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FULLY ORDINARY SUBFAMILIES AND THE FONTAINE–MAZUR CONJECTURE IN THE RESIDUALLY REDUCIBLE MIXED CASE

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Abstract. Let F be a totally real field, abelian over $\mathbb{Q}$, in which an odd prime p splits completely, and let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous, absolutely irreducible, totally odd representation, unramified outside finitely many places and de Rham with distinct Hodge–Tate weights at the places above p, with reducible residual representation. We prove that ρ is modular in the mixed case, in which ρ is reducible at some places above p and irreducible at the others, provided that the ratio of the two residual characters extends to $G_{\mathbb{Q}}$ and its restriction to $G_{\mathbb{Q}_p}$ is neither 1 nor $\omega^{\pm 1}$. No assumption is made on the splitness of ρ at the reducible places, on potential modularity, or on the local behaviour of ρ away from p. A suitable irreducible component through the point of ρ in a partially ordinary deformation space contains a fully ordinary subfamily of maximal dimension. A classical point of this subfamily is a regular point of the deformation space, by the vanishing of adjoint Selmer groups, and a dimension estimate for a partially ordinary Hecke algebra then identifies the component of ρ with a component of the Hecke algebra.

1. Introduction

1.1. **Setting.** Let p be an odd prime and F a totally real field, abelian over $\mathbb{Q}$, in which p splits completely, so that $F_v = \mathbb{Q}_p$ for every place $v \mid p$. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field $\mathbb{F}$, which we enlarge when necessary. Let $\rho : G_F \to \mathrm{GL}_2(E)$ be a continuous, absolutely irreducible, totally odd representation, unramified outside finitely many places. Let Σ_p be the set of places above p, fix a partition $\Sigma_p = \Sigma_{\mathrm{n}} \sqcup \Sigma_{\mathrm{o}}$, and consider, as in [Die26, §1.2], the conditions:

- (N) for $v \in \Sigma_{\mathrm{n}}$, $\rho_v = \rho|_{G_{F_v}}$ is irreducible and de Rham with distinct Hodge–Tate weights;
- (O) for $v \in \Sigma_{\mathrm{o}}$, ρ_v is reducible and de Rham with distinct Hodge–Tate weights;
- (R) $\bar{\rho}^{ss} = \bar{\chi}_1 \oplus \bar{\chi}_2$, the character $\bar{\chi} = \bar{\chi}_1 \bar{\chi}_2^{-1}$ extends to a character of $G_{\mathbb{Q}}$, and $\bar{\chi}|_{G_{F_v}} \neq 1$ for every $v \mid p$.

After enlarging E, irreducibility of a local representation always means absolute irreducibility. Under (O), the Jordan–Hölder character of ρ_v with the smaller Hodge–Tate number is a subcharacter of ρ_v [Die26, Lemma 2.1]. It is the *classical character* ψ_v^{cl}, the one which occurs for ordinary modular forms. Since $\bar{\chi}$ comes from $G_{\mathbb{Q}}$ and p splits completely, $\bar{\chi}|_{G_{F_v}}$ is the same character of $G_{\mathbb{Q}_p}$ for all $v \mid p$.

The case $\Sigma_{\mathrm{o}} = \emptyset$ is due to Pan [Pan22] and Zhang [Zha25], and the case $\Sigma_{\mathrm{n}} = \emptyset$ to Pan [Pan22, Theorem 5.1.2]. In the mixed case, in which both are non-empty, the following result is proved in [Die26].

Theorem ([Die26, Theorem A])**.** *Assume* (N)*,* (O) *and* (R)*, and that ρ_v is non-split for every $v \in \Sigma_{\mathrm{o}}$. Then ρ is modular.*

1.2. **Main result.**

Theorem A. *Assume* (N)*,* (O)*,* (R)*, and*

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(a) $\bar{\chi}|_{G_{F_v}} \neq \omega^{\pm 1}$ *for one, equivalently every, place* $v \mid p$.

Then ρ *is modular.*

No assumption is made on the splitness of ρ_v for $v \in \Sigma_{\mathrm{o}}$, on the Hodge–Tate weights beyond (N) and (O), on the degree of F, or on the restrictions of ρ to the decomposition groups away from p. Condition (a) is a single condition on the restriction to $G_{\mathbb{Q}_p}$ of the extension of $\bar{\chi}$ to $G_{\mathbb{Q}}$. It says that $\bar{\rho}^{ss}|_{G_{\mathbb{Q}_p}}$ is not a twist of $1 \oplus \omega$, the case excluded by Kisin [Kis09a] and later treated by Hu–Tan [HT15] and Tung [Tun21]. Its essential use is to make the reducibility ideal at the places of Σ_{n} principal (Lemma 3.1); it also simplifies the local smoothness and global dimension estimates of Proposition 3.3. See Remark 8.1.

1.3. **The method.** Let R^{o} be the pseudo-deformation ring of $\bar{\rho}^{ss}$ which is ordinary at the places of Σ_{o}, with the reduction of the classical character prescribed as subcharacter, and unrestricted at the places of Σ_{n}. Let x be the point defined by ρ. Let $\mathbb{T}^{\mathrm{o}}$ be the Hecke algebra of the part of the completed cohomology of a definite quaternion algebra which is ordinary at the places of Σ_{o} [Die26, §3.3]. It is a quotient of R^{o}, and it carries the ordinary characters at Σ_{o} through its diamond operators. If x lies in $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}$, then the corresponding ordinary eigenvectors exist (Proposition 6.1), and the classicality criterion [Die26, Theorem 4.2] shows that ρ is modular. The proof that it does has four steps. Write $d = [F : \mathbb{Q}]$.

(1) A Greenberg–Wiles estimate gives an irreducible component C of $\mathrm{Spec}R^{\mathrm{o}}$ through x of dimension at least $1 + 2d - |\Sigma_{\mathrm{o}}|$ (Proposition 3.3).
(2) Under (a), the reducible locus at each place of Σ_{n} is cut out by one equation. Cutting C by these equations gives an irreducible closed subset $D \subset C$ which is reducible at every place above p, with ordinary subcharacters chosen uniformly on D, of dimension $1 + d$. By Pan's finiteness theorem D is finite and surjective over the Iwasawa algebra, so its classical points are dense. By Pan's theorem in the fully ordinary case and Hida theory, they lie in $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}$ (Proposition 5.4).
(3) Let y be a classical point of D of sufficiently regular weight which is not of CM type. By the Greenberg–Wiles formula, the tangent space of R^{o} at y has the expected dimension $2d - |\Sigma_{\mathrm{o}}|$ as soon as the Bloch–Kato Selmer group $H^1_f(F, \mathrm{ad}^0\rho_y(1))$ vanishes, and this follows from the theorem of Newton and Thorne [NT23, Theorem B] together with global duality. Hence R^{o} is regular at y, C is the only irreducible component of $\mathrm{Spec}R^{\mathrm{o}}$ through y, and $\dim C = 1 + 2d - |\Sigma_{\mathrm{o}}|$ (Proposition 5.5). At the places away from p where ρ has Steinberg type, we use local–global compatibility at y (Lemma 4.1).
(4) Every irreducible component of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}$ has dimension at least $1 + 2d - |\Sigma_{\mathrm{o}}|$, by an adaptation of Pan's Gelfand–Kirillov dimension argument (Proposition 6.1). Hence an irreducible component of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}$ through y is contained in C, and equal to it. Thus $x \in \mathrm{Spec}\mathbb{T}^{\mathrm{o}}$.

The deep inputs are Pan's theorem in the fully ordinary case, the theorem of Newton–Thorne, local–global compatibility for Hilbert modular forms, and the theory of ordinary parts at the places of Σ_{o} together with the classicality criterion of [Die26, §§3–4]. Theorem A of [Die26] is not used. There are no nice primes, no connectedness arguments, no auxiliary primes, no local indecomposability theorems, and no Taylor–Wiles patching beyond the one contained in these results.

The pattern of steps (2)–(4) is that of Böckle [Böc01]: one produces a modular point of C at which the deformation ring is regular, and combines it with a lower bound for the dimension of the Hecke algebra. In the residually reducible case this was carried out by Deo [Deo23] over $\mathbb{Q}$ and by Zhang [Zha26], over $\mathbb{Q}$ and, when $\Sigma_{\mathrm{o}} = \emptyset$, over abelian totally real fields in which p splits completely. In particular, the cut of the unrestricted locus by the principal reducibility ideal, the use of Pan's ordinary finiteness theorem and the Newton–Thorne vanishing theorem already

appear there. What is new here is the partially ordinary setting: the deformation ring R^{o} and the Hecke algebra $\mathbb{T}^{\mathrm{o}}$ with ordinary conditions at Σ_{o}, the dimension estimate of Proposition 6.1 for $\mathbb{T}^{\mathrm{o}}$, and the local analysis at the places of Σ_{o} and at the places of Steinberg type (Lemma 4.1 and Proposition 5.5).

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Notation. We use the conventions of [Die26]: $\mathrm{HT}(\varepsilon) = -1$, characters of $G_{\mathbb{Q}_p}$ are identified with characters of $\mathbb{Q}_p^\times$ via the local Artin map normalized as in [Pan22, Zha25], and ω is the mod p cyclotomic character. We write $G_v = G_{F_v}$, I_v for its inertia subgroup, ϕ_v for a lift of the arithmetic Frobenius and $q_v = \mathrm{N}(v)$. For a pseudo-representation T with coefficients in a ring R and a prime $\mathfrak{r}$ of R, $\rho(\mathfrak{r})$ is the semisimple representation over an algebraic closure of $k(\mathfrak{r})$ with trace T mod $\mathfrak{r}$; for a point z we also write ρ_z. A prime $\mathfrak{r}$ is *irreducible* if $\rho(\mathfrak{r})$ is absolutely irreducible. We write $h^i(V) = \dim H^i(V)$.

2. Setup

2.1. **The base field.** Twisting ρ by the Teichmüller lift of $\bar\chi_2^{-1}$, we may assume $\bar\rho^{ss} = 1 \oplus \bar\chi$. Put $\chi = \det\rho$, a totally odd de Rham character lifting $\bar\chi$. We first normalize the determinant as in [Zha25, Remark 5.5.2]. Write $\chi = \alpha\varepsilon^k$ with α of finite order, and put $\chi_0 = \widetilde{\bar\alpha}\varepsilon^k$, where $\bar\alpha = \bar\chi\omega^{-k}$ extends to $G_{\mathbb{Q}}$ by (R) and $\widetilde{\bar\alpha}$ is the Teichmüller lift. The character $\chi_0\chi^{-1}$ has finite p-power order, so it has a unique square root φ of p-power order. Replacing ρ by $\rho\otimes\varphi$, which has trivial reduction and Hodge–Tate weight 0 and preserves (N), (O), (R), (a), local splitness and modularity, we may assume $\chi = \chi_0$. By [Zha25, Theorem 5.5.1 and Remark 5.5.2], there is a totally real extension F_1/F, abelian over $\mathbb{Q}$, of even degree, in which p splits completely, such that $\rho|_{G_{F_1}}$ satisfies the conditions of [Zha25, §4.1] recalled in (F2) below. We replace F by F_1 and ρ by $\rho|_{G_{F_1}}$, which is absolutely irreducible by [Die26, Lemma 4.4]. The hypotheses of Theorem A are preserved, since $F_{1,w} = F_v = \mathbb{Q}_p$ for $w \mid v \mid p$, and ρ is modular if $\rho|_{G_{F_1}}$ is, by solvable descent [BLGGT14, Lemma 2.2.2], [AC89]. We keep the notation Σ_{o}, Σ_{n} for the sets of places of F_1 above the original ones, write $d = [F:\mathbb{Q}]$, and let Σ be the set of places above p and those at which ρ is ramified. Thus:

- (F1) F is totally real, abelian over $\mathbb{Q}$, of even degree $d \geq 2$, and p splits completely in F; $\bar\rho^{ss} = 1 \oplus \bar\chi$;
- (F2) for $v \in \Sigma\setminus\Sigma_p$: $p \mid q_v - 1$, χ is unramified at v, $\chi(\phi_v) \equiv 1 \pmod{\varpi}$, and there is a character ξ_v of p-power order of $k(v)^\times$ with $\operatorname{tr}\rho|_{I_v} = \xi_v + \xi_v^{-1}$ [Zha25, Remark 4.1.2].

Here Σ is chosen after the twists and the base change. For the inertial assertion in (F2): the residual representation at v is trivial, so $\rho(I_v)$ is pro-p, hence factors through tame inertia; by quasi-unipotence and since χ is unramified at v, its eigencharacters are ξ_v, ξ_v^{-1} of p-power order, and Frobenius does not interchange them when $\xi_v \neq 1$, since $q_v \equiv 1 \pmod p$ and $p > 2$. Since p is odd, $\xi_v^2 \neq 1$ unless $\xi_v = 1$. If $\xi_v \neq 1$, then $\rho|_{G_v} \simeq \mu_1 \oplus \mu_2$ with $\mu_1|_{I_v} = \xi_v$ and $\mu_2|_{I_v} = \xi_v^{-1}$, since these restrictions are distinct. If $\xi_v = 1$, then $\rho|_{I_v}$ is unipotent and non-trivial, as ρ is ramified at v. Hence $\rho|_{G_v}$ is of Steinberg type: its Weil–Deligne representation is $(\nu\oplus\nu|\cdot|, N)$ with $N \neq 0$. Let Σ_{St} be the set of these places. For $v \in \Sigma_{\mathrm{St}}$ we fix $\sigma_v \in I_v$ mapping to a topological generator of the p-part of the tame inertia group.

2.2. **Deformation rings.** Let R^{ps} be the universal deformation ring of the pseudo-representation $1 + \bar\chi$ of $G_{F,\Sigma}$ with determinant χ. For $v \mid p$, on the locus where $T|_{G_v}$ is reducible we have $T|_{G_v} = \psi + \chi\psi^{-1}$ with ψ lifting 1 or $\bar\chi|_{G_v}$, and the decomposition is unique since $\bar\chi|_{G_v} \neq 1$. For

$v \in \Sigma_o$ let $\bar\psi_v \in \{1, \bar\chi|_{G_v}\}$ be the reduction of ψ_v^{cl}. Let Ξ be the finite set of tuples $\xi' = (\xi'_v)_{v\in\Sigma\setminus\Sigma_p}$ with $\xi'_v \in \{\xi_v, \xi_v^{-1}\}$.

Definition 2.1. (1) Let R^{o} be the quotient of R^{ps} by the conditions:
- $T|_{I_v} = \xi_v + \xi_v^{-1}$ for $v \in \Sigma \setminus \Sigma_p$;
- for $v \in \Sigma_o$, $T|_{G_v} = \psi_v + \chi\psi_v^{-1}$ is reducible with ψ_v lifting $\bar\psi_v$, and T is ψ_v-ordinary in the sense of [Pan22, Definition 5.3.2] (recalled in [Die26, §2.2]).

(2) Let $R^{\mathrm{ps,ord}}$ be the quotient of R^{o} by the condition that $T|_{G_v}$ is reducible for every $v \in \Sigma_n$.

The point $x \in \mathrm{Spec}R^{\mathrm{o}}[1/p]$ defined by ρ lies in $\mathrm{Spec}R^{\mathrm{o}}$ by [Die26, Lemma 2.3(1)], and $\psi_v(x) = \psi_v^{\mathrm{cl}}$ for $v \in \Sigma_o$: the two Jordan–Hölder characters of ρ_v have distinct reductions, and ψ_v^{cl} is the one lifting $\bar\psi_v$.

On $\mathrm{Spec}R^{\mathrm{ps,ord}}$, for every $v \mid p$ write $T|_{G_v} = \psi_v + \chi\psi_v^{-1}$, with ψ_v lifting $\bar\psi_v$ if $v \in \Sigma_o$ and lifting 1 if $v \in \Sigma_n$. Let $\Lambda = \mathcal{O}[[\prod_{v|p}(1 + p\mathbb{Z}_p)]]$, a power series ring over $\mathcal{O}$ in d variables. Through the restrictions of the characters ψ_v to $\mathrm{Art}(1 + p\mathbb{Z}_p) \subset I_v$, $R^{\mathrm{ps,ord}}$ is a Λ-algebra.

Lemma 2.2. *$R^{\mathrm{ps,ord}}$ is finite over Λ. In particular, every irreducible closed subset of* $\mathrm{Spec}R^{\mathrm{ps,ord}}$ *has dimension at most $1+d$, and one of dimension $1+d$ is finite over, and surjective onto,* $\mathrm{Spec}\Lambda$.

Proof. $R^{\mathrm{ps,ord}}$ is a quotient of the ring $R^{\mathrm{ps,ord}}$ of [Pan22, 5.1.1], which is finite over $\Lambda_F = \mathcal{O}[[\prod_{v|p}\mathbb{Z}_p^\times]]$ by [Pan22, Theorem 5.1.2(1)] (see [Zha25, Theorem 4.2.3] for $p = 3$). The hypotheses of loc. cit. hold by (F1), (R) and the oddness of χ. Our Λ-structure differs from the restriction of that of loc. cit. by the automorphism induced by $\psi \mapsto \chi\psi^{-1}$ at the places of Σ_o where $\bar\psi_v = \bar\chi|_{G_v}$, and Λ_F is finite over Λ. As Λ is a domain of dimension $1+d$, the last assertions follow. □

The pseudo-representations which are reducible as pseudo-representations of G_F form a closed subset of $\mathrm{Spec}R^{\mathrm{ps}}$ of dimension at most 2. Indeed, they are of the form $\psi+\chi\psi^{-1}$ with ψ a character of $G_{F,\Sigma}$ lifting 1, and the pro-p completion of $G_{F,\Sigma}^{\mathrm{ab}}$ has $\mathbb{Z}_p$-rank 1, by Leopoldt's conjecture for the abelian field F.

2.3. **Partially ordinary Hecke algebras.** We use the setting of [Die26, §3]. Let B be the quaternion algebra over F ramified exactly at the infinite places, $\psi = \chi\varepsilon$, and $U^p = \prod_{v\nmid p} U_v$, with $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v \notin \Sigma$ and U_v the Iwahori subgroup for $v \in \Sigma \setminus \Sigma_p$. For $\xi' \in \Xi$ and $v \in \Sigma \setminus \Sigma_p$ we use the character $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \mapsto \xi'_v(a/d \bmod v)$ of U_v, which is trivial on the centre and corresponds to the inertial pair $\xi'_v, \xi_v'^{-1}$ [Pan22, 4.1.2]. Let $S_{\xi'} = S_{\psi,\xi'}(U^p)_{\mathfrak{m}}$ be the completed cohomology with these characters, localized at the Eisenstein maximal ideal of $1 + \bar\chi$, and let $S^{\mathrm{o}}_{\xi'} = \mathrm{Ord}_P(S_{\xi'})$ be its ordinary part at the places of Σ_o, where P is the upper Borel subgroup at the places of Σ_o and the whole group at those of Σ_n [Die26, §3.3]. Let $\mathbb{T}^{\mathrm{o}}_{\xi'}$ be the closed image of the algebra generated by the operators T_w, $w \notin \Sigma$, and the full torus operators at the places of Σ_o (including the uniformizer operators U_v), acting on $S^{\mathrm{o}}_{\xi'}$, and let $\mathfrak{m}'$ be the maximal ideal at which the first torus character $\psi_{v,1}$ of [Die26, §3.3] reduces to $\bar\psi_v$ for every $v \in \Sigma_o$. The comparison of $S^{\mathrm{o}}_{\xi'}$ with torsion ordinary parts and the finiteness properties of $\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$ are proved in Proposition 6.1.

The ordinary Galois identities are [Die26, Lemma 3.4]: there is a pseudo-representation over $\mathbb{T}^{\mathrm{o}}_{\xi'}$ with determinant χ which is $\psi_{v,1}$-ordinary at every $v \in \Sigma_o$. Since ψ need not be locally constant at p, its proof first passes to the finite order central character ψ_0, the Teichmüller lift of $\bar\psi$, by the twist by $\theta \circ \mathrm{Nrd}$ with $\theta^2 = \psi_0\psi^{-1}$, and then transports the identities back; θ is not assumed to be de Rham. By (F2), ψ, hence θ, is unramified at the places of $\Sigma \setminus \Sigma_p$, so the same U^p and ξ' can be used. Together with local–global compatibility at the places of $\Sigma \setminus \Sigma_p$ [Pan22, 4.1.3], this gives a surjection $R^{\mathrm{o}} \twoheadrightarrow \mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$ sending ψ_v to $\psi_{v,1}$ for $v \in \Sigma_o$, surjective because $\mathbb{T}^{\mathrm{o}}_{\xi'}$ is topologically generated by the T_w and the torus operators. A prime of R^{o} is *o-pro-modular* if it

lies in the image of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$ for some $\xi' \in \Xi$. An o-pro-modular prime thus records both the trace and the ordinary characters at the places of Σ_{o}.

3. Local results and a lower bound for C

Lemma 3.1. *Let $v \in \Sigma_{\mathrm{n}}$, and let R^{ps}_v be the deformation ring of the pseudo-representation $(1+\bar\chi)|_{G_v}$ with determinant $\chi|_{G_v}$. There is $f_v \in R^{\mathrm{ps}}_v$ such that a deformation of $(1+\bar\chi)|_{G_v}$ with values in a field is reducible if and only if it kills f_v.*

Proof. Since $\bar\chi|_{G_v} \neq 1$, the universal Cayley–Hamilton algebra over R^{ps}_v is a generalized matrix algebra $\left(\begin{smallmatrix} R^{\mathrm{ps}}_v & B \\ C & R^{\mathrm{ps}}_v \end{smallmatrix}\right)$. A pseudo-representation with values in a field is reducible if and only if it kills the image BC of $B \otimes C$ [BC09, §1.3–1.5]. The minimal numbers of generators of B and C are at most $h^1(G_{\mathbb{Q}_p}, \bar\chi)$ and $h^1(G_{\mathbb{Q}_p}, \bar\chi^{-1})$ [BC09, Theorem 1.5.5], [Deo, Lemma 2.5]. By the local Euler characteristic formula, (R) and (a), $h^1(G_{\mathbb{Q}_p}, \bar\chi^{\pm 1}) = 1 + h^0(\bar\chi^{\pm 1}) + h^0(\bar\chi^{\mp 1}\omega) = 1$. Hence $B = R^{\mathrm{ps}}_v b$ and $C = R^{\mathrm{ps}}_v c$, and $f_v = bc$ works. Compare [Zha26, Lemma 5.3.3], where the same principal ideal is used to cut out the ordinary locus. □

Lemma 3.2. *Let G be a group, k a field, $\alpha \neq \beta$ and $\eta \neq 1$ characters $G \to k^\times$, and $r = \left(\begin{smallmatrix} \alpha & * \\ 0 & \beta \end{smallmatrix}\right)$ a non-split extension. If $H^0(G, \mathrm{ad}^0 r \otimes \eta) \neq 0$, then $\alpha\beta^{-1} = \eta^{-1}$.*

Proof. A non-zero invariant is a G-equivariant map $X : V \to V \otimes \eta$, where $V = k^2$ with the action r. The representations V and $V \otimes \eta$ have unique stable lines, both equal to the first basis line L, with characters α and $\alpha\eta$. If X were invertible, it would map L to L, so $\alpha = \alpha\eta$, which is impossible. Hence X has rank 1, its kernel and image are stable lines, so both are L. Then X is a multiple of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, on which G acts through r by $\alpha\beta^{-1}$, so $\alpha\beta^{-1} = \eta^{-1}$. □

Proposition 3.3. *There is an irreducible component C of $\mathrm{Spec}R^{\mathrm{o}}$ containing x with $\dim C \geq 1 + 2d - |\Sigma_{\mathrm{o}}|$.*

Proof. We follow the proof of [Zha25, Lemma 5.4.1]. Let $\mathfrak{r}$ be the prime of R^{o} defined by x. It is one-dimensional and irreducible. After enlarging $\mathcal{O}$, choose a $G_{F,\Sigma}$-stable lattice ρ° in ρ whose reduction $\bar\rho_b$ is a non-split extension of $\bar\chi$ by 1 [SW99, Lemma 2.13]. By Lemma 3.2 with $\eta = \omega$, applied to $\alpha\beta^{-1} = \bar\chi^{-1}$, and by (a), we have $H^0(G_{F,\Sigma}, \mathrm{ad}^0\bar\rho_b(1)) = 0$; here $\omega|_{G_F} \neq 1$, since $F \cap \mathbb{Q}(\zeta_p)$ is unramified at p by (F1), hence equal to $\mathbb{Q}$. For $v \in \Sigma_{\mathrm{o}}$, saturate in ρ° the stable line of ρ_v with character ψ^{cl}_v, and let $\bar L_v$ be its reduction. Let R_b be the deformation ring of $\bar\rho_b$ with determinant χ which is ordinary along $\bar L_v$ for $v \in \Sigma_{\mathrm{o}}$ (that is, admits a stable direct summand of rank one lifting $\bar L_v$), unrestricted at Σ_{n}, and with $\mathrm{tr}|_{I_v} = \xi_v + \xi_v^{-1}$ for $v \in \Sigma \setminus \Sigma_p$.

The local framed rings are as follows.

- For $v \in \Sigma_{\mathrm{o}}$: the ordinary ring $R^\Delta_v \simeq R^B_v[[n]]$ of [Pan22, 5.3.8(1)]. Here R^B_v parametrizes lifts to the upper Borel subgroup. Its obstruction group $H^2(G_{\mathbb{Q}_p}, \mathfrak{b}^0)$ is dual to $H^0(G_{\mathbb{Q}_p}, (\mathfrak{b}^0)^\vee(1))$, whose Jordan–Hölder factors are ω and $\omega\bar\chi^{\pm 1}$, both non-trivial by (a). Hence R^Δ_v is formally smooth over $\mathcal{O}$ of relative dimension 5 (compare [Pan22, Lemma 5.3.9]).
- For $v \in \Sigma_{\mathrm{n}}$: the unrestricted framed ring, formally smooth of relative dimension 6, since the Jordan–Hölder factors of $\mathrm{ad}^0\bar\rho_b(1)$ are ω, $\omega\bar\chi$ and $\omega\bar\chi^{-1}$, all non-trivial by (a).
- For $v \in \Sigma \setminus \Sigma_p$: the ring R_{v,ξ_v} of [Pan22, 4.2.1], equidimensional of dimension 4 and $\mathcal{O}$-flat [Pan22, Lemma 4.2.10].

By [CHT08, Corollary 2.3.5], as in [Pan22, 5.4.2], the framed version of R_b is a quotient of a power series ring in g variables over the completed tensor product R_{loc} of these rings by r relations, with $g - r = |\Sigma| - 1 - d - h^0(G_{F,\Sigma}, \mathrm{ad}^0\bar\rho_b(1)) = |\Sigma| - 1 - d$. Removing the $4|\Sigma| - 1$ framing variables, which is possible since $\bar\rho_b$ is a non-split extension of distinct characters, so that its endomorphisms are scalars, every minimal prime Q of R_b satisfies

$$\dim R_b/Q \geq 1 + 5|\Sigma_{\mathrm{o}}| + 6|\Sigma_{\mathrm{n}}| + 3(|\Sigma| - d) + (|\Sigma| - 1 - d) - (4|\Sigma| - 1) = 1 + 2d - |\Sigma_{\mathrm{o}}|.$$

Let $\mathfrak{r}_b$ be the prime of R_b defined by ρ°, and choose a minimal prime $Q \subset \mathfrak{r}_b$. Since the reducible locus is closed and does not contain $\mathfrak{r}_b$, the representation at Q is irreducible. By [Zha25, Proposition 2.2.5(2)] (that is, [Pan22, Corollary 2.3.6]), applied to the inverse image of Q in the deformation ring of $\bar\rho_b$ without local conditions, the pull-back P of Q to R^{o} satisfies $\dim R^{\mathrm{o}}/P \geq \dim R_b/Q \geq 1+2d-|\Sigma_{\mathrm{o}}|$. As $P \subset \mathfrak{r}$, any irreducible component of $\mathrm{Spec}R^{\mathrm{o}}$ containing $V(P)$ has the required properties. □

We fix such a component C, with its reduced structure. Since $x \in C$ has characteristic 0, C is flat over $\mathcal{O}$, and $C[1/p]$ is irreducible.

4. Places of Steinberg type

Lemma 4.1. *Let $v \in \Sigma_{\mathrm{St}}$ and put $P_v = (1+q_v)^2\chi(\phi_v) - q_v\,T(\phi_v)^2$, a function on $\mathrm{Spec}R^{\mathrm{o}}$.*

(1) P_v vanishes on C.

(2) Let $y \in C[1/p]$ be a closed point such that ρ_y is irreducible and arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$. Then π_v is a twist of the Steinberg representation, $\rho_y|_{G_v}$ is of Steinberg type, and $H^0(G_v, \mathrm{ad}^0\rho_y(1)) = 0$.

Proof. For a closed point z of $\mathrm{Spec}R^{\mathrm{o}}[1/p]$, $\rho_z|_{I_v}$ has trace 2, hence is unipotent, and it factors through the tame inertia group since the wild inertia group is pro-ℓ. If a, b are the eigenvalues of $\rho_z(\phi_v)$, then $ab = \chi(\phi_v)$ and

$$P_v(z) = (1+q_v)^2ab - q_v(a+b)^2 = (a - q_vb)(b - q_va).$$

(1) Let $Z_N \subset C[1/p]$ be the closed subset of points z with $T_z(\sigma_v g) = T_z(g)$ for all $g \in G_{F,\Sigma}$. If ρ_z is irreducible, then $\rho_z(G_{F,\Sigma})$ spans M_2, and the trace form is non-degenerate, so $z \in Z_N$ if and only if $\rho_z(\sigma_v) = 1$, that is, if and only if the monodromy of $\rho_z|_{G_v}$ vanishes. Hence $x \notin Z_N$. Let $U \subset C[1/p]$ be the open subset of irreducible points not in Z_N. It contains x, so it is dense in $C[1/p]$. For $z \in U$, the Weil–Deligne representation of $\rho_z|_{G_v}$ has non-zero monodromy, so the ratio of the eigenvalues of $\rho_z(\phi_v)$ is $q_v^{\pm1}$, and $P_v(z) = 0$. As $C[1/p]$ is reduced and Jacobson, P_v vanishes on $C[1/p]$, hence on C, which is $\mathcal{O}$-flat.

(2) By (1), the semisimplification of $\rho_y|_{G_v}$ is $\nu_1 \oplus \nu_2$ with ν_1, ν_2 unramified and $\nu_1\nu_2^{-1} = \varepsilon^{\pm1}$. By local–global compatibility up to semisimplification [Car86, Tay89, Jar97], the Langlands parameter of π_v has semisimplification a sum of two characters with ratio $|\cdot|^{\pm1}$. As π_v is irreducible and generic, it is a twist of the Steinberg representation, which is square integrable. Since π is then square integrable at a finite place, full local–global compatibility at v, including the monodromy operator, holds by Carayol's theorem [Car86, Théorème (A)]. Hence $\rho_y|_{G_v}$ is a non-split extension $\left(\begin{smallmatrix}\nu\varepsilon & * \\ 0 & \nu\end{smallmatrix}\right)$. By Lemma 3.2 with $\eta = \varepsilon|_{G_v}$, a non-zero element of $H^0(G_v, \mathrm{ad}^0\rho_y(1))$ would force $\varepsilon = \varepsilon^{-1}$ on G_v, which is false since $\varepsilon(\phi_v) = q_v$ and $q_v^2 \neq 1$. □

5. Classical points of the fully ordinary subfamily

5.1. **The subfamily.** Let $A_C = \mathcal{O}(C) = R^{\mathrm{o}}/\mathfrak{q}$, where $\mathfrak{q}$ is the minimal prime of R^{o} defining C; it is a complete local domain of dimension at least $1 + 2d - |\Sigma_{\mathrm{o}}|$ by Proposition 3.3. For $v \in \Sigma_{\mathrm{n}}$, the restriction $T|_{G_v}$ of the universal pseudo-representation is a deformation of $(1+\bar\chi)|_{G_v}$ with determinant $\chi|_{G_v}$. By the universal property of R_v^{ps}, it is induced by a unique local homomorphism $r_v : R_v^{\mathrm{ps}} \to R^{\mathrm{ps}}$. Denote also by $f_v \in A_C$ the image of the element f_v of Lemma 3.1 under the composite

$$R_v^{\mathrm{ps}} \xrightarrow{r_v} R^{\mathrm{ps}} \twoheadrightarrow R^{\mathrm{o}} \twoheadrightarrow R^{\mathrm{o}}/\mathfrak{q} = A_C.$$

By Lemma 3.1, a point z of C lies in $V(f_v)$ if and only if $T_z|_{G_v}$ is reducible. The element f_v lies in the maximal ideal of A_C, since the residual pseudo-representation is reducible. By Krull's height

theorem, a minimal prime $\mathfrak{D}$ of $(f_v)_{v\in\Sigma_\mathrm{n}}$ satisfies

$$\dim A_C/\mathfrak{D} \geq \dim A_C - |\Sigma_\mathrm{n}| \geq 1 + 2d - |\Sigma_\mathrm{o}| - |\Sigma_\mathrm{n}| = 1 + d.$$

Let $D = \mathrm{Spec}A_C/\mathfrak{D}$, with its reduced structure. Its points are reducible at every place above p: at the places of Σ_o because $C \subset \mathrm{Spec}R^\mathrm{o}$, and at those of Σ_n by Lemma 3.1. Hence $D \subset \mathrm{Spec}R^{\mathrm{ps,ord}}$, and by Lemma 2.2, $\dim D = 1+d$ and D is finite over, and surjective onto, $\mathrm{Spec}\Lambda$. Since $1+d > 2$, the irreducible points form a dense open subset D^irr of D.

Lemma 5.1. *For every $v \in \Sigma_\mathrm{n}$ there is $\psi_{v,\mathrm{s}} \in \{\psi_v, \chi\psi_v^{-1}\}$ such that T is $\psi_{v,\mathrm{s}}$-ordinary on D. For $v \in \Sigma_\mathrm{o}$ put $\psi_{v,\mathrm{s}} = \psi_v$.*

Proof. For $z \in D^\mathrm{irr}$ and $v \in \Sigma_\mathrm{n}$, $\rho_z|_{G_v}$ is reducible, so one of $\psi_v(z)$, $\chi\psi_v(z)^{-1}$ is a subcharacter. By [Die26, Lemma 2.3], T_z is ψ_v-ordinary or $\chi\psi_v^{-1}$-ordinary. Both conditions are closed, and D^irr is irreducible. □

Definition 5.2. A closed point y of $D[1/p]$ is *good* if:

(1) for every $v \mid p$, $\psi_{v,\mathrm{s}}(y)$ is de Rham, of Hodge–Tate number h_v, and $\mathrm{HT}(\chi\psi_{v,\mathrm{s}}(y)^{-1}) \geq h_v + 2$;
(2) ρ_y is absolutely irreducible and not induced from a character of an index two subgroup of G_F.

Lemma 5.3. *Good points are Zariski dense in D.*

Proof. The characters $\psi_{v,\mathrm{s}}|_{\mathrm{Art}(1+p\mathbb{Z}_p)}$ are, up to the automorphism of Lemma 2.2, the coordinates of $D \to \mathrm{Spec}\Lambda$. Their restrictions to $\mathrm{Art}(\mathbb{Z}_p^\times)$ differ from them by a fixed character of finite order. Hence condition (1) holds at every point above the set of points of $\mathrm{Spec}\Lambda[1/p]$ at which $\psi_{v,\mathrm{s}}|_{1+p\mathbb{Z}_p}$ is $u \mapsto u^{-h_v}$ with $h_v \in \mathbb{Z}$ and $2h_v \leq \mathrm{HT}(\chi) - 2$. This set is Zariski dense in $\mathrm{Spec}\Lambda$. Since $D \to \mathrm{Spec}\Lambda$ is finite and surjective and D is irreducible of the same dimension, its inverse image is dense in D: if it were contained in a proper closed subset, the image of that subset would be closed of dimension $< 1 + d$ and would contain a dense set.

The points with ρ_z reducible form a proper closed subset. So do the induced points, since only finitely many quadratic extensions K/F unramified outside Σ occur, and $\rho_z \simeq \rho_z \otimes \eta_{K/F}$ is a closed condition on traces. It is proper because a point z satisfying (1) is not induced. Indeed, if $\rho_z \simeq \mathrm{Ind}_{G_K}^{G_F}\theta$, then $\bar\rho^{ss} \simeq \bar\rho^{ss} \otimes \eta_{K/F}$, so $\eta_{K/F} \equiv \bar\chi$ and, by (R), no place above p splits in K. For $v \mid p$, $\rho_z|_{G_v}$ is then induced from a quadratic extension of $\mathbb{Q}_p$ and is reducible, so it is $\mu \oplus \mu\eta$ with η quadratic. This contradicts the distinctness of the Hodge–Tate numbers in (1). □

5.2. **Good points are o-pro-modular.**

Proposition 5.4. *Let y be a good point. Then:*

(1) ρ_y is the Galois representation of a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, not of CM type, ordinary at every $v \mid p$;
(2) for every $v \in \Sigma_\mathrm{o}$, $\psi_v(y)$ is the classical character of $\rho_y|_{G_v}$, that is, the Jordan–Hölder character with the smaller Hodge–Tate number;
(3) y is o-pro-modular.

Proof. For $v \mid p$, $\rho_y|_{G_v}$ is an extension of $\chi\psi_{v,\mathrm{s}}(y)^{-1}$ by $\psi_{v,\mathrm{s}}(y)$, by Lemma 5.1 and [Die26, Lemma 2.3]. It is de Rham by [Die26, Lemma 2.1], since both characters are de Rham and the subcharacter has the smaller Hodge–Tate number. This proves (2), as $\psi_{v,\mathrm{s}} = \psi_v$ for $v \in \Sigma_\mathrm{o}$. By [Pan22, Theorem 5.1.2(2)] (see [Zha25, Theorem 4.2.3] for $p = 3$), which allows $\rho_y|_{G_v}$ to be split, ρ_y comes from a twist of a Hilbert modular form. This gives π as in (1), and π is not of CM type since ρ_y is not induced.

For (3), choose at each $v \in \Sigma_{\mathrm{o}}$ the nearly ordinary refinement of π_v corresponding to the Jordan–Hölder character of smaller Hodge–Tate number, using the weight-normalized torus operators at sufficiently deep p-power level [Pan22, §5.2.1]. By ordinary local–global compatibility [Pan22, §5.2.2], [Wil88, Theorem 2], the full torus eigencharacter, including its value on a uniformizer, is $\psi_v(y)$; the distinct Hodge–Tate numbers select this character even when $\rho_y|_{G_v}$ is split, and the argument applies to ramified as well as unramified local characters. By local–global compatibility at the places of $\Sigma \setminus \Sigma_p$, π_v has nonzero vectors under the Iwahori subgroup with character ξ'_v, for some $\xi' \in \Xi$. As d is even, the Jacquet–Langlands correspondence gives automorphic forms on $B^\times$. Their ordinary stabilization at the places of Σ_{o} gives, by Hida theory [Hid89], [Pan22, §5.2] and the comparison with torsion ordinary parts of Proposition 6.1 (see also [Die26, Lemma 3.3]), an eigenvector in $S^{\mathrm{o}}_{\xi'} \otimes E$ with Hecke eigenvalues $\operatorname{tr}\rho_y(\mathrm{Frob}_w)$ and diamond characters $\psi_v(y)$, $v \in \Sigma_{\mathrm{o}}$. At the places of Σ_{n} it contributes locally algebraic vectors [Pan22, 3.2.10]. Hence y is o-pro-modular. □

5.3. **Good points are regular.**

Proposition 5.5. *Let y be a good point and $\mathfrak{p}_y$ the corresponding prime of R^{o}. Then the local ring $R^{\mathrm{o}}_{\mathfrak{p}_y}$ is regular of dimension $2d - |\Sigma_{\mathrm{o}}|$. Consequently C is the unique irreducible component of* $\mathrm{Spec}R^{\mathrm{o}}$ *containing y, and* $\dim C = 1 + 2d - |\Sigma_{\mathrm{o}}|$.

Proof. Let $k = k(y)$ and $V = \mathrm{ad}^0\rho_y$. Since $y \in C$, $\dim R^{\mathrm{o}}_{\mathfrak{p}_y} \geq \dim C - 1 \geq 2d - |\Sigma_{\mathrm{o}}|$, as C is the spectrum of an $\mathcal{O}$-flat complete local domain and $\mathfrak{p}_y$ has coheight 1. It therefore suffices to show that the tangent space t_y of $R^{\mathrm{o}}_{\mathfrak{p}_y}$, a k-vector space, has dimension at most $2d - |\Sigma_{\mathrm{o}}|$. A regularity statement of the same kind at ordinary regular automorphic points, also based on [NT23], is [Zha26, Corollary 5.3.9 and Lemma 5.3.11].

Step 1: t_y as a Selmer group. Elements of t_y are continuous homomorphisms $R^{\mathrm{o}} \to k[\epsilon]$ lifting y (cf. [Kis09b, §2.3]), that is, pseudo-representations $G_{F,\Sigma} \to k[\epsilon]$ with determinant χ lifting $\operatorname{tr}\rho_y$ and satisfying the conditions of Definition 2.1. As ρ_y is absolutely irreducible, these are traces of deformations of ρ_y to $k[\epsilon]$, unique up to conjugation [BC09, Theorem 1.4.4]. For $v \in \Sigma_{\mathrm{o}}$, the ψ_v-ordinarity of the trace is equivalent, at the irreducible prime $\mathfrak{p}_y$, to the existence of a G_v-stable direct summand of rank one with character lifting $\psi_v(y)$ [Pan22, Lemma 5.3.3 and 5.3.8(2)]. Hence t_y is contained in the Selmer group $H^1_{\mathcal{L}}(F, V)$ of classes unramified outside Σ with local conditions:

- $\mathcal{L}_v = \mathrm{im}\big(H^1(G_v, \mathfrak{b}^0_v) \to H^1(G_v, V)\big)$ for $v \in \Sigma_{\mathrm{o}}$, where $\mathfrak{b}^0_v \subset V$ is the stabilizer of the $\psi_v(y)$-line;
- $\mathcal{L}_v = H^1(G_v, V)$ for $v \in \Sigma \setminus \Sigma_{\mathrm{o}}$.

Step 2: local terms. Let $v \mid p$ and write $\rho_y|_{G_v}$ as an extension of $\psi' = \chi\psi_{v,\mathrm{s}}(y)^{-1}$ by $\psi = \psi_{v,\mathrm{s}}(y)$, and put $m = \mathrm{HT}(\psi') - \mathrm{HT}(\psi) \geq 2$ (Definition 5.2(1)). The Jordan–Hölder factors of $V(1)$ have Hodge–Tate numbers -1, $-m-1$ and $m-1$, all non-zero, so $H^0(G_v, V(1)) = 0$. For $v \in \Sigma_{\mathrm{n}}$ this gives $h^1(G_v, V) - h^0(G_v, V) = 3$. For $v \in \Sigma_{\mathrm{o}}$, $Q_v = V/\mathfrak{b}^0_v \simeq \psi'\psi^{-1}$ has Hodge–Tate number $m > 0$, so $H^0(G_v, Q_v) = 0$, $\mathcal{L}_v \simeq H^1(G_v, \mathfrak{b}^0_v)$ and $H^0(G_v, \mathfrak{b}^0_v) = H^0(G_v, V)$. The factors of $(\mathfrak{b}^0_v)^\vee(1)$ have Hodge–Tate numbers -1 and $m-1$, so $H^2(G_v, \mathfrak{b}^0_v) = 0$, and $\dim \mathcal{L}_v - h^0(G_v, V) = 2$ by the local Euler characteristic formula. Moreover $D_{\mathrm{dR}}(Q_v)/\mathrm{Fil}^0 = 0$, so $H^1_f(G_v, Q_v) = 0$ [BK90, Corollary 3.8.4], and by functoriality $H^1_f(G_v, V) \subset \ker\big(H^1(G_v, V) \to H^1(G_v, Q_v)\big) = \mathcal{L}_v$.

Let $v \in \Sigma \setminus \Sigma_p$. If $\xi_v \neq 1$, then $\rho_y|_{G_v} \simeq \mu_1 \oplus \mu_2$ with $\mu_1\mu_2^{-1}$ ramified, so the factors ε, $\varepsilon\mu_1\mu_2^{-1}$, $\varepsilon\mu_2\mu_1^{-1}$ of $V(1)$ are non-trivial and $H^0(G_v, V(1)) = 0$. If $v \in \Sigma_{\mathrm{St}}$, then $H^0(G_v, V(1)) = 0$ by Lemma 4.1(2). In both cases $h^1(G_v, V) - h^0(G_v, V) = h^2(G_v, V) = 0$.

Step 3: the global estimate. Let $\mathcal{L}^\perp_v \subset H^1(G_v, V(1))$ be the annihilators under local Tate duality, and $H^1_{\mathcal{L}^\perp}(F, V(1))$ the dual Selmer group, with unramified conditions at the places outside Σ. By Step 2, $\mathcal{L}^\perp_v = 0$ for $v \in \Sigma \setminus \Sigma_{\mathrm{o}}$, and $\mathcal{L}^\perp_v \subset H^1_f(G_v, V)^\perp = H^1_f(G_v, V(1))$ for $v \in \Sigma_{\mathrm{o}}$ [BK90,

Proposition 3.8]. Hence $H^1_{\mathcal{L}^\perp}(F,V(1)) \subset H^1_f(F,V(1))$. By the Greenberg–Wiles formula [DDT97, Theorem 2.19], using $H^0(F,V) = H^0(F,V(1)) = 0$ and $h^0(G_v,V) = 1$ for the d real places v,

$$h^1_{\mathcal{L}}(F,V) - h^1_{\mathcal{L}^\perp}(F,V(1)) = \sum_{v\in\Sigma\setminus\Sigma_\infty} \big(\dim \mathcal{L}_v - h^0(G_v,V)\big) - d = 2|\Sigma_{\mathrm{o}}| + 3|\Sigma_{\mathrm{n}}| - d = 2d - |\Sigma_{\mathrm{o}}|.$$

The same formula, applied to the Bloch–Kato conditions, which are mutually orthogonal, gives

$$h^1_f(F,V) - h^1_f(F,V(1)) = \sum_{v\mid p} \dim D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 - d = 0,$$

using $\dim H^1_f(G_v,V) = h^0(G_v,V) + \dim D_{\mathrm{dR}}(V)/\mathrm{Fil}^0$ for $v \mid p$ and $\dim H^1_f(G_v,V) = h^0(G_v,V)$ for $v \nmid p$ [BK90, Corollary 3.8.4], and $\dim D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 = 1$ for $v \mid p$, since the Hodge–Tate weights of $\rho_y|_{G_v}$ are distinct. By [NT23, Theorem B], $H^1_f(F,\mathrm{ad}\rho_y) = 0$, as π is regular algebraic and not of CM type. Hence $H^1_f(F,V) = 0$, so $H^1_f(F,V(1)) = 0$ and $H^1_{\mathcal{L}^\perp}(F,V(1)) = 0$. Therefore $\dim t_y \le h^1_{\mathcal{L}}(F,V) = 2d - |\Sigma_{\mathrm{o}}|$.

Conclusion. $R^{\mathrm{o}}_{\mathfrak{p}_y}$ is a regular local ring, hence a domain, so exactly one minimal prime $\mathfrak{q}$ of R^{o} is contained in $\mathfrak{p}_y$. Since $y \in C$, $C = \mathrm{Spec} R^{\mathrm{o}}/\mathfrak{q}$. As $R^{\mathrm{o}}/\mathfrak{q}$ is a complete local domain, it is catenary, and $\dim C = \dim R^{\mathrm{o}}_{\mathfrak{p}_y} + 1 = 1 + 2d - |\Sigma_{\mathrm{o}}|$. □

6. The dimension of partially ordinary Hecke algebras

Proposition 6.1. *Let $\xi' \in \Xi$. Every minimal prime $\mathfrak{p}$ of $\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$ has characteristic 0, and $\dim \mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}/\mathfrak{p} \ge 1 + 2d - |\Sigma_{\mathrm{o}}|$. Moreover $S^{\mathrm{o}}_{\xi'}/\varpi^a \simeq \mathrm{Ord}_P(S_{\xi'}/\varpi^a)$ for every $a \ge 1$, and every closed point $\mathfrak{P}$ of $\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}[1/p]$ occurs: $(S^{\mathrm{o}}_{\xi'} \otimes E)[\mathfrak{P}] \ne 0$, after enlarging E.*

Proof. This is [Pan22, Theorem 3.6.1], which gives the bound $1 + 2d$ for the full Hecke algebra, with completed cohomology replaced by its ordinary part at Σ_{o}; we give the details of the changes. Write $n = |\Sigma_{\mathrm{n}}|$, $o = |\Sigma_{\mathrm{o}}|$, $\mathbb{T} = \mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$ and $M = (S^{\mathrm{o}}_{\xi'} \otimes E/\mathcal{O})^\vee$, localized at $\mathfrak{m}'$. For $p = 3$, one uses [Zha25, Theorem 3.3.3 and Corollary 3.3.10] instead of [Pan22, Theorems 3.5.5 and 3.6.1], which applies since the blocks at the places of Σ_{n} are generic by (R) and (a).

Freeness and reduction. Put $V = S_{\psi,\xi'}(U^p, E/\mathcal{O})_{\mathfrak{m}}$ and $W = \mathrm{Ord}_P V$. Choose sufficiently small compact open subgroups at p, with an Iwahori decomposition at the places of Σ_{o}; the upper unipotent subgroup N_0 may be taken as small as necessary without changing Ord_P [Eme10, Lemma 3.1.11]. After a continuous determinant twist trivializing the central character on the compact centres (it exists since p is odd, commutes with Ord_P and changes each ordinary operator by a unit), and identifying the compact quotients by the centres with determinant-one complements via $g \mapsto (\det g)^{-1/2} g$, the compact group has an Iwahori decomposition $I = \bar{N}_* H N_0$ with

$$H = (K_{\mathrm{n}}/Z_{\mathrm{n}}) \times (T_{\mathrm{o}}/Z_{\mathrm{o}}), \qquad \dim H = 3n + o,$$

where K_{n} is compact open in $\prod_{v\in\Sigma_{\mathrm{n}}} \mathrm{GL}_2(\mathbb{Q}_p)$, T_{o} is a compact open subgroup of the torus at Σ_{o}, $Z_{\mathrm{n}}, Z_{\mathrm{o}}$ are their scalar subgroups, and H is chosen uniform pro-p. By [Pan22, Proposition 3.2.4 and §3.7], $V|_I$ is a direct summand of $C^\infty(I, E/\mathcal{O})^r$. As a right HN_0-space $I = \bar{N}_* \times HN_0$, so this module is smoothly coinduced from the divisible, hence injective, $\mathcal{O}$-module $C^\infty(\bar{N}_*, E/\mathcal{O})$; hence $V|_{HN_0}$ is injective, and V^{N_0} is injective over H, since taking N_0-invariants is right adjoint to the exact inflation from H. The finite modules $V[\varpi^a]^{\bar{N}_i H_j N_0}$ exhaust V^{N_0} and are stable under the commuting operators U_v, $v \in \Sigma_{\mathrm{o}}$ [Eme10, Lemma 3.3.2], which commute with H; the Fitting projections $\lim_k U_v^{k!}$ are compatible, and their product is an H-equivariant idempotent e of V^{N_0} whose image is the canonical image of W [Eme10, Lemmas 3.1.5–3.1.7 and Theorem 3.3.3]. Hence W is an injective direct summand of V^{N_0}. By admissibility $W^\vee$ is finitely generated over $\mathcal{O}[[H]]$,

and by injectivity it is projective, hence free, as $\mathcal{O}[[H]]$ is local. In particular W is divisible, so left exactness and $S^{\mathrm{o}}_{\xi'} = T_{\varpi}W$ [Eme10, Definition 3.4.1] give

$$S^{\mathrm{o}}_{\xi'}/\varpi^a \simeq W[\varpi^a] = \mathrm{Ord}_P(S_{\xi'}/\varpi^a).$$

Localization at $\mathfrak{m}'$ is a direct summand, so $M/\varpi \simeq \Lambda'^{\oplus r}$ with $\Lambda' = \mathbb{F}[[H]]$.

The relative block module. Let $\mathcal{B}_{\mathrm{n}}$ be the product of the blocks of $\prod_{v\in\Sigma_{\mathrm{n}}}\mathrm{GL}_2(\mathbb{Q}_p)$ attached to the residual pseudo-representation, with projective generator $P_{\mathcal{B}_{\mathrm{n}}}$ and endomorphism ring $E_{\mathcal{B}_{\mathrm{n}}}$, and put $\mathbf{m}^{\mathrm{o}} = \mathrm{Hom}(P_{\mathcal{B}_{\mathrm{n}}}, M)$, so that $M \simeq \mathbf{m}^{\mathrm{o}}\widehat{\otimes}_{E_{\mathcal{B}_{\mathrm{n}}}}P_{\mathcal{B}_{\mathrm{n}}}$, the equivalence being exact, faithful and compatible with the torus action. By admissibility, applied after quotienting by the augmentation ideal of the torus, and topological Nakayama, $\mathbf{m}^{\mathrm{o}}$ is finite over $E_{\mathcal{B}_{\mathrm{n}}}\widehat{\otimes}\mathcal{O}[[T_{\mathrm{o}}/Z_{\mathrm{o}}]]$ (the relative form of [Zha25, Proposition A.1.4 and Corollary 3.3.7]), hence over

$$A_0 = \Big(\widehat{\bigotimes}_{v\in\Sigma_{\mathrm{n}}} R_v^{\mathrm{ps}}\Big)\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[T_{\mathrm{o}}/Z_{\mathrm{o}}]],$$

since $E_{\mathcal{B}_{\mathrm{n}}}$ is finite over its centre. The local centres at Σ_{n} commute with N_0, the U_v and their projectors, so local–global compatibility passes to ordinary parts [Die26, Lemma 3.5], and the action of A_0 factors through $\mathbb{T}$. As $\mathbf{m}^{\mathrm{o}}$ is faithful over $\mathbb{T}$, we get $\mathbb{T}\hookrightarrow \mathrm{End}_{A_0}(\mathbf{m}^{\mathrm{o}})$, so $\mathbb{T}$ is finite over the noetherian ring A_0. Since $\mathbf{m}^{\mathrm{o}}$ is $\mathcal{O}$-torsion free, ϖ is a non-zero-divisor on $\mathbb{T}$, and the minimal primes of $\mathbb{T}$ have characteristic 0.

The dimension estimate. Put $A = A_0/\varpi$. Pan's dimension formula [Pan22, Proposition 3.6.12], with the torus variables kept in the central coefficient ring, gives

$$\dim_{\Lambda'}(N\widehat{\otimes}_{E_{\mathcal{B}_{\mathrm{n}}}}P_{\mathcal{B}_{\mathrm{n}}}) \le \dim_A N + n$$

for finitely generated modules N over $(E_{\mathcal{B}_{\mathrm{n}}}/\varpi)\widehat{\otimes}\mathbb{F}[[T_{\mathrm{o}}/Z_{\mathrm{o}}]]$, with the same proof [Pan22, Lemmas 3.6.8–3.6.11]: if $\dim_A N = 0$, the torus acts through a finite quotient and $N\widehat{\otimes}P_{\mathcal{B}_{\mathrm{n}}}$ is dual to a smooth representation of finite length of $\prod_{v\in\Sigma_{\mathrm{n}}}\mathrm{GL}_2(\mathbb{Q}_p)$, of Gelfand–Kirillov dimension at most n [Pan22, Lemma 3.6.14], and the induction on a central parameter is unchanged. Let $\mathfrak{p}$ be a minimal prime of $\mathbb{T}$. Then $\mathbf{m}^{\mathrm{o}}[\mathfrak{p}] \neq 0$, and it is ϖ-saturated by torsion-freeness, so $N = \mathbf{m}^{\mathrm{o}}[\mathfrak{p}]/\varpi \hookrightarrow \mathbf{m}^{\mathrm{o}}/\varpi$ is non-zero. Its image $N\widehat{\otimes}P_{\mathcal{B}_{\mathrm{n}}}$ is a non-zero submodule of $M/\varpi \simeq \Lambda'^{\oplus r}$; as Λ' is a domain of dimension $3n+o$, it has dimension $3n+o$, and so $\dim_A N \ge 2n+o$. Since A acts on N through $\mathbb{T}/(\mathfrak{p},\varpi)$,

$$\dim \mathbb{T}/\mathfrak{p} = 1 + \dim \mathbb{T}/(\mathfrak{p},\varpi) \ge 1 + 2n + o = 1 + 2d - o.$$

Occurrence. Let $\mathfrak{P}$ be a closed point of $\mathbb{T}[1/p]$ and $\mathfrak{q}$ its contraction to $\mathbb{T}$. As $\mathbf{m}^{\mathrm{o}}$ is finite and faithful, $\mathbf{m}^{\mathrm{o}}\otimes_{\mathbb{T}} k(\mathfrak{P}) \neq 0$, and $\mathbf{m}^{\mathrm{o}}/\mathfrak{q}\mathbf{m}^{\mathrm{o}}$ is finite over $\mathcal{O}$. After enlarging $\mathcal{O}$, its quotient by its (bounded) $\mathcal{O}$-torsion has a non-zero torsion-free quotient Q on which $\mathbb{T}$ acts through $\mathfrak{P}$. By the block equivalence, $Q\widehat{\otimes}P_{\mathcal{B}_{\mathrm{n}}}$ is a non-zero compact torsion-free quotient of $M/\mathfrak{q}M$, and its continuous dual gives $(S^{\mathrm{o}}_{\xi'}\otimes E)[\mathfrak{P}] \neq 0$. □

7. Proof of Theorem A

We keep the setting of Section 2 and the component C of Proposition 3.3.

By Lemma 5.3 there is a good point $y \in D \subset C$. By Proposition 5.4(3), y is the image of a prime of $\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$ for some $\xi' \in \Xi$. Let W be an irreducible component of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$ containing it, viewed as a closed subset of $\mathrm{Spec}R^{\mathrm{o}}$ through the surjection $R^{\mathrm{o}} \twoheadrightarrow \mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$. Then $W = V(\mathfrak{p}_W)$, where $\mathfrak{p}_W$ is the inverse image in R^{o} of a minimal prime of $\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}$; it is an irreducible closed subset of $\mathrm{Spec}R^{\mathrm{o}}$ containing y, though a priori not an irreducible component of it. Let $\mathfrak{q}' \subset \mathfrak{p}_W$ be a minimal prime of R^{o}. Since $y \in W$, we have $\mathfrak{q}' \subset \mathfrak{p}_W \subset \mathfrak{p}_y$. By Proposition 5.5, $\mathfrak{q}$ is the unique minimal prime of R^{o} contained in $\mathfrak{p}_y$, so $\mathfrak{q}' = \mathfrak{q}$ and $W \subset C$. By Propositions 6.1 and 5.5,

$$\dim W \ge 1 + 2d - |\Sigma_{\mathrm{o}}| = \dim C,$$

so $W = C$. In particular $x \in W$: there is a maximal ideal $\mathfrak{P}$ of $\mathbb{T}^{\mathrm{o}}_{\xi',\mathfrak{m}'}[1/p]$ above x, with $\psi_{v,1}(\mathfrak{P}) = \psi_v(x) = \psi_v^{\mathrm{cl}}$ for $v \in \Sigma_{\mathrm{o}}$. Let $\mathfrak{p}$ be the corresponding maximal ideal of the big Hecke algebra and $\mu = \bigotimes_{v\in\Sigma_{\mathrm{o}}} \mu_v$ the character of the torus at Σ_{o} given by $\mathfrak{P}$, so that $\mu_{v,1} \circ \mathrm{Art}^{-1} = \psi_v^{\mathrm{cl}}$. By Proposition 6.1, $(S^{\mathrm{o}}_{\xi'} \otimes E)[\mathfrak{P}] \neq 0$, and since Ord_P is left exact this says that $\mathrm{Ord}_P((S_{\xi'} \otimes E)[\mathfrak{p}])[T_{\mathrm{o}} = \mu] \neq 0$. Then $\rho(\mathfrak{p}) \simeq \rho$ is irreducible, and the hypotheses of [Die26, Theorem 4.2], including this occurrence of μ, hold by (N) and (O). Hence ρ is modular over the field of §2.1, and over the original field by solvable descent. This proves Theorem A. □

8. Remarks

Remark 8.1. The essential use of condition (a) is Lemma 3.1, and through it the equality $\dim D = 1 + d$. In Proposition 3.3, (a) also gives the vanishing of $H^0(G_{F,\Sigma}, \mathrm{ad}^0\bar{\rho}_b(1))$ and the smoothness of the local rings, but there one could argue as in [Zha25, Lemma 5.4.1] with weaker information. If $\bar{\chi}|_{G_{\mathbb{Q}_p}} = \omega^{\pm 1}$ and $p > 3$, the reducible locus at a place of Σ_{n} is cut out by two equations: $R_v^{\mathrm{ps}} \simeq \mathcal{O}[[x_0, x_1, y_0, y_1]]/(x_0y_1 - x_1y_0)$ and the reducible quotient is $R_v^{\mathrm{ps}}/(x_0, x_1)$ [Pan22, Lemma 7.4.4]. Then D only has dimension at least $1 + |\Sigma_{\mathrm{o}}|$. It lies in the fully ordinary locus, of dimension at most $1 + d$, but its map to $\mathrm{Spec}\Lambda$ need not be dominant, so the density argument for de Rham points of Section 5 does not apply.

Remark 8.2. Since $\bar{\rho}^{ss}|_{G_v} \simeq \bar{\chi}_1|_{G_v} \oplus \bar{\chi}_2|_{G_v}$ for every $v \mid p$, condition (a) can be checked at a single place above p, of Σ_{n} or of Σ_{o}, from the local behaviour of ρ there. It suffices that the ratio of the two residual characters restricted to the inertia group I_v is not $\omega^{\pm 1}$. Condition (a) excludes some natural cases. If ρ_v is crystalline with consecutive Hodge–Tate weights, the ratio is $\omega^{\pm 1}$ on inertia, and (a) can only hold through the unramified parts of the two characters. If ρ_v is of Steinberg type with consecutive Hodge–Tate weights, then $\bar{\rho}_v^{ss}$ is a twist of $1 \oplus \omega$, and (a) fails. On the other hand, there are cases in which (a) can be verified directly.

- *Fontaine–Laffaille case.* Assume that ρ_v is crystalline with Hodge–Tate weights $\{0, k-1\}$, twisted by a character, with $2 \leq k \leq p-1$. By Fontaine–Laffaille theory [FL82], if $\bar{\rho}_v$ is reducible, then the characters of $\bar{\rho}_v^{ss}|_{I_v}$ are, up to a common twist, 1 and ω^{k-1}. Hence (a) holds as soon as $k - 1 \not\equiv \pm 1 \pmod{p-1}$, that is, for $3 \leq k \leq p-2$.
- *Principal series types.* Assume that ρ_v is potentially crystalline of tame principal series type, with Hodge–Tate weights $\{0, k-1\}$ up to twist, becoming crystalline over a tamely ramified extension of ramification index e with $e(k-1) < p-1$. Caruso's description of torsion semistable representations in the range $er < p-1$ [Car06] can then be combined with the descent data of the given type to determine the two residual inertial characters. The bound alone does not exclude consecutive exponents: one must check, for the particular type, that their ratio is not $\omega^{\pm 1}$. A check of this kind was used in [Die12, §4] to verify the corresponding hypothesis in Kisin's modularity lifting theorem [Kis09a], which at that time was only available under this condition.

Remark 8.3. The argument identifies C with a component of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}$ by means of a single regular point. This is the strategy of Böckle [Böc01] and, in a related form, of Khare–Wintenberger: a lower bound for the dimension of the Hecke algebra, together with the regularity of the deformation ring at one modular point, replaces an $R = \mathbb{T}$ theorem. In the residually reducible case it was used by Deo [Deo23] over $\mathbb{Q}$, with Pan's ordinary finiteness [Pan22, Theorem 5.1.1] providing the modular points, and by Zhang [Zha26], with Newton–Thorne [NT23] providing the regularity; steps (2)–(4) follow the same pattern. Here the lower bound comes from Pan's Gelfand–Kirillov dimension argument, adapted to the partially ordinary Hecke algebra in Proposition 6.1, and the regular point comes from the fully ordinary subfamily D. At these points the ordinary character carried by the Hecke algebra is the classical one even when the local representation

is split (Proposition 5.4). This is why no non-split points are needed, in contrast with the density argument based on [Die26, Theorem A].

Remark 8.4. In [Die26, Theorem A], the non-split hypothesis is used to pass from pro-modularity for the big Hecke algebra to o-pro-modularity at x, by Emerton's adjunction [Die26, Proposition 4.1]. Here o-pro-modularity is obtained directly for the whole component C, and no pro-modularity theorem for the big Hecke algebra is used.

Remark 8.5. When $\Sigma_{\mathrm{o}} = \emptyset$, that is, when ρ_v is irreducible for every $v \mid p$, the argument above is essentially that of Zhang [Zha26, Theorem 5.5.1]. Under the condition $\bar{\chi}|_{G_{F_v}} \neq \omega^{\pm 1}$ for $v \mid p$, he identifies the union of the irreducible components of the pseudo-deformation ring of dimension at least $1 + 2d$ with the reduced big Hecke algebra of completed cohomology. Combined with the classicality criterion ([Die26, Theorem 4.2] with no ordinary places, which rests on [CDP14]), this recovers Pan's theorem [Pan22, Theorem 7.1.1] under condition (a), with the $R = \mathbb{T}$ theorem at nice primes [Pan22, §4] replaced by the theorem of Newton–Thorne. In that case R^{o} is the pseudo-deformation ring with the tame conditions, $\mathbb{T}^{\mathrm{o}}$ is the big Hecke algebra, Proposition 6.1 is [Pan22, Theorem 3.6.1], Proposition 3.3 is replaced by [Pan22, Lemma 7.1.3], and the cut of Section 5 is [Pan22, Lemma 7.2.1 and Corollary 7.2.3]. Condition (a) is essential here as well: it is Pan's generic case [Pan22, §7.2], and the method does not reach the case $\bar{\chi}|_{G_{\mathbb{Q}_p}} = \omega^{\pm 1}$ of [Pan22, §7.4] (Remark 8.1). The contribution of the present paper is the partially ordinary setting $\Sigma_{\mathrm{o}} \neq \emptyset$.

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