r(m_k)| \to 0$.

(b) Θ_r is continuous, and $\Lambda^{\mathrm{R}} + I$ is Hölder of exponent $1/2$ on $[-1,1]$.

(c) $\sup_k |p_N(k) - \Lambda^{\mathrm{R}}(m_k)| \to 0$. So if $m_{k_N} \to m$,

$$\lim_N p_N(k_N) = \Lambda^{\mathrm{R}}(m) = \sup_{r \ge 1} \sup_{\boldsymbol{t} \in P_r(m)} \Big\{ \frac{\beta}{2}\big(m^2 + |\boldsymbol{\omega}(\boldsymbol{t})|^2\big) + \Phi^{(r)}(\boldsymbol{t}) \Big\} \ge \frac{\beta m^2}{2} + \Phi(m).$$

(iv) (Concentration.) For every ρ and $t > 0$,

$$\mathbb{P}\Big(|f^\rho(\varepsilon) - \mathbb{E} f^\rho(\varepsilon)| > t + C_1 e^{-c_1 N} \Big) \le C_1 e^{-c_1 N} + C_1 e^{-c_1 N t^2}. \tag{3.4}$$

With $f^{\mu,k} := \frac{1}{N} \log 2^{-N} \sum_{m_\mu(\sigma) = m_k} \exp\big(\frac{\beta}{2N} \sum_{\mu' \ne \mu} (\xi^{\mu'} \cdot \sigma)^2\big)$, a union over all patterns and all slices costs a factor $M(N+1)$ in (3.4). At zero temperature, for every nonempty $A \subset \Sigma_N$,

$$\mathbb{P}\big(|L_A(\varepsilon) - \mathbb{E} L_A(\varepsilon)| > t\big) \le C_1 e^{-c_1 N t^2}, \qquad 0 \le \frac{1}{2}\mathbb{E} L_A(\varepsilon)^2 - \frac{1}{2}\big(\mathbb{E} L_A(\varepsilon)\big)^2 \le \frac{C_1}{N}, \tag{3.5}$$

and $\mathbb{E} L_A(\varepsilon) \le C_1$.

3.3. **Inputs.** For an $N \times K'$ matrix A, a weight $w : \Sigma_N \to [0,\infty)$ with $\int w \, \mathrm{d}P_N > 0$, and a Borel set $D \subset \mathbb{R}^{K'}$ of positive Gaussian measure, we put [1, (6.1)]

$$\mathsf{F}(A, w, D) := \frac{1}{N} \log \iint_{\Sigma_N \times D} w(\sigma) \exp\Big(\sqrt{\frac{\beta}{N}}\, \sigma^T A z \Big) \, \mathrm{d}P_N(\sigma) \, \mathrm{d}\gamma_{K'}(z).$$

By the Gaussian integral, $\mathsf{F}(A, 2^N\rho, \mathbb{R}^{K'}) = f^\rho(A)$, and for random weights we assume that $\log \int w \, \mathrm{d}P_N$ is integrable. With $\mathbb{B}(L) := \{|z| \le L\sqrt{N}\}$, let $\mathbb{D}(\delta, L)$ be the intersection of $\mathbb{B}(L)$ with the cube $[-\delta\sqrt{N}, \delta\sqrt{N}]^{K'}$. Let A have independent entries, uniform on $\{-1,1\}$ or standard Gaussian, and $K' \le \alpha_+ N$. We use the following from [1, Section 6].

- (6.3). $\mathbb{P}(\|A\|_{\mathrm{op}} > C(\sqrt{N} + \sqrt{K'}) + t) \le 2e^{-ct^2}$, and $\mathbb{E}\|A\|_{\mathrm{op}}^q \le C_q (N + K')^{q/2}$.
- (6.4), (6.5). If U is a convex, globally ℓ-Lipschitz function of independent signs, then $\mathbb{P}(|U - \mathbb{E}U| > t) \le Ce^{-ct^2/\ell^2}$. For J such functions, with no independence between them, $\mathbb{E} \max_j U_j \le \max_j \mathbb{E} U_j + C\ell\sqrt{\log(2J)}$.
- Lemma 6.1. If w is independent of ε and G, $\delta \in (0,1]$, $L > 0$ and $D \subset \mathbb{D}(\delta, L)$, then $|\mathbb{E}\mathsf{F}(\varepsilon, w, D) - \mathbb{E}\mathsf{F}(G, w, D)| \le C_\beta L^2 \delta^2$.
- Lemma 6.2. There are $L_0 \ge 1$, $\Lambda < \infty$, depending only on β, α_+, such that $\mathbb{P}(\|A\|_{\mathrm{op}} > \Lambda\sqrt{N}) \le Ce^{-cN}$, and such that for $L \ge L_0$ and every w, on $\{\|A\|_{\mathrm{op}} \le \Lambda\sqrt{N}\}$,

$$0 \le \mathsf{F}(A, w, \mathbb{R}^{K'}) - \mathsf{F}(A, w, \mathbb{B}(L)) \le Ce^{-cN}. \tag{3.6}$$

  The difference is at most $\frac{\beta}{2N}\|A\|_{\mathrm{op}}^2 + \frac{\log 2}{N}$ for all A, and its expectation is at most Ce^{-cN}, also for random w.
- (6.20). If $\alpha_- N \le K'$, $\delta\sqrt{N} \ge 2$, $L \ge L_0$ and w is independent of G, then $0 \le \mathbb{E}\mathsf{F}(G, w, \mathbb{R}^{K'}) - \mathbb{E}\mathsf{F}(G, w, \mathbb{D}(\delta, L)) \le C/(\delta^2 N) + Ce^{-cN}$.

From now on, we fix $L := L_0$.

We will also need a Gaussian form of the maximum bound. Let $\boldsymbol{k}$ run over a finite set of size J, let $w_{\boldsymbol{k}}$ be deterministic weights and $a_{\boldsymbol{k}}$ real numbers. Each $\mathsf{F}(G, w_{\boldsymbol{k}}, \mathbb{B}(L))$ is globally Lipschitz in G, with constant $\ell = L\sqrt{\beta/N}$. Indeed, its gradient has entries $N^{-1}\sqrt{\beta/N}\langle \sigma_i z_\mu\rangle$, so its squared norm is at most $\beta N^{-2}\langle |z|^2\rangle \le \beta L^2/N$. Using Gaussian concentration, (3.6) and the deterministic bound on its complement, we obtain

$$\mathbb{E}\max_{\boldsymbol{k}}\{a_{\boldsymbol{k}} + \mathsf{F}(G, w_{\boldsymbol{k}}, \mathbb{R}^{K'})\} \le \max_{\boldsymbol{k}}\{a_{\boldsymbol{k}} + \mathbb{E}\mathsf{F}(G, w_{\boldsymbol{k}}, \mathbb{R}^{K'})\} + C\sqrt{\frac{\log(2J)}{N}}. \tag{3.7}$$

3.4. **The lower and upper bounds.**

Proof of Theorem 3.1(i). We keep $r-1$ columns inside the weight and compare the others under a cutoff, which only lowers the free energy on the Rademacher side.

We split $\varepsilon = (\zeta, \varepsilon')$, with ζ the first $r-1$ columns and ε' the other $M-r$, and put $w(\sigma) := 2^N \rho(\sigma) \exp(\frac{\beta}{2N}\sum_j (\zeta^j \cdot \sigma)^2)$, which is a function of ζ only, with $\log \int w \,\mathrm{d}P_N$ bounded. Then $f^\rho(\varepsilon) = \mathsf{F}(\varepsilon', w, \mathbb{R}^{M-r})$ and $f^\rho(\zeta, G') = \mathsf{F}(G', w, \mathbb{R}^{M-r})$, so, with $\delta := N^{-1/4}$,

$$\begin{aligned}\mathbb{E}f^\rho(\varepsilon) &\ge \mathbb{E}\mathsf{F}(\varepsilon', w, \mathbb{D}(\delta, L)) \ge \mathbb{E}\mathsf{F}(G', w, \mathbb{D}(\delta, L)) - C_\beta L^2\delta^2 \\ &\ge \mathbb{E}\mathsf{F}(G', w, \mathbb{R}^{M-r}) - C_\beta L^2\delta^2 - \frac{C}{\delta^2 N} - Ce^{-cN} = p^{(r)}(\rho) - O(N^{-1/2}).\end{aligned}$$

The first step restricts a positive integrand, the second is Lemma 6.1 of [1], and the third is (6.20) of [1]. The constants do not depend on ρ, and the statement on p_N follows from (3.2). □

Proof of Theorem 3.1(ii). Large hidden coordinates cannot be replaced by a Lindeberg step, but there are few of them. We thus cut z to a ball and split the ball according to the set S of large coordinates; the columns in S go into the weight, and the others meet only small coordinates.

Let $w := 2^N\rho$. By Lemma 6.2 of [1], $\mathbb{E}f^\rho(\varepsilon) \le \mathbb{E}\mathsf{F}(\varepsilon, w, \mathbb{B}(L)) + Ce^{-cN}$. For $z \in \mathbb{B}(L)$, the set $S(z) := \{\mu : |z_\mu| > \delta\sqrt{N}\}$ has at most K_δ elements, so $\mathbb{B}(L)$ is covered by the sets $D_S := \{|z_S| \le L\sqrt{N}\} \times \mathbb{D}_{S^c}(\delta, L)$, $S \subset \{2, \dots, M\}$, $|S| \le K_\delta$, where $\mathbb{D}_{S^c}$ is the set $\mathbb{D}$ in the coordinates outside S. Let $\mathcal{S}$ be the family of these S, of size $J \le (K_\delta+1)M^{K_\delta}$, and put $F_S(X) := \mathsf{F}(X, w, D_S)$. Then

$$\mathsf{F}(\varepsilon, w, \mathbb{B}(L)) \le \max_{S\in\mathcal{S}} F_S(\varepsilon) + \frac{\log J}{N}.$$

Each F_S is convex in X, as a log-Laplace transform of functions linear in X. As before, its gradient has entries $N^{-1}\sqrt{\beta/N}\langle\sigma_i z_\mu\rangle$, with squared norm at most $\beta N^{-2}\langle|z|^2\rangle \le 2\beta L^2/N$, because $|z|^2 \le 2L^2N$ on D_S. Hence (6.5) of [1] gives $\mathbb{E}\max_S F_S(\varepsilon) \le \max_S \mathbb{E}F_S(\varepsilon) + CL\sqrt{\beta\log(2J)/N}$.

We next fix S with $|S| = s$ and put

$$w_S(\sigma) := w(\sigma)\int_{|z_S|\le L\sqrt{N}} \exp\Big(\sqrt{\frac{\beta}{N}}\,\sigma^T\varepsilon_S z_S\Big)\,\mathrm{d}\gamma_S(z_S),$$

which is a function of ε_S only, and so is independent of ε_{S^c} and of a Gaussian matrix G_{S^c} in the other columns. By Jensen and the symmetry of the ball, $\int w_S \,\mathrm{d}P_N \ge \gamma_S(|z_S| \le L\sqrt{N}) \int w \,\mathrm{d}P_N$, and ε_S takes finitely many values, so $\log\int w_S\,\mathrm{d}P_N$ is bounded. By Fubini, $F_S(\varepsilon) = \mathsf{F}(\varepsilon_{S^c}, w_S, \mathbb{D}_{S^c}(\delta, L))$. Lemma 6.1 of [1], and then enlarging the domain to the whole space twice, give

$$\mathbb{E}F_S(\varepsilon) \le \mathbb{E}\mathsf{F}(G_{S^c}, w_S, \mathbb{D}_{S^c}(\delta, L)) + C_\beta L^2\delta^2 \le \mathbb{E}f^\rho(\varepsilon_S, G_{S^c}) + C_\beta L^2\delta^2.$$

Since the columns are i.i.d. and ρ does not see their labels, we have $\mathbb{E}f^{\rho}(\varepsilon_S, G_{S^c}) = p^{(s+1)}(\rho)$. Collecting the terms gives (ii), with $\mathrm{err}_N(\delta) = Ce^{-cN} + \frac{\log J}{N} + CL\sqrt{\beta \log(2J)/N}$. □

3.5. **Concentration.**

Proof of Theorem 3.1(iv). Let $w = 2^N\rho$ and $F^L(X) := \mathsf{F}(X, w, \mathbb{B}(L))$. As in the last proof, F^L is convex and globally $L\sqrt{\beta/N}$-Lipschitz, so (6.4) of [1] applies. By Lemma 6.2 of [1], $0 \le f^{\rho} - F^L \le Ce^{-cN}$ on $\{\|\varepsilon\|_{\mathrm{op}} \le \Lambda\sqrt{N}\}$ and $0 \le \mathbb{E}f^{\rho} - \mathbb{E}F^L \le Ce^{-cN}$, which gives (3.4).

Under the gauge $\mathrm{diag}(\xi^{\mu})$ for pattern μ, the other patterns become i.i.d. Rademacher, so $f^{\mu,k}$ has the law of $f^{\rho_k}(\varepsilon)$. A union bound over $M(N+1)$ events gives the second claim, and no independence between patterns is used. The term of pattern μ itself is the constant $\beta m_k^2/2$ and does not fluctuate.

The function L_A is convex, as a maximum of norms of linear maps, and $|L_A(X) - L_A(X')| \le \|X - X'\|_{\mathrm{HS}}/\sqrt{N}$ globally. Hence (6.4) of [1] with $\ell = 1/\sqrt{N}$ gives the tail in (3.5), and integrating the tail gives the variance. Finally, $L_A(\varepsilon) \le \|\varepsilon\|_{\mathrm{op}}/\sqrt{N}$, and (6.3) of [1] bounds its mean. □

3.6. **The limit.** We now prove Theorem 3.1(iii). Given the retained patterns, the sites split into d_r types, and the retained term depends on σ only through the magnetizations of the types, so a slice splits into polynomially many type slices. On each type slice, only the Gaussian bulk remains and Theorem 2.3 applies, giving Θ_r for fixed r. The two sides of (i) and (ii) then squeeze $p_N(k)$ between Θ_r and $\sup_r \Theta_r$.

Lemma 3.2. *$m \mapsto P_r(m)$ is Lipschitz in Hausdorff distance for the sup norm, with constant d_r. So Θ_r is uniformly continuous on $[-1, 1]$.*

Proof. For $\boldsymbol{t} \in P_r(m)$ and $m' > m$, the vector $\boldsymbol{t}' := \boldsymbol{t} + \frac{m'-m}{1-m}(\mathbf{1} - \boldsymbol{t})$ lies in $P_r(m')$. Since $0 \le 1 - t_\eta \le d_r(1-m)$, we get $|\boldsymbol{t}' - \boldsymbol{t}|_\infty \le d_r(m'-m)$, and the case $m' < m$ is symmetric. Moreover, $\mathcal{F}_r$ is uniformly continuous, by Theorem 2.3(d) and continuity of I. □

Lemma 3.3. *For fixed r, $\sup_k |p_N^{(r)}(k) - \Theta_r(m_k)| \to 0$.*

Proof. We condition on $\zeta = (\zeta^2, \dots, \zeta^r)$, and we denote by $\eta(i) := (\zeta_i^j)_j$ the type of site i. Put $I_\eta := \{i : \eta(i) = \eta\}$, $n_\eta := |I_\eta|$ and $w_\eta^N := n_\eta/N$, and let $\mathrm{Good}_N := \{|w_\eta^N - 1/d_r| \le N^{-1/3} \text{ for all } \eta\}$. By Hoeffding, $\mathbb{P}(\mathrm{Good}_N^c) \le 2d_r \exp(-2N^{1/3})$. We take N so large that $n_\eta \ge N/(2d_r)$ on Good_N.

We call an integer vector $\boldsymbol{k} = (k_\eta)$ admissible if $0 \le k_\eta \le n_\eta$ and $\sum_\eta k_\eta = k$. The admissible vectors form a nonempty set of at most $(N+1)^{d_r}$ elements. With $t_\eta := 2k_\eta/n_\eta - 1$, let $S(\boldsymbol{k})$ be the type slice. On $S(\boldsymbol{k})$, we have $\zeta^j \cdot \sigma = N\boldsymbol{\omega}_j^N(\boldsymbol{t})$ with $\boldsymbol{\omega}_j^N(\boldsymbol{t}) := \sum_\eta w_\eta^N \eta_j t_\eta$. We set

$$\Gamma_N(\boldsymbol{k}) := \frac{1}{N}\log 2^{-N} \sum_{S(\boldsymbol{k})} \exp\Big(\frac{\beta}{2N}|G'^T\sigma|^2\Big), \qquad \Upsilon_N := \frac{\beta m_k^2}{2} + f^{\rho_k}(\zeta, G'),$$

$$V_N(\boldsymbol{k})^2 := m_k^2 + |\boldsymbol{\omega}^N(\boldsymbol{t})|^2, \qquad B_N(\boldsymbol{k}) := \frac{\beta}{2}V_N(\boldsymbol{k})^2 + \Gamma_N(\boldsymbol{k}).$$

Pointwise,

$$\max_{\boldsymbol{k}} B_N(\boldsymbol{k}) \le \Upsilon_N \le \max_{\boldsymbol{k}} B_N(\boldsymbol{k}) + \frac{d_r \log(N+1)}{N}. \tag{3.8}$$

For uniformity in the type proportions, let err'_N be the supremum over counts (n_η) in Good_N and over $\boldsymbol{k}$ of $|\mathbb{E}_{G'}\Gamma_N(\boldsymbol{k}) - \Phi^{(r)}(\boldsymbol{t})|$. Since the rows of G' are exchangeable, the law of Γ_N depends on the partition only through (n_η). Choosing for each N counts that nearly attain

the supremum gives a sequence of partitions of $[N]$ with $|I_\eta|/N \to 1/d_r$, and $(M-r)/N \to \alpha$. Hence Theorem 2.3(a) applies, and $\mathrm{err}'_N \to 0$.

For the upper bound on Good_N, note that $|\boldsymbol{\omega}_j^N(\boldsymbol{t}) - \boldsymbol{\omega}_j(\boldsymbol{t})| \le d_r N^{-1/3}$ and $|\mathbb{E}_\eta t_\eta - m_k| \le d_r N^{-1/3}$, so $\mathbb{E}_{G'} B_N(\boldsymbol{k}) \le \mathcal{F}_r(\boldsymbol{t}) + C_r N^{-1/3} + \mathrm{err}'_N$, with $\mathcal{F}_r(\boldsymbol{t}) \le \Theta_r(\mathbb{E}_\eta t_\eta) \le \Theta_r(m_k) + \varpi_r(d_r N^{-1/3})$, where ϖ_r is the modulus of continuity of Θ_r from Lemma 3.2. Given ζ, the type slices are deterministic and only G' is random, so using (3.7) with $J \le (N+1)^{d_r}$ and (3.8), we obtain

$$\mathbb{E}_{G'} \Upsilon_N \le \max_{\boldsymbol{k}} \mathbb{E}_{G'} B_N(\boldsymbol{k}) + O(\sqrt{\log N/N}) \le \Theta_r(m_k) + o(1),$$

uniformly in k and in good ζ.

For the lower bound on Good_N, let $\boldsymbol{t}^o$ attain $\Theta_r(m_k)$ and put $\bar{k}_\eta := n_\eta(1 + t^o_\eta)/2$, so $|\sum_\eta \bar{k}_\eta - k| \le (d_r/2)N^{2/3}$. Rounding with $\sum_\eta k_\eta = k$ inside the box is possible since $0 \le k \le \sum_\eta n_\eta$, with $\sum_\eta |k_\eta - \bar{k}_\eta| \le (d_r/2)N^{2/3} + d_r$, and hence $|\boldsymbol{t}(\boldsymbol{k}) - \boldsymbol{t}^o|_\infty \le C_r N^{-1/3}$. By uniform continuity of $\mathcal{F}_r$, we obtain $\mathbb{E}_{G'} \Upsilon_N \ge \max_{\boldsymbol{k}} \mathbb{E}_{G'} B_N(\boldsymbol{k}) \ge \Theta_r(m_k) - o(1)$.

It remains to treat bad partitions, for which (3.1) gives $-\log 2 \le \Upsilon_N \le \beta r/2 + \frac{\beta}{2N}\|G'\|^2_{\mathrm{op}}$. Hence $\mathbb{E}[(|\Upsilon_N| + |\Theta_r(m_k)|)\mathbf{1}_{\mathrm{Good}_N^{\mathrm{c}}}] \le C\,\mathbb{P}(\mathrm{Good}_N^c)^{1/2} \to 0$, uniformly in k, which proves the lemma. □

Lemma 3.4. *Put $U_N(k) := p_N(k) - \beta m_k^2/2 + I(m_k)$ and $C_3 := \sup_N \mathbb{E}\|\varepsilon\|^2_{\mathrm{op}}/N$, which is finite by (6.3) of* [1]*. Then for all k, k',*

$$|U_N(k) - U_N(k')| \le \sqrt{2}\,\beta C_3 |m_k - m_{k'}|^{1/2} + \frac{2\log(N+1)}{N}.$$

Proof. For a matrix X, let $\mathrm{Av}_k(X)$ be the uniform average over $S_N(k)$ of $\exp(\frac{\beta}{2N}|X^T\sigma|^2)$. If $k < k'$ and $\sigma \le \sigma'$ coordinatewise, with $\sigma \in S_N(k)$ and $\sigma' \in S_N(k')$, then

$$||X^T\sigma|^2 - |X^T\sigma'|^2| \le 4\|X\|^2_{\mathrm{op}}\sqrt{(k'-k)N}.$$

Since the bipartite graph $\{\sigma \le \sigma'\}$ between $S_N(k)$ and $S_N(k')$ is biregular, averaging over neighbors gives

$$|\log \mathrm{Av}_k(X) - \log \mathrm{Av}_{k'}(X)| \le 2\beta\|X\|^2_{\mathrm{op}}\sqrt{(k'-k)/N}, \qquad (k'-k)/N = |m_k - m_{k'}|/2.$$

By (3.2), $U_N(k) = \frac{1}{N}\log(2^{-N}|S_N(k)|) + I(m_k) + \mathbb{E}\frac{1}{N}\log \mathrm{Av}_k(\varepsilon)$, and $|\frac{1}{N}\log(2^{-N}|S_N(k)|) + I(m_k)| \le \log(N+1)/N$. □

We now complete the proof of (iii). Part (a) is Lemma 3.3, and Θ_r is continuous by Lemma 3.2. Let $m_{k_N} \to m$. It follows from (i) and Lemma 3.3 that $\liminf_N p_N(k_N) \ge \Theta_r(m)$ for every r, so $\liminf_N p_N(k_N) \ge \Lambda^{\mathrm{R}}(m)$. Conversely, for fixed δ, applying (ii) and Lemma 3.3 to the finitely many $r \le 1 + K_\delta$, we obtain

$$\limsup_N p_N(k_N) \le \max_{r \le 1+K_\delta} \Theta_r(m) + C_1\delta^2 \le \Lambda^{\mathrm{R}}(m) + C_1\delta^2.$$

Letting $\delta \to 0$, we obtain $p_N(k_N) \to \Lambda^{\mathrm{R}}(m)$, which is finite by (3.1). Taking two such sequences in Lemma 3.4 shows that $\Lambda^{\mathrm{R}} - \beta m^2/2 + I$ is Hölder of exponent $1/2$, which gives (b). If (c) failed, there would be $\epsilon > 0$ and k_N, along a subsequence, with $|p_N(k_N) - \Lambda^{\mathrm{R}}(m_{k_N})| \ge \epsilon$ and $m_{k_N} \to m$, which contradicts the limit we just proved and the continuity of Λ^{R}. This completes the proof of (iii). □

Corollary 3.5. $\sup_k |e_N(k) - e^{\mathrm{R}}(m_k)| \to 0$, *where $e^{\mathrm{R}} := \lim_{\beta\to\infty} \Lambda^{\mathrm{R}}_\beta/\beta$, the limit being uniform on $[-1,1]$.*

Proof. By (3.3), $\beta e_N(k) - \log 2 \le p_N(k) \le \beta e_N(k)$, and Theorem 3.1(iii)(c) gives

$$\limsup_N \sup_k |e_N(k) - \Lambda^{\mathrm{R}}_\beta(m_k)/\beta| \le \frac{\log 2}{\beta}.$$

Comparing two values of β and using continuity of $\Lambda^{\mathrm{R}}_\beta$ on $[-1,1]$ gives

$$\sup_m |\Lambda^{\mathrm{R}}_\beta(m)/\beta - \Lambda^{\mathrm{R}}_{\beta'}(m)/\beta'| \le \log 2 \Big(\frac{1}{\beta} + \frac{1}{\beta'}\Big).$$

Hence the limit exists uniformly. □

4. A strictly convex problem and the capacity for Gaussian rows

In this section, we show that the Gaussian-row problem of Section 2, at fixed overlap and with the radius eliminated in closed form, is a strictly convex minimization in the law of the overlap path:

$$\Phi(m) = \min_{x \in \mathcal{Q}_1} \{J_m(x) + \alpha \mathcal{S}_\beta(x)\}, \qquad \mathcal{S}_\beta(x) = \frac{\beta}{2} \int_0^1 \frac{\mathrm{d}w}{D_x(w)}.$$

We then deduce that the optimizer is unique at every (α, T, m) and that the free energy Λ is C^1 with $\Lambda'(m) = \beta m - h(m)$, where h is the multiplier at the optimizer. This characterizes the capacity $\alpha_{\mathrm{c}}(T)$ of Gaussian rows at every $T > 0$, including the replica symmetry breaking region, through a generalized Mattis equation.

Strict convexity and uniqueness at fixed radius and field, with the spherical law as variable, are proved in [1, Proposition 1.2], while our form keeps the Ising law and removes the radius.

4.1. **Paths, the Parisi equation and the quoted facts.** Let $\mathcal{Q}_1$ be the set of nondecreasing right-continuous paths $[0,1) \to [0,1]$, and $\mathcal{Q}_\infty$ the set of bounded nondecreasing right-continuous paths with values in $\mathbb{R}_+$. A path $x \in \mathcal{Q}_1$ and its law $\mathcal{L}_x$ on $[0,1]$, the law of $x(U)$ with U uniform, determine each other. Since L^1 convergence of paths in $\mathcal{Q}_1$ is weak convergence of laws, $\mathcal{Q}_1$ is compact and convex in the variable $\mathcal{L}$. Throughout, convexity in the law refers to mixtures of laws, not to the pointwise sum of paths. We write $\bar{x} := \int_0^1 x$, $\hat{y} := y(1) := \lim_{\theta \to 1} y(\theta)$ and $\check{y} := y(0)$, and z denotes a standard Gaussian variable.

For a law $\mathcal{L}$ on $[0,1]$, the law of a path y, we set

$$\delta_{\mathcal{L}}(w) := \mathbb{E}[1 - y(U) \vee w] = \int_w^1 \mathcal{L}([0,w'])\, \mathrm{d}w'.$$

It is affine and injective in $\mathcal{L}$, with $\delta_{\mathcal{L}}(w) \le 1 - w$ and $\delta_{\mathcal{L}}(w) = 1 - w$ above $\hat{y}$. We write $\delta_x := \delta_{\mathcal{L}_x}$ and

$$D_x(w) := 1 - \beta \delta_x(w),$$

which is increasing in w, with $D_x(1) = 1$ and $D_x(0) = 1 - \beta + \beta \bar{x}$.

For $\gamma > 0$ we put $\tau := \gamma\alpha/2$ and $c := \gamma/2$, so $\tau = \alpha c$. At radius v, that is at $\gamma = \beta v$, we write $\tau_v := \alpha\beta v/2$ and $c_v := \beta v/2$. Recall that $\mathrm{rad}(v) = \frac{\alpha}{2}(1 + \log v - v)$. By (2.6) and (2.7),

$$\Phi^{\mathrm{mix}}(\gamma; m) = \min_{x,y \in \mathcal{Q}_1} H(x,y), \qquad H(x,y) := \tau\Big(1 - \int xy\Big) + J_m(x) + \alpha \psi^*_{\mathrm{S}}(y), \tag{4.1}$$

and $\Phi(m) = \sup_{v>0}\{\mathrm{rad}(v) + \Phi^{\mathrm{mix}}(\beta v; m)\}$, with the supremum attained at a unique v^*. Recall also that $\Lambda(m) = \beta m^2/2 + \Phi(m)$.

For types $\eta \in \{-1,1\}^{r-1}$ with uniform weights and $\boldsymbol{t} = (t_\eta) \in [-1,1]^{d_r}$, Theorem 2.3 gives $\Phi^{(r)}(\boldsymbol{t}) = \sup_{v>0}\{\mathrm{rad}(v) + \Phi^{\mathrm{mix}}_r(\beta v; \boldsymbol{t})\}$, with the supremum attained at a unique v. For one type, this gives back Φ and Φ^{mix}.

On the Ising side, let ν be a law on $\mathbb{R}_+$ with bounded support, $a \in \mathcal{Q}_\infty$ its path, and $S \geq \sup\operatorname{supp}\nu$ a horizon. Recall that the function ϕ_ν solves the Parisi equation (1.4), whose space variable we now call u, because x is a path:

$$-\partial_s\phi_\nu = \partial_u^2\phi_\nu + \nu([0,s])(\partial_u\phi_\nu)^2 \quad \text{on } [0,S]\times\mathbb{R}, \qquad \phi_\nu(S,\cdot) = \log\cosh .$$

In this notation, $\mathcal{M}_\nu(u) = \partial_u\phi_\nu(0,u)$. Along $\mathrm{d}X_s = 2\nu([0,s])\partial_u\phi_\nu(s,X_s)\,\mathrm{d}s + \sqrt{2}\,\mathrm{d}B_s$, we put $\mathsf{m}_s := \partial_u\phi_\nu(s,X_s)$. Recall that $g_m(a) = g_m(\nu)$, with g_m as in [6, (1.7)], $J_m(x) = \sup_{a\in\mathcal{Q}_\infty}\{\int xa - g_m(a)\}$ [6, (9.1)], and $I(m) = \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m)$.

Above the support of ν, we have $\nu([0,s]) = 1$ and $\phi_\nu(s,u) = \log\cosh u + S - s$. Hence g_m does not depend on the horizon $S \geq \sup\operatorname{supp}\nu$, nor does m_s on $[0,S]$, and we continue m_s to all $s \geq 0$ accordingly, setting $\mathsf{m}_s = \tanh X_s$ above the support. The potential of [6, (6.19)] is stopped at the horizon; it agrees with the continued one on $[0,S]$, and we always take S above the supports of all laws in play.

Proposition 4.1. *Let $|m| < 1$, and let ν be a law on $\mathbb{R}_+$ with bounded support and path a.*

(a) [6, (1.7), (6.14), (6.15)] *One has $g_m(\nu) = S + \sup_{u\in\mathbb{R}}\{um - \phi_\nu(0,u)\}$. The maximizer $\lambda(\nu,m)$ is unique, and $\mathcal{M}_\nu(\lambda(\nu,m)) = m$.*

(b) [6, Lemma 6.7, (6.25)–(6.29)] *$\partial_m g_m(\nu) = \lambda(\nu,m)$. The function $m \mapsto g_m(\nu) - I(m)$ is even, convex and $2\int s\,\mathrm{d}\nu(s)$-Lipschitz.*

(c) [6, Corollary 6.8], [1, (2.9)] *$\nu \mapsto g_m(\nu)$ is concave. Put $z_\nu(s) := \mathbb{E}[\mathsf{m}_s^2] \in [m^2,1)$, with z_ν nondecreasing, increasing to 1, by the continuation above.*

(d) [6, Proposition 6.10, (6.45), (6.46)] *g_m is convex in the path and 1-Lipschitz for L^1, with supporting path $z_a(\theta) := z_\nu(a(\theta))$:*

$$g_m(a') \geq g_m(a) + \int (a'-a)z_a. \tag{4.2}$$

(e) [6, Lemmas 9.1 and 9.5] *$J_m(z_a) = \int z_a a - g_m(a)$, and J_m is convex in the law. $J_m(x) = -I(m)$ for constant $x \leq m^2$.*

(f) [6, Lemma 6.3] *If $\nu_n \to \nu$ in W_1 with supports in a common interval $[0,S]$, then $\phi_{\nu_n} \to \phi_\nu$ and $\partial_u\phi_{\nu_n} \to \partial_u\phi_\nu$ uniformly. Also $|g_m(\nu) - g_m(\nu')| \leq W_1(\nu,\nu')$, which is the Lipschitz bound of* (d)*, since $W_1(\nu,\nu')$ is the L^1 distance of the paths.*

At $|m| = 1$ one has $g_{\pm1}(a) = \int a + \log 2$ [6, (6.22)], $z_a \equiv 1$ and $J_{\pm1} \equiv -\log 2$ on $\mathcal{Q}_1$, by Lemma 2.2. When the slice varies we write $z_{\nu,m}$, $z_{a,m}$.

On the sphere side, we use the functional of Section 2, changing its notation in this section for convenience: its path is π, written ϑ there, and $\mathcal{D}$, K, ζ_π, $\mathcal{R}_y$ below are T, k, z_ϑ, r_y there. For $\pi \in \mathcal{Q}_\infty$ let $d_{\pi,\ell}(p) := \ell + 2\int_0^1 (p - \pi(\theta))_+\,\mathrm{d}\theta$, $K(\pi) := \pi(1) - \int\pi$ and $K_\pi(p) := \int_0^1 (\pi(1) - \pi(\theta)\vee p)\,\mathrm{d}\theta$. As in [9, (2.23)–(2.25)],

$$\psi_{\mathrm{S}}(\pi) = -\inf_{b>2K(\pi)} \mathcal{D}(\pi,b), \qquad \mathcal{D}(\pi,b) := \int_0^{\pi(1)} \frac{\mathrm{d}p}{b - 2K_\pi(p)} + \frac{1}{2}(b - 1 - \log b) - \pi(1).$$

By [9, Lemma 2.4], there is a unique $\ell_0(\pi) > 0$ with $2\int_0^\infty d_{\pi,\ell_0}^{-2}\,\mathrm{d}p = 1$, and the minimizer is $b_0 := \ell_0 + 2K(\pi) = d_{\pi,\ell_0}(\pi(1))$ [9, (2.35)]. We put $\mathsf{y}_\pi(p) := 1 - 2\int_p^\infty d_{\pi,\ell_0(\pi)}^{-2}\,\mathrm{d}p'$ and $\zeta_\pi := \mathsf{y}_\pi \circ \pi$ [9, (2.37)], so $\zeta_\pi \in \mathcal{Q}_1$ and $\zeta_\pi(1) = 1 - 1/b_0 < 1$. The functional ψ_{S} is convex and 1-Lipschitz in the path, with supporting path ζ_π [9, Proposition 2.8, (2.52), Lemma 2.9]:

$$\psi_{\mathrm{S}}(\pi') \geq \psi_{\mathrm{S}}(\pi) + \int (\pi' - \pi)\zeta_\pi. \tag{4.3}$$

Recall that its conjugate is $\psi_{\mathrm{S}}^*(y) = \sup_{\pi\in\mathcal{Q}_\infty}\{\int y\pi - \psi_{\mathrm{S}}(\pi)\}$ for $y \in \mathcal{Q}_1$.

4.2. **Reduced forms and the radius-free formula.** Eliminating one of the two paths in (4.1) leaves a problem in the other.

Proposition 4.2. *For $|m| \le 1$ and $\gamma > 0$,*

$$\Phi^{\mathrm{mix}}(\gamma; m) = \tau + \min_{y \in \mathcal{Q}_1} \{\alpha \psi_{\mathrm{S}}^*(y) - g_m(\tau y)\} \tag{4.4}$$

$$= \tau + \min_{x \in \mathcal{Q}_1} \{J_m(x) - \alpha \psi_{\mathrm{S}}(cx)\}. \tag{4.5}$$

For types,

$$\Phi_r^{\mathrm{mix}}(\gamma; \boldsymbol{t}) = \tau + \min_{y \in \mathcal{Q}_1} \Big\{\alpha \psi_{\mathrm{S}}^*(y) - \mathbb{E}_\eta g_{t_\eta}(\tau y)\Big\}. \tag{4.6}$$

Proof. For $a := \tau y$ we have $J_m(x) - \int xa \ge -g_m(a)$ by the definition of J_m, with equality at $x = z_a \in \mathcal{Q}_1$ by Proposition 4.1(e), so minimizing over x in (4.1) gives (4.4). Similarly, for $\pi := cx$ we have $\alpha[\psi_{\mathrm{S}}^*(y) - \int y\pi] \ge -\alpha\psi_{\mathrm{S}}(\pi)$, with equality at $y = \zeta_\pi \in \mathcal{Q}_1$ by (4.3), and since $\tau = \alpha c$, minimizing over y gives (4.5). For types, we minimize over each x_η separately in the first step. □

Proposition 4.3. *For fixed γ, the function $m \mapsto \Phi^{\mathrm{mix}}(\gamma; m) + I(m)$ is concave and even on $[-1, 1]$. So $m \mapsto \Phi^{\mathrm{mix}}(\gamma; m)$ is concave.*

Proof. In (4.4), $-g_m(a) + I(m)$ is concave in m by Proposition 4.1(b), and an infimum of concave functions is concave. □

In the next lemma, we write the spherical terms in the law, the variable in which they are convex. We set

$$\mathcal{R}_{\mathcal{L}}(w) := \frac{1}{2} \int_0^w \frac{\mathrm{d}w'}{\delta_{\mathcal{L}}(w')^2},$$

and $\mathcal{R}_y := \mathcal{R}_{\mathcal{L}_y}$.

Lemma 4.4. *(i) For every $y \in \mathcal{Q}_1$, with values in $[0, +\infty]$,*

$$\psi_{\mathrm{S}}^*(y) = \mathcal{C}(\mathcal{L}_y) := \frac{1}{2} \int_0^1 \Big[\frac{1}{\delta_{\mathcal{L}_y}(w)} - \frac{1}{1 - w}\Big] \mathrm{d}w. \tag{4.7}$$

So ψ_{S}^* is a strictly convex function of the law.

(ii) ψ_{S} is strictly concave in the law of its argument.

(iii) For $\pi \in \mathcal{Q}_\infty$ and $y := \zeta_\pi$ one has $\mathcal{R}_y(\mathsf{y}_\pi(p)) = p$ for all $p \ge 0$. In particular $\mathcal{R}_y(y) = \pi$.

Proof. (i) For $\hat{y} < 1$, Lemma 2.1 reads $\psi_{\mathrm{S}}^*(y) = \frac{1}{2}[\int_0^{\hat{y}} \mathrm{d}w/\delta_y(w) + \log(1 - \hat{y})]$. Moreover, $\int_0^{\hat{y}} \mathrm{d}w/(1 - w) = -\log(1 - \hat{y})$, and the integrand of (4.7) vanishes above $\hat{y}$. For $\hat{y} = 1$, we use $y_\varepsilon := (1 - \varepsilon)y$: path convexity and $\psi_{\mathrm{S}}^*(0) = 0$ give $\psi_{\mathrm{S}}^*(y_\varepsilon) \le (1 - \varepsilon)\psi_{\mathrm{S}}^*(y)$, and lower semicontinuity gives $\psi_{\mathrm{S}}^*(y_\varepsilon) \to \psi_{\mathrm{S}}^*(y)$. Since δ_{y_ε} decreases to δ_y, we obtain (4.7) by monotone convergence. Strictness holds because $\mathcal{L} \mapsto \delta_{\mathcal{L}}$ is affine and injective and $d \mapsto 1/d$ is strictly convex.

(ii) Let $\pi \in \mathcal{Q}_\infty$. The function $d_{\pi,\ell}(p) = \ell + 2\int (p - p')_+ \, \mathrm{d}\mathcal{L}_\pi(p')$ is affine in $(\mathcal{L}_\pi, \ell)$, and with $\ell = b - 2K(\pi) > 0$ we have $b - 2K_\pi(p) = d_{\pi,\ell}(p)$ on $[0, \pi(1)]$ and $d_{\pi,\ell}(p) = b + 2(p - \pi(1))$ above, by [9, Lemma 2.4]. Thus for any $r_0 \ge \pi(1)$,

$$\mathcal{D}(\pi, b) = \mathcal{E}(\mathcal{L}_\pi, \ell) := \int_0^{r_0} \frac{\mathrm{d}p}{d_{\pi,\ell}(p)} - \frac{1}{2} \log d_{\pi,\ell}(r_0) + \frac{1}{2} d_{\pi,\ell}(r_0) - r_0 - \frac{1}{2}. \tag{4.8}$$

For r_0 above the supports of the laws in play, each term is convex in $(\mathcal{L}_\pi, \ell)$, and the first strictly so. Hence $\inf_\ell \mathcal{E}(\mathcal{L}_\pi, \ell)$, which is attained by [9, Lemma 2.4], is strictly convex in $\mathcal{L}_\pi$, and $\psi_{\mathrm{S}}(\pi) = -\inf_\ell \mathcal{E}(\mathcal{L}_\pi, \ell)$.

(iii) We put $\ell := \ell_0(\pi)$ and $d := d_{\pi,\ell}$. The map y_π is an increasing bijection $[0,\infty) \to [0,1)$ with $\mathsf{y}'_\pi = 2/d^2$, and $\{y \le \mathsf{y}_\pi(p)\} = \{\pi \le p\}$. Since $d'(p) = 2\mathbb{P}(\pi(U) \le p)$ and $\delta'_y(w) = -\mathbb{P}(y(U) \le w)$ almost everywhere, the functions $p \mapsto \delta_y(\mathsf{y}_\pi(p))$ and $p \mapsto 1/d(p)$ have the same derivative $-2\mathbb{P}(\pi(U) \le p)/d(p)^2$, and both tend to 0 at infinity. It follows that $\delta_y(\mathsf{y}_\pi(p)) = 1/d(p)$. The substitution $w' = \mathsf{y}_\pi(p')$ gives $\mathcal{R}_y(\mathsf{y}_\pi(p)) = \frac{1}{2}\int_0^p \mathsf{y}'_\pi(p')d(p')^2\,\mathrm{d}p' = p$. □

Theorem 4.5. *Let $\alpha, \beta > 0$ and $|m| < 1$. For $x \in \mathcal{Q}_1$ let*

$$\mathcal{S}_\beta(x) := \frac{\beta}{2}\int_0^1 \frac{\mathrm{d}w}{D_x(w)} \quad \text{if } D_x(0) > 0, \qquad \mathcal{S}_\beta(x) := +\infty \quad \text{otherwise.} \tag{4.9}$$

Then

$$\Phi(m) = \min_{x\in\mathcal{Q}_1}\{J_m(x) + \alpha\mathcal{S}_\beta(x)\}. \tag{4.10}$$

The objective is strictly convex in the law of x, so the optimizer x^* is unique. The optimal radius is

$$v^* = 1 + \beta R_{x^*}(1), \qquad R_x(w) := \int_0^w \frac{\mathrm{d}w'}{D_x(w')^2}. \tag{4.11}$$

The proof is based on the fact that at fixed radius the objective is convex in the law of x, and at fixed law concave in the radius, so that a minimax theorem exchanges the two. The supremum over the radius is then explicit, because the radius and the spherical multiplier enter through one affine term once the multiplier is scaled by v.

Proof. Step 1: minimax. By (2.7) and (4.5), $\Phi(m) = \sup_v \min_{\mathcal{L}} f(v,\mathcal{L})$ with

$$f(v,\mathcal{L}) := \mathrm{rad}(v) + \alpha c_v + J_m(\mathcal{L}) - \alpha\psi_{\mathrm{S}}(c_v x_{\mathcal{L}}),$$

where $x_{\mathcal{L}}$ is the path with law $\mathcal{L}$. For fixed v, f is convex by Proposition 4.1(e) and Lemma 4.4(ii), and lower semicontinuous in $\mathcal{L}$ on the compact convex set of laws on $[0,1]$, while for fixed $\mathcal{L}$ with $J_m(\mathcal{L}) < \infty$, f is concave and continuous in v, since rad is concave, c_v is affine and ψ_{S} is convex in the path. As f takes the value $+\infty$ where $J_m = +\infty$, we apply Sion's theorem [18] on $v \in [1/n, n]$ and $\mathcal{L}$ in the convex set $\{J_m < \infty\}$, where f is real, and let $n \to \infty$, using lower semicontinuity on the compact space of laws. This gives $\Phi(m) = \min_{\mathcal{L}} \sup_v f(v,\mathcal{L})$, with attainment.

Step 2: the supremum over v. We fix x with law $\mathcal{L}$ and let $\Delta_x(w) := \int_0^1 (w - x(\theta))_+\,\mathrm{d}\theta$, so $\Delta_x(1) = 1 - \bar{x}$ and $\delta_x(w) = 1 - \bar{x} - \Delta_x(w)$. The substitution $p = cw$ in (4.8) with $r_0 = c$ gives, for $\ell > 0$ and $b = \ell + 2K(cx)$,

$$\mathcal{D}(cx,b) = c\int_0^1 \frac{\mathrm{d}w}{\ell + 2c\Delta_x(w)} - \frac{1}{2}\log(\ell + 2c(1-\bar{x})) + \frac{\ell}{2} - c\bar{x} - \frac{1}{2}.$$

With $\ell = v\vartheta$ and $2c = \beta v$, where $\vartheta > 0$, the two terms in $\log v$ cancel, and

$$\frac{1}{2}(1 + \log v - v) + c + \mathcal{D}(cx,b) = \Gamma_x(\vartheta) + \frac{v}{2}(\vartheta - D_x(0)),$$
$$\Gamma_x(\vartheta) := \frac{\beta}{2}\int_0^1 \frac{\mathrm{d}w}{\vartheta + \beta\Delta_x(w)} - \frac{1}{2}\log(\vartheta + \beta(1-\bar{x})).$$

Hence, for each fixed $x \in \mathcal{Q}_1$ and every $v > 0$,

$$\mathsf{k}_x(v) := \mathrm{rad}(v) + \alpha c_v - \alpha\psi_{\mathrm{S}}(c_v x) = \alpha \inf_{\vartheta>0}\Big\{\Gamma_x(\vartheta) + \frac{v}{2}(\vartheta - D_x(0))\Big\}. \tag{4.12}$$

The function Γ_x is convex and strictly decreasing, and $\Gamma_x(0+) = +\infty$ since $\Delta_x(w) \le w$. Also, $\sup_v f(v,\mathcal{L}) - J_m(\mathcal{L}) = \sup_v \mathsf{k}_x(v)$.

If $D_x(0) > 0$, the choice $\vartheta = D_x(0)$ shows that $\sup_v \inf_\vartheta\{\cdots\} \le \Gamma_x(D_x(0))$, and $v_x := -2\Gamma_x'(D_x(0))$ makes $D_x(0)$ the minimizer in ϑ, so the value equals $\Gamma_x(D_x(0))$. Since $D_x(0) + \beta\Delta_x(w) = D_x(w)$ and $D_x(0) + \beta(1-\bar{x}) = 1$, this value is $\mathcal{S}_\beta(x)$, and $v_x = 1 + \beta\int_0^1 \mathrm{d}w/D_x(w)^2$.

If $D_x(0) < 0$, then for $v \ge 1$, $\Gamma_x(\vartheta) + v\vartheta/2 \ge -C_x$ with C_x finite, so the value is at least $-C_x + \frac{v}{2}|D_x(0)| \to \infty$.

Finally, if $D_x(0) = 0$, then for each n we pick $\vartheta_n > 0$ with $\Gamma_x \ge n$ on $(0, \vartheta_n)$. Since $\Gamma_x(\vartheta) + v\vartheta/2 \ge -C_x + (v-1)\vartheta_n/2$ on $\vartheta \ge \vartheta_n$, the value tends to infinity with v.

Thus $\sup_v f(v, \mathcal{L}) = J_m(\mathcal{L}) + \alpha\mathcal{S}_\beta(x)$, which with Step 1 gives (4.10). At the optimum, $(v^*, \mathcal{L}_{x^*})$ is a saddle point of f, so v^* is the maximizer v_{x^*} of $f(\cdot, \mathcal{L}_{x^*})$, which is (4.11).

Step 3: convexity and attainment. The function D_x is affine in $\mathcal{L}$ and positive where $\mathcal{S}_\beta < \infty$, $d \mapsto 1/d$ is strictly convex, $\mathcal{L} \mapsto \delta_{\mathcal{L}}$ is injective, and J_m is convex in $\mathcal{L}$. The objective is lower semicontinuous, being a supremum over v of lower semicontinuous functions, and finite at the constant $x = q$ with $q \in (\max(m^2, 1 - 1/\beta), 1)$, since $J_m(q) < \infty$ for $|m| < 1$ by Lemma 2.2. This completes the proof. □

4.3. **The optimal paths.** We fix (γ, m) with $|m| < 1$. For a law $\mathcal{L}$ on $[0,1]$, let $\nu_{\mathcal{L}}$ be the law of τy, where y has law $\mathcal{L}$, and put $\mathcal{G}(\mathcal{L}) := \alpha\mathcal{C}(\mathcal{L}) - g_m(\nu_{\mathcal{L}})$.

Theorem 4.6. *$\mathcal{G}$ is strictly convex and lower semicontinuous, and it has a unique minimizer $\mathcal{L}^*$ on the laws on $[0,1]$, with $\hat{y}^* < 1$. Its value is $\Phi^{\mathrm{mix}}(\gamma; m) - \tau$. If (x, y) minimizes H, then $y = \zeta_{cx}$.*

Proof. Strict convexity follows from (4.7) and Proposition 4.1(c). Compactness and semicontinuity give a minimizer, and (4.4) gives the value. By Proposition 4.2, there is a minimizer (x, y) of H, and then y minimizes $\mathcal{G}$. Also, y minimizes $y' \mapsto \alpha\psi_{\mathrm{S}}^*(y') - \tau\int xy'$, which has the minimizer ζ_{cx} by (4.3). Hence (x, ζ_{cx}) minimizes H too, and ζ_{cx} minimizes $\mathcal{G}$, so $y = \zeta_{cx}$ by strict convexity. In particular, $\hat{y} = 1 - 1/b_0 < 1$. □

Lemma 4.7. *Let $x \in \mathcal{Q}_1$ with $D_x(0) > 0$, and put $v := 1 + \beta R_x(1)$ and $y := \beta R_x(x)/v$. Then $y = \zeta_{c_v x}$.*

Proof. We put $\ell := vD_x(0)$ and $\pi := c_v x$. Then $d_{\pi,\ell}(c_v q) = vD_x(q)$ for all $q \ge 0$, with $D_x(q) = 1 + \beta(q-1)$ above 1. Thus $2\int_0^\infty d_{\pi,\ell}^{-2}\,\mathrm{d}p = \frac{\beta}{v}(R_x(1) + \frac{1}{\beta}) = 1$, which says that ℓ is the multiplier $\ell_0(\pi)$, and

$$\mathsf{y}_\pi(c_v q) = 1 - \frac{\beta}{v}\int_q^\infty \frac{\mathrm{d}q'}{D_x(q')^2} = 1 - \frac{1}{v} - \frac{\beta}{v}(R_x(1) - R_x(q)) = \frac{\beta R_x(q)}{v},$$

where we used $v - 1 = \beta R_x(1)$ in the last step. Hence $\zeta_\pi = \beta R_x(x)/v = y$. □

4.4. **Uniqueness and the derivative.** We will obtain the derivative of Φ in m from an envelope argument with the unique optimizer, and control its size by the slope of the magnetization response, which has an explicit formula. Let ν be a law on $\mathbb{R}_+$ with bounded support, and write $A_\nu := \mathcal{M}_\nu'$.

Lemma 4.8. *(i) $\mathcal{M}_\nu$ is odd and strictly increasing, $\mathcal{M}_\nu(\pm\infty) = \pm 1$, and $\mathcal{M}_\nu$ is concave on $[0, \infty)$. So A_ν is even and nonincreasing in $|u|$.*

(ii) If $m = \mathcal{M}_\nu(u)$ then $A_\nu(u) = 1 - \int z_{\nu,m}\,\mathrm{d}\nu$. So $\partial_m\lambda(\nu, m) = 1/(1 - \bar{x}(\nu, m))$, where $\bar{x}(\nu, \tilde{m}) := \int z_{a,\tilde{m}}$ is the mean of the supporting path at slice $\tilde{m}$ for the fixed law ν.

(iii) $\nu \mapsto A_\nu$ is continuous for W_1, uniformly on compacts, on sets of ν with a common bound on the support.

Proof. It suffices to treat finite-step ν: the general case follows from Proposition 4.1(f), since concavity and monotonicity pass to uniform limits, and (ii), (iii) will follow from the bound on $\partial_u^3\phi_\nu$ below.

We write $\phi = \phi_\nu$ and $\phi^{(k)} := \partial_u^k \phi$. On each piece, $\nu([0,s]) =: \varsigma$ is constant. Evenness of ϕ in u follows from the terminal datum and uniqueness. Differentiating the equation, we obtain

$$-\partial_s\phi^{(1)} = \partial_u^2\phi^{(1)} + 2\varsigma\phi^{(1)}\phi^{(2)}, \qquad -\partial_s\phi^{(2)} = \partial_u^2\phi^{(2)} + 2\varsigma\big(\phi^{(1)}\partial_u\phi^{(2)} + (\phi^{(2)})^2\big),$$
$$-\partial_s\phi^{(3)} = \partial_u^2\phi^{(3)} + 2\varsigma\phi^{(1)}\partial_u\phi^{(3)} + 6\varsigma\phi^{(2)}\phi^{(3)}.$$

Thus $\phi^{(3)}$ is a bounded solution of a linear backward parabolic equation with bounded coefficients on $u > 0$, with $\phi^{(3)}(s,0) = 0$ by oddness. At the top of the piece, $\phi^{(3)} \le 0$ on $u \ge 0$. Indeed, this holds initially, as $\tanh'' = -2\tanh\operatorname{sech}^2 \le 0$, and then by induction over pieces, since at a breakpoint the function is continuous and only ς changes. The maximum principle then gives $\phi^{(3)} \le 0$ on $u \ge 0$, which is the concavity in (i). Oddness is the evenness of ϕ, and strict monotonicity is $\phi^{(2)} > 0$, which we will obtain from the representation below. The limits at $\pm\infty$ hold because $\mathcal{M}_\nu$ takes values in $(-1,1)$ and hits every $m \in (-1,1)$, by Proposition 4.1(a).

(ii) Let $\mathsf{p}_s := \phi^{(2)}(s, X_s)$. With $\mathcal{A} := \partial_s + \partial_u^2 + 2\varsigma\phi^{(1)}\partial_u$, the generator along X, we have $\mathcal{A}\phi^{(1)} = 0$ and $\mathcal{A}\phi^{(2)} = -2\varsigma(\phi^{(2)})^2$. Hence $\mathrm{d}\mathsf{p}_s = -2\varsigma\mathsf{p}_s^2\,\mathrm{d}s +$ martingale, and $\mathrm{d}\,\mathbb{E}\mathsf{m}_s^2 = 2\mathbb{E}\mathsf{p}_s^2\,\mathrm{d}s$ as $\mathrm{d}\mathsf{m}_s = \sqrt{2}\mathsf{p}_s\,\mathrm{d}B_s$. Since $\mathsf{p}_S = \operatorname{sech}^2 X_S = 1 - \mathsf{m}_S^2$, with S above the support, taking expectations gives

$$A_\nu(u) = \mathsf{p}_0 = 1 - z_\nu(S) + \int_0^S \nu([0,s])\,\mathrm{d}z_\nu(s) = 1 - z_\nu(S) + \int \big(z_\nu(S) - z_\nu(s)\big)\,\mathrm{d}\nu(s) = 1 - \int z_\nu\,\mathrm{d}\nu.$$

The same computation started at (s,u) gives $\phi^{(2)}(s,u) \in (0,1]$, and the formula for $\partial_m\lambda$ follows as $\lambda(\nu,\cdot)$ is the inverse of $\mathcal{M}_\nu$.

(iii) By the maximum principle applied to its linear equation, $\phi^{(3)}$ is bounded on $[0,S]\times\mathbb{R}$ by $C\mathrm{e}^{6S}$, uniformly over ν with support in $[0,S]$. With Proposition 4.1(f) and Arzelà–Ascoli this gives (iii). □

Theorem 4.9. *Let $\alpha, \beta > 0$ and $|m| < 1$.*

(a) (Uniqueness.) *At fixed radius, the minimizer (x,y) of H is unique. At the optimal radius, the triple (x^*, y^*, v^*) is unique, with*

$$y^* = \beta R_{x^*}(x^*)/v^*, \qquad a^* := \frac{\alpha\beta^2}{2} R_{x^*}(x^*) = \tau_{v^*} y^*, \qquad x^* = z_{a^*,m}. \tag{4.13}$$

In all cases $\operatorname{supp}\mathcal{L}_{x^*} \subset [m^2, 1)$.

(b) (Continuity.) *The map $(m,\alpha,\beta) \mapsto (\mathcal{L}_{x^*}, \mathcal{L}_{y^*}, v^*)$ is continuous on $(-1,1)\times(0,\infty)^2$, and the optimal law of y at fixed radius is continuous in (γ, m).*

(c) (Differentiability.) *$\Phi^{\mathrm{mix}}(\gamma;\cdot)$ and Φ are C^1 on $(-1,1)$, with*

$$\Phi'(m) = -h(m), \qquad h(m) := \lambda(\nu^*(m), m), \qquad \Lambda'(m) = \beta m - h(m), \tag{4.14}$$

where $\nu^*(m)$ is the law of a^* at slice m. The function h is odd and continuous, and for $m \in [0,1)$,

$$\max\Big\{\operatorname{atanh} m, \frac{m}{1 - \bar{x}(\nu^*(m), 0)}\Big\} \le h(m) \le \min\Big\{\operatorname{atanh} m + 2\int s\,\mathrm{d}\nu^*(m)(s), \frac{m}{1 - \bar{x}(\nu^*(m), m)}\Big\}, \tag{4.15}$$

with $\bar{x}(\nu^*(m), m) = \int x^*$. So $\Lambda'(m) \to -\infty$ as $m \to 1$.

(d) ($m = 0$.) $m = 0$ *is a strict local maximum of* Λ *at every* $T > 0$*: with* $\bar{x}_0$ *the mean of* x^* *at slice* $m = 0$*, one has* $1 - \beta + \beta\bar{x}_0 > 0$ *and*

$$\limsup_{m\to 0+} \frac{\Lambda'(m)}{m} \le \beta - \frac{1}{1-\bar{x}_0} = -\frac{1-\beta+\beta\bar{x}_0}{1-\bar{x}_0} < 0. \tag{4.16}$$

Proof. (a) At fixed radius, we use Theorem 4.6 for y, and for x the objective in (4.5) is convex in $\mathcal{L}_x$ by Proposition 4.1(e), and strictly so by Lemma 4.4(ii), since τ, c are fixed and cx has an injective law. For the triple at the optimal radius, Theorem 4.5 gives the unique x^* with $v^* = 1 + \beta R_{x^*}(1)$, and by Lemma 4.7, we have $\zeta_{c_{v^*}x^*} = \beta R_{x^*}(x^*)/v^*$, which is y^* by Theorem 4.6. Finally, $(z_{\tau_{v^*}y^*}, y^*)$ minimizes H at v^* by Proposition 4.2, so $x^* = z_{a^*,m}$ by uniqueness of x, and the values of x^* lie in $[m^2, 1)$ by Proposition 4.1(c).

(b) By Proposition 4.1(b),(f), and since $\nu_{\mathcal{L}}$ has support in $[0,\tau]$, $g_m(\nu_{\mathcal{L}})$ is jointly continuous in $(m, \tau, \mathcal{L})$ on $(-1,1) \times \mathbb{R}_+ \times \{\text{laws on } [0,1]\}$. The functional $\mathcal{C}$ is lower semicontinuous and independent of the parameters, so $\mathcal{G}$ Γ-converges when the parameters converge. As the domain is compact and limits of minimizers are minimizers, uniqueness gives continuity of the optimal law of y. The function $v \mapsto \mathrm{rad}(v) + \Phi^{\mathrm{mix}}(\beta v; m)$ is strictly concave in v and jointly continuous in all parameters, with a unique maximizer by Theorem 2.3. We pick $a < v^* < b$ where it lies strictly below its maximum. These inequalities persist at nearby parameters, and concavity then confines the nearby maximizers to (a, b), so v^* is continuous. Finally, $\mathcal{L}_{x^*}$ is the law of $z_{\tau_{v^*}y^*}$, which is continuous by Proposition 4.1(f) and the continuity of z_ν.

(c) We fix γ, and let y_m be the minimizer of (4.4) and ν_m the law of τy_m. For $m, m' \in (-1,1)$, by (4.4) and minimality,

$$g_m(\nu_m) - g_{m'}(\nu_m) \ge \Phi^{\mathrm{mix}}(\gamma; m') - \Phi^{\mathrm{mix}}(\gamma; m) \ge g_m(\nu_{m'}) - g_{m'}(\nu_{m'}).$$

By Proposition 4.1(b) and the mean value theorem, each side is $-(m'-m)\lambda(\nu, \tilde{m})$ for some $\tilde{m}$ between m and m', with $\nu = \nu_m$ or $\nu_{m'}$. The function $\lambda(\nu, \tilde{m})$ is jointly continuous: for $\nu_n \to \nu$, the convex functions $g_\cdot(\nu_n)$ converge uniformly to the differentiable $g_\cdot(\nu)$, and derivatives of convex functions then converge locally uniformly [17, Theorem 25.7]. Hence $\Phi^{\mathrm{mix}}(\gamma; \cdot)$ is differentiable with derivative $-\lambda(\nu_m, m)$, which is continuous in (γ, m) by (b). For Φ, with $v_m := v^*(m)$, by (2.7),

$$\Phi^{\mathrm{mix}}(\beta v_{m'}; m') - \Phi^{\mathrm{mix}}(\beta v_{m'}; m) \ge \Phi(m') - \Phi(m) \ge \Phi^{\mathrm{mix}}(\beta v_m; m') - \Phi^{\mathrm{mix}}(\beta v_m; m),$$

and joint continuity of $\partial_m \Phi^{\mathrm{mix}}(\gamma; m)$ gives (4.14). Oddness of h follows from evenness of g_m in m and uniqueness.

For (4.15), we have $\operatorname{atanh} m \le \lambda(\nu, m) \le \operatorname{atanh} m + 2\int s \,\mathrm{d}\nu$ by Proposition 4.1(b), as $g_m - I(m)$ is even, convex and $2\int s\,\mathrm{d}\nu$-Lipschitz. By Lemma 4.8, $\lambda(\nu, m) = \int_0^m \mathrm{d}\tilde{m}/A_\nu(\lambda(\nu, \tilde{m}))$, with $A_\nu(\lambda(\nu, \tilde{m}))$ between $A_\nu(\lambda(\nu, m)) = 1 - \bar{x}(\nu, m)$ and $A_\nu(0) = 1 - \bar{x}(\nu, 0)$. The identity $\bar{x}(\nu^*(m), m) = \int x^*$ is (4.13). This finishes the proof of (c).

(d) At $m = 0$, $D_{x^*}(0) = 1 - \beta + \beta\bar{x}_0 > 0$ since $\mathcal{S}_\beta(x^*) < \infty$ with $\alpha > 0$, so $\beta < 1/(1-\bar{x}_0)$. As $m \to 0+$, $\nu^*(m) \to \nu^*(0)$ by (b), and $\bar{x}(\nu^*(m), 0) \to \bar{x}(\nu^*(0), 0)$ by Lemma 4.8(iii),(ii), and this limit equals $\bar{x}_0$ by (4.13). Then (4.15) gives $h(m) \ge m/(1 - \bar{x}(\nu^*(m), 0))$, which is (4.16). Since Λ is even and $\Lambda' < 0$ on a right neighborhood of 0, $m = 0$ is a strict local maximum. □

4.5. **The potential of a path.** In Section 6, we will compare a configuration of several retained patterns with one pattern at larger overlap through a bound linear in a path. For $a \in \mathcal{Q}_\infty$ with law ν, we put

$$LC_a(m) := -g_m(a) = \inf_u \{\Psi_a(u) - um\}, \qquad \Psi_a(u) := \phi_\nu(0, u) - S, \qquad \mathcal{M}_a := \Psi_a' = \mathcal{M}_\nu. \tag{4.17}$$

Proposition 4.1 shows that LC_a is concave and $-\partial_m LC_a(m) = \lambda(\nu, m)$, and Lemma 4.8 that it is C^2 on $(-1, 1)$, with $\partial_m^2 LC_a(m) = -1/A_\nu(\lambda(\nu, m))$. By (4.6), for every $y \in \mathcal{Q}_1$ and $\gamma > 0$,

$$\Phi_r^{\mathrm{mix}}(\gamma; \boldsymbol{t}) \le \tau + \alpha\psi_{\mathrm{S}}^*(y) + \mathbb{E}_\eta LC_{\tau y}(t_\eta), \tag{4.18}$$

with equality when $r = 1$ and y is the optimal path at the slice.

4.6. **The capacity for Gaussian rows.** We now define the capacity of Gaussian rows with one binary pattern, at every $T > 0$, by a condition on the multiplier h of Theorem 4.9.

Definition 4.10. For $\alpha > 0$ and $T = 1/\beta > 0$, let $h(m) = h(m; \alpha, \beta)$ and

$$\mathsf{h}(\alpha, \beta) := \sup_{m \in (0,1)} \big(\beta m - h(m; \alpha, \beta)\big) \ge 0, \quad \mathcal{W}_T := \{\alpha > 0 : \mathsf{h}(\alpha, 1/T) > 0\}, \quad \alpha_{\mathrm{c}}(T) := \sup \mathcal{W}_T, \tag{4.19}$$

with the supremum of the empty set equal to 0. A *well* at (α, T) means: there exist $0 < m_b < m^* < 1$, with m_b a local minimum and m^* a local maximum of Λ, such that $\Lambda(m_b) < \Lambda(m^*)$ and $\Lambda(m_b) = \min_{[0,m^*]} \Lambda$.

Unwinding the definitions, we obtain

$$\alpha_{\mathrm{c}}(T) = \sup\big\{\alpha > 0 : \exists m \in (0, 1),\ \beta m > \lambda(\nu^*(m), m)\big\},$$

where $\nu^*(m)$ is the law of $a^* = \frac{\alpha\beta^2}{2} R_{x^*}(x^*)$ and x^* is the unique minimizer of the strictly convex functional $x \mapsto J_m(x) + \alpha\mathcal{S}_\beta(x)$. By the mean value theorem and (4.14), $\mathcal{W}_T$ is the union over m_s of the sets (1.11), and $\mathcal{W}_T(m_s) = \{\alpha > 0 : \exists m \in [m_s, 1),\ \beta m > h(m; \alpha, \beta)\}$.

Remark 4.11. By Theorem 4.9(d), the state $m = 0$ is always a strict local maximum for $\alpha > 0$. Hence a point with $\Lambda' > 0$ always lies on the inner wall of a true double well, however small m is, and no window near $m = 1$ is needed.

Theorem 4.12. *For every $T > 0$ the following hold.*

(a) $\alpha \in \mathcal{W}_T$ if and only if Λ has a well at (α, T), if and only if there is $m \in (0, 1)$ with $\mathcal{M}_{a^*(m)}(\beta m) > m$, where $a^*(m)$ is the optimal path of Theorem 4.9 at slice m. So the stationary points $m > 0$ of Λ are exactly the solutions of the generalized Mattis equation

$$m = \mathcal{M}_{a^*(m)}(\beta m). \tag{4.20}$$

(b) $\mathcal{W}_T$ is open in $(0, \infty)$.

(c) For $T \ge 1$, $\mathcal{W}_T$ is empty and $\alpha_{\mathrm{c}}(T) = 0$. For $T < 1$, $\mathcal{W}_T$ contains $(0, \alpha_0(T))$ for some $\alpha_0(T) > 0$, so $\alpha_{\mathrm{c}}(T) > 0$.

(d) At a positive boundary point $\alpha_b > 0$ of $\mathcal{W}_T$, in particular at $\alpha_{\mathrm{c}}(T)$ when $T < 1$ by Corollary 4.15, one has $\mathsf{h}(\alpha_b, 1/T) = 0$, and there is $m_c \in (0, 1)$ with $h(m_c) = \beta m_c$ and $\Lambda' \le 0$ on $[0, 1)$. So at the boundary Λ has an interior stationary contact: the well disappears at positive overlap, with no collapse to $m = 0$ or $m = 1$.

For a constant path $a \equiv a_0$, we have $\mathcal{M}_a(u) = \mathbb{E}\tanh(u + \sqrt{2a_0}\, z)$, and with $2a_0 = \alpha\beta^2\mathsf{r}$, equation (4.20) becomes the Mattis equation $m = \mathbb{E}\tanh(\beta m + \beta\sqrt{\alpha\mathsf{r}}\, z)$ of [2, 3], which is the replica symmetric case.

We will use three facts about Λ': it is negative just above 0, it tends to $-\infty$ at 1, and it depends continuously on the load. Hence a single point with $\Lambda' > 0$ produces a true double well, and at a boundary load the contact stays inside $(0, 1)$.

Proof. (a) If there is a well, the mean value theorem gives m with $\Lambda'(m) > 0$. Conversely, let $\Lambda'(m_\circ) > 0$ with $m_\circ \in (0, 1)$. Since Λ is strictly decreasing near 0+ by Theorem 4.9(d) and $\Lambda'(m_\circ) > 0$, the minimum of Λ on $[0, m_\circ]$ is attained at an interior point. By (4.15),

$\Lambda' \to -\infty$ at 1, so the maximum of Λ on $[m_\circ, 1)$ is attained at some $m^* \in (m_\circ, 1)$, with $\Lambda(m^*) > \Lambda(m_\circ)$. We let m_b be a minimizer of Λ on $[0, m^*]$; it is interior and a local minimum, and $\Lambda(m_b) \le \Lambda(m_\circ) < \Lambda(m^*)$. It remains to treat the last form. Here $\mathcal{M}_{a^*(m)}$ is strictly increasing with $\mathcal{M}_{a^*(m)}(h(m)) = m$, so $\beta m > h(m)$ if and only if $\mathcal{M}_{a^*(m)}(\beta m) > m$, and likewise with equalities. This gives (4.20), by (4.14).

(b) The map $(m, \alpha) \mapsto h$ is continuous by Theorem 4.9(b),(c) and Proposition 4.1(f), so a point with $\beta m - h(m) > 0$ persists under small changes of α.

(c) For $T \ge 1$, $h(m) \ge \operatorname{atanh} m > m \ge \beta m$ for $m \in (0, 1)$, by (4.15). If $T < 1$, let m_{CW} be the positive maximizer of $m \mapsto \beta m^2/2 - I(m)$, so $\beta(1 - m_{\mathrm{CW}}^2) < 1$, as the Curie–Weiss root is stable, and pick $m_\circ \in (0, m_{\mathrm{CW}})$ with $D_{m_\circ^2}(0) = 1 - \beta(1 - m_\circ^2) > 0$. We have $\Phi(m) \ge -I(m)$, because $J_m \ge -g_m(0) = -I(m)$ and $\mathcal{S}_\beta > 0$. On the other hand, by (4.10) with $x = m_\circ^2$ and $J_{m_\circ}(m_\circ^2) = -I(m_\circ)$, we have $\Phi(m_\circ) \le -I(m_\circ) + \alpha \mathcal{S}_\beta(m_\circ^2)$. Hence for α below $[\beta m_{\mathrm{CW}}^2/2 - I(m_{\mathrm{CW}}) - \beta m_\circ^2/2 + I(m_\circ)]/\mathcal{S}_\beta(m_\circ^2)$, we get $\Lambda(m_\circ) < \beta m_{\mathrm{CW}}^2/2 - I(m_{\mathrm{CW}}) \le \Lambda(m_{\mathrm{CW}})$, and the mean value theorem gives a point with $\Lambda' > 0$ in $(m_\circ, m_{\mathrm{CW}})$.

(d) By (b), $\alpha_b \notin \mathcal{W}_T$, so $\mathsf{h}(\alpha_b, 1/T) = 0$, that is, $\Lambda' \le 0$ on $[0, 1)$ at α_b. We take $\alpha_n \to \alpha_b$ in $\mathcal{W}_T$, with m_n such that $\beta m_n - h(m_n; \alpha_n) > 0$. By (4.15), m_n stays below $1 - \varepsilon$, since $h \ge \operatorname{atanh} m > \beta$ for m near 1. Also, m_n stays away from 0. Indeed, as in the proof of Theorem 4.9(d), $\bar{x}(\nu^*(m; \alpha), 0)$ tends to $\bar{x}_0(\alpha_b)$ as $(m, \alpha) \to (0, \alpha_b)$, so (4.15) and (4.16) give $h(m; \alpha) \ge (\beta + \varepsilon)m$ near $(0, \alpha_b)$. Hence (m_n) has limit points in $(0, 1)$, and every limit point m_c has $\beta m_c - h(m_c; \alpha_b) \ge 0$, and ≤ 0 since $\mathsf{h} = 0$ at α_b. □

It remains to bound the load. The next lemma is an estimate at every fixed radius: the optimal spherical path has a top value $\hat{y}$ bounded below explicitly, which bounds the slope of the response from above. Using $v^* = 1 + \beta R_{x^*}(1) \ge 1 + \beta$, which holds because $0 < D_{x^*} \le 1$, we will deduce from it a criterion against low-overlap wells and the uniform bound $\mathcal{W}_T \subset (0, 2)$.

We fix $\gamma > 0$ and $m \in (0, 1)$, and let y be the optimal path of (4.4) at slice m, $a := \tau y$, and ν its law. Let $\mathsf{f}(s) := \mathbb{E} \tanh^2(\sqrt{s}\, z)$, which is increasing in $s \ge 0$, and

$$\hat{y}_{\mathrm{low}}(\gamma, \alpha) := \inf\Big\{ w \in (0, 1) : \frac{w}{(1 - w)^2} \ge \gamma \mathsf{f}(\gamma \alpha w) \Big\}.$$

Lemma 4.13. *For all $\alpha, \gamma > 0$ and $m \in (0, 1)$:*

(a) $z_\nu(\tau w) = \frac{2}{\gamma} \mathcal{R}_y(w)$ on the closure of the range of y. In particular $\check{y} > 0$, $z_\nu(\tau \check{y}) = \check{y}/(\gamma \delta_y(0)^2)$ and $z_\nu(\tau \hat{y}) = \frac{2}{\gamma} \mathcal{R}_y(\hat{y})$.

(b) $\alpha\gamma A_\nu(\lambda(\nu, m))^2 \le \dfrac{1}{\gamma \delta_y(0)^2} - \dfrac{m^2}{\check{y}} \le \dfrac{1}{\hat{y}}$.

(c) $\hat{y}/(1 - \hat{y})^2 \ge \gamma z_\nu(\tau \hat{y}) \ge \gamma \mathsf{f}(\gamma \alpha \hat{y})$. So $\hat{y} \ge \hat{y}_{\mathrm{low}}(\gamma, \alpha)$. The function $\hat{y}_{\mathrm{low}}$ increases in γ and in α.

(d) $-\partial_m \Phi^{\mathrm{mix}}(\gamma; m) \ge m \sqrt{\alpha \gamma \hat{y}_{\mathrm{low}}(\gamma, \alpha)}$.

Proof. (a) By Theorem 4.6 and Proposition 4.2, $y = \zeta_{cx}$ and $x = z_a$, and by Lemma 4.4(iii), $cx = \mathcal{R}_y(y)$, with $c = \gamma/2$. Hence $z_\nu(\tau y(\theta)) = \frac{2}{\gamma} \mathcal{R}_y(y(\theta))$ for all θ, and both sides are continuous in the value of y. At $w = 0$, $z_\nu(0) = m^2 > 0 = \mathcal{R}_y(0)$, so 0 is not in the closure of the range. Also, $\delta_y = \delta_y(0)$ on $[0, \check{y}]$.

(b) On $[0, \tau\check{y})$ we have $\nu([0, s]) = 0$, so there $X_s = \lambda(\nu, m) + \sqrt{2} B_s$, the function $\partial_u \phi_\nu$ is a bounded heat flow, and $\mathsf{p}_s := \partial_u^2 \phi_\nu(s, X_s)$ is a martingale with $\mathbb{E}\mathsf{p}_s = A_\nu(\lambda(\nu, m))$. By Cauchy–Schwarz, $\partial_s z_\nu(s) = 2\mathbb{E}\mathsf{p}_s^2 \ge 2A_\nu(\lambda(\nu, m))^2$, so, using (a) and $2\tau = \alpha\gamma$, we obtain

$$\frac{\check{y}}{\gamma \delta_y(0)^2} = z_\nu(\tau \check{y}) \ge m^2 + \alpha\gamma A_\nu(\lambda(\nu, m))^2 \check{y}.$$

For the last inequality of (b), $\delta_y \le \delta_y(0)$ on $[0,\hat y]$ gives $\hat y/(\gamma\delta_y(0)^2) \le \frac{2}{\gamma}\mathcal{R}_y(\hat y) = z_\nu(\tau\hat y) \le 1$.

(c) As $\delta_y \ge \delta_y(\hat y) = 1-\hat y$ on $[0,\hat y]$, we have $\frac{2}{\gamma}\mathcal{R}_y(\hat y) \le \hat y/(\gamma(1-\hat y)^2)$, which with (a) gives the first inequality. Above $\tau\hat y$, the top of the support of ν, $\partial_u\phi_\nu = \tanh$, so $z_\nu(\tau\hat y) = \mathbb{E}\tanh^2 X_{\tau\hat y}$. On $[0,\tau\hat y)$ the drift of X has the sign of X, by Lemma 4.8(i). Thus, by Tanaka's formula, $|X|$ dominates reflected Brownian motion of variance $2s$ started from $|\lambda(\nu,m)|$, hence the one started from 0, and as $\tanh^2$ increases in $|\cdot|$, we obtain $z_\nu(\tau\hat y) \ge \mathsf{f}(2\tau\hat y) = \mathsf{f}(\gamma\alpha\hat y)$.

Since $\hat y \ge \check y > 0$, $\hat y$ lies in the set defining $\hat y_{\mathrm{low}}$, and monotonicity holds because f increases. This completes the proof of (c).

(d) The function $\Phi^{\mathrm{mix}}(\gamma;\cdot)$ is concave and even by Proposition 4.3, and differentiable by Theorem 4.9(c), so $\partial_m\Phi^{\mathrm{mix}}(\gamma;0) = 0$. Let y° be optimal at $m_0 \in (0,1)$, and ν° the law of τy°. By (4.4), $\Phi^{\mathrm{mix}}(\gamma;\cdot) \le \tau + \alpha\psi_{\mathrm{S}}^*(y^\circ) + LC_{\tau y^\circ}$ on $[-1,1]$, with equality at m_0, and $LC_{\tau y^\circ}$ is C^2 with second derivative $-1/A_{\nu^\circ}(\lambda(\nu^\circ,\cdot))$. Hence $\partial_m^2\Phi^{\mathrm{mix}}(\gamma;m_0) \le -1/A_{\nu^\circ}(\lambda(\nu^\circ,m_0))$ at every m_0 where $\partial_m\Phi^{\mathrm{mix}}(\gamma;\cdot)$ is differentiable, and by (b) and (c), $1/A_{\nu^\circ}(\lambda(\nu^\circ,m_0)) \ge \sqrt{\alpha\gamma\hat y^\circ} \ge \sqrt{\alpha\gamma\hat y_{\mathrm{low}}(\gamma,\alpha)}$. Finally, $-\partial_m\Phi^{\mathrm{mix}}(\gamma;\cdot)$ is nondecreasing, so it is at least the integral of its almost everywhere derivative, which proves (d). □

Proposition 4.14. *For $m \in (0,1)$, with $v^*(m)$ the optimal radius and $\gamma^* := \beta v^*(m)$,*

$$\Lambda'(m) \le m\Big(\beta - \sqrt{\alpha\beta v^*(m)\hat y_{\mathrm{low}}(\gamma^*,\alpha)}\Big). \tag{4.21}$$

So $\Lambda'(m) < 0$ at every $m > 0$ with $\alpha v^*(m)\hat y_{\mathrm{low}}(\gamma^*,\alpha) > \beta$. At a contact point m_c of Theorem 4.12(d)*, $\alpha v^*(m_c)\hat y_{\mathrm{low}}(\gamma^*,\alpha) \le \beta$.*

Proof. We have $\Lambda' = \beta m - h(m)$, and $h(m) = \lambda(\nu^*(m),m) = -\partial_m\Phi^{\mathrm{mix}}(\gamma^*;m)$ at this m, by the proof of Theorem 4.9(c), so Lemma 4.13(d) applies at γ^*. □

Corollary 4.15. *For every $T > 0$, $\mathcal{W}_T \subset (0,2)$. So $\alpha_{\mathrm{c}}(T) \le 2$ uniformly in T.*

Proof. For $T \ge 1$, $\mathcal{W}_T$ is empty. Let $T < 1$. By (4.11), $v^* \ge 1+\beta$, so $\gamma^* \ge \beta(1+\beta) \ge 2$ and $\alpha v^*/\beta \ge \alpha(1+T)$. For $\alpha \ge 2$, monotonicity gives $\hat y_{\mathrm{low}}(\gamma^*,\alpha) \ge \hat y_{\mathrm{low}}(\beta(1+\beta),2)$. In ball arithmetic, we have $\hat y_{\mathrm{low}}(2,2) > .35$ and $\hat y_{\mathrm{low}}(4.44,2) > .5$; the computation will be given in Subsection A.2. If $T \ge .6$, then $\alpha(1+T)\hat y_{\mathrm{low}}(\gamma^*,\alpha) > 2(1+.6)(.35) > 1$, while if $T \le .6$, then $\gamma^* \ge \frac{5}{3}\cdot\frac{8}{3} > 4.44$, so $\hat y_{\mathrm{low}}(\gamma^*,\alpha) > .5$ and $\alpha(1+T)\hat y_{\mathrm{low}}(\gamma^*,\alpha) > 1$. In both cases Proposition 4.14 gives $\Lambda' < 0$ on $(0,1)$, so $\alpha \notin \mathcal{W}_T$. □

5. Zero temperature

In this section, we show that the energy at fixed overlap $e(m) = m^2/2 + L(m)^2/2$ exists and that L is the minimum of an explicit convex functional with a unique minimizer and is jointly concave and C^1 in (α,m). We define the capacity at $T = 0$ and characterize it by a one-dimensional condition, namely the sign of e' in the overlap window, and Section 7 will turn this into digits. The function space and the zero-temperature Parisi recursion, used here through step functions, are those of Auffinger and Chen [4].

5.1. **Setting and existence of the energy.** Throughout, G is an $M \times N$ matrix with i.i.d. $N(0,1)$ entries, $M/N \to \alpha > 0$, and z is a standard Gaussian. Recall that $I(m) = \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m)$, and note that $2^{-N}|S_N(k)| \le e^{-NI(m_k)}$ exactly. Let $\lambda_0(m) := \mathsf{G}^{-1}((1+m)/2)$ and

$$\mathsf{w}(m) := 2\mathsf{g}(\lambda_0(m)) = \inf_\lambda\{\mathbb{E}|z+\lambda| - \lambda m\}.$$

The function w is concave on $[-1,1]$, with $\mathsf{w}' = -\lambda_0$.

For nonempty $A \subset \Sigma_N$, let $L_A(G) := \frac{1}{N}\max_{\sigma\in A}|G\sigma|$ and

$$L_N(k) := \mathbb{E}L_{S_N(k)}(G), \qquad e_N(k) := \frac{m_k^2}{2} + \frac{1}{2}\mathbb{E}L_{S_N(k)}(G)^2.$$

In this section, e_N denotes this energy for Gaussian rows, not the energy (1.12) for binary patterns. With a binary pattern ξ^1 gauged to all ones, $e_N(k) = \frac{1}{2N^2}\mathbb{E}\max_{m_1(\sigma)=m_k}[(\xi^1\cdot\sigma)^2 + |G\sigma|^2]$.

We write $\Phi_\beta, \Lambda_\beta$ for Φ, Λ when β varies, and we will use three facts from Section 2. First, by Theorem 2.3, $\sup_k |\frac{1}{N}\mathbb{E}\log Y_{N,k} - \Phi_\beta(m_k)| \to 0$, and $\Phi_\beta + I$ is Lipschitz. Second, let μ_M be the uniform measure on the sphere of radius $\sqrt{M}$ in $\mathbb{R}^M$ and

$$Z_N^{\mathrm{mix}}(\gamma;k) := 2^{-N}\sum_{\sigma\in S_N(k)}\int \mu_M(\,\mathrm{d}\tau)\exp\big(\sqrt{\gamma/N}\,\tau\cdot G\sigma\big).$$

This is $Z_{N,k}(s)$ of (2.9) at $\gamma = \beta s^2$, so by (2.12), for each γ_0, $\sup_{\gamma\le\gamma_0}\sup_k|\frac{1}{N}\mathbb{E}\log Z_N^{\mathrm{mix}}(\gamma;k) - \Phi^{\mathrm{mix}}(\gamma;m_k)| \to 0$. Third, by (2.6) and (2.7),

$$\begin{aligned}\Phi^{\mathrm{mix}}(\gamma;m) &= \inf_{x,y\in\mathcal{Q}_1}\Big\{\frac{\gamma\alpha}{2}\Big(1-\int_0^1 xy\Big) + J_m(x) + \alpha\psi_{\mathrm{S}}^*(y)\Big\},\\ \Phi_\beta(m) &= \sup_{v>0}\{\mathrm{rad}(v) + \Phi^{\mathrm{mix}}(\beta v;m)\}.\end{aligned} \tag{5.1}$$

Let $\mathrm{Sph}(\theta^2) := b - 1 - \frac{1}{2}\log b$ with $b := \frac{1}{2}(1+\sqrt{1+4\theta^2})$. By Laplace's method with exponent $\sup_u\{\theta u + \frac{1}{2}\log(1-u^2)\}$, this is the limit of $\frac{1}{M}\log\int\mu_M(\,\mathrm{d}\tau)\exp(\theta\sqrt{M}\tau_1)$. Since $b\in[\theta+\frac{1}{2},\theta+1]$,

$$\theta - \frac{1}{2} - \frac{1}{2}\log(1+\theta) \le \mathrm{Sph}(\theta^2) \le \theta. \tag{5.2}$$

Theorem 5.1. *There are continuous functions L, e on $[-1,1]$, with $e = m^2/2 + L^2/2$ and $L \ge \sqrt{\alpha}$, such that the following hold.*

(a) $\sup_k|L_N(k) - L(m_k)| \to 0$ *and* $\sup_k|e_N(k) - e(m_k)| \to 0$.

(b) For all $\beta > 0$ and $m\in[-1,1]$, $\beta L(m)^2/2 - \log 2 \le \Phi_\beta(m) \le \beta L(m)^2/2 - I(m)$. So $e(m) - \log 2/\beta \le \Lambda_\beta(m)/\beta \le e(m) - I(m)/\beta$, and $\sup_m|\Lambda_\beta/\beta - e| \le \log 2/\beta$.

(c) For all $\gamma > 0$ and m,

$$-\log 2 + \alpha\,\mathrm{Sph}\left(\gamma L(m)^2/\alpha\right) \le \Phi^{\mathrm{mix}}(\gamma;m) \le -I(m) + \alpha\,\mathrm{Sph}\left(\gamma L(m)^2/\alpha\right).$$

So, with $\mathsf{b}(\gamma) := \log 2 + \frac{\alpha}{2} + \frac{\alpha}{2}\log\left(1+\sqrt{\gamma}(1+1/\sqrt{\alpha})\right)$,

$$-\mathsf{b}(\gamma) \le \Phi^{\mathrm{mix}}(\gamma;m) - \sqrt{\gamma\alpha}\,L(m) \le -I(m) \le 0. \tag{5.3}$$

(d) $\mathbb{P}(|L_A(G) - \mathbb{E}L_A(G)| > t) \le 2e^{-Nt^2/2}$ for every nonempty A, and $0 \le \frac{1}{2}\mathbb{E}L_A^2 - \frac{1}{2}(\mathbb{E}L_A)^2 \le \frac{1}{2N}$. So $L_{S_N(k)}(G) \to L(m_k)$ almost surely, uniformly in k.

(e) (Types.) For a partition (I_η) of $[N]$ with $|I_\eta|/N \to 1/d_r$ and $\boldsymbol{t} = (t_\eta)\in[-1,1]^{d_r}$, the limit $L^{(r)}(\boldsymbol{t}) := \lim\frac{1}{N}\mathbb{E}\max|G\sigma|$ over the type slice with magnetizations $\boldsymbol{t}$ exists, is continuous in $\boldsymbol{t}$, and is uniform in the slices. Moreover $\frac{\beta}{2}L^{(r)2} - \log 2 \le \Phi^{(r)}(\boldsymbol{t}) \le \frac{\beta}{2}L^{(r)2} - \mathbb{E}_\eta I(t_\eta)$, and (c) holds with I replaced by $\mathbb{E}_\eta I(t_\eta)$.

In particular, $\lim_{\gamma\to\infty}\Phi^{\mathrm{mix}}(\gamma;m)/\sqrt{\gamma} = \sqrt{\alpha}L(m)$, *uniformly in m, with error at most* $\mathsf{b}(\gamma)/\sqrt{\gamma}$.

Proof. The partition function on a set is close to its largest term, up to the entropy $\log 2$, which gives existence by a Cauchy argument in β.

We start with (d). The function L_A is $N^{-1/2}$-Lipschitz in G for the Hilbert–Schmidt norm: $|L_A(G) - L_A(G')| \le \frac{1}{N}\max_A|(G-G')\sigma| \le \|G-G'\|_{\mathrm{HS}}/\sqrt{N}$, so Gaussian concentration gives the tail, and the Poincaré inequality gives $\mathrm{Var}\,L_A \le 1/N$.

For (a) and (b), keeping the largest term or bounding all terms by it gives, pointwise in G,

$$\frac{\beta}{2}L_A(G)^2 - \log 2 \le \frac{1}{N}\log 2^{-N}\sum_{\sigma\in A}\exp\Big(\frac{\beta}{2N}|G\sigma|^2\Big) \le \frac{\beta}{2}L_A(G)^2 + \frac{1}{N}\log(2^{-N}|A|). \tag{5.4}$$

With $A = S_N(k)$, taking expectations gives $\frac{\beta}{2}\mathbb{E}L^2_{S_N(k)} - \log 2 \le \frac{1}{N}\mathbb{E}\log Y_{N,k} \le \frac{\beta}{2}\mathbb{E}L^2_{S_N(k)} - I(m_k)$, so $\limsup_N \sup_k |\frac{1}{2}\mathbb{E}L^2_{S_N(k)} - \Phi_\beta(m_k)/\beta| \le \log 2/\beta$. Comparing two values of β gives $\limsup_N \sup_k |\Phi_\beta(m_k)/\beta - \Phi_{\beta'}(m_k)/\beta'| \le \log 2\,(1/\beta + 1/\beta')$. As $\Phi_\beta + I$ is Lipschitz, Φ_β is continuous, and since $\{m_k\}$ becomes dense, we get $\sup_m |\Phi_\beta/\beta - \Phi_{\beta'}/\beta'| \le \log 2\,(1/\beta + 1/\beta')$. Hence Φ_β/β converges uniformly to a continuous $\mathcal{G}_0$, and $\sup_k |\frac{1}{2}\mathbb{E}L^2_{S_N(k)} - \mathcal{G}_0(m_k)| \to 0$. Passing to the limit along $m_k \to m$ gives $\beta\mathcal{G}_0(m) - \log 2 \le \Phi_\beta(m) \le \beta\mathcal{G}_0(m) - I(m)$. For fixed σ, $\mathbb{E}|G\sigma|/N = \mathbb{E}|g|/\sqrt{N} \to \sqrt{\alpha}$ with g standard in $\mathbb{R}^M$, so $\mathcal{G}_0 \ge \alpha/2 > 0$. We set $L := \sqrt{2\mathcal{G}_0}$, and with (d) we get $L_N(k) \to L(m_k)$ uniformly. This proves (a) and (b).

(c). Let $\mathcal{E}_M(r) := \int \mu_M(\mathrm{d}\tau)e^{r\tau_1}$, which is even, log-convex, and nondecreasing on $[0,\infty)$. For fixed σ, $\int \mu_M(\mathrm{d}\tau)\exp(\sqrt{\gamma/N}\,\tau\cdot G\sigma) = \mathcal{E}_M(\sqrt{\gamma/N}|G\sigma|)$, so with $A = S_N(k)$ and $r_* := \sqrt{\gamma/N}\,NL_A(G)$,

$$2^{-N}\mathcal{E}_M(r_*) \le Z_N^{\mathrm{mix}}(\gamma;k) \le 2^{-N}|A|\mathcal{E}_M(r_*).$$

The function $\Pi_M(\theta) := \frac{1}{M}\log\mathcal{E}_M(\sqrt{M}\theta)$ is convex, $|\Pi_M'| \le 1$ because μ_M lives on the sphere of radius $\sqrt{M}$, and $\Pi_M \to \mathrm{Sph}(\theta^2)$ locally uniformly. Here $r_* = \sqrt{M}\theta_*$ with $\theta_* := \sqrt{\gamma N/M}\,L_A(G)$, and $\frac{1}{N}\log\mathcal{E}_M(r_*) = \frac{M}{N}\Pi_M(\theta_*)$ is Lipschitz in L_A with constant $\sqrt{\gamma M/N}$, so by (d) its expectation tends to $\alpha\,\mathrm{Sph}(\gamma L(m)^2/\alpha)$ along $m_k \to m$. We take logarithms in the previous display, divide by N, and take expectations and limits. Then (5.3) follows from (5.2) with $\theta = \sqrt{\gamma/\alpha}\,L$, using $\sqrt{\alpha} \le L \le 1 + \sqrt{\alpha}$, where the upper bound holds because $L_A(G) \le \|G\|_{\mathrm{op}}/\sqrt{N}$ and $\mathbb{E}\|G\|_{\mathrm{op}} \le \sqrt{N} + \sqrt{M}$.

(e). The same proof works for types, with $\Phi^{(r)}$ in place of Φ and the type slices in place of $S_N(k)$. □

5.2. **The zero-temperature recursion.** Let $\mathcal{U}$ be the set of nondecreasing, right-continuous $\kappa : [0,1) \to [0,\infty)$ with $\int_0^1 \kappa < \infty$, which is the space of [4], and let $\mathcal{V} := \mathcal{U}\times[0,\infty)$. For $(\kappa, C) \in \mathcal{V}$,

$$\delta_\kappa(w) := \int_w^1 \kappa, \qquad \delta(w) := \delta_{\kappa,C}(w) := C + \delta_\kappa(w), \qquad \Psi^{\mathrm{S}}(\kappa, C) := C + \int_0^1 \frac{\mathrm{d}w}{\delta(w)} \in (0,\infty].$$

We fix α. For $c, b \ge 0$ and f 1-Lipschitz, let $\mathcal{T}_{c,b}f(x) := \frac{1}{c}\log\mathbb{E}\exp(cf(x+bz))$ for $c > 0$, and $\mathcal{T}_{0,b}f(x) := \mathbb{E}f(x+bz)$.

Lemma 5.2. *(a) $\mathcal{T}_{c,b}$ is monotone and $\mathcal{T}_{c,b}(f+K) = \mathcal{T}_{c,b}f + K$, so $\sup|\mathcal{T}_{c,b}f - \mathcal{T}_{c,b}g| \le \sup|f-g|$.*

(b) $\mathcal{T}_{c,b}$ preserves the 1-Lipschitz property, convexity (by Hölder) and evenness, and $\mathcal{T}_{c,b}\mathcal{T}_{c,b'} = \mathcal{T}_{c,\sqrt{b^2+b'^2}}$.

(c) $\mathcal{T}_{c,b}(sx+K) = sx + K + cs^2b^2/2$.

(d) For f 1-Lipschitz and $c' \ge c \ge 0$, $0 \le \mathcal{T}_{c',b}f - \mathcal{T}_{c,b}f \le (c'-c)b^2/2$.

Proof. Parts (a)–(c) are straightforward. For (d), let $c > 0$ and $\vartheta := cf(x+b\cdot)$, a function of z. Then $\partial_c\mathcal{T}_{c,b}f = c^{-2}\,\mathrm{Ent}(e^\vartheta)/\mathbb{E}e^\vartheta \ge 0$. The Gaussian log-Sobolev inequality gives $\mathrm{Ent}(e^\vartheta) \le \frac{1}{2}\mathbb{E}[|\partial_z\vartheta|^2e^\vartheta] \le \frac{1}{2}c^2b^2\mathbb{E}e^\vartheta$, so $0 \le \partial_c\mathcal{T}_{c,b}f \le b^2/2$. Finally, we let $c \downarrow 0$, using continuity. □

For a step function κ on $[w,w')$, with partition $w=w_0<\cdots<w_n=w'$, value c_i on $[w_i,w_{i+1})$ and $b_i^2:=\alpha(w_{i+1}-w_i)$, let

$$\mathcal{P}_\kappa^{w,w'}:=\mathcal{T}_{c_0,b_0}\cdots\mathcal{T}_{c_{n-1},b_{n-1}},$$

which does not depend on the partition, by Lemma 5.2(b).

Lemma 5.3. *For f,g 1-Lipschitz and step functions κ,κ':*

(i) $\mathcal{P}_\kappa^{w,w'}$ is monotone, commutes with constants, preserves the 1-Lipschitz property, convexity and evenness, and $\sup|\mathcal{P}_\kappa^{w,w'}f-\mathcal{P}_\kappa^{w,w'}g|\le\sup|f-g|$.

(ii) (L^1 contraction.) $\sup_x|\mathcal{P}_\kappa^{w,w'}f-\mathcal{P}_{\kappa'}^{w,w'}f|\le\frac{\alpha}{2}\int_w^{w'}|\kappa-\kappa'|$. If $\kappa\le\kappa'$ then $\mathcal{P}_\kappa^{w,w'}f\le\mathcal{P}_{\kappa'}^{w,w'}f$.

(iii) For $s=\pm1$, $\mathcal{P}_\kappa^{w,w'}(sx+K)=sx+K+\frac{\alpha}{2}\int_w^{w'}\kappa$.

(iv) With $\bar z:=\sqrt{\alpha(w'-w)}\,z$,

$$\max\Big\{|x|+\frac{\alpha}{2}\int_w^{w'}\kappa,\ \mathbb{E}|x+\bar z|\Big\}\le\mathcal{P}_\kappa^{w,w'}|\cdot|(x)\le\mathbb{E}|x+\bar z|+\frac{\alpha}{2}\int_w^{w'}\kappa\le|x|+\sqrt{\frac{2\alpha(w'-w)}{\pi}}+\frac{\alpha}{2}\int_w^{w'}\kappa.$$

Proof. Parts (i) and (iii) follow from Lemma 5.2, and (ii) follows on a common partition by induction, with Lemma 5.2(a),(d). For the lower bound in (iv), we write $|x|=\max_{s=\pm1}sx$ and use monotonicity and (iii); the second term follows from $\kappa\ge0$ and (ii). For the upper bound, we induct from the top: if $\mathsf{u}(x):=\mathbb{E}|x+\sqrt{v'}z|+K$ is the bound at a level, then by monotonicity and Lemma 5.2(d) with $c=0$, $\mathcal{T}_{c,b}\mathsf{u}\le\mathcal{T}_{0,b}\mathsf{u}+cb^2/2=\mathbb{E}|x+\sqrt{v'+b^2}\,z|+K+cb^2/2$. □

By (ii), $\mathcal{P}_\kappa^{w,w'}$ extends to every $\kappa\in\mathcal{U}$ (truncate to $\kappa\wedge n$ and take step functions below it), and (i)–(iv) still hold; compare [4, Corollary 2]. For $\kappa\in\mathcal{U}$ and f 1-Lipschitz, let

$$\Psi_\kappa^f(w,x):=\mathcal{P}_\kappa^{w,1}f(x),\quad\Psi_\kappa:=\Psi_\kappa^{|\cdot|},\quad W_m^{[f]}(\kappa):=\inf_\lambda\{\Psi_\kappa^f(0,\lambda)-\lambda m\},\quad W_m:=W_m^{[|\cdot|]},$$

and $W_m^{[0,w_0]}(\kappa):=\inf_\lambda\{\mathcal{P}_\kappa^{0,w_0}|\cdot|(\lambda)-\lambda m\}$. This is the zero-temperature Parisi recursion, in the level variable w, with steps $\frac{1}{c}\log\mathbb{E}\exp(c\,\cdot)$ and terminal function $|x|$. For step κ and $w<1$, Ψ_κ is smooth on each slab, its x-derivatives are continuous across slabs and bounded for $w\le w_0<1$, and by Cole–Hopf

$$-\partial_w\Psi=\frac{\alpha}{2}(\partial_x^2\Psi+\kappa(\partial_x\Psi)^2).\tag{5.5}$$

By Lemma 5.3(iv), for every $\kappa\in\mathcal{U}$, $|m|\le1$ and $w_0<1$,

$$W_m(\kappa)\ge\frac{\alpha}{2}\delta_\kappa(0),\tag{5.6}$$

$$\frac{\alpha}{2}\delta_\kappa(w_0)+W_m^{[0,w_0]}(\kappa)\le W_m(\kappa)\le\frac{\alpha}{2}\delta_\kappa(w_0)+W_m^{[0,w_0]}(\kappa)+\sqrt{\frac{2\alpha(1-w_0)}{\pi}}.\tag{5.7}$$

5.3. **The optimally controlled process and the supporting hyperplane.** The recursion has a stochastic control representation, compare [4, Theorem 2], whose value is the supremum of functions convex in (κ,λ); this gives the convexity of W_m in κ and a supporting hyperplane with slope the overlap function Q.

Lemma 5.4. *Let $\kappa_\circ$ be a step function, $\lambda_\circ$ real, and $u(w,x):=\partial_x\Psi_{\kappa_\circ}(w,x)$, so $|u|\le1$. Let X solve $\mathrm{d}X_w=\alpha\kappa_\circ(w)u(w,X_w)\,\mathrm{d}w+\sqrt{\alpha}\,\mathrm{d}B_w$, $X_0=\lambda_\circ$, and put $u_w:=u(w,X_w)$, $Q(w):=Q_{\kappa_\circ,\lambda_\circ}(w):=\mathbb{E}u_w^2$.*

(a) (Verification.) For $\kappa \in \mathcal{U}$, λ real and any bounded adapted control ϑ,

$$\Psi_\kappa(0,\lambda) \ge \mathbb{E}\Big|\lambda + \alpha\int_0^1 \kappa\vartheta_w\,\mathrm{d}w + \sqrt{\alpha}B_1\Big| - \frac{\alpha}{2}\int_0^1 \kappa\,\mathbb{E}\vartheta_w^2\,\mathrm{d}w, \tag{5.8}$$

with equality when $\kappa = \kappa_\circ$, $\lambda = \lambda_\circ$, $\vartheta = u_w$. Consequently $(\kappa,\lambda) \mapsto \Psi_\kappa(0,\lambda)$ is convex on $\mathcal{U}\times\mathbb{R}$.

(b) u_w is a bounded martingale on $[0,1)$ with $u_{1-} = \operatorname{sgn}(X_1)$ almost surely and $u_0 = \partial_x\Psi_{\kappa_\circ}(0,\lambda_\circ)$. So Q is nondecreasing, $Q(0) = u_0^2$, $Q(1-) = 1$, and $\mathbb{E}[\operatorname{sgn}(X_1)u_w] = Q(w)$.

(c) (Supporting hyperplane.) For $\kappa \in \mathcal{U}$ and λ real,

$$\Psi_\kappa(0,\lambda) \ge \Psi_{\kappa_\circ}(0,\lambda_\circ) + u_0(\lambda-\lambda_\circ) + \frac{\alpha}{2}\int_0^1 (\kappa-\kappa_\circ)Q\,\mathrm{d}w. \tag{5.9}$$

Proof. (a). Let κ be a step function and $w \le w_0 < 1$, let X^ϑ denote the process inside (5.8), and put $\mathsf{p} := \partial_x\Psi_\kappa(w, X_w^\vartheta)$. By Itô and (5.5),

$$\mathrm{d}\Psi_\kappa(w,X_w^\vartheta) = \frac{\alpha}{2}\kappa(2\vartheta_w\mathsf{p} - \mathsf{p}^2)\,\mathrm{d}w + \text{martingale} \le \frac{\alpha}{2}\kappa\vartheta_w^2\,\mathrm{d}w + \text{martingale},$$

with equality when $\vartheta_w = \mathsf{p}$. We take expectations and let $w_0 \to 1$, using $\sup_x|\Psi_\kappa(w_0,x) - |x|| \to 0$ from Lemma 5.3(iv). The case of general κ follows from Lemma 5.3(ii). For step functions, the right side of (5.8) is convex in (κ,λ) for each control, and $\Psi_\kappa(0,\lambda)$ is the supremum over controls; convexity is preserved in the limit.

(b). Differentiating (5.5) gives $-\partial_w u = \frac{\alpha}{2}\partial_x^2 u + \alpha\kappa_\circ u\partial_x u$, so by Itô $\mathrm{d}u_w = \sqrt{\alpha}\,\partial_x u(w,X_w)\,\mathrm{d}B_w$. As $w\to 1$, $u(w,x)\to\operatorname{sgn}(x)$ locally uniformly on $x\neq 0$, by the closed form of $\mathcal{T}_{c,b}|\cdot|$, and $\mathbb{P}(X_1 = 0) = 0$ by Girsanov.

(c). By (5.8) with $\vartheta = u_w$, we have $\Psi_\kappa(0,\lambda) \ge \mathbb{E}|X_1^\kappa| - \frac{\alpha}{2}\int\kappa Q$, where $X_1^\kappa = X_1 + (\lambda-\lambda_\circ) + \alpha\int(\kappa-\kappa_\circ)u_w\,\mathrm{d}w$. Convexity of $|\cdot|$ at X_1 gives

$$\begin{aligned}\mathbb{E}|X_1^\kappa| &\ge \mathbb{E}|X_1| + \mathbb{E}[\operatorname{sgn}(X_1)](\lambda-\lambda_\circ) + \alpha\int(\kappa-\kappa_\circ)\mathbb{E}[\operatorname{sgn}(X_1)u_w]\,\mathrm{d}w\\ &= \mathbb{E}|X_1| + u_0(\lambda-\lambda_\circ) + \alpha\int(\kappa-\kappa_\circ)Q.\end{aligned}$$

By the equality case of (a), $\mathbb{E}|X_1| = \Psi_{\kappa_\circ}(0,\lambda_\circ) + \frac{\alpha}{2}\int\kappa_\circ Q$, and summing gives (c). □

For $\kappa\in\mathcal{U}$ let $\mathcal{M}_\kappa := \partial_x\Psi_\kappa(0,\cdot)$.

Lemma 5.5. *Let $\kappa\in\mathcal{U}$.*

(a) $\Psi_\kappa(0,x) > |x| + \frac{\alpha}{2}\delta_\kappa(0)$ for all x.

(b) For $w<1$, $\partial_x\Psi_\kappa(w,\cdot)$ exists, is odd, nondecreasing, Lipschitz, and concave on $[0,\infty)$. For $x\ge 0$, $\partial_x\Psi_\kappa(w,x) \ge 2\mathsf{G}(x/\sqrt{\alpha(1-w)}) - 1$.

(c) $\mathcal{M}_\kappa$ is a strictly increasing bijection $\mathbb{R}\to(-1,1)$. So for $|m|<1$ the infimum in $W_m(\kappa)$ is attained only at $\lambda(\kappa,m) := \mathcal{M}_\kappa^{-1}(m)$, and $m\mapsto W_m(\kappa)$ is concave and C^1 on $(-1,1)$ with derivative $-\lambda(\kappa,m)$ and slope tending to $\mp\infty$ at ± 1.

Proof. (a). Let $x\ge 0$. Taking $\vartheta = 1$ in (5.8), with $\bar\delta := \delta_\kappa(0)$ and $Y := x + \alpha\bar\delta + \sqrt{\alpha}z$, we get $\Psi_\kappa(0,x) \ge \mathbb{E}|Y| - \frac{\alpha}{2}\bar\delta$, and $\mathbb{E}|Y| = \mathbb{E}Y + 2\mathbb{E}Y_- > x + \alpha\bar\delta$. The case $x<0$ follows by evenness.

(b). We first take step κ, with terminal function $f_n(x) := n^{-1}\log\cosh(nx)$, and let $u := \partial_x\Psi_\kappa^{f_n}$, which is odd. Then $\mathsf{s} := \partial_x^2 u$ solves the linear equation $-\partial_w\mathsf{s} = \frac{\alpha}{2}\partial_x^2\mathsf{s} + \alpha\kappa(u\partial_x\mathsf{s} + 3(\partial_x u)\mathsf{s})$, with $\mathsf{s}(w,0) = 0$ and terminal data $f_n''' \le 0$ on $x>0$, so $\mathsf{s}\le 0$ on $x\ge 0$ by the maximum principle on the half line. We next prove the lower bound. Let u^{h} be the heat flow of $\tanh(nx)$. Then $\mathsf{v} := u - u^{\mathrm{h}}$ solves $-\partial_w\mathsf{v} = \frac{\alpha}{2}\partial_x^2\mathsf{v} + \alpha\kappa u\partial_x u$, whose source is nonnegative on $x>0$ because

$u, \partial_x u \geq 0$ there, and $\mathsf{v}(w,0) = 0$, $\mathsf{v}(1,\cdot) = 0$, so $\mathsf{v} \geq 0$ on $x \geq 0$. We then let $n \to \infty$, using that $|f_n - |x|| \leq \log 2/n$ and that derivatives of convex functions converge where the limit is differentiable.

For the Lipschitz bound, let $w \leq w_0 < 1$, so $\kappa \leq K := \kappa(w_0)$ on $[0, w_0]$. Then u solves the linear equation $-\partial_w u = \frac{\alpha}{2}\partial_x^2 u + \mathsf{c}\,\partial_x u$ with $|\mathsf{c}| \leq \alpha K$. Duhamel from level w_0, the bound on the x-derivative of the heat kernel and a singular Gronwall inequality give $\sup_x |\partial_x u(w,x)| \leq \mathrm{const}(\alpha, K, w_0)(w_0 - w)^{-1/2}$, independent of the step approximation. Hence the same bound holds for $\kappa \in \mathcal{U}$, and (b) passes to the limit.

(c). The function $\mathcal{M}_\kappa$ is odd, concave on $[0,\infty)$, and bounded by 1. If it were not strictly increasing on $[0,\infty)$, it would be constant on some $[x_0,\infty)$, so $\Psi_\kappa(0,x) = a_\infty x + b_\infty$ there. By Lemma 5.3(iv), $a_\infty = 1$ and $b_\infty = \frac{\alpha}{2}\delta_\kappa(0)$, which contradicts (a). The limit at infinity is 1, again by Lemma 5.3(iv). The rest follows by Legendre duality. □

Lemma 5.6. *For $\kappa \in \mathcal{U}$ and λ real, the equation of Lemma 5.4 has a unique strong solution on $[0,1)$, and $Q_{\kappa,\lambda}(w) := \mathbb{E}u_w^2$ is nondecreasing with values in $[\mathcal{M}_\kappa(\lambda)^2, 1]$. If $\kappa_n \to \kappa$ in L^1 within $\mathcal{U}$ and $\lambda_n \to \lambda$, then $Q_{\kappa_n,\lambda_n} \to Q_{\kappa,\lambda}$ almost everywhere on $[0,1)$. The inequality (5.9) holds for every $\kappa_\circ \in \mathcal{U}$. For $|m| < 1$ put $Q_{\kappa,m} := Q_{\kappa,\lambda(\kappa,m)}$. Then W_m is convex on $\mathcal{U}$, and for $\kappa, \kappa' \in \mathcal{U}$,*

(5.10)
$$W_m(\kappa') \geq W_m(\kappa) + \frac{\alpha}{2}\int (\kappa' - \kappa) Q_{\kappa,m}, \qquad \lim_{t \downarrow 0} \frac{W_m(\kappa + t(\kappa' - \kappa)) - W_m(\kappa)}{t} = \frac{\alpha}{2}\int (\kappa' - \kappa) Q_{\kappa,m}.$$

Proof. The drift is Lipschitz on $w \leq w_0$ by Lemma 5.5(b). On $[0, w_0]$ the κ_n are uniformly bounded, $\kappa_n \to \kappa$ almost everywhere and $\Psi_{\kappa_n} \to \Psi_\kappa$ uniformly, and by Lemma 5.5(b) the derivatives converge locally uniformly, with a uniform Lipschitz constant. By SDE stability, we have $X_w^n \to X_w$ in L^2, so $Q_n \to Q$ on $[0, w_0]$. The inequality (5.9) for step approximations of $\kappa_\circ$ passes to the limit, since Q_n is bounded and $\kappa - \kappa_\circ^n$ converges in L^1. The convexity of W_m follows by partial minimization of a jointly convex function. The first part of (5.10) is (5.9) with $u_0 = m$. For the derivative, the inequality $\geq$ follows from the first part. For $\leq$, we apply the first part at $\kappa_t := \kappa + t(\kappa' - \kappa)$ with κ' replaced by κ, and use $Q_{\kappa_t,m} \to Q_{\kappa,m}$ almost everywhere, which holds as $\lambda(\kappa_t, m) \to \lambda(\kappa, m)$ by Lemma 5.5(c) and uniform convergence; here Q is bounded and $\kappa' - \kappa \in L^1$. □

5.4. **The variational formula.** The main result of this subsection is the following.

Theorem 5.7. *For all $\alpha > 0$ and $m \in [-1,1]$,*

$$\sqrt{\alpha} L(m) = \lim_{\gamma \to \infty} \frac{\Phi^{\mathrm{mix}}(\gamma; m)}{\sqrt{\gamma}} = \min_{(\kappa,C) \in \mathcal{V}} \mathcal{B}_m(\kappa, C), \qquad \mathcal{B}_m(\kappa, C) := \frac{\alpha}{2}\Psi^{\mathrm{S}}(\kappa, C) + W_m(\kappa). \tag{5.11}$$

The minimum is attained at a unique $(\kappa^*, C^*) = (\kappa_m^*, C_m^*)$. Bounded step functions with $C > 0$ are enough for the infimum. For types, in the notation of Theorem 5.1(e), the same proof gives for $\boldsymbol{t} \in [-1,1]^{d_r}$

$$\sqrt{\alpha} L^{(r)}(\boldsymbol{t}) = \min_{(\kappa,C) \in \mathcal{V}} \Big\{ \frac{\alpha}{2}\Psi^{\mathrm{S}}(\kappa, C) + \mathbb{E}_\eta W_{t_\eta}(\kappa) \Big\}. \tag{5.12}$$

We prove the theorem by taking the limit $\gamma \to \infty$ in (5.1). We will eliminate the Ising path x exactly by conjugacy, and the two remaining terms then become functions of the rescaled path $\kappa = \sqrt{\gamma} F$, with F the distribution function of y. The upper bound will come from scaled trials, and the lower bound from compactness, in which the mass of κ that moves to the top level $w = 1$ lands in C.

Lemma 5.8. *Let $\mathbb{F}$ be the set of $F : [0,1] \to [0,1]$ nondecreasing, right-continuous, with $F(1) = 1$. Such an F is the distribution function of a path $y \in \mathcal{Q}_1$ with $\hat{y} := y(1-)$, and $\delta_F(w) := \int_w^1 F = \int_0^1 (1 - y(u) \vee w)\,\mathrm{d}u \le 1 - w$. For $\gamma > 0$ and $m \in (-1,1)$,*

$$\Phi^{\mathrm{mix}}(\gamma; m) = \inf\{\mathcal{A}_\gamma(F; m) : F \in \mathbb{F}, \hat{y} < 1\},$$

where, with $\kappa := \sqrt{\gamma} F$ and $\ell_\gamma(x) := \gamma^{-1/2} \log\cosh(\sqrt{\gamma} x)$,

$$\frac{\mathcal{A}_\gamma(F; m)}{\sqrt{\gamma}} = \frac{\alpha}{2} \int_0^1 \Big[\frac{1}{\delta_\kappa(w)} - \frac{1}{\sqrt{\gamma}(1-w)}\Big]\,\mathrm{d}w + W_m^{[\ell_\gamma]}(\kappa). \tag{5.13}$$

The integrand is nonnegative. The right side is a term independent of m plus an infimum over λ of functions affine in m, so $\Phi^{\mathrm{mix}}(\gamma; \cdot)$ is concave for every γ. Also $|x| - \log 2/\sqrt{\gamma} \le \ell_\gamma(x) \le |x|$.

Proof. We first eliminate x. By (2.13) with one type,

$$\Phi^{\mathrm{mix}}(\gamma; m) = \inf_{y \in \mathcal{Q}_1} \Big\{\frac{\gamma\alpha}{2} - g_m\Big(\frac{\gamma\alpha}{2} y\Big) + \alpha \psi_{\mathrm{S}}^*(y)\Big\},$$

and by Theorem 2.3(b), the path y may be restricted to $\hat{y} < 1$.

For the sphere term, by Lemma 2.1, we have $\psi_{\mathrm{S}}^*(y) = \frac{1}{2}[\int_0^{\hat{y}} \mathrm{d}w/\delta_F(w) + \log(1-\hat{y})] = \frac{1}{2}\int_0^1 [1/\delta_F(w) - 1/(1-w)]\,\mathrm{d}w$, because $\delta_F(w) = 1 - w$ on $[\hat{y}, 1]$ and $\int_0^{\hat{y}} \mathrm{d}w/(1-w) = -\log(1-\hat{y})$; also $\delta_F = \delta_\kappa/\sqrt{\gamma}$.

For the Ising term, we take in the definition of g_m in [6, (1.7), (1.8)] the horizon $S := \gamma\alpha/2$ and $\nu([0,v]) = F(2v/(\gamma\alpha))$, so that $-\partial_v \phi_\nu = \partial_x^2 \phi_\nu + F(2v/(\gamma\alpha))(\partial_x \phi_\nu)^2$, $\phi_\nu(S, \cdot) = \log\cosh$. For step F with values f_i on levels w_i, and $v_i = S w_i$, we have $\phi_\nu(v_i, x) = \frac{1}{f_i} \log \mathbb{E} \exp(f_i \phi_\nu(v_{i+1}, x + \sqrt{2(v_{i+1} - v_i)} z))$. Writing $\phi_\nu(v, x) = \sqrt{\gamma}\hat{\phi}(2v/(\gamma\alpha), x/\sqrt{\gamma})$, $c_i := \sqrt{\gamma} f_i$, $b_i^2 := \alpha(w_{i+1} - w_i)$, we get $\hat{\phi}(w_i, \cdot) = \mathcal{T}_{c_i, b_i} \hat{\phi}(w_{i+1}, \cdot)$ and $\hat{\phi}(1, \cdot) = \ell_\gamma$, so $\phi_\nu(0, \sqrt{\gamma}\lambda) = \sqrt{\gamma} \Psi_\kappa^{\ell_\gamma}(0, \lambda)$ and

$$g_m(a) = \sup_\lambda \{S - \phi_\nu(0, \sqrt{\gamma}\lambda) + \sqrt{\gamma}\lambda m\} = \frac{\gamma\alpha}{2} - \sqrt{\gamma} W_m^{[\ell_\gamma]}(\kappa).$$

The term $\gamma\alpha/2$ then cancels. Since both sides are L^1-Lipschitz in F, by [6, (6.43)], $|y_F - y_{F'}|_{L^1} = |F - F'|_{L^1}$ and Lemma 5.3(ii), this covers general F. □

Proof of the upper bound in (5.11). We fix $m \in (-1,1)$, $C > 0$ and a bounded $\kappa \le K$. For $\sqrt{\gamma} > \max(K, C)$, let $\hat{y}_\gamma := 1 - C/\sqrt{\gamma}$, $\kappa_\gamma := \kappa$ on $[0, \hat{y}_\gamma)$, $\kappa_\gamma := \sqrt{\gamma}$ on $[\hat{y}_\gamma, 1)$, $F_\gamma := \kappa_\gamma/\sqrt{\gamma}$.

For the sphere term, $\delta_{\kappa_\gamma}(w) = C + \int_w^{\hat{y}_\gamma} \kappa$ for $w < \hat{y}_\gamma$, so the first term of (5.13) is $\frac{\alpha}{2}[\int_0^{\hat{y}_\gamma} \mathrm{d}w/(C + \int_w^{\hat{y}_\gamma} \kappa) + \gamma^{-1/2} \log(C/\sqrt{\gamma})]$, which tends to $\frac{\alpha}{2}\int_0^1 \mathrm{d}w/\delta(w)$.

For the Ising term, $\ell_\gamma \le |\cdot|$, and by Lemma 5.2, bounding the exponential of a maximum by the sum, $\mathcal{T}_{c,b}|\cdot|(x) \le |x| + \log 2/c + cb^2/2$. With $c = \sqrt{\gamma}$, $b^2 = \alpha C/\sqrt{\gamma}$ we get $cb^2/2 = \alpha C/2$. By Lemma 5.3(iv), $|x| \le \Psi_\kappa(\hat{y}_\gamma, x)$, so $\Psi_{\kappa_\gamma}^{\ell_\gamma}(0, \lambda) \le \Psi_\kappa(0, \lambda) + \alpha C/2 + \log 2/\sqrt{\gamma}$ for all λ, and $W_m^{[\ell_\gamma]}(\kappa_\gamma) \le W_m(\kappa) + \alpha C/2 + \log 2/\sqrt{\gamma}$. Using Lemma 5.8 and Theorem 5.1(c), we obtain $\sqrt{\alpha} L(m) \le \mathcal{B}_m(\kappa, C)$.

For general (κ, C) let $\kappa_K := \kappa \wedge K$, $C_K := C + \int(\kappa - K)_+$. Then $C_K + \delta_{\kappa_K} \ge C + \delta_\kappa$, so $\int \mathrm{d}w/\delta$ does not increase; $W_m(\kappa_K) \le W_m(\kappa)$; and $C_K \to C$. Hence $\mathcal{B}_m(\kappa_K, C_K) \le \mathcal{B}_m(\kappa, C) + \frac{\alpha}{2}\int(\kappa - K)_+$, and bounded κ with $C > 0$ suffices (if $C_K = 0$, then κ is bounded and $C = 0$, and $\mathcal{B}_m = +\infty$). Step functions suffice by Lemma 5.3(ii) and continuity of Ψ^{S} at $C > 0$. □

Lemma 5.9. *Let $(\kappa_n, C_n) \in \mathcal{V}$ with $\delta_n := C_n + \delta_{\kappa_n}$ and $\delta_n(0) \le \bar{\delta}$, and let f_n be 1-Lipschitz with $|x| - \epsilon_n \le f_n$, $\epsilon_n \to 0$. Along a subsequence $\delta_n(0) \to \delta_\infty$, $\kappa_n \to \kappa$ almost everywhere, $\kappa \in \mathcal{U}$, and with $C := \delta_\infty - \int \kappa \ge 0$:*

(a) $\delta_n(w) \to C + \delta_\kappa(w)$ *for all* $w \in [0,1)$*;*
(b) for $\lambda_n \to \lambda$, $\liminf\{\frac{\alpha}{2}C_n + \Psi^{f_n}_{\kappa_n}(0,\lambda_n)\} \geq \frac{\alpha}{2}C + \Psi_\kappa(0,\lambda)$*;*
(c) for $m \in [-1,1]$, $\liminf\{\frac{\alpha}{2}C_n + W_m^{[f_n]}(\kappa_n)\} \geq \frac{\alpha}{2}C + W_m(\kappa)$.

Proof. Since $\kappa_n(w)(1-w) \leq \delta_{\kappa_n}(w) \leq \bar{\delta}$, the κ_n are uniformly bounded and nondecreasing on $[0,w_0]$. We obtain the almost-everywhere limit from Helly's theorem and a diagonal argument, and Fatou gives $\int \kappa \leq \delta_\infty$. For $w<1$, $\int_0^w \kappa_n \to \int_0^w \kappa$, which is (a). To prove (b) and (c), we fix $w_0<1$. By Lemma 5.3(iii),(i), $\Psi^{f_n}_{\kappa_n}(w_0,x) \geq |x| + \frac{\alpha}{2}\delta_{\kappa_n}(w_0) - \epsilon_n$. Hence, for all λ,

$$\frac{\alpha}{2}C_n + \Psi^{f_n}_{\kappa_n}(0,\lambda) - \lambda m \geq \frac{\alpha}{2}\delta_n(w_0) - \epsilon_n + \mathcal{P}^{0,w_0}_{\kappa_n}|\cdot|(\lambda) - \lambda m.$$

By Lemma 5.3(ii), $\mathcal{P}^{0,w_0}_{\kappa_n}|\cdot| \to \mathcal{P}^{0,w_0}_{\kappa}|\cdot|$ uniformly, with error $\frac{\alpha}{2}\int_0^{w_0}|\kappa_n - \kappa| \to 0$. Thus the liminf of the left side, also after the infimum over λ, is at least $\frac{\alpha}{2}(C+\delta_\kappa(w_0)) + \mathcal{P}^{0,w_0}_{\kappa}|\cdot|(\lambda) - \lambda m$. By (5.7) and Lemma 5.3(iv) this is at least $\frac{\alpha}{2}C + \Psi_\kappa(0,\lambda) - \lambda m - \sqrt{2\alpha(1-w_0)/\pi}$. Letting $w_0 \to 1$ gives (b) and (c). □

The mass that κ_n carries to $w=1$ lands in C, which is why C belongs to the space: $\mathcal{U}$ alone is not closed for this convergence, while $\mathcal{U}\times[0,\infty)$ with the total mass is.

End of the proof of Theorem 5.7. For the lower bound, let $m \in (-1,1)$ and take F_γ with $\mathcal{A}_\gamma(F_\gamma;m) \leq \Phi^{\mathrm{mix}}(\gamma;m)+1$. We put $\kappa_\gamma := \sqrt{\gamma}F_\gamma$ and let $\gamma_n \to \infty$. By (5.6) for ℓ_γ, $W_m^{[\ell_\gamma]}(\kappa_\gamma) \geq \frac{\alpha}{2}\delta_{\kappa_\gamma}(0) - \log 2/\sqrt{\gamma}$. The sphere term in (5.13) is nonnegative, so with (5.3) we get $\delta_{\kappa_\gamma}(0) \leq 2L/\sqrt{\alpha} + o(1)$, which is bounded. Applying Lemma 5.9 with $C_n = 0$ and $f_n = \ell_{\gamma_n}$, we obtain a limit (κ, C). The integrand in (5.13) is nonnegative and for $w<1$ tends to $1/(C+\delta_\kappa(w))$, or $+\infty$ if the limit is 0, so Fatou gives $\liminf \geq \int_0^1 \mathrm{d}w/\delta(w)$. Lemma 5.9(c) handles the Ising term, and summing gives $\sqrt{\alpha}L(m) \geq \frac{\alpha}{2}\int \mathrm{d}w/\delta + \frac{\alpha}{2}C + W_m(\kappa) = \mathcal{B}_m(\kappa,C)$, which is the lower bound.

For existence, the same steps apply to a minimizing sequence of $\mathcal{B}_m$. Indeed, its masses are bounded by (5.6), and Lemma 5.9 with Fatou for $1/\delta$ gives lower semicontinuity under the convergence of that lemma.

For uniqueness, note that Ψ^{S} is convex in (κ,C), since δ is linear in (κ,C) and $t \mapsto 1/t$ is strictly convex, and W_m is convex by Lemma 5.6 (at $m=\pm1$, see (5.14) below). Therefore, two minimizers would have $\delta_1 = \delta_2$ almost everywhere, hence everywhere on $[0,1)$ by continuity, so $\kappa_1 = \kappa_2$ and $C_1 = C_2$.

We next treat $m = \pm 1$. Here $L(\pm1) = \sqrt{\alpha}$, as $S_N(N)$ is a single point. By (5.6) and $\lambda \to \infty$ in Lemma 5.3(iv),

$$W_1(\kappa) = \frac{\alpha}{2}\delta_\kappa(0), \qquad \mathcal{B}_1(\kappa,C) = \frac{\alpha}{2}\Big[\delta(0) + \int_0^1 \frac{\mathrm{d}w}{\delta(w)}\Big] \geq \frac{\alpha}{2}\Big(\delta(0) + \frac{1}{\delta(0)}\Big) \geq \alpha, \tag{5.14}$$

with equality only at $\kappa = 0$, $C = 1$. This is the formula at $m=1$, and $m=-1$ follows by evenness.

For types, the elimination of the Ising paths runs type by type and gives $\mathbb{E}_\eta g_{t_\eta}$ in place of g_m. The rest of the proof is unchanged. □

Lemma 5.10. *For* $m \in [0,1]$,

$$\sqrt{\alpha}\, m + \mathsf{w}(m) \leq L(m) \leq \sqrt{\alpha} + \mathsf{w}(m). \tag{5.15}$$

Proof. For the upper bound, we take $\kappa = 0$, $C = 1$ in (5.11), so $\mathcal{B}_m(0,1) = \alpha + W_m(0)$, where $W_m(0) = \inf_\lambda\{\mathbb{E}|\lambda + \sqrt{\alpha}z| - \lambda m\} = \sqrt{\alpha}\,\mathsf{w}(m)$. For the lower bound, let G_i be the columns of G, $G_\Sigma := \sum_i G_i$, and $\mathsf{e} := G_\Sigma/|G_\Sigma|$. Given G_Σ, the variables $h_i := (G_i - G_\Sigma/N)\cdot \mathsf{e}$ are

centered Gaussian with variance $1-1/N$, exchangeable, with sum 0, and independent of G_Σ. If σ flips a set F of size pN, $p=(1-m)/2$, then $|G\sigma| \ge G\sigma\cdot \mathsf{e} = |G_\Sigma|(1-2p) - 2\sum_F h_i$. With F the pN smallest h_i, we get $-\frac{2}{N}\sum_F h_i \to 2\mathsf{g}(\mathsf{G}^{-1}(p)) = \mathsf{w}(m)$, and $|G_\Sigma|/N \to \sqrt{\alpha}$. □

5.5. **Optimality conditions.** We now state the optimality conditions for the minimizer.

Proposition 5.11. *Let $m\in(-1,1)$. $\mathcal{B}_m$ is convex on $\mathcal{V}$, strictly in δ. For $(\kappa,C)\in\mathcal{V}$ with $\mathcal{B}_m(\kappa,C)<\infty$ put $Q:=Q_{\kappa,m}$, $R(s):=R_{\kappa,C}(s):=\int_0^s \mathrm{d}w/\delta(w)^2$, $\Gamma(s):=\int_s^1 (Q-R)\,\mathrm{d}w$. If (κ,C) is the minimizer, then*

(K1) $R(1-)\le 1$, and $C(1-R(1-))=0$;
(K2) $\Gamma(s)\ge 0$ for all $s\in[0,1)$;
(K3) $\int_{[0,1)}\Gamma\,\mathrm{d}\kappa = 0$, where $\mathrm{d}\kappa$ includes the atom $\kappa(0)$ at $s=0$.

On the interior of $\operatorname{supp}\mathrm{d}\kappa$, *$Q=R$. At the minimizer, $\delta(0)\ge 1$, by (K1) and $\delta\le\delta(0)$.*

Proof. By Fubini, the directional derivative of Ψ^{S} along $(\kappa'-\kappa, C'-C)$ is $(C'-C)(1-R(1-)) - \int(\kappa'-\kappa)R$, so with (5.10) the derivative of $\mathcal{B}_m$ is $\frac{\alpha}{2}[\int(\kappa'-\kappa)(Q-R)+(C'-C)(1-R(1-))]$, and at the minimizer it is nonnegative in every admissible direction. Taking $C'=C+t$, we see by monotone convergence that the derivative in C is $\frac{\alpha}{2}(1-R(1-))$, so $R(1-)\le 1$. If $C>0$, decreasing C is also allowed, which gives (K1). With $\kappa'=\kappa+\theta\mathbf{1}_{[s,1)}$, we get (K2). Finally, $\kappa'=(1\pm\epsilon)\kappa$ gives $\int\kappa(Q-R)=0$, and $\int\kappa(Q-R)=\int\Gamma\,\mathrm{d}\kappa$ by Fubini, the integrand being integrable since $\int\kappa<\infty$ and $|Q-R|\le 1+R(1-)$; this is (K3). □

5.6. **Unit scale and joint concavity in (α,m).** Rescaling removes α from the Ising part. More precisely, we have $\Psi_\kappa(w,x)=\sqrt{\alpha}\tilde\Psi_{\sqrt{\alpha}\kappa}(w,x/\sqrt{\alpha})$, where $\tilde\Psi$ is the recursion at $\alpha=1$: $-\partial_w\tilde\Psi = \frac12(\partial_x^2\tilde\Psi + \tilde\kappa(\partial_x\tilde\Psi)^2)$. Let $\tilde\kappa:=\sqrt{\alpha}\kappa$, $\tilde C:=\sqrt{\alpha}C$, $\tilde\delta:=\sqrt{\alpha}\delta$, and let W_m^1 denote W_m at $\alpha=1$. Then $W_m(\kappa)=\sqrt{\alpha}W_m^1(\tilde\kappa)$, and Q and R are unchanged, with $R'=\alpha/\tilde\delta^2$.

Corollary 5.12. *For all $\alpha>0$ and $m\in[-1,1]$,*

$$L(\alpha,m) = \min_{(\tilde\kappa,\tilde C)\in\mathcal{V}}\Big\{\frac{\tilde C}{2} + \frac{\alpha}{2}\int_0^1 \frac{\mathrm{d}w}{\tilde\delta(w)} + W_m^1(\tilde\kappa)\Big\}. \tag{5.16}$$

The functional involves α only through the factor $\alpha/2$. So:

(a) $L(\alpha,m)=\inf_{\tilde\kappa,\tilde C,\tilde\lambda}\{\frac{\tilde C}{2}+\frac{\alpha}{2}\int_0^1\mathrm{d}w/\tilde\delta + \tilde\Psi_{\tilde\kappa}(0,\tilde\lambda)-\tilde\lambda m\}$ is an infimum of functions affine in (α,m), and L is jointly concave on $(0,\infty)\times[-1,1]$.
(b) L is C^1 on $(0,\infty)\times(-1,1)$, with $\partial_m L = -\tilde\lambda^*(\alpha,m)$ and $\partial_\alpha L = \frac12\Upsilon(\alpha,m)$, $\Upsilon := \int_0^1\mathrm{d}w/\tilde\delta^*(w)$. If $(\alpha_n,m_n)\to(\alpha,m)$ in this set, the minimizers converge: $\tilde\delta_n\to\tilde\delta^*$ pointwise on $[0,1)$, $\tilde\kappa_n\to\tilde\kappa^*$ almost everywhere, $\tilde\lambda_n\to\tilde\lambda^*$, and $\Upsilon_n\to\Upsilon$.
(c) Put

$$P^*(m) := \inf_{\tilde\kappa\in\mathcal{U}}\Big\{W_m^1(\tilde\kappa) - \frac12\int_0^1 w\tilde\kappa(w)\,\mathrm{d}w\Big\}. \tag{5.17}$$

P^* is even and concave, $P^*\ge 0$ and $P^*(1)=0$. So P^* is nonincreasing on $[0,1]$. And $L\ge\sqrt{\alpha}+P^*(m)$ for all α.

By [4, Theorem 1] *and a Legendre transform in the field, P^* is the ground state energy of the Sherrington–Kirkpatrick model on the slice. This is why we call it the slice energy. No proof below uses that identification.*

Proof. Part (a) follows by writing out W_m^1 in (5.16).

(b). By (a), at each (α,m) with $|m|<1$ the vector $(\Upsilon/2,-\tilde\lambda^*)$ is a supergradient of L, where $(\tilde\kappa^*,\tilde C^*)$ is the minimizer and $\tilde\lambda^*:=\lambda(\tilde\kappa^*,m)$ at $\alpha=1$. Let $(\alpha_n,m_n)\to(\alpha,m)$, with minimizing

triples $(\tilde\kappa_n, \tilde C_n, \tilde\lambda_n)$. The masses are bounded, since (5.6) gives $\frac12\tilde C_n + \frac12\delta_{\tilde\kappa_n}(0) \le L(\alpha_n, m_n)$, and so is $|\tilde\lambda_n|$, since $\tilde\Psi(0,\lambda) - \lambda m \ge |\lambda|(1-|m|)$. By Lemma 5.9(b) and Fatou, a limit point (κ', C', λ') has value at most $\lim L(\alpha_n, m_n) = L(\alpha, m)$, so it is the minimizer, by Theorem 5.7 and Lemma 5.5(c), and the whole sequence converges. Also $\int \mathrm{d}w/\tilde\delta_n^2 \le 1/\alpha_n$ by (K1), so $1/\tilde\delta_n$ is uniformly integrable and $\Upsilon_n \to \Upsilon$. Hence every limit of supergradients at nearby points is $(\Upsilon/2, -\tilde\lambda^*)$. The superdifferential of a concave function is the closed convex hull of limits of gradients at nearby points [17, Theorem 25.6], so it is a singleton, and L is differentiable. A differentiable concave function on an open convex set is C^1 [17, Corollary 25.5.1].

(c). The function P^* is an infimum of functions affine in m, hence concave, and it is even because $\tilde\Psi_{\tilde\kappa}(0,\cdot)$ is even. By Lemma 5.3(iv), $\tilde\Psi_{\tilde\kappa}(0,\lambda) - \lambda m - \frac12\int w\tilde\kappa \ge |\lambda|(1-|m|) + \frac12\int(1-w)\tilde\kappa \ge 0$, so $P^* \ge 0$. At $m = 1$, we take $\tilde\kappa = 0$ and let $\lambda \to \infty$: $\mathbb{E}|\lambda + z| - \lambda \to 0$, so $P^*(1) = 0$.

It remains to prove the lower bound on L. Since $\tilde\delta/2 + (\alpha/2)/\tilde\delta \ge \sqrt\alpha$ pointwise and $\frac{\tilde C}{2} = \frac12\int_0^1(\tilde\delta - \delta_{\tilde\kappa})\,\mathrm{d}w$, we get $\frac{\tilde C}{2} + \frac{\alpha}{2}\int \mathrm{d}w/\tilde\delta \ge \sqrt\alpha - \frac12\int\delta_{\tilde\kappa}$, with $\int_0^1\delta_{\tilde\kappa} = \int w\tilde\kappa$, so (5.16) and (5.17) give $L \ge \sqrt\alpha + P^*(m)$. This completes the proof of (c). □

5.7. **Shape of L and e.** At fixed load, the previous results give the following.

Theorem 5.13. *Fix $\alpha > 0$.*

- *(a) L is concave and even on $[-1,1]$, and $L(m) = \inf_{\kappa,C,\lambda}\alpha^{-1/2}\{\frac{\alpha}{2}\Psi^{\mathrm S}(\kappa,C) + \Psi_\kappa(0,\lambda) - \lambda m\}$ is an infimum of functions affine in m.*
- *(b) L is C^1 on $(-1,1)$ with $L'(m) = -h(m)$, $h(m) := \lambda(\kappa^*_m, m)/\sqrt\alpha = \tilde\lambda^*(m)$. The function h is continuous and nondecreasing, $h(0) = 0$, and $L'(1-) = -\infty$.*
- *(c) e is C^1 on $(-1,1)$, $e'(m) = m + LL' = m - L(m)h(m)$, $e'(0) = 0$, and $e'(m) \to -\infty$ as $m \to 1$.*

Proof. Part (a) follows from Theorem 5.7 and evenness under $\sigma \mapsto -\sigma$.

For (b), we apply Corollary 5.12(b), and for $L'(1-) = -\infty$ we use (5.15), concavity and $L(1) = \sqrt\alpha$: for $q < 1$, $L'(1-) \le (L(1) - L(q))/(1-q) \le \sqrt\alpha - \mathsf w(q)/(1-q)$, and the right side tends to $-\infty$ as $q \to 1$.

Part (c) is straightforward, using $L \ge \sqrt\alpha$. □

5.8. **The energy for binary patterns.** We recall the objects of Section 3. For $r \ge 1$, $d_r = 2^{r-1}$, and η is uniform on $\{-1,1\}^{r-1}$. For $\boldsymbol t = (t_\eta) \in [-1,1]^{d_r}$, $\boldsymbol\omega(\boldsymbol t) = (\mathbb{E}_\eta\eta_a t_\eta)_{a=2,\dots,r}$, $V(\boldsymbol t)^2 = (\mathbb{E}_\eta t_\eta)^2 + |\boldsymbol\omega(\boldsymbol t)|^2$, $P_r(m) = \{\boldsymbol t : \mathbb{E}_\eta t_\eta = m\}$, $\Theta_r(m) = \max_{P_r(m)}\mathcal F_r$, $\Lambda^{\mathrm R} = \sup_r\Theta_r$, and $e^{\mathrm R} = \lim_{\beta\to\infty}\Lambda^{\mathrm R}_\beta/\beta$, uniformly on $[-1,1]$ (Corollary 3.5). With $L^{(r)}$ from Theorem 5.1(e), let

$$\Theta^0_r(m) := \max_{\boldsymbol t\in P_r(m)}\Big\{\frac{V(\boldsymbol t)^2}{2} + \frac{L^{(r)}(\boldsymbol t)^2}{2}\Big\}. \tag{5.18}$$

Lemma 5.14. *For all r and β, $\sup_m|\Theta_r(m)/\beta - \Theta^0_r(m)| \le \log 2/\beta$. So $e^{\mathrm R} = \sup_r\Theta^0_r$. Moreover $e^{\mathrm R} \ge \Theta^0_1 = e$.*

Proof. By Theorem 5.1(e), $\frac\beta2 L^{(r)}(\boldsymbol t)^2 - \log 2 \le \Phi^{(r)}(\boldsymbol t) \le \frac\beta2 L^{(r)}(\boldsymbol t)^2$, with rates independent of r, so $|\mathcal F_r(\boldsymbol t)/\beta - \frac12 V(\boldsymbol t)^2 - \frac12 L^{(r)}(\boldsymbol t)^2| \le \log 2/\beta$ for all $\boldsymbol t$. We take the maximum on $P_r(m)$, then the supremum over r, and let $\beta \to \infty$. For $r = 1$ the polytope is a point. □

5.9. **The capacity at zero temperature.**

Definition 5.15. For $m_s \in (0,1)$,

$$\begin{aligned}\mathcal W_0(m_s) &:= \{\alpha > 0 : e_\alpha(m_2) > e_\alpha(m_1) \text{ for some } m_s \le m_1 < m_2 \le 1\}\\ &= \{\alpha > 0 : \sup_{m\in(m_s,1)} e'_\alpha(m) > 0\},\end{aligned}$$

and $\alpha_c(0;m_s) := \sup \mathcal{W}_0(m_s)$, $\mathcal{W}_0 := \bigcup_{m_s\in(0,1)} \mathcal{W}_0(m_s)$, $\alpha_c(0) := \sup \mathcal{W}_0$. For binary patterns, $\mathcal{W}_0^{\mathrm{R}}(m_s)$, $\alpha_c^{\mathrm{R}}(0;m_s)$, $\mathcal{W}_0^{\mathrm{R}}$ and $\alpha_c^{\mathrm{R}}(0)$ are defined by the first form, with e^{R} in place of e.

The two forms agree by the mean value theorem and Theorem 5.13(c), which also gives $e'_\alpha(m) = m - L_\alpha(m)h_\alpha(m)$, where (L,h) come from the convex program (5.16). In particular, $\alpha \in \mathcal{W}_0(m_s)$ if and only if $m > L_\alpha(m)h_\alpha(m)$ somewhere in the window. This is a variational characterization of $\alpha_c(0;m_s)$.

For the landscape statement, H_N denotes the energy (1.1) for binary patterns, and $H_N(\sigma) := -\frac{1}{2N}[(\xi^1\cdot\sigma)^2 + |G\sigma|^2]$ for Gaussian rows with one binary pattern.

Theorem 5.16. *Fix $m_s \in (0,1)$ and write $\mathsf{d}(\alpha) := \max_{[m_s,1)} e'_\alpha$.*

(a) $\mathcal{W}_0(m_s)$ is open. The function d is continuous, $e'_\alpha \to -\infty$ at $m \to 1$ locally uniformly in α, and $\mathcal{W}_0(m_s) = \{\mathsf{d} > 0\}$.

(b) (Landscape, for all patterns.) Let $e_\bullet$ be e for Gaussian rows with one binary pattern and e^{R} for binary patterns, and let $\mathcal{W}_\bullet(m_s)$ be the corresponding well set. Let $m_1 < m_2$ with $\Delta := e_\bullet(m_2) - e_\bullet(m_1) > 0$, and $m_{k_i} \to m_i$. For some $c > 0$ and N large, with probability at least $1 - e^{-cN}$, for all patterns μ simultaneously (for Gaussian rows, only $\mu = 1$),

$$\min\{H_N(\sigma) : m_\mu(\sigma) = m_{k_1}\} \ge \min\{H_N(\sigma) : m_\mu(\sigma) = m_{k_2}\} + \frac{\Delta}{3}N.$$

If $\alpha \notin \mathcal{W}_\bullet(m_s)$, then for every $\epsilon > 0$, with probability tending to 1, for all μ (for Gaussian rows, only $\mu = 1$) and all $k_1 < k_2$ with $m_{k_1} \ge m_s$,

$$\min\{H_N : m_\mu = m_{k_1}\} \le \min\{H_N : m_\mu = m_{k_2}\} + \epsilon N.$$

So the profile $k \mapsto \min\{H_N : m_\mu = m_k\}$ has no barrier of extensive depth on the window, and for each α, membership in the well set is exactly the existence of an extensive well in this profile.

(c) (Binary patterns.) By Lemma 5.14, $e^{\mathrm{R}} \ge e$, and $\mathcal{W}_0^{\mathrm{R}}(m_s)$ and $\alpha_c^{\mathrm{R}}(0;m_s)$ are read from $e^{\mathrm{R}} = \sup_r \Theta_r^0$. Section 6 identifies the two well sets, Theorem 6.9.

(d) ($T \to 0$.) For $T = 1/\beta > 0$, let $\mathcal{W}_T(m_s)$ and $\alpha_c(T;m_s)$ be as in (1.11), for Gaussian rows. If $\alpha \in \mathcal{W}_0(m_s)$, with $m_1 < m_2$ and $\Delta := e(m_2) - e(m_1) > 0$, then $\alpha \in \mathcal{W}_T(m_s)$ for $T < 1/\beta_0$, $\beta_0 := (\log 2 - I(m_1))/\Delta$. So

$$\liminf_{T\to 0} \alpha_c(T;m_s) \ge \alpha_c(0;m_s).$$

Let $v^*(m)$ be the optimal radius in (5.1). Then, locally uniformly in α, $v^*(m)/\beta \to L(m)^2/\alpha$ uniformly in $m \in [-1,1]$, and $\Lambda'_\beta(m)/\beta \to e'(m)$ uniformly on compact subsets of $(-1,1)$. For every compact $A \subset (0,\infty)$ there is $\bar m < 1$ such that $\Lambda'_\beta < 0$ on $[\bar m, 1)$ for all $\alpha \in A$ and all large β.

Proof. (a). Continuity follows from Corollary 5.12(b) and compactness once m near 1 is excluded uniformly. By concavity $L'(m) \le L'(\bar m)$ for $m \ge \bar m$, and $L \ge \sqrt{\alpha}$, so $e'_\alpha(m) \le 1 + \sqrt{\alpha}L'(\bar m)$ once $L'(\bar m) < 0$. For $q < \bar m < 1$, concavity and (5.15) give

$$L'(\bar m) \le \frac{L(\bar m) - L(q)}{\bar m - q} \le \frac{\sqrt{\alpha}(1-q) + \mathsf{w}(\bar m) - \mathsf{w}(q)}{\bar m - q}.$$

Taking q near 1 and then $\bar m$ near 1 makes this as negative as we like, locally uniformly in α. Openness and $\mathcal{W}_0(m_s) = \{\mathsf{d} > 0\}$ follow from the second form of the definition.

(b). We fix μ. By the gauge of Section 3, $\max_{m_\mu(\sigma)=m_k}(-H_N(\sigma))/N = m_k^2/2 + \frac{1}{2}L^\mu(k)^2$, where $L^\mu(k)$ has the law of $L_{S_N(k)}$ on the Gaussian rows or on the Rademacher matrix

of the gauged patterns. Let $\mathsf{q}^\mu(k) := \frac{1}{2}L^\mu(k)^2$. For every $\tau > 0$ there is $c(\tau) > 0$ with $\mathbb{P}(|\mathsf{q}^\mu(k) - \mathbb{E}\mathsf{q}^\mu(k)| > \tau) \le e^{-c(\tau)N}$ for N large, uniformly in k; for binary patterns this follows from (3.5), and for Gaussian rows from Theorem 5.1(d), as $\mathbb{E}L_A$ is bounded. By Corollary 3.5, respectively Theorem 5.1(a), $\sup_k |m_k^2/2 + \mathbb{E}\mathsf{q}^\mu(k) - e_\bullet(m_k)| \to 0$. We take $\tau = \Delta/7$ and a union bound over the M patterns and the two slices. On that event the difference of the two minima, divided by N, is at least $\Delta - 2\tau - o(1) \ge \Delta/3$, by continuity of $e_\bullet$. For the second statement, $e_\bullet$ is nonincreasing on the window, and the union bound runs over $(N+1)M$ events, with $\tau = \epsilon/3$.

Part (c) follows from Lemma 5.14.

(d). For the first part, by Theorem 5.1(b), we have $\Lambda_\beta(m_2) - \Lambda_\beta(m_1) \ge \beta\Delta - \log 2 + I(m_1)$; the endpoint $m_2 = 1$ makes no difference, by continuity of Λ_β.

For the rest, we show that the optimal radius converges after scaling and that $\Phi^{\mathrm{mix}}(\gamma;\cdot)/\sqrt{\gamma}$ converges with its derivative because it is concave. Let $\tau := v/\beta$. Since $\mathrm{rad}(v) = \frac{\alpha}{2}(1+\log v - v)$, (5.3) gives, uniformly for $\tau \in (0,\infty)$ in compact sets, in m, and in α in a compact subset of $(0,\infty)$,

$$\frac{\mathrm{rad}(\beta\tau) + \Phi^{\mathrm{mix}}(\beta^2\tau; m)}{\beta} \to \sqrt{\tau\alpha}\, L(m) - \frac{\alpha}{2}\tau,$$

which is strictly concave in τ with maximum at L^2/α. As the left side is concave in τ, its maximizer $\tau^* = v^*/\beta$ converges to $L(m)^2/\alpha$ uniformly. Let $m < m'$. Since $\Phi_\beta(m) \ge \Phi^{\mathrm{mix}}(\beta v^*(m'); m) + \mathrm{rad}(v^*(m'))$,

$$\Phi_\beta(m') - \Phi_\beta(m) \le \Phi^{\mathrm{mix}}(\beta v^*(m'); m') - \Phi^{\mathrm{mix}}(\beta v^*(m'); m) \le (m'-m)\partial_m^+ \Phi^{\mathrm{mix}}(\beta v^*(m'); m).$$

The function $f_\gamma := \Phi^{\mathrm{mix}}(\gamma;\cdot)/\sqrt{\gamma}$ is concave by Lemma 5.8, and by (5.3) it converges to $\sqrt{\alpha}L$ uniformly, also uniformly on compact α sets. Since L is C^1, a chord comparison gives $\partial_m^\pm f_\gamma \to \sqrt{\alpha}L'$ uniformly on compact subsets of $(-1,1)$. With $\gamma = \beta v^*(m') = \beta^2\tau^*(m')$ we have $\sqrt{\gamma}/\beta \to L(m')/\sqrt{\alpha}$, so

$$\frac{\Lambda_\beta(m') - \Lambda_\beta(m)}{\beta(m'-m)} \le \frac{m+m'}{2} + L(m')L'(m) + o(1).$$

The same argument, with $v^*(m)$ in place of $v^*(m')$ and ∂_m^- at m', gives the lower bound $\frac{m+m'}{2} + L(m)L'(m') - o(1)$. Since Λ_β is C^1 by Theorem 4.9, letting $m' \downarrow m$ gives $\Lambda_\beta'(m)/\beta \to m + L(m)L'(m) = e'(m)$, with the stated uniformity.

It remains to treat the region near 1. For $\bar m \le m < m' \le 1$, concavity gives $\partial_m^+ f_\gamma(m) \le \partial_m^+ f_\gamma(\bar m)$, and $\sqrt{\gamma}/\beta \ge 1 - o(1)$ since $L \ge \sqrt{\alpha}$, so once $L'(\bar m) < 0$ the quotient is at most $1 + \sqrt{\alpha}L'(\bar m) + o(1)$. As we saw in the proof of (a), $\bar m$ can be chosen with $1 + \sqrt{\alpha}L'(\bar m) \le -1$ for all $\alpha \in A$, which proves the last claim of (d). □

5.10. **Structure of the minimizer.** We show that the support of the minimizer stays away from level 0. Below its first level the flow is the pure heat flow, which gives closed bounds.

Lemma 5.17. *Let $\alpha > 0$, $m \in (0,1)$. Let (κ^*, C^*) be the minimizer, with δ^*, $M_* = \delta^*(0)$, $Q^* = Q_{\kappa^*,m}$, R^*, Γ^* as in Proposition 5.11, and $\lambda^* = \lambda(\kappa^*, m)$. Let $w_1 := \min(\mathrm{supp}\ \mathrm{d}\kappa^* \cup \{1\})$, where $\mathrm{d}\kappa^*$ includes the atom $\kappa^*(0)$ at 0. Then:*

- *(a) $w_1 > 0$, $\delta^* = M_*$ on $[0, w_1]$, and $M_* \ge 1$;*
- *(b) $Q^*(w_1) = R^*(w_1) = w_1/M_*^2$, so $w_1 = M_*^2 Q^*(w_1) \ge Q^*(w_1) \ge m^2$ (at $w_1 = 1$ read $Q^*(1-)$, $R^*(1-)$);*
- *(c) $\sqrt{\alpha w_1}\,\lambda_0(m) \le \lambda^* \le \sqrt{\alpha}\,\lambda_0(m)$;*

(d) $w_1 \ge w_{\mathrm{fl}}(m)$, *where* $w_{\mathrm{fl}}(m)$ *is the limit of the monotone iteration* $w^{(0)} = m^2$, $w^{(n+1)} = Q_{\mathrm{fl}}(w^{(n)})$, *and*

$$Q_{\mathrm{fl}}(w) := \mathbb{E}\Big(2\mathsf{G}\Big(\sqrt{\frac{w}{1-w}}\,|\lambda_0(m)+z|\Big)-1\Big)^2.$$

The function w_{fl} *is nondecreasing in* m*;*

(e) let $\mathsf{f} := \partial_x \Psi_{\kappa^*}(w_1,\cdot)$, *with* $\mathsf{f} = \operatorname{sgn}$ *if* $w_1 = 1$, *and let* ς *be the probability measure* $\frac{1}{2}\,\mathrm{d}\mathsf{f}$. *Then*

$$\mathcal{M}_{\kappa^*}(x) = \int \Big(2\mathsf{G}\Big(\frac{x-t}{\sqrt{\alpha w_1}}\Big)-1\Big)\varsigma(\,\mathrm{d}t), \qquad \mathcal{M}'_{\kappa^*} \le \frac{\mathsf{w}(\mathcal{M}_{\kappa^*})}{\sqrt{\alpha w_1}}.$$

Proof. (a), (b). If $w_1 = 1$, then $\kappa^* = 0$, $\delta^* \equiv C^* > 0$, and (K1) forces $C^* = 1$, so $R^*(1-) = 1 = Q^*(1-)$ and both items hold. Now let $w_1 < 1$. Since $(Q^* - R^*)(0) = m^2 > 0$ and Q^*, R^* are continuous, Γ^* is strictly decreasing near 0, and (K2) gives $\Gamma^*(0) > 0$. Hence $\Gamma^* > 0$ on some $[0, s_0)$, and (K3) excludes support there. By (K1), $R^*(1-) \le 1$, while $\delta^* \le M_*$ gives $1/M_*^2 \le R^*(1-)$, so $M_* \ge 1$. By (K2) and (K3), we have $\Gamma^* = 0$ at w_1, an interior minimum, so $Q^*(w_1) = R^*(w_1)$. Also $R^*(w_1) = w_1/M_*^2$ as $\delta^* = M_*$ on $[0, w_1]$, and $Q^* \ge m^2$.

(e). Below w_1, the flow is the pure heat flow, so $\mathcal{M}_{\kappa^*}(x) = \mathbb{E}\mathsf{f}(x + \sqrt{\alpha w_1}z)$. By Lemma 5.5(b), f is odd and nondecreasing with limits ± 1, so ς is a probability measure and the representation follows. Each $\mathsf{M}_t(x) := 2\mathsf{G}((x-t)/\sqrt{\alpha w_1}) - 1$ satisfies $\mathsf{M}'_t = \mathsf{w}(\mathsf{M}_t)/\sqrt{\alpha w_1}$, since $\mathsf{w}(2\mathsf{G}(y)-1) = 2\mathsf{g}(y)$, and as w is concave, Jensen gives the bound on $\mathcal{M}'_{\kappa^*}$.

(c). For $x \ge 0$, $t \mapsto \mathsf{M}_t(x) + \mathsf{M}_{-t}(x)$ is largest at $t = 0$, and ς is symmetric, so $m = \mathcal{M}_{\kappa^*}(\lambda^*) \le \mathsf{M}_0(\lambda^*) = 2\mathsf{G}(\lambda^*/\sqrt{\alpha w_1}) - 1$. For the upper bound, $m \ge 2\mathsf{G}(\lambda^*/\sqrt{\alpha}) - 1$ by Lemma 5.5(b).

Part (d) is trivial if $w_1 = 1$. Otherwise $X_{w_1} \sim N(\lambda^*, \alpha w_1)$, and by Lemma 5.5(b), $\mathsf{f}(x)^2 \ge (2\mathsf{G}(|x|/\sqrt{\alpha(1-w_1)}) - 1)^2$. By (c), and since $\mathbb{E}\mathsf{n}(|a+z|)$ is nondecreasing in $a \ge 0$ for nondecreasing n, we get $w_1 \ge Q^*(w_1) \ge Q_{\mathrm{fl}}(w_1)$. The function Q_{fl} is nondecreasing in w and in m, so $Q_{\mathrm{fl}}(w) \le Q_{\mathrm{fl}}(w_1) \le w_1$ whenever $w \le w_1$. Since $w^{(0)} = m^2 \le w_1$, all iterates are at most w_1, and they are monotone because Q_{fl} is nondecreasing. By induction each $w^{(n)}$ is nondecreasing in m. □

In unit scale (Corollary 5.12), w_1 and Q^* are unchanged, $M_* = \tilde{\delta}^*(0)/\sqrt{\alpha}$, and the optimal field is $\tilde{\lambda}^* = \lambda^*/\sqrt{\alpha} = h(m)$. Hence (c) and (e) read $\sqrt{w_1}\lambda_0(m) \le h(m) \le \lambda_0(m)$, $\mathcal{M}_{\tilde{\kappa}^*}(y) = \int (2\mathsf{G}((y-t)/\sqrt{w_1}) - 1)\tilde{\varsigma}(\,\mathrm{d}t)$ with $\tilde{\varsigma}$ the image of ς under $t \mapsto t/\sqrt{\alpha}$, and

$$\mathsf{A}_0(m) := \mathcal{M}'_{\tilde{\kappa}^*}(h(m)) \le \frac{\mathsf{w}(m)}{\sqrt{w_1}}. \tag{5.19}$$

6. Merging retained patterns

In this section, we prove the inequality that decides the capacity for binary patterns. By Theorem 3.1, the free energy at fixed overlap is a supremum over states correlated with finitely many retained patterns, and we show that such a state is never better than a Gaussian-row state with the single overlap V. The inequality is Jensen's inequality on the sphere of overlap vectors. Its field form is due to [1, Lemma 2.4, Proposition 5.2], and at a fixed slice it takes two lines from the bound (4.18), at the path that is optimal at V. What this section adds is the strict inequality with a gap uniform in r, the identity of the well sets, and the zero-temperature theorem.

6.1. **Notation.** Let $r \geq 1$, $d_r = 2^{r-1}$, $\eta \in \{-1,1\}^{r-1}$, and let $\mathbb{E}_\eta$ be the uniform average. For $\boldsymbol{t} \in [-1,1]^{d_r}$ let $m = \mathbb{E}_\eta t_\eta$ and $\boldsymbol{\omega} = \mathbb{E}_\eta[\eta t_\eta] \in \mathbb{R}^{r-1}$. We write $x_\eta := \boldsymbol{\omega} \cdot \eta$, $R_\eta := m + x_\eta$, and $V = V(\boldsymbol{t}) = (m^2 + |\boldsymbol{\omega}|^2)^{1/2}$. Since $1, \eta_2, \dots, \eta_r$ are orthonormal in $L^2(\eta)$,

$$\mathbb{E}_\eta R_\eta^2 = V^2, \qquad \mathbb{E}_\eta[R_\eta t_\eta] = V^2, \qquad V^2 \leq \mathbb{E}_\eta t_\eta^2 \leq 1. \tag{6.1}$$

Recall that $\Lambda'(m) = \beta m - h(m)$ and $h(m) \geq \operatorname{atanh} m$ by (4.15), so $\Lambda' \to -\infty$ as $m \to 1$, and $\Lambda(1) < \max_{[m,1]} \Lambda$ for every $m < 1$.

For a law ν with bounded support and quantile path a, we will use Ψ_a, also written Ψ_ν, $\mathcal{M}_a = \Psi_a'$ and $LC_a(s) = \inf_u\{\Psi_a(u) - us\} = -g_s(a)$ from Section 4.

6.2. **Jensen on the sphere of overlap vectors.**

Lemma 6.1. *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be even and C^1, with $\mathcal{M} := \Psi'$ nondecreasing, concave on $[0,\infty)$, and $\mathcal{M}(0) = 0$. Let $LC(s) := \inf_u\{\Psi(u) - us\}$. Then for all r and $\boldsymbol{t} \in [-1,1]^{d_r}$,*

$$\mathbb{E}_\eta LC(t_\eta) \leq LC(V(\boldsymbol{t})). \tag{6.2}$$

More precisely, if $V > 0$, then for every $\lambda \geq 0$,

$$\mathbb{E}_\eta LC(t_\eta) \leq \mathbb{E}_\eta \Psi(\lambda R_\eta / V) - \lambda V \leq \Psi(\lambda) - \lambda V. \tag{6.3}$$

Proof. Since $\mathcal{M}$ is concave with $\mathcal{M}(0) = 0$, the function $\mathcal{M}(u)/u$ is nonincreasing on $(0,\infty)$. Hence $G(q) := \Psi(\sqrt{q})$ has $G'(q) = \mathcal{M}(\sqrt{q})/(2\sqrt{q})$ nonincreasing, so G is concave on $[0,\infty)$. If $V > 0$, we use the trial $u_\eta := \lambda R_\eta / V$ in the definition of $LC(t_\eta)$ and (6.1) to obtain

$$\mathbb{E}_\eta LC(t_\eta) \leq \mathbb{E}_\eta \Psi(\lambda R_\eta/V) - \frac{\lambda}{V}\mathbb{E}_\eta[R_\eta t_\eta] = \mathbb{E}_\eta G(\lambda^2 R_\eta^2/V^2) - \lambda V \leq G(\lambda^2) - \lambda V = \Psi(\lambda) - \lambda V.$$

Since Ψ is even, the infimum defining $LC(V)$, $V \geq 0$, may be restricted to $u \geq 0$, because $\Psi(-u) + uV \geq \Psi(u) - uV$. Taking the infimum over $\lambda \geq 0$ gives (6.2). If $V = 0$, then $m = 0$, and concavity of LC gives $\mathbb{E}_\eta LC(t_\eta) \leq LC(m) = LC(0)$. □

The function Ψ_a satisfies these hypotheses by Lemma 4.8(i), and so does $\Psi_\kappa(0,\cdot)$ by Lemma 5.5.

Theorem 6.2. *For $\alpha, \beta, \gamma > 0$, $r \geq 1$ and $\boldsymbol{t} \in [-1,1]^{d_r}$,*

$$\Phi_r^{\mathrm{mix}}(\gamma; \boldsymbol{t}) \leq \Phi^{\mathrm{mix}}(\gamma; V(\boldsymbol{t})), \qquad \Phi^{(r)}(\boldsymbol{t}) \leq \Phi(V(\boldsymbol{t})), \qquad \mathcal{F}_r(\boldsymbol{t}) \leq \Lambda(V(\boldsymbol{t})). \tag{6.4}$$

Hence for $m \in [0,1]$,

$$\Lambda(m) \leq \Lambda^{\mathrm{R}}(m) \leq \max_{V \in [m,1]} \Lambda(V), \qquad \Lambda^{\mathrm{R}}(0) = \max_{[0,1]} \Lambda. \tag{6.5}$$

Proof. We bound the multi-type problem by a linear functional of the path, and choose the path that is optimal at the merged overlap V, not at m.

First let $V := V(\boldsymbol{t}) < 1$ and $\gamma > 0$. We put $\tau = \gamma\alpha/2$ and $a := \tau y$, where y denotes the optimal path at the slice V. By (4.18), used for r types and then for one type with equality, and Lemma 6.1, we have

$$\Phi_r^{\mathrm{mix}}(\gamma; \boldsymbol{t}) \leq \tau + \alpha\psi_{\mathrm{S}}^*(y) + \mathbb{E}_\eta LC_a(t_\eta) \leq \tau + \alpha\psi_{\mathrm{S}}^*(y) + LC_a(V) = \Phi^{\mathrm{mix}}(\gamma; V).$$

At $\gamma = \beta v$ we add $\operatorname{rad}(v)$, take the supremum over v using (2.7), and add $\beta V^2/2$. If $V = 1$, we apply the above to $(1-\varepsilon)\boldsymbol{t}$, whose V is $1 - \varepsilon$, and let $\varepsilon \to 0$ by continuity, using (2.8) for the first inequality.

It remains to prove (6.5). The bounds $V(\boldsymbol{t}) \geq |m|$ and $V(\boldsymbol{t}) \leq 1$ give the upper bound, and the case $r = 1$ gives the lower bound. At $m = 0$, for $s \in [0,1]$, the choice $r = 2$, $\boldsymbol{t} = (s, -s)$ has $m = 0$ and $\boldsymbol{\omega} = s$. Since $g_{-s} = g_s$, by Proposition 4.1(b) for $s < 1$ and by (2.2) for $s = 1$, (2.13)

gives $\Phi_2^{\mathrm{mix}}(\gamma;\boldsymbol{t}) = \Phi^{\mathrm{mix}}(\gamma;s)$, so $\Phi^{(2)}(s,-s) = \Phi(s)$ by (2.7), and $\mathcal{F}_2(\boldsymbol{t}) = \Lambda(s)$. This completes the proof. □

Remark 6.3. In [1], Lemma 2.4 is the concavity in the square of the field, and Proposition 5.2, (5.21), compares k retained fields with one at fixed radius. Thus (6.4) at a fixed slice also follows from (5.21), their Proposition 5.1, (5.3), and convexity in m. The upper bound in $\Lambda^{\mathrm{R}}(0) = \max\Lambda$ is their Theorem 1.4 and Corollary 1.5 with the gauge, and the same holds at $T = 0$ as $\beta \to \infty$.

Corollary 6.4. *Let* $E := \{m \in [0,1] : \Lambda(m) = \max_{[m,1]}\Lambda\}$. *On* E, $\Lambda^{\mathrm{R}} = \Lambda$. *If* $\alpha \notin \mathcal{W}_T$, *then* $E = [0,1]$, *so* $\Lambda^{\mathrm{R}} = \Lambda$ *on* $[0,1]$ *and* Λ^{R} *is nonincreasing there.*

Proof. This is (6.5). If $\alpha \notin \mathcal{W}_T$, then Λ is nonincreasing on $(0,1)$, and it is continuous. □

6.3. **The strict form.** The bound (6.5) alone does not give $\mathcal{W}_T(m_s) \subset \mathcal{W}_T^{\mathrm{R}}(m_s)$, since the value $\Lambda^{\mathrm{R}}(m_1)$ could rise up to $\max_{[m_1,1]}\Lambda$ and fill the well. We will need strictness at the barrier, with a gap that does not disappear as $r \to \infty$. The gap comes from the difference between Jensen's inequality for the pair $\pm\eta$ and for the mean, and from the fact that a Rademacher sum cannot concentrate near zero.

Lemma 6.5. *Let* $\Psi : \mathbb{R} \to \mathbb{R}$ *be even and* C^1, *with* $\mathcal{M} := \Psi'$ *real analytic, strictly increasing, bounded, and concave on* $[0,\infty)$. *Then* $\mathcal{M}(u)/u$ *is strictly decreasing on* $(0,\infty)$, *and* $q \mapsto \Psi(\sqrt{q})$ *is strictly concave on* $[0,\infty)$.

Proof. By oddness, $\mathcal{M}(0) = 0$, so $\mathcal{M}(u)/u$ is nonincreasing by concavity. If it were constant, equal to c, on an interval in $(0,\infty)$, then $\mathcal{M} = cu$ on $\mathbb{R}$ by analyticity, and boundedness gives $c = 0$, contrary to strict monotonicity. The last claim follows as in the proof of Lemma 6.1, since $\frac{\mathrm{d}}{\mathrm{d}q}\Psi(\sqrt{q}) = \mathcal{M}(\sqrt{q})/(2\sqrt{q})$. □

Theorem 6.6. *Let* $0 < m_b < 1$ *and* $L_* := \max_{[m_b,1]}\Lambda > \Lambda(m_b)$. *Then* $\Lambda^{\mathrm{R}}(m_b) < L_*$.

Proof. Fix r, $\boldsymbol{t}$ with $\mathbb{E}_\eta t_\eta = m_b$ and $V := V(\boldsymbol{t}) \in (0,1)$. Let v be the maximizer of (2.7) at $\boldsymbol{t}$, let y be the optimal path at the slice V, and put $\gamma = \beta v$ and $a := \tau_v y$. We write ν for its law, and we set $\lambda := \lambda(\nu, V) = \mathcal{M}_\nu^{-1}(V) > 0$ and $G := \Psi_\nu(\sqrt{\cdot})$. By (4.18) and (6.3) at this λ, we have

$$\mathcal{F}_r(\boldsymbol{t}) \le \Lambda(V) - g(\boldsymbol{t}), \qquad g(\boldsymbol{t}) := G(\lambda^2) - \mathbb{E}_\eta G(\lambda^2 R_\eta^2/V^2).$$

We pair η with $-\eta$: since $R_{\pm\eta} = m_b \pm x_\eta$, we set

$$\Delta(x) := G\Big(\frac{\lambda^2(m_b^2+x^2)}{V^2}\Big) - \frac{1}{2}\Big[G\Big(\frac{\lambda^2(m_b+x)^2}{V^2}\Big) + G\Big(\frac{\lambda^2(m_b-x)^2}{V^2}\Big)\Big] \ge 0.$$

Then $\mathbb{E}_\eta G(\lambda^2 R_\eta^2/V^2) = \mathbb{E}_\eta G(\lambda^2(m_b^2+x_\eta^2)/V^2) - \mathbb{E}_\eta\Delta(x_\eta)$, and $\mathbb{E}_\eta G(\lambda^2(m_b^2+x_\eta^2)/V^2) \le G(\lambda^2)$ by Jensen, since $\mathbb{E}_\eta x_\eta^2 = |\boldsymbol{\omega}|^2$. Thus $g(\boldsymbol{t}) \ge \mathbb{E}_\eta\Delta(x_\eta)$.

Suppose $\Lambda^{\mathrm{R}}(m_b) \ge L_*$. By (6.5) equality holds, and there are $(r_n, \boldsymbol{t}_n)$ with $\mathcal{F}_{r_n}(\boldsymbol{t}_n) \to L_*$. A pair with $V(\boldsymbol{t}) = 1$ has $\mathcal{F}_r(\boldsymbol{t}) \le \Lambda(1) < L_*$, so $V_n \in [m_b, 1)$ for n large and the bound above applies. Since $\Lambda(V_n) \le L_*$, along a subsequence $V_n \to V_\infty$ with $\Lambda(V_\infty) = L_*$, and $g(\boldsymbol{t}_n) \to 0$. Now $\Lambda(m_b) < L_*$ and $\Lambda(1) < L_*$ give $V_\infty \in (m_b, 1)$, hence $|\boldsymbol{\omega}_n|^2 \to \varsigma^2 := V_\infty^2 - m_b^2 > 0$. Theorem 2.3(c) gives $v_n \le v_1$, so the laws ν_n have support in $[0,S]$ with $S := \alpha\beta v_1/2$, and along a subsequence $v_n \to v_\infty$. By Theorem 4.9(b), the optimal path at $(\beta v_n, V_n)$ converges to the one at $(\beta v_\infty, V_\infty)$, so $\nu_n \to \nu_\infty$ in W_1, where ν_∞ is the law of $\tau_{v_\infty} y_\infty$ with y_∞ optimal at $(\beta v_\infty, V_\infty)$. Moreover, $g_m - I(m)$ is even, convex and $2\int s\,\mathrm{d}\nu_n$-Lipschitz in m by Proposition 4.1, so $\operatorname{atanh} V_n \le \lambda_n \le \operatorname{atanh} V_n + 2S$. Thus $\lambda_n \to \lambda_\infty > 0$ along a subsequence, with $\mathcal{M}_{\nu_\infty}(\lambda_\infty) = V_\infty$.

The anti-concentration step is independent of r. Indeed, $x_\eta = \boldsymbol{\omega}\cdot\eta$ is a Rademacher sum, with $\mathbb{E}_\eta x^2 = |\boldsymbol{\omega}|^2$ and $\mathbb{E}_\eta x^4 \le 3|\boldsymbol{\omega}|^4$, so the Paley–Zygmund inequality gives $\mathbb{P}(x^2 \ge |\boldsymbol{\omega}|^2/2) \ge 1/12$, and Markov's inequality gives $\mathbb{P}(x^2 > 24|\boldsymbol{\omega}|^2) \le 1/24$. Thus x_η lies in the set $\mathcal{K}_n := \{|\boldsymbol{\omega}_n|^2/2 \le x^2 \le 24|\boldsymbol{\omega}_n|^2\}$ with weight at least $1/24$, and for n large, $\mathcal{K}_n$ lies in the compact set $\mathcal{K} := \{\varsigma^2/3 \le x^2 \le 25\varsigma^2\}$.

By Proposition 4.1, $(\nu,u)\mapsto\Psi_\nu(u)$ is jointly continuous, so $\Delta_n\to\Delta_\infty$ uniformly on $\mathcal{K}$, where Δ_∞ denotes Δ for $\nu_\infty,\lambda_\infty,V_\infty$. Using Lemma 4.13(a), we obtain $\check{y}_\infty>0$, so ν_∞ puts no mass on $[0,s_0)$, $s_0:=\tau_{v_\infty}\check{y}_\infty>0$. There the Parisi equation is the heat equation, whence $\phi_{\nu_\infty}(0,u)=\mathbb{E}\phi_{\nu_\infty}(s_0,u+\sqrt{2s_0}z)$ and $\mathcal{M}_{\nu_\infty}$ is real analytic. With Lemma 4.8(i), Lemma 6.5 applies, so G_∞ is strictly concave. Hence $\Delta_\infty>0$ on $\mathcal{K}$, because $m_b>0$ and $x\neq 0$ make $(m_b+x)^2\neq(m_b-x)^2$. Thus $\liminf g(\boldsymbol{t}_n)\ge\frac{1}{24}\min_{\mathcal{K}}\Delta_\infty>0$, which contradicts $g(\boldsymbol{t}_n)\to 0$. □

6.4. **The capacity for binary patterns.**

Theorem 6.7. *Let $T>0$ and $m_s\in(0,1)$.*

(a) *$\mathcal{W}_T^{\mathrm{R}}(m_s)=\mathcal{W}_T(m_s)$. So $\alpha_{\mathrm{c}}^{\mathrm{R}}(T)=\alpha_{\mathrm{c}}(T)$ and*

$$\alpha_{\mathrm{c}}^{\mathrm{R}}(T;m_s)=\alpha_{\mathrm{c}}(T;m_s)=\sup\{\alpha:\ \exists\, m\in[m_s,1),\ \beta m>\lambda(\nu^*(m;\alpha,\beta),m)\}.$$

For $T\in(0,1)$ there is $m_0(T)>0$ with $\alpha_{\mathrm{c}}^{\mathrm{R}}(T;m_0)=\alpha_{\mathrm{c}}(T)$.

(b) *If $\alpha\in\mathcal{W}_T(m_s)$, there are $m_s\le m_b<m_*<1$ with $\Delta^{\mathrm{R}}:=\Lambda^{\mathrm{R}}(m_*)-\Lambda^{\mathrm{R}}(m_b)>0$, and for N large, with probability at least $1-e^{-cN}$ for some $c>0$, for all μ simultaneously, $Z^\mu(k_b)\le\exp(-N\Delta^{\mathrm{R}}/3)Z^\mu(k_*)$.*

Proof. (a) For $\mathcal{W}_T^{\mathrm{R}}(m_s)\subset\mathcal{W}_T(m_s)$, let $\Lambda^{\mathrm{R}}(m_2)>\Lambda^{\mathrm{R}}(m_1)$ with $m_s\le m_1<m_2<1$. By (6.5) there is $V_0\in[m_2,1]$ with $\Lambda(V_0)\ge\Lambda^{\mathrm{R}}(m_2)>\Lambda^{\mathrm{R}}(m_1)\ge\Lambda(m_1)$. If $V_0=1$, we replace it by a point of $(m_1,1)$ with a larger value, as $\Lambda'\to-\infty$ at 1. Thus $\alpha\in\mathcal{W}_T(m_s)$.

For $\mathcal{W}_T(m_s)\subset\mathcal{W}_T^{\mathrm{R}}(m_s)$, let $\Lambda(m_2)>\Lambda(m_1)$ with $m_s\le m_1<m_2<1$, and let m_* be a maximizer of Λ on $[m_1,1]$, so $m_*\in(m_1,1)$. Theorem 6.6 with $m_b=m_1$ gives $\Lambda^{\mathrm{R}}(m_1)<\Lambda(m_*)\le\Lambda^{\mathrm{R}}(m_*)$, hence $\alpha\in\mathcal{W}_T^{\mathrm{R}}(m_s)$.

The formula is the second form of the definition (Theorem 4.12), and taking unions gives $\alpha_{\mathrm{c}}^{\mathrm{R}}(T)=\alpha_{\mathrm{c}}(T)$. For m_0, let $T<1$. Then $\alpha_{\mathrm{c}}(T)$ is positive by Theorem 4.12 and finite by Corollary 4.15, so Theorem 4.12 gives a contact point $m_c>0$, the limit of points m_n with $\beta m_n>h(m_n;\alpha_n)$, $\alpha_n\to\alpha_{\mathrm{c}}(T)$, $\alpha_n\in\mathcal{W}_T$. With $m_0:=m_c/2$, for n large $\alpha_n\in\mathcal{W}_T(m_0)$, hence $\alpha_{\mathrm{c}}(T;m_0)=\alpha_{\mathrm{c}}(T)$. This proves (a).

(b) We apply Theorem 3.1(iii) and the concentration bound (3.4) to the two slices k_b, k_*, for each pattern, with deviation parameter $\Delta^{\mathrm{R}}/4$, and take a union bound over M patterns. □

Proof of Theorem 1.1. Part (a) and the statements on (1.6) are Theorem 2.3 with Theorems 4.5 and 4.9; part (b) is Theorem 3.1(iii); part (c) is Theorem 6.2, Corollary 6.4 and Theorem 6.6. □

Proof of Theorem 1.3. Parts (a), (b), (c) are Theorem 4.12 and Corollary 4.15; part (d) is Theorem 6.7(b); part (e) is Corollary 6.4. □

Remark 6.8. Here Δ^{R} comes from compactness and is not explicit. The identity $\Lambda^{\mathrm{R}}=\Lambda$ fails at $m=0$ as soon as a state at positive overlap beats $m=0$, and by continuity on an interval next to 0. The set where the identity holds contains E, and the capacity does not depend on it.

6.5. **Zero temperature.** With the notation of Section 5, let $L^{(r)}(\boldsymbol{t})$ be the limit of $\frac{1}{N}\mathbb{E}\max|G\sigma|$ on the type slice. For simplicity of notation, in this subsection we write $\Psi_\kappa:=\Psi_\kappa(0,\cdot)$ and $\mathcal{M}_\kappa:=\Psi_\kappa'$, and we put $W_s(\kappa)=\inf_\lambda\{\Psi_\kappa(\lambda)-\lambda s\}$. Recall from (5.18) that $e^{\mathrm{R}}=\sup_r\Theta_r^0$.

Theorem 6.9. *Fix $\alpha>0$.*

(a) *For all r and $\boldsymbol{t}$: $L^{(r)}(\boldsymbol{t}) \le L(V(\boldsymbol{t}))$, so $\frac{1}{2}(m^2+|\boldsymbol{\omega}|^2)+\frac{1}{2}L^{(r)}(\boldsymbol{t})^2 \le e(V(\boldsymbol{t}))$, and*

$$e(m) \le e^{\mathrm{R}}(m) \le \max_{[m,1]} e, \qquad e^{\mathrm{R}}(0) = \max_{[0,1]} e.$$

On $\{m : e(m) = \max_{[m,1]} e\}$, $e^{\mathrm{R}} = e$. If $\alpha \notin \mathcal{W}_0$, then $e^{\mathrm{R}} = e$ on $[0,1]$ and e^{R} is nonincreasing.

(b) *If $0 < m_b < 1$ and $e_* := \max_{[m_b,1]} e > e(m_b)$, then $e^{\mathrm{R}}(m_b) < e_*$.*

(c) *$\mathcal{W}_0^{\mathrm{R}}(m_s) = \mathcal{W}_0(m_s)$ for all $m_s \in (0,1)$, so $\alpha_{\mathrm{c}}^{\mathrm{R}}(0; m_s) = \alpha_{\mathrm{c}}(0; m_s)$. If $\alpha \in \mathcal{W}_0(m_s)$, Theorem 5.16(b) gives the landscape statement for all patterns with $\Delta = e^{\mathrm{R}}(m_*) - e^{\mathrm{R}}(m_b) > 0$.*

Proof. (a) By Theorem 6.2, $\Phi^{(r)}(\boldsymbol{t}) \le \Phi(V(\boldsymbol{t}))$ at every β, and by Theorem 5.1(b), (e), $\frac{\beta}{2}L^{(r)}(\boldsymbol{t})^2 - \log 2 \le \Phi^{(r)}(\boldsymbol{t})$ and $\Phi(V) \le \frac{\beta}{2}L(V)^2$. We divide by β and let $\beta \to \infty$. The bounds on e^{R} follow in the same way from (6.5), since $\Lambda_\beta/\beta \to e$ and $\Lambda_\beta^{\mathrm{R}}/\beta \to e^{\mathrm{R}}$ uniformly on $[0,1]$ (Theorem 5.1(b), Corollary 3.5). If $\alpha \notin \mathcal{W}_0$, then e is nonincreasing on $[0,1]$ by Definition 5.15 and continuity, so $e(m) = \max_{[m,1]} e$ for all m, which proves (a).

(b) We follow the proof of Theorem 6.6, with these changes. Let (κ_n, C_n) be the minimizer at V_n, and put $\lambda_n := \lambda(\kappa_n, V_n)$ and $G_n := \Psi_{\kappa_n}(\sqrt{\cdot})$. Then $L^{(r_n)}(\boldsymbol{t}_n) \le L(V_n) - g(\boldsymbol{t}_n)/\sqrt{\alpha}$ and $g(\boldsymbol{t}_n) \ge \mathbb{E}_\eta \Delta_n(x_\eta)$. If the value tends to e_*, then, as before, $V_n < 1$ for n large and $V_n \to V_\infty \in (m_b, 1)$, because $e' \to -\infty$ at 1 (Theorem 5.13(c)) gives $e(1) < e_*$, and $g(\boldsymbol{t}_n) \to 0$ since $L \ge \sqrt{\alpha} > 0$.

By the proof of Corollary 5.12(b), $(\kappa_n, C_n) \to (\kappa_\infty, C_\infty)$, the minimizer at V_∞, in the sense used there, and $\lambda_n \to \lambda_\infty > 0$. Hence $\kappa_n \to \kappa_\infty$ a.e., with $\int_0^{w_0} |\kappa_n - \kappa_\infty| \to 0$ for $w_0 < 1$. By Lemma 5.3(iv) with $w = w_0$, $w' = 1$, we have $|x| \le \Psi_\kappa(w_0, x) - \frac{\alpha}{2}\delta_\kappa(w_0) \le |x| + \sqrt{2\alpha(1-w_0)/\pi}$ for every $\kappa \in \mathcal{U}$. Since the flow is a sup-norm contraction which commutes with constants, and Lemma 5.3(ii) compares κ_n with κ_∞ on $[0, w_0]$, we get $\Psi_{\kappa_n}(x) - \Psi_{\kappa_n}(0) \to \Psi_{\kappa_\infty}(x) - \Psi_{\kappa_\infty}(0)$ uniformly in x. As Δ_n depends only on such differences, $\Delta_n \to \Delta_\infty$ uniformly on $\mathcal{K}$.

We next prove strictness of Δ_∞. With w_1 as in Lemma 5.17 for the minimizer at V_∞, part (b) there gives $w_1 \ge V_\infty^2 > 0$. Below w_1 the flow is pure heat, so $\mathcal{M}_{\kappa_\infty}$ is real analytic, and Lemma 6.5 applies by Lemma 5.5. Hence G_∞ is strictly concave and $\Delta_\infty > 0$ on $\mathcal{K}$. The anti-concentration step is unchanged. This completes the proof of (b).

(c) We argue as for Theorem 6.7. Since $e' \to -\infty$ at 1, the value $e(1)$ is exceeded just below 1, which handles the replacement of $V_0 = 1$. The landscape statement is Theorem 5.16(b) with e^{R}. □

7. Certified digits at zero temperature

In this section, we prove $0.138198129 \le \alpha_{\mathrm{c}}(0) \le 0.13820106$, with no well at any overlap above the upper end, no hole below the lower end, the same statements for binary patterns, and the same bracket for every limit point of $\alpha_{\mathrm{c}}(T)$ as $T \to 0$.

We write $\underline{\alpha} = 138198129/10^9$ and $\overline{\alpha} = 13820106/10^8$, so the bracket has width $2.931 \cdot 10^{-6}$ and rounds to 0.13820.

The capacity is the sign of e', by Theorem 5.16. One Ising evaluation of the recursion, in ball arithmetic, gives an upper bound on $L(\alpha, m)$ for all α through a trial, and a lower bound for all α through a tangent of the sphere problem. Thus $e' > 0$ at a certified pair of overlaps at the lower end, and $e' < 0$ on $[0.89, 1)$ at the upper end. A separate argument, the slope of the response at the optimizer, excludes wells on $(0, 0.89)$.

Appendix A states the ball evaluation of the recursion and what was computed. We will bound the slice energy from below in Subsection 7.2 (Theorem 7.2) and give the upper and

lower curves for L and the strong concavity that controls L' in Subsections 7.3 and 7.4. In Subsections 7.5, 7.6, 7.7 and 7.8, we then prove the bracket on a window, exclude holes, remove the window, and pass to $T \to 0$.

7.1. **Setting.** We work in the unit scale of Corollary 5.12 and drop its tildes, so $\kappa, C, \delta, \lambda$ are the variables of (5.16), and Ψ_κ is the recursion at $\alpha = 1$: $-\partial_w \Psi_\kappa = \frac{1}{2}(\partial_x^2 \Psi_\kappa + \kappa(\partial_x \Psi_\kappa)^2)$, $\Psi_\kappa(1, x) = |x|$. We will use Lemmas 5.2, 5.3, 5.4 and 5.5 at $\alpha = 1$. On a level $[w, w')$ where $\kappa = c$, we have $\Psi_\kappa(w, \cdot) = \mathcal{T}_{c,\sqrt{w'-w}} \Psi_\kappa(w', \cdot)$, where $\mathcal{T}_{c,b} f(x) = \frac{1}{c} \log \mathbb{E} \exp(cf(x + bz))$ and $\mathcal{T}_{0,b} f(x) = \mathbb{E} f(x + bz)$; here and below, z denotes a standard Gaussian. By Corollary 5.12, we have

$$L(\alpha, m) = \min_{\kappa, C, \lambda} \Big\{ \frac{C}{2} + \frac{\alpha}{2} \int_0^1 \frac{\mathrm{d}w}{\delta_{\kappa,C}(w)} + \Psi_\kappa(0, \lambda) - \lambda m \Big\}, \tag{7.1}$$

with $h = -\partial_m L$ the optimal λ, and $\partial_\alpha L = \frac{1}{2} \int_0^1 \mathrm{d}w/\delta^* > 0$. Moreover, L is jointly concave, $L \geq \sqrt{\alpha}$ and $L(\alpha, 1) = \sqrt{\alpha}$. Recall that $e = m^2/2 + L^2/2$ and $e' = m - Lh$ (Theorem 5.13).

We will use two bounds throughout:

$$\sqrt{\alpha} + P^*(m) \leq L(\alpha, m) \leq \sqrt{\alpha} + \mathsf{w}(m). \tag{7.2}$$

The upper bound is Lemma 5.10, and the lower one is Corollary 5.12(c), with the slice energy

$$P^*(m) = \inf_{\kappa, \lambda} \mathcal{P}_m(\kappa, \lambda), \qquad \mathcal{P}_m(\kappa, \lambda) := \Psi_\kappa(0, \lambda) - \lambda m - \frac{1}{2} \int_0^1 w\kappa(w) \, \mathrm{d}w. \tag{7.3}$$

By the L^1 contraction, Lemma 5.3(ii), the infimum may be taken over step functions. The function P^* is even, concave and nonincreasing on $[0, 1]$, and $P^*(1) = 0$.

For a step function κ_0 and real λ, let $u = \partial_x \Psi_{\kappa_0}$ and let X solve $\mathrm{d}X_w = \kappa_0(w) u(w, X_w) \, \mathrm{d}w + \mathrm{d}B_w$, $X_0 = \lambda$. We then set $Q(w) = Q_{\kappa_0,\lambda}(w) = \mathbb{E} u(w, X_w)^2$ and $u_0 = u(0, \lambda)$. By Lemma 5.4, Q is nondecreasing, $Q(0) = u_0^2$, $Q(1-) = 1$, and for $\kappa \in \mathcal{U}$ and real λ',

$$\Psi_\kappa(0, \lambda') \geq \Psi_{\kappa_0}(0, \lambda) + u_0(\lambda' - \lambda) + \frac{1}{2} \int_0^1 (\kappa - \kappa_0) Q \, \mathrm{d}w. \tag{7.4}$$

The well sets are those of Definition 5.15. A ball $[c \pm r]$ is the set of reals within r of c.

All certified numbers below are obtained from the evaluation of the recursion described in Subsection A.1. For a step path with rational data, this evaluation produces balls containing $\Psi_\kappa(0, \lambda)$, $\partial_x \Psi_\kappa(0, \lambda)$, the values of Q at the level boundaries, and the integrals of $1 - Q$ over the levels, through (A.3).

7.2. **A lower bound for the slice energy.** A trial path gives an upper bound on $P^*(m)$ at once, by (7.3), and the lower bound comes from convexity of $\mathcal{P}_m$. More precisely, the tangent (7.4) at the trial bounds every competitor from below up to the term $\frac{1}{2} \int \kappa(Q - w)$, which involves the competitor's mass. We will control that mass by the value itself through a short a priori bound, and the inequality then closes.

Lemma 7.1. *Let $n \geq 1$ and $|m| < 1$. The function $\mathcal{P}_m$ has a minimizer (κ, λ) over real λ and step functions with at most n steps, and at a minimizer $\int_0^1 w\kappa(w) \, \mathrm{d}w \leq \mathcal{P}_m(\kappa, \lambda)$.*

Proof. By Lemma 5.3(iv), $\Psi_\kappa(w, x) \geq |x| + \frac{1}{2} \int_w^1 \kappa$, so $\mathcal{P}_m(\kappa, \lambda) \geq \frac{1}{2} \int (1-w)\kappa + |\lambda|(1 - |m|)$, which bounds λ and κ on every $[0, 1 - \epsilon]$. A minimizing sequence with levels squeezed to $w = 1$ loses nothing in the limit. Indeed, $\Psi_\kappa(w, x) - \frac{1}{2} \int_w^1 w'\kappa(w') \, \mathrm{d}w' \geq |x| \geq \mathbb{E}|x + B_{1-w}| - \sqrt{2(1-w)/\pi}$, and monotonicity of the recursion in the terminal data lets us drop those levels at cost $\sqrt{2(1-w)/\pi}$. Thus the infimum is attained on a path with at most n active steps.

Let the minimizer have active levels $[s_j, s_{j+1})$ with values c_j, and let $Q = Q_{\kappa,\lambda}$. Then $\partial_\lambda \mathcal{P}_m = u_0 - m$, $\partial_{c_j}\mathcal{P}_m = \frac{1}{2}\int_{s_j}^{s_{j+1}}(Q-w)$, and $\partial_{s_j}\mathcal{P}_m = \frac{1}{2}(c_{j-1}-c_j)(Q(s_j)-s_j)$, so $u_0 = m$, $\int_{s_j}^{s_{j+1}}(Q-w) = 0$ on positive active levels, and $Q(s_j) = s_j$ at internal positive jumps. An initial zero level satisfies only a one-sided condition, but it contributes nothing when multiplied by κ.

Let $u_w = u(w, X_w)$, $\mathsf{p}_w = \partial_x^2 \Psi_\kappa(w, X_w) \geq 0$, $\mathsf{a}(w) = \mathbb{E}\mathsf{p}_w$. Differentiating the recursion in x and Itô's formula give $\mathrm{d}u_w = \mathsf{p}_w\,\mathrm{d}B_w$ and $\mathrm{d}\mathsf{p}_w = -\kappa\mathsf{p}_w^2\,\mathrm{d}w + \text{martingale}$, so $Q' = \mathbb{E}\mathsf{p}_w^2$ and $\mathsf{a}' = -\kappa Q'$. The equality case of Lemma 5.4(a) gives $\Psi_\kappa(0,\lambda) = \mathbb{E}|X_1| - \frac{1}{2}\int \kappa Q$. By Lemma 5.4(b), $\mathbb{E}|X_1| = \mathbb{E}[\mathrm{sgn}(X_1)X_1] = \lambda m + \int \kappa Q + \mathbb{E}[\mathrm{sgn}(X_1)B_1]$, where $\mathbb{E}[\mathrm{sgn}(X_1)B_1] = \int_0^1 \mathsf{a}$ by the Itô isometry. Thus $\mathcal{P}_m(\kappa,\lambda) = \frac{1}{2}\int \kappa(Q-w) + \int_0^1 \mathsf{a} = \int_0^1 \mathsf{a}$. Integrating by parts, we obtain $\int_0^t \mathsf{a} = t\mathsf{a}(t) + \int_0^t w\kappa Q' \geq \int_0^t w\kappa Q'$. On each active level $\int wQ' = [wQ] - \int Q = (s_{j+1}^2 - s_j^2)/2 = \int w\,\mathrm{d}w$, by the stationarity conditions and $Q(1-) = 1$. Letting $t \to 1$ completes the proof. □

Theorem 7.2. *Let κ be a step function in $\mathcal{U}$, λ real, $|m| < 1$, $Q = Q_{\kappa,\lambda}$, and let*

$$\Gamma_{\mathrm{id}}(s) = \int_s^1 (Q(w) - w)\,\mathrm{d}w, \qquad D = \min\Big(0, \inf_{s\in[0,1)} \frac{2\Gamma_{\mathrm{id}}(s)}{1-s^2}\Big),$$

$$\mathcal{P}_{\tan} = \mathcal{P}_m(\kappa,\lambda) - \frac{1}{2}\int_0^1 \kappa(Q-w)\,\mathrm{d}w, \qquad \Xi = |\lambda| + \frac{\mathcal{P}_m(\kappa,\lambda)}{1-|m|}.$$

Then

$$P^*(m) \geq \frac{\mathcal{P}_{\tan} - |u_0 - m|\Xi}{1 + |D|/2}. \tag{7.5}$$

Proof. Let (κ_n, λ_n) minimize over at most n steps, with n at least the number of steps of κ, and let $\mathcal{P}_n$ be its value. Then $\mathcal{P}_n \leq \mathcal{P}_m(\kappa,\lambda)$ and $|\lambda_n|(1-|m|) \leq \mathcal{P}_n$, so $|\lambda_n - \lambda| \leq \Xi$. By Fubini, with the Stieltjes measure including the atom $\kappa_n(0)\delta_0$,

$$\int \kappa_n(Q-w) = \int_{[0,1)} \Gamma_{\mathrm{id}}\,\mathrm{d}\kappa_n, \qquad \int w\kappa_n = \int_{[0,1)} \frac{1-s^2}{2}\,\mathrm{d}\kappa_n(s).$$

Thus $\int \kappa_n(Q-w) \geq D\int w\kappa_n \geq D\mathcal{P}_n$, by Lemma 7.1 and $D \leq 0$. Subtracting $\frac{1}{2}\int w\kappa_n$ from the tangent (7.4) gives $\mathcal{P}_n \geq \mathcal{P}_{\tan} + \frac{1}{2}D\mathcal{P}_n - |u_0 - m|\Xi$, and we conclude by letting $n \to \infty$ and using (7.3). □

At an exactly stationary trial, $\mathcal{P}_{\tan} = \mathcal{P}_m(\kappa,\lambda)$, while for a rational trial, the difference is first order in the distance to the stationary one and is included in the ball evaluation.

The quantity D is an infimum over a continuum, which we will reduce to finitely many values of Γ_{id} in the next lemma. These come from (A.3) with no quadrature in w: for levels cut at $s_0 < \cdots < s_M = 1$, we have $\Gamma_{\mathrm{id}}(s_i) = \sum_{j\geq i}[b_j^2 - \int_{s_j}^{s_{j+1}}(1-Q) - (s_{j+1}^2 - s_j^2)/2]$.

Lemma 7.3. *Let $s_i < s_{i+1}$ be consecutive grid points and $t_i = 1 - s_i$. On $[s_i, s_{i+1}]$,*

$$\frac{\Gamma_{\mathrm{id}}(s)}{1-s} \geq \min\Big(\frac{\Gamma_{\mathrm{id}}(s_i)}{t_i}, \frac{\Gamma_{\mathrm{id}}(s_{i+1})}{t_{i+1}}\Big) - \frac{1}{2}(\sqrt{t_i} - \sqrt{t_{i+1}})^2.$$

Let ℓ_i be the right side, replaced by 0 if it is positive. Then $2\Gamma_{\mathrm{id}}(s)/(1-s^2) \geq 2\ell_i/(1+s_i)$ on the cell. Moreover:

(a) *if $Q(s_i) \geq s_{i+1}$, then $\Gamma_{\mathrm{id}} \geq \Gamma_{\mathrm{id}}(s_{i+1})$ on the cell, and if $Q(s_{i+1}) \leq s_i$, then $\Gamma_{\mathrm{id}} \geq \Gamma_{\mathrm{id}}(s_i)$ on the cell;*
(b) *on the last cell $[s_{M-1}, 1)$, $2\Gamma_{\mathrm{id}}(s)/(1-s^2) \geq Q(s_{M-1}) - 1$;*
(c) *below the first point, $\Gamma_{\mathrm{id}}(s) \geq \Gamma_{\mathrm{id}}(s_0) - (s_0 - u_0^2)_+^2/2$.*

Proof. As $\Gamma_{\mathrm{id}}(s) - s^2/2$ has derivative $-Q$, it is concave, and $\Gamma_{\mathrm{id}}(s) \geq \mathrm{chord}(s) - (s - s_i)(s_{i+1} - s)/2$. With $t = 1 - s$, the chord divided by t is monotone in t, and $(t_i - t)(t - t_{i+1})/t \leq (\sqrt{t_i} - \sqrt{t_{i+1}})^2$. Multiplying by $2/(1+s) \leq 2/(1+s_i)$ keeps a lower bound only for a nonpositive bound, which is why ℓ_i is capped at 0. For (a), the sign of $\Gamma'_{\mathrm{id}} = s - Q(s)$ is fixed on the cell; for (b), we have $\Gamma_{\mathrm{id}}(s) \geq Q(s_{M-1})(1-s) - (1-s^2)/2$; and for (c), we use $Q \geq u_0^2$. □

Uniform spacing in $\sqrt{1-s}$ makes the slack in Lemma 7.3 constant. We take the largest available lower bound on each cell and evaluate (7.5) in balls. In this way, (7.5) with Lemma 7.3 gives a certified lower bound on $P^*(m)$ at a rational m, which we denote by $P(m)$, and we call such a rational m an SK node.

7.3. **Upper and lower curves for L.** We fix a rational $m \in (0,1)$ and a rational step trial $(\kappa_0, \lambda_{\mathrm{t}})$ with values $0 < c_1 \leq \cdots \leq c_k$ on $[\hat{w}_j, \hat{w}_{j+1})$, $\hat{w}_j = 1 - t_j$, $t_1 > \cdots > t_k > t_{k+1} := 0$, and $\kappa_0 = 0$ below $\hat{w}_1$. Let $Q_0 = Q_{\kappa_0, \lambda_{\mathrm{t}}}$, $u_0 = \partial_x \Psi_{\kappa_0}(0, \lambda_{\mathrm{t}})$, and

$$\mathsf{W} = \Psi_{\kappa_0}(0, \lambda_{\mathrm{t}}) - \lambda_{\mathrm{t}} m, \qquad \mathsf{I} = \mathsf{W} - \frac{1}{2}\int_0^1 \kappa_0 Q_0.$$

By (7.1), for every $C' > 0$ and every α,

$$L(\alpha, m) \leq U(\alpha, m) := \frac{C'}{2} + \frac{\alpha}{2}\mathcal{I}(C') + \mathsf{W}, \tag{7.6}$$

where, with $\delta_j = \delta_{\kappa_0, C'}(\hat{w}_j)$ and $\delta_{k+1} = C'$,

$$\mathcal{I}(C') = \int_0^1 \frac{\mathrm{d}w}{\delta_{\kappa_0, C'}(w)} = \frac{1 - t_1}{\delta_1} + \sum_{j=1}^{k} \frac{1}{c_j} \log \frac{\delta_j}{\delta_{j+1}}.$$

Thus one Ising evaluation gives U for all α, affine in α at fixed C'.

For the lower curve, we take one tangent of the Ising part at the trial and one of the convex function $\delta \mapsto 1/\delta$ for the sphere part. The second is a function ϑ on $[0,1)$, admissible as long as a one-dimensional inequality holds, and no bound on the mass of the unknown minimizer is needed. The choice $\vartheta \equiv 1$ corresponds to the lower bound in (7.2).

Lemma 7.4. *Let $\vartheta > 0$ be measurable on $[0,1)$, $R_\vartheta(s) = \int_0^s \vartheta^2$, with $R_\vartheta(1) \leq 1$ and $\Gamma_\vartheta(s) := \int_s^1 (Q_0 - R_\vartheta) \geq 0$ on $[0,1)$. Then for every $\alpha > 0$,*

$$L(\alpha, m) \geq \sqrt{\alpha}\int_0^1 \vartheta + \mathsf{I} - |u_0 - m|\Xi(\alpha), \qquad \Xi(\alpha) := |\lambda_{\mathrm{t}}| + \frac{\sqrt{\alpha} + 1}{1 - |m|}. \tag{7.7}$$

Proof. We take the minimizer (κ, C, λ) of (7.1). By (7.4),

$$\Psi_\kappa(0, \lambda) - \lambda m \geq \mathsf{W} + (u_0 - m)(\lambda - \lambda_{\mathrm{t}}) + \frac{1}{2}\int (\kappa - \kappa_0) Q_0.$$

Also $|\lambda|(1 - |m|) \leq \Psi_\kappa(0, \lambda) - \lambda m \leq L \leq \sqrt{\alpha} + \mathsf{w}(m) \leq \sqrt{\alpha} + 1$, since $\Psi_\kappa(0, x) \geq |x|$, the sphere part is nonnegative, and by (7.2). Thus the second term is at least $-|u_0 - m|\Xi(\alpha)$. For the sphere part, we use $1/\delta \geq 2\theta - \delta\theta^2$ for any $\theta > 0$, and with $\theta = \vartheta/\sqrt{\alpha}$ this reads $(\alpha/2)/\delta \geq \sqrt{\alpha}\vartheta - \delta\vartheta^2/2$. By Fubini $\int_0^1 \delta\vartheta^2 = C R_\vartheta(1) + \int \kappa R_\vartheta$, so

$$\frac{C}{2} + \frac{\alpha}{2}\int \frac{\mathrm{d}w}{\delta} + \frac{1}{2}\int \kappa Q_0 \geq \sqrt{\alpha}\int \vartheta + \frac{C}{2}(1 - R_\vartheta(1)) + \frac{1}{2}\int \kappa (Q_0 - R_\vartheta).$$

Finally $\int \kappa(Q_0 - R_\vartheta) = \int_{[0,1)} \Gamma_\vartheta \, \mathrm{d}\kappa \geq 0$ by Fubini with the atom of $\mathrm{d}\kappa$ at 0; the integrals converge absolutely since $|Q_0 - R_\vartheta| \leq 1$ and $\kappa \in L^1$. Collecting the above bounds proves the lemma. □

The numbers $S_\vartheta := \int \vartheta$, I, W, u_0 do not depend on α, so we write

$$\mathrm{Lo}(\alpha, m) := \sqrt{\alpha} S_\vartheta + \mathsf{I} - |u_0 - m| \Xi(\alpha) = S' \sqrt{\alpha} + I', \tag{7.8}$$

with $S' = S_\vartheta - |u_0 - m|/(1-m)$ and $I' = \mathsf{I} - |u_0 - m|(|\lambda_{\mathrm{t}}| + 1/(1-m))$. The curve is affine in $\sqrt{\alpha}$. Below, ϑ is rational and piecewise constant, proposed by a floating-point solver of the sphere problem; any ϑ is allowed, since the next proposition rechecks it.

Let $s_0 < \cdots < s_{M-1} < 1$ be rational nodes with all $\hat{w}_j$ among them, and cut the levels at the nodes. We take ϑ constant on $[0, s_0]$, on each $[s_i, s_{i+1}]$ with value ϑ_i, and on $[s_{M-1}, 1)$. The recursion gives balls for $Q_i = Q_0(s_i)$, for the cell integrals of Q_0, and for W, I, u_0. Then $R_i = R_\vartheta(s_i)$ are exact rationals, and $\Gamma_i = \Gamma_\vartheta(s_i)$ are balls, through $\Gamma_i = \Gamma_{i+1} + \int_{s_i}^{s_{i+1}} Q_0 - (s_{i+1} - s_i)(R_i + R_{i+1})/2$.

Proposition 7.5. *Suppose the following hold.*

(i) *On each cell $[s_i, s_{i+1}]$, with $\mathsf{D}_i = s_{i+1} - s_i$, the largest of the applicable lower bounds below is nonnegative: Γ_{i+1} if $Q_i \ge R_{i+1}$; Γ_i if $Q_{i+1} \le R_i$; and $\min(\Gamma_i, \Gamma_{i+1}) - (\vartheta_i^2 - k_i)_+ \mathsf{D}_i^2/8$, where k_i is any lower bound of Q_0' on the cell.*

(ii) *$Q_{M-1} \ge R_\vartheta(1)$.*

(iii) *$u_0^2 \ge R_0$ and $\Gamma_0 \ge 0$.*

Then ϑ satisfies the hypotheses of Lemma 7.4.

Proof. As Γ_ϑ is C^1 with $\Gamma_\vartheta' = R_\vartheta - Q_0$, and Q_0, R_ϑ are nondecreasing, Γ_ϑ is nonincreasing on the cell if $Q_i \ge R_{i+1}$, and nondecreasing if $Q_{i+1} \le R_i$. Always $\Gamma_\vartheta'' = \vartheta_i^2 - Q_0' \le \bar{k} := (\vartheta_i^2 - k_i)_+$, so $x \mapsto \Gamma_\vartheta(x) + \bar{k}(x - s_i)(s_{i+1} - x)/2$ is concave. This gives (i). On $[s_{M-1}, 1)$, it follows from (ii) that $Q_0 \ge R_\vartheta$, so $\Gamma_\vartheta \ge \Gamma_\vartheta(1) = 0$; on $[0, s_0]$, $Q_0 \ge u_0^2 \ge R_0$, so Γ_ϑ is nonincreasing there. Finally $R_\vartheta(1) \le Q_{M-1} \le 1$. □

All comparisons in Proposition 7.5 are made in balls. Since Q_0 is steep near the top of a step path, a good lower bound k_i matters, and the next lemma provides one.

Lemma 7.6. *For a step path, let $\mathsf{p} = \partial_x^2 \Psi_{\kappa_0}$ and $\bar{\mathsf{p}}(w) = \sup_x \mathsf{p}(w, x) = \mathsf{p}(w, 0)$. Then $Q_0' = \mathbb{E}\mathsf{p}(w, X_w)^2$ is continuous across levels, and on each level $Q_0'' \ge -2\kappa_0 \bar{\mathsf{p}} Q_0'$. Hence, on consecutive cells $i-1, i$ with widths $\mathsf{D}_{i-1}, \mathsf{D}_i$, values c_{i-1}, c_i, and upper bounds $\bar{\mathsf{p}}_{i-1}, \bar{\mathsf{p}}_i$ of $\bar{\mathsf{p}}$, the number*

$$k_i := \frac{Q_i - Q_{i-1}}{\mathsf{D}_{i-1}} \exp\bigl(-2c_{i-1}\bar{\mathsf{p}}_{i-1}\mathsf{D}_{i-1} - 2c_i \bar{\mathsf{p}}_i \mathsf{D}_i\bigr)$$

is a lower bound of Q_0' on cell i. On a level with constant $c > 0$, with ℓ the distance from w to the top of the level and $\bar{\mathsf{p}}_+$ the bound at the top,

$$\bar{\mathsf{p}}(w) \le \frac{c + \sqrt{c^2 + 4/\ell}}{2}, \qquad \bar{\mathsf{p}}(w) \le \bar{\mathsf{p}}_+ + c\,\bar{\mathsf{p}}_+^2 \ell\, \mathsf{k}(c\sqrt{\ell})^2, \qquad \mathsf{k}(\eta) := \frac{\eta + \sqrt{\eta^2 + 4}}{2}.$$

Proof. From $-\partial_w \mathsf{p} = \frac{1}{2}\partial_x^2 \mathsf{p} + \kappa_0(\mathsf{p}^2 + u\partial_x \mathsf{p})$ and Itô, $\mathrm{d}\mathsf{p}_w = -\kappa_0 \mathsf{p}_w^2\,\mathrm{d}w + \partial_x \mathsf{p}(w, X_w)\,\mathrm{d}B_w$. Hence $Q_0'' = \mathbb{E}[(\partial_x \mathsf{p})^2 - 2\kappa_0 \mathsf{p}^3] \ge -2\kappa_0 \bar{\mathsf{p}} Q_0'$, because $0 \le \mathsf{p}(w, x) \le \mathsf{p}(w, 0)$ by Lemma 5.5(b). Thus $\log Q_0'$ has derivative at least $-2\kappa_0 \bar{\mathsf{p}}$, and the mean slope of Q_0 on cell $i-1$ is attained at some point of that cell.

For the first bound, let $f = \Psi_{\kappa_0}$ at the top of the level and $\mathsf{x}(x) = \frac{1}{c}\log \mathbb{E}\exp(cf(x + \sqrt{\ell} z))$. Completing the square gives $\mathsf{x}''(0) = (\mathbb{E}_{\mathrm{t}} z^2 - 1)/(c\ell)$, with $\mathbb{E}_{\mathrm{t}}$ the tilted law at $x = 0$. Since f is even and 1-Lipschitz, $\exp(\eta|z| - cf(\sqrt{\ell} z))$ is nondecreasing in $|z|$, with $\eta = c\sqrt{\ell}$. Hence $\mathbb{E}_{\mathrm{t}} z^2 \le \mathbb{E}_\eta z^2$ under the density proportional to $\exp(\eta|z| - z^2/2)$. Integration by parts gives $\mathbb{E}_\eta z^2 = \eta \mathbb{E}_\eta |z| + 1 \le \eta\sqrt{\mathbb{E}_\eta z^2} + 1$, so $\mathbb{E}_{\mathrm{t}} z^2 \le \mathsf{k}(\eta)^2$ and $\bar{\mathsf{p}} \le \mathsf{k}(\eta)/\sqrt{\ell}$. For the second, we use $\mathsf{x}'' = \mathbb{E}_{\mathrm{t}} f'' + c\,\mathrm{Var}_{\mathrm{t}}\, f'$ and $|f'(y)| \le \bar{\mathsf{p}}_+ |y|$. □

On a cell we take the smaller of the first bound at the lowest ℓ in the cell and the second at the largest, and we use $k = 0$ in the first cell and no k in the last.

7.4. **Strong concavity.** To bound e', we need a bound on L'. Concavity of L gives L' from a chord, with an error of the order of the square root of $U - \mathrm{Lo}$, which is too large on a mesh of reasonable size. The next lemma shows that L is strongly concave near overlap one, with a constant independent of α. The idea is that $h' = 1/\mathcal{M}'_{\kappa^*}$, up to the regularity needed, and that $\mathcal{M}'_{\kappa^*}$ is small at large fields because the optimal path is flat below a point w_1 close to 1.

Recall from Lemma 5.17(c), (d), in unit scale, that with $w_1 = w_1(m)$ as there, we have $w_1 \ge w_{\mathrm{fl}}(m)$, w_{fl} is nondecreasing in m, and $h(m) \ge \sqrt{w_1}\lambda_0(m)$. Thus a lower bound of $w_{\mathrm{fl}}(y)$ is a lower bound of $w_1(m')$ for all $m' \ge y$. Let $\mathsf{b}_u(T) = \frac{1}{2}[\mathsf{g}(u-T) + \mathsf{g}(u+T)]$ for $T \le u-1$, and $\mathsf{b}_u(T) = \frac{1}{2}[\mathsf{g}(0) + \mathsf{g}(u)]$ for $T > u-1$.

Lemma 7.7. *Let $y \in (0,1)$, $\underline{w} \in (0,1)$ with $\underline{w} \le w_1(m')$ for all $m' \ge y$, and $u \ge 1$ with $u \le h(y)$. Then for $y \le m_1 < m_2 < 1$,*

$$h(m_2) - h(m_1) \ge \frac{m_2 - m_1}{\mathsf{K}(u,\underline{w})}, \qquad \mathsf{K}(u,\underline{w}) := \frac{2}{\sqrt{\underline{w}}}\mathbb{E}\mathsf{b}_u\Big(\sqrt{\frac{1-\underline{w}}{\underline{w}}}|z|\Big).$$

So $L + \mu m^2/2$ is concave on $[y,1)$, with $\mu = 1/\mathsf{K}(u,\underline{w})$.

Proof. Let (κ_i, C_i, h_i) be the minimizers at m_i, $\Psi_i^0 = \Psi_{\kappa_i}(0,\cdot)$, $\mathcal{M}_i = \mathcal{M}_{\kappa_i}$. Testing (κ_1, C_1, h_2) at m_2 and (κ_2, C_2, h_1) at m_1 in (7.1) and adding gives $\Psi_1^0(h_1) + \Psi_2^0(h_2) \le \Psi_1^0(h_2) + \Psi_2^0(h_1)$, that is $\int_{h_1}^{h_2} \mathcal{M}_2 \le \int_{h_1}^{h_2} \mathcal{M}_1$. Let K bound $\mathcal{M}_i'$ on $[h_1, h_2]$. As $\mathcal{M}_1(h_1) = m_1$, $\mathcal{M}_2(h_2) = m_2$, we have $\mathcal{M}_1(x) \le m_1 + \mathsf{K}(x - h_1)$ and $\mathcal{M}_2(x) \ge m_2 - \mathsf{K}(h_2 - x)$, so $(m_2 - m_1)\mathsf{D} \le \mathsf{K}\mathsf{D}^2$ with $\mathsf{D} = h_2 - h_1$. Here $\mathsf{D} > 0$. Indeed h is nondecreasing, and if $h_1 = h_2 =: h$, the two tests are equalities, so both (κ_i, C_i) minimize $C/2 + (\alpha/2)\int \mathrm{d}w/\delta + \Psi_\kappa(0,h)$, which is strictly convex in δ. Then $\kappa_1 = \kappa_2$, and $m_1 = \mathcal{M}_1(h) = \mathcal{M}_2(h) = m_2$, a contradiction.

We now bound $\mathcal{M}_i'$. Let $\kappa = \kappa_i$, $w_1 = w_1(m_i)$, $\bar{r}_1^2 = (1 - w_1)/w_1$. By Lemma 5.17(e) in unit scale, with $\mathsf{f} = \partial_x \Psi_\kappa(w_1, \cdot)$ and the probability measure $\varsigma = \frac{1}{2}\,\mathsf{df}$,

$$\mathcal{M}_\kappa(x) = \int \Big(2\mathsf{G}\Big(\frac{x-t}{\sqrt{w_1}}\Big) - 1\Big)\varsigma(\,\mathrm{d}t), \qquad \mathcal{M}'_\kappa(x) = \frac{2}{\sqrt{w_1}}\int \frac{1}{2}\,[\mathsf{g}(x'-t') + \mathsf{g}(x'+t')]\varsigma(\,\mathrm{d}t),$$

where $x' = x/\sqrt{w_1}$, $t' = t/\sqrt{w_1}$, using the symmetry of ς. By Lemma 5.5(b), $\varsigma(|t| > T) = 1 - \mathsf{f}(T) \le \mathbb{P}(\sqrt{1-w_1}|z| > T)$, so $|t'|$ is stochastically below $\bar{r}_1|z|$. Also $x \ge h_1 \ge h(y) \ge u$ and $w_1 \le 1$, so $x' \ge u$. Let $\mathsf{v}(x',t') = \frac{1}{2}[\mathsf{g}(x'-t') + \mathsf{g}(x'+t')]$, $x' \ge u$, $t' \ge 0$. Since $x\mathsf{g}(x)$ decreases on $x \ge 1$, $\partial_{t'}\mathsf{v} \ge 0$ when $t' \le x' - 1$, and $\partial_{x'}\mathsf{v} \le 0$ when $x' \ge t'$. Thus for $T \le u - 1$, $\sup_{x' \ge u, t' \le T} \mathsf{v} = \mathsf{v}(u,T)$, and always $\mathsf{v} \le \frac{1}{2}[\mathsf{g}(0) + \mathsf{g}(u)]$, because $x' + t' \ge u$. The function b_u is nondecreasing in T, where at $T = u - 1$ we use $\mathsf{g}(1) + \mathsf{g}(2u-1) \le \mathsf{g}(0) + \mathsf{g}(u)$. It follows that $\mathcal{M}'_\kappa(x) \le (2/\sqrt{w_1})\mathbb{E}\mathsf{b}_u(\bar{r}_1|z|) \le \mathsf{K}(u,\underline{w})$, where in the last step we used that $w_1 \ge \underline{w}$ decreases both $2/\sqrt{w_1}$ and $\bar{r}_1$. □

Of the two sources for u, the universal one, $u = \sqrt{w_{\mathrm{fl}}(y)}\lambda_0(y)$, holds at any α. The other holds at a given α and comes from a chord: with μ_{u} the universal μ on $[y', 1)$,

$$h(y) \ge \frac{L(y') - L(y)}{y - y'} + \frac{\mu_{\mathrm{u}}(y - y')}{2}, \qquad y' < y. \tag{7.9}$$

This follows by integrating $h(t) \le h(y) - \mu_{\mathrm{u}}(y - t)$ over $[y', y]$.

In balls, we bound w_{fl} from below by lower Riemann sums of Q_{fl}, with λ_0 replaced by a rational below it within $2 \cdot 10^{-9}$ and the iterates rounded down to multiples of 10^{-7}, bound K

from above by a Riemann sum in $|z|$ on $|z| \le z_{\max}$, with $\sqrt{(1-\underline{w})/\underline{w}}\, z_{\max} \le u - 1$, and by the second branch of b_u on the tail, and we check $u \ge 1$.

Let $\tilde{\mu} \ge 0$ be a step function such that $h(t_2) - h(t_1) \ge \int_{t_1}^{t_2} \tilde{\mu}$ for $a \le t_1 < t_2 < 1$, and set $\bar{\mu}(a,b) = (b-a)^{-1} \int_a^b (t-a)\tilde{\mu}(t)\,\mathrm{d}t$. Integrating over $[a,b]$ gives, for $a < b \le m < 1$,

$$L'(m) \le \frac{L(b) - L(a)}{b-a} - \bar{\mu}(a,b) - \int_b^m \tilde{\mu}. \tag{7.10}$$

7.5. **The bracket on a window.** The main result of this subsection is the following.

Theorem 7.8. *Consider Gaussian rows.*

(a) *At $\alpha = \underline{\alpha}$, $e(0.9668708) > e(0.9666236)$. So $\underline{\alpha} \in \mathcal{W}_0(m_s)$ for every $m_s \le 0.9666236$.*
(b) *For every $\alpha \ge \overline{\alpha}$, $e' < 0$ on $[0.89, 1)$.*
(c) *For every $m_s \in [0.89, 0.9666236]$, $\underline{\alpha} \le \alpha_{\mathrm{c}}(0; m_s) \le \overline{\alpha}$.*

Part (c) follows from (a), (b) and the two forms of $\mathcal{W}_0(m_s)$. Part (a) is one evaluation at two overlaps, while (b) needs a mesh. A trial node is a rational y with a rational trial and a ϑ that passes Proposition 7.5; it provides $U(\alpha, y)$ and $\mathrm{Lo}(\alpha, y)$ for all α. An SK node provides, by (7.2), two bounds on L that differ by $\mathsf{w}(y) - P(y)$, independently of α. Trial nodes are tight and serve near the threshold, while SK nodes are cruder and serve elsewhere.

Proof of (a). We set $e_{\mathrm{up}} = m^2/2 + U^2/2$ and $e_{\mathrm{low}} = m^2/2 + \mathrm{Lo}^2/2$. If $\mathrm{Lo} > 0$, then $e \ge e_{\mathrm{low}}$ by Lemma 7.4, and $e \le e_{\mathrm{up}}$ by (7.6). At $\alpha = \underline{\alpha}$, with $m_1 = 0.9666236$, $m_2 = 0.9668708$, the computation gives

$$\begin{aligned} U(\underline{\alpha}, m_1) &= [0.4547205791577915 \pm 7.78 \cdot 10^{-17}], \\ e_{\mathrm{up}}(m_1) &= [0.5705659945932787 \pm 7.26 \cdot 10^{-17}], \\ \mathrm{Lo}(\underline{\alpha}, m_2) &= [0.4541947217816 \pm 3.86 \cdot 10^{-14}], \\ e_{\mathrm{low}}(m_2) &= [0.5705659945935 \pm 4.63 \cdot 10^{-14}], \end{aligned}$$

and $e_{\mathrm{low}}(m_2) - e_{\mathrm{up}}(m_1) = [2 \cdot 10^{-13} \pm 2.50 \cdot 10^{-14}] > 0$. □

Proof of (b). On a mesh cell, (7.10) bounds L' by a chord through two nodes, corrected by strong concavity, and the lower curve bounds L. Their product bounds $e' = m + LL'$ by a convex function of the position in the cell, so the endpoints suffice. Since the chord numerator is convex in α, the endpoints of a load interval suffice too, and SK nodes need no upper limit on α.

For the slopes, let $\tilde{\mu}$ be a step function on the mesh with value μ_i on $[y_i, y_{i+1})$, where $\mu_i = 1/\mathsf{K}(u, \underline{w})$ of Lemma 7.7, $\underline{w}$ is a rational lower bound of w_{fl} at the point of the lattice $\frac{1}{2000}\mathbb{Z}$ below y_i, and u is a rational lower bound of $h(y_i)$. For SK nodes u is the universal one, and for trial nodes on a load interval u is the larger of the universal one and the chord value (7.9) with $\mathrm{Lo}(\alpha, y')$ and $U(\alpha, y)$. Since the numerator $\mathrm{Lo}(\alpha, y') - U(\alpha, y)$ is concave in α, the endpoints of the interval give a lower bound.

Next, for trial nodes on a load interval $[\alpha_{\mathrm{l}}, \alpha_{\mathrm{h}}]$, we fix C' at each node for the interval. Then U is affine in α, and $\mathrm{Lo} = S'\sqrt{\alpha} + I'$ with $S' > 0$ is concave in α, so $U - \mathrm{Lo}$ is convex in α. For nodes $y_j < y_i$ put

$$\bar{\mathsf{s}} = \max_{\alpha \in \{\alpha_{\mathrm{l}}, \alpha_{\mathrm{h}}\}} \frac{U(\alpha, y_i) - \mathrm{Lo}(\alpha, y_j)}{y_i - y_j} - \bar{\mu}(y_j, y_i).$$

By (7.10), for α in the interval and $m = y_i + x$ with $0 \le x \le \nu := y_{i+1} - y_i$, we have $L' \le \bar{\mathsf{s}} - \mu_i x$. Also $L(\alpha, m) \ge L(\alpha_{\mathrm{l}}, m) \ge \ell(x)$ by $\partial_\alpha L > 0$ and concavity, where ℓ is the chord of $\mathrm{Lo}(\alpha_{\mathrm{l}}, \cdot)$ on

the cell. If $\bar{\mathsf{s}} < 0$, then

$$e'(m) \le q(x) := y_i + x + \ell(x)(\bar{\mathsf{s}} - \mu_i x).$$

When ℓ also has negative slope, $q'' = -2\mu_i \ell' \ge 0$, so q is convex and $q(0), q(\nu) < 0$ suffices. The inequalities $\bar{\mathsf{s}} < 0$, $\ell' < 0$, $q(0) < 0$, $q(\nu) < 0$ are checked in balls, at an anchor j chosen in floating point.

For SK nodes and $\alpha \ge \alpha_1$, we obtain from (7.2) that $L(\alpha, b) - L(\alpha, a) \le \mathsf{w}(b) - P(a)$, independent of α. Thus, for nodes $y_j < y_i$ and $\bar{\mathsf{s}} = (\mathsf{w}(y_i) - P(y_j))/(y_i - y_j) - \bar{\mu}(y_j, y_i)$, we have $L' \le \bar{\mathsf{s}} - \mu_i x$, and $L(\alpha, m) \ge \sqrt{\alpha_1} + P^*(m) \ge \ell(x) := \sqrt{\alpha_1}+$ chord of P on the cell, by concavity of P^*. The same q applies, with $\ell' < 0$ checked, and one check at α_1 covers all larger α. On the final cell $[m_{\text{top}}, 1)$, concavity with $L(\alpha, m_{\text{top}}) \ge \sqrt{\alpha}$ and $L(\alpha, 1) = \sqrt{\alpha}$ gives $L \ge \sqrt{\alpha}$, and $L' \le \bar{\mathsf{s}} < 0$, so $e' \le 1 + \sqrt{\alpha_1}\bar{\mathsf{s}}$.

The certificate uses 164 trial nodes on $[0.931, 0.985]$ and 48 SK nodes on $[0.88, 0.9975]$, and the load range $[\overline{\alpha}, 164868419/10^9]$ is split into 35 intervals with exactly matching ends. With these nodes and intervals, we find that all the checks above pass, which proves (b). Table 1 lists the largest upper end of the ball for q on each part, rounded to nearest. The worst cell at $\overline{\alpha}$ is $[0.9667225, 0.9667349]$, with anchor 0.9666856. □

TABLE 1. Coverage of $\{(\alpha, m) : \alpha \ge \overline{\alpha},\ m \in [0.89, 1)\}$ in the proof of Theorem 7.8(b), and the largest value of q found. The final cell $[0.9975, 1)$ has $q \le -0.1093$ at $\overline{\alpha}$.

nodes	load	overlap	worst q
trial	$[\overline{\alpha},\ 164868419/10^9]$	$[0.9322, 0.985]$	$-2.041 \cdot 10^{-8}$
SK	$[\overline{\alpha}, \infty)$	$[0.89, 0.9325]$	$-6.506 \cdot 10^{-3}$
SK	$[\overline{\alpha}, \infty)$	$[0.985, 1)$	$-1.784 \cdot 10^{-2}$
SK	$[164868419/10^9, \infty)$	$[0.89, 1)$	$-6.221 \cdot 10^{-2}$

7.6. **No holes below the lower end.**

Theorem 7.9. *Every $\alpha \in (0, \underline{\alpha}]$ lies in $\mathcal{W}_0(m_s) \cap \mathcal{W}_0^{\mathrm{R}}(m_s)$ for all $m_s \in (0, 0.9616]$. Explicitly, for Gaussian rows, with $\lambda_b = 2.0709518$ and $m_b = 2\mathsf{G}(\lambda_b) - 1 = [0.961636699353 \pm 1.20 \cdot 10^{-14}]$:*

(a) *$e(0.974) > e(m_b)$ for every $\alpha \in (0, 0.13755521]$;*
(b) *$e(0.9668835) > e(0.9666236)$ for every $\alpha \in [0.13755521, \underline{\alpha}]$.*

Proof. Both left overlaps are at least 0.9616, so (a) and (b) give the membership in $\mathcal{W}_0(m_s)$. For binary patterns, $\mathcal{W}_0^{\mathrm{R}}(m_s) = \mathcal{W}_0(m_s)$ by Theorem 6.9(c).

For (a), let $m^* = 0.974$, let P be a lower bound of $P^*(m^*)$, and write $w_b = 2\mathsf{g}(\lambda_b) = \mathsf{w}(m_b)$. By (7.2),

$$e(m^*) - e(m_b) \ge \frac{m^{*2} - m_b^2}{2} + \frac{(\sqrt{\alpha} + P)^2 - (\sqrt{\alpha} + w_b)^2}{2} = \frac{m^{*2} - m_b^2}{2} - \frac{w_b - P}{2}(2\sqrt{\alpha} + P + w_b).$$

For $w_b > P$ this is positive exactly when

$$\sqrt{\alpha} < \sqrt{\alpha_{\mathrm{D}}} := \frac{1}{2}\Big[\frac{m^{*2} - m_b^2}{w_b - P} - P - w_b\Big], \tag{7.11}$$

which is monotone in α, so no grid in α is needed. Theorem 7.2 with a 12-step trial, cut into 215 levels, gives $\mathcal{P}_m = [0.066934867211846 \pm 5.15 \cdot 10^{-16}]$, $\mathcal{P}_{\tan} = [0.066934867141065 \pm 4.66 \cdot 10^{-16}]$, $D = -1.634 \cdot 10^{-7}$, $|u_0 - m^*| \le 2.6 \cdot 10^{-10}$, and

$$P^*(0.974) \ge P = [0.06693486043888495 \pm 1.53 \cdot 10^{-18}], \quad \text{so } P^*(0.974) \ge 0.066934860438.$$

With $w_b = [0.09346090751068000 \pm 2.78 \cdot 10^{-18}]$, the right side of (7.11) gives

$$\alpha_{\mathrm{D}} \in [0.1375552194487222 \pm 9.37 \cdot 10^{-18}],$$

which exceeds 0.13755521. This proves (a).

For (b), we make the argument of Theorem 7.8(a) uniform in α. With trial nodes $m_1 = 0.9666236$, $m_2 = 0.9668835$ and $C' = 167090722627/549755813888$, and $x = \sqrt{\alpha}$, we have $U(\alpha, m_1) = \mathsf{c}_0 + \mathsf{c}_1 x^2$, where $\mathsf{c}_0 = C'/2 + \mathsf{W}(m_1)$ and $\mathsf{c}_1 = \mathcal{I}(C')/2$, and $\mathrm{Lo}(\alpha, m_2) = S'x + I'$, with $S' > 0$ and $I' > 0$ in balls, so $\mathrm{Lo} > 0$ and

$$f(x) := e_{\mathrm{low}}(m_2) - e_{\mathrm{up}}(m_1) = \frac{m_2^2 - m_1^2}{2} + \frac{1}{2}\big[(S'x + I')^2 - (\mathsf{c}_0 + \mathsf{c}_1 x^2)^2\big]$$

has $f''(x) = S'^2 - 2\mathsf{c}_1(\mathsf{c}_0 + \mathsf{c}_1 x^2) - 4\mathsf{c}_1^2 x^2$, which decreases in $x > 0$. In balls, we find that $f \ge 2.103 \cdot 10^{-8}$ at the lower end of $[0.13755521, \underline{\alpha}]$, $f \ge 9.521 \cdot 10^{-14}$ at the upper end, and $f'' \le -1.216$ at the lower end, hence throughout. Thus f is concave in x and positive on the interval, and this completes the proof. □

7.7. **No well below the window.** Theorem 7.8(b) says nothing below 0.89, where the constants of Lemma 7.7 fail, because they need $h \ge 1$ and a large $w_{\mathfrak{fl}}$. A cruder fact works at every overlap, namely that the response of the optimal path has slope less than one, by an explicit margin, at its own field. Thus h grows at least like $\int_0^m (1 + s^2/\mathsf{w}(s)^2)^{1/2}\,\mathrm{d}s$, which, combined with $L \ge \sqrt{\alpha} + P^*(m)$, makes $e' = m - Lh$ negative, down to $m = 0$.

We fix $\alpha > 0$ and $m \in (0,1)$ and use the notation of Lemma 5.17 in unit scale: $M^* = \delta^*(0)/\sqrt{\alpha}$, w_1 as there, and $A_0(m) = \mathcal{M}'_{\kappa^*}(h(m))$.

Lemma 7.10. *For all $\alpha > 0$, $m \in (0,1)$,*

$$A_0(m)^2 \le \frac{1}{M^{*2}} - \frac{m^2}{w_1}, \qquad A_0(m) \le \frac{\mathsf{w}(m)}{\sqrt{w_1}}, \qquad \textit{hence} \quad A_0(m) \le \frac{\mathsf{w}(m)}{\sqrt{m^2 + \mathsf{w}(m)^2}} < 1. \tag{7.12}$$

Proof. On $[0, w_1)$ the path vanishes, so u_w and $\mathsf{p}_w = \partial_x^2 \Psi_{\kappa^*}(w, X_w)$ are martingales for the heat flow, with $Q' = \mathbb{E}\mathsf{p}_w^2$ and $\mathbb{E}\mathsf{p}_w = A_0$. By Cauchy–Schwarz $Q' \ge A_0^2$, so $Q(w_1) \ge m^2 + A_0^2 w_1$. By Lemma 5.17(b), $Q(w_1) = w_1/M^{*2}$, which gives the first inequality, and the second is in (5.19).

For the third, $M^* \ge 1$ by Lemma 5.17(a), so $A_0^2 \le 1 - m^2/w_1$ and $A_0^2 \le \mathsf{w}^2/w_1$. The first bound increases in w_1, the second decreases, and they cross at $w_1 = m^2 + \mathsf{w}^2$, where both equal $\mathsf{w}^2/(m^2 + \mathsf{w}^2)$. □

Lemma 7.11. *h is nondecreasing, $h(0) = 0$, and for $0 \le a < b < 1$,*

$$h(b) - h(a) \ge \int_a^b \mu_0(s)\,\mathrm{d}s, \qquad \mu_0(s) := \sqrt{1 + \frac{s^2}{\mathsf{w}(s)^2}}. \tag{7.13}$$

So $h(m) \ge \mathsf{H}(m) := \int_0^m \mu_0 > m$ on $(0,1)$. H is convex with $\mathsf{H}(0) = 0$. Neither μ_0 nor H depends on α.

Proof. By Theorem 5.13(b), $h = -L'$ is continuous and nondecreasing, and $h(0) = 0$. Thus h is differentiable at almost every m_0, and at such a point L has a second-order Taylor expansion. We fix such $m_0 \in (0,1)$, with minimizer (κ_0, C_0). For all m,

$$L(\alpha, m) \le \mathsf{l}(m) := \frac{C_0}{2} + \frac{\alpha}{2}\int \frac{\mathrm{d}w}{\delta_{\kappa_0, C_0}} + \inf_\lambda\{\Psi_{\kappa_0}(0,\lambda) - \lambda m\},$$

with equality at m_0. Below w_1 the flow is the heat flow, so $\mathcal{M}_{\kappa_0}$ is real analytic, and $\mathcal{M}'_{\kappa_0} > 0$, since $\mathcal{M}_{\kappa_0}$ is concave and strictly increasing on $[0,\infty)$ (Lemma 5.5(b),(c)). It follows that l is C^2 near m_0, with $\mathsf{l}' = -\mathcal{M}_{\kappa_0}^{-1}$ and $\mathsf{l}''(m_0) = -1/A_0(m_0)$. Since $\mathsf{l} - L \ge 0$ has a minimum

at m_0, we have $L''(m_0) \le \mathsf{l}''(m_0)$, that is $h'(m_0) \ge 1/A_0(m_0) \ge \mu_0(m_0)$ by (7.12). For a nondecreasing function $h(b)-h(a) \ge \int_a^b h'$. Finally μ_0 is continuous and nondecreasing, because $(s/\mathsf{w}(s))' = (\mathsf{w}+s\lambda_0)/\mathsf{w}^2 > 0$. □

Corollary 7.12. *For $\alpha \ge 1$, $e' < 0$ on $(0,1)$. So $\mathcal{W}_0 \subset (0,1)$ and $\alpha_{\mathrm{c}}(0) \le 1$.*

Proof. Since $L \ge \sqrt{\alpha}$, we have $e' = m - Lh \le m - \sqrt{\alpha}\mathsf{H}(m) < 0$. □

Theorem 7.13. *For every $\alpha \ge \overline{\alpha}$ and every $m \in (0,1)$, $e'(m) < 0$. Consequently:*

(a) *$\mathcal{W}_0 \subset (0, \overline{\alpha})$, $\underline{\alpha} \le \alpha_{\mathrm{c}}(0; m_s) \le \overline{\alpha}$ for all $m_s \in (0, 0.9666236]$, and $\underline{\alpha} \le \alpha_{\mathrm{c}}(0) \le \overline{\alpha}$.*
(b) *The same holds for binary patterns, with $\mathcal{W}_0^{\mathrm{R}}$ and $\alpha_{\mathrm{c}}^{\mathrm{R}}$. For $\alpha \ge \overline{\alpha}$, $e^{\mathrm{R}} = e$ on $[0,1]$, and e^{R} is strictly decreasing on $(0,1)$.*
(c) *For $\alpha \ge \overline{\alpha}$ the slice-minimum profile has no barrier of extensive depth at any overlap, in the sense of Theorem 5.16*(b).

Proof. On $[0.89, 1)$ this is Theorem 7.8(b), and it remains to treat smaller overlaps. Recall that $P(m)$ denotes the certified lower bound of $P^*(m)$ at SK nodes. We will use three facts, which hold for all $\alpha \ge \alpha_{\mathrm{l}} := \overline{\alpha}$:

(i) $L(\alpha, m) \ge \sqrt{\alpha_{\mathrm{l}}} + P^*(m)$, and P^* is nonincreasing on $[0,1]$ and at least the chord of P on a cell;
(ii) $L(\alpha, b) - L(\alpha, a) \le \mathsf{w}(b) - P^*(a)$ for $a < b$, independent of α, by (7.2);
(iii) (7.13), independent of α.

Part A, $m \in (0, 0.65]$. On a cell $[a,b]$,

$$\frac{e'(m)}{m} = 1 - \frac{Lh}{m} \le 1 - (\sqrt{\alpha_{\mathrm{l}}} + P(b))\frac{\mathsf{H}(a)}{a},$$

because $\mathsf{H}(m)/m$ is nondecreasing. On the first cell, with $a = 0$, we use $\mathsf{H}(m) \ge m$. It is enough that the right side is negative in balls.

Part B, $m \in [0.65, 0.89]$. For nodes $y_j < y_i$, integrating (7.13) gives $L(y_j) - L(y_i) = \int_{y_j}^{y_i} h \le (y_i - y_j)h(y_i) - \int_{y_j}^{y_i}(t - y_j)\mu_0(t)\,\mathrm{d}t$, so $-h(y_i) \le \mathsf{s}_{ij}$, where

$$\mathsf{s}_{ij} := \frac{\mathsf{w}(y_i) - P(y_j)}{y_i - y_j} - \frac{1}{y_i - y_j}\int_{y_j}^{y_i}(t - y_j)\mu_0(t)\,\mathrm{d}t.$$

For $m = y_i + x$, $x \in [0, \nu]$, we have $L' \le \mathsf{s}_{ij} - \mu_0(y_i)x$ and $L \ge \ell(x) := \sqrt{\alpha_{\mathrm{l}}} +$ chord of P. If $\mathsf{s}_{ij} < 0$, then $e' \le q(x) := y_i + x + \ell(x)(\mathsf{s}_{ij} - \mu_0(y_i)x)$, which is convex when $\ell' \le 0$, so $q(0), q(\nu) < 0$ suffices. These are checked in balls, with the anchor j chosen in floating point, μ_0 bounded below on the lattice $\frac{1}{800}\mathbb{Z}$ by its value at the left end, and w from above.

The certificate uses 43 SK nodes in $[0.05, 0.9]$, in 160-bit balls, and Part A holds on 25 cells, up to 0.65. On $[0.65, 0.675]$ its expression is positive, and Part B applies there. Part B holds on 14 cells, with $\mathsf{s}_{ij} < 0$ and P decreasing in every cell, and $\max q < -0.0284228$ over all cells, which proves the first statement. Table 2 shows samples.

For (a), the first statement gives $\mathcal{W}_0 \subset (0, \overline{\alpha})$, $\alpha_{\mathrm{c}}(0) \le \overline{\alpha}$, and $\alpha_{\mathrm{c}}(0; m_s) \le \overline{\alpha}$ for all $m_s \in (0,1)$, and the lower end is Theorem 7.8(a). Part (b) follows from Theorem 6.9(a), (c), and $e' < 0$, and (c) is Theorem 5.16(b). □

7.8. **The limit $T \to 0$.** Theorem 5.16(d) compares $\alpha_{\mathrm{c}}(T; m_s)$ with the capacity at $T = 0$ on a window. For the unrestricted $\alpha_{\mathrm{c}}(T)$, we must also exclude a contact point that sinks to $m = 0$ as $T \to 0$, where $e'(0) = 0$ and the convergence of Λ'/β to e' says nothing. We will do this with Proposition 4.14.

TABLE 2. Samples from the proof of Theorem 7.13. Decimals are midpoints of balls, and acceptance uses ball comparisons only. Left: the bound $1-(\sqrt{\overline{\alpha}}+P(b))\mathsf{H}(a)/a$ in Part A. Right: Part B, with y_i, anchor y_j, s_{ij}, $q(0)$ and $q(\nu)$.

$[a,b]$	bound
$[0,0.05]$	-0.13310
$[0.25,0.275]$	-0.11322
$[0.5,0.525]$	-0.05332
$[0.6,0.625]$	-0.01467
$[0.625,0.65]$	-0.00336

y_i	y_j	s_{ij}	$q(0)$	$q(\nu)$
0.65	0.425	-0.8132	-0.06530	-0.05614
0.75	0.625	-1.0581	-0.07494	-0.05853
0.85	0.775	-1.3853	-0.05549	-0.02842
0.875	0.825	-1.4902	-0.04433	-0.03919
0.8875	0.85	-1.5476	-0.03736	-0.03475

Theorem 7.14. *For T small enough, $\alpha_{\mathrm{c}}(T)\le\overline{\alpha}$. Also $\liminf_{T\to0}\alpha_{\mathrm{c}}(T)\ge\underline{\alpha}$. So every limit point of $\alpha_{\mathrm{c}}(T)$ as $T\to0$ lies in $[\underline{\alpha},\overline{\alpha}]$, and by Theorem 6.7 the same holds for $\alpha_{\mathrm{c}}^{\mathrm{R}}(T)$. The same holds for $\alpha_{\mathrm{c}}(T;m_s)$ and $\alpha_{\mathrm{c}}^{\mathrm{R}}(T;m_s)$, for every $m_s\in(0,0.9666236]$.*

Proof. The lower bound is Theorem 5.16(d) with Theorem 7.8(c), since $\mathcal{W}_T$ contains $\mathcal{W}_T(m_s)$.

For the upper bound, we may take $\alpha\in[\overline{\alpha},2]$ by Corollary 4.15. By Theorem 7.13 and continuity of e' in (α,m), we have $e'<0$ uniformly on $[\overline{\alpha},2]\times[0.4,\bar{m}]$, for each $\bar{m}<1$. Theorem 5.16(d) gives $\Lambda'/\beta\to e'$ uniformly there, and $\Lambda'<0$ on $[\bar{m},1)$ for some $\bar{m}<1$ and β large. Thus $\Lambda'<0$ on $[0.4,1)$ for β large, uniformly on the load interval.

On $(0,0.4]$, let $v^*(m)$ be the optimal radius and $\gamma^*=\beta v^*(m)$. By Theorem 5.16(d), $\alpha v^*/\beta\to L(m)^2$ uniformly, and $L(m)\ge L(0.4)\ge\sqrt{\overline{\alpha}}+P^*(0.4)>1.04$, by $P^*(0.4)\ge0.6747288067$ at the SK node 0.4. The same limit and $L\ge\sqrt{\alpha}$ give $\gamma^*=\beta^2v^*/\beta\ge\beta^2/2$ for β large. Hence $\hat{y}_{\mathrm{low}}(\gamma^*,\alpha)\ge\hat{y}_{\mathrm{low}}(\beta^2/2,\overline{\alpha})$ by Lemma 4.13(c), and this tends to 1. Indeed, since $\mathsf{f}(s)\ge c\min(s,1)$ for some $c>0$, for each $w_0<1$ the inequality defining $\hat{y}_{\mathrm{low}}$ fails on $(0,w_0]$ once γ is large. Therefore $\alpha v^*\hat{y}_{\mathrm{low}}(\gamma^*,\alpha)>\beta$ on $(0,0.4]$ for β large, and Proposition 4.14 gives $\Lambda'<0$ there, so no $\alpha\ge\overline{\alpha}$ lies in $\mathcal{W}_T$ for T small. This proves the upper bound.

For a window, $\alpha_{\mathrm{c}}(T;m_s)\le\alpha_{\mathrm{c}}(T)$ gives the upper side, and Theorem 5.16(d) with Theorem 7.13(a) gives $\liminf_{T\to0}\alpha_{\mathrm{c}}(T;m_s)\ge\alpha_{\mathrm{c}}(0;m_s)\ge\underline{\alpha}$. Finally, Theorem 6.7 covers binary patterns. □

Proof of Theorem 1.2. Part (a) is Corollary 3.5, Theorem 5.1, Corollary 5.12 and Theorem 5.13(c); (b) is Theorem 6.9(c) with Theorem 5.13(c) and Definition 5.15; (c) is Theorem 7.13(a) with Theorem 6.9(c); (d) is Theorem 7.13(b) and Theorem 7.9; (e) is Theorem 5.16(b); and (f) is Theorem 7.14. □

Proof of Theorem 1.4. Part (a) is Theorem 6.7(a) with Theorem 1.3(a), (b) is Theorem 1.3(b), and (c) and (d) are Theorems 7.13 and 7.9. □

USE OF AI

The statements, proofs, code, Lean formalization and text of this paper were generated mainly by AI systems: Claude Opus 5.5 (Anthropic) derived the proofs and wrote the rest, and GPT-6.1 Sol (OpenAI) wrote an independent report on each result. A final audit by fresh AI readers covered the longer version in the repository, and the certified computations were rebuilt from a clean start.

Code and data

The code, the certificates of Section 7, the longer version and the dependencies between the results are at `https://github.com/OctopusTide/hopfield-capacity`. The repository also contains a Lean 4 proof of Theorem 1.4(a),(b), Theorem 1.1(a),(b), and the statements in Theorem 1.3(b) that $\mathcal{W}_T$ is empty for $T \geq 1$ and that $\mathcal{W}_T \subset (0, 2)$. This proof is conditional on the results used from [6, 9, 1], two classical Gaussian facts, and 802 load inequalities behind Corollary 4.15, computed outside Lean in ball arithmetic.

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Appendix A. The certificate

In this appendix, we use the notation of Section 7. We give the error terms of the ball evaluation of the recursion, on which the code is based, and describe what was computed.

A.1. **The recursion in ball arithmetic.** For a step path with rational data, the evaluation produces balls containing $\Psi_\kappa(0,\lambda)$, $\partial_x\Psi_\kappa(0,\lambda)$, the values of Q at the level boundaries, and the integrals of $1-Q$ over the levels. Since the recursion is a chain of Gaussian convolutions of analytic integrands, the trapezoid rule on a lattice converges exponentially fast, and analytic continuation into a strip bounds its error. We now give the error terms.

In this subsection, M denotes the number of levels, not of patterns. Fix rationals $0 < s_0 < \cdots < s_M = 1$, $0 \leq c_0 \leq \cdots \leq c_{M-1}$ with $c_{M-1} > 0$, and λ, and let $\kappa = c_i$ on $[s_i, s_{i+1})$ and $\kappa = 0$ on $[0, s_0)$. We write $b_i^2 = s_{i+1} - s_i$, $\tau_i^2 = 1 - s_i$, $\mathsf{d}_i = \frac{1}{2}\int_{s_i}^1 \kappa$ and $\Psi_i = \Psi_\kappa(s_i, \cdot)$, and we let φ_b denote the density of $N(0, b^2)$. Then $\Psi_M = |\cdot|$ and $\Psi_\kappa(0,\lambda) = \mathbb{E}\Psi_0(\lambda + \sqrt{s_0}z)$. By Lemma 5.2(b), a level may be cut into several with the same c without changing anything, so the grid of s_i can be refined freely.

For $c_i > 0$, we put $\mathsf{A}_i = \exp(c_i\Psi_{i+1})$ and $\mathsf{B}_i = \exp(c_i\Psi_i)$, so that $\mathsf{B}_i(x) = \mathbb{E}\mathsf{A}_i(x + b_i z)$ and $\mathsf{A}_i = \mathsf{B}_{i+1}^{\theta_i}$ with $\theta_i = c_i/c_{i+1} \in [0, 1]$. The law of $X_{s_{i+1}}$ given $X_{s_i} = x$ is that of $x + W$, where W has density proportional to $\varphi_{b_i}(y)\mathsf{A}_i(x + y)$, and we write $\mathbb{E}_x^i$ for this tilted law. For $c_i = 0$, the tilted law is the Gaussian law, with $\mathsf{A}_i = 1$. Put $\mathsf{E}_i(x) = \mathbb{E}_x^i\Psi_{i+1}(x + W)$.

Lemma A.1. (a) *Ψ_i is even, convex and 1-Lipschitz.*

(b) *$\Psi_i(x) \geq |x| + \mathsf{d}_i$, and $\psi_i(x) := \Psi_i(x) - |x| - \mathsf{d}_i$ is nonnegative and nonincreasing on $x \geq 0$.*

(c) *$\mathsf{B}_i \geq \mathsf{A}_i$.*

(d) *$\Psi_i(x) \leq |x| + \bar{X}_i$, where $\bar{X}_i := \sqrt{2\tau_i^2/\pi} + \mathsf{d}_i$.*

(e) *For $x \geq 0$, $1 - \Psi_i'(x) \leq 2\mathsf{G}(-x/\tau_i) \leq \exp(-x^2/(2\tau_i^2))$. So $1 - \Psi_i'(x)^2 \leq 2\exp(-x^2/(2\tau_i^2))$ for all x.*

Proof. Parts (a), (b), (d) are Lemma 5.3(i), (iv), with $\Psi_i' \leq 1$ for the monotonicity in (b). For (c), convexity gives $\Psi_{i+1}(x + y) \geq \Psi_{i+1}(x) + \Psi_{i+1}'(x)y$, and $\mathbb{E}\exp(tz) \geq 1$. Part (e) is Lemma 5.5(b), and for the last claim we use that Ψ_i' is odd and $1 - \Psi_i'^2 \leq 2(1 - |\Psi_i'|)$. □

Let $\mathsf{k}(\eta) = (\eta + \sqrt{\eta^2 + 4})/2$, so that $\mathsf{k}^2 = \eta\mathsf{k} + 1$.

Lemma A.2. *Let $f > 0$ on $\mathbb{R}$ with $\log f$ being c-Lipschitz, let $b > 0$, $\mathcal{H}(x) = \mathbb{E} f(x + bz)$, and let $\mathbb{E}_{\mathrm{t}}$ be the tilted law at x, under which W has density proportional to $\varphi_b(y) f(x + y)$. For $\zeta = x + \mathrm{i}y$:*

(i) $\mathcal{H}(\zeta) = \exp(y^2/(2b^2))\mathcal{H}(x)\mathbb{E}_{\mathrm{t}} \exp(\mathrm{i}yW/b^2)$, *so* $|\mathcal{H}(\zeta)| \le \exp(y^2/(2b^2))\mathcal{H}(x)$.
(ii) $\mathbb{E}_{\mathrm{t}} W^2 \le \mathsf{k}(cb)^2 b^2$.
(iii) $\operatorname{Re} \mathbb{E}_{\mathrm{t}} \exp(\mathrm{i}yW/b^2) \ge \varpi(y) := 1 - y^2 \mathsf{k}(cb)^2/(2b^2)$, *which is positive for* $|y| < \sqrt{2}b/\mathsf{k}(cb)$. *On that strip* $\log \mathcal{H}$ *is analytic,* $|\operatorname{Im} \log(\mathcal{H}(\zeta) \exp(-y^2/(2b^2))/\mathcal{H}(x))| < \pi/2$, *and* $0 \le -\log |\mathbb{E}_{\mathrm{t}} \exp(\mathrm{i}yW/b^2)| \le -\log \varpi(y)$.

Proof. For (i), we complete the square in $\varphi_b(y' - \zeta)$. For (ii), Gaussian integration by parts gives $\mathbb{E}_{\mathrm{t}} W^2 = b^2 + b^2 \mathbb{E}_{\mathrm{t}}[W(\log f)'(x + W)] \le b^2 + b^2 c \sqrt{\mathbb{E}_{\mathrm{t}} W^2}$, and we solve the quadratic. Part (iii) follows from $\cos t \ge 1 - t^2/2$ and (ii). □

We apply this with $f = \mathsf{A}_{i+1}$, $b = b_{i+1}$, $c = c_{i+1} > 0$, since $\log \mathsf{A}_{i+1}$ is c_{i+1}-Lipschitz by Lemma A.1(a), also at the top level. Fix a strip half-width $a_i < \sqrt{2} b_{i+1}/\mathsf{k}_{i+1}$, with $\mathsf{k}_{i+1} = \mathsf{k}(c_{i+1} b_{i+1})$, and write $\varpi_i = \varpi(a_i)$. For $\zeta = x + \mathrm{i}y$, $|y| \le a_i$, and $\theta \in [0, 1]$, we have

$$
\begin{aligned}
&|\mathsf{B}_{i+1}(\zeta)^\theta| \le \exp(\theta a_i^2/(2b_{i+1}^2)) \mathsf{B}_{i+1}(x)^\theta, \\
&|\Psi_{i+1}(\zeta)| \le \Psi_{i+1}(x) + K_i, \qquad && K_i := \big(a_i^2/(2b_{i+1}^2) + \pi/2 - \log \varpi_i\big)/c_{i+1}, \\
&|\Psi'_{i+1}(\zeta)| \le \bar{\mathsf{B}}_i, && \bar{\mathsf{B}}_i := (\mathsf{k}_{i+1} b_{i+1} + a_i)/(c_{i+1} b_{i+1}^2 \varpi_i).
\end{aligned} \tag{A.1}
$$

The first line is Lemma A.2(i), and the second is (iii), with $\Psi_{i+1} > 0$ on $\mathbb{R}$. The third follows from $\mathcal{H}'(\zeta) = \int ((y' - \zeta)/b^2) \varphi_b(y' - \zeta) f(y') \, \mathrm{d}y'$ and $\mathbb{E}_{\mathrm{t}}|W| \le \mathsf{k}_{i+1} b_{i+1}$. Let $H = N/\mathsf{B}_{i+1}$ with $N(\zeta) = \int \varphi_{b_{i+1}}(y' - \zeta) \mathsf{A}_{i+1}(y') \phi(y') \, \mathrm{d}y'$ and ϕ real, $|\phi| \le S$ on $\mathbb{R}$. Then

$$
|H(\zeta)| \le S/\varpi_i, \qquad |\mathsf{A}_i(\zeta) H(\zeta)| \le \exp(\theta_i a_i^2/(2b_{i+1}^2)) \mathsf{A}_i(x) S/\varpi_i. \tag{A.2}
$$

If $c_{i+1} = 0$, then Ψ_{i+1} is entire, and with $\mathsf{f}_i := \exp(a_i^2/(2b_{i+1}^2))$ we have $|\Psi_{i+1}(\zeta)| \le \mathsf{f}_i \Psi_{i+1}(x)$, $|\Psi'_{i+1}(\zeta)| \le \mathsf{f}_i$ and $|H(\zeta)| \le \mathsf{f}_i S$. In that case we put $\varpi_i = 1$, $K_i = 0$, $\bar{\mathsf{B}}_i = \mathsf{f}_i$, and take $a_i < \sqrt{2} b_{i+1}$.

Lemma A.3. *If f is analytic on $|\operatorname{Im} y| < a'$, $a < a'$, $\int |f(y + \mathrm{i}y')| \, \mathrm{d}y \le \mathsf{F}$ for $|y'| \le a$, and f decays as $|\operatorname{Re} y| \to \infty$ uniformly in the strip, then*

$$
\Big| \mathsf{h} \sum_j f(j\mathsf{h}) - \int f \Big| \le \frac{2\mathsf{F}}{\exp(2\pi a/\mathsf{h}) - 1}.
$$

For $J\mathsf{h} > cb^2$, the terms $t_j := \mathsf{h} \varphi_b(j\mathsf{h}) \exp(cj\mathsf{h})$ satisfy $t_{j+1}/t_j \le \mathsf{q} := \exp(-J\mathsf{h}^2/b^2 + c\mathsf{h}) < 1$ for $j \ge J$, so $\sum_{|j|>J} t_{|j|} \le 2 t_J \mathsf{q}/(1 - \mathsf{q}) =: \mathsf{t}(c)$.

Proof. The first bound is Poisson summation with the Fourier transform of f bounded by $\mathsf{F} \exp(-a|\omega|)$ after moving the contour, and the geometric series over $\omega = 2\pi n/\mathsf{h}$; see [24, Theorem 5.1]. The second is a geometric series. □

The tables use nested lattices: h_i is an integer multiple of h_{i+1}, so no interpolation occurs. The table of Ψ_i holds $x \in \mathsf{h}_i \mathbb{Z}$, $0 \le x \le Y_i$, with $Y_i \ge Z\tau_i$, $Z = 10$. For the quadrature at level i, let $\mathsf{t}_i(c)$ be $\mathsf{t}(c)$ of Lemma A.3 with $b = b_i$, $\mathsf{h} = \mathsf{h}_{i+1}$, $J = J_i$, and put

$$
E_i := \frac{2 \exp(a_i^2/(2b_i^2))}{\exp(2\pi a_i/\mathsf{h}_{i+1}) - 1}, \qquad E'_i := E_i \exp\Big(\frac{\theta_i a_i^2}{2b_{i+1}^2}\Big), \qquad \mathsf{r}_i := E'_i + \mathsf{t}_i(c_i).
$$

Let $\mathsf{T}_i, \mathsf{N}_i, \mathsf{S}_i$ be the sums $\sum_{|j| \le J_i} \mathsf{h}_{i+1} \varphi_{b_i}(j\mathsf{h}_{i+1}) \mathsf{A}_i(y_j) \phi(y_j)$ with $y_j = x + j\mathsf{h}_{i+1}$, for $\phi = 1$, Ψ'_{i+1}, Ψ_{i+1} respectively.

Proposition A.4. *Let $x \in \mathsf{h}_i\mathbb{Z}$.*

(a) *If $c_i > 0$, then $|\mathsf{T}_i - \mathsf{B}_i| \le \mathsf{r}_i\mathsf{B}_i$, so $c_i\Psi_i(x)$ lies in $\log \mathsf{T}_i \pm (-\log(1-\mathsf{r}_i))$. Further*

$$|\mathsf{N}_i - \mathsf{B}_i\Psi_i'| \le (E_i'\bar{\mathsf{B}}_i + \mathsf{t}_i(c_i))\mathsf{B}_i,$$
$$|\mathsf{S}_i - \mathsf{B}_i\mathsf{E}_i| \le (E_i'(|x| + \mathsf{k}_i b_i + \bar{X}_{i+1} + K_i) + (|x| + \bar{X}_{i+1} + 1)\mathsf{t}_i(c_i+1))\mathsf{B}_i.$$

A ratio of a sum N with error $\mathsf{r}_\mathrm{n}\mathsf{B}_i$ to the denominator T_i has error at most $(\mathsf{r}_\mathrm{n} + \mathsf{r}_i|\mathsf{N}/\mathsf{T}_i|)/(1-\mathsf{r}_i)$.

(b) *If $c_i = 0$, put $\mathsf{C} = |x| + \bar{X}_{i+1}$. The trapezoid sums of φ_{b_i} times $\Psi_{i+1}, \Psi_{i+1}^2, \Psi_{i+1}'$ have errors at most*

$$E_i\mathsf{C}_1 + (\mathsf{C}+1)\mathsf{t}_i(1), \qquad E_i\mathsf{C}_2 + (\mathsf{C}+1)^2\mathsf{t}_i(2), \qquad E_i\mathsf{C}_3 + \mathsf{t}_i(0),$$

where $(\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3) = (\mathsf{C}+b_i+K_i, (\mathsf{C}+b_i+K_i)^2, \bar{\mathsf{B}}_i)$ if $c_{i+1} > 0$, and $(\mathsf{f}_i(\mathsf{C}+b_i), \mathsf{f}_i^2(\mathsf{C}+b_i)^2, \mathsf{f}_i)$ if $c_{i+1} = 0$. The outer integral for $\Psi_\kappa(0,\lambda)$ and u_0 is this case with variance s_0.

(c) *At the top level $i = M-1$, with $c = c_{M-1}$, $b = b_{M-1}$, $x \ge 0$, $\mathsf{p}_\pm = \mathsf{G}(\pm x/b + cb)$ and $\mathsf{Q} = \mathrm{e}^{cx}\mathsf{p}_+ + \mathrm{e}^{-cx}\mathsf{p}_-$, one has the closed forms*

$$\Psi_{M-1} = \frac{cb^2}{2} + \frac{\log \mathsf{Q}}{c}, \quad \Psi_{M-1}' = \frac{\mathrm{e}^{cx}\mathsf{p}_+ - \mathrm{e}^{-cx}\mathsf{p}_-}{\mathsf{Q}}, \quad \mathsf{E}_{M-1} = cb^2 + x\Psi_{M-1}' + \frac{2b\mathsf{g}(x/b)\mathrm{e}^{-c^2b^2/2}}{\mathsf{Q}}.$$

(d) *For $|y| > Y_i$, $\Psi_i(y)$ lies in $|y| + \mathsf{d}_i + [0, \bar{\psi}_i]$, with $\bar{\psi}_i$ the upper end of the enclosure of $\psi_i(Y_i)$, and $\Psi_i'(y)$ lies in $\operatorname{sgn}(y)[1 - \exp(-Y_i^2/(2\tau_i^2)), 1]$.*

Proof. (a) Let $\phi = 1$. By (A.1) with $\theta = \theta_i$, the function $y \mapsto \varphi_{b_i}(y-x)\mathsf{A}_i(y)$ has, on $|\operatorname{Im} y| \le a_i$, L^1 norm at most $\exp(a_i^2/(2b_i^2) + \theta_i a_i^2/(2b_{i+1}^2))\mathsf{B}_i(x)$, so Lemma A.3 gives the error $E_i'\mathsf{B}_i(x)$. Since $\log \mathsf{A}_i$ is c_i-Lipschitz, the tail is at most $\mathsf{A}_i(x)\mathsf{t}_i(c_i) \le \mathsf{B}_i(x)\mathsf{t}_i(c_i)$ by Lemma A.1(c). The other two numerators are bounded similarly, using (A.1) for ϕ, Lemma A.1(d), $\mathbb{E}_x^i|W| \le \mathsf{k}_i b_i$, and $\mathsf{C}' + |y| \le (\mathsf{C}'+1)\mathrm{e}^{|y|}$. For the ratio, we write $\mathsf{T}_i = \mathsf{B}_i(1+\mathsf{y}_1)$, $|\mathsf{y}_1| \le \mathsf{r}_i$, and $\mathsf{N} = N + \mathsf{y}_2\mathsf{B}_i$, $|\mathsf{y}_2| \le \mathsf{r}_\mathrm{n}$. Then $N/\mathsf{B}_i - \mathsf{N}/\mathsf{T}_i = \mathsf{y}_1\mathsf{N}/\mathsf{T}_i - \mathsf{y}_2$, which is at most $\mathsf{r}_i|\mathsf{N}/\mathsf{T}_i| + \mathsf{r}_\mathrm{n}$ in absolute value, and hence at most the stated bound. This completes the proof of (a).

Part (b) follows as in (a), with $\mathsf{A}_i = 1$ and $\mathbb{E}(\mathsf{C}' + |W|)^2 \le (\mathsf{C}' + b_i)^2$.

(c) We differentiate $\mathbb{E}\exp(c|x + bz|) = \mathrm{e}^{c^2b^2/2}\mathsf{Q}$ in x and in c.

Part (d) follows from Lemma A.1(b), (e). □

The exact recursion is a sup-norm contraction by Lemma 5.2(a), so errors add along the levels without amplification. In the computation this is not a separate estimate: every table entry is a ball, and its width passes through the exponentials, products and logarithms of the next level.

The functionals, namely the values of Q and its integrals, come from a forward pass. Put $f_l^{\mathrm{Q}} := 1 - \Psi_l'^2$, and $f_l^{\mathrm{I}} := b_l^2 - (2/c_l)(\mathsf{E}_l - \Psi_l)$ for $c_l > 0$, $f_l^{\mathrm{I}}(x) := b_l^2 - \operatorname{Var}\Psi_{l+1}(x + b_l z)$ for $c_l = 0$. Then

$$Q(s_l) = 1 - \mathbb{E}f_l^{\mathrm{Q}}(X_{s_l}), \qquad \int_{s_l}^{s_{l+1}} (1-Q)\,\mathrm{d}w = \mathbb{E}f_l^{\mathrm{I}}(X_{s_l}). \tag{A.3}$$

The first is the definition of Q. For the second, we have $\partial_{c_l}\Psi_l = (\mathsf{E}_l - \Psi_l)/c_l$ (and $\frac{1}{2}\operatorname{Var}$ at $c_l = 0$), the chain rule through the lower levels gives $\partial_{c_l}\Psi_\kappa(0,\lambda) = \mathbb{E}[\partial_{c_l}\Psi_l(X_{s_l})]$, and (7.4) gives $\partial_{c_l}\Psi_\kappa(0,\lambda) = \frac{1}{2}\int_{s_l}^{s_{l+1}} Q$. The same computation from $X_{s_l} = x$ gives $f_l^{\mathrm{I}}(x) = \int_{s_l}^{s_{l+1}} \mathbb{E}[1 - u(w, X_w)^2 \mid X_{s_l} = x]\,\mathrm{d}w$, so $0 \le f_l^{\mathrm{I}} \le b_l^2$.

Let π_0 be the outer trapezoid weights on the table of Ψ_0, and let $\pi_{i+1}(y) = \sum_x \pi_i(x)\mathsf{h}_{i+1}\varphi_{b_i}(y-x)\mathsf{A}_i(y)/\mathsf{T}_i(x)$, folded to $y \ge 0$ and restricted to the table of Ψ_{i+1}. Put $\mathrm{mass}_i = \sum \pi_i$. Let $\bar{S}$ bound f on the strip of the step into level l.

Lemma A.5. *For $f = f_l^{\mathrm{Q}}$ with $S = 1$, or $f = f_l^{\mathrm{I}}$ with $S = b_l^2$, so that $|f| \le S$ on $\mathbb{R}$, and with $\mathsf{e} = 2\exp(-Z^2/2)S$,*

$$\Big|\mathbb{E}f(X_{s_l}) - \sum_y \pi_l(y)f(y)\Big| \le \sum_{i\le l-2} \mathrm{mass}_i\Big[S\,\frac{E_i'/\varpi_i + \mathsf{t}_i(c_i) + \mathsf{r}_i}{1-\mathsf{r}_i} + \mathsf{e}\Big] + \mathrm{mass}_{l-1}\Big[\frac{E_{l-1}'\bar S + (\mathsf{t}_{l-1}(c_{l-1}) + \mathsf{r}_{l-1})S}{1-\mathsf{r}_{l-1}} + \mathsf{e}\Big] + \mathsf{o},$$

where o is the term of the outer integral, of the same form, with mass $= 1$. *One may take $\bar S = 1 + \bar{\mathsf{B}}_{l-1}^2$ for f_l^{Q}. For f_l^{I}, with $\mathsf{C}_b = Y_{l-1} + \mathsf{k}_{l-1}b_{l-1} + \mathsf{k}_l b_l + \bar X_l$, one may take $\bar S = b_l^2 + (2/c_l)(\mathsf{C}_b/\varpi_{l-1} + \mathsf{C}_b + K_{l-1})$ if $c_l > 0$, and $\bar S = b_l^2 + \mathsf{f}_{l-1}(\mathsf{C}_b + b_l)^2 + \mathsf{f}_{l-1}^2(\mathsf{C}_b + b_l)^2$ if $c_l = 0$.*

Proof. Put $H_l = f$, $H_i(x) = \mathbb{E}_x^i H_{i+1}(x+W)$, $\mathsf{J}_i = \sum \pi_i H_i$, so that $\mathbb{E}f(X_{s_l})$ is the outer integral of H_0 and $|H_i| \le S$. We telescope the differences

$$\mathsf{J}_{i+1} - \mathsf{J}_i = \sum_x \pi_i(x)\Big[\sum_y \frac{\mathsf{h}_{i+1}\varphi_{b_i}(y-x)\mathsf{A}_i(y)}{\mathsf{T}_i(x)} H_{i+1}(y) - H_i(x)\Big].$$

The bracket is a ratio of truncated trapezoid sums against the true integrals, so Lemma A.3 with (A.2) bounds it for $i \le l-2$, where H_{i+1} has the form N/B_{i+1}, and with $\bar S$ for $i = l-1$. The bound $\bar S$ on the strip follows from (A.1), (A.2), Lemma A.1(d), and $\mathbb{E}_{\mathsf{t}}|W| \le \mathsf{k}b$.

It remains to treat the nodes $|y| > Y_{i+1}$ dropped from the table, for which we need $\sup_{|y|>Y_{i+1}} |H_{i+1}| \le \mathsf{e}$. By Lemma A.1(e), integrated over the level for f_l^{I}, we have $|f(y)| \le 2S\exp(-y^2/(2\tau_l^2))$. Since the drift κu has the sign of X, by Tanaka's formula $|X|$ is pathwise above the reflected Brownian motion from $|x|$. Hence $|H_i(x)| \le 2S\mathbb{E}\exp(-(x+B)^2/(2\tau_l^2)) \le 2S\exp(-x^2/(2\tau_i^2))$, where B has variance $s_l - s_i$, and $\tau_l^2 + s_l - s_i = \tau_i^2$. With $Y_i \ge Z\tau_i$ this is e, and the proof is complete. □

The sums of the recursion are evaluated by products of ball polynomials, and every error term above is evaluated in ball arithmetic and added to the radius. The evaluation also verifies, in balls or exact rationals, the nesting of the lattices, $N_i^2\mathsf{h}_i^2 \ge Z^2\tau_i^2$ where $Y_i = N_i\mathsf{h}_i$, $\varpi_i > 0$, $\mathsf{r}_i < 1/2$, $\mathsf{q} < 1$ for every tail, and that c_i is nondecreasing. The forward pass uses $4\exp(-Z^2/2)S$, unweighted, in place of $\mathrm{mass}_i\mathsf{e}$, which absorbs a mass above 1. Floating point chooses $\mathsf{h}_i, N_i, a_i, J_i$ and has no other role.

A.2. **What is computed.** All certified numbers are outputs of FLINT/Arb [13] ball arithmetic, through python-flint 0.9.0, at 160 bits. Every parameter reaches a ball as an exact rational. Floating point proposes trials, grids, table sizes, the function ϑ and anchors. It enters no bound: every proposed object is rechecked in balls or exact rationals before use. A bound written here with $\le$ or $\ge$ is rounded in the safe direction, and a printed ball contains the unrounded value. Other decimals, such as the entries of Tables 1 and 2, are rounded to nearest. Acceptance uses ball comparisons only.

The trial nodes have 31 levels on $[0.9555, 0.9785]$ and 25 levels outside, down to t_k about $2\cdot10^{-9}$, with mesh $1.23\cdot10^{-5}$ near $m = 0.9667$. The levels are cut at between 177 and 628 nodes s_i. The 48 SK nodes of Theorem 7.8 and the 34 further SK nodes of Theorem 7.13 use 8-step trials, the former cut at between 29 and 353 nodes. The value at 0.974 uses a 12-step trial cut at 215 nodes.

We compute the two values in Corollary 4.15 as follows. Let $\mathsf{f}(s) = \mathbb{E}\tanh^2(\sqrt{s}\,z)$, $\alpha = 2$, and $(\gamma, y_{\mathrm{h}}) = (2, 0.35)$ or $(4.44, 0.5)$. It is enough that $y/(1-y)^2 < \gamma\mathsf{f}(\gamma\alpha y)$ on $(0, y_{\mathrm{h}}]$. On $(0, 1/100]$, the bound $\tanh^2 s \ge s^2 - \frac{2}{3}s^4$ gives $\mathsf{f}(x) \ge x - 2x^2$, so it is enough that

$\gamma^2\alpha(1-2\gamma\alpha/100) > (1-1/100)^{-2}$. For $[1/100, y_{\mathrm{h}}]$ we use 400 cells. Since both sides increase in y, on each cell we compare the left side at the right end with a lower bound of the right side at the left end, namely a lower Riemann sum in $|z| \le 8$, with step $1/200$ and Gaussian masses in balls.

Code and certificates are in the repository given under Code and data. Every certified number has been rebuilt from a clean start, and the computed values agree with the stored certificates. Two separately written checkers, also written by AI, share only Arb with this code, each with its own remainder bounds and its own grids.

The first recomputed the pair at $\underline{\alpha}$, the pair of Theorem 7.9(b), the worst cell at $\overline{\alpha}$, the value at 0.974, and its own recursions at 22 of the 164 trial nodes, and it reran the mesh logic over all 35 load intervals. All balls overlap. The second evaluated Theorem 7.2 at all 82 SK nodes on the stored rational trials, where its values of $\mathcal{P}_m$ overlap the stored ones and its lower bounds are at least the stored ones. With its own values it passes all cells of Theorem 7.13, and it confirms the two computations behind Corollary 4.15. At the other 142 trial nodes the recursion has been reproduced only by the rebuild, not by separate code.

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