Noncommutative von Neumann-Riesz and von Neumann-Riesz-Lorch Mean Ergodic Theorems

K. Mahesh Krishna
School of Mathematics and Natural Sciences
Chanakya University Global Campus
NH-648, Haraluru Village
Devanahalli Taluk, Bengaluru North District
Karnataka State, 562 110, India
Email: kmaheshak@gmail.com
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Abstract: We derive noncommutative von Neumann-Riesz and von Neumann-Riesz-Lorch mean ergodic theorems for adjointable contractions on a Hilbert C*-module. We crucially use a result obtained by Paschke [*Trans. Amer. Math. Soc., 1973*]. We also note that the exact version of the noncommutative von Neumann–Riesz mean ergodic theorem fails.
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1. Introduction

In 1931, Koopman showed that the *quasi-ergodic hypothesis* of classical Hamiltonian dynamics can be interpreted in Hilbert spaces [10]. Motivated by this, in 1932, von Neumann proved the following mean ergodic theorem for unitary operators on Hilbert spaces [26].

Theorem 1.1. *[2–7, 11, 13, 15, 21, 24, 26, 27] (**von Neumann Mean Ergodic Theorem**) Let $\mathcal{H}$ be a Hilbert space. Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Then for every $h \in \mathcal{H}$, the mean ergodic sequence*

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} U^k h\right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{H}$.

In 1938, Riesz extended Theorem 1.1 to contractions on Hilbert spaces [18].

Theorem 1.2. *[2, 3, 7, 11, 15, 17–20, 24] (**von Neumann-Riesz Mean Ergodic Theorem**) Let $\mathcal{H}$ be a Hilbert space. Let $T : \mathcal{H} \to \mathcal{H}$ be a contraction. Then for every $h \in \mathcal{H}$, the mean ergodic sequence*

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k h\right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{H}$.

In 1939, Birkhoff extended Theorem 1.2 to uniformly convex Banach spaces [1]. In 1939, Lorch derived following result for power bounded operators on Hilbert spaces [12].

Theorem 1.3. *[12] (**von Neumann-Riesz-Lorch Mean Ergodic Theorem**) Let $\mathcal{H}$ be a Hilbert space. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded operator such that*

$$\sup_{n\in\mathbb{N}} \|T^n\| < \infty.$$

Then for every $h \in \mathcal{H}$, the mean ergodic sequence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k h\right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{H}$.

In 1953, Sz.-Nagy gave a new proof of Theorem 1.2 using his unitary dilation of contraction [25].
We are fundamentally concerned with the question: What is the noncommutative analogue of Theorem 1.2? For this, we need the noncommutative version of Hilbert spaces known as Hilbert C*-modules which were first introduced by Kaplansky [9] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [14] and Rieffel [16].

Definition 1.4. *[9, 14, 16] Let $\mathcal{A}$ be a unital C*-algebra. A right module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (right) **Hilbert C*-module** if there exists a map $\langle\cdot,\cdot\rangle : \mathcal{E}\times\mathcal{E} \to \mathcal{A}$ such that the following hold.*

(i) $\langle x, x\rangle \geq 0$, $\forall x \in \mathcal{E}$. *If $x \in \mathcal{E}$ satisfies $\langle x, x\rangle = 0$, then $x = 0$.*
(ii) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$, $\forall x, y, z \in \mathcal{E}$.
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$, $\forall x, y \in \mathcal{E}$, $\forall a \in \mathcal{A}$.
(iv) $\langle x, y\rangle = \langle y, x\rangle^*$, $\forall x, y \in \mathcal{E}$.
(v) *$\mathcal{E}$ is complete w.r.t. the norm $\|x\| := \sqrt{\|\langle x, x\rangle\|}$, $\forall x \in \mathcal{E}$.*

2. Noncommutative von Neumann-Riesz and von Neumann-Riesz-Lorch Mean Ergodic Theorems

We need the following important result of Paschke [14].

Theorem 2.1. *[14] Let $\mathcal{E}$ be a Hilbert C*-module. Let $T : \mathcal{E} \to \mathcal{E}$ be a bounded adjointable morphism. Then*

$$\langle Tx, Tx\rangle \leq \|T\|\langle x, x\rangle, \quad \forall x \in \mathcal{E}.$$

Following is a noncommutative version of Theorem 1.2.

Theorem 2.2. *(**Noncommutative von Neumann-Riesz Mean Ergodic Theorem**) Let $\mathcal{E}$ be a Hilbert C*-module. Let $T : \mathcal{E} \to \mathcal{E}$ be an adjointable contraction. Then for every*

$$x \in \overline{(I_{\mathcal{E}} - T)(\mathcal{E})} \oplus \ker(I_{\mathcal{E}} - T^*),$$

the mean ergodic sequence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{E}$.

Proof. Our proof is motivated from the arguments by Riesz for the Hilbert space [2,3,7,11,15,24]. Define

$$S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k, \quad \forall n \in \mathbb{N}.$$

Case 1: Let $x \in \ker(I_{\mathcal{E}} - T^*)$. Then $T^*x = x$. Since T is a contraction, using Theorem 2.1, we get

$$\begin{aligned}\langle x - Tx, x - Tx\rangle &= \langle x, x\rangle - \langle T^*x, x\rangle - \langle x, T^*x\rangle + \langle Tx, Tx\rangle \\ &= \langle x, x\rangle - \langle x, x\rangle - \langle x, x\rangle + \langle Tx, Tx\rangle \\ &= -\langle x, x\rangle + \langle Tx, Tx\rangle \\ &\leq -\langle x, x\rangle + \|T\|\langle x, x\rangle \\ &\leq -\langle x, x\rangle + \langle x, x\rangle \\ &\leq 0.\end{aligned}$$

Therefore $Tx = x$. So we have $S_n x = nx/n = x$ for all n. Therefore $\{S_n x\}_{n=1}^n$ converges.

Case 2: Let $x \in (I_{\mathcal{E}} - T)(\mathcal{E})$. Choose $y \in \mathcal{E}$ such that $x = y - Ty$. We then have

$$\begin{aligned}S_n x &= \frac{1}{n}\sum_{k=0}^{n-1} T^k x \\ &= \frac{1}{n}\sum_{k=0}^{n-1} T^k (y - Ty) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} T^k y - \frac{1}{n}\sum_{k=0}^{n-1} T^{k+1} y \\ &= \frac{1}{n}(y - T^n y), \quad \forall n \in \mathbb{N}.\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} \|S_n x\| &= \lim_{n\to\infty} \frac{1}{n}\|y - T^n y\| \\ &\leq \lim_{n\to\infty} \frac{1}{n}\|I_{\mathcal{E}} - T^n\|\|y\| \\ &\leq \lim_{n\to\infty} \frac{1}{n}(\|I_{\mathcal{E}}\| + \|T^n\|)\|y\| \\ &\leq \lim_{n\to\infty} \frac{1}{n}(\|I_{\mathcal{E}}\| + \|T\|^n)\|y\| \\ &\leq 2\lim_{n\to\infty} \frac{1}{n}\|y\| = 0.\end{aligned}$$

Case 3: Let $x \in \overline{(I_{\mathcal{E}} - T)(\mathcal{E})} \setminus (I_{\mathcal{E}} - T)(\mathcal{E})$. Let $\varepsilon > 0$. Choose $y \in (I_{\mathcal{E}} - T)(\mathcal{E})$ such that $\|y - x\| < \varepsilon$. Let $z \in \mathcal{E}$ be such that $y = (I_{\mathcal{E}} - T)z$. Since $\{S_n y\}_{n=1}^{\infty}$ converges to zero, there exists N such that $\|S_n y\| < \varepsilon$ for all $n \geq N$. Then

$$\begin{aligned}\|S_n x\| &\leq \|S_n x - S_n y\| + \|S_n y\| \\ &\leq \|S_n\|\|x - y\| + \|S_n y\| \\ &\leq \|x - y\| + \varepsilon \\ &< 2\varepsilon, \quad \forall n \geq N.\end{aligned}$$

Therefore $\lim_{n\to\infty} \|S_n x\| = 0$.

□

Corollary 2.3. *Let $\mathcal{E}$ be a Hilbert C*-module. Let $T : \mathcal{E} \to \mathcal{E}$ be an adjointable contraction. For a given $x \in \mathcal{E}$,*

$$Tx = x \iff T^*x = x.$$

In particular,

$$\text{Fix}(T) \coloneqq \{x \in \mathcal{E} : Tx = x\} = \{x \in \mathcal{E} : T^*x = x\} = \text{Fix}(T^*).$$

Corollary 2.4. *Let $\mathcal{E}$ be a Hilbert C*-module. Let $T : \mathcal{E} \to \mathcal{E}$ be an adjointable contraction such that $\overline{(I_{\mathcal{E}} - T)(\mathcal{E})}$ or $\ker(I_{\mathcal{E}} - T^*)$ is orthogonally complementable in $\mathcal{E}$. Then for every $x \in \mathcal{E}$, the mean ergodic sequence*

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{E}$.

Proof. $\mathcal{E} = \overline{(I_{\mathcal{E}} - T)(\mathcal{E})} \oplus \ker(I_{\mathcal{E}} - T^*)$. □

A careful inspection of the proof of Theorem 2.2 gives the following result for contractions which need not be adjointable.

Theorem 2.5. *(**Noncommutative von Neumann-Riesz Mean Ergodic Theorem**) Let $\mathcal{E}$ be a Hilbert C*-module. Let $T : \mathcal{E} \to \mathcal{E}$ be a contraction (need not be adjointable). Then for every*

$$x \in \overline{(I_{\mathcal{E}} - T)(\mathcal{E})} \oplus \ker(I_{\mathcal{E}} - T),$$

the mean ergodic sequence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{E}$.

By seeing the proof of Theorem 2.2 we also get following result for power bounded morphisms.

Theorem 2.6. *(**Noncommutative von Neumann-Riesz-Lorch Mean Ergodic Theorem**) Let $\mathcal{E}$ be a Hilbert C*-module. Let $T : \mathcal{E} \to \mathcal{E}$ be a bounded morphism such that*

$$\sup_{n\in\mathbb{N}} \|T^n\| < \infty.$$

Then for every

$$x \in \overline{(I_{\mathcal{E}} - T)(\mathcal{E})} \oplus \ker(I_{\mathcal{E}} - T),$$

the mean ergodic sequence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{E}$.

Following example shows that exact noncommutative version of von Neumann–Riesz mean ergodic theorem fails.

Example 2.7. *Let $\mathbb{T}$ be the standard unit circle and $\mathcal{C}(\mathbb{T})$ be the C*-algebra of continuous complex valued functions on $\mathbb{T}$ in the sup norm. Define $\mathcal{A} \coloneqq \mathcal{C}(\mathbb{T})$. Define $\mathcal{E} \coloneqq \mathcal{A}$. Define*

$$U : \mathcal{E} \ni f \mapsto Uf \in \mathcal{E};\ (Uf)(z) \coloneqq zf(z), \quad \forall z \in \mathbb{T}.$$

Then U is unitary. Now

$$(S_n\chi_{\mathbb{T}})(z) = \frac{1}{n}\sum_{k=0}^{n-1}(U^k\chi_{\mathbb{T}})(z) = \frac{1}{n}\sum_{k=0}^{n-1} z^k = \frac{1}{n}\sum_{k=0}^{n-1} z^k, \quad \forall n \in \mathbb{N}, \quad \forall z \in \mathbb{T}.$$

Hence

$$(S_n\chi_{\mathbb{T}})(z) = \frac{1-z^n}{n(1-z)}, \quad \forall n \in \mathbb{N}, \quad \forall z \in \mathbb{T} \setminus \{1\}$$

and

$$(S_n\chi_{\mathbb{T}})(1) = 1.$$

We therefore have

$$\lim_{n\to\infty}(S_n\chi_{\mathbb{T}})(z) = 0, \quad \forall z \in \mathbb{T} \setminus \{1\}$$

and

$$\lim_{n\to\infty}(S_n\chi_{\mathbb{T}})(1) = 1.$$

Therefore

$$\lim_{n\to\infty} S_n\chi_{\mathbb{T}} = \chi_{\{1\}} \notin \mathcal{C}(\mathbb{T}).$$

Hence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} U^k\chi_{\mathbb{T}}\right\}_{n=1}^{\infty}$$

does not converge in the norm on $\mathcal{E}$.

3. Non-Archimedean von Neumann-Riesz Mean Ergodic Problem

We recall the definition of non-Archimedean Banach and p-adic Hilbert spaces.

Definition 3.1. *[22] Let $\mathbb{K}$ be a field. A map $|\cdot| : \mathbb{K} \to [0,\infty)$ is said to be a **non-Archimedean valuation** if the following conditions hold:*

(i) *If $\lambda \in \mathbb{K}$ satisfies $|\lambda| = 0$, then $\lambda = 0$.*
(ii) *$|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.*
(iii) *(Ultratriangle inequality) $|\lambda + \mu| \leq \max\{\lambda, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.*

We say $\mathbb{K}$ is non-Archimedean complete valued field if it is complete w.r.t. the ultrametric $d(\lambda,\mu) \coloneqq |\lambda - \mu|$ for all $\lambda, \mu \in \mathbb{K}$.

Definition 3.2. *[23] Let $\mathbb{K}$ be a non-Archimedean complete valued field. A vector space $\mathcal{X}$ over $\mathbb{K}$ is said to be a **non-Archimedean Banach space** if there is a map (called as norm) $\|\cdot\| : \mathcal{X} \to [0,\infty)$ satisfying the following:*

(i) *If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.*
(ii) *$\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.*
(iii) *(Ultranorm inequality) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$.*

(iv) $\mathcal{X}$ *is complete w.r.t. the ultrametric* $d(x,y) := \|x-y\|$ *for all* $x, y \in \mathcal{X}$.

Definition 3.3. *[8] Let* $\mathbb{K}$ *be a non-Archimedean valued field (with valuation* $|\cdot|$*) and* $\mathcal{X}$ *be a non-Archimedean Banach space (with norm* $\|\cdot\|$*) over* $\mathbb{K}$*. We say that* $\mathcal{X}$ *is a* ***p-adic Hilbert space*** *if there is a map (called as* ***p-adic inner product****)* $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ *satisfying the following.*

(i) *If* $x \in \mathcal{X}$ *is such that* $\langle x, y \rangle = 0$ *for all* $y \in \mathcal{X}$*, then* $x = 0$.
(ii) $\langle x, y \rangle = \langle y, x \rangle$ *for all* $x, y \in \mathcal{X}$.
(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ *for all* $\alpha \in \mathbb{K}$*, for all* $x, y \in \mathcal{X}$.
(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ *for all* $x, y, z \in \mathcal{X}$.
(v) $|\langle x, y \rangle| \leq \|x\|\|y\|$ *for all* $x, y \in \mathcal{X}$.

Let $\mathcal{X}, \mathcal{Y}$ be p-adic Hilbert spaces. A bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is said to be adjointable if there exists a linear operator (which is unique) $T^* : \mathcal{Y} \to \mathcal{X}$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}.$$

We say that an adjointable linear operator $T : \mathcal{X} \to \mathcal{Y}$ is an isometry if $\|Tx\| = \|x\|$ for all $x \in \mathcal{X}$. An invertible isometry $T : \mathcal{X} \to \mathcal{Y}$ is said to be unitary if $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x \in \mathcal{X}$, for all $y \in \mathcal{Y}$.

Problem 3.4. ***(Non-Archimedean von Neumann-Riesz Mean Ergodic Problem)*** *Let* $\mathcal{X}$ *be p-adic Hilbert space. Let* $T : \mathcal{X} \to \mathcal{X}$ *be an adjointable contraction or unitary operator. For which* $x \in \mathcal{X}$*, the p-adic mean ergodic sequence*

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} T^k x \right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{X}$.

Problem 3.5. ***(Non-Archimedean von Neumann-Riesz-Lorch Mean Ergodic Problem)*** *Let* $\mathcal{X}$ *be p-adic Hilbert space. Let* $T : \mathcal{X} \to \mathcal{X}$ *be a bounded operator such that*

$$\sup_{n \in \mathbb{N}} \|T^n\| < \infty.$$

For which $x \in \mathcal{X}$*, the p-adic mean ergodic sequence*

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} T^k x \right\}_{n=1}^{\infty}$$

converges in the norm on $\mathcal{X}$.

4. Conclusions

(1) In 1932, von Neumann derived the mean ergodic theorem for unitary operators on Hilbert spaces [26].
(2) In 1938, Riesz derived the mean ergodic theorem for contractions on Hilbert spaces [18].
(3) In 1939, Lorch derived the mean ergodic theorem for power bounded operators on Hilbert spaces [12].
(4) In this article, we derived noncommutative version of von Neumann-Riesz mean ergodic theorem.

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