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REAL ZEROS OF L-FUNCTION DERIVATIVES: DENSITY-ONE ASYMPTOTICS IN ARITHMETIC FAMILIES

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Abstract. We study real zeros of the derivative of an uncompleted L-function on $[1/2, 1]$. The limiting stationary process has covariance $\operatorname{sech}^2((t-s)/2)$ after the change of variables $\sigma = 1/2 + e^{-t}$. Its nonzero-level Rice intensity gives the constant $e^{-1/2}/(\pi\sqrt{2})$ for quadratic Dirichlet characters, a specified family of quadratic Hecke characters over $\mathbb{Q}(i)$, and the unweighted level-one holomorphic family; the resulting density-one asymptotics are unconditional. For a fixed holomorphic modular form and its quadratic twists, and for non-Galois cubic Artin L-functions, we obtain the same constant under the respective family GRH. The real cyclic cubic augmentation instead has constant $1/(\pi\sqrt{2})$. In these three GRH families a uniform quantile estimate controls all three counts: distinct real zeros, zeros with multiplicity, and strict sign changes. For an individual complex cubic character, the derivative has $o(\log\log q)$ real zeros in density under cubic-character GRH. We also prove a uniform mean asymptotic and exponential concentration for the random Euler model.

Contents

1. Introduction and statements 2
1.1. Unconditional arithmetic asymptotics 3
1.2. The three further arithmetic families under GRH 4
1.3. Individual complex cubic derivatives 4
1.4. The random theorem 5
1.5. What is averaged and where the proof changes scale 5
1.6. Relation to earlier results and limitations 6
2. Random Euler products and arithmetic transfer 6
2.1. Analytic construction and Gaussian comparison 7
2.2. Uniform density from discrete prime variables 10
2.3. Joint large and small values in the symmetric model 11
2.4. The strong random zero theorem 15
2.5. A finite-block polynomial test 18
2.6. Finite Euler polynomials and finite-word orthogonality 20
2.7. The abstract two-sided order theorem 23
3. Analytic approximation and local zero counting 24
3.1. Normalized analytic hypotheses 25
3.2. Normalized polynomial moments 26
3.3. A local meromorphic Jensen estimate 27
3.4. Eight fixed anchors 28
3.5. Retained local moments in the bulk 29
4. Crossing meshes and the uniform quantile estimate 30

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4.1. Two nearby crossings of the Gaussian process 30
4.2. Sign tests without a separation restriction 31
4.3. Finite Euler changes and summing dyadic families 33
5. Three unconditional families 34
5.1. The arithmetic probability spaces 34
5.2. A central mollified bound for all signed discriminants 36
5.3. Zero detectors in the three families 40
5.4. A zero-free complex approximation 44
5.5. Model bounds and the central interval 46
5.6. The short bridge 48
5.7. The bulk count with retained local moments 50
5.8. Assembly and normalization 53
6. Three arithmetic families under GRH 54
6.1. Fixed modular forms and all quadratic discriminants 54
6.2. Non-Galois cubic fields 58
6.3. Oriented cubic characters: a uniform arithmetic lemma 59
6.4. Real cyclic cubic augmentation 62
7. Real zeros for individual complex cubic characters 64
7.1. The two-dimensional random limit 64
7.2. Approximation of the actual complex function 66
7.3. A grid argument and domination by the real product 67
8. Numerical diagnostics and reproducibility 68
8.1. The Gaussian crossing calculation 68
8.2. Local quadratic distributions and finite root counts 69
8.3. An actual modular twist 70
8.4. The cost of eight fixed anchors 71
Appendix A. Assembly of the abstract order-of-magnitude theorem 71
A.1. The zero detector and the off-line anchor 72
A.2. A lower bound without a discrepancy axiom 73
A.3. Bulk growth with one fixed choice of constants 73
A.4. Retained small values and the tip coefficient vector 74
A.5. Depth allocation and the bulk upper bound 76
A.6. The central interval and root number 76
References 77

1. Introduction and statements

For a real entire L-function L, write R_L^{dist} and R_L^{mult} for the number of zeros of L' on the *closed* real interval $[1/2, 1]$, without and with complex-analytic multiplicity. Write S_L for the number of strict sign changes of L' on $(1/2, 1)$. All derivatives in this paper are derivatives of the uncompleted function. These conventions matter at the central point and for families with root number -1.

The phrase "in density" always refers to the original family measure. Thus $R_L/H_L \to c$ in density means that, for every fixed $\eta > 0$, the proportion with $|R_L/H_L - c| > \eta$ tends to zero. This does not assert an asymptotic for every member, or an asymptotic for the family mean of the unbounded count.

Our constants are

$$c_{\mathrm{BM}} = \frac{e^{-1/2}}{\pi\sqrt{2}}, \qquad c(m,v) = \frac{e^{-m^2/(2v)}}{\pi\sqrt{2}} \quad (v>0). \tag{1.1}$$

The parameters m and v are, respectively, the weighted mean of the prime-square logarithmic coefficient and the weighted variance of the prime coefficient. They are defined precisely in the Euler-model section. The variance of the derivative of a stationary Gaussian process with covariance $\operatorname{sech}^2(t/2)$ is $1/2$. Its crossings of the level $-m/\sqrt{v}$ have intensity $c(m,v)$. This is why changing the drift from $1/2$ to $-1/2$ preserves the constant, whereas removing the drift changes it.

1.1. **Unconditional arithmetic asymptotics.** Let $\mathcal{F}(X)$ be the set of fundamental discriminants D with $1<|D|\le X$, of both signs, and let χ_D be the primitive quadratic character of conductor $|D|$.

Theorem 1.1 (Quadratic Dirichlet family, unconditional)**.** *For each fixed $\eta>0$, all but $o_\eta(X)$ elements of $\mathcal{F}(X)$ satisfy, simultaneously,*

$$\max_{C\in\{R^{\mathrm{dist}},R^{\mathrm{mult}},S\}}\left|\frac{C_{L(s,\chi_D)}}{\log\log|D|}-c_{\mathrm{BM}}\right|\le\eta.$$

No hypothesis on zeros or central values is imposed.

The full signed family in this theorem requires an arithmetic argument at the prime 2. A theorem for the subfamily $D=8d$ cannot simply be relabelled as a theorem for all fundamental discriminants. We give the complete extension of the mollified estimate, including the other residue classes and the cancellation of the paired primary terms, in the unconditional-family section.

For the Gaussian-integer family put $\pi=1+i$. Let $d\in\mathbb{Z}[i]$ range over elements that are prime to 2 and squarefree as ideals, with $X/2<N(d)\le X$. Use the primitive quadratic Hecke character $\chi_{\pi^5 d}$. The finite conductor has norm $32N(d)$, and the normalization used for the completion has conductor parameter $Q_d=128N(d)$. Each character occurs with exactly the two labels $d,-d$.

Theorem 1.2 (Quadratic Hecke family over $\mathbb{Q}(i)$, unconditional)**.** *For the preceding family, the three counts $R^{\mathrm{dist}},R^{\mathrm{mult}},S$ of $L'(s,\chi_{\pi^5 d})$ each equal*

$$(c_{\mathrm{BM}}+o(1))\log\log Q_d$$

in density. More explicitly, at each fixed relative tolerance $\eta>0$, the exceptional number of labels is $o_\eta(X)$. The same statement holds when each character is counted once and for cumulative norm cutoffs.

This is a specified quadratic Hecke family, rather than all Hecke characters over $\mathbb{Q}(i)$.

Let $\mathcal{H}_k$ be the set of normalized level-one holomorphic Hecke eigenforms of even weight k, each counted once. The L-function is normalized to have central point $1/2$.

Theorem 1.3 (All even weights at level one, unconditional)**.** *As even $k\to\infty$, for each fixed $\eta>0$, all but $o_\eta(|\mathcal{H}_k|)$ forms satisfy*

$$\max_{C\in\{R^{\mathrm{dist}},R^{\mathrm{mult}},S\}}\left|\frac{C_{L(s,f)}}{\log\log(k^2)}-c_{\mathrm{BM}}\right|\le\eta.$$

The measure is unweighted. Both residue classes of even k modulo 4 are included, and no bound on the central multiplicity is assumed.

1.2. **The three further arithmetic families under GRH.** The following families are considered separately; GRH for one is not used to prove a statement about another.

(i) Fix a primitive holomorphic cuspidal eigenform f of even weight at least 2, fixed level, and trivial nebentypus. Let D range over all signed fundamental discriminants. The target is the primitive automorphic twist $L(s, f \otimes \chi_D)$, whose conductor $Q_D \asymp_f D^2$. The form f may have complex multiplication.

(ii) Let K range over non-Galois cubic fields of either fixed discriminant sign, without weights, and put $L_K(s) = \zeta_K(s)/\zeta(s)$. This is the entire degree-two standard Artin L-function of the S_3-closure, with conductor $Q_K = |D_K|$.

(iii) Let K range over cyclic cubic fields, each once. If $\chi, \bar{\chi}$ are its two primitive cubic Dirichlet characters of conductor q, put $Z_K(s) = L(s,\chi)L(s,\bar{\chi}) = \zeta_K(s)/\zeta(s)$. The conductor and field discriminant are $Q_K = D_K = q^2$.

In (i), the parameter is $|D|$; in (ii) and (iii), it is $|D_K|$. A dyadic family means $X/2 <$ parameter $\le X$. Write $L_X = \log X$, $H = \log\log X$ and $H_Q = \log\log Q$. Constants below may depend on the fixed form in (i).

Theorem 1.4 (Uniform full-count estimate under family GRH)**.** *Assume GRH for the primitive L-functions of the selected family (i), (ii), or (iii). In (iii) this means GRH for the primitive cubic characters. Set $c = c_{\mathrm{BM}}$ in (i) and (ii), and $c = 1/(\pi\sqrt{2})$ in (iii). For every fixed $0 < \varepsilon < 1/16$ there are constants $A_\varepsilon, C_\varepsilon, X_\varepsilon > 0$ such that, for $X \ge X_\varepsilon$ and every deterministic real z with*

$$H^{-1/2+\varepsilon} \le z \le 1,$$

the original unweighted dyadic measure satisfies

$$\mathbb{P}_X\left\{\max_{C\in\{R^{\mathrm{dist}},R^{\mathrm{mult}},S\}} |C - cH_Q| > A_\varepsilon z H_Q\right\} \le C_\varepsilon \frac{\log H}{Hz^2}. \tag{1.2}$$

The constants are independent of z. Central nonvanishing and simple zeros are not assumed.

The quantifier in (1.2) allows a choice $z = z(X)$ in the indicated interval. It does not assert a single sample event for all z. It concerns the full count in probability, not its variance.

For fixed $\eta > 0$, this gives an exceptional proportion $O_\eta(\log H/H)$ outside relative tolerance η. Two useful shrinking tolerances are

$$|C - cH_Q| \le AH_Q^{2/3}(\log H_Q)^{1/3}, \qquad \text{exception } O((\log H/H)^{1/3}), \tag{1.3}$$

$$|C - cH_Q| \le A_\varepsilon H_Q^{1/2+\varepsilon}, \qquad \text{exception } O_\varepsilon((\log H)H^{-2\varepsilon}). \tag{1.4}$$

These corollaries also hold for cumulative cutoffs: the exceptional counts are the displayed proportions times X in (i),(ii) and times $\sqrt{X}$ in (iii). We do not assert an arbitrary-$z(X)$ cumulative estimate without specifying its dyadic envelope.

The coefficientwise twist $\sum_{n\ge1}\lambda_f(n)\chi_D(n)n^{-s}$ also satisfies the conclusions for (i). Its difference from the primitive twist is a fixed finite set of Euler factors. We retain the primitive conductor and the primitive GRH in this version; we do not assign the modified function a new primitive functional equation.

1.3. **Individual complex cubic derivatives.** The augmentation in (iii) is real. An individual cubic character instead gives a complex function on the real axis. A real zero of its derivative means a simultaneous zero of its real and imaginary parts; there is no associated sign-change statistic.

Theorem 1.5 (Complex cubic characters under cubic-family GRH)**.** *Let $\mathcal{C}(X)$ contain all primitive exact-order-three Dirichlet characters with $X/2 < q_\chi \le X$, including both conjugates.*

Assume GRH for this family. Let R_χ^{mult} count the real zeros of $L'(s,\chi)$ on $[1/2,1]$ with complex-analytic multiplicity. For every fixed $0<\varepsilon<1/16$,

$$\mathbb{P}_{\mathcal{C}(X)}\{R_\chi^{\mathrm{mult}} > A_\varepsilon(\log\log q_\chi)^{1/2+\varepsilon}\} \ll_\varepsilon \frac{\log H}{H^{1/2+\varepsilon}} + H^{-2}\log H.$$

In particular $R_\chi^{\mathrm{dist}} \le R_\chi^{\mathrm{mult}} = o(\log\log q_\chi)$ in density. The same proportion holds if each conjugate pair is counted once. The cumulative conductor and field-discriminant versions have the corresponding family cardinalities $\asymp X$ and $\asymp \sqrt{X}$.

This conclusion is an upper bound, not an asymptotic of the form "$R \sim 0$". In the bulk used in the proof there is no real zero with probability $1-O(H^{-3})$; the entire closed interval need not be zero-free.

1.4. **The random theorem.** Fix a class of independent variables $X(p) \in \{0,1,-1\}$ with $\mathbb{P}(X(p)=\pm1)=\theta_p/2$, where $0<\theta_*\le\theta\le 1$ and $|\theta_p-\theta|\le A_* p^{-\varepsilon_*}$. Define the nonvanishing random Euler product in $\Re s>1/2$ and

$$G_E(t) = -e^{-t}\frac{L_E'}{L_E}(1/2+e^{-t}).$$

Let $N_E^\bullet(u,T)$ count its real zeros on $[u,u+T]$.

Theorem 1.6 (Uniform random mean and exponential concentration)**.** *Uniformly in the fixed class, deterministic $u\ge\log 2$ and $T\ge 1$,*

$$\mathbb{E}N_E^\bullet(u,T) = \frac{e^{-\theta/2}}{\pi\sqrt{2}}T + O(e^{-5u/56}), \qquad \bullet\in\{\mathrm{dist},\mathrm{mult}\}.$$

For every fixed $\eta>0$ there are positive c_η, C_η, depending only on η and the class, such that

$$\mathbb{P}\left\{\left|N_E^\bullet(u,T) - \frac{e^{-\theta/2}}{\pi\sqrt{2}}T\right| > \eta T\right\} \le C_\eta e^{-c_\eta T}, \qquad \bullet\in\{\mathrm{dist},\mathrm{mult}\}.$$

1.5. **What is averaged and where the proof changes scale.** The prime variables are independent in the model. Members of an arithmetic family are not independent random samples of an Euler product. The connection is a finite-polynomial orthogonality estimate with a power saving in the family parameter. We apply it to a finite Taylor polynomial of a characteristic function; its degree and its Euler cutoff are chosen so that their product is $o(\log X)$. This produces the required two-point and four-point comparisons without comparing exponentially many sign rectangles.

The variable $t=-\log(\sigma-1/2)$ turns a multiplicative interval of distances from the central point into an interval of fixed length. In the bulk, a fine mesh counts crossings, and Rolle's theorem controls the roots it misses. Near the central point a direct Gaussian comparison alone does not control poles of L'/L. There we combine the positive paired zero kernel, a meromorphic Jensen formula, and finitely many possible anchors. A single anchor whose value is not small suffices. Eight fixed anchors turn the bad event into an eight-dimensional small-ball event; independence of those anchors is neither true nor required.

For the unconditional families, the role of GRH is replaced by specifically proved density and mollified moment estimates. Their uniform ranges are kept separate. In particular, a power loss X^ε cannot be substituted for a logarithmic loss in a detector whose saving is only a power of $\log X$.

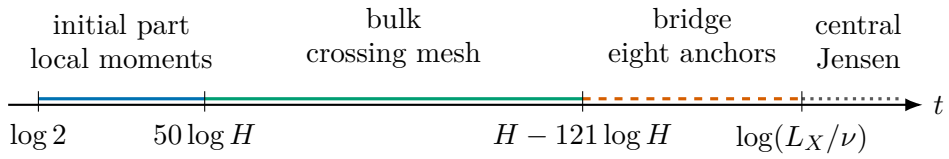


FIGURE 1. The proof regions for the quantitative GRH theorem, with $L_X = \log X$ and $\nu = H^{1/2+\varepsilon/4}$. This is a scale diagram, not a plot of numerical data or of region lengths. The central endpoint $\sigma = 1/2$ corresponds to $t = +\infty$ and is treated in the s-plane.

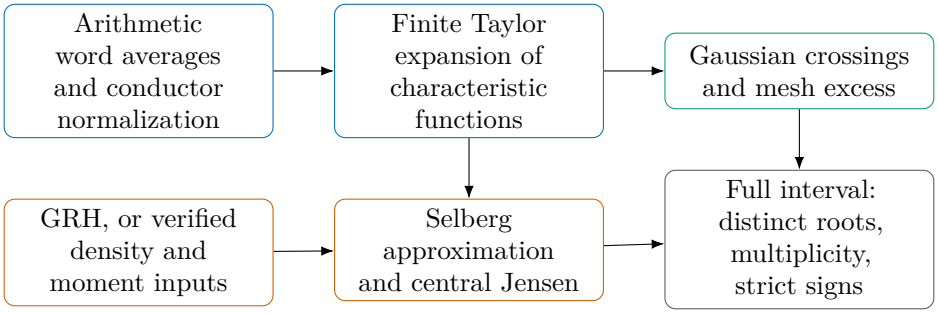


FIGURE 2. The logical inputs to the full-count results. This schematic separates the arithmetic transfer from the analytic control of central zeros. The unconditional families supply their own density and mollified moment estimates; finite-word averages alone do not replace those inputs.

1.6. **Relation to earlier results and limitations.** Baker and Montgomery formulated the expected log log order of the oscillation of quadratic L-function derivatives [5]. Their formulation includes a broader character convention; our main statement uses primitive characters indexed by fundamental discriminants. The Fekete-polynomial problem of Conrey, Granville, Poonen and Soundararajan [9] is related through character sums, but its unit-circle zero constant is not the Rice constant in (1.1).

Related frameworks for using local splitting statistics in Artin-family moment problems appear in Cho and Kim [7], and for conductor-ordered cyclic extensions in Cho and Oh [8]. These works provided background for the extensions considered here. The quantitative field and oriented-character inputs used in our proofs are established separately in Section 6.

None of the arithmetic statements here asserts a central nonvanishing theorem, a central simplicity theorem, a full-count central limit theorem, or a family mean asymptotic. The quantitative conductor thresholds are not made effective. The constants arising from fixed-anchor covariance matrices can be very large. Actual low-conductor computations therefore test local identities and sign brackets, not the asymptotic Rice constant. Removing GRH for the three further families requires new estimates in the central mesoscopic range; currently available estimates with conductor-power losses do not supply those inputs by direct substitution.

2. Random Euler products and arithmetic transfer

All random variables indexed by distinct primes are independent unless a different dependence is explicitly specified. Constants are uniform in the parameters appearing in a statement; dependence on a fixed number field is allowed. We use closed intervals when counting real zeros, and count a zero at an endpoint once.

Work	Family and interval	Conclusion	Method or qualification
Baker–Montgomery	quadratic characters; $(1/2, 1)$	conjectured $\log\log$ order of strict oscillation	character sums and logarithmic derivatives
Conrey–Granville–Poonen–Soundararajan	Fekete polynomials; unit circle and real intervals	zero-distribution results in a different normalization	finite character sums; no derivative Rice constant
Klurman–Lamzouri–Munsch	fundamental quadratic family; $(0, 1)$	almost-all lower bound of order $\log_2 \lvert D\rvert / \log_4 \lvert D\rvert$	joint discrepancy; exception $O_A(Xe^{-(\log_3 X)^A})$, fixed $A \geq 1$
Lamzouri–Nath	$D = 8m$; $[1/2, 1]$	almost-all upper bound $O(\log_2 X \log_3 X)$	Jensen and zero density; exception $O(X \log_3 X/\sqrt{\log_2 X})$
This paper	the families of the theorems above; $[1/2, 1]$	exact density-one intensities; simultaneous three counts	finite-dimensional transfer, mesh excess, and central anchors; exception $o(\#\mathcal{F})$ unconditionally, $O_\eta(\log H/H)$ under stated GRH

TABLE 1. Comparison of the strength and setting of the statements. The intervals and counting conventions are not identified across different works. The quadratic lower bound uses [18, Theorem 1.8], version 2; the upper bound uses [20, Theorem 1.2], version 2.

2.1. **Analytic construction and Gaussian comparison.** We first give a construction which also applies to the prime ideals of a fixed number field. Write $q_\mathfrak{p} = N\mathfrak{p}$, and assume

$$\Theta_K(y) := \sum_{q_\mathfrak{p} \leq y} \log q_\mathfrak{p} = y + O_K(ye^{-c_K\sqrt{\log y}}). \tag{2.1}$$

Repeated norms are counted with their multiplicities. For $K = \mathbb{Q}$ this follows, for example, from [17, Theorem 1.1]: the contribution of proper prime powers to $\psi(y)$ is $O(\sqrt{y}\log^2 y)$, and any fixed power of $\log y$ is absorbed by decreasing the positive constant in the exponential.

Fix $0 < \theta_* \leq 1$, $\varepsilon_* > 0$, and $A_* < \infty$. Our symmetric Euler class consists of

$$\mathbb{P}(X_\mathfrak{p} = 1) = \mathbb{P}(X_\mathfrak{p} = -1) = \theta_\mathfrak{p}/2, \quad \mathbb{P}(X_\mathfrak{p} = 0) = 1-\theta_\mathfrak{p}, \quad |\theta_\mathfrak{p}-\theta| \leq A_* q_\mathfrak{p}^{-\varepsilon_*}, \quad \theta_* \leq \theta \leq 1. \tag{2.2}$$

Put

$$H_X(s) = \sum_\mathfrak{p} \sum_{k\geq 1} \frac{X_\mathfrak{p}^k}{k q_\mathfrak{p}^{ks}}, \qquad L_X(s) = e^{H_X(s)}, \qquad G_X(w) = -e^{-w}\frac{L_X'}{L_X}(1/2 + e^{-w}). \tag{R.1}$$

Lemma 2.1 (Analytic versions)**.** *There is a single event of probability one on which these functions are holomorphic in $\Re s > 1/2$, respectively $|\Im w| < \pi/2$, and*

$$-\frac{L_X'}{L_X}(s) = \sum_{\mathfrak{p},k\geq 1} (\log q_\mathfrak{p}) X_\mathfrak{p}^k q_\mathfrak{p}^{-ks}.$$

The function L_X never vanishes, and L_X' and G_X are not identically zero. Their real zeros correspond under $s = 1/2 + e^{-w}$, with multiplicities preserved.

Proof. Let Γ be a circle with closed interior in $\Re s > 1/2$, and $\sigma_0 = \min_\Gamma \Re s > 1/2$. In the deterministic Hilbert space $L^2(\Gamma, |ds|)$, independence and zero means give

$$\mathbb{E}\left\| \sum_{M < q_\mathfrak{p} \le N} X_\mathfrak{p} q_\mathfrak{p}^{-s} \right\|_2^2 \le |\Gamma| \sum_{q_\mathfrak{p} > M} q_\mathfrak{p}^{-2\sigma_0} \longrightarrow 0.$$

The last series converges by (2.1). For Hilbert-valued independent centered sums S_j, conditional Jensen and the first crossing of a level λ imply $\lambda \mathbb{P}(\max_{j \le n} \|S_j\| > \lambda) \le \mathbb{E}[\|S_n\| \mathbf{1}_{\{\max \|S_j\| > \lambda\}}]$. Integrating against $2\,d\lambda$ and using Cauchy–Schwarz proves $\mathbb{E} \max_{j \le n} \|S_j\|^2 \le 4\mathbb{E}\|S_n\|^2$. Choose successive tails whose variances are at most 2^{-4j}. The probability that such a tail has supremum exceeding 2^{-j} is $O(2^{-2j})$. Borel–Cantelli therefore gives almost sure convergence in the boundary Hilbert space. Cauchy's formula bounds the supremum on each smaller disc by its boundary L^2 norm. A countable covering gives local uniform convergence throughout the half-plane.

On each smaller half-plane, all terms with $k \ge 2$, together with any fixed number of their derivatives, converge absolutely: their majorants are $C_j (\log q_\mathfrak{p})^j q_\mathfrak{p}^{-2\sigma_1}$ with $\sigma_1 > 1/2$. This proves the analytic assertions and permits differentiation. Exponentiation proves nonvanishing.

There is almost surely an active prime ideal, since eventually $\theta_\mathfrak{p} \ge \theta_*/2$. If an active prime lies above the rational prime p, let $f_1, \dots, f_r$ be the residue degrees of all primes above p, and set $M = 2\,\mathrm{lcm}(f_1, \dots, f_r)$. The coefficient of $(p^M)^{-s}$ in the absolutely convergent logarithmic derivative on $\Re s > 1$ is

$$(\log p) \sum_{i : X_{\mathfrak{p}_i} \ne 0} f_i > 0.$$

Thus cancellation between equal norms cannot make that derivative identically zero. Uniqueness here follows directly by multiplying a Dirichlet series by its least nonzero index to the power s and letting real s tend to infinity. Finally $ds/dw = -e^{-w}$, $L_X(s)$, and e^{-w} are nonzero, which proves the assertion about multiplicity. The same coefficient argument is deterministic: it also proves $L_f' \not\equiv 0$ whenever the ideal Euler coefficients take values in $\{0, \pm 1\}$ and at least one prime ideal is active. □

The following more general model will be needed for real coefficients which need not have the symmetric three-point law. Let $a_{p,k} = a_{p,k}(Z_p)$ be real, $|a_{p,k}| \le D$, and put

$$b_p = \mathbb{E} a_{p,1} = O(p^{-1}), \qquad v_p = \mathrm{Var}(a_{p,1}), \qquad c_p = \mathbb{E} a_{p,2},$$
$$\sum_{p \le x} v_p \log p = vx + O(xe^{-c\sqrt{\log x}}), \quad \sum_{p \le x} c_p \log p = mx + O(xe^{-c\sqrt{\log x}}), \quad v > 0. \tag{2.3}$$

Here m may have either sign. Define

$$F(a) = \sum_p (\log p) \sum_{k \ge 1} a_{p,k} p^{-k(1/2+a)}, \qquad G(w) = e^{-w} F(e^{-w}).$$

Lemma 2.2 (General real model)**.** *These series have holomorphic versions on $\Re a > 0$ and $|\Im w| < \pi/2$. In every fixed closed right sector, for bounded a, b,*

$$\mathbb{E}[aF(a)] = m/2 + O(|a|), \qquad \mathrm{Cov}(aF(a), bF(b)) = \frac{vab}{(a+b)^2} + O(|ab|). \tag{2.4}$$

The covariance displayed is bilinear; for the conjugated covariance replace b by $\bar{b}$. For $R < \pi/2$, $t \ge \log 2$, and $q \ge 2$,

$$\left\| \sup_{|w-t| \le R} |G(w)| \right\|_q \le C_R \sqrt{q}. \tag{2.5}$$

The same conclusion holds for each fixed derivative on a smaller disc. For the symmetric model, these statements hold with primes replaced by prime ideals and $m = v = \theta$.

Proof. Apply the boundary Hilbert-space argument above to the centered prime terms. The prime means and the terms with $k \geq 2$ are absolutely convergent on compact subsets. To calculate their singular parts, write $\Theta(t) = \alpha t + E(t)$, where $E(t) \ll te^{-c\sqrt{\log t}}$. Partial integration gives

$$\int_{1^-}^{\infty} t^{-1-z}\, d\Theta(t) = \frac{\alpha}{z} + \alpha + (1+z)\int_1^{\infty} E(t)t^{-2-z}\, dt.$$

The error and every fixed derivative are bounded for $\Re z \geq 0$ and bounded z, since $\int_0^{\infty}(1+u)^j e^{-c\sqrt{u}}\, du < \infty$. Differentiation gives the corresponding z^{-2} formula with $(\log p)^2$. In the mean only $k = 2$ contributes a pole. In the covariance only $k = l = 1$ contributes a pole; all other terms are bounded by $C\sum_p (\log p)^2 p^{-3/2}$. This proves (2.4). The perturbation $\theta_{\mathfrak{p}} - \theta$ is absolutely summable after multiplication by $q_{\mathfrak{p}}^{-1}(\log q_{\mathfrak{p}})^j$, proving the ideal version.

For clarity, the elementary bounded-variable inequality used for moments is as follows. If a centered real variable U lies in an interval of length d, then the second derivative of $\log \mathbb{E}e^{tU}$ is a tilted variance, at most $d^2/4$. Integrating twice gives $\mathbb{E}e^{tU} \leq e^{t^2d^2/8}$. For any real or imaginary projection of a centered prime block of $aF(a)$, the interval length is at most $C|a|(\log p)p^{-1/2-\Re a}$. The sum of its squares is bounded in a fixed right sector, by partial integration of the prime theorem. Independence, the preceding exponential bound, and tail integration therefore give L^q norm $O(\sqrt{q})$. The means are bounded by the formula already proved. Choose $R < R_+ < \pi/2$. Cauchy's formula bounds the supremum on the disc of radius R by a constant times the angular average on the circle of radius R_+. Minkowski gives (2.5). The derivative assertion follows from Cauchy's formula once more. □

Let W be the isonormal Gaussian process on $L^2(0,\infty)$: if (e_j) is an orthonormal basis and Z_j are independent standard Gaussians, set $W(f) = \lim_n \sum_{j\leq n}\langle f, e_j\rangle Z_j$ in L^2. Parseval proves convergence and $\mathbb{E}W(f)W(h) = \langle f, h\rangle$; applying the characteristic function formula to every finite linear combination proves joint Gaussianity. For $f_\delta(x) = 2\delta\sqrt{x}\, e^{-\delta x}$, put $g(t) = W(f_{e^{-t}})$. Thus

$$\mathbb{E}g(t)g(s) = r(t-s), \qquad r(h) = \operatorname{sech}^2(h/2), \qquad \mathcal{G}(t) = m/2 + (\sqrt{v}/2)g(t). \tag{2.6}$$

The next lemma proves directly the special case of the Gaussian Rice formula used here; see also [3, Theorem 3.2] for the general formula.

Lemma 2.3 (Gaussian paths and exact level intensity)**.** *The process g has a C^2 version. For every fixed real level u and compact interval I of length T, its level-u zeros are almost surely finite and simple, no fixed endpoint is a zero, and*

$$\mathbb{E}N_u(I) = \frac{e^{-u^2/2}}{\pi\sqrt{2}}\, T.$$

Proof. Write $\phi_t = f_{e^{-t}}$. For each compact interval J and $0 \leq k \leq 3$, $\partial_t^k \phi_t(x)$ is $2e^{-t}\sqrt{x}e^{-e^{-t}x}$ times a polynomial in $e^{-t}x$, so $\int_J \|\partial_t^k \phi_t\|_2^2\, dt < \infty$. Parseval and dominated convergence show that the Gaussian partial sums are Cauchy in $L^2(\Omega; H^3(J))$, where $\|h\|_{H^3(J)}^2 = \sum_{k=0}^3 \int_J |h^{(k)}|^2$. The elementary inequality

$$\|h\|_\infty \leq |J|^{-1/2}\|h\|_2 + |J|^{1/2}\|h'\|_2 \qquad (h \in H^1(J))$$

follows by integrating from a point with $|h(t_0)| \leq |J|^{-1/2}\|h\|_2$. Applying it to the first three derivatives, starting at order zero, gives a C^2 representative. Versions on nested integer intervals agree on their overlaps, first at rational points and then by continuity.

On the event $\sup_J |g''| \le K$, a level-u zero with zero derivative forces $|g(t_j) - u| \le Kh^2/2$ at a point of a mesh of spacing h. The marginal density is bounded by $(2\pi)^{-1/2}$, so the probability is $O_J(Kh)$ by the union bound. Let $h \downarrow 0$, then $K \to \infty$. An infinite level set in a compact interval would have an accumulation point with zero derivative; thus all zeros are finite and simple. Every fixed endpoint, and simultaneously every dyadic mesh point, avoids the level almost surely.

Let C_n count strict crossings on the 2^n equal subintervals of I. Subdivision adds zero or two crossings. Finitely many simple zeros imply $C_n \uparrow N_u(I)$, and stationarity and monotone convergence give $\mathbb{E}N_u(I) = \lim_n 2^n p_u(T2^{-n})$. For completeness, set $U_h = (g(0) + g(h))/2$ and $V_h = (g(h) - g(0))/h$. These are independent Gaussians with variances $(1 + r(h))/2 \to 1$ and $2(1-r(h))/h^2 \to 1/2$. A crossing occurs exactly when $|U_h - u| < h|V_h|/2$. Integrating the density of U_h over this interval and using domination by $C|Z|$ for one standard Gaussian Z gives

$$\lim_{h\downarrow 0} \frac{p_u(h)}{h} = \mathbb{E}|N(0,1/2)| \, \frac{e^{-u^2/2}}{\sqrt{2\pi}} = \frac{e^{-u^2/2}}{\pi\sqrt{2}}.$$

□

Lemma 2.4 (Smooth comparison)**.** *If $t_1, \dots, t_d \ge \log 2$, $b = \max_j e^{-t_j}$, and Φ is bounded and C^3, let M_k be the largest supremum norm of a specified k-th partial derivative. Then*

$$|\mathbb{E}\Phi((G(t_j))) - \mathbb{E}\Phi((\mathcal{G}(t_j)))| \le C(dbM_1 + d^2b^2M_2 + d^3b^3M_3). \tag{R.2}$$

No separation or nonsingularity condition is imposed on the times.

Proof. Center the vectors of whole prime blocks, calling them U_p. Their coordinates have absolute value at most $Cb(\log p)p^{-1/2}$, so $\sum_p \mathbb{E}\|U_p\|_1^3 \le Cd^3b^3$. Independent Gaussian blocks with the same covariances satisfy the same bound, by Minkowski and the one-dimensional Gaussian third moment. Replace finite blocks successively. In Taylor's formula the terms of orders zero, one, and two cancel; the remainder is at most $M_3\|U_p\|_1^3/6$. Passing to infinite sums uses L^2 convergence and bounded continuity. The means differ by $O(b)$ in each coordinate, giving the first error. For Gaussian vectors of covariance $\Sigma_u = (1-u)\Sigma_0 + u\Sigma_1$, integration by parts gives

$$\frac{d}{du}\mathbb{E}\Phi(Z_u) = \frac{1}{2}\sum_{j,k}(\Sigma_1 - \Sigma_0)_{jk}\mathbb{E}\partial_{jk}\Phi(Z_u).$$

One first adds ϵI to both matrices, differentiates their densities, and then lets $\epsilon \downarrow 0$. Equation (2.4) gives the second error. The prime-ideal proof is identical, with its sums evaluated by (2.1). □

2.2. **Uniform density from discrete prime variables.**

Lemma 2.5 (A two-atom density criterion)**.** *Suppose $G_\delta = \sum_p V_{p,\delta}(Z_p)$, $0 < \delta \le 1/2$, converges in probability, with deterministic centering permitted. Assume that for every sufficiently large p two distinct atoms x_p, y_p have probabilities at least $\eta > 0$, and, for bounded real numbers d_p,*

$$c\delta|d_p|(\log p)p^{-1/2-\delta} \le |V_{p,\delta}(x_p) - V_{p,\delta}(y_p)| \le C\delta(\log p)p^{-1/2-\delta}.$$

Assume also $\sum_{u<p\le 2u} d_p^2 \log p \ge cu$ for every sufficiently large u. Then G_δ has a continuous density whose supremum norm is bounded independently of δ. Every fixed derivative of this density is uniformly bounded. In particular

$$\sup_{\delta,z}\mathbb{P}(|G_\delta - z| \le a) \le Ca, \qquad \sup_{\delta,z}\mathbb{E}(\log^+|G_\delta - z|^{-1})^q \le C\Gamma(q+1) \quad (q > 0).$$

Proof. If $\phi_p(\tau) = \mathbb{E}e^{i\tau V_{p,\delta}}$ and $D_{p,\delta}$ is the difference of the two displayed values, an independent copy gives

$$1 - |\phi_p(\tau)|^2 = \sum_{x,y} \mathbb{P}(Z_p = x)\mathbb{P}(Z_p = y)[1 - \cos(\tau(V_{p,\delta}(x) - V_{p,\delta}(y)))].$$

Retaining the two ordered atom pairs proves, when $|\tau D_{p,\delta}| \le 1$,

$$|\phi_p(\tau)| \le \exp\{-c\tau^2\delta^2 d_p^2 (\log p)^2 p^{-1-2\delta}\}.$$

Put $s = 1/2 + \delta$, $T = 2 + |\tau|$, $B = C_1 T(\log T)^2$, and $P = \max(P_0, B^{1/s})$. For large T, $\log P \ll \log T$, and $(\log u)u^{-s}$ decreases for $u \ge P$. Taking C_1 large ensures the local restriction for every $p > P$. The dyadic lower bound yields

$$\sum_{p>P} d_p^2 (\log p)^2 p^{-1-2\delta} \ge cP^{-2\delta} \sum_{j\ge 0} (\log P + j\log 2)\, 2^{-2\delta j}.$$

For $q_\delta = 2^{-2\delta} \in [1/2, 1)$, $\delta^2 \sum j q_\delta^j = \delta^2 q_\delta/(1-q_\delta)^2 \ge c$. Consequently the full characteristic function satisfies

$$|\phi(\tau)| \le e^{-c\tau^2 P^{-2\delta}} \le \exp\{-c|\tau|/\log^2(2+|\tau|)\}.$$

Here $2\delta/s \le 1$; finite products pass to the limit by convergence in probability. Bounded frequencies cause no problem. The last majorant, multiplied by any fixed power of $|\tau|$, is integrable uniformly in δ. Fourier inversion and differentiation under the integral prove the density assertions. Integrating the density over an interval proves the small-ball bound, and layer-cake integration of $qu^{q-1}\mathbb{P}(|G_\delta - z| < e^{-u})$ proves the last assertion. □

2.3. **Joint large and small values in the symmetric model.** We return to (2.2). All statements in this subsection hold for fixed K under (2.1). Define $A(t) = \max_{|w-t|\le 4/5} |G_X(w)|$.

Lemma 2.6 (Joint large-value moments)**.** *For $1/2$-separated $t_1, \dots, t_d \ge \log 2$ and every fixed $\lambda > 0$,*

$$\mathbb{E}\exp\{\lambda \sum_j A(t_j)\} \le C_\lambda^d.$$

Proof. Write $Z(w) = e^{-w}\sum_{\mathfrak{p}} X_{\mathfrak{p}}(\log q_{\mathfrak{p}}) q_{\mathfrak{p}}^{-1/2-e^{-w}}$. On the larger discs $|w - t_j| \le 9/10$, the sum of the absolute values of all higher-power terms is at most

$$C\delta_j \sum_{\mathfrak{p}} \frac{\log q_{\mathfrak{p}}}{q_{\mathfrak{p}}^{1+2c\delta_j}(1 - q_{\mathfrak{p}}^{-1/2-c\delta_j})} \le C, \qquad \delta_j = e^{-t_j}.$$

If $b_{\mathfrak{p}}(w)$ is the coefficient of $X_{\mathfrak{p}}$ in $Z(w)$, partial integration gives, for points in those discs,

$$\sum_{\mathfrak{p}} |b_{\mathfrak{p}}(w_j) b_{\mathfrak{p}}(w_k)| \le C\frac{\delta_j\delta_k}{(\delta_j+\delta_k)^2} \le Ce^{-|t_j - t_k|}.$$

The row sums are bounded. Since $\mathbb{E}e^{aX_{\mathfrak{p}}} \le e^{a^2/2}$, every real signed combination of real and imaginary parts of $Z(w_j)$ has exponential moment at most $e^{C\lambda^2 d}$. Summing over the 4^d choices of signs gives $\mathbb{E}e^{\lambda\sum|Z(w_j)|} \le 4^d e^{C\lambda^2 d}$. Cauchy's formula bounds each supremum on the smaller disc by nine times the angular average on the larger circle. Jensen on the product of these angular probability spaces, followed by the bound just obtained, proves the result. Finite sums pass to the limit by Fatou. □

Lemma 2.7 (Joint small values over a fixed number field)**.** *Let K be fixed and assume (2.1) and the prime-ideal law (2.2). Distinct prime ideals, including those of equal norm, have independent variables. There are $h_0 \ge 1$, $a_0 > 0$, and C_0 such that for h_0-separated $t_1, \dots, t_d \ge \log 2$,*

$$\mathbb{E} \exp\left\{ a_0 \sum_j \log^+ |G_X(t_j)|^{-1} \right\} \le C_0^d. \tag{2.7}$$

The following two quantitative estimates hold. If $\varepsilon_0 \le \delta_j = e^{-t_j} \le 1/2$ and the logarithms are 1-separated, the joint density is bounded by

$$\exp\{Cd^5 \log^2(2d/\varepsilon_0)\}. \tag{2.8}$$

If the times are h_0-separated, $Q \ge Q_0$, $\delta_j \le 1/Q$, and $n_j \ge 1$ are integers with $\sum n_j \le N$, then, conditionally on all $X_{\mathfrak{p}}$ with $q_{\mathfrak{p}} \le e^Q$,

$$\mathbb{P}(|G_X(t_j)| \le e^{-n_j} \ (1 \le j \le d) \mid (X_{\mathfrak{p}})_{q_{\mathfrak{p}} \le e^Q}) \le C_1^d e^{-\sum n_j} + C_2 d^3 (Q+1)^2 e^{-Q/2+3N}. \tag{2.9}$$

Proof. Write $X_{\mathfrak{p}} = B_{\mathfrak{p}} R_{\mathfrak{p}}$, where B is Bernoulli of mean $\theta_{\mathfrak{p}}$ and R is an independent symmetric sign. The entire block is

$$Y_{\mathfrak{p}}(\delta) = B_{\mathfrak{p}}(c_q(\delta) + R_{\mathfrak{p}} a_q(\delta)), \quad c_q(\delta) = \frac{\delta \log q}{q^{1+2\delta} - 1}, \quad a_q(\delta) = \frac{\delta (\log q) q^{1/2+\delta}}{q^{1+2\delta} - 1}.$$

First consider the Gram matrix $K_{jk} = \int_0^1 e^{-(\delta_j + \delta_k)v}\, dv$. Its least eigenvalue is at least $\lambda_0 = \exp\{-Cd^2 \log(2d/\varepsilon_0)\}$. Here are details of this potentially small eigenvalue. Order $x_j = 1/2 - \delta_j$ increasingly. Their gaps are at least $(1 - e^{-1})\varepsilon_0$. The power-series expansion and Cauchy–Binet give

$$\det(e^{v_i x_j}) \ge \frac{\prod_{i<j}(v_j - v_i) \prod_{i<j}(x_j - x_i)}{\prod_{k=0}^{d-1} k!} \quad (0 \le v_1 < \cdots < v_d).$$

Every generalized Vandermonde in this expansion is nonnegative: a linear combination of d distinct monomials has at most $d-1$ positive zeros, as follows inductively by dividing by the smallest power and applying Rolle's theorem. Thus its determinant never changes sign on ordered positive arguments; coalescing the arguments determines the positive sign. Zero arguments follow by continuity. The Gram determinant identity, restricted to $v_i \in [(i-1)/d, (i-1)/d + 1/(3d)]$, now gives

$$\det K \ge \frac{e^{-d}(3d)^{-d}}{d!} \frac{(2/(3d))^{d(d-1)} ((1 - e^{-1})\varepsilon_0)^{d(d-1)}}{(\prod_{k=0}^{d-1} k!)^2}.$$

The factor e^{-d} accounts for the row factors $e^{-v_i/2}$ after squaring. Since $\operatorname{tr} K \le d$, the asserted least-eigenvalue bound follows.

For the joint characteristic function, the phase difference between the atoms $X_{\mathfrak{p}} = 1, -1$ is

$$D_{\mathfrak{p}} = 2(\log q_{\mathfrak{p}}) \sum_j \frac{\xi_j \delta_j q_{\mathfrak{p}}^{-1/2-\delta_j}}{1 - q_{\mathfrak{p}}^{-1-2\delta_j}}.$$

Their probabilities are eventually at least $\theta_*/4$; hence the local characteristic modulus is at most $e^{-cD_{\mathfrak{p}}^2}$ when $|D_{\mathfrak{p}}| \le 1$. Put $L_0 = \log(2d/\varepsilon_0)$, $q_0 = C_3 d^4 L_0^2$. For $\|\xi\|_\infty > e^{q_0}/\varepsilon_0$, choose $q > q_0$ by

$$d(q+1) \max_j (|\xi_j| \delta_j e^{-(1/2+\delta_j)q}) = c_0.$$

The left side is strictly decreasing for $q \ge 3$, starts above c_0, and tends to zero. Set $b_j = \xi_j \delta_j e^{-\delta_j q}$, $B = \max |b_j| = c_0 e^{q/2}/(d(q+1))$, and $g(v) = \sum b_j e^{-\delta_j v}$. In $q < \log q_{\mathfrak{p}} \le q+1$ all local phase

differences have size at most one. Partial integration of (2.1), using $|g| + |g'| \le CdB$, gives

$$\sum_{e^q < q_{\mathfrak{p}} \le e^{q+1}} \frac{(\log q_{\mathfrak{p}})^2}{q_{\mathfrak{p}}} g(\log q_{\mathfrak{p}} - q)^2 = \int_0^1 (q+v) g(v)^2 \, dv + O(qd^2B^2 e^{-c_K\sqrt{q}}).$$

The integral is at least $q\lambda_0 B^2$. For the denominators in $D_{\mathfrak{p}}$, write the perturbed sum as $g + e$, with $|e| \ll dBe^{-q}$, and use $(g+e)^2 \ge g^2/2 - e^2$. The lower bound therefore loses at most $O(qd^2e^{-2q}B^2)$, apart from a fixed factor in its main term. The choice of q_0 absorbs both errors, so $\sum D_{\mathfrak{p}}^2 \ge c\lambda_0 e^q/(d^2(q+1))$. The coordinate of largest $|\xi_j|$ also gives $e^q \ge c_0^{-1} d(q+1)\varepsilon_0 \|\xi\|_\infty$. The full characteristic function is therefore bounded, outside the cube just specified, by $\exp(-c\lambda_0\varepsilon_0\|\xi\|_\infty/d)$. Its integral is at most

$$(2e^{q_0}/\varepsilon_0)^d + 2^d d! \left(\frac{d}{c\lambda_0\varepsilon_0}\right)^d \le \exp(Cd^5 L_0^2).$$

Fourier inversion proves (2.8).

Next condition on $q_{\mathfrak{p}} \le e^Q$ and center the remaining blocks. Their covariance matrix has diagonal bounded below, because

$$\delta_j^2 \int_{e^Q}^\infty (\log u) u^{-1-2\delta_j} \, du = \tfrac{1}{4}(1 + 2\delta_j Q) e^{-2\delta_j Q} \ge e^{-2}/4,$$

up to $O(e^{-c_K\sqrt{Q}})$, when $\delta_j Q \le 1$. The exact covariance is $\theta_{\mathfrak{p}} a_q(\delta_j) a_q(\delta_k) + \theta_{\mathfrak{p}}(1-\theta_{\mathfrak{p}}) c_q(\delta_j) c_q(\delta_k)$. It is nonnegative, and its sum over any subset of primes is $O(e^{-|t_j - t_k|})$. Choose h_0 so that the sum of all off-diagonal bounds in each row is less than half the diagonal lower bound. The covariance is then at least cI, as is seen by expanding its quadratic form and using $2|z_j z_k| \le z_j^2 + z_k^2$. The sum of third ℓ^1-moments of the tail blocks and their matching Gaussians is

$$\le Cd^3 \sum_{q_{\mathfrak{p}} > e^Q} (\log q_{\mathfrak{p}})^3 q_{\mathfrak{p}}^{-3/2} \le Cd^3 (Q+1)^2 e^{-Q/2}.$$

Apply the Taylor replacement argument to $\prod_j \psi(e^{n_j} z_j)$, where $\psi = 1$ on $[-1,1]$, vanishes outside $[-2,2]$, and is a fixed smooth function between zero and one. Its specified third derivatives are $O(e^{3N})$. The Gaussian density is at most C^d, uniformly in the conditioned translation, and the support has volume $4^d e^{-\sum n_j}$. This proves (2.9).

Finally put $V = \sum_j \log^+ |G_X(t_j)|^{-1}$, $R = d + v$, $v \ge 0$, $N = \lceil BR \rceil$, and $Q = DR$. Split the coordinates according as $\delta_j > 1/Q$ or $\delta_j \le 1/Q$. There are $k = O(\log R)$ coordinates in the first group. If their total loss exceeds $N + k$, their integer parts admit an allocation of total mass N. Omitting zero entries, the union bound and (2.8), with $\varepsilon_0 = 1/Q$, give

$$\mathbb{P}(\text{first loss} > N + k) \le \exp\{-BR + C_{B,D}(1 + \log R)^7\}.$$

Indeed there are at most $(N+k)^k$ allocations, each box has volume at most $2^k e^{-N}$, and the density exponent is $O(k^5 \log^2(2kQ))$. For the second group of size d_2, a loss exceeding $N + d_2$ likewise admits total integer mass N. The number of allocations is at most $\binom{N+d}{d} \le e^{E_B R}$, $E_B = 1 + \log(B+3)$. Conditioning and (2.9) bound its probability by

$$e^{E_B R}\{C_1^d e^{-N} + Cd^3(Q+1)^2 e^{-Q/2+3N}\}.$$

Choose first $B > E_B + \log C_1 + 4$, and then $D/2 > E_B + 3(B+1) + 5$, also $D \ge Q_0$. If $V > 2N + d$, one of the two group thresholds is exceeded. Thus $\mathbb{P}(V > C_4(d+v)) \le C_5 e^{-c_4(d+v)}$, after increasing constants for bounded R. The density estimate in dimension one excludes atoms at zero. Integrating the last tail with $0 < a_0 < c_4/(2C_4)$ proves (2.7). □

Corollary 2.8 (One-point negative moment with fixed omissions)**.** *Let K be a fixed number field satisfying (2.1), and let $\mathcal{S}$ be a fixed finite set of prime ideals. For each prime ideal take an independent variable*

$$\mathbb{P}(X_{\mathfrak{p}} = \pm 1) = \theta_{\mathfrak{p}}/2, \qquad \mathbb{P}(X_{\mathfrak{p}} = 0) = 1 - \theta_{\mathfrak{p}}.$$

Suppose $0 \le \theta_{\mathfrak{p}} \le 1$ for all $\mathfrak{p}$, and $|\theta_{\mathfrak{p}} - \theta| \le A_*(N\mathfrak{p})^{-\varepsilon_*}$ outside $\mathcal{S}$, with $0 < \theta_* \le \theta \le 1$. Define the analytic normalized logarithmic derivative G_X by (R.1), with the sum over prime ideals. There are $a_0 > 0$ and $C < \infty$, depending only on these fixed data, such that

$$\sup_{t \ge \log 2} \mathbb{E}|G_X(t)|^{-a_0} \le C. \tag{2.10}$$

In particular one may set $\theta_{\mathfrak{p}} = 0$ at every prime in $\mathcal{S}$, so the corollary permits fixed ramified omissions, including the ramified prime of $\mathbb{Q}(i)$. No independence is imposed on rational norms: the independent coordinates are the distinct prime ideals.

Proof. Enlarge the class constant to

$$A'_* = \max\left(A_*, \max_{\mathfrak{p} \in \mathcal{S}} (N\mathfrak{p})^{\varepsilon_*}\right),$$

with the second maximum omitted when $\mathcal{S}$ is empty. Since $|\theta_{\mathfrak{p}} - \theta| \le 1$, the full prime-ideal law now satisfies (2.2) with A'_*. Apply Lemma 2.7 with $d = 1$; its separation condition is then empty. The point has no atom at zero, and

$$|G_X(t)|^{-a_0} \le \exp\{a_0 \log^+ |G_X(t)|^{-1}\},$$

which proves (2.10). To make explicit why finite omissions cause no hidden rational-prime reduction, the proof of that lemma uses the prime-ideal counting measure $\sum_{\mathfrak{p}} (\log N\mathfrak{p}) \delta_{N\mathfrak{p}}$. Equal norms contribute separate independent characteristic factors. Its large-prime Fourier windows may start beyond all of $\mathcal{S}$, and its conditional tail estimate is uniform in every value of the finitely many conditioned blocks. Thus zero probabilities at the omitted primes do not enter any tail covariance lower bound. This argument uses the fixed-field prime-ideal theorem; it does not infer the corollary from the rational-prime density criterion in Lemma 2.5. □

Lemma 2.9 (Jensen bounds and exponential zero moments)**.** *Let $N_X^{\mathrm{mult}}(u, T)$ count the real zeros of G_X in $[u, u + T]$, with multiplicity. There are $a, b > 0$ such that*

$$\mathbb{E} e^{a N_X^{\mathrm{mult}}(u,T)} \le e^{b(T+1)} \quad (u \ge \log 2,\ T \ge 1).$$

The constants are uniform over (2.2).

Proof. For a holomorphic $f \not\equiv 0$, let m and M be its maxima on the concentric discs of radii $1/8$ and $3/4$ about t. Choose z in the smaller disc attaining $m > 0$. The interval $[t - 1/4, t + 1/4]$ is contained in $|w - z| < 1/2$, and $|w - z| \le 5/8$ lies in the larger disc. Jensen on radii approaching $5/8$, avoiding boundary zeros, gives

$$N_f([t - 1/4, t + 1/4]) \le \frac{\log(M/m)}{\log(5/4)}. \tag{R.3}$$

This includes endpoints and all multiplicities. For $f = G_X$, the numerator is at most $A(t) + \log^+ |G_X(t)|^{-1}$. Take $t_j = u + j/2$, $0 \le j \le \lceil 2T \rceil$, which cover the desired interval. Partition these indices into a fixed number L of residue classes with $L/2 \ge h_0$. Hölder across classes and (2.7) give

$$\mathbb{E} \exp\left\{\frac{a_0}{L} \sum_j \log^+ |G_X(t_j)|^{-1}\right\} \le C^{2T+2}.$$

Lemma 2.6 controls the sum of the $A(t_j)$. Cauchy–Schwarz and a sufficiently small a now prove the assertion. □

2.4. **The strong random zero theorem.**

Theorem 2.10. *For the symmetric Euler class over the rational primes, set*

$$c_\theta = \frac{e^{-\theta/2}}{\pi\sqrt{2}}.$$

For every deterministic $u \geq \log 2$, $T \geq 1$, and $\diamond \in \{\mathrm{dist}, \mathrm{mult}\}$,

$$\left|\mathbb{E}N_X^\diamond(u,T) - c_\theta T\right| \leq Ce^{-5u/56}. \tag{2.11}$$

For every $\eta > 0$ there are positive c_η, C_η such that

$$\mathbb{P}(|N_X^\diamond(u,T) - c_\theta T| > \eta T) \leq C_\eta e^{-c_\eta T}. \tag{2.12}$$

Moreover the limit

$$B_X^\diamond(u) = \lim_{T\to\infty}\{\mathbb{E}N_X^\diamond(u,T) - c_\theta T\}$$

exists, has absolute value at most $Ce^{-5u/56}$, and the error in this limit is $O(e^{-5(u+T)/56})$. The notation $B_X^\diamond$ denotes a deterministic quantity depending on the law. All constants are uniform over the fixed Euler class. These estimates concern each deterministic interval; they do not assert a single event valid for every interval.

Proof of the expectation estimate. We record both the Gaussian crossing calculation and the control of multiple zeros, since sign changes alone would not prove (2.11). Let $p_\theta(h)$ be the probability of opposite strict signs for $\mathcal{G}(0), \mathcal{G}(h)$, where now $m = v = \theta$. Put

$$U = (g(0) + g(h))/2, \qquad V = (g(h) - g(0))/h.$$

They are independent centered Gaussians, with variances $(1 + r(h))/2 = 1 + O(h^2)$ and $2(1 - r(h))/h^2 = 1/2 + O(h^2)$. The crossing event is $|U + \sqrt{\theta}| < h|V|/2$. If q_σ is the density of a centered Gaussian of variance σ, then

$$\frac{p_\theta(h)}{h} = \mathbb{E}\left[|V|\int_{-1/2}^{1/2} q_{(1+r(h))/2}(-\sqrt{\theta} + h|V|z)\,dz\right] = c_\theta + O(h^2), \quad 0 < h \leq 1/2.$$

The linear Taylor term integrates to zero, the second derivative of the density is uniformly bounded, and $\mathbb{E}|V|^3 = O(1)$. The leading term is $(2\pi)^{-1/2}e^{-\theta/2}/\sqrt{\pi} = c_\theta$. All bounds are uniform in $\theta \in [\theta_*, 1]$. The same calculation for the general process in (2.6) gives

$$p_{m,v}(h) = c(m,v)h + O(h^3), \qquad c(m,v) = \frac{1}{\pi\sqrt{2}}\exp(-m^2/(2v)). \tag{2.13}$$

For one or two Euler coordinates at times at least t, smooth a half-line or a strict crossing test on scale $\epsilon = e^{-t/2}$. Each specified k-th derivative is $O(\epsilon^{-k})$. The Gaussian marginals have bounded densities, and (R.2) therefore gives

$$\mathbb{P}(|G_X(v)| \leq a) \leq Ca + Ce^{-v/2}, \qquad \mathbb{P}(G_X(v) = 0) \leq Ce^{-v/2}, \tag{2.14}$$

$$|\mathbb{P}(G_X(v)G_X(v+h) < 0) - p_\theta(h)| \leq Ce^{-v/2}. \tag{2.15}$$

For example, the smooth crossing test is $a_\epsilon(x)(1 - a_\epsilon(y)) + (1 - a_\epsilon(x))a_\epsilon(y)$, where a_ϵ is a smooth step between $-\epsilon$ and ϵ. Its disagreement with strict crossing is supported on the two marginal strips. Thus no lower bound on the two-point covariance eigenvalue is used.

Lemma 2.2 and Cauchy's formula give

$$\sup_{v\geq\log 2}\mathbb{E}\sup_{[v,v+1]}|G_X''|^8 < \infty.$$

Lemma 2.9 gives a uniform fourth moment for the multiplicity count on a unit interval. For a cell $I = [v, v + d]$, $0 < d \leq 1/2$, write N_I for its multiplicity count, $D_I = \sup_I |G_X''|$, and choose

$K = d^{-1/4}$. If $N_I \geq 2$, Rolle's theorem, including a repeated root, gives a point with derivative zero and a point with value zero. Hence, when $D_I \leq K$, $|G_X(v)| \leq Kd^2$. Markov and (2.14) imply

$$\mathbb{P}(N_I \geq 2) \leq C(d^{7/4} + e^{-v/2}), \qquad \mathbb{E}[N_I \mathbf{1}_{\{N_I \geq 2\}}] \leq C(d^{21/16} + e^{-3v/8}). \tag{R.4}$$

The second bound uses Hölder with the fourth zero-count moment. At a deterministic point v, its root multiplicity $Z(v)$ similarly satisfies $\mathbb{E}Z(v) \leq Ce^{-3v/8}$.

Consider $[t, t+L]$, $0 < L \leq 1$. Take $h = \frac{1}{2}e^{-2t/7}$, $n = \lceil L/h \rceil \geq 1$, and $d = L/n$. Let S be the strict sign-change count on this mesh. Every simple interior zero which is the only zero in its cell is detected by S; all other zeros are accounted for either by crowded cells or by mesh endpoints. Thus

$$S \leq N^{\mathrm{dist}} \leq N^{\mathrm{mult}} \leq S + \sum_I N_I \mathbf{1}_{\{N_I \geq 2\}} + 2 \sum_{\text{mesh } v} Z(v).$$

Equations (2.15), (R.4), and the crossing calculation give

$$|\mathbb{E}S - c_\theta L| \leq C(h^2 + h^{-1}e^{-t/2}), \qquad \mathbb{E}(N^{\mathrm{mult}} - S) \leq C(h^{5/16} + h^{-1}e^{-3t/8}).$$

Here $n \leq 1 + h^{-1}$, and $nd^{21/16} = Ld^{5/16} \leq h^{5/16}$. Both leading exponents in the second bound equal $5t/56$. Consequently

$$|\mathbb{E}N^\diamond(t, L) - c_\theta L| \leq Ce^{-5t/56}, \qquad 0 \leq L \leq 1. \tag{2.16}$$

For $L = 0$ use the point-multiplicity estimate.

Partition a longer closed interval into unit intervals and a last fraction. The sum of their counts overcounts precisely the shared endpoint roots; their expectations form a convergent geometric sum. Summing (2.16) proves (2.11). The same argument applied to $[u + T_1, u + T_2]$, with its common endpoint correction, bounds the difference of the two centered expectations by $Ce^{-5(u+T_1)/56}$, for $T_2 \geq T_1$. The Cauchy criterion proves the asserted limit and remainder. No equality between the two possible limits is asserted. □

For the concentration proof we give the late-interval estimates explicitly. They use a fixed mesh, chosen after the required tolerance.

Lemma 2.11 (Crowded cells and small mesh values)**.** *Fix $\tau, \zeta > 0$. There is a sufficiently small fixed $h > 0$ such that, for any mesh of spacing h in an interval of length at most T starting at $v \geq \log 2 + \tau T$, with at least one cell,*

$$\mathbb{P}\left(\sum_I N_I \mathbf{1}_{\{N_I \geq 2\}} > \zeta T\right) \leq Ce^{-cT}.$$

For this h, there is a fixed $\epsilon > 0$ such that

$$\mathbb{P}\left(\sum_{\text{mesh } x} \mathbf{1}_{\{|G_X(x)| \leq \epsilon\}} > \zeta T\right) \leq Ce^{-cT}.$$

One may decrease h, and then decrease ϵ, while retaining these conclusions.

Proof. Fix $H_0 \geq \max(h_0, 1)$. For cells of length at most $1/2$ whose centers c_i are H_0-separated, Jensen and Lemmas 2.6–2.7 imply

$$\mathbb{E}e^{a_N \sum_i N_i} \leq C_N^k, \qquad \mathbb{E}e^{\sum_i D_i} \leq C_D^k, \quad D_i = \sup_{I_i} |G_X''|. \tag{R.5}$$

Indeed Jensen at the center uses inner radius $1/4$ and outer radius $3/4$, or the weaker constant $\log(5/4)^{-1}$; Cauchy gives $D_i \leq 8A(c_i)$. Choose $a_N > 0$ small enough to use Cauchy–Schwarz and (2.7).

For every fixed $0 < z < e^{-1}$ and arbitrarily large fixed B, the conditional estimate (2.9) gives

$$\mathbb{P}(|G_X(c_i)| \le z \text{ for all } i \in S) \le (C_s z)^{|S|} + C_{B,z,\tau} e^{-BT}. \tag{2.17}$$

To verify uniformity, put $n_z = \lfloor \log(1/z) \rfloor \ge 1$, take $Q = DT$, and $N = n_z|S|$. For large T, $Qe^{-c_i} \le DTe^{-\tau T}/2 \le 1$. As $|S| \le T+2$, the error in (2.9) is at most $C(T+2)^3(DT+1)^2 e^{-DT/2+3n_z(T+2)}$. Choose $D/2 > B + 3n_z + 2$; bounded T is absorbed in the constant. Also $e^{-n_z} \le ez$, giving the first term.

For crowded cells choose $K = h^{-1/4}$ and $z = h^{7/4}$. The Rolle argument gives $\{N_i \ge 2\} \subset \{|G_X(c_i)| \le z\} \cup \{D_i > K\}$. For any separated collection of k cells, expand this union over subsets and apply Hölder with exponents $8/7$ and 8. Using (2.17) and (R.5), the probability that all cells are crowded is at most

$$(p_0^{7/8} + p_D^{1/8})^k + \epsilon_T^{7/8}(1 + p_D^{1/8})^k, \quad p_0 = C_s h^{7/4}, \quad p_D = C_D e^{-h^{-1/4}}, \quad \epsilon_T = C_B e^{-BT}. \tag{R.6}$$

The two displayed contributions are a sum, including for an empty subset. Because B can be arbitrarily large, this is $p(h)^k + C_B e^{-BT}$, where

$$p(h) = O(h^{49/32}) + O(e^{-h^{-1/4}/8}).$$

Hölder once more, now with exponents 4 and $4/3$, yields

$$\mathbb{E}\left[e^{(a_N/4)\sum_i N_i} \mathbf{1}_{\{N_i \ge 2 \ \forall i\}}\right] \le q(h)^k + C_N^{k/4} C_B^{3/4} e^{-3BT/4},$$

where $q(h) = C_N^{1/4} p(h)^{3/4} = O(h^{147/128}) + O(e^{-3h^{-1/4}/32}) = o(h)$. Expand the exponential of $\sum Y_i$, $Y_i = N_i \mathbf{1}_{\{N_i \ge 2\}}$, as a product of $1 + (e^{a_N Y_i/4} - 1)$, and use the last bound for every subset. For a separated class of m cells this gives

$$\mathbb{E}e^{(a_N/4)\sum_i Y_i} \le (1 + q(h))^m + C_B^{3/4} e^{-3BT/4}(1 + C_N^{1/4})^m \le 2e^{q(h)m},$$

by choosing B sufficiently large. Partition the complete mesh into $L = \lceil H_0/h \rceil$ residue classes. Hölder across classes gives $\mathbb{E}e^{a_N \sum Y_i/(4L)} \le 2e^{q(h)n/L}$. Since $n \le T/h$, its Markov exponent at ζT is at most

$$\left[-\frac{a_N \zeta h}{4(H_0+1)} + \frac{q(h)}{H_0}\right] T.$$

Choose h so that this is strictly negative.

For the small mesh indicators J_ϵ, the same subset expansion using (2.17) gives

$$\mathbb{E}e^{J_\epsilon/L} \le 2\exp\{(e-1)C_s\epsilon(n+1)/L\}.$$

Choose $\epsilon \le \zeta h H_0/[2(e-1)C_s(H_0+1)]$. Markov proves the second assertion, again with bounded T absorbed. □

Lemma 2.12 (Concentration of the smooth sign count)**.** *Fix $h, \epsilon, \tau, \zeta > 0$. On the mesh $x_j = v + jh$, $0 \le j \le n$, where $n \ge 1$, $nh \le T$, and $v \ge \log 2 + \tau T$, let*

$$S_\epsilon = \sum_{j<n} \{a_\epsilon(G_X(x_j))(1 - a_\epsilon(G_X(x_{j+1}))) + (1 - a_\epsilon(G_X(x_j)))a_\epsilon(G_X(x_{j+1}))\}.$$

If m_ϵ is the mean of the analogous Gaussian expression, then

$$\mathbb{P}(|S_\epsilon - m_\epsilon| > \zeta T) \le Ce^{-cT}, \qquad |m_\epsilon - c_\theta nh| \le CT(h^2 + \epsilon/h).$$

Proof. The Gaussian covariance on this mesh has operator norm at most C/h: the row sum of $|r((j-k)h)|$ has this bound. The squared gradient norm of the smooth sign sum in its $n+1$ coordinates is at most $C(n+1)/\epsilon^2$, because each coordinate occurs in at most two summands. After writing the Gaussian vector as its mean plus a covariance square root times standard Gaussians, its squared Lipschitz constant is therefore at most $CT/(h^2\epsilon^2)$.

For completeness, the mean-centered Gaussian concentration used here is obtained from the Gaussian logarithmic Sobolev inequality $\mathrm{Ent}_\gamma(f^2) \le 2\int |\nabla f|^2 d\gamma$ [21, Theorem 2.5, pp. 18–19]. For an L-Lipschitz function F, apply it to $f = e^{\lambda F/2}$. Writing $\Lambda(\lambda) = \mathbb{E}e^{\lambda F}$ gives $\lambda\Lambda' - \Lambda\log\Lambda \le \lambda^2 L^2\Lambda/2$. Integrate the derivative of $\lambda^{-1}\log\Lambda$, after subtracting $\mathbb{E}F$, to obtain $\mathbb{E}e^{\lambda(F-\mathbb{E}F)} \le e^{\lambda^2L^2/2}$. Truncation and smooth approximation extend this to every Lipschitz F. Chernoff applied with both signs yields

$$\mathbb{P}(|S_{\epsilon,\mathrm{G}} - m_\epsilon| > sT) \le 2e^{-cs^2h^2\epsilon^2T}.$$

This is also the argument in [21, pp. 20–21].

To transfer this event, majorize each tail by a smooth function which changes from zero to one over a band of width $\zeta T/2$. Its composition with the sign sum has every specified derivative of order $k \le 3$ bounded by $C_\zeta\epsilon^{-k}$, independently of the dimension: a fixed coordinate enters only two adjacent terms. In (R.2), $d = n+1 = O(T/h)$ and $b \le e^{-\tau T}/2$. The resulting error is $C(T+1)^3e^{-\tau T} \le Ce^{-\tau T/2}$. This proves the probability assertion. A smooth sign test differs from strict crossing only in marginal strips of probability $O(\epsilon)$. Equation (2.13), multiplied by n, proves the estimate for the mean. □

Proof of (2.12). We specify the deterministic counting step. For a real analytic function not identically zero, a mesh of $n \ge 1$ cells satisfies, with S its strict sign changes and J_ϵ its small-value mesh count,

$$S \le N^{\mathrm{dist}} \le N^{\mathrm{mult}} \le S + \sum_I N_I\mathbf{1}_{\{N_I\ge 2\}} + 2\sum_{\text{mesh } x}\mathbf{1}_{\{G_X(x)=0\}}.$$

Every endpoint root of multiplicity at least two is already included in a crowded adjacent cell; a simple endpoint needs at most the displayed indicator correction. Also $|S - S_\epsilon| \le 2J_\epsilon$. It follows that

$$N^{\mathrm{mult}} \le S_\epsilon + \sum_I Y_I + 4J_\epsilon, \qquad N^{\mathrm{dist}} \ge S_\epsilon - 2J_\epsilon. \tag{2.18}$$

The condition $n \ge 1$ is essential here: an interval consisting of one point may have arbitrary root multiplicity.

Let $\eta' = \min(\eta, 1)$, $\zeta = \eta'/100$, and let a, b be the constants of Lemma 2.9. Choose $\tau \le \min(1/4, a\zeta/(4b), \zeta/c_{\max})$, where $c_{\max} = 1/(\pi\sqrt{2})$. Next choose h small enough for Lemma 2.11 and $Ch^2 \le \zeta/2$; then choose ϵ small enough for that lemma and $C\epsilon/h \le \zeta/2$. All these choices are fixed before T. For sufficiently large T, put

$$v = u + \tau T, \quad n = \lfloor (1-\tau)T/h \rfloor \ge 1, \quad w = v + nh.$$

The exponential zero moment shows that the front interval $[u, v]$ and the last interval $[w, u+T]$, of length less than h, each contain at most ζT zeros with multiplicity, except with probability Ce^{-cT}. For the front interval, for example, the Markov bound is $e^be^{(b\tau - a\zeta)T}$. On the late mesh, the two previous lemmas and (2.18) give, outside another exponentially small set,

$$N^{\mathrm{mult}}([v,w]) \le c_\theta nh + 7\zeta T, \qquad N^{\mathrm{dist}}([v,w]) \ge c_\theta nh - 4\zeta T.$$

The full multiplicity count is at most the sum of the three counts, so is at most $c_\theta T + 9\zeta T$. The full distinct count is at least the late one, hence at least $c_\theta T - 6\zeta T$, once $c_{\max}h \le \zeta T$. Both deviations are less than ηT. Increasing C_η covers bounded T, and proves the theorem. □

2.5. **A finite-block polynomial test.**

Lemma 2.13. *Let $X_\alpha \in \{0, \pm 1\}$ be independent symmetric variables with $\mathbb{E}X_\alpha^2 = \theta_\alpha$, indexed by a finite set. For $a_\alpha, b_\alpha, \mu \in \mathbb{R}^r$ put*

$$\xi_\alpha = a_\alpha X_\alpha + b_\alpha(X_\alpha^2 - \theta_\alpha), \qquad Y = \mu + \sum_\alpha \xi_\alpha, \qquad \Sigma = \mathrm{Cov}\, Y.$$

Suppose, for fixed $c_0, C_0 > 0$,

$$c_0 I \le \Sigma \le C_0 I, \quad \|\mu\|^2 \le C_0 r, \quad \sum_\alpha (a_\alpha a_\alpha^{\mathsf T} + b_\alpha b_\alpha^{\mathsf T}) \le C_0 I, \quad \beta_\alpha := \|a_\alpha\| + \|b_\alpha\| \le 1.$$

There is a sufficiently small fixed $\epsilon_0 > 0$ such that, for all sufficiently large r, with $A = \operatorname{tr}\Sigma + \|\mu\|^2$, $m = \lfloor \epsilon_0 r \rfloor$, and

$$\mathcal{Q}(z) = \left(\frac{A - \|z\|^2}{A - 1} \right)^{2m}, \qquad Z \sim N(\mu, \Sigma),$$

one has $\mathcal{Q} \ge 0$, $\mathcal{Q}(z) \ge 1$ for $\|z\| \le 1$, and

$$\mathbb{E}\mathcal{Q}(Z) \le e^{-cr}, \qquad |\mathbb{E}\mathcal{Q}(Y) - \mathbb{E}\mathcal{Q}(Z)| \le e^{Cr} \sum_\alpha \beta_\alpha^3. \tag{2.19}$$

The constants depend only on c_0, C_0 and the fixed choice of ϵ_0.

Proof. We have $c_0 r \le A \le 2C_0 r$; take r large enough that $A - 1 \ge c_0 r/2$ and $m \ge \epsilon_0 r/2 \ge 1$. Nonnegativity and the unit-ball majorization follow from the even power. Diagonalize the Gaussian as $Z_i = u_i + \sqrt{\lambda_i} g_i$, where $c_0 \le \lambda_i \le C_0$. For $V = \|Z\|^2 - A$, direct Gaussian integration gives

$$\log \mathbb{E} e^{tV} = \sum_i \left\{ -\tfrac{1}{2} \log(1 - 2t\lambda_i) - t\lambda_i + \frac{2t^2 \lambda_i u_i^2}{1 - 2t\lambda_i} \right\} \le Ct^2 r, \quad |t| \le (4C_0)^{-1}.$$

Chernoff with both signs, choosing t proportional to $\min(v/r, 1)$, gives

$$\mathbb{P}(|V| > v) \le 2e^{-c \min(v^2/r, v)}.$$

Tail integration therefore gives $\|V\|_q \le C(\sqrt{rq} + q)$, $q \ge 2$. At $q = 2m$, its ratio to $A - 1$ is at most $C(\sqrt{\epsilon_0} + \epsilon_0)$. Choose ϵ_0 so that this is at most $1/2$. Then

$$\mathbb{E}\mathcal{Q}(Z) \le 2^{-2m} \le e^{-cr}.$$

Replace the ξ_α successively by independent centered Gaussian blocks g_α with matching covariances. If W is a centered intermediate sum with the current block omitted, the bounded-variable inequality proved in Lemma 2.2 gives $\mathbb{E} e^{\langle h, W \rangle} \le e^{C\|h\|^2}$. Indeed the squared range lengths sum to at most $8C_0\|h\|^2$; the Gaussian replacements satisfy the same bound. A maximal $1/2$-separated set on the unit sphere is a $1/2$-net of at most 5^r points, by disjoint radius-$1/4$ balls and volume comparison. The norm is at most twice the largest absolute projection on this net. Scalar Chernoff and a union bound give $\mathbb{P}(\|W\| > C\sqrt{r} + s) \le 2e^{-cs^2}$. Consequently, uniformly throughout the replacement,

$$(\mathbb{E}\|W + \mu\|^q)^{1/q} \le C(\sqrt{r} + \sqrt{q}), \qquad q \ge 2. \tag{2.20}$$

Write $p = 2m$ and $q(z) = A - \|z\|^2$. Differentiating the polynomial three times gives

$$D^3 \mathcal{Q}(z)[h,h,h] = (A-1)^{-p} \{ p(p-1)(p-2) q^{p-3} (Dq[h])^3 + 3p(p-1) q^{p-2} Dq[h] D^2 q[h,h] \}.$$

When $p = 2$, the first term is absent, so no negative power is evaluated at $q = 0$. Since $|Dq[h]| \le 2\|z\|\|h\|$, $|D^2 q[h,h]| \le 2\|h\|^2$, $p = O(r)$, and $A - 1 \ge cr$,

$$|D^3 \mathcal{Q}(z)[h,h,h]| \le e^{Cr} \|h\|^3 (1 + \|z\|/\sqrt{r})^{2p+3}. \tag{R.7}$$

For $z = W + \mu + th$, $0 \le t \le 1$, bound the last factor by the product of the corresponding factors for $W + \mu$ and h. The first has expectation at most e^{Cr} by (2.20), since its power is $O(r)$. For $h = \xi_\alpha$, $\|h\| \le \beta_\alpha \le 1$, so

$$\mathbb{E}[\|h\|^3 (1 + \|h\|/\sqrt{r})^{2p+3}] \le e^{Cr} \beta_\alpha^3.$$

For $h = g_\alpha$, its covariance trace is at most β_α^2, and Gaussian diagonalization gives $\mathbb{E}\|g_\alpha\|^k \le (C\sqrt{k}\,\beta_\alpha)^k$. Expanding the power $K = 2p+3$ proves the same bound:

$$\sum_{j=0}^{K}\binom{K}{j} r^{-j/2}\mathbb{E}\|g_\alpha\|^{j+3} \le \beta_\alpha^3 \sum_{j=0}^{K}\binom{K}{j} C^{j+3}(j+3)^{(j+3)/2} r^{-j/2} \le e^{Cr}\beta_\alpha^3.$$

The omitted block is independent of W. Taylor's integral remainder, (R.7), and the equality of means and covariances show that a single replacement costs at most $e^{Cr}\beta_\alpha^3$. Summing proves (2.19). □

2.6. **Finite Euler polynomials and finite-word orthogonality.** We use the explicit weight

$$w_y(n) = \frac{\log_+^2(y/n) - 2\log_+^2(y^2/n) + \log_+^2(y^3/n)}{2(\log y)^2}, \qquad y = e^\ell, \quad \ell \ge 8. \tag{2.21}$$

It is between zero and one, equals one for $n \le y$, vanishes for $n > y^3$, and is nonincreasing. These facts follow by evaluating the three quadratic polynomials on the intervals cut out by $\log n = \ell, 2\ell, 3\ell$. For the real model of (2.3) put

$$P_{y,E}(s) = \sum_{p^k \le y^3} w_y(p^k)(\log p) a_{p,k} p^{-ks}.$$

An arithmetic family $\mathcal{F}_X$, equipped with its original probability measure, has corresponding coefficients $a_{p,k}(f)$. A word $W = \prod_{j=1}^r a_{p_j,k_j}$ is allowed to repeat primes and indices; set $n(W) = \prod_j p_j^{k_j}$. Our orthogonality hypothesis is

$$|\mathbb{E}_X W - \mathbb{E}_E W| \le C X^{-\kappa} C_0^r n(W)^A, \qquad \kappa > 0, \quad A \ge 0, \quad C_0 \ge 1. \tag{2.22}$$

The outside constant is uniform in every word and X. No conditioning of the arithmetic family is implicit in this hypothesis.

Lemma 2.14 (The arithmetic norm)**.** *Fix a formal representation $P = \sum_W c_W W$, and set $\|P\|_{\mathrm{ar}} = \sum_W |c_W| C_0^{r(W)} n(W)^A$. Products are represented by word concatenation. Then*

$$\|PQ\|_{\mathrm{ar}} \le \|P\|_{\mathrm{ar}}\|Q\|_{\mathrm{ar}}, \qquad |\mathbb{E}_X P - \mathbb{E}_E P| \le C X^{-\kappa}\|P\|_{\mathrm{ar}}.$$

For $\Re s \ge 1/2$ and bounded $|a|$,

$$\|aP_y(s)\|_{\mathrm{ar}} \le C e^{C_A \ell}. \tag{2.23}$$

There is no need to minimize over equivalent formal representations.

Proof. Both $r(W)$ and $\log n(W)$ are additive under concatenation. The triangle inequality proves submultiplicativity; summing (2.22) proves the expectation bound. Each term in aP_y is a word of length one, so its cost contains one factor C_0. Its coefficient has modulus at most $C\log(p^k)(p^k)^{-1/2}$. Enlarge the sum to all integers $n \le e^{3\ell}$ and use $\sum_{n \le e^{3\ell}} (\log n) n^{A-1/2} \le C e^{C_A \ell}$. This proves (2.23). □

Lemma 2.15 (Random polynomial moments and tails)**.** *In a fixed right sector and for bounded a, put $Q_E(a) = aP_{y,E}(1/2+a)$. Then, uniformly for $q \ge 2$,*

$$\|Q_E(a)\|_q \le C\sqrt{q}.$$

If $\Re(a)\ell \ge 1$, then

$$\|aF(a) - Q_E(a)\|_2 \le C(1+|a|\ell)^C e^{-c\Re(a)\ell}. \tag{2.24}$$

For real $a_j = e^{-t_j}$, $t_j \ge \log 2$, let $V_f = \sum_j \tau_j a_j P_{y,f}(1/2+a_j)$, $V_E = \sum_j \tau_j Q_E(a_j)$, and $T = \sum_j |\tau_j|$. For every integer $M \ge 1$,

$$\mathbb{E}_X|V_f|^{2M} \le (CT\sqrt{M})^{2M} + CX^{-\kappa}(CTe^{C_A\ell})^{2M}. \tag{2.25}$$

This moment bound also holds for complex τ_j and bounded complex a_j in a fixed right sector, with the same definition of T.

Proof. The squared range lengths of the centered prime blocks of $Q_E(a)$ are at most those used for (2.5). The bounded-variable exponential estimate therefore gives L^q norm $O(\sqrt{q})$. Its mean is bounded as well: the prime-linear means and terms of degree at least three are absolutely bounded, while partial integration of the c_p-sum in (2.3) against the nonincreasing weight $w_y(p^2)$ gives

$$\left| a \sum_p w_y(p^2) c_p (\log p) p^{-1-2a} \right| \le C.$$

Explicitly the partial sums of $c_p \log p$ are $O(x)$; the integral of the derivative of x^{-1-2a} is $O(1/\Re a)$, and the weight has total variation one. Multiplication by $|a|$ is harmless in a fixed right sector.

For the tail, the centered linear variance is bounded by

$$C|a|^2 \sum_{p>y} (\log p)^2 p^{-1-2\Re a} \le C(1+|a|\ell) e^{-2\Re(a)\ell}.$$

The centered degree-two terms are bounded in L^2 by their absolute sum, which is $O((1+|a|\ell)^C e^{-c\Re(a)\ell})$, since their support has $p > \sqrt{y}$. The same partial integration, with the weight $1 - w_y(p^2)$, treats their mean. The linear mean is absolutely summable and has an exponentially small tail in ℓ. For completeness, all terms of degree at least three have $p^k > y$. Split

$$(p^k)^{-1/2-\Re a} \le y^{-1/12} (p^k)^{-5/12-\Re a}.$$

The sum over $p, k \ge 3$ of $(\log p)(p^k)^{-5/12-\Re a}$ is bounded, because its first exponent is $5/4$. This yields $Ce^{-\ell/12}$, including the terms at finitely many bad primes. Combining these estimates proves (2.24), with a possibly smaller sector constant c.

Minkowski gives $\|V_E\|_{2M} \le CT\sqrt{M}$. Since V_f is real, its $2M$-th absolute moment equals the expectation of the polynomial V_f^{2M}. Apply Lemma 2.14 and (2.23) to this polynomial to obtain (2.25). For complex values apply the identical argument to $V_f^M \overline{V}_f^M$; the coefficients $a_{p,k}(f)$ are real, so conjugation changes only the deterministic coefficients and leaves their arithmetic norm unchanged. The same argument applies to mixed moments by expanding the specified finite polynomial. □

Proposition 2.16 (Characteristic-function transfer)**.** *In the notation above, for every integer $M \ge 1$,*

$$|\mathbb{E}_X e^{iV_f} - \mathbb{E}_E e^{iV_E}| \le 2(CT/\sqrt{M})^{2M} + CX^{-\kappa} \exp\{CM(\ell + \log(2+T))\}. \tag{2.26}$$

This applies, in particular, to every fixed-dimensional vector. The displayed bound also keeps all dimension dependence through T. If a test function Φ has integrable Fourier transform, any frequency region Ω gives the rigorous transfer bound

$$|\mathbb{E}_X \Phi(A_f) - \mathbb{E}_E \Phi(A_E)| \le (2\pi)^{-d} \left\{ \int_\Omega |\widehat{\Phi}(\tau)|\, \Delta(\tau)\, d\tau + 2 \int_{\Omega^c} |\widehat{\Phi}(\tau)|\, d\tau \right\},$$

where $\Delta(\tau)$ is the right side of (2.26). Thus a growing Fourier norm or tail is never discarded.

Proof. Taylor-expand e^{iV} only to degree $2M-1$. For real V its remainder is bounded by $|V|^{2M}/(2M)!$. The two moment bounds in Lemma 2.15 and $(2M)! \ge (2M/e)^{2M}$ control both remainders. The polynomial part has arithmetic error at most

$$CX^{-\kappa} \sum_{j<2M} \frac{\|V\|_{\mathrm{ar}}^j}{j!}.$$

Using $\|V\|_{\mathrm{ar}} \le CTe^{C_A\ell}$ bounds this sum and the arithmetic part of the remainder by the second term in (2.26). No norm of an infinite exponential is used. Fourier inversion and Fubini give the final statement; outside Ω each characteristic function has modulus at most one. □

Lemma 2.17 (The ideal arithmetic norm)**.** *Suppose a_f is completely multiplicative on integral ideals and*

$$|\mathbb{E}_{\mu_x} a_f(\mathfrak{a}) - \mathbb{E}X(\mathfrak{a})| \le Cx^{-\kappa_o}(N\mathfrak{a})^{\lambda}.$$

For $P = \sum_{\mathfrak{a}} c_{\mathfrak{a}} a_f(\mathfrak{a})$, set $\|P\|_{\mathrm{ar},K} = \sum |c_{\mathfrak{a}}|(N\mathfrak{a})^{\lambda}$. This norm is submultiplicative and controls expectation transfer with factor $Cx^{-\kappa_o}$. The ideal versions of $P_y(s)$ and of $\sum_{q_{\mathfrak{p}}\le y^3} w_y(q_{\mathfrak{p}})a_f(\mathfrak{p})q_{\mathfrak{p}}^{-s}$ have norm at most $Ce^{C\ell}$ for $\Re s \ge 1/2$. For a finite prime-ideal sum $S_f = \sum c_{\mathfrak{p}} a_f(\mathfrak{p})$,

$$\mathbb{E}_{\mu_x}|S_f|^{2m} \le (Cm)^m \left(\sum |c_{\mathfrak{p}}|^2\right)^m + Cx^{-\kappa_o}\left(\sum |c_{\mathfrak{p}}| q_{\mathfrak{p}}^{\lambda}\right)^{2m}.$$

Proof. Complete multiplicativity makes the product coefficients the convolution over $\mathfrak{ab} = \mathfrak{c}$. Multiplicativity of the norm and the triangle inequality prove the first assertions without identifying ideals of equal norm. If $n_K = [K:\mathbb{Q}]$, the number of ideals of norm n is at most the divisor function $d_{n_K}(n)$: locally, $\prod_{i=1}^{r}(1-z^{f_i})^{-1}$ is bounded coefficientwise by $(1-z)^{-n_K}$. Counting divisor tuples gives

$$\#\{\mathfrak{a} : N\mathfrak{a} \le Y\} \le Y(1+\log Y)^{n_K-1}.$$

Since the ideal von Mangoldt function is at most $\log N\mathfrak{a}$, the polynomial norm is at most $(y^3)^{\lambda+1}\log(y^3)(1+3\ell)^{n_K-1} \le Ce^{C\ell}$. For the random moment, expansion of a real prime-ideal sum leaves only terms in which every prime ideal occurs an even number of times. Pairing the $2m$ positions bounds it by $(2m)!/(2^m m!)$ times the m-th power of the coefficient square sum. Split real and imaginary parts for complex coefficients. Apply the norm transfer to $S_f^m \overline{S}_f^m$ to get the last display. Equal norms still correspond to independent coordinates: $\mathfrak{pq}$ need not be a square ideal when $N\mathfrak{p} = N\mathfrak{q}$. □

Lemma 2.18 (Analytic coefficients near an endpoint)**.** *For the symmetric ideal model, $0 < \Delta \le 1/2$, and $|z|, |w| \le 1/2$, put*

$$a(z) = \Delta\frac{1+z}{1-z}, \qquad H_\Delta(z) = \frac{\Delta}{(1-z)^2}\left(-\frac{L_X'}{L_X}\right)(1/2 + a(z)).$$

Then

$$\mathbb{E}H_\Delta(z) = \frac{\theta}{2(1-z^2)} + O(\Delta), \qquad \mathrm{Cov}(H_\Delta(z), H_\Delta(w)) = \frac{\theta}{4(1-zw)^2} + O(\Delta^2).$$

In particular, if $Z_k = 2[z^k]H_\Delta(z)/\sqrt{\theta(k+1)}$, then

$$\mathrm{Cov}(Z_k, Z_l) = \mathbf{1}_{k=l} + O(\Delta^2 2^{k+l}), \qquad \sum_{k<r} |\mathbb{E}Z_k|^2 \le C(1+\log(r+1)) + C\Delta^2 4^r.$$

Replacing the logarithmic derivative in H_Δ by P_y gives an L^2 error at every $|z| \le 1/2$ of squared size at most $Ce^{-\Delta\ell/3}$.

Proof. Divide (2.4) by its normalizing factors. The two exact identities

$$\frac{\Delta}{(1-z)^2}\frac{1}{2a(z)} = \frac{1}{2(1-z^2)}, \qquad \frac{\Delta^2}{(1-z)^2(1-w)^2(a(z)+a(w))^2} = \frac{1}{4(1-zw)^2}$$

give the first formulas. Cauchy on circles of radius $1/2$ gives the coefficient errors; the main means have squares $\theta/(k+1)$ for even k and zero for odd k.

For the sharper squared tail bound in this symmetric model, $\mathbb{E}[X(\mathfrak{a})X(\mathfrak{b})] \ge 0$ for all ideals. Let $r_z = \Re a(z) \ge \Delta/3$. In the finite expansion of the second moment of the tail, both ideal

norms exceed y. Remove half the real offset from each norm. The nonnegative expectations and $0 \le 1 - w_y \le 1$ give

$$\mathbb{E}|\mathcal{L}_X(1/2 + a(z)) - P_{y,X}(1/2 + a(z))|^2 \le y^{-r_z}\mathbb{E}|\mathcal{L}_X(1/2 + r_z/2)|^2 \le Cy^{-r_z}r_z^{-2}.$$

The last inequality follows from the prime variance and the absolute higher-power sum computed above. Finite sums converge in L^2; multiplication by $\Delta/(1-z)^2$ proves the asserted bound. □

2.7. **The abstract two-sided order theorem.** We state precisely the arithmetic hypotheses used by the order theorem. They concern a fixed number field; none asserts uniformity as the field or its discriminant varies. The measure of the full finite family is kept throughout, including in estimates containing indicators.

Definition 2.19 (Abstract Euler-family hypotheses)**.** Let K be a fixed number field satisfying (2.1). For every sufficiently large x, let $(\mathcal{F}_x, \mu_x)$ be a finite probability space. Fix a symmetric prime-ideal law (2.2). We impose the following conditions, with constants independent of x and f.

(1) Each a_f takes values in $\{0, \pm 1\}$, is completely multiplicative on integral ideals, and satisfies $a_f(\mathcal{O}_K) = 1$. At least one prime ideal is active for each f. For $\Re s > 1$,

$$L_f(s) = \sum_{\mathfrak{a}} \frac{a_f(\mathfrak{a})}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - a_f(\mathfrak{p})q_{\mathfrak{p}}^{-s})^{-1}.$$

(2) There are $\kappa_o > 0, \lambda \ge 0$ such that, for every ideal,

$$|\mathbb{E}_{\mu_x} a_f(\mathfrak{a}) - \mathbb{E}X(\mathfrak{a})| \le Cx^{-\kappa_o}(N\mathfrak{a})^{\lambda}, \qquad \mathbb{E}X(\mathfrak{a}) = \mathbf{1}_{\{\mathfrak{a}=\mathfrak{b}^2\}} \prod_{\mathfrak{p}|\mathfrak{a}} \theta_{\mathfrak{p}}.$$

The square condition is on the ideal, rather than its norm. The conductors satisfy, for fixed $0 < b \le B$, $\eta_c > 0$,

$$3 \le Q_f \le x^B, \qquad \mu_x\{Q_f < x^b\} \le Cx^{-\eta_c}.$$

(3) There are finite ideal Dirichlet polynomials $M_f(s)$ and fixed $C_m, C_t, \kappa_m > 0$ such that, for $0 \le v \le 3/2$ and all real t,

$$\mathbb{E}_{\mu_x}|L_f(1/2 + v + it)M_f(1/2 + v + it) - 1|^2 \le C(\log x)^{C_m}(1 + |t|)^{C_t}e^{-\kappa_m v \log x}. \tag{2.27}$$

In particular the mollifier is holomorphic at the zeros whose detection is required.

(4) For every sufficiently large fixed A, some finite $C_d(A)$ and C_A satisfy, with $H = \log\log x$, $\delta = AH/\log x$,

$$\sup_{|s-1/2|\le 8\delta} \mathbb{E}_{\mu_x}|L_f'(s)|^2 \le C_A e^{C_d(A)H}. \tag{2.28}$$

(5) The function L_f is entire, real on the real axis, and nonvanishing on $\Re s \ge 1$. Its nontrivial zeros lie in $0 < \Re\rho < 1$, and conjugation and $\rho \mapsto 1 - \bar{\rho}$ preserve their multiplicities. The finite-order completion $\Lambda_f(s) = Q_f^{s/2}\gamma_f(s)L_f(s)$ satisfies

$$\Lambda_f(s) = w_f\Lambda_f(1-s), \qquad w_f \in \{1, -1\}.$$

The factor γ_f has no zero or pole on $|s - 1/2| \le 1/4$, and $\gamma_f'/\gamma_f = O(1)$ there, uniformly in f. The archimedean remainder in the real-part identity below is uniformly bounded on $1/2 \le \sigma \le 2$, $|t| \le 1$. The nontrivial zero count in each unit height interval is $O(\log(Q_f(|T| + 2)))$. Horizontal contours avoiding zeros have the logarithmic-derivative

bounds, and left contours have the decay, needed for the following explicit formulas, which are included as hypotheses:

$$\mathcal{L}_f(s) = P_{e^\ell,f}(s) + \sum_{\rho \text{ nontrivial}} K_\ell(\rho - s) + E_{\mathrm{tr},f}(s), \qquad \mathcal{L}_f = -L_f'/L_f, \tag{2.29}$$

$$K_\ell(z) = \frac{e^{\ell z} - 2e^{2\ell z} + e^{3\ell z}}{\ell^2 z^3}, \qquad |E_{\mathrm{tr},f}(s)| \le Ce^{-\ell \Re s}/\ell^2,$$

$$\sum_{\rho \text{ nontrivial}} \frac{\sigma - \Re\rho}{|s-\rho|^2} = \tfrac{1}{2}\log Q_f + O(1) - \Re\mathcal{L}_f(s). \tag{2.30}$$

Here $\ell \ge 8$, $L_f(s) \ne 0$, and the first identity holds for $\Re s \ge 1/2$, $|\Im s| \le 1$, with absolutely convergent zero sum. The last identity is required on $1/2 \le \sigma \le 2$, $|t| \le 1$, with its paired zero sum permitting the positive absolutely convergent comparisons used in the proof. In these formulas

$$P_{y,f}(s) = \sum_{N\mathfrak{a} \le y^3} \frac{\Lambda_K(\mathfrak{a}) w_y(N\mathfrak{a}) a_f(\mathfrak{a})}{(N\mathfrak{a})^s}, \qquad \Lambda_K(\mathfrak{p}^k) = \log N\mathfrak{p},$$

and Λ_K is zero on the other ideals.

Theorem 2.20 (Abstract two-sided order)**.** *Under Definition 2.19, there are positive constants c, C, c_-, C_+ such that, outside a set of μ_x-measure at most $Ce^{-c\sqrt{\log\log x}}$,*

$$c_- \log\log Q_f \le N^{\mathrm{dist}}_{L_f'}([1/2,1]) \le N^{\mathrm{mult}}_{L_f'}([1/2,1]) \le C_+ \log\log Q_f.$$

Zeros at $1/2$ are included with their actual multiplicities. Both root numbers are allowed. If μ_x is weighted, the conclusion is for that same weighted measure.

Proof. The independent probabilistic and algebraic inputs have been proved above: analytic nontriviality in Lemma 2.1, smooth Gaussian comparison in Lemma 2.4, joint growth and negative moments in Lemmas 2.6 and 2.7, the finite-dimensional polynomial test in Lemma 2.13, the coefficient covariance and tail in Lemma 2.18, and arithmetic norm transfer in Lemma 2.17. The analytic assembly from the displayed family hypotheses, including the zero detector, retained and endpoint intervals, and the central functional-equation argument, is proved in Section A. That proof gives both bounds with $\log\log x$ and an exceptional measure $Ce^{-c\sqrt{\log\log x}}$. Remove also the set $Q_f < x^b$; on the remainder $\log\log Q_f = \log\log x + O_{b,B}(1)$. Its measure $O(x^{-\eta_c})$ is absorbed, and changing the two positive constants proves the displayed conductor-normalized bounds. □

Remark 2.21. The hypotheses do not claim that the mollified estimate follows from orthogonality, or that the explicit formulas follow merely from a functional equation. Replacing the logarithmic loss in (2.27) by x^ϵ would not suffice at $v = AH/\log x$. The central derivative bound is required for each sufficiently large fixed A, after which A is chosen once. These distinctions are needed in the analytic proof.

3. Analytic approximation and local zero counting

We first give the analytic argument for real families under GRH. The arithmetic inputs are the finite-word orthogonality, weighted prime sums, and model density established in the preceding section. All expectations, including those with an indicator of a good event, use the original family denominator.

3.1. **Normalized analytic hypotheses.** Write $L_X = \log X$ and $H = \log L_X$. We consider real entire functions $L_D(s)$ with normalized Euler product

$$L_D(s) = \exp\left(\sum_{p,k\ge 1} \frac{a_{D,p,k}}{kp^{ks}}\right), \qquad |a_{D,p,k}| \le d, \quad \Re s > 1. \tag{3.1}$$

The degree bound d is fixed and $L_D' \not\equiv 0$. Suppose

$$\Lambda_D(s) = Q_D^{s/2} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s+\mu_j) L_D(s), \qquad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$

is entire of order one and satisfies $\Lambda_D(s) = \epsilon_D \Lambda_D(1-s)$, $\epsilon_D \in \{1,-1\}$. The nonnegative shifts μ_j range over a fixed finite set, and $cL_X \le \log Q_D \le CL_X$. For every fixed vertical strip we require

$$|L_D(s)| \le C_M[Q_D(2+|\Im s|)]^{C_M} \quad (-M \le \Re s \le M). \tag{3.2}$$

GRH in this section means that every zero of Λ_D lies on $\Re s = 1/2$. Put $\mathcal{L}_D = -L_D'/L_D$ and $G_D(w) = e^{-w}\mathcal{L}_D(1/2+e^{-w})$. This process is holomorphic for $|\Im w| < \pi/2$.

The smoothing weight below is the classical Selberg weight; see [20, Section 3 and Lemma 3.1] for its quadratic Dirichlet formulation and the zero-dependent anchor. We give the contour and anchor estimates needed for the present real families.

Lemma 3.1 (Paired zero formula and Selberg approximation)**.** *Under these hypotheses, if $\ell \ge 8$, $a \ge 4/\ell$, $a \le 3/2$ and $|t| \le 1$, then*

$$|\mathcal{L}_D(1/2+a+it) - P_{\ell,D}(1/2+a+it)| \le Ce^{-a\ell}\{L_X + |P_{\ell,D}(1/2+4/\ell+it)|\}. \tag{3.3}$$

Here

$$P_{\ell,D}(s) = \sum_{p^k \le e^{3\ell}} w_\ell(p^k) a_{D,p,k} (\log p) p^{-ks},$$

and, with $x = \log n/\ell$, the weight is

$$w_\ell(n) = \tfrac{1}{2}\{(3-x)_+^2 - 2(2-x)_+^2 + (1-x)_+^2\}. \tag{3.4}$$

In particular $0 \le w_\ell \le 1$, and $w_\ell = 1$ for $n \le e^\ell$.

Proof. Normalization gives $|L_D(2+iT)| \ge \exp(-d\sum_{p,k} p^{-2k}/k) > 0$. Jensen's formula on the discs centered at $2+iT$ with radii 2 and 3, together with (3.2), bounds the number of critical-line zeros with $|\gamma - T| \le 1$ by $C\log(Q_D(|T|+2))$. The gamma factors have no zeros, so these are actual zeros of L_D. Consequently the paired inverse-square sum converges.

Write $z = s - 1/2$. Pair the noncentral zeros $\alpha, -\alpha$ of the completion, retaining one representative in a multiset $\mathcal{P}_D$, and let m_D be the central multiplicity. The canonical product, paired in this order, converges normally away from its poles. The remaining exponential has the form e^{az+b}; parity in the functional equation forces $a = 0$. Thus

$$\mathcal{L}_D(1/2+z) = B_D(z) - \frac{m_D}{z} - \sum_{\alpha\in\mathcal{P}_D} \frac{2z}{z^2-\alpha^2}, \quad B_D(z) = \tfrac{1}{2}\log Q_D + \sum_j \frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(1/2+z+\mu_j). \tag{3.5}$$

The function B_D is holomorphic and $O(L_X)$ on $|z| \le 1/4$. On $0 < a \le 3/2$, $|t| \le 1$, its real part is $\frac{1}{2}\log Q_D + O(1)$. Under GRH, the real part of (3.5) becomes the positive identity

$$\sum_\rho \frac{a}{a^2+(t-\gamma)^2} = \Re B_D(a+it) - \Re\mathcal{L}_D(1/2+a+it). \tag{3.6}$$

Zeros in this sum are counted with multiplicity, including $\gamma = 0$.

For completeness the smoothing kernel is

$$K_\ell(z) = \frac{e^{\ell z} - 2e^{2\ell z} + e^{3\ell z}}{\ell^2 z^3}.$$

Inverse Mellin inversion of $(u_+)^2/2$ gives (3.4); the identity $(3-x)^2 - 2(2-x)^2 + (1-x)^2 = 2$ explains its normalization at small n. Integrating the absolutely convergent logarithmic Euler derivative on the right and moving the contour yields

$$\mathcal{L}_D(s) = P_{\ell,D}(s) + \sum_\rho K_\ell(\rho - s) + E_{\mathrm{tr},D}(s), \qquad |E_{\mathrm{tr},D}(s)| \le C\ell^{-2}e^{-\ell\Re s}. \tag{3.7}$$

Here the last sum of residues is from the fixed gamma progressions $-\mu_j - 2n$. Its bound follows by summing their geometric decay; multiplicities are at most the fixed degree. The sign of the nontrivial-zero term follows from using $K_\ell(\rho - s)$, whose pole has residue $+1$ in its argument.

We justify the contour limits explicitly. In each unit interval at large height choose a horizontal line at distance at least $c/\log(Q_D(T+2))$ from the zero ordinates. Subtracting the Hadamard logarithmic derivative at $2+it$ cancels the exponential constant. On these lines the logarithmic derivative on a fixed strip is $O_M(\log^2(Q_D(T+2)))$; distant zeros contribute an absolutely convergent inverse-square tail. The factor $|T|^{-3}$ in the kernel makes the horizontal integrals tend to zero. Choose the left lines a fixed distance from all of the finitely many gamma progressions. The functional equation, the Euler bound on the reflected right line, and the gamma reflection formula give $O(\log(Q_D(M+|t|+2)))$ there. The left exponential in the kernel then makes the remaining integral tend to zero. GRH ensures that a gamma pole is not also a zero of the completion, so its trivial-zero multiplicity is exactly the stated gamma multiplicity.

Set $a_0 = 4/\ell$. For real u,

$$|K_\ell(-a_0 + iu)| \le \frac{4e^{-4}}{\ell^2(a_0^2+u^2)^{3/2}} \le \frac{e^{-4}}{4}\frac{a_0}{a_0^2+u^2}.$$

Use (3.6) at a_0 in (3.7). Since $e^{-4}/4 < 1$, its logarithmic-derivative term can be absorbed, giving $|\mathcal{L}_D(1/2+a_0+it)| \le C(L_X + |P_{\ell,D}(1/2+a_0+it)|)$. For $a \ge a_0$,

$$|K_\ell(-a+iu)| \le \frac{4e^{-a\ell}}{\ell^2 a a_0}\frac{a_0}{a_0^2+u^2}.$$

Another use of the positive identity, and $\ell^2 a a_0 \ge 16$, proves (3.3). □

3.2. **Normalized polynomial moments.** We will repeatedly use the following consequence of finite-word transfer. In a fixed sector $|\arg a| \le \alpha < \pi/2$, all fixed moments of $aP_{\ell,D}(1/2+a)$ are bounded when its random-model moments are bounded and $\ell = o(L_X)$. Indeed the transfer of a fixed even moment has error $O_q(X^{-\kappa}e^{C_q\ell})$. One must normalize before making this assertion.

At the auxiliary anchor $a_0 = 4/\ell$, the bound also holds uniformly in every vertical height t. The range of a centered real or imaginary prime projection of $\ell^{-1}P_{\ell,E}(1/2+a_0+it)$ is bounded by

$$C\ell^{-1}(\log p)p^{-1/2-a_0}/(1-p^{-1/2-a_0}).$$

The sum of its squares is $O(\ell^{-2}a_0^{-2}) = O(1)$. The mean from $k \ge 2$ is $O(\ell^{-1}a_0^{-1}) = O(1)$ and the prime-linear mean is $O(\ell^{-1})$. Hoeffding's inequality for real projections therefore gives $\|P_{\ell,E}(1/2+a_0+it)\|_q \le C\sqrt{q}\,\ell$. The phases have modulus one, so the constants are independent of t. Fixed-moment transfer gives the same estimate with C_q in the arithmetic family.

In particular, after multiplication by $|a|$, (3.3) gives the precise normalized error

$$\|a(\mathcal{L}_D - P_{\ell,D})(1/2+a)\|_q \le C_q e^{-\Re(a)\ell}(|a|L_X + |a|\ell). \tag{3.8}$$

If $\ell = C_y L_X/H^{120}$ and $\Re(a)\ell \ge cC_yH$ in a fixed sector, put $x = \Re(a)\ell$. Its right side is at most $C_qH^{120}xe^{-x} \le C_qH^{121}e^{-c'C_yH}$. Cauchy's formula on a slightly larger fixed circle then bounds every fixed moment of analytic suprema and derivatives in the bulk. The weaker condition $\Re(a)\ell \to \infty$ alone would not justify this conclusion.

3.3. **A local meromorphic Jensen estimate.** The next lemma is also useful without GRH and under finite Euler modifications. It is formulated locally so that no unproved global symmetry of a modified function is required.

Lemma 3.2 (Moving-anchor local Jensen)**.** *Let F_D be real entire with $F_D' \not\equiv 0$. Suppose on $|z| \le 1/4$,*

$$-F_D'/F_D(1/2+z) = B_D^*(z) - m_D/z - \sum_{\alpha\in\mathcal{P}_D} 2z/(z^2-\alpha^2),$$

where B_D^* is holomorphic and $O(L_X)$, and the paired series and its inverse-square tail converge locally. The near poles, with their indicated multiplicities, must agree with the actual zeros of F_D. Put $\Delta = \nu/L_X$, $1 \ll \nu = o(H)$, and

$$N_D = m_D + 2\#\{\alpha : |\alpha| \le 12\Delta\}, \qquad W_D = \Delta^2 \sum_{|\alpha|>12\Delta} |\alpha|^{-2}.$$

If $\mathbb{E}(N_D+W_D) \le C(\nu+1)$, then, for U independently uniform on $[1,2]$, $A_{D,U} = -\Delta F_D'/F_D(1/2+U\Delta)$, and all $a,\eta > 0$,

$$\mathbb{P}\{R^{\mathrm{mult}}_{F_D'}([1/2,1/2+\Delta]) > \eta H\} \le \mathbb{P}\{|A_{D,U}| < e^{-a}\} + \frac{C(\nu+1+\log(\nu+2)+a)}{\eta H}. \tag{3.9}$$

Restricted estimates on a good set hold with its indicator and the original denominator.

Proof. Take $c = U\Delta$, $z = c + 4\Delta e^{i\theta}$. For every $0 < p < 1$,

$$\int_0^{2\pi} \left|\frac{\Delta}{z-\alpha}\right|^p \frac{d\theta}{2\pi} \le C_p$$

uniformly in α. To see uniformity, rescale the circle to unit radius; away from the closest point the denominator is bounded below by a fixed multiple of angular distance, whose negative p-power is integrable. Near poles contribute at most C_pN_D by subadditivity. For far paired poles, $|z| \le 6\Delta$ and $|\alpha| > 12\Delta$, so $|z^2-\alpha^2| \ge 3|\alpha|^2/4$ and their entire normalized sum is at most $16W_D$. The analytic term is $O(\nu)$. Concavity of the logarithm gives a positive boundary-log mean bounded by $C\log(2+\nu+N_D+W_D)$; its expectation is $O(\log(\nu+2))$.

Every actual pole in the outer disc is a near zero of F_D. Its meromorphic Jensen weight satisfies

$$\int_1^2 \log^+ \frac{4}{|u-\pi/\Delta|}\,du \le 8.$$

For example integrate the positive logarithm on the larger interval $|u - \Re(\pi/\Delta)| < 4$; discarding the imaginary part only increases it. The expected total pole cost is thus at most $8\mathbb{E}N_D$. Meromorphic Jensen between radii 2Δ and 4Δ has denominator $\log 2$. At a zero of F_D of order r, the additional zero of F_D' has order $r-1$, which is bounded by the near-zero count. At all other points zeros of F_D' equal zeros of $-F_D'/F_D$ with the same multiplicities. We obtain pointwise

$$R^{\mathrm{mult}}_{F_D'} \le \frac{B^+_{D,U} + \log^-|A_{D,U}| + J_{D,U}}{\log 2} + N_D.$$

The target interval is contained in the inner disc for all $U \in [1,2]$. Remove the event $|A_{D,U}| < e^{-a}$, and apply Markov to the remaining nonnegative terms. This proves (3.9). Boundary poles and

zeros follow by radial limits of the circle-average formula; their logarithmic singularities are integrable. Tonelli applies throughout to nonnegative majorants. □

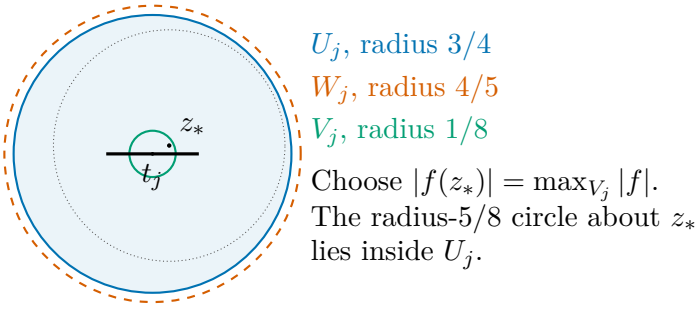


FIGURE 3. The shifted Jensen construction in the w-plane. The thick segment is $[t_j - 1/4, t_j + 1/4]$; it lies inside the radius-1/2 circle about any $z_* \in V_j$. The dotted circle is one possible outer Jensen circle. The figure is geometric and shows no numerical zero locations.

3.4. **Eight fixed anchors.** In the GRH setting the near poles satisfy a useful deterministic gap: if $c \in [\Delta, 2\Delta]$, then $|c - i\gamma| \geq \Delta$. We use eight possible centers instead of averaging over all centers.

Lemma 3.3 (Finite anchors for the central interval and bridge)**.** *Fix $0 < \varepsilon < 1/16$. Under the real Euler-model and analytic hypotheses, including GRH, put*

$$\nu = H^{1/2+\varepsilon/4}, \quad \Delta = \nu/L_X, \quad \ell_c = L_X/H^{1/2+\varepsilon/8}, \quad r = H^{-1/4}, \tag{3.10}$$
$$T_F = H^{1/4+\varepsilon/32}, \qquad M = \lceil H^{1/2+3\varepsilon/32} \rceil.$$

For $H^{-1/2+\varepsilon} \leq z \leq 1$, the number, with multiplicity, of zeros of L'_D on $[1/2, 1/2 + e^5 H^{121}/L_X]$ exceeds $C_\varepsilon z H$ with probability at most

$$C_\varepsilon H^{-2} \log H + C_\varepsilon H^{-6}. \tag{3.11}$$

The same conclusion holds with a fixed finite Euler modification of the type treated below.

Proof. First take eight distinct fixed shifts $v_1, \dots, v_8$, and consider locations $\delta_j = e^{-t-v_j}$ in $[c\Delta, CH^{121}/L_X]$. The positive majorant $f_r(x) = \exp(8C_0^2)\exp(-\|x\|^2/r^2)$ bounds the indicator of $|x_j| \leq C_0 r$. With the Fourier convention $\widehat{f}(\tau) = \int e^{-i\tau\cdot x} f(x)\,dx$, its *normalized* Fourier L^1 norm is $(2\pi)^{-8}\|\widehat{f}_r\|_1 = f_r(0)$, independent of r. The Fourier tail outside the box of width T_F is $O(e^{-c(rT_F)^2})$. The transfer parameters satisfy

$$\Delta\ell_c = H^{\varepsilon/8}, \quad rT_F = H^{\varepsilon/32}, \quad T_F/\sqrt{M} = O(H^{-\varepsilon/64}), \quad M\ell_c/L_X = O(H^{-\varepsilon/32}). \tag{3.12}$$

Also $M \log(2 + T_F) = o(L_X)$. The finite Taylor remainder and the arithmetic error, even after Fourier integration, are smaller than every fixed power of H^{-1}. The normalized approximation error is bounded by a fixed power of H times $\exp(-cH^{\varepsilon/8})$; multiplying by the Lipschitz norm $O(r^{-1})$ preserves this property. The random truncation tail has the same property.

The smooth Gaussian comparison has maximum distance $\delta_j \ll H^{121}e^{-H}$ and derivative norms $O(r^{-j})$, $j \leq 3$, so again its error is smaller than every fixed power of H^{-1}. The limiting covariance is $(v/4)\operatorname{sech}^2((v_i - v_j)/2)$. This matrix is positive definite: for $\lambda_j = e^{-v_j}$ it is a fixed positive multiple of the Gram matrix of $\lambda_j e^{-\lambda_j x}$ in $L^2(x\,dx)$. Distinct exponential functions are linearly

independent, as follows by successively removing the smallest decay exponent at infinity. Thus the joint Gaussian density is bounded by a fixed constant and

$$\mathbb{P}\{|G_D(t+v_j)| \le C_0 r \ (1 \le j \le 8)\} \le Cr^8 + O_A(H^{-A}) = O(H^{-2}). \tag{3.13}$$

No independence among the eight observations is assumed.

At scale Δ define the positive zero statistic

$$K_D(\Delta) = \sum_\rho \frac{\Delta^2}{\Delta^2+\gamma^2} = \Delta\{B_D(\Delta) - \mathcal{L}_D(1/2+\Delta)\}.$$

The polynomial moments and (3.8) at cutoff ℓ_c give $\|K_D(\Delta)\|_q \le C_q(\nu+1)$ for every fixed q. The near count and far paired tail of Lemma 3.2 satisfy pointwise

$$N_D \le 145 K_D(\Delta), \qquad W_D \le (145/288) K_D(\Delta).$$

The second constant accounts for the fact that a representative α corresponds to two zeros. Hence $\|N_D\|_q + \|W_D\|_q \le C_q(\nu+1)$.

Choose $U_j = 1 + j/9$ and $c_j = U_j\Delta$. The eight-anchor estimate applies with $v_j = -\log U_j$. In the local Jensen proof the boundary term at each such center is pointwise at most $C\log(2+\nu+N_D+W_D)$. Since $|c_j - i\gamma| \ge \Delta$, every pole Jensen weight is at most $\log 8$ (the weaker gap $\Delta/2$ would suffice). If one anchor $|\Delta\mathcal{L}_D(1/2+c_j)| \ge r$, then

$$R_D^{\mathrm{mult}}([1/2, 1/2+\Delta]) \le C(1+\nu+N_D+W_D+\log(1/r)).$$

The probability that no anchor works is $O(H^{-2})$. Markov in the fixed qth moment bounds the other failure probability by $C_q((\nu+1+\log H)/T)^q$ at threshold CT.

The bridge from Δ to $e^5 H^{121}/L_X$ is covered in the t-variable by $O(\log H)$ fixed-radius discs. In each small disc choose eight distinct real points. A small maximum below r forces all eight values below r, so (3.13) and a union bound cost $O(H^{-2}\log H)$. Otherwise the sum of the negative logarithms of these maxima is $O((\log H)^2)$. The sum of large-disc suprema has qth norm $O_q(\log H)$ by Cauchy, the normalized moment bound, and Minkowski. Local shifted Jensen now bounds the bridge count, with failure at threshold CT, by $CH^{-2}\log H + C_q(\log H/T)^q$. All the larger discs remain in the same fixed-sector approximation range of (3.12).

Finally choose $q = \lceil 8/\varepsilon \rceil$ before taking $X \to \infty$ and put $T = zH$. Uniformly in the stated range, $\nu/T \le H^{-3\varepsilon/4}$ and $T \gg (\log H)^2$. The high-moment terms are $O_\varepsilon(H^{-6})$. This proves (3.11). □

3.5. **Retained local moments in the bulk.** Let

$$\ell_b = C_y L_X/H^{120}, \quad d_X = H^{121}/L_X, \quad b = H - 121\log H, \quad t_* = 50\log H. \tag{3.14}$$

Use a fixed-radius disc cover of $[\log 2, b]$, with slightly larger discs still inside $|\Im w| < \pi/2$. Their real distances satisfy $\Re(e^{-w})\ell_b \ge cC_yH$. The constants in this construction are fixed before X.

Lemma 3.4 (Local moments on one retained event)**.** *There is an event $\mathcal{R}_X$ with $\mathbb{P}(\mathcal{R}_X^c) \ll H^{-7}$ such that, on every interval of length at most one in the indicated cover, every fixed moment of its real-root count is bounded after multiplication by $\mathbf{1}_{\mathcal{R}_X}$. Every fixed moment of each analytic supremum and fixed derivative supremum is also bounded. The event is constructed from $O(H)$ fixed discs and is independent of the fine mesh chosen below.*

Proof. The normalized moments and Cauchy's formula give the supremum and derivative assertions directly. The uniform density of the full random model, the positive Gaussian test, and finite characteristic-function transfer imply

$$\mathbb{P}\{|G_D(t)| \le r\} \le Cr + O_A(H^{-A}), \qquad H^{-8} \le r \le 1/4, \tag{3.15}$$

uniformly in t. One permissible frequency cutoff and Taylor half-degree are H^{10} and $\lceil H^{22}\rceil$: their ratio is H^{-1}, their degree-cutoff product divided by L_X is $O(H^{-98})$, and $rH^{10} \ge H^2$. At the bounded initial times we use the full random density, rather than a Gaussian approximation requiring $t \to \infty$.

Let m_j and M_j be maxima on a small and a larger disc in the fixed cover. Remove the events $m_j < H^{-8}$; each is contained in a scalar small-value event at its center. The union has probability $O(H^{-7})$. On the retained event, $V_j = \log^+(1/m_j) \le 8\log H$, and

$$\mathbb{P}(\mathcal{R}_X \cap \{V_j > u\}) \le Ce^{-u} + O_A(H^{-A}) \quad (0 \le u \le 8\log H).$$

Integrating this truncated tail gives $\mathbb{E}(\mathbf{1}_{\mathcal{R}_X} V_j^q) \le C_q$ for every fixed q. Shifted Jensen bounds the roots in a unit interval by a fixed sum of $C(M_j + V_j)$. This proves the claim. The unretained tail of a root count is never integrated. □

4. Crossing meshes and the uniform quantile estimate

We give the quantitative argument because it identifies both the sampling variance and the loss from the central region.

4.1. **Two nearby crossings of the Gaussian process.** Let g be stationary with covariance $r(t) = \operatorname{sech}^2(t/2)$ and unit variance. Fix a level u. By Lemma 2.3, its paths may be taken C^2, its zeros at a fixed level are almost surely finite and simple on compact intervals, and the one-point Rice intensity is $c_u = e^{-u^2/2}/(\pi\sqrt{2})$.

Lemma 4.1 (Two-point crossing bound)**.** *For intervals I, J with disjoint interiors,*

$$\mathbb{E}(N_u(I)N_u(J)) \le C_u \int_I \int_J \min(|s-t|, 1)\, ds\, dt.$$

For two distinct cells of length $h \le 1$, their crossing indicators have covariance in absolute value at most $C_u h^2$. A diagonal variance is at most $C_u h$.

Proof. For $t > 0$, set $r = r(t)$. Then $r'^2 = r^2(1-r)$ and $\operatorname{Var} g'(0) = 1/2$. Conditional on $g(0) = g(t) = u$, the derivative means are $-ur'/(1+r)$ and $ur'/(1+r)$, and each variance equals

$$v_c = \frac{1}{2} - \frac{r'^2}{1-r^2} = \frac{(1-r)(1+2r)}{2(1+r)}.$$

Conditional Cauchy–Schwarz bounds the absolute product of the two derivatives in expectation by $v_c + u^2 r'^2/(1+r)^2 \le C_u(1-r)$. The two-value density at (u,u) is

$$p_t(u,u) = \frac{e^{-u^2/(1+r)}}{2\pi\sqrt{1-r^2}}.$$

Thus its product with the conditional derivative moment, denoted $\rho_2(t)$, satisfies

$$\rho_2(t) \le C_u\sqrt{1-r(t)} \le C_u \min(t, 1).$$

To justify the required Rice upper bound without presupposing a second-moment identity, replace the root count by $(2\delta)^{-1}\int_I \mathbf{1}_{|g-u|<\delta}|g'|$. On every fixed compact interval, the inverse function theorem at the finitely many simple zeros shows that this converges almost surely to the count. If I, J have a positive separation, Gaussian conditioning and dominated convergence evaluate the limit of the expected product as the double integral of ρ_2. The covariance matrix is uniformly nonsingular at positive separation, which supplies the domination. Fatou gives the stated upper bound. If the intervals touch, trim them from inside and use monotone convergence; a fixed endpoint is a zero with probability zero. Finally a crossing indicator is bounded by its

cell root count, so its product expectation is $O(h^2)$ and its individual mean is $O(h)$. These give the covariance and diagonal bounds. □

The probability $p_u(h)$ of opposite signs at two points distance h is

$$p_u(h) = c_u h + O_u(h^3). \tag{4.1}$$

Indeed the average $(g(0)+g(h))/2$ and difference $(g(h)-g(0))/2$ are independent Gaussians of variances $(1+r(h))/2$ and $(1-r(h))/2$. Opposite signs at level u occur exactly when the average lies within the absolute difference of u. Taylor expansion of the density of the average over this symmetric interval has error $O(\mathbb{E}|\text{difference}|^3) = O(h^3)$. Its variance differs from one by $O(h^2)$, and the standard deviation of the difference is $h/(2\sqrt{2}) + O(h^3)$. The leading term is $c_u h$.

For cells separated by distance $R \geq 1$, whiten each of the two Gaussian value pairs. The smallest eigenvalue of a within-cell covariance is $1 - r(h) \asymp h^2$, while the cross covariance entries are $O(e^{-R})$. The whitened cross matrix is therefore $O(h^{-2}e^{-R})$. Once this is small, the relative entropy from the product Gaussian law is bounded by its squared Hilbert–Schmidt norm: expand $-\log\det(I - AA^*)$ as a convergent power series. The elementary entropy bound on total variation, or its direct Gaussian-density proof by Cauchy–Schwarz, gives a crossing-indicator covariance $O(h^{-2}e^{-R})$. We only use this bound when $R = 16\log H$ and $h = H^{-3/2}$.

4.2. **Sign tests without a separation restriction.** In the bulk $[t_*, b]$, smooth the sign at scale $\rho = H^{-5}$ and truncate each coordinate outside $[-K, K]$, $K = H$. Choose the smooth factors constant off the transition intervals and with jth derivative bounded by $C_j\rho^{-j}$; use a compact cutoff with the usual scaled derivative bounds at K. For $d \leq 4$ coordinates, integration by parts in each variable gives

$$\|\widehat{f}\|_1 \leq CK^d\rho^{-2d}, \qquad \int_{\|\tau\|_\infty > T_F} |\widehat{f}(\tau)|\, d\tau \leq CK^d\rho^{-4d}T_F^{-2}. \tag{4.2}$$

For example apply $1 - \partial_{x_j}^2$ in every coordinate for the first bound and two further derivatives in a coordinate exceeding T_F for the second; the supports have volume $O(K^d)$ and all relevant derivatives have the displayed bounds. At $d = 4$, $T_F = H^{50}$, $M = \lceil H^{102} \rceil$, these costs are at most CH^{44} and CH^{-16}. Moreover

$$T_F/\sqrt{M} = O(H^{-1}), \qquad M\ell_b/L_X = O(H^{-18}), \qquad M\log(2+T_F) = o(L_X).$$

Finite characteristic-function transfer thus pays the entire Fourier norm. Approximation errors are $H^{O(1)}e^{-cH}$ and can be made smaller than the required power by fixing C_y first. The full random-to-Gaussian comparison has maximum distance $e^{-t_*} = H^{-50}$; its first three derivative costs are H^5, H^{10}, H^{15}, so the respective errors are $O(H^{-45}), O(H^{-90}), O(H^{-135})$. Scalar boundary layers cost $O(\rho)$ by positive small-value tests. The fixed eighth moment pays $O(K^{-8})$ for the tails, and retention costs $O(H^{-7})$. Consequently every two-point or four-point bounded sign test differs from its Gaussian counterpart by

$$\delta_H \leq CH^{-5}. \tag{4.3}$$

This proof uses no inverse of the covariance of the observation vector. It remains valid for close or repeated times.

Theorem 4.2 (Quantile estimate for real Euler families under GRH)**.** *Under the real model, finite-word, weighted-prime, density, and analytic hypotheses above, for each fixed* $0 < \varepsilon < 1/16$ *and all* $H^{-1/2+\varepsilon} \leq z \leq 1$,

$$\mathbb{P}_X\left\{\max_{C \in \{R^{\mathrm{dist}}, R^{\mathrm{mult}}, S\}} |C - c(m,v)H_Q| > A_\varepsilon z H_Q\right\} \leq C_\varepsilon \frac{\log H}{Hz^2}.$$

The conclusion persists under the finite Euler modifications described below.

Proof. Set $h = H^{-3/2}$, $R = 16 \log H$. Shift the deterministic mesh by an arbitrarily small offset so that its endpoints are not zeros of any member of the finite arithmetic family. There are finitely many forbidden endpoints in a compact bulk interval because all derivatives are nontrivial analytic functions. All previous estimates are uniform in this offset. Use cells wholly inside $[t_*, b-2]$; the fixed terminal remainder lies in the bridge overlap, or is bounded by a fixed number of retained unit-window counts.

Let S_X^{mesh} be the crossing sum of the process equal to G_D on $\mathcal{R}_X$ and to the constant one outside. There are $O(H/h)$ cells. The Gaussian diagonal variance costs $O(H)$; Lemma 4.1 bounds pairs within distance R by a total $O(H(R+1))$. Far pairs cost, after division by H^2, $O(h^{-4}e^{-R})$. The actual two- and four-point comparison adds $O(\delta_H/h^2)$. Hence

$$\frac{\operatorname{Var} S_X^{\mathrm{mesh}}}{H^2} \ll \frac{\log H}{H} + h^{-4}e^{-R} + \delta_H/h^2 \ll \frac{\log H}{H}. \tag{4.4}$$

This is a variance bound for a bounded mesh statistic. It does not bound the variance of the full root count. From (4.1) and (4.3),

$$\left| \frac{\mathbb{E} S_X^{\mathrm{mesh}}}{H} - c(m,v) \right| \ll h^2 + \frac{\log H}{H} + \frac{\delta_H}{h} \ll \frac{\log H}{H}. \tag{4.5}$$

Here the Gaussian level is $u = -m/\sqrt{v}$; multiplication by the positive scale $\sqrt{v}/2$ does not change crossings.

It remains to bound roots missed by the mesh, including multiple roots. On a cell containing at least two roots counted with multiplicity, Rolle's theorem and Taylor's formula give

$$|G_D(t_i)| \le CK_1h^2 \quad \text{or} \quad \sup_{I_i} |G_D''| > K_1, \qquad K_1 = h^{-1/8}.$$

The assertion includes a double zero, where the first derivative vanishes. The small-value radius is $Ch^{15/8} = CH^{-45/16}$, which exceeds H^{-8}. Equation (3.15) and the sixteenth derivative moment bound the event by $Ch^{15/8} + Ch^2 \ll h^{15/8}$. Hölder with the retained sixteenth root-count moment yields

$$\mathbb{E}(\mathbf{1}_{\mathcal{R}_X} N(I_i) \mathbf{1}_{N(I_i)\ge 2}) \ll (h^{15/8})^{15/16} = h^{225/128}.$$

Summing the cells and the $O(\log H)$ initial segment gives a nonnegative total excess E_X with

$$H^{-1}\mathbb{E}E_X \ll h^{97/128} + \log H/H = H^{-291/256} + O(\log H/H). \tag{4.6}$$

At the initial segment we use retained local root moments, not a Gaussian limit. A cell with precisely one simple root contributes a crossing; all other discrepancies are bounded by the indicated excess. Endpoint corrections are bounded because the shifted mesh avoids roots.

Uniformly for $z \ge H^{-1/2+\varepsilon}$, the bias in (4.5) is $o(z)$. Chebyshev and Markov, applied respectively to (4.4) and (4.6), give a bulk failure probability at tolerance CzH of at most

$$C\left\{ z^{-2}\frac{\log H}{H} + z^{-1}H^{-291/256} + z^{-1}\frac{\log H}{H} + H^{-7} \right\} \ll \frac{\log H}{Hz^2}.$$

For the absorption, the ratio of the second term to the first is at most $H^{-35/256}/\log H$, and that of the third is $z \le 1$. Lemma 3.3 bounds the missing central and bridge multiplicity by $C_\varepsilon zH$, except on the smaller event (3.11). Thus the closed distinct and multiple counts both lie within $C_\varepsilon zH$ of $c(m,v)H$.

The strict-sign count follows from Lemma 4.3 below. The bounded conductor comparison $|H_Q - H| \le C$ and $zH \to \infty$ permit replacement by the actual H_Q on the good event. No bound on every exceptional sample is used. This proves the theorem. □

Lemma 4.3 (Closed counts and strict signs)**.** *For a nonzero real analytic function on a neighborhood of $[a,b]$, with q distinct zeros and total multiplicity M on the closed interval and S strict sign changes on the open interval,*

$$2q - M - 2 \le S \le q.$$

Proof. An interior zero has a strict sign change exactly when its multiplicity is odd. If there are e even-multiplicity interior zeros, their extra multiplicities give $M - q \ge e$. At most two distinct zeros are endpoints. Therefore $S \ge q - 2 - e \ge 2q - M - 2$, and $S \le q$. This proof permits arbitrarily high multiplicity. □

4.3. **Finite Euler changes and summing dyadic families.**

Proposition 4.4 (Finite Euler stability)**.** *Suppose the hypotheses above hold for L_D. Fix a finite prime set and a finite collection of real Euler polynomials*

$$E(s) = \prod_{p \in S} \prod_{j=1}^{d_p} (1 - \alpha_{p,j} p^{-s}), \qquad |\alpha_{p,j}| \le 1.$$

For a fixed such E, the full quantile conclusion holds for $(L_D E)'$, with the same constant and the reference conductor Q_D. If E varies over the finite collection, the conclusion holds provided every nonempty class, each of positive limiting density, has the stated arithmetic inputs under its own original conditional measure.

Proof. All zeros of E have $\Re s \le 0$. Thus $g_E = -E'/E$ and all fixed derivatives are bounded on the compact regions used here. Its logarithmic coefficients are bounded and supported on finitely many primes. We have

$$-\frac{(L_D E)'}{L_D E} = \mathcal{L}_D + g_E, \quad \widetilde{G}_D(w) = G_D(w) + e^{-w} g_E(1/2 + e^{-w}).$$

The weighted means m, v and the limiting Gaussian covariance are unchanged, and the full-model density proof uses arbitrarily large primes outside S. Expanding a modified word into old words costs at most a fixed exponential in its degree, so finite-word transfer and the moment bounds persist. The finite-prime Selberg truncation error for g_E is $O_E(e^{-\ell \Re s})$, since the geometric tails start at $p^k > e^\ell$ and the weight equals one below that point. This gives the same approximation (3.3) with modified polynomials.

In the central disc replace B_D in (3.5) by $B_D + g_E(1/2 + z)$. The actual near zeros and their multiplicities are exactly those of L_D. The reference positive zero statistic still supplies the near and tail moments; local Jensen applies without assigning a global functional equation to $L_D E$. Its pole gap is unchanged. All bulk and finite-anchor estimates can therefore be applied to the modified process itself. We have not asserted that an analytic perturbation changes the number of zeros by only $O(1)$.

Finally $L_D E$ is nonconstant. Otherwise its limit at $+\infty$ forces $L_D E \equiv 1$. If E is nonconstant it has a zero, contradicting the entireness of $L_D = 1/E$; if $E \equiv 1$, use $L_D' \not\equiv 0$. A finite union of verified conditional classes proves the last assertion. An arbitrary thin subfamily is not covered by this argument. □

For a family of cardinality $\asymp X^a$, $a > 0$, a fixed-tolerance dyadic estimate sums without an extra logarithmic factor. Discard parameters at most $\sqrt{X}$, whose total size is $O(X^{a/2})$. All remaining dyadic blocks have $H_{\text{block}} = H + O(1)$, and their sizes form a geometric sum $O(X^a)$. This proves the cumulative fixed-tolerance estimate and the two regular shrinking-tolerance corollaries (1.3)–(1.4). An arbitrary function $z(X)$ may behave differently on the dyadic blocks; no unstated uniform envelope is used.

5. Three unconditional families

We count zeros of the derivative of the uncompleted L-function. In particular, differentiating a gamma factor would give a different problem. For a real analytic function F, let $R^{\mathrm{dist}}(F)$ and $R^{\mathrm{mult}}(F)$ denote its numbers of distinct zeros and zeros counted with multiplicity in $[1/2,1]$, and let $S^-(F)$ denote its strict changes of sign in $(1/2,1)$. Put

$$c_* = \frac{e^{-1/2}}{\pi\sqrt{2}}.$$

All the probability measures in this section are the original uniform counting measures. Restricting an expectation to a good event always means inserting its indicator, without renormalizing the measure.

Theorem 5.1. *Let $\mathcal{F}(X)$ be the set of fundamental discriminants D with $1<|D|\le X$. For each fixed $\eta>0$, all but $o_\eta(X)$ members satisfy, simultaneously for $C=R^{\mathrm{dist}},R^{\mathrm{mult}},S^-$,*

$$\bigl|C(L'(\cdot,\chi_D))-c_*\log\log|D|\bigr|\le\eta\log\log|D|.$$

Both signs of the discriminant and every possible conductor at 2 are included.

Theorem 5.2. *Let $K=\mathbb{Q}(i)$ and*

$$\mathcal{D}_X=\{d\in\mathbb{Z}[i]:(d,2)=1,\ (d)\ \text{square-free},\ X<N(d)\le 2X\}.$$

For $d\in\mathcal{D}_X$, let $L_d(s)=L(s,\chi_{(1+i)^5d})$. Its finite conductor has norm $32N(d)$; write $Q_d=128N(d)$ for the conductor including the field discriminant. For each fixed $\eta>0$, all but $o_\eta(X)$ labels satisfy

$$\max_{C=R^{\mathrm{dist}},R^{\mathrm{mult}},S^-}\bigl|C(L_d')-c_*\log\log Q_d\bigr|\le\eta\log\log Q_d.$$

The same assertion holds with uniform measure on the distinct primitive characters occurring in this family.

Theorem 5.3. *Let k tend to infinity through all even integers and let H_k be the normalized level-one Hecke eigenbasis of weight k. Use the unitary normalization*

$$L(s,f)=\prod_p(1-\lambda_f(p)p^{-s}+p^{-2s})^{-1}.$$

For each fixed $\eta>0$, all but $o_\eta(|H_k|)$ forms satisfy

$$\max_{C=R^{\mathrm{dist}},R^{\mathrm{mult}},S^-}\bigl|C(L'(\cdot,f))-c_*\log\log(k^2)\bigr|\le\eta\log\log(k^2).$$

The measure is unweighted, and the assertion includes both root numbers $i^k=1$ and $i^k=-1$.

No theorem above assumes the Riemann hypothesis, nonvanishing at the central point, or simplicity of a zero. We give the arithmetic and analytic inputs in some detail, since the passage from a mollified moment to the very short interval next to the centre is sensitive to the family and to the range of the shift.

5.1. **The arithmetic probability spaces.** For the rational quadratic family the independent model has variables $X_p\in\{0,1,-1\}$ with

$$\mathbb{P}(X_p=1)=\mathbb{P}(X_p=-1)=\frac{p}{2(p+1)},\qquad \mathbb{P}(X_p=0)=\frac{1}{p+1}.$$

This includes $p=2$. For every positive integer n,

$$\mathbb{E}_{\mathcal{F}(X)}\chi_D(n)=\mathbf{1}_{n=\square}\prod_{p\mid n}\frac{p}{p+1}+O(X^{-1/2}n). \tag{5.1}$$

Here and below sufficiently large implied constants also cover bounded parameters. One elementary derivation, which will also fix the normalization needed for the mollifier, is as follows. Write $D = \epsilon cd$, where $d > 0$ is odd and square-free, $\epsilon = \pm 1$, $c \in \{1, 4, 8\}$, and $d \equiv r \pmod 8$. The allowed classes are

$$c = 1 : \ \epsilon r \equiv 1 \pmod 4; \qquad c = 4 : \ \epsilon r \equiv 3 \pmod 4; \qquad c = 8 : \ r \text{ arbitrary odd.}$$

Expanding $\mu^2(d) = \sum_{\alpha^2 | d} \mu(\alpha)$ shows that each odd class contains $Y/\pi^2 + O(\sqrt{Y})$ elements up to Y. For odd nonsquare n the complete character sum in the progression modulo 8 vanishes; summing the incomplete-period error over $\alpha \le \sqrt{Y}$ costs $O(\sqrt{Y} n)$. For square n, inclusion-exclusion for $(d, n) = 1$ gives the local factors $p/(p+1)$ and the same error. The classes with $c = 1, 4, 8$ contribute respectively $4X/\pi^2, X/\pi^2, X/\pi^2$. In the first group $\chi_D(2) = (-1)^{(r^2-1)/8}$; in the other two it is zero. Summing the classes proves (5.1), including the factor $2/3$ at 2, and

$$|\mathcal{F}(X)| = \frac{6X}{\pi^2} + O(\sqrt{X}). \tag{5.2}$$

For $K = \mathbb{Q}(i)$, use independent variables $X_{\mathfrak{p}}$ with the same ternary law with p replaced by $N\mathfrak{p}$, except that $X_{(1+i)} = 0$. Distinct prime ideals of the same norm are distinct independent coordinates. We have

$$|\mathcal{D}_X| = \frac{2\pi X}{3\zeta_K(2)} + O(\sqrt{X}\log(2X)), \qquad |\mathbb{E}_{\mathcal{D}_X} \chi_d(\mathfrak{n}) - \mathbb{E} X_{\mathfrak{n}}| \ll X^{-1/3} N\mathfrak{n}. \tag{5.3}$$

For completeness, the number of points of one residue class modulo an ideal $\mathfrak{q}$ in a norm annulus $(Y, 2Y]$ is $\pi Y/N\mathfrak{q} + O(\sqrt{Y/N\mathfrak{q}} + 1)$. This follows by covering the boundary circle with fundamental squares of the rotated lattice $\mathfrak{q}$. A nonprincipal character modulo $\mathfrak{n}$ therefore has annular sum $O(\sqrt{YN\mathfrak{n}} + N\mathfrak{n})$, also with the oddness condition. Apply this after the ideal-Möbius expansion and sum $N\mathfrak{a} \le \sqrt{2X}$; the error is $O(\sqrt{X} N\mathfrak{n} \log(2X))$. In the square-ideal case the main Euler product is

$$\frac{\pi X}{2} \prod_{\mathfrak{p} | \mathfrak{n}} (1 - (N\mathfrak{p})^{-1}) \prod_{\mathfrak{p} \nmid 2\mathfrak{n}} (1 - (N\mathfrak{p})^{-2}) = \frac{2\pi X}{3\zeta_K(2)} \prod_{\mathfrak{p} | \mathfrak{n}} \frac{N\mathfrak{p}}{N\mathfrak{p} + 1}.$$

The elementary bound $2^{\omega(\mathfrak{n})} \le N\mathfrak{n}$ suffices for inclusion-exclusion. Ideals divisible by $(1+i)$ have character value zero. Division by the first formula in (5.3) proves the second.

Gao's family definition and functional equation give

$$\Lambda_d(s) = (128 N(d))^{s/2} (2\pi)^{-s} \Gamma(s) L_d(s) = \Lambda_d(1 - s) \tag{5.4}$$

and primitivity of conductor $((1+i)^5 d)$ [14, §2.3, (2.1)–(2.4)]. Equal primitive characters must have equal conductor ideals, so their labels differ by a unit. The supplementary quadratic law gives $(-1/\mathfrak{n}) = 1$ for odd ideals, whereas $(i/n) = -1$ for a primary $n \equiv -1 + 2i \pmod 4$ [13, (2.2)]. The Chinese remainder theorem permits the additional condition $n \equiv 1 \pmod{\mathrm{rad}(d)}$. Consequently the fibers of $d \mapsto \chi_d$ are exactly $\{d, -d\}$. Thus passing to distinct characters divides both the numerator and the denominator of every character-dependent event by two.

The weight family has a different local law. If $U_j = U_j(x/2)$ are the Chebyshev polynomials, set

$$d\mu_p(x) = \frac{1 + p^{-1}}{(1 + p^{-1})^2 - p^{-1} x^2} \frac{\sqrt{4 - x^2}}{2\pi}\, dx, \quad -2 \le x \le 2. \tag{5.5}$$

The identity $\sum_{j \ge 0} p^{-j} U_{2j}(x/2) = (1 + p^{-1})/((1 + p^{-1})^2 - p^{-1} x^2)$ and Sato–Tate orthogonality show that $\mathbb{E} U_j(X_p/2) = p^{-j/2}$ for even j and zero otherwise. The density relative to Sato–Tate lies between $2/3$ and 6. Put $A_1(x) = x$ and $A_j(x) = U_j(x/2) - U_{j-2}(x/2)$ for $j \ge 2$. Then $|A_j| \le 2$, $\mathbb{E} X_p^2 = 1 + p^{-1}$, and

$$\mathbb{E} A_{2j}(X_p) = -(1 - p^{-1}) p^{-j+1}, \qquad \mathbb{E} A_{2j-1}(X_p) = 0. \tag{5.6}$$

Hough's unweighted coefficient estimate says that, for fixed $0 < \epsilon < 2$ and $n < k^{2-\epsilon}$,

$$\frac{1}{|H_k|}\sum_{f\in H_k}\lambda_f(n) = \frac{\mathbf{1}_{n=\square}}{\sqrt{n}} + O_\epsilon(k^{-\gamma(\epsilon)}), \qquad |H_k| = (k-1)/12 + O(1).$$

See [16, Lemma 2.2]. There is no additional factor of n in this error. This coefficient lemma holds for all even weights; a restriction to root number $+1$ in the central-value application of that paper is not a restriction of the coefficient lemma.

To multiply finite polynomials, use the Hecke algebra $e_m e_n = \sum_{d|(m,n)} e_{mn/d^2}$ and the norm

$$\Big\|\sum a_n e_n\Big\|_{\mathrm{H}} = \sum |a_n|\tau(n)n.$$

It is submultiplicative, since $\sum_{d|(m,n)}\tau(mn/d^2) = \tau(m)\tau(n)$, as is checked prime by prime. Taking $\epsilon = 1$ in Hough's estimate and using $|\lambda_f(n)|, |\lambda_X(n)| \le \tau(n)$ for $n \ge k$ proves, with some fixed $\kappa > 0$,

$$|\mathbb{E}_{H_k}P - \mathbb{E}P(X)| \ll k^{-\kappa}\|P\|_{\mathrm{H}} \tag{5.7}$$

for every finite Hecke polynomial, with no support restriction. In the two quadratic families the analogous submultiplicative norm is $\sum |a_n|n$ or $\sum |a_{\mathfrak{n}}|N\mathfrak{n}$, and the exponents are $1/2$ and $1/3$ by (5.1) and (5.3).

5.2. **A central mollified bound for all signed discriminants.** The following extension of the quadratic mollifier calculation is needed on the complete family, rather than only on χ_{-8d}.

Proposition 5.4. *Write $L = \log X$, $H = \log L$, $M = X^{1/100}$, and let $Q(v) = 0$ for $v \le 0$, $Q(v) = 3(2v)^2 - 2(2v)^3$ for $0 < v < 1/2$, and $Q(v) = 1$ for $v \ge 1/2$. The entire mollifier*

$$M_D(s) = (1-\chi_D(2)2^{-s})\sum_{\substack{n\le M\\ n\ \mathrm{odd}}}\frac{\mu(n)Q(\log(M/n)/\log M)\chi_D(n)}{n^s}$$

satisfies

$$\mathbb{E}_{\mathcal{F}(X)}|L(1/2+it,\chi_D)M_D(1/2+it)|^2 \ll H^{20} \qquad (|t| \le \tfrac{1}{2}L^{-5/6}). \tag{5.8}$$

Proof. We retain the class notation $D = \epsilon cd$, $d \equiv r \pmod 8$ used above, and work first with a fixed nonnegative smooth weight on $Y < d < 2Y$. Write $a = (1-\epsilon)/2$ and $\zeta = \chi_D(2)$. The completed primitive function is $(cd/\pi)^{(s+a)/2}\Gamma((s+a)/2)L(s,\chi_D)$ with root number $+1$. Removing the Euler factor at 2 does not give it a new primitive functional equation; we keep that factor explicitly throughout.

Here are the modifications of the calculation in [10, §§3–6]. They are recorded also to identify the secondary terms which would be lost by replacing a fixed congruence class by the sum of its classes. For odd n, let $\tau_j(n) = \sum_{b \bmod n}(b/n)e(jb/n)$ and $\widehat{F}(\xi) = \int F(v)e(-v\xi)\,dv$. The Chinese remainder theorem and Poisson summation give exactly

$$\sum_{d\equiv r(8)}(d/n)F(d/Y) = \frac{Y}{8n}(8/n)\sum_{j\in\mathbb{Z}}e(jrn/8)\tau_j(n)\widehat{F}(jY/(8n)). \tag{5.9}$$

After truncating the square-free expansion at $\alpha \le V$, the normalized expression has the factor $(8/n)/(8n)\sum_{\alpha\le V,(\alpha,2n)=1}\mu(\alpha)/\alpha^2$, with Fourier argument $jY/(8\alpha^2 n)$. Including the factor $(\epsilon c/n)$ from $\chi_D(n)$, the total character multiplier is $(8\epsilon c/n)$.

Set $A = (1+i)/2$, $B = (1-i)/2$, and $\epsilon_n = 1$ or i according as $n \equiv 1$ or $3 \pmod 4$. Writing $G_j(n) = \tau_j(n)/\epsilon_n$, the phase $e(jrn/8)\epsilon_n$ is

$$\begin{cases} \chi_8(jr)e^{is\pi/4}(2s/n), & j \text{ odd},\ s = \chi_4(jr), \\ i^{jr/2}\{B + A\chi_4(n)\}, & j \equiv 2 \pmod 4, \\ (-1)^{j/4}\{A + B\chi_4(n)\}, & 4 \mid j. \end{cases}$$

At an unramified odd prime, $G_j(p) = (j/p)\sqrt{p}$. The resulting quadratic character is principal precisely on the following branches:

c	j
1	$\pm m^2$ (m odd), $\pm 2m^2$
4	$\pm 2m^2$
8	$\pm 4m^2$.

Indeed, the mod-8 phases above can only alter the sign and the power of 2 in the radicand, never a nontrivial odd square-free part.

Let $T_\epsilon(u) = \cos(\pi u/2) + \epsilon \sin(\pi u/2)$. Summing the principal branches, including their Fourier Mellin phases, gives

$$\begin{aligned} S_8(u) &= -\frac{2^{-2u}(1-2^{1-2u})}{1-2^{-2u}} T_\epsilon(u), \\ S_4(u) &= 2^u S_8(u), \qquad S_1(u) = \left(\sqrt{2}\zeta + \frac{2^{-u}}{1-2^{-2u}}\right) T_\epsilon(u). \end{aligned} \tag{5.10}$$

For example the $c = 8$ branch with 2-adic valuation 2 has scalars $-A, -B$ for signs $\epsilon, -\epsilon$, and the branches of valuations $4, 6, \ldots$ have A, B. For $c = 4$ these valuations are 1 and $3, 5, \ldots$; for $c = 1$ the odd branch contributes $\zeta e^{i\pi/4}, \zeta e^{-i\pi/4}$ and the odd positive valuations contribute A, B. These geometric series give (5.10).

Put $\tau = (\delta_1 + \delta_2)/2$, $\delta = (\delta_1 - \delta_2)/2$ and

$$\Gamma_{a,z}(s) = \Gamma((1/2 + a + s + z)/2)\Gamma((1/2 + a + s - z)/2).$$

The parity change in the secondary integral follows from the exact reflection identity

$$\frac{\Gamma_{0,z}(s)T_+(s-z)}{\Gamma_{1,z}(s)T_-(s-z)} = \frac{\Gamma_{0,s}(z)}{\Gamma_{1,s}(z)}. \tag{5.11}$$

To check it, apply reflection to $p = (1/2 + s + z)/2$, $q = (1/2 + s - z)/2$; the gamma ratio is $\cot(\pi q)$, while $T_+ = \sqrt{2}\sin(\pi q)$ and $T_- = \sqrt{2}\cos(\pi q)$.

In the odd-conductor case define

$$B_2 = (1 - \zeta 2^{-1/2-\tau-\delta})(1 - \zeta 2^{-1/2-\tau+\delta}), \qquad E_2(s) = \frac{B_2}{(1 - \zeta 2^{-1/2-s+\delta})(1 - \zeta 2^{-1/2-s-\delta})}.$$

At either secondary residue $z = \delta$ or $z = -\delta$, put $q = 2^s$, $t = 2^z$. The relative secondary factor is

$$R_\zeta(s,z) = 8^{-s}\frac{S_1(s-z)}{S_8(s-z)}E_2(s) = \frac{-\sqrt{2}\zeta t^{-2}B_2}{q + q^{-1} - \zeta(\sqrt{2}t + (\sqrt{2}t)^{-1})}. \tag{5.12}$$

The equality with $E_2(s)$ in (5.12) is asserted only for $z = \pm\delta$. For other z we define $R_\zeta(s,z)$ by the last rational expression; this is the continuation used below. It is even in s. For $c = 4$ the relative factor is 2^{-z}, and for $c = 8$ it is 1. The apparent poles in (5.12) cancel against the 2-Euler factor

$$\eta_{2,s}(1+2z) = (1 - (2t^2q^2)^{-1})(1 - (2t^2)^{-1})(1 - q^2/(2t^2)).$$

Their product is the entire, even expression

$$\frac{\zeta}{\sqrt{2}}t^{-4}B_2(1-(2t^2)^{-1})\{q+q^{-1}+\zeta(\sqrt{2}t+(\sqrt{2}t)^{-1})\}. \tag{5.13}$$

Thus this cancellation removes all the periodic denominator zeros, not just their real representatives.

An additional precaution is necessary in the even-character classes. Removing the square-free condition can produce the principal character when $d=l^2$. Use the even polynomial

$$H_{\delta,\tau}(s)=\frac{\{s^2-(1/2+\delta)^2\}\{s^2-(1/2-\delta)^2\}}{\{\tau^2-(1/2+\delta)^2\}\{\tau^2-(1/2-\delta)^2\}} \tag{5.14}$$

in the approximate functional equation, whose Mellin kernel is $\Gamma_{a,\delta}(s)H_{\delta,\tau}(s)2s/(s^2-\tau^2)$. Since $H(\tau)=H(-\tau)=1$, the functional-equation contour proves the same exact product formula. In its odd-series representation the kernel also has the factor $E_c(s)$: for $c=1$ this is obtained by multiplying the primitive product formula by B_2 and removing the two Euler factors inside the integral. No functional equation for $L_D^{(2)}$ is assumed. It kills the artificial principal poles $\pm(1/2\pm\delta)$, with their required multiplicities. The possible extra gamma poles of $\Gamma_{0,s}(z)$ at $s=\pm(1/2+z)$ are killed either by (5.13) when $\zeta=-1$, or by H when $\zeta=1$. No such poles occur for $a=1$ in $|\Re s|<3/4$.

Here is a precise form of the resulting four main terms. Define $r_v(n)=\sum_{ab=n}(a/b)^v$, $Z(s;v)=\zeta(s-2v)\zeta(s)\zeta(s+2v)$, write $l=l_1l_2^2$ with l_1 square-free, and put $\check{\Psi}(w)=\int_0^\infty\Psi(x)x^w\,dx$. For odd p, the Euler factors of $\eta_v(s;l)$ are

$$\frac{p}{p+1}\begin{cases}(1-p^{-s})\{1+p^{-1}+p^{-s}-(p^{2v}+p^{-2v})p^{-s-1}+p^{-2s-1}\}, & p\nmid l,\\ 1-p^{-s}, & p\mid l_1,\\ 1-p^{-2s}, & p\mid l, p\nmid l_1,\end{cases}$$

and the factor at 2 is $(1-2^{-s-2v})(1-2^{-s})(1-2^{-s+2v})$. Let $E_c(s)=1$ for $c=4,8$ and $E_1(s)=E_2(s)$; let $R_8=1$, $R_4(s,z)=2^{-z}$, and $R_1=R_\zeta$. More precisely, on square-free d put $\mathcal{A}_{\delta,\tau,c}(d)=(cd/\pi)^\tau\Gamma_{a,\delta}(\tau)L_D^{(2)}(1/2+\delta_1)L_D^{(2)}(1/2+\delta_2)$, and extend it to the square-free expansion by the exact H-kernel formula above. The sum $Y^{-1}\sum_{d\equiv r(8)}(\sum_{\alpha^2\mid d,\,\alpha\le V}\mu(\alpha))\chi_D(l)\mathcal{A}_{\delta,\tau,c}(d)\Psi(d/Y)$ equals

$$\frac{1}{\pi^2\sqrt{l_1}}\Big[\sum_{\mu=\pm1}r_\delta(l_1)\Gamma_{a,\delta}(\mu\tau)(cY/(\pi l_1))^{\mu\tau}E_c(\mu\tau)\check{\Psi}(\mu\tau)Z(1+2\mu\tau;\delta)\eta_\delta(1+2\mu\tau;l)$$
$$+\sum_{z=\pm\delta}r_\tau(l_1)\Gamma_{a,\tau}(z)(8Y/(\pi l_1))^zR_c(\tau,z)\check{\Psi}(z)Z(1+2z;\tau)\eta_\tau(1+2z;l)\Big] \tag{5.15}$$

with error

$$\ll_\epsilon Y^\epsilon l^\epsilon\{l_1^{-1/4}Y^{-1/4}+l_1^{-1/2}V^{-1}+l^{1/2}VY^{-1/2}\}. \tag{5.16}$$

Coincident residues in (5.15) mean their joint holomorphic continuation.

We justify both this continuation and the error bounds, rather than appealing to a change of conductor in the original calculation. The zero Fourier mode in (5.9) has ln square, so $n=l_1m^2$; its Euler product is exactly the η product above. Moving its contour to $\Re s=-1/4+\epsilon$ gives the first error in (5.16) and the residues at $\pm\tau,\pm\delta$. On the principal nonzero branches, (5.11) and (5.13) make the secondary integrand even. At $s=z=\pm\delta$ one has $R_\zeta(z,z)=8^{-z}E_2(z)$; these residues cancel those of the first contour. The remaining residues are the four terms displayed above, and (5.14) equals one there.

For nonprincipal branches, start the Fourier–Mellin calculation with $w=-1/2+\epsilon+iv$ and an s-line $b=\epsilon$; this is the common absolute-convergence strip $\max(|\Re\tau|,\Re w)<b<1+\Re w$. Only

after this identity has been obtained do we move to $b = \epsilon + A_0$. All poles of $\Gamma(s-w)$, $\Gamma_{a,\delta}(s)$, $E_2(s)$ and $s^2-\tau^2$ lie to the left; the gamma decay remains after the Fourier phase is included. The Mellin transform is bounded by

$$Y^{-1/2+\epsilon}\frac{\alpha\sqrt{l}}{\sqrt{|j|}}(1+|w|)^C|\check{\Psi}(w)|(1+|j|/(\alpha^2 l))^{-A_0}.$$

The odd Euler series factors as two quadratic L-functions and a remainder bounded by

$$\alpha^\epsilon |j|^\epsilon l^{1/2+\epsilon}(l, j_2^2)^{1/2},$$

where $j = j_0 j_2^2$ and j_0 is square-free, with a bounded number of 2-adic twists. Sum j_2 first, obtaining $O(\alpha\sqrt{l/|j_0|})$, and apply Cauchy–Schwarz and the all-real- primitive fourth-moment bound [10, Lemma 3.5] to the dyadic j_0-sum. It costs $(\alpha^2 l)^\epsilon(1+|w|)^C$. Summing $\alpha \le V$ gives the last error in (5.16). The principal zeta factor, when present in the square-free tail, is handled with (5.14) before shifting contours.

For clarity, the shifts here have imaginary parts as large as $L^{-5/6}$, not just $O(L^{-1})$. Fix the final ϵ first and choose $0 < \vartheta < \min(10^{-4}, \epsilon/100)$. For large Y the shifts have modulus at most $\vartheta/4$. Extend the α Euler sum using $\Re s = \vartheta + 1/\log Y$ and the circle $|z| = \vartheta + 1/(2\log Y)$. The factor $2z$ cancels the zeta pole at $z=0$, and $\Re(s-z) \ge 1/(2\log Y)$. The omitted tail is

$$\ll l^{\epsilon_1} l_1^{2\vartheta-1/2} Y^{\vartheta+\epsilon_1} V^{-1+4\vartheta}.$$

With $V = Y^{1/20}$, $l \le M^2 \le Y^{.04}$, and $\epsilon_1 < \epsilon/10$, this is absorbed by the middle term of (5.16). This Y^ϵ occurs only in a power-saving error, never in the main bound (5.8).

The full square-free remainder can be checked separately. Write $d = l^2 m$ with m odd and square-free and $l > V$. With $Y_l = Y/l^2$, the large sieve [10, Lemma 3.4] bounds the fourth mollifier sum by $Y^\epsilon(Y_l + M^2)$, and the product approximate functional equation and [10, Lemma 3.5] bound the second L-sum by $Y^\epsilon Y_l$. The principal case $m=1$ is bounded separately and is included in M^2. Two applications of Cauchy–Schwarz and summation over l give $Y^\epsilon(V^{-1} + MY^{-1/2}\log Y)$ after normalization. Summing (5.16) against the two mollifier coefficients therefore leaves at most

$$Y^\epsilon\{MY^{-1/4} + MV^{-1} + M^2VY^{-1/2} + V^{-1} + MY^{-1/2}\}, \tag{5.17}$$

a fixed negative power when $.009 \le \log M/\log Y \le .02$.

It remains to bound the main terms uniformly through colliding shifts. We give the cancellation estimate used for this purpose. For a generic term (w,v) in (5.15), its mollifier sum is

$$\mathcal{A}(w,v) = \sum_{r,s \text{ odd}} \frac{\lambda(r)\lambda(s)}{r^{1/2+\delta_1}s^{1/2+\delta_2}}\frac{r_v(l_1)}{l_1^{1/2+w}}\eta_v(1+2w; rs), \quad \lambda(n) = \mu(n)Q(\log(M/n)/\log M).$$

The square-free rearrangement of the Euler factors gives

$$\mathcal{A}(w,v) = \eta_v(1+2w;1)\sum_{\gamma \text{ odd}} \frac{H_{w,v}(\gamma)}{\gamma^{1+2\tau}h_{w,v}(\gamma)} S_\gamma(\delta_1+w, v)S_\gamma(\delta_2+w,v), \tag{5.18}$$

where

$$\begin{aligned} h_{w,v}(p) &= 1 + p^{-1} + p^{-1-2w} - (p^{2v}+p^{-2v})p^{-2-2w} + p^{-3-4w},\\ H_{w,v}(p) &= 1 + p^{-1-2w} - r_v(p)^2 p^{-1-2w}/h_{w,v}(p),\\ S_\gamma(u,v) &= \sum_{\substack{a \le M/\gamma\\ a \text{ odd}}} \frac{r_v(a)\mu(a\gamma)}{a^{1+u}h_{w,v}(a)} Q(\log(M/(a\gamma))/\log M). \end{aligned}$$

Let $E_\gamma = \prod_{p|\gamma}(1 + C/\sqrt{p})$. The residue formula of [10, Lemma 6.1], applied to these explicit Euler factors, gives

$$|S_\gamma(u,v)| \ll E_\gamma H^6(|u+v| + L^{-1})(|u-v| + L^{-1}). \tag{5.19}$$

Here are the estimates within that application. Its generating series is $\mu(\gamma)G(1+s,\gamma;u,v)J(s+u+v)J(s+u-v)$, with $J(z) = 1/\zeta(1+z) = z + O(z^2)$. The derivatives of G through order three are $O(E_\gamma H^{2j})$: local ratios are $1 + O(p^{-1})$ and their logarithmic derivatives cost $\sum_{p|\gamma}(\log p)^j/p \ll 1 + H^{j+1}$. The cubic residues are divided by $(\log M)^j$, and the error is $O(E_\gamma/L^2)$. For $\gamma > M/2$ use $Q(v) = O(v^2)$ directly. For $\gamma \le M^{1/2}$ apply the residue formula at M and at $M^{1/2}$ to the two pieces of the flat cutoff; their derivative residues cancel. These observations prove (5.19) also across the join of Q. Finally

$$\sum_{\gamma \le M} \frac{|H_{w,v}(\gamma)/h_{w,v}(\gamma)|E_\gamma^2}{\gamma} \ll \prod_{p \le M}(1 + p^{-1} + Cp^{-3/2}) \ll L.$$

For $\delta_1 = it$, $\delta_2 = -it$, $|t| \ge 4/L$, the two primary residues must be taken together on $|w| = 1/L$. There $\mathcal{A} \ll H^{12}(|t| + L^{-1})^2/L$ and $Z(1+2w; it) \ll L/(|t| + L^{-1})^2$; the kernel $2/w$ and the contour length cancel. Each secondary term at $w = \pm it$, $v = 0$ is $O(H^{12}/(|t|L))$. Thus the main contribution is $O(H^{12})$. For $|t| \le 4/L$, apply two-variable Cauchy to the entire mollified moment on $|\delta_1| = 10/L$, $|\delta_2| = 20/L$. The boundary shifts are separated, $Z = O(L^3)$, and $\mathcal{A} = O(H^{12}L^{-3})$. The Cauchy kernels have bounded total norm. This proves the same bound inside. Bounding individual primary residues at $\tau = 0$ would not be legitimate.

Divide by $(cY/\pi)^\tau \Gamma_{a,\delta}(\tau)$, take $\Psi(x) = x^{-\tau}\Phi(x)$, and normalize by the actual class mass $\check{\Phi}(0)/\pi^2 + O(Y^{-1/2})$. Equations (5.17)–(5.19) prove the weighted bound, weakened to H^{20}. Fixed plateaus which are at least one on $[6/5, 9/5]$, at scales $5Y/6$ and $10Y/9$, remove the weight. For $|D| \ge X^{.9}$ the single global choice $M = X^{.01}$ remains in the compact interval $.009 \le \log M/\log Y \le .02$. The sum of the dyadic conductor masses is $O(X)$, so no logarithm is lost in combining the blocks. For $|D| < X^{.9}$, the fourth-moment and large-sieve bounds in [10, Lemmas 3.4–3.5] give, by Cauchy–Schwarz, a normalized contribution $O(X^{-.08})$. This proves (5.8) on the original family. □

5.3. **Zero detectors in the three families.** In this subsection a zero is written $\rho = 1/2 + \beta + i\gamma$. The notation $N(\sigma, T)$ counts, with multiplicity, zeros with $\beta > \sigma$ and $|\gamma| \le T$.

We first record the logarithmic fourth-moment bound needed to turn Proposition 5.4 into a zero detector:

$$\mathbb{E}_{\mathcal{F}(X)}|L(1/2 + it, \chi_D)|^4 \ll (\log X)^{10}(1 + |t|)^8. \tag{5.20}$$

Here is a derivation with a harmless, but explicit, height exponent. For the fixed smooth partition function G in [26, Lemma 2.9(ii)], Proposition 1.3 of that paper bounds the dyadic conductor sum of $|\sum \chi_D(n)n^{-1/2-it}G(n/N)|^4$ by

$$C(1+|t|)^2\{M + N^2\log(2 + N^2/M)\}\log^6(2 + MN).$$

Put $T = 1 + |t|$. If $N \le \sqrt{MT}$ this is $O(MT^4\log^6(2+M))$, since $\log^7(2+T) \ll T$. For $N > \sqrt{MT}$ set $N_1 = MT/N$. The functional-equation Mellin transform has the gamma quotient

$$\frac{\Gamma(1/4 + (s - it + a)/2)}{\Gamma(1/4 - (s - it - a)/2)}, \qquad a = 0, 1.$$

Decompose the transformed series into dyadic blocks N_0 and insert $V(x) = G(2x) + G(x) + G(x/2)$, which equals one on the support of G. For $N_0 \le N_1$ move the Mellin line only to $\Re s = -1/8$;

the first possible numerator pole has real part $-1/2$ and is not crossed. For $N_0 > N_1$ move it to $\Re s = 4$. Stirling's formula and rapid Mellin decay give the pointwise majorant

$$C \sum_{N_0 \le N_1}^{\text{dyad}} (N_0/N_1)^{1/8} \int_{\mathbb{R}} \frac{|A_D(N_0, v-t)|}{(1+|v|)^9}\, dv + C \sum_{N_0 > N_1}^{\text{dyad}} (N_1/N_0)^4 \int_{\mathbb{R}} \frac{|A_D(N_0, v-t)|}{(1+|v|)^9}\, dv,$$

where $A_D(N_0, w) = \sum \chi_D(n) n^{-1/2-iw} V(n/N_0)$. The lower-line bound uses $T \le (1+|v|)(1+|v-t|)$; two Mellin factors with decay of order 24 leave the displayed integrable weight. All contour shifts can first be made on finite blocks. The high-block tail is absolutely summable using $|A_D| \ll \sqrt{N_0}$, so passage to the full series is justified. Weighted Hölder and Proposition 1.3 now give

$$CT^2 \left\{ \sum_{N_0 \le N_1} (N_0/N_1)^{1/8} F_M(N_0) + \sum_{N_0 > N_1} (N_1/N_0)^4 F_M(N_0) \right\},$$

where $F_M(v) = \{M + v^2 \log(2 + v^2/M)\} \log^6(2 + Mv)$. For $N_1 \ge 1$ this is $O(T^2 F_M(N_1))$ by geometric summation; for $N_1 < 1$ it is $O(MT^2 \log^6(2+M))$. Thus the same $MT^4 \log^6(2+M)$ bound holds for every N. Summing conductor blocks and dividing by (5.2) bounds each smooth block in L^4 by $CL^{3/2}T$. Take a partition cutoff $Y \asymp X^4 T^4$. Periodicity and the vanishing complete sum of a nonprincipal primitive character show by partial summation that the remaining tail is $O(XTY^{-1/2}) = O(1)$. There are $O(L + \log(2+T))$ blocks. Minkowski proves (5.20). This argument avoids moving a localized even-character gamma quotient past its pole at $-1/2 + it$.

Lemma 5.5. *For the rational family let $r = L/\ell \to \infty$, $\ell \to \infty$, and $0 \le T_* \le L^{-5/6}/32$. With probability at least*

$$1 - CH^{20}(1 + T_*\ell)e^{-cr} - Ce^{-KH}, \tag{5.21}$$

there is no zero with $\beta > 2/\ell$ and $|\gamma| \le T_* + \ell^{-1} e^{3\ell\beta}$. Here K is any fixed constant, and the constants in the estimate may depend on K. For $\nu = A\sqrt{H}$, the probability of a zero with $\beta > \nu/(2L)$ and $|\gamma| \le 2\beta$ is

$$\ll H^{20} e^{-c\nu} + e^{-KH}. \tag{5.22}$$

There is also a coarse detector, valid for $8 \le \ell \le c_0 L$, $\alpha \ge 1/L$, and $T_* \le 1$, with error $CL^{C_G} e^{-c_G \alpha L}$ for the same region with $\beta > \alpha$.

Proof. Put $B_D(s) = L(s, \chi_D) M_D(s) - 1$. The mollifier fourth moment is $O(L^6)$: the square diagonal is bounded by $\prod_{p \le M}(1 + 6/p + 1/p^2) \ll L^6$, and the error from (5.1) is $O(X^{-1/2} M^6)$. Cauchy–Schwarz and (5.20) give $\mathbb{E}|B_D(1/2+it)|^2 \ll L^8(1+|t|)^C$. The flat part of the mollifier annihilates coefficients $1 < n \le M^{1/2}$ of $LM - 1$ and the remaining coefficients are at most $\tau(n)$. Thus $|B_D(2+it)| \ll LX^{-1/200}$. Apply the vector-valued three-lines theorem after division by a fixed power of $s+3$. For $0 \le x \le 3/2$ it gives

$$\mathbb{E}|B_D(1/2 + x + it)|^2 \ll L^8(1+|t|)^C e^{-(2/300)xL}. \tag{5.23}$$

At an L-zero, $B_D = -1$; no assertion that the mollifier is zero-free is needed. Submean disks of radius comparable to $1/L$ and dyadic height bands in (5.23) give the coarse detector. More explicitly the area and height weights cost a fixed power of L and a fixed power of $e^{3\ell\beta}$, absorbed by the exponential in βL after choosing c_0 sufficiently small. The resulting geometric sum is $CL^{C_G} e^{-c_G \alpha L}$.

To remove the power of L near the centre, let $h = L^{-5/6}/2$, $b = B_0 H/L$, and $u(x,t) = \log \|B_D(1/2 + x + it)\|_{L^2(\mathcal{F}(X))}$. Proposition 5.4 bounds $u(0,t)$ by $Q_0 = \frac{1}{2}\log(C'H^{20})$ for $|t| \le h$. Choose $B_0 \ge 16 C_0/\kappa_0$, where $u \le C_0 H - \kappa_0 x L$ follows from (5.23). The subharmonic function

$u - Q_0 + (\kappa_0/2)Lx$ is nonpositive on the two vertical sides of $(0,b)\times(-h,h)$ and at most C_0H on the horizontal sides. Their harmonic measure satisfies

$$p(x,t) = \sum_{n\ \text{odd}} \frac{4}{\pi n}\sin(n\pi x/b)\frac{\cosh(n\pi t/b)}{\cosh(n\pi h/b)} \le \frac{9x}{b}e^{-\pi h/(2b)} \quad (|t|\le h/2).$$

Since $h/b\to\infty$, it follows that

$$\mathbb{E}|B_D(1/2+x+it)|^2 \ll H^{20}e^{-\kappa_0 xL/2} \quad (0\le x\le b, |t|\le h/2). \tag{5.24}$$

Submean disks of radius $\sigma/4$ centred at zeros with $\sigma<\beta\le 2\sigma$, $|\gamma|\le T$, lie in $3\sigma/4\le x\le 9\sigma/4$, $|t|\le T+\sigma/4$. Integrating (5.24) gives probability

$$\ll H^{20}(1+T/\sigma)e^{-c\sigma L} \tag{5.25}$$

provided this rectangle lies in the preceding one.

Take $a_c = A_cH/L$ with $A_c = (K+C_G+2)/c_G$ and increase B_0 to at least $8A_c$. Zeros with $\beta > a_c$ cost $O(e^{-KH})$ by the coarse detector. Below a_c use bins $j/\ell<\beta\le(j+1)/\ell$, $j\ge 2$, and height $2(T_*+\ell^{-1}e^{3(j+1)})$ in (5.25). If $2/\ell\ge a_c$ there are no such bins. Otherwise their nonconstant heights have logarithm $-H+\log r+3A_cH/r+O(1) = -H+o(H)$ and hence are $o(h)$. The sum is bounded by $CH^{20}(1+T_*\ell)\sum_{j\ge2}(1+e^{3(j+1)})e^{-crj}$, which proves (5.21). For the wedge use the same high-zero cutoff and, below it, the bands $\sigma = 2^{j-1}\nu/L$, $T=4\sigma$. Their geometric sum proves (5.22). □

We next derive the Hecke-family detector from the precise range of Gao's mollified moment. Fix $\kappa = 10^{-3}$, $m = 1/2-5\kappa$, $b=1/2$, $M = X^m$, and the cubic flat cutoff Q used above, with join at b. The mollifier is the sum over primary Gaussian integers $n\equiv 1 \pmod{(1+i)^3}$, with coefficient $\mu_K(n)Q(\log(M/Nn)/\log M)$ and denominator $(Nn)^s$. For a fixed nonnegative weight Φ on $[1,2]$, let $W(z) = \mathbb{E}_\Phi|L_d(1/2+z)M_d(1/2+z)|^2$. Gao's Proposition 3.5 [14] states, with $z=\tau+it$, $L_0 = \log(32X/\pi^2)$,

$$W(z) = 1+\left\{\frac{1-e^{-2\tau L_0}}{2\tau mL} - e^{-\tau L_0}\frac{\sin(tL_0)}{tmL}\right\}\int_0^b e^{-2\tau mL(1-v)}\left|Q'(v)+\frac{Q''(v)}{2zmL}\right|^2 dv + O_\Phi(X^{-\kappa-\epsilon}+e^{-2m(1-b)\tau L}|z|^6L^5), \tag{5.26}$$

for fixed $\epsilon>0$, $\tau\ge -1/(\epsilon L)$ and $\epsilon/L\le|z|\le\kappa$. Quotients at zero are interpreted by continuity. This is the second moment of LM, not of $LM-1$. Put $\eta_0 = m(1-b)$. For $\tau\ge 1/L$, the prefactor is nonnegative and bounded: use $\sinh y/y\ge 1$ and $\sin y/y\le 1$. Since $|z|L\ge 1$, the integral is $O(e^{-2\eta_0\tau L})$, and the τ^6L^5 part of the error is absorbed into that exponential. Consequently

$$|W(\tau+it)-1| \ll (1+|t|^6L^5)e^{-\eta_0\tau L} + X^{-\kappa}. \tag{5.27}$$

In particular the loss is logarithm-free only at height $|t|\le L^{-5/6}$.

Lemma 5.6. *There are fixed positive σ_0,τ_0,c such that*

$$\mathbb{E}_{\mathcal{D}_X}N(\sigma,T) \ll (1+TL)(1+T^6L^5)e^{-c\sigma L} \quad (4/L\le\sigma\le\sigma_0, 0\le T\le\tau_0). \tag{5.28}$$

For every fixed $\alpha>0$ there is $\theta_\alpha>0$ such that

$$\mathbb{P}_{\mathcal{D}_X}\{N(\alpha,X^{\theta_\alpha})>0\} \ll X^{-c_\alpha}. \tag{5.29}$$

Also, for $8\le\nu=o(L)$, the probability of a zero with $\beta>\nu/(2L)$, $|\gamma|\le 2\beta$ is

$$\ll \nu e^{-c\nu} + X^{-c_0}. \tag{5.30}$$

Proof. We spell out the application of Selberg's rectangle lemma, because discarding its signed logarithm would lose (5.28). Take $\lambda = \eta_0/100$, $S = \pi/(2\lambda)$, and rectangle half-height $h = S/L$, with left side $1/2+\sigma/2$ and right side 2. A zero with $\beta > \sigma$ and $|\gamma - t_0| \le h/2$ contributes at least $4h\cos(\pi/4)\sinh(\lambda\sigma L/2)$. On each boundary average the signed $\log|LM|$ and use $\mathbb{E}_\Phi \log|LM| \le \frac{1}{2}\log W \le \frac{1}{2}|W-1|$. The left side is bounded by $Ch\{(1+(T+2h)^6L^5)e^{-\eta_0\sigma L/2} + X^{-\kappa}\}$. On the horizontal sides split the offset u into $[\sigma/2,\kappa/2]$, $[\kappa/2,.6]$, and $[.6,1.5]$. The first uses (5.27) and integrates to $C(1+(T+2h)^6L^5)L^{-1}e^{-\eta_0\sigma L/2}$ plus a fixed power error. On the second use Gao's Proposition 3.2 with a fixed $\epsilon_0 < \eta_0\kappa/100$:

$$|W(u+it)-1| \ll X^{\epsilon_0}\{e^{-2\eta_0 uL} + e^{(\eta_0(1/2-u)-1/2)L}\}.$$

After multiplication by the boundary weight $e^{\lambda uL}$ both exponents are negative fixed multiples of L. For the last interval the flat mollifier gives directly $|LM(1/2+u+it)-1| \ll Le^{-\eta_0(u-1/2)L}$. It also makes LM zero-free near the right edge and gives $|\log LM(2+it)| \ll Le^{-\eta_0 L}$. Thus the last interval and the right edge contribute fixed negative powers of X. Divide by the zero weight and cover $[-T,T]$ with $O(1+TL)$ rectangles. Shrinking fixed σ_0,τ_0 absorbs all these fixed-power errors and proves the weighted form of (5.28). Two fixed plateaus at scales $5X/6$ and $10X/9$ remove the weight; using $4/L$ in the stated lower endpoint allows for the bounded change of logarithmic scale. The denominator is still $|\mathcal{D}_X|$.

For (5.29) a polynomial height bound is needed, rather than the compact-height statement of Proposition 3.2 alone. Its proof, using Gao's Lemmas 2.7–2.8, decomposes $B = LM - 1$ as $T_1 - T_2$. The large sieve gives $\mathbb{E}|T_1|^2 \ll X^\epsilon e^{-2\eta_0 uL}$ uniformly in the phases n^{-it}. The second term is a Mellin integral on $\Re w = -u$; the inverse Mellin weight is $e^{-Nn/X}$. For $\alpha/2 \le u \le .7$, the fourth-moment bounds

$$\mathbb{E}|L(1/2+iv)|^4 \ll X^\epsilon(1+v^2)^{1+\epsilon}, \qquad \mathbb{E}|M(1/2+iv)|^4 \ll X^\epsilon(2+M^2/X)$$

and gamma decay imply $\mathbb{E}|T_2|^2 \ll X^{-2u+\epsilon}(1+|t|)^4$. Hence $\mathbb{E}|B(1/2+u+it)|^2 \ll X^\epsilon(1+|t|)^4e^{-2\eta_0 uL}$. Cover $\beta \ge \alpha$, $|\gamma| \le T$ by $O_\alpha(T+1)$ submean disks of fixed radius $\alpha/8$. Since $B = -1$ at an L-zero, the probability is at most $C_\alpha X^{\epsilon-\eta_0\alpha}(1+T)^5$. Take $\epsilon = \theta_\alpha = \eta_0\alpha/100$. For larger fixed u the deterministic right-hand estimate applies. This proves (5.29).

Finally take $\beta_0 < \min(\sigma_0/2, \tau_0/16)$. The part $\beta > \beta_0$ of the wedge is controlled by (5.29). Below it use the bands $b_j = 2^{j-1}\nu/L$, with height $8b_j$, in (5.28). Writing $v_j = b_jL$, their bound is $Cv_j(1+v_j^6/L)e^{-cv_j} \ll Cv_je^{-cv_j/2}$. Its geometric sum proves (5.30). □

For the weight family the needed zero density is a direct literature input. Hough's Main Theorem 1.1 [15] states, for some sufficiently small fixed positive θ,δ,

$$\mathbb{E}_{H_k}N(\sigma,T) \ll T\log k\, e^{-\theta\sigma\log k}, \quad \frac{2}{\log k} < \sigma < \tfrac{1}{2}, \quad \frac{10}{\log k} < T < k^\delta. \tag{5.31}$$

Its measure is unweighted and its range includes all even weights. Set $L = \log k$. Binning β into intervals of length $1/\ell$ shows that, for $r = L/\ell \to \infty$, $\ell \to \infty$, $T_* \le 1$,

$$\mathbb{P}\{\exists\rho : \beta > 2/\ell, |\gamma| \le T_* + \ell^{-1}e^{3\ell\beta}\} \ll (T_*L + r)e^{-cr}. \tag{5.32}$$

Indeed the jth bin uses $T_j = 2(T_* + \ell^{-1}e^{3(j+1)})$, $j \ge 2$. Its smallest height is greater than $10/L$, its smallest offset is greater than $2/L$, and its largest height is $e^{3\ell/2+O(1)} = k^{o(1)}$. Thus every bin is within (5.31), and summing $C(T_*L + re^{3(j+1)})e^{-crj}$ proves the assertion. The same estimate on the bands $2^{j-1}\nu/L$, with height eight times the offset, proves the wedge bound

$$\mathbb{P}\{\exists\rho : \beta > \nu/(2L), |\gamma| \le 2\beta\} \ll \nu e^{-c\nu}, \qquad \nu \to \infty, \quad \nu = o(L). \tag{5.33}$$

5.4. **A zero-free complex approximation.** Write $\mathcal{L} = -L'/L$. For the rational and weight families use

$$K_\ell(z) = \frac{e^{\ell z}(e^{\ell z}-1)^2}{\ell^2 z^3}, \qquad P_\ell(s) = \sum_{n \le e^{3\ell}} \Lambda_L(n) w_\ell(n) n^{-s},$$

where $\Lambda_L(p^j) = \chi_D(p)^j \log p$ in the rational family and $\Lambda_L(p^j) = A_j(\lambda_f(p)) \log p$ in the weight family, and

$$w_\ell(n) = \frac{(3\ell - \log n)_+^2 - 2(2\ell - \log n)_+^2 + (\ell - \log n)_+^2}{2\ell^2}.$$

The weight lies in $[0,1]$, is one for $n \le e^\ell$, and is zero for $n > e^{3\ell}$. The arithmetic norms of P_ℓ are at most $Ce^{9\ell/2}$ and $Ce^{7\ell}$, respectively. In the second case expand A_j as $e_{p^j} - e_{p^{j-2}}$ and use $\tau(n) \le 2\sqrt{n}$ to bound the norm by $C\sum_{n \le e^{3\ell}} n \log n$.

The Gaussian Hecke family needs a kernel with stronger vertical decay. Choose a fixed $\tau > 0$ smaller than $\tau_0/10$, put $a = 4/\tau$, $k_\ell = \lfloor \ell/a \rfloor$, and define

$$K_\ell^*(z) = K_\ell(z) \left(\frac{\sinh(az/2)}{az/2} \right)^{k_\ell}. \tag{5.34}$$

If U_1, U_2 are uniform on $[0,\ell]$ and the V_j are uniform on $[-a/2, a/2]$, then $K_\ell^*(z) = \mathbb{E}e^{Vz}/z$ for $V = \ell + U_1 + U_2 + \sum V_j \in [\ell/2, 7\ell/2]$. Thus its Dirichlet weight $w_\ell^*(n) = \mathbb{P}(V \ge \log n)$ is nonnegative, at most one, equals one up to $e^{\ell/2}$, and vanishes above $e^{7\ell/2}$. The corresponding ideal polynomial P_ℓ^* has arithmetic norm at most $Ce^{8\ell}$. The kernel satisfies

$$|K_\ell^*(u+iv)| \le \frac{4}{\ell^2 |u+iv|^3} \begin{cases} e^{7\ell u/2}, & u \ge 0, \\ e^{\ell u/2}, & u \le 0. \end{cases} \tag{5.35}$$

Moreover, for $\beta \le \beta_0 = \tau/1000$, $\delta \ge 0$, $|v| \ge \tau$,

$$|K_\ell^*(\beta - \delta + iv)| \le \frac{4e^{-\ell\delta/2 - \tau\ell/20}}{\ell^2 |v|^3} (\tau/|v|)^{k_\ell}. \tag{5.36}$$

Indeed the extra factor is at most $\{\frac{1}{2} e^{a|u|/2} \tau/|v|\}^{k_\ell}$ and $k_\ell \ge \tau\ell/8$ for large ℓ. The loss $7\beta_0\ell/2$ is smaller than the gain from 2^{-k_ℓ}.

We use the following basic analytic facts in their unconditional form. The primitive completed functions are entire of order one, their zeros are in $0 \le \Re s \le 1$, and their real functional equations pair ρ with $1-\rho$ and $\bar{\rho}$. These facts follow from the usual theta-integral continuation for the two degree-one families, and from the Mellin integral of the cusp form in the weight family. Euler coefficients bound the function on a right-hand line; the functional equation, Stirling's formula and the maximum principle give polynomial bounds on every fixed vertical strip. Jensen at $2 + iT$, with inner radius 3 and outer radius 4, then gives

$$\#\{\rho : |\Im\rho - T| \le 1\} \ll \log(Q(|T|+2)) \tag{5.37}$$

in the degree-one families. The lower bound at the centre of the Jensen disk is supplied by the absolutely convergent Euler product. For weight k, the same bound is $O(\log(k + |T| + 2))$; alternatively take the real part of the completed logarithmic derivative at $2 + iT$, where each zero in the indicated band has weight at least $1/5$ and the Euler logarithmic derivative is bounded.

Writing $z = s - 1/2$, the order-one product and functional equation give

$$\mathcal{L}(1/2 + z) = B(z) - \frac{m_0}{z} - \sum_{\alpha \in \mathcal{P}} \frac{2z}{z^2 - \alpha^2}. \tag{5.38}$$

Here m_0 is the central multiplicity, $\mathcal{P}$ contains one representative of each pair of noncentral completed zeros, and

$$B(z) = \begin{cases} \frac{1}{2}\log(|D|/\pi) + \frac{1}{2}\psi((1/2 + a_D + z)/2), & \chi_D, \\ \frac{1}{2}\log Q_d + \psi(1/2 + z) - \log(2\pi), & K, \\ \psi(k/2 + z) - \log(2\pi), & H_k. \end{cases}$$

In particular $B = O(L)$ near $z = 0$. The paired series and its inverse-square tail converge by (5.37). For root number -1, the completed function is z^{m_0} times an even nonzero function at zero, with m_0 odd; its logarithmic derivative is still odd after the m_0/z term is retained. No assumption $m_0 = 1$ has been made.

The contour formula is

$$\mathcal{L}(s) = P_\ell(s) + \sum_\rho K_\ell(\rho - s) + E_{\mathrm{tr}}(s), \tag{5.39}$$

with starred versions for K. Start the convolution integral on $\Re z = 2$, so that its Dirichlet series is absolutely convergent. Move left across the completed zeros and the gamma poles, using horizontal edges a distance $\gg 1/\log(Q(2 + T))$ from zero ordinates. The local count (5.37) and the order-one product bound the logarithmic derivative there by a fixed power of the logarithm. The kernels have vertical decay $|z|^{-3}$, or $|z|^{-3-k_\ell}$. The functional equation on left vertical edges, chosen a fixed distance from gamma poles, and exponential kernel decay let the edges tend to infinity. The residue of $-L'/L$ at a zero is its negative multiplicity; consequently the sign in (5.39) is positive when the argument is $\rho - s$. This also fixes the sign if the denominator is instead written $(s - \rho)^3$. The trivial-zero contribution is $O(\ell^{-2}e^{-\ell\Re s})$ in the unstarred cases, and $O(\ell^{-2}e^{-\ell\Re s/2})$ in the starred case. For the latter the gamma poles are $0, -1, -2, \ldots$; a completed zero at a boundary point, if present, is counted separately with its multiplicity.

Lemma 5.7. *For the rational and weight families, outside the detector in Lemma 5.5 or (5.32), the function L is zero-free at $s = 1/2 + \delta + it$, simultaneously for $|t| \le T_*$, $4/\ell \le \delta \le 3/2$, and*

$$\mathcal{L}(s) = P_\ell(s) + O\left(e^{-\delta\ell}\{L + |P_\ell(1/2 + 4/\ell + it)|\}\right). \tag{5.40}$$

For K, outside a set of probability

$$C(T_*L + r)(1 + T_*^6L^5)e^{-cr} + CX^{-c_0}, \qquad r = L/\ell \to \infty, \quad T_* \le \tau/4, \tag{5.41}$$

the analogous assertion holds for $8/\ell \le \delta \le 3/2$, with P_ℓ^* and $e^{-\delta\ell/2}$ in place of P_ℓ and $e^{-\delta\ell}$. If $T_* \le L^{-5/6}$ the factor $1 + T_*^6L^5$ in (5.41) is bounded.

Proof. At the anchor $s_0 = 1/2 + a + it$, set $S_0 = \sum_\rho a/|s_0 - \rho|^2$ and $T_0 = \sum_\rho(a - \beta)/|s_0 - \rho|^2$. The ratio of the contribution of a horizontal pair to T_0 and to S_0 is

$$\frac{a^2 + (\gamma - t)^2 - \beta^2}{a^2 + (\gamma - t)^2 + \beta^2} \ge 3/5$$

whenever $|\beta| \le a/2$ or $|\gamma - t| > 2|\beta|$. For $a = 4/\ell$, the unstarred detector gives exactly these alternatives. In the variables $A = \ell a = 4$, $B = \ell\beta$, $C = \ell(\gamma - t)$, the sum of the three exponential terms in K_ℓ divided by the anchor summand is, for $B \le 2$, at most

$$\frac{2}{A^2}(e^{-A/2} + 2e^{-A} + e^{-3A/2}) < 1/32.$$

For $B > 2$, $|C| > e^{3B}$ and it is at most $(e^{-2A} + 2e^{-5A/2} + e^{-3A})/A < 1/32$. The real part of (5.38) and (5.39) therefore absorb the zero sum with coefficient $3/5 - 1/32 = 91/160$, giving $S_0 \ll L + |P_\ell(s_0)|$. At a rightward point with $A_1 = \ell\delta \ge 4$, the same comparison gives Ce^{-A_1} times the anchor sum: if $B \le 2$ use $|s - \rho| \ge |s_0 - \rho|$ and $\ell|s - \rho| \ge A_1/2$; if $B > 2$ use $|C| > e^{3B}$

to remove the positive exponential. This proves (5.40). The inequalities hold up to $\delta = 3/2$; this endpoint is useful for the first complex disks.

For the starred kernel exclude the following two sets of zeros:

$$\begin{aligned} &4/\ell < \beta \le \beta_0, && |\gamma| \le T_* + \min(\tau, \ell^{-1}e^{4\ell\beta}),\\ &\beta > \beta_0, && |\gamma| \le Y = X^{\theta_{\beta_0}}. \end{aligned}$$

The second has probability $O(X^{-c_0})$ by (5.29). Apply (5.28) to the first in bins $j/\ell < \beta \le (j+1)/\ell$, $j \ge 4$. Their heights are at most $2(T_* + \min(\tau, \ell^{-1}e^{4(j+1)}))$. The two prefactors are bounded by $C(T_*L + r)e^{4(j+1)}$ and $C(1 + T_*^6L^5)(1 + r^6/L)e^{24(j+1)}$. The geometric sum and $r \to \infty$ absorb $1 + r^6/L$, proving (5.41).

Now take $a = 8/\ell$. The pair ratio is again at least $3/5$. Zeros with $\beta \le \beta_0$, $|\gamma - t| > \tau$ contribute, by (5.36) and (5.37), $O(Le^{-\ell\delta/2-\tau\ell/20})$. Those with $\beta > \beta_0$ have $|\gamma - t| > Y/2$ and contribute

$$O\left(\ell^{-2}e^{7\ell/4-\ell\delta/2}Y^{-2}\log(QY)\right) \ll Le^{-\ell\delta/2}X^{-c},$$

because $\ell = o(L)$. For the remaining zeros put $A = 8$, and use (5.35). If $B \le 4$ the kernel-to-anchor ratio is at most $8A^{-2}e^{-A/4} = e^{-2}/8 < 1/32$. If $B > 4$, then $|C| > e^{4B}$, and the ratio is at most $4A^{-1}e^{-9A/4} < 1/32$. The same absorption works. At $A_1 = \ell\delta \ge 8$ the ratio is $Ce^{-A_1/2}$, using $|s - \rho| \ge |s_0 - \rho|$ in the first case and $|C| > e^{4B}$ in the second. This proves the starred assertion, including zero-freeness throughout the stated complex region. □

5.5. **Model bounds and the central interval.** In all three families set

$$G_D(w) = e^{-w}\mathcal{L}_D(1/2 + e^{-w}), \qquad G_E(w) = e^{-w}\sum_{p,j\ge1} a_{p,j}(\log p)p^{-j(1/2+e^{-w})},$$

with prime ideals and their norms in the Hecke case. In the quadratic cases $a_{p,j} = X_p^j$; in the weight case $a_{p,j} = A_j(X_p)$. The analytic construction, moment bounds and Gaussian comparison of Lemmas 2.1, 2.2 and 2.4 apply. Here their parameters are

$$\begin{aligned} \mathbb{E}G_E(t) &= \mu + O(e^{-t}),\\ \operatorname{Cov}(G_E(t), G_E(s)) &= \frac{1}{4}\operatorname{sech}^2((t-s)/2) + O(e^{-t-s}),\\ \mu &= \begin{cases} 1/2, & \text{quadratic cases},\\ -1/2, & H_k. \end{cases} \end{aligned} \tag{5.42}$$

We indicate the checks that distinguish these applications. The prime sum is an analytic L^2 martingale; Cauchy's formula on a slightly larger fixed sector gives local uniform convergence and derivative moments. Higher prime powers converge absolutely there. The prime number theorem, or the prime-ideal theorem for the fixed field K, gives $\sum(\log p)p^{-1-z} = z^{-1} + O(1)$ and $\sum(\log p)^2p^{-1-z} = z^{-2} + O(1)$ in the required small sectors. The second prime powers have mean $1 + O(p^{-1})$ in the quadratic cases and $-1 + O(p^{-1})$ by (5.6); these give the two means in (5.42). The first powers give the covariance. All remaining covariance terms are absolutely summable. Whole-prime blocks have range at most $C|a|(\log p)p^{-1/2-\Re a}/(1 - p^{-1/2-\Re a})$. The squared ranges sum to $O(1)$ in each fixed sector, and their cubes sum to $O(|a|^3)$. Thus the moment and Lindeberg estimates also hold for the continuous local laws (5.5).

In particular, for a fixed finite number of points and every $q \ge 2$, the normalized model polynomials have L^q norm $O(\sqrt{q})$. The same holds for the actual polynomials for fixed q, by the arithmetic norm estimates above. At an anchor $a_0 = 4/\ell$ or $8/\ell$, the precise bound is

$$\|\ell^{-1}P_\ell(1/2 + a_0 + it)\|_q \ll_q 1, \tag{5.43}$$

uniformly in the height t; the unnormalized norm is $O_q(\ell)$. Phases do not affect the range estimates. Finite-word transfer bounds the error for a product of degree $2m$ by $X^{-\kappa}\exp(Cm\ell+O(m\log m))$, with X replaced by k in the weight family. Taylor expansion of e^{ivY} to order $2m-1$ then gives characteristic-function error

$$2(C|v|/\sqrt{m})^{2m}+X^{-\kappa}\exp(Cm(\ell+\log(2+|v|)+\log m)). \tag{5.44}$$

The actual even remainder is transferred before it is bounded; this is why the estimate is valid on the original family. This is the argument of Proposition 2.16, with the actual arithmetic norms verified here.

The ternary models have a uniform negative one-point moment, by Lemma 2.7 and Corollary 2.8:

$$\mathbb{E}|G_E(t)|^{-a_0}\le C \qquad (t\ge\log 2) \tag{5.45}$$

for some $a_0>0$. The prime-ideal version is obtained by precisely the same prime-window argument: the fixed-field prime-ideal theorem counts every logarithmic window, the active local variances are at least $3/4$, and prime ideals of equal norm remain independent. The single inactive ideal $(1+i)$ is omitted. No assertion about independence of equal norms is used. For the weight model the following substitutes suffice:

$$\mathbb{P}(|G_E(t)-v|\le h)\ll e^t h, \qquad \mathbb{P}(|G_E(t)-v|\le h)\ll h+e^{-t/4}. \tag{5.46}$$

For the first inequality condition on all primes except 2. If $a=e^{-t}$ and $r=2^{-1/2-a}$, its exact block is $a\log 2\,(rx-2r^2)/(1-rx+r^2)$, whose derivative divided by a is $\log 2\,r(1-r^2)/(1-rx+r^2)^2\gg 1$ on $[-2,2]$. The density in (5.5) is bounded, giving density $O(a^{-1})$. For the second inequality smooth an interval at width $e^{-t/4}$ and use the Gaussian comparison; its first three derivative costs are absorbed by e^{-t}, e^{-2t} and e^{-3t}, respectively. The Gaussian marginal density is bounded.

Take now $\nu=A\sqrt{H}$, $\Delta=\nu/L$, with fixed $A>0$, and $\ell=L/\sqrt{\nu}$. On an interval $c\Delta\le\delta\le C\Delta$ the detectors and Lemma 5.7 imply

$$\|\mathbf{1}_{\mathrm{good}}\,\delta(\mathcal{L}-P_\ell)(1/2+\delta)\|_1\ll \nu e^{-c\sqrt{\nu}}=o(1).$$

Here (5.43) was used with its normalization. The detector probabilities tend to zero, including the factor H^{20} in the rational case. In (5.44) choose $m=\lfloor\nu^{1/8}\rfloor$ and $|v|\le\nu^{1/64}$. Then $|v|/\sqrt{m}\ll\nu^{-3/64}$ and $m\ell/L\ll\nu^{-3/8}$. The model tail tends to zero, so uniformly in these real shifts

$$\delta\mathcal{L}_D(1/2+\delta)\Longrightarrow\ N(\mu,1/4). \tag{5.47}$$

For U uniform on $[1,2]$, the same statement implies

$$\mathbb{P}(|\Delta\mathcal{L}_D(1/2+U\Delta)|<e^{-\sqrt{H}})=o(1). \tag{5.48}$$

To justify the shrinking threshold, first bound it by the probability of a fixed interval of radius ε, pass to the limit using (5.47), and then let ε decrease to zero. Uniformity for $U\in[1,2]$ follows from the uniform estimates just proved. No quantitative conclusion is being extracted from weak convergence.

Proposition 5.8. *In each of the three families,*

$$R^{\mathrm{mult}}_{L'_D}([1/2,1/2+\Delta])=o_{\mathbb{P}}(H).$$

Proof. Remove the wedge exceptional sets in (5.22), (5.30), or (5.33), and, in the rational case, the $O(X^{-1/2})$ proportion $|D|\le\sqrt{X}$. Every remaining zero has $|\beta|\le\Delta/2$ or $|\gamma|>2|\beta|$. For a horizontal pair its real resolvent at Δ is at least

$$\frac{6}{5}\frac{\Delta}{\Delta^2+\beta^2+\gamma^2}.$$

This follows by writing $D_0 = \Delta^2 + \beta^2 + \gamma^2$: the numerator is at least $3D_0/5$ and the denominator at most D_0^2. Zeros on the symmetry line give nonnegative terms directly. Thus the positive statistic

$$K_D = \sum_\rho \frac{\Delta^2}{\Delta^2 + |\rho - 1/2|^2} \le \frac{5}{3}\Delta \Re\{B(\Delta) - \mathcal{L}_D(1/2 + \Delta)\}$$

has restricted expectation $O(\nu + 1)$ by the anchor estimate and $B = O(L)$. If N_D counts the near zeros $|\rho - 1/2| \le 12\Delta$ and $W_D = \Delta^2 \sum_{|\alpha|>12\Delta} |\alpha|^{-2}$ is over the paired representatives in (5.38), then $N_D \le 145 K_D$ and $W_D \le 145 K_D/288$. All hypotheses of Lemma 3.2 now hold with the original denominator and its good-set indicator. Using (5.48), that lemma bounds the probability of a count exceeding ηH by

$$o(1) + \frac{C\{\nu + 1 + \log(\nu + 2) + \sqrt{H}\}}{\eta H} = o(1).$$

The central pole m_0/z is included in K_D and in N_D. At a central zero of order m_0, the derivative has order $m_0 - 1$; the lemma counts this contribution. The argument consequently applies unchanged to every root-number -1 form, without a rank-one assumption. Finally $L' \not\equiv 0$: a quadratic character has an active prime, and for a level-one form $\lambda_f(2) = 0$ implies $\lambda_f(4) = -1$. □

5.6. **The short bridge.** We will need a growing-dimensional small-ball test for the weight family as well as for the ternary models. The following variant makes the necessary hypothesis on continuous blocks explicit.

Lemma 5.9. *Let $Y = \mu + \sum \xi_p \in \mathbb{R}^r$ be a finite sum of independent centred blocks, with covariance $cI \le \Sigma \le CI$ and $\|\mu\|^2 \le Cr$. Suppose $\|\xi_p\| \le \beta_p \le 1$ and, for every $h \in \mathbb{R}^r$, the directional ranges satisfy*

$$\sum_p \big(\operatorname{ess\,sup} h \cdot \xi_p - \operatorname{ess\,inf} h \cdot \xi_p\big)^2 \le C\|h\|^2.$$

For all sufficiently large r, and a sufficiently small fixed $\epsilon_0 > 0$, put $A_0 = \operatorname{tr}\Sigma + \|\mu\|^2$ and $m = \lfloor \epsilon_0 r \rfloor$ and $\mathcal{Q}(y) = ((A_0 - \|y\|^2)/(A_0 - 1))^{2m}$. Then

$$\mathbf{1}_{\{\|y\| \le 1\}} \le \mathcal{Q}(y), \qquad \mathbb{E}\mathcal{Q}(Y) \le e^{-cr} + e^{Cr} \sum_p \beta_p^3.$$

Proof. For sufficiently large r, $A_0 - 1 \ge cr/2$ and $m \ge 1$. The Gaussian calculation is that in Lemma 2.13: its centred quadratic norm has L^q norm $O(\sqrt{rq} + q)$, so the expected polynomial is at most 2^{-2m} when ϵ_0 is small. For completeness, the replacement argument uses only the displayed directional-range condition. Hoeffding bounds every mixed partial sum, including Gaussian replacement blocks, by $\mathbb{E}e^{h\cdot W} \le e^{C\|h\|^2}$. A $1/2$-net of the sphere with at most 5^r points gives $\|W\|_{L^q} \ll \sqrt{r} + \sqrt{q}$. Write $p = 2m$ and $q(y) = A_0 - \|y\|^2$. The third directional derivative is

$$(A_0 - 1)^{-p}\{p(p-1)(p-2)q(y)^{p-3}(-2y\cdot h)^3 + 3p(p-1)q(y)^{p-2}(-2y\cdot h)(-2\|h\|^2)\}.$$

Its absolute value is bounded by $e^{Cr}\|h\|^3(1 + \|y\|/\sqrt{r})^{2p+3}$. Taylor's formula to order two cancels the matching means and covariances. Independence of the omitted block and the partial sum, the preceding norm bounds, and $\|g_p\|_{L^q} \ll \sqrt{q}\,\beta_p$ for its Gaussian replacement bound each remainder by $e^{Cr}\beta_p^3$. For example the expansion of the power in the derivative reduces to $r^{-j/2}(j+3)^{(j+3)/2} \le C^j (Cr)^{3/2}$ for $j = O(r)$; the polynomial factor is absorbed in e^{Cr}. Sum over the finite blocks. The even power makes $\mathcal{Q}$ nonnegative, and $A_0 - \|y\|^2 \ge A_0 - 1 > 0$ on the unit ball. □

Put $U = H^{1/4}$ and choose

$$u = \begin{cases} \lfloor \sqrt{H} \rfloor, & \mathcal{F}(X), \\ \lfloor H^{1/8} \rfloor, & \mathcal{D}_X, H_k, \end{cases} \qquad d = \frac{H^8 e^{2u}}{L}, \qquad \Delta = \frac{AU^2}{L}. \tag{5.49}$$

Here A will be a sufficiently large fixed constant. The bridge is $[1/2 + \Delta, 1/2 + e^5 d]$.

Proposition 5.10. *For this choice of A the multiplicity count on the bridge is $o_{\mathbb{P}}(H)$ in all three families.*

Proof. Cover its logarithmic image by half-step cells, with centres t_j and $\delta_j = e^{-t_j}$. Extend the endpoint centres by at most $1/2$. The number of cells is $n = O(u + \log H)$ and, writing $S_j = \delta_j L$,

$$e^{-1/2} AU^2 \le S_j \le e^{11/2} H^8 e^{2u}. \tag{5.50}$$

The required heights are $T_* = 3\delta_j$. In the rational and Hecke families these are $o(L^{-5/6})$, because $\delta_j L^{5/6} \le \exp(-H/6 + O(u + \log H))$. Thus every detector is used within its actual height range.

First use $\ell_j = L/\sqrt{S_j}$ to control large maxima. The detector errors are bounded respectively by

$$CH^{20}(1 + 3\sqrt{S_j})e^{-c\sqrt{S_j}} + Ce^{-4H}, \quad C(S_j + \sqrt{S_j})e^{-c\sqrt{S_j}} + CX^{-c_0}, \quad C(S_j + \sqrt{S_j})e^{-c\sqrt{S_j}}.$$

Their sum over the cells is $o(1)$. The normalized approximation error has every fixed restricted L^q norm at most $C_q S_j e^{-c\sqrt{S_j}}$, by (5.43). Cauchy's formula on fixed disks and the polynomial moment bounds therefore show that the sum of the positive logarithms of the large disk maxima has expectation $O(n) = o(H)$.

The small disk maximum requires a longer polynomial. Choose $r = \lfloor \epsilon U \rfloor$, with fixed $\epsilon \log 64 < 1/4$, put $v_j = r + \log(2 + S_j)$, and take

$$\ell_j' = C_y v_j / \delta_j. \tag{5.51}$$

The ratio $L/\ell_j' = S_j/(C_y v_j)$ tends to infinity uniformly and is $\gg U$ after A is chosen. More precisely,

$$\sup_j \frac{\log H + \log S_j}{S_j/(C_y v_j)} \longrightarrow 0.$$

This follows from the monotonicity of $S/(r + \log(2 + S))$ for large S and (5.50). It is needed in the rational case, where $\log S_j$ is not $o(r)$. The detector errors for (5.51), including the H^{20} factor, again sum to $o(1)$.

Use the Cayley map $a(z) = \delta_j(1 + z)/(1 - z)$ and define

$$\mathcal{H}_D(z) = \frac{\delta_j}{(1 - z)^2} \mathcal{L}_D(1/2 + a(z)), \qquad Z_k = \frac{2}{\sqrt{k + 1}} [z^k] \mathcal{H}_D(z), \quad 0 \le k < r.$$

The map is related to logarithmic time by $\mathcal{H}_D(z) = G_D(t_j - \log((1 + z)/(1 - z)))/(1 - z^2)$. On $|z| \le 1/64$ the time displacement is at most $\log(65/63) < 1/8$. If the maximum m_j of $|G_D|$ on the disk of radius $1/8$ is at most e^{-U}, Cauchy's estimate gives $\|Z\| \le C\sqrt{r}\, 64^r e^{-U} < 1/4$.

Replace $\mathcal{L}_D$ by the polynomial with cutoff ℓ_j' to obtain Y_D. On $|z| \le 1/2$, $\Re a(z) \ge \delta_j/3$ and $|\Im a(z)| \le 3\delta_j$. Cover the anchor height segment by $O(v_j)$ polynomial disks of radius comparable to $1/\ell_j'$. Equation (5.43) and Cauchy's formula give

$$\mathbb{E} \sup_{|t| \le 3\delta_j} |P(1/2 + a_0 + it)|^2 \ll C_y v_j^2 (\ell_j')^2.$$

For fixed large b_0, Markov shows that $\delta_j \sup |P| > e^{b_0 v_j}$ has probability at most $C v_j^4 e^{-2b_0 v_j} \le e^{-b_0 v_j}$. Choose C_y after b_0. The normalized analytic error is then at most $Ce^{-C_y v_j/6}(e^{b_0 v_j} + S_j) \le e^{-30 v_j}$ (the denominator 6 allows the starred kernel). Another Cauchy estimate, costing $C\sqrt{r}\, 2^r$,

gives $\|Z - Y_D\| < 1/4$. Thus a small actual disk maximum implies $\|Y_D\| \le 1$ outside the stated exceptional set.

We verify the model-vector hypotheses instead of assuming independent Taylor coefficients. Expansion of (5.42) under the Cayley map gives

$$\mathbb{E}Y_k = \frac{2\mu\,\mathbf{1}_{2|k}}{\sqrt{k+1}} + O(\delta_j 2^k), \qquad \operatorname{Cov}(Y_k, Y_l) = \mathbf{1}_{k=l} + O(\delta_j^2 2^{k+l}). \tag{5.52}$$

The squared polynomial-tail contribution is $O(r4^r e^{-\delta_j \ell_j'/6})$. Also $\delta_j^2 r 4^r \to 0$ uniformly. Hence $\frac{1}{2}I \le \Sigma \le 2I$ and $\|\mu_Y\|^2 \le Cr$. For a ternary prime block write $\xi_p = a_p X_p + b_p(X_p^2 - \theta_p)$. Odd and even parts are uncorrelated, $\theta_p \ge 2/3$ in the rational case and $\theta_p \ge 3/4$ for active ideals. Thus $\sum a_p a_p^{\mathsf{T}} \ll I$, while the higher-power estimate gives $\sum \|b_p\|^2 = o(1)$. Moreover $\max \|\xi_p\| \le \delta_j e^{Cr} = o(1)$ and $\sum \|\xi_p\|^3 \le \delta_j^3 e^{Cr}$. Lemma 2.13 applies.

For the weight model write instead $\xi_p = a_p X_p + \eta_p(X_p)$, with η_p the centred higher powers. Cauchy's coefficient estimate on $|z| = 1/2$ gives

$$\|\eta_p\|_\infty \le b_p = C\delta_j \sqrt{r}\, 2^r (\log p) p^{-1-2\delta_j/3}, \qquad \|a_p\| \le C\delta_j \sqrt{r}\, 2^r (\log p) p^{-1/2-\delta_j/3}.$$

Here $\sum b_p^2 = o(1)$. The L^2 triangle inequality, used separately at each prime, implies

$$\sum_p \mathbb{E}X_p^2 (h \cdot a_p)^2 \le 2 \operatorname{Var}(h \cdot Y) + 2 \sum_p \mathbb{E}|h \cdot \eta_p|^2 \ll \|h\|^2.$$

Since $\mathbb{E}X_p^2 \ge 1$, the linear Gram matrix is bounded. The directional range is at most $4|h \cdot a_p| + 2\|h\| b_p$, proving the hypothesis of Lemma 5.9. The block envelopes have $\sum \beta_p^3 \le \delta_j^3 e^{Cr}$ and maximum $o(1)$. Correlation between X_p and its higher powers has not been discarded.

It follows in every family that $\mathbb{E}\mathcal{Q}(Y_E) \le e^{-cr} + \delta_j^3 e^{Cr}$. The polynomial $\mathcal{Q}$ has degree $4m$, not $2m$. The norm of each coefficient coordinate is bounded by $C2^r \exp(C_N \ell_j')$ with $C_N = 9/2, 8, 7$ respectively. Consequently its arithmetic norm has logarithm at most

$$4C_N m \ell_j' + O(mr + m \log r + m). \tag{5.53}$$

Choose constants in this order: the arithmetic exponent κ, then ϵ, then the small ϵ_0 in the polynomial test, then b_0, then C_y, and finally A. Indeed,

$$\frac{m\ell_j'}{L} \ll \frac{r(r + \log(2 + S_j))}{S_j} \ll \frac{\epsilon^2}{A} + o(1).$$

Increasing A makes the leading term in (5.53) smaller than $\kappa L/4$, and its remaining terms are $o(L)$. The arithmetic error is therefore at most $X^{-\kappa/2}$ (or $k^{-\kappa/2}$). This proves, for every cell, the small-maximum probability bound $Ce^{-cU} + e^{-H} + X^{-\kappa/2}$, including the analytic exceptions already summed. Its sum over n tends to zero.

On the retained set all depths $\log^+(1/m_j)$ are at most U. Their sum is at most $nU = o(H)$: in the rational case this is $O(H^{3/4} + H^{1/4} \log H)$, and in the other cases it is smaller. The large-log sum has expectation $O(n)$, as proved at the start. The shifted Jensen inequality (R.3) on each cell now proves the assertion by Markov. The approximation supplies zero-freeness of L on the surrounding complex disks, so the zeros and multiplicities of G_D there are exactly those of L_D'. □

5.7. **The bulk count with retained local moments.** We provide the transfer and counting argument on the unconditional good sets just constructed. In particular an approximation under GRH is not being used for this step.

Let d, u be as in (5.49), and set

$$b_H = -\log d = H - 8\log H - 2u, \qquad \ell_B = \frac{C_y H}{d} = \frac{C_y L}{H^7 e^{2u}}, \qquad a_H = H^{1/4}. \tag{5.54}$$

Consider the disks $|w - t| \le .9$ about half-step centres. For $a = e^{-w}$, $\Re a \ge c_+ e^{-t}$ with $c_+ = e^{-.9}\cos(.9) > 0$, and $|a| < 3/2$ for $t \ge \log 2$. The rational coarse detector, with $T_* = 1$, and the weight detector therefore give one common zero-free event on all disks with centres $\log 2 \le t \le b_H$. For the Hecke family take first $t \ge t_0 = \log(16e^{.9}/\tau)$ and use the starred detector with $T_* = \tau/4$. In all cases

$$\frac{L}{\ell_B} = \frac{H^7 e^{2u}}{C_y}, \qquad \frac{L/\ell_B}{H} \longrightarrow \infty.$$

This pays any fixed power of L in the detector, including the L^5 factor in (5.41) at fixed height.

The anchor line segment is covered by $O(\ell_B + 1)$ disks of radius $1/(2\ell_B)$ and slightly larger disks of radius $1/\ell_B$. Equation (5.43), finite-moment transfer, and Cauchy's formula show that

$$\mathbb{P}\left(\sup_{|t| \le T_*} |P(1/2 + a_0 + it)| > L e^{4H} \right) \ll e^{-7H}.$$

Indeed the second moment costs $O((\ell_B + 1)\ell_B^2)$ and division by $L^2 e^{8H}$ gives the assertion since $\ell_B \le L$. For each prescribed fixed $E > 0$, choose C_y sufficiently large. Lemma 5.7 then gives, on this common event,

$$\sup_{t,w} |G_D(w) - e^{-w} P(1/2 + e^{-w})| \le e^{-EH}. \tag{5.55}$$

In the starred case the exponent in the kernel is halved, which is allowed in the choice of C_y. The squared random tail is $O(e^{-c\Re(a)\ell_B})$; its L^2 norm has half this exponent, again paid by the same choice. For every fixed q the normalized actual polynomial moments are bounded, since $q\ell_B = o(L)$. Applying Cauchy's formula to each good path and then taking expectations bounds all fixed moments of maxima and fixed derivatives on the smaller disks. These are restricted moments with the original denominator.

In the Hecke family the fixed interval $[\log 2, t_0 + 2]$ is treated without using a shrinking-height detector. Let $a_* > 0$ be a lower bound for $\Re(e^{-w})$ on its disks, and take $\alpha = a_*/2$ in (5.29). Outside probability $O(X^{-c})$, zeros with $\beta \ge \alpha$ have ordinate larger than $Y = X^{\theta_\alpha}$. Use the exact starred identity with $\ell = C_y H$. Zeros with $\beta < \alpha$ contribute at most $C\ell^{-2} L e^{-a_*\ell/4}$, by (5.37) and the negative-real kernel bound. The others contribute at most $C\ell^{-2} e^{7\ell/4} Y^{-2} \log(QY) = O(X^{-c})$. Thus (5.55) and its moment consequences hold on these finitely many disks as well, with constants depending on t_0.

We next control the lower tails of their small-disk maxima. Write $m_j = \max_{|w - t_j| \le 1/8} |G_D(w)|$ and $v_j = \log^+(1/m_j)$. A convenient nonnegative majorant is

$$g_n(x) = \exp(1 - e^{2n} x^2/4), \qquad \mathbf{1}_{\{|x| \le e^{-n}\}} \le g_n(x).$$

Its Fourier transform has bounded normalized L^1 norm, and its tail outside $|v| \le C_E \sqrt{H} e^n$ is $O(e^{-EH})$. For the two quadratic models (5.45) gives $\mathbb{E} g_n(G_E(t)) \ll e^{-a_0 n}$. Use $m = \lceil H^4 e^{2u} \rceil$ in (5.44), for every $n \le u$. Uniformly over the disks,

$$\frac{|v|}{\sqrt{m}} = O(H^{-3/2}), \qquad \frac{m\ell_B}{L} = O(H^{-3}), \qquad m(u + \log H) = o(L). \tag{5.56}$$

Thus the finite Taylor error and the arithmetic norm error are smaller than every fixed negative power of e^H. The Lipschitz cost $O(e^n)$ in (5.55) and in the random tail is also absorbed. Positivity permits us to drop the good-set indicator when bounding the polynomial majorant, giving

$$\mathbb{P}(\text{good}, v_j \ge n) \ll e^{-a_0 n} + O(e^{-EH}), \qquad 1 \le n \le u. \tag{5.57}$$

For the finitely many Hecke front disks, the same calculation uses $m\ell/L = O(H^5 e^{2u-H}) \to 0$. Retain only samples with all these depths below u. The discarded probability is $O(He^{-a_0 u}) + o(1) = o(1)$. Summing the truncated tails in (5.57) gives uniformly bounded restricted fourth moments of v_j. The shifted Jensen inequality, the positive maximum moments, and Cauchy's formula therefore give

$$\mathbb{E}[\mathbf{1}_{\text{retained}} N_j^4] \ll 1, \qquad \mathbb{E}[\mathbf{1}_{\text{retained}} \sup_{[t_j-1/4, t_j+1/4]} |G_D''|^8] \ll 1, \tag{5.58}$$

where N_j is the multiplicity count on the half-cell. In particular the fixed Hecke front interval has count $O_{\mathbb{P}}(1)$.

The weight model requires a separate early interval because the first bound in (5.46) grows with t. On $[\log 2, a_H + 2]$ use $d_F = e^{-a_H - 3}$, $\ell_F = C_y H/d_F$, and cap $v_F = 4a_H$. Use $m = \lceil H^4 e^{2v_F} \rceil$ for its depth tests. Then $m\ell_F/L = O(H^5 e^{9a_H + 3 - H}) \to 0$, and the other two Taylor budgets are as in (5.56). The first estimate in (5.46) gives

$$\mathbb{P}(\text{good}, v_j \geq n) \ll \min(1, e^{t_j - n}) + e^{-EH}.$$

The union of the early cap failures is $O(a_H e^{-3a_H+3}) + o(1)$, and the restricted expected depth is $O(1 + t_j)$. Jensen summed over the early cells gives expected count $O(a_H^2) = O(H^{1/2}) = o(H)$. On the late interval $[a_H - 2, b_H - 2]$ use ℓ_B and cap u. The second estimate in (5.46) gives instead $\mathbb{P}(\text{good}, v_j \geq n) \ll e^{-n} + e^{-t_j/4} + e^{-EH}$. Its union of cap failures is $O(H(e^{-u} + e^{-a_H/4} + e^{-EH})) = o(1)$, and its restricted fourth depth moment is bounded by $C + Cu^4(e^{-a_H/4} + e^{-EH}) = O(1)$. Thus (5.58) holds on the late interval also in the weight family. No uniform negative moment for the weight model has been assumed.

Proposition 5.11. *For each family and each fixed $\eta > 0$, outside a proportion $o_\eta(1)$, both the distinct and the multiplicity counts on $[1/2 + e^2 d, 1]$ are between $(c_* - \eta)H$ and $(c_* + \eta)H$.*

Proof. Work on the retained set just constructed, putting the real process equal to 1 on its complement whenever a bounded sign test is used. Its exceptional probability tends to zero. The counts on the time interval before a_H are $o_{\mathbb{P}}(H)$: for the quadratic cases this follows from (5.58) and $O(a_H)$ cells; for weight it was proved separately above.

For at most four time points in $[a_H, b_H - 2]$, Taylor transfer with $m = \lceil H^4 \rceil$ and Fourier cutoff H, followed by fixed smooth Fourier approximation, gives uniform convergence of bounded smooth tests to the Gaussian process

$$\mathcal{G}(t) = \mu + \tfrac{1}{2} g(t), \qquad \mathbb{E} g(t) g(s) = \operatorname{sech}^2((t-s)/2).$$

The budgets are $H/\sqrt{m} = H^{-1}$ and $m\ell_B/L = O(H^{-3} e^{-2u})$. The model Gaussian comparison is uniform since the smallest time tends to infinity. Repeated and arbitrarily close points are allowed: no inverse covariance matrix enters the smooth comparison. To pass to signs, truncate values at height K, costing $O(K^{-2})$, and smooth the sign at width ε, costing $O(\varepsilon) + o(1)$ by the Gaussian marginal density. Let X or k tend to infinity first, then $K \to \infty$ and $\varepsilon \to 0$. Thus two-point and four-point strict-sign tests converge uniformly.

Fix a small mesh $h > 0$. Choose a deterministic offset, depending on the finite arithmetic family if necessary, so that no retained function vanishes at an internal mesh point. This is possible because there are finitely many functions and each has only finitely many zeros on the compact interval. Let S_h count crossing cells. The two-point convergence and stationarity give

$$\frac{\mathbb{E} S_h}{H} \longrightarrow \frac{p(h)}{h}, \qquad p(h) = \mathbb{P}(\mathcal{G}(0)\mathcal{G}(h) < 0). \tag{5.59}$$

For pairs of cells separated by at most R there are only $O_h(H(R+1))$ pairs. For the others the off-diagonal Gaussian covariance block is $O(e^{-R})$. Interpolate this block for smoothed sign

tests. Gaussian integration by parts bounds the difference from the product of expectations by $O(\varepsilon^{-2}e^{-R})$; the four marginal smoothing strips cost $O(\varepsilon)$. With $\varepsilon = e^{-R/3}$ the total is $O(e^{-R/3})$. The uniform four-point arithmetic convergence now implies

$$\limsup_{X\to\infty} \operatorname{Var}(S_h/H) \ll_h e^{-R/3}.$$

Let $R \to \infty$. The same statement holds with k as the family parameter. Therefore $S_h/H - p(h)/h \to 0$ in probability.

It remains to recover all multiplicities from mesh signs. If a cell of length h contains at least two zeros counted with multiplicity, Rolle's theorem, including a repeated zero, gives

$$|G_D(t_{\text{left}})| \le h^2 \sup_I |G_D''|.$$

Set $K = h^{-1/4}$. One-point transfer and the second bound in (5.58) imply

$$\mathbb{P}(\text{retained}, N_I \ge 2) \ll h^{7/4} + h^2 + o(1).$$

Hölder and the first bound in (5.58) give $\mathbb{E}[\mathbf{1}_{\text{retained}} N_I \mathbf{1}_{N_I \ge 2}] \ll h^{21/16} + o(1)$, uniformly over cells. Summing $O(H/h)$ cells bounds the expected multiplicity excess by $CHh^{5/16} + o(H)$. Cells meeting the two ends cost $O_{\mathbb{P}}(1)$ by the same local moments. A cell with one simple zero has a strict sign change, while all failures of this correspondence are bounded by the preceding excess. Hence the distinct and multiplicity counts, divided by H, differ from S_h/H by a quantity with expectation at most $Ch^{5/16} + o(1)$ on the retained set.

Finally the Gaussian calculation (2.13), with the parameters of (5.42), gives $p(h) = c_* h + O(h^3)$. The sign change $\mu = 1/2$ to $-1/2$ does not alter this probability. Choose h sufficiently small for the prescribed η, then let the arithmetic parameter tend to infinity. This proves the proposition. The interval ends at $b_H - 2$ in time, which is exactly the physical lower offset $e^2 d$. □

5.8. **Assembly and normalization.** The three physical intervals

$$[1/2, 1/2 + \Delta], \qquad [1/2 + \Delta, 1/2 + e^5 d], \qquad [1/2 + e^2 d, 1]$$

cover $[1/2, 1]$, with overlap between the bridge and bulk. Propositions 5.8 and 5.10 make the first two multiplicity counts $o_{\mathbb{P}}(H)$, while Proposition 5.11 gives both bulk counts $c_* H + o_{\mathbb{P}}(H)$. An upper bound uses the sum of these three counts; a lower bound uses just the bulk. Thus on the full closed interval

$$R^{\text{dist}}(L') = c_* H + o_{\mathbb{P}}(H), \qquad R^{\text{mult}}(L') = c_* H + o_{\mathbb{P}}(H).$$

The deterministic multiplicity/sign inequality of Lemma 4.3 is

$$2R^{\text{dist}}(L') - R^{\text{mult}}(L') - 2 \le S^-(L') \le R^{\text{dist}}(L').$$

The subtraction of two allows for the closed endpoints, and gives the same asymptotic for strict changes of sign on the open interval.

For the rational family discard $|D| \le \sqrt{X}$, which is a proportion $O(X^{-1/2})$ by (5.2). On the remaining family, $H - \log 2 \le \log\log|D| \le H$. Given the requested tolerance, use a smaller fixed tolerance in the H-normalized statement; the bounded difference is then absorbed for all sufficiently large X. This proves Theorem 5.1 with the full cumulative denominator, not a conditional dyadic denominator. For the Gaussian family $128X < Q_d \le 256X$, so $\log\log Q_d = H + O(1/L)$ and $|\mathcal{D}_X| \asymp X$. The exact two-label fibers proved after (5.4) preserve every probability, proving Theorem 5.2 also for distinct characters. The cumulative cutoff follows by discarding $\mathrm{N}d \le 2^{-J}Y$, summing over the remaining J dyadic blocks, and then letting $Y \to \infty$ followed by $J \to \infty$. For the weight family $L = \log k$, so $\log\log(k^2) = H + \log 2$. Its arithmetic and density inputs were unweighted throughout. Its gamma factor was $(2\pi)^{-s}\Gamma(s + (k-1)/2)$

and its root number was i^k; the central argument retained the arbitrary central multiplicity for either sign. This proves Theorem 5.3 along each even residue class modulo four, and hence along all even weights.

6. Three arithmetic families under GRH

We verify the arithmetic and analytic hypotheses of the preceding transfer results for three real families. All averages in this section are counting averages. In particular, no harmonic weight, automorphism weight, or choice of one member of a conjugate pair is implicit. We subsequently treat complex cubic characters separately; their real zeros are not sign changes of a real-valued process.

For a real L-function put

$$A_F(p^j) = \text{the coefficient of } (\log p)p^{-js} \text{ in } -L_F'/L_F(s), \qquad G_F(t) = e^{-t}\frac{-L_F'}{L_F}(1/2+e^{-t}).$$

A word $W = \prod_{i=1}^r A_F(p_i^{j_i})$ has length r and support cost $n(W) = \prod_i p_i^{j_i}$. Factors at the same prime retain their dependence. We use independent copies of the entire local block at different primes. The estimates proved below have the form

$$|\mathbb{E}_{\mathcal{F}(Y)}W - \mathbb{E}W_E| \le CY^{-\kappa}C_0^r n(W)^A. \tag{6.1}$$

For a formal polynomial $P = \sum_W c_W W$, the norm

$$\|P\|_* = \sum_W |c_W| C_0^{|W|} n(W)^A \tag{6.2}$$

is submultiplicative, and the error in its arithmetic mean is at most $CY^{-\kappa}\|P\|_*$. Formal words, rather than their possibly nonunique representations as functions, suffice for this assertion. These are the inputs of Lemmas 2.14 and 2.15 and Proposition 2.16.

The local parameters needed for the limiting real process are as follows. Here Y is the size parameter in the second column, Q_F is the analytic conductor, and g denotes the centered stationary Gaussian process with covariance $\operatorname{sech}^2((t-s)/2)$.

Family	parameter	Q_F	limiting G_F	(κ, A)
Fixed modular quadratic twists	$\lvert D\rvert \asymp Y$	$\asymp_f D^2$	$-1/2+g/2$	$(1/2,1)$
Non-Galois cubic fields	$\lvert D_K\rvert \asymp Y$	$\lvert D_K\rvert$	$1/2+g/2$	$(1/6,1)$
Cyclic cubic augmentation	$q_K \asymp Y$	q_K^2	$g/\sqrt{2}$	$(1/4,2)$

The last exponent is expressed in the conductor parameter Y; in the discriminant parameter Y^2 it is $1/8$.

6.1. Fixed modular forms and all quadratic discriminants.

Let f be a normalized primitive holomorphic newform of even weight $k \ge 2$, level N, and trivial nebentypus. Write

$$f(z) = \sum_{n\ge1} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}.$$

The coefficients are real. No non-CM hypothesis is imposed. At $p \nmid N$ write $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = 1$. Deligne's theorem [12, Théorème 8.2] gives $|\alpha_p| = |\beta_p| = 1$. Hence the logarithmic-derivative coefficients are

$$b_f(p^j) = \alpha_p^j + \beta_p^j, \qquad b_f(p) = \lambda_f(p), \qquad b_f(p^2) = \lambda_f(p)^2 - 2. \tag{6.3}$$

The last expression is not the Hecke coefficient $\lambda_f(p^2)$. At primes dividing N, the local factor is $(1-\lambda_f(p)p^{-s})^{-1}$, where

$$\lambda_f(p) = 0 \ (p^2 \mid N), \qquad |\lambda_f(p)| = p^{-1/2} \ (p\|N), \tag{6.4}$$

by [1, Theorem 3]. Thus all the finitely many bad-prime blocks and their derivatives are bounded on a fixed neighborhood of the real interval $[1/2, 1]$.

Lemma 6.1 (Quadratic averages in a fixed progression)**.** *Fix M divisible by $8N$, a unit class $a \pmod M$ with $a \equiv 1 \pmod 4$, and a sign $\eta \in \{1,-1\}$. Let $\mathcal{D}(Y)$ consist of fundamental discriminants satisfying $Y < |D| \le 2Y$, $\eta D > 0$, and $D \equiv a \pmod M$. For $p \nmid M$ let X_p be independent, with*

$$\mathbb{P}(X_p = 0) = \frac{1}{p+1}, \qquad \mathbb{P}(X_p = 1) = \mathbb{P}(X_p = -1) = \frac{p}{2(p+1)}.$$

At $p \mid M$, set $X_p = \chi_D(p)$, the value fixed by the progression. Then, uniformly for all positive integers n,

$$\mathbb{E}_{D\in\mathcal{D}(Y)}\chi_D(n) = \mathbb{E}X(n) + O_M(Y^{-1/2}n), \qquad X(n) = \prod_p X_p^{v_p(n)}. \tag{6.5}$$

Also $|\mathcal{D}(Y)| = c_M Y + O_M(\sqrt{Y})$, where $c_M = M^{-1}\prod_{p\nmid M}(1-p^{-2}) > 0$.

Proof. Write $D = \eta d$, $d > 0$, and use $\mu^2(d) = \sum_{h^2\mid d}\mu(h)$. Since the progression is a unit class, only $(h, M) = 1$ occur. Summing h^{-2} proves the count, with an $O_M(\sqrt{Y})$ error. The part of n supported on M contributes a fixed factor. Denote the remaining part by n_0.

If n_0 is not a square, the Jacobi character $(\cdot/n_0)$ is nonprincipal modulo n_0. For $(h, n_0) = 1$, its sum on the progression $h^2 r \equiv \eta a \pmod M$ vanishes on each complete block of n_0 terms; the remainder is $O(n_0)$. For $(h, n_0) > 1$ the character is zero. Summing $h \le \sqrt{2Y}$ gives $O_M(n_0\sqrt{Y})$, while the random mean is zero.

If n_0 is a square, the character equals $\mathbf{1}_{(d,n_0)=1}$. For each h coprime to Mn_0, inclusion and exclusion in r gives the main term

$$\frac{Y}{Mh^2}\prod_{p\mid n_0}(1-p^{-1})$$

and an error $O(2^{\omega(n_0)})$. Summing h and extending the absolutely convergent main-term sum gives, after division by $c_M Y$,

$$\prod_{p\mid n_0}\frac{1-p^{-1}}{1-p^{-2}} = \prod_{p\mid n_0}\frac{p}{p+1}.$$

The error is $O_M(Y^{-1/2}2^{\omega(n_0)})$, which is bounded by the error in (6.5). The signs introduced by $D = \eta d$ are fixed and are 1 in the square case. □

For $(D, N) = 1$, the primitive twist has conductor $Q_D = ND^2$ and completion

$$\begin{aligned}\Lambda_D(s) &= \left(\frac{\sqrt{Q_D}}{2\pi}\right)^s \Gamma(s + (k-1)/2)L(s, f\otimes\chi_D),\\ \Lambda_D(s) &= \epsilon_f\chi_D(-N)\Lambda_D(1-s).\end{aligned} \tag{6.6}$$

These are entire functions of order one, with the usual fixed-degree polynomial growth in vertical strips. Both root-number signs are allowed; a forced central zero is dealt with by the central zero estimate, and is not removed from the family. The primitive twisting statements used here are [2, §3]; see also the normalization in [24, pp. 1–2].

Lemma 6.2 (The modular local process)**.** *The model of Lemma 6.1 satisfies, with $a = e^{-t}$,*

$$\mathbb{E}G_E(t) = -\tfrac{1}{2} + O_{f,M}(a), \tag{6.7}$$

$$\operatorname{Cov}(G_E(-\log a), G_E(-\log b)) = \frac{ab}{(a+b)^2} + O_{f,M}(ab). \tag{6.8}$$

The estimates extend to fixed closed subsectors of $\Re a > 0$ in their analytic form. Normalized local analytic moments of every fixed order are bounded, with constants allowed to depend on the fixed modulus M. For $p \nmid M$, the full local blocks have a two-atom difference sufficient for the uniform small-value estimate of Lemma 2.5.

Proof. The Rankin–Selberg prime number theorem of [22, Theorem 2.3], applied to the self-contragredient representation of f, gives

$$\sum_{n\le x} \Lambda(n)|a_f(n)|^2 = x + O_f(xe^{-c_f\sqrt{\log x}}).$$

Deligne's bound removes the good prime powers of exponent at least two with error $O(\sqrt{x}\log^2 x)$; (6.4) handles the finitely many remaining primes. Consequently

$$\sum_{\substack{p\le x\\ p\nmid N}} \lambda_f(p)^2 \log p = x + O_f(xe^{-c_f\sqrt{\log x}}). \tag{6.9}$$

This argument includes CM forms. Partial summation, and the ordinary prime number theorem, now give

$$\sum_p \lambda_f(p)^2 \log p\, p^{-1-z} = z^{-1} + O_f(1), \qquad \sum_p \lambda_f(p)^2 (\log p)^2 p^{-1-z} = z^{-2} + O_f(1).$$

The second formula follows either by differentiating the convergent remainder integral or by a second partial summation; both are uniform in a fixed closed subsector of $\Re z > 0$.

For $p \nmid M$, odd moments of X_p vanish and each positive even moment is $p/(p+1)$. The finitely many fixed blocks at $p \mid M$ contribute $O_{f,M}(a)$ to the normalized mean and zero to the covariance. The $j = 2$ terms outside M, by (6.3), therefore have mean $-1/(2a) + O_{f,M}(1)$ before multiplication by a. The terms $j \ge 4$ have bounded total mean. The centered terms with $j \ge 2$ have bounded L^2 norm before multiplication by a: their individual bounds are $O_f((\log p)/p)$, whose squares are summable. The $j = 1$ terms and (6.9) give (6.8); covariance cross terms are summable as well. The bounded independent prime blocks give the fixed-order moment bounds by the exponential-moment inequality, followed by Cauchy's formula on a slightly larger sectorial disc.

For clarity, the atom calculation concerns the full block, not just its first term. Put $u = p^{-1/2-a}$ and

$$B_p(x) = (\log p)\frac{\lambda_f(p)xu - 2x^2u^2}{1 - \lambda_f(p)xu + x^2u^2}.$$

Then

$$B_p(1) - B_p(-1) = \frac{2\lambda_f(p)u(1-u^2)\log p}{(1+u^2)^2 - \lambda_f(p)^2u^2}. \tag{6.10}$$

The factor multiplying $\lambda_f(p)u\log p$ is bounded above and below by positive absolute constants for $p \ge 2$, $a > 0$. For $p \nmid M$, both atoms have probability bounded below. Although $\lambda_f(p)$ can vanish, (6.9) supplies the required weighted prime mass on dyadic intervals. It is thus unnecessary to impose a pointwise lower bound on $|\lambda_f(p)|$ or to invoke Sato–Tate. □

Lemma 6.1 and $|b_f(p^j)| \le 2$ give (6.1) with $(\kappa, A, C_0) = (1/2, 1, 2)$. For a Selberg polynomial supported on $p^j \le e^{3\ell}$, the norm (6.2) is at most $e^{C_{f,M}(\ell+1)}$ throughout the fixed region under consideration. Its normalized $2m$th moment is bounded by

$$(C_{f,M}\sqrt{m})^{2m} + Y^{-1/2}\exp(C_{f,M}m(\ell+1)). \tag{6.11}$$

Indeed, the random estimate follows from independent bounded prime blocks, and expansion of the finite power followed by (6.1) proves the arithmetic estimate. This also proves the required weighted-polynomial input when the number of primes grows; there is no fixed-cylinder restriction.

Proposition 6.3 (All discriminants and coefficientwise twists)**.** *The preceding inputs apply to all signed fundamental discriminants, with $L(s, f \otimes \chi_D)$ interpreted as the primitive twist and $Q_D \asymp_f D^2$. They also apply to the coefficientwise twist*

$$L_D^{\mathrm{coeff}}(s) = \sum_{n\ge 1} \lambda_f(n)\chi_D(n)n^{-s},$$

using the conductor of its primitive representative as the reference conductor. The GRH assumption is imposed on the primitive twists.

Proof. Let S consist of the primes dividing $2N$, and set $p^* = (-1)^{(p-1)/2}p$ for odd p. The prime-discriminant decomposition is unique:

$$D = D_2 \prod_{\substack{p\mid D\\ p \text{ odd}}} p^*, \qquad D_2 \in \{1, -4, 8, -8\}.$$

Separate the factors supported on S, obtaining $D = D_S d$ and $\chi_D = \theta\chi_d$, where $\theta = \chi_{D_S}$ belongs to a fixed finite set and d is an odd fundamental discriminant prime to S. Successive prime-primary twists in [2, Theorem 3.2] produce a fixed primitive form f_θ of the same weight and trivial nebentypus, whose level N_θ is supported on S. Uniqueness of the primitive representative follows from [1, Theorem 4], since its Hecke eigenvalues away from S are $\lambda_f(p)\theta(p)$. Twisting by χ_d is now coprime twisting, so

$$L(s, f \otimes \chi_D) = L(s, f_\theta \otimes \chi_d), \qquad Q_D = N_\theta d^2 = \frac{N_\theta}{|D_S|^2}D^2. \tag{6.12}$$

Partition the remaining d into finitely many unit classes modulo a multiple of $8\prod_{p\in S} p$ and their two signs, and apply the preceding lemmas to f_θ. All constants remain fixed. Possible CM self-twists do not affect this argument: the discriminant labels, which are the objects being averaged, have a unique decomposition.

The coefficientwise twist differs at only the common ramified primes:

$$L_D^{\mathrm{coeff}}(s) = L(s, f_\theta \otimes \chi_d)E_D(s), \qquad E_D(s) = \prod_{p\mid (D,N)} L_p(s, f_\theta \otimes \chi_d)^{-1}. \tag{6.13}$$

This is also the depletion assertion of [2, Theorem 3.2]. The local inverse factors have real coefficients and all reciprocal roots of modulus at most one. After a further finite partition, E_D ranges over a fixed finite set. Its zeros lie in $\Re s \le 0$, and all derivatives of E_D'/E_D are bounded on a neighborhood of $[1/2, 1]$. Thus it adds $O_f(|a|)$, with the same bound for fixed analytic derivatives, to the normalized process. The Selberg approximation and its moments acquire only this bounded term; the local limiting law and the bulk small-value and sign comparison estimates are unchanged. The local Jensen and central estimates apply to the nonvanishing finite factor in exactly the same way. This proves the required stability, without asserting equality of the two derivative zero counts or assigning a primitive functional equation to L_D^{coeff}; equivalently one can apply Proposition 4.4 to each of the finite classes. □

6.2. **Non-Galois cubic fields.** Let $\mathcal{F}_{\pm}(Y)$ be the isomorphism classes of non-Galois cubic fields K with $Y < |D_K| \le 2Y$ and prescribed discriminant sign. The standard two-dimensional Artin representation has

$$L_K(s) = \frac{\zeta_K(s)}{\zeta(s)}, \qquad Q_K = |D_K|. \tag{6.14}$$

This Artin L-function is entire of order one unconditionally. Its completion has root number $+1$, with archimedean factor $\Gamma_{\mathbb{R}}(s)^2$ for $D_K > 0$ and $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$ for $D_K < 0$, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. These facts and the local factors below are recorded in [25, Propositions 3.1–3.2 and (22)]. GRH here concerns L_K itself; no auxiliary assumption on ζ is made.

Put $D_p = p^2 + p + 1$. The five aggregate local types and their limiting probabilities are

Type	inverse Euler factor, $x = p^{-s}$	$A(p^j)$	probability
(111)	$(1-x)^2$	2	$p^2/(6D_p)$
(12)	$1-x^2$	$1+(-1)^j$	$p^2/(2D_p)$
(3)	$1+x+x^2$	$3\mathbf{1}_{3\mid j} - 1$	$p^2/(3D_p)$
$(1^2 1)$	$1-x$	1	p/D_p
(1^3)	1	0	$1/D_p$

The last two rows include all their local-algebra refinements, also at 2 and 3.

Lemma 6.4 (Uniform cubic-field word averages)**.** *For the independent five-state local model in the table,* (6.1) *holds on either* $\mathcal{F}_{\pm}(Y)$ *with* $(\kappa, A, C_0) = (1/6, 1, 10)$.

Proof. Theorem 1.3 and Table 1 of [6] count cubic fields with finitely many local specifications. In its notation the count up to Y is

$$A_{\pm} Y \prod_{p\in S} C_p + B_{\pm} Y^{5/6} \prod_{p\in S} K_p + O_{\delta}\left(Y^{2/3+\delta} \prod_{p\in S} E_p \right), \tag{6.15}$$

where

$$A_{\pm} = \frac{C_{\pm}}{12\zeta(3)}, \quad C_+ = 1,\ C_- = 3, \qquad B_{\pm} = \frac{4K_{\pm}\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}, \quad K_+ = 1,\ K_- = \sqrt{3}.$$

For the five aggregate types, C_p is the displayed probability, the K_p are nonnegative and sum to one, and $E_p \le p^{2/3}$. We only use these aggregate types; the larger error for an arbitrary single local algebra is not being suppressed.

Subtracting the formula at $2Y$ and Y retains the secondary term, with its factor $2^{5/6} - 1$. The positive-discriminant count can include cyclic cubic fields. They contribute $O(\sqrt{Y}\log(2Y))$: their conductor is at most $\sqrt{2Y}$ and the number of cubic characters of a given conductor is at most $2\tau(q)$. Removing them therefore gives the required non-Galois family.

If F depends on the types at S, summing the $5^{|S|}$ cylinders and using positivity of both main-term measures yields

$$|\mathbb{E}_{\mathcal{F}_{\pm}(Y)} F - \mathbb{E} F_E| \ll_{\delta} \|F\|_{\infty} \big(Y^{-1/6} + Y^{-1/3+\delta} 5^{|S|} \big(\prod_{p\in S} p \big)^{2/3} + Y^{-1/2}\log(2Y) \big). \tag{6.16}$$

The secondary term costs $\|F\|_{\infty}$, not $5^{|S|}\|F\|_{\infty}$, since the K_p form probability measures. For a word of length r, $\|F\|_{\infty} \le 2^r$, $|S| \le r$, and $\prod_{p\in S} p \le n(W)$. Fixing $\delta = 1/12$ in (6.16) proves the claim. In particular, the secondary term prevents replacing $\kappa = 1/6$ here by the exponent suggested solely by the error in (6.15). □

Lemma 6.5 (The cubic-field local process)**.** *The five-state model has limiting normalized process* $1/2 + g/2$*, bounded analytic moments of every fixed order, and the uniform small-value input required by the transfer theorem.*

Proof. Directly from the table,

$$\mathbb{E}A(p) = \frac{p}{D_p}, \qquad \mathbb{E}A(p)^2 = \mathbb{E}A(p^2) = \frac{p^2+p}{D_p}, \qquad \operatorname{Var} A(p) = 1 - \frac{1}{D_p} - \frac{p^2}{D_p^2}. \tag{6.17}$$

More generally, $\mathbb{E}A(p^j) = [p^2(\mathbf{1}_{2|j} + \mathbf{1}_{3|j}) + p]/D_p$. The first-prime mean is absolutely summable after multiplication by $p^{-1/2-a}\log p$. The prime-square mean is $1 + O(p^{-2})$, and hence produces drift $1/2$. The first-prime variance is $1 + O(p^{-2})$, giving $ab/(a+b)^2 + O(ab)$ for the normalized covariance. All other mean and covariance terms are absolutely summable. Bounded local coefficients give the analytic moment estimates as in Lemma 6.2.

The identity and transposition atoms have probabilities at least $2/21$ and $2/7$, respectively. Their full normalized block difference is, for $u = p^{-1/2-a}$,

$$2a(\log p)\frac{u}{1-u^2}. \tag{6.18}$$

It lies between $2a(\log p)u$ and $4a(\log p)u$. This verifies the two-atom hypothesis without a truncation of the local Euler factor. Lemma 2.5 therefore applies. In its characteristic-function estimate a valid prime cutoff is $P = (CT\log^2(2T))^{1/(1/2+a)}$, where $T = 2 + |\tau|$ and τ is the frequency of the normalized process. Retaining this a-dependence ensures uniformity near the end of the interval. □

The word norm of a Selberg polynomial is again $e^{C(\ell+1)}$; Lemma 6.4 thus supplies all polynomial and characteristic-function comparisons, including a growing set of primes. The functional equation in (6.14), its fixed gamma parameters, and GRH supply the hypotheses of Lemma 3.1 and the central estimates.

6.3. **Oriented cubic characters: a uniform arithmetic lemma.** Let $\mathcal{C}(Y)$ be all primitive Dirichlet characters of exact order three and conductor at most Y. Both conjugates are included. Let $\mathcal{C}_0(Y)$ be its subfamily with conductor prime to 3, and write N, N_0 for the two cardinalities. Put $\omega = e^{2\pi i/3}$.

Proposition 6.6 (Oriented cubic orthogonality)**.** *Let* Z_p *be independent and supported on* $\{0, 1, \omega, \omega^2\}$*, with*

$$\mathbb{P}(Z_p = 0) = 1 - \theta_p, \qquad \mathbb{P}(Z_p = \omega^j) = \theta_p/3 \quad (0 \le j \le 2),$$
$$\theta_p = \begin{cases} 9/11, & p = 3, \\ p/(p+2), & p \equiv 1 \pmod 3, \\ 1, & p \equiv 2 \pmod 3. \end{cases} \tag{6.19}$$

Uniformly for every positive integer n*,*

$$\frac{1}{N(Y)} \sum_{\chi \in \mathcal{C}(Y)} \chi(n) = \mathbf{1}_{n \text{ is a cube}} \prod_{p|n} \theta_p + O(Y^{-1/4} n). \tag{6.20}$$

The same assertion holds for $Y < q_\chi \le 2Y$*. A mixed word in* $\chi(p^j)$ *and* $\overline{\chi(p^j)}$*, of support cost* $n(W)$*, has error* $O(Y^{-1/4} n(W)^2)$*.*

Proof. We give the noncube estimate directly, since a splitting-type count alone does not determine oriented character values. Let $\mathcal{O} = \mathbb{Z}[\omega]$. By [4, Lemma 2.1 and §2.1], characters in $\mathcal{C}_0(Y)$ are parametrized by primary elements $c \equiv 1 \pmod 3$ that are squarefree, have no rational prime

divisor, and satisfy $N(c) \le Y$; the character is $(\cdot/c)_3$. The norm-one parameter is to be removed. The same source gives the finite-order Hecke character

$$\psi_n((c)) = (n/c)_3 \qquad (c \equiv 1 \pmod 3)$$

of modulus $9n\mathcal{O}$, nonprincipal when n is not a cube. For a rational positive integer, being a cube in $\mathbb{Q}(\omega)$ implies being a rational cube: the valuations above $p \neq 3$ are $v_p(n)$ and that above 3 is $2v_3(n)$.

We retain the original imprimitive support: the value is zero whenever $(c, n) \neq 1$. With $R = 9n$, the function $f(c) = \mathbf{1}_{c\equiv 1(3)}(n/c)_3$ is periodic modulo $R\mathcal{O}$, and its sum on a complete residue cell is zero. Here is the unit convention that justifies the cancellation. The six units of $\mathcal{O}$ give the six distinct units modulo 3. Thus, for $U = (\mathcal{O}/R\mathcal{O})^\times$ and $H = \ker(U \to (\mathcal{O}/3\mathcal{O})^\times)$, every element of U is a unique product of a global unit and an element of H. The ray character is trivial on global units and nonprincipal on U modulo them, so its restriction to H is nonprincipal. The nonunits contribute zero. This is a statement at modulus $9n$, and does not replace its support by that of a primitive inducing character.

Tiling the disc $N(c) \le V$ by cells of $R\mathcal{O}$ leaves $O(\sqrt{V}/R + 1)$ boundary cells, each containing $O(R^2)$ lattice points. If $V \ge R^2$ this is $O(n\sqrt{V})$; if $V < R^2$ use the trivial $O(V)$ bound. We have proved, uniformly for $V \ge 1$,

$$B_n(V) := \sum_{\substack{c\equiv 1(3)\\ N(c)\le V}} (n/c)_3 \ll \min(V, n\sqrt{V}), \qquad n \text{ not a cube.} \tag{6.21}$$

We spell out how the restrictions on c affect this estimate. Detect the absence of a rational prime divisor by rational Möbius inversion, using the unique signed representative $d \equiv 1 \pmod 3$ of each positive integer $|d|$ prime to 3. For squarefree $c = db$ this imposes $(d, b) = 1$. Detect squarefreeness of b by writing $b = \mathfrak{l}^2 b_1$ in the ideal notation, using the unique primary generator of each ideal prime to 3. Finally detect $(b_1, d) = 1$ by ideal inclusion and exclusion over $\mathfrak{e} \mid (d)$. The factors extracted from the cubic symbol have modulus at most one and vanish if they meet n. In particular, for a rational d prime to n, $(n/d)_3 = 1$: the factors at a split pair are conjugate and the symbol of a rational numerator at an inert prime is one. Taking absolute values and discarding additional coprimality conditions gives

$$\left|\sum_{\chi\in\mathcal{C}_0(Y)} \chi(n)\right| \ll 1 + \sum_{\substack{d\equiv 1(3)\\ |d|\le\sqrt{Y}\\ \mu^2(|d|)=1}} \sum_{\mathfrak{e}\mid(d)} \sum_{\mathfrak{l}} \min\left(\frac{Y}{d^2 N\mathfrak{e}(N\mathfrak{l})^2}, \frac{n\sqrt{Y}}{|d|\sqrt{N\mathfrak{e}}\, N\mathfrak{l}}\right). \tag{6.22}$$

Only ideals prime to 3 need occur, but dropping this restriction is harmless. Terms whose implied norm bound is below one are zero.

The number of ideals of norm at most U in this fixed quadratic field is $O(U)$, for example by counting primary generators in a disc. Set $y = \sqrt{Y}/(|d|\sqrt{N\mathfrak{e}})$. Splitting the sum over $N\mathfrak{l}$ at y/n, partial summation gives

$$\sum_{\mathfrak{l}} \min\{y^2/(N\mathfrak{l})^2, ny/N\mathfrak{l}\} \ll ny\log(2Y).$$

When $y < n$, the convergent sum of $y^2/(N\mathfrak{l})^2$ already gives $O(y^2) \le O(ny)$. The remaining divisor sum satisfies

$$\sum_{\substack{d\le D\\ \mu^2(d)=1}} \frac{1}{d} \sum_{\mathfrak{e}\mid(d)} (N\mathfrak{e})^{-1/2} \ll \log(2D). \tag{6.23}$$

Indeed the inner multiplicative function has local value $(1+p^{-1/2})^2$ at a split prime and $1+p^{-1}$ at an inert prime. Writing this function on squarefree integers as $1*h$, the product $\prod_p(1+h(p)/p)$ converges. Summing first over the divisor carrying h proves (6.23). Thus

$$S_0(Y;n) := \sum_{\chi\in\mathcal{C}_0(Y)} \chi(n) \ll n\sqrt{Y}\log^2(2Y) \qquad (n \text{ not a cube}). \tag{6.24}$$

For cubes we use only the unconditional counting assertion of [11, Proposition 2.1], specialized to its family of conductors prime to 3:

$$\begin{aligned} N_0(Y) &= K_0 Y + O_\delta(Y^{5/8+\delta}),\\ S_0(Y;n) &= K_0 Y \prod_{\substack{p\mid n\\ p\equiv 1(3)}} \frac{p}{p+2} + O_\delta(3^{\omega(n)}Y^{5/8+\delta}) \quad (n \text{ a cube}), \end{aligned} \tag{6.25}$$

where

$$K_0 = \frac{2}{3}L(1,\chi_{-3}) \prod_{p\equiv 1(3)} (1-3p^{-2}+2p^{-3}) \prod_{p\equiv 2(3)} (1-p^{-2}) > 0.$$

The constant is the residue at one of $\prod_{p\equiv1(3)}(1+2p^{-s})$; imposing $(q,n)=1$ removes the corresponding Euler factors and gives the displayed local ratios. No noncube character-sum estimate from that paper is required here.

It remains to include the exact 3-part of the conductor. Let $\xi,\bar\xi$ be the two primitive cubic characters of conductor 9. Away from 3, a nontrivial primitive cubic local character has conductor $p\equiv 1 \pmod 3$. At 3 every character of order three factors through modulus 9, and a nontrivial one has exact conductor 9. Unique factorization into these local characters gives the exact identities, with $b(Y)=\mathbf{1}_{Y\ge 9}$,

$$\begin{aligned} S(Y;n) &= S_0(Y;n) + [\xi(n)+\bar\xi(n)]\{S_0(Y/9;n)+b(Y)\},\\ N(Y) &= N_0(Y) + 2N_0(Y/9) + 2b(Y). \end{aligned} \tag{6.26}$$

In particular $N(Y) = KY + O_\delta(Y^{5/8+\delta})$, with $K = 11K_0/9$. If n is a cube, the bracket is $2\mathbf{1}_{3\nmid n}$, so the normalized local factor at 3 is 1 for $3\nmid n$ and $9/11$ for $3\mid n$. If n is not a cube, (6.24) and (6.26) give the same noncube bound for $S(Y;n)$. Choose $\delta = 1/16$, use $3^{\omega(n)}\le n$ for a cube, and use $Y^{-1/2}\log^2(2Y) \ll Y^{-1/4}$. Dividing by $N(Y)$ proves (6.20) with one absolute constant for all n.

Finally, $\bar z = z^2$ for every $z\in\{0,1,\omega,\omega^2\}$, including $z=0$. Thus a mixed word is $\chi(n_W)$ with $n_W \le n(W)^2$. This proves the last assertion without any coprimality restrictions. Subtracting the cumulative formulas at bY and aY proves the same estimates on $aY < q_\chi \le bY$, for every fixed $0<a<b$, with constants allowed to depend on a,b and with the actual interval cardinality as denominator. In particular this gives the stated dyadic versions. For a cyclic cubic field the discriminant is q_χ^2, so a dyadic field-discriminant interval corresponds to a conductor interval with endpoint ratio $\sqrt{2}$, which is covered by this fixed-ratio formulation. □

For completeness, this lemma also determines joint phases, rather than only joint splitting types. On the four-point support,

$$\mathbf{1}_{z=0} = 1-z^3, \qquad \mathbf{1}_{z=\zeta} = \tfrac{1}{3}(z^3+\bar\zeta z+\bar\zeta^2 z^2) \quad (\zeta^3=1).$$

Consequently a phase specification on S has error $O(2^{|S|}(\prod_{p\in S}p)^3 Y^{-1/4})$. The mixed-word estimate is more efficient for polynomial comparison. It is not valid for an arbitrary selection of one conjugate from each pair, or for a family restricted to prime conductors, without further proof.

6.4. **Real cyclic cubic augmentation.** Each cyclic cubic field K corresponds to exactly two primitive cubic characters $\chi, \bar{\chi}$ of the same conductor q_K. The conductor-discriminant formula and the abelian factorization give

$$D_K = q_K^2, \qquad Z_K(s) := \frac{\zeta_K(s)}{\zeta(s)} = L(s,\chi)L(s,\bar{\chi}), \qquad \Lambda_K(s) = q_K^s \Gamma_{\mathbb{R}}(s)^2 Z_K(s) = \Lambda_K(1-s). \tag{6.27}$$

Every cubic character is even. Thus Z_K is real on the real axis, entire of order one, and $Z_K(x) = |L(x,\chi)|^2$ for real x. The reference conductor is $Q_K = q_K^2$. Its nontrivial zeros are the union, with multiplicity, of the zeros of the two character L-functions. The GRH assumption on the primitive cubic characters therefore gives precisely the GRH needed for Z_K.

Proposition 6.7. *The uniform family of cyclic cubic fields with $Y < q_K \le 2Y$ satisfies (6.1) with $(\kappa, A, C_0) = (1/4, 2, 2)$. Its random local model has limiting normalized process $g/\sqrt{2}$, and satisfies the analytic-moment and uniform small-value inputs. The number of fields of conductor at most Y is $(11K_0/18)Y + O(Y^{11/16})$.*

Proof. The local coefficients are $A_K(p^j) = \chi(p)^j + \bar{\chi}(p)^j$. A word of length r expands into 2^r mixed character words of the same support cost. Apply Proposition 6.6. Every field has exactly two oriented characters and the trace word has the same value on them; division by two in both numerator and denominator proves the asserted field average and the cardinality formula.

At a prime, the split, inert, and ramified states have respective probabilities $\theta_p/3$, $2\theta_p/3$, and $1-\theta_p$. For $3 \nmid j$ their coefficients are $2, -1, 0$, and for $3 \mid j$ they are $2, 2, 0$. In particular

$$\mathbb{E}A(p) = \mathbb{E}A(p^2) = 0, \qquad \mathbb{E}A(p)^2 = 2\theta_p = 2 + O(p^{-1}). \tag{6.28}$$

The mean starts with $j = 3$, whose sum is absolutely convergent. The usual prime number theorem gives normalized covariance $2ab/(a+b)^2 + O(ab)$, and all higher local terms have bounded unnormalized variance. This proves drift zero and variance parameter two. The moment argument again uses bounded independent prime blocks. The full split-minus-inert block difference is

$$3a(\log p)\frac{u(1+u)}{1-u^3}, \qquad u = p^{-1/2-a},$$

which is comparable to $a(\log p)u$. Both atom probabilities are bounded below, including at $p = 3$. This proves the required small-value hypothesis. Finally, the norm of the Selberg polynomial for the word cost $n(W)^2$ is still $e^{C(\ell+1)}$. Hence this larger support exponent does not change the polynomial-length condition $m\ell = o(\log Y)$. □

Lemma 6.8 (Uniform analytic specifications)**.** *Every primitive function used above has an entire completion of order one. Its gamma parameters belong to a fixed finite set, and on every fixed strip $-B \le \Re s \le B$ it satisfies*

$$|L_F(s)| \le C_B[Q_F(2 + |\Im s|)]^{C_B}. \tag{6.29}$$

The same assertions, with a contragredient functional equation, hold for each individual primitive cubic character. These statements do not require GRH.

Proof. For the modular and S_3 functions, the cited primitive or cuspidal realizations give the entire completion and its functional equation. The standard finite-order property of cuspidal standard L-functions is recorded in [23, §2.14.5]; we only use its individual order statement, not a claim of uniformity in the representation for its displayed strip estimate. For a primitive nonprincipal Dirichlet character, the theta transformation and Mellin integral prove entireness and (7.1); see [19, Chapter 11]. Splitting that integral at one and transforming the lower half

expresses the completion as two integrals over $[1,\infty)$. For fixed q, the theta sums there are bounded by $C_q e^{-c_q y^2}$. Therefore, on $|s|\le R$, the integrals are at most

$$C_q\int_1^\infty e^{-c_q y^2}y^{R+C}\,dy \le \exp(C_q R\log(R+2)).$$

This proves order at most one directly; division by the gamma factor preserves finite order. It also proves finite order for the real product of two conjugates.

We obtain the required uniformity independently. At a fixed right boundary $b>\max(B,2)$ the bounded logarithmic Euler coefficients give $|L_F(b+it)|\le\zeta(b)^d$. At a fixed left boundary $a<\min(-B,-1)$ avoiding the finitely many gamma progressions, the functional equation and Stirling's formula give

$$|L_F(a+it)|\le CQ_F^{1/2-a}(2+|t|)^C.$$

The contragredient has the same right-boundary Euler bound. Choose a fixed polynomial $(s+c)^J$, with its zero to the left of the strip, that dominates the latter height factor. The strip Phragmén–Lindelöf principle applies to $L_F(s)/(s+c)^J$: finite order is smaller than the exponential of an exponential permitted by that principle. The resulting bound depends only on the boundary constants, and gives (6.29). The individual constants in the initial finite-order bound do not survive this argument.

Finally Stirling's formula and the Euler product bound the completion in the right half-plane by $\exp(C_F R\log(R+2))$ on $|s|\le R$. Use the functional equation in the left half-plane and (6.29) in the intervening fixed strip. The finitely many apparent gamma poles there are removable and are filled by the maximum principle. The completion thus has order at most one. Along the positive real axis $L_F(\sigma)\to 1$ and

$$\log|\Lambda_F(\sigma)| = (d/2)\sigma\log\sigma + O_F(\sigma),$$

so the order is exactly one. For modular twists the gamma shifts are $(k-1)/2,(k+1)/2$; for S_3 they are $(0,0)$ or $(0,1)$; for the cyclic augmentation they are $(0,0)$. The finite twist partition keeps all modular parameters in a fixed finite set. □

Corollary 6.9 (The real-family constants)**.** *Assume GRH for the primitive L-functions in the relevant family. The real-family quantile theorem 4.2 applies to all three families above, and to both conventions for modular twists in Proposition 6.3. In particular, outside a subset of density zero, each of the distinct-zero count, the count with multiplicity on $[1/2,1]$, and the strict-sign-change count on $(1/2,1)$ is*

$$(c+o(1))\log\log Q_F,\qquad c=\begin{cases}e^{-1/2}/(\pi\sqrt{2}), & \text{modular twists and non-Galois cubic fields,}\\ 1/(\pi\sqrt{2}), & \text{real cyclic cubic augmentation.}\end{cases}$$

The same conclusion holds in cumulative conductor or discriminant families, with their original counting measure. More precisely, on each dyadic family, put $H=\log\log Y$ and $H_F=\log\log Q_F$. For each fixed $0<\varepsilon<1/16$, uniformly for $H^{-1/2+\varepsilon}\le z\le 1$,

$$\mathbb{P}\left\{\max_{C\in\{R^{\mathrm{dist}},R^{\mathrm{mult}},S\}}|C-cH_F|>A_\varepsilon zH_F\right\}\le C_\varepsilon\frac{\log H}{Hz^2}. \tag{6.30}$$

The constants may depend on the fixed modular form or the chosen discriminant sign, but not on z. The denominator is the cardinality of the original dyadic family.

Proof. The arithmetic word and polynomial estimates, local Gaussian limits, uniform density inputs, and fixed-degree analytic specifications have all been verified above. The common Selberg and central estimates under GRH therefore apply. The Rice intensity for a process $m/2+\sqrt{v}\,g/2$ is $e^{-m^2/(2v)}/(\pi\sqrt{2})$, giving the displayed constants for $(m,v)=(-1,1),(1,1),(0,2)$. For cyclic

fields $\log\log D_K = \log\log q_K + \log 2$; for modular twists (6.12) gives the same bounded change. These changes do not affect the leading term. Cumulative families follow by discarding the parameters below $\sqrt{Y}$ and summing the uniform dyadic conclusions over the rest. All three L-functions are nonconstant: at an unramified prime the modular first or second coefficient is nonzero, the S_3 first or second coefficient is nonzero, and a cyclic trace is 2 or -1. □

7. Real zeros for individual complex cubic characters

The real augmentation $L(s,\chi)L(s,\bar\chi)$ and the individual complex function $L(s,\chi)$ have different derivative statistics. For the latter we count actual zeros on the real axis, with their complex analytic multiplicities. There is no sign-change statistic. Throughout this section GRH is assumed for all primitive cubic Dirichlet L-functions in question. We use the full GRH assumption, including zeros at all heights.

Let $\mathcal{C}_X = \{\chi : X/2 < q_\chi \le X,\ \chi \text{ primitive of exact order } 3\}$, with both conjugates counted, and put $L = \log X$, $H = \log\log X$, and $H_\chi = \log\log q_\chi$. All probabilities below have denominator $|\mathcal{C}_X|$. The standard primitive Dirichlet functional equation is

$$\Lambda(s,\chi) = q_\chi^{s/2}\Gamma_{\mathbb{R}}(s)L(s,\chi) = \epsilon_\chi\Lambda(1-s,\bar\chi), \qquad |\epsilon_\chi| = 1. \tag{7.1}$$

The functions are entire of order one and have polynomial growth in fixed vertical strips. Cubic characters are even, so there is no parity shift in (7.1). Put

$$\mathcal{L}_\chi(s) = -L'/L(s,\chi), \qquad G_\chi(w) = e^{-w}\mathcal{L}_\chi(1/2+e^{-w}).$$

Under GRH, G_χ is holomorphic on $|\Im w| < \pi/2$.

Theorem 7.1 (A sublinear real-zero bound)**.** *For every fixed $0 < \varepsilon < 1/16$ there is A_ε such that*

$$\mathbb{P}_{\chi\in\mathcal{C}_X}\left\{R^{\mathrm{mult}}_{L'(\cdot,\chi)}([1/2,1]) > A_\varepsilon H_\chi^{1/2+\varepsilon}\right\} \ll_\varepsilon \frac{\log H}{H^{1/2+\varepsilon}} + H^{-2}\log H. \tag{7.2}$$

In particular the distinct and multiple real-zero counts are $o(\log\log q_\chi)$ in density. Moreover, with probability $1 - O(H^{-3})$ there are no real zeros of $L'(s,\chi)$ in the interval corresponding to

$$50\log H \le -\log(s-1/2) \le H - 121\log H. \tag{7.3}$$

The full-interval estimate also holds in the cumulative family $q_\chi \le X$. Since conjugates have identical real zeros with identical multiplicities, the same proportions hold when each conjugate pair is counted once.

7.1. **The two-dimensional random limit.** Use the independent variables in (6.19) and define, in $\Re a > 0$,

$$\mathcal{G}_E(a) = a\sum_p\sum_{j\ge1} Z_p^j(\log p)p^{-j(1/2+a)}.$$

The centered prime-linear series converges locally in L^2; its analytic version follows from Cauchy's formula on compact discs. The mean and all higher-prime-power remainder series have the convergence asserted in the next lemma. We distinguish the covariance from the pseudocovariance of complex random variables.

Lemma 7.2 (Complex Gaussian comparison)**.** *For positive real a, b,*

$$\begin{aligned}
\mathbb{E}\mathcal{G}_E(a) &= O(a),\\
\mathbb{E}[(\mathcal{G}_E(a)-\mathbb{E}\mathcal{G}_E(a))\overline{(\mathcal{G}_E(b)-\mathbb{E}\mathcal{G}_E(b))}] &= \frac{ab}{(a+b)^2} + O(ab),\\
\mathbb{E}[(\mathcal{G}_E(a)-\mathbb{E}\mathcal{G}_E(a))(\mathcal{G}_E(b)-\mathbb{E}\mathcal{G}_E(b))] &= O(ab).
\end{aligned} \tag{7.4}$$

In the variable $t = -\log a$, the limiting real and imaginary parts are $g_1/(2\sqrt{2})$ and $g_2/(2\sqrt{2})$, where g_1, g_2 are independent copies of g. In particular the limiting one-point covariance matrix is $\frac{1}{8}I_2$.

For a fixed collection of d real coordinates of these processes and a C^3 test function Φ, let M_j bound all order-j partial derivatives of Φ, and let $b_* = \max a_i$. Then the random to Gaussian comparison error is bounded by

$$C(db_*M_1 + d^2b_*^2M_2 + d^3b_*^3M_3). \tag{7.5}$$

Normalized analytic suprema and fixed derivatives on fixed sectorial discs have bounded moments of every fixed order.

Proof. The elementary identities are $\mathbb{E}Z_p = \mathbb{E}Z_p^2 = 0$ and $\mathbb{E}|Z_p|^2 = \theta_p = 1 + O(p^{-1})$. The mean begins at $j = 3$ and is $O(a)$. For the covariance, the prime-linear terms contribute

$$ab\sum_p \theta_p(\log p)^2 p^{-1-a-b} = \frac{ab}{(a+b)^2} + O(ab).$$

The error from $\theta_p - 1$ is absolutely summable. All terms with the sum of the two prime-power exponents at least three are also absolutely summable. The prime-linear pseudocovariance is zero because $\mathbb{E}Z_p^2 = 0$. This proves (7.4). The real and imaginary covariance matrices now give the stated limit; their cross covariance tends to zero.

For the smooth comparison, regard all the coordinates from a given prime as a single real vector. Subtract its mean and replace this whole vector by a Gaussian with the same covariance. A third-order Taylor expansion, with its integral remainder, gives an error at most Cd^3M_3 times the sum of third absolute block moments. This sum is $O(b_*^3)$, since $\sum_p(\log p)^3p^{-3/2} < \infty$ and the higher terms have the same bound. Correcting the mean costs Cdb_*M_1. Gaussian interpolation corrects the $O(b_*^2)$ covariance discrepancy at cost $Cd^2b_*^2M_2$. Passing from finite prime sums to their L^2 limit is justified by bounded-derivative truncations; the resulting estimate is (7.5). The argument never assumes independence of real and imaginary parts at the same prime.

Each real projection of the centered prime sum is a sum of independent bounded variables whose squared ranges have bounded sum after normalization. The exponential-moment inequality gives the required fixed-order moments. Apply Cauchy's formula on a slightly larger disc, then Minkowski's inequality, to obtain analytic supremum and derivative moments. □

Lemma 7.3 (Finite characteristic-function comparison)**.** *Let $\mathbf{P}_\chi$ consist of a fixed number of real and imaginary coordinates of normalized Selberg polynomials with cutoff $p^j \le e^{3\ell}$. Let $\mathbf{P}_E$ be the corresponding random vector. If $\|\tau\|_1 \le T$ and $M \ge 1$, then*

$$\left|\mathbb{E}_X e^{i\langle\tau,\mathbf{P}_\chi\rangle} - \mathbb{E}e^{i\langle\tau,\mathbf{P}_E\rangle}\right| \le 2\left(\frac{CT}{\sqrt{M}}\right)^{2M} + CX^{-1/4}\exp\{CM(\ell + \log(2+T))\}. \tag{7.6}$$

The same comparison gives normalized polynomial moments at every fixed complex location in a fixed sector. At the auxiliary height anchor $a_0 = 4/\ell$, it gives

$$\|aP_\chi(1/2+a)\|_q \le C_q, \qquad \|a_0P_{Z_\chi}(1/2+a_0+it)\|_q \le C_q \tag{7.7}$$

for fixed q, uniformly in the ranges used below and in $|t| \le 1$. Here $Z_\chi = L(\cdot,\chi)L(\cdot,\bar\chi)$.

Proof. Expand each real or imaginary coordinate as a linear combination of $\chi(p^j)$ and its conjugate. Proposition 6.6 gives the finite-word estimate with support cost $n(W)^2$; the coefficient cost is at most a fixed exponential in the word length. The norm of each polynomial is $e^{C(\ell+1)}$. For the real projection $V = \langle\tau,\mathbf{P}\rangle$, the random $2M$th moment is at most $(CT\sqrt{M})^{2M}$. Transfer its finite expansion with the word estimate. Taylor's formula for the real exponential e^{iV} through degree $2M-1$ has remainder at most $|V|^{2M}/(2M)!$. The bound $(2M)! \ge (2M/e)^{2M}$ and the

polynomial norm give (7.6). No growth bound for an exponential of a complex projection is used. The same expansion for fixed even moments proves the first part of (7.7). For the second, apply the trace-word estimate in Proposition 6.7. The factors p^{-ijt} have modulus one; the sum of squared normalized block ranges is bounded uniformly in t. The arithmetic error is $X^{-1/4}e^{C_q\ell} = o(1)$ for the cutoffs below. □

7.2. **Approximation of the actual complex function.**

Lemma 7.4 (Complex approximation from a real product)**.** *Put $\ell = C_yL/H^{120}$, where C_y is fixed sufficiently large. On the fixed sectorial discs covering $[50\log H, H - 121\log H]$ in the w variable, the actual process $G_\chi(w)$ and every fixed analytic derivative have bounded moments of every fixed order. More precisely, for $a = e^{-w}$ on a slightly larger such disc,*

$$\|a(\mathcal{L}_\chi - P_\chi)(1/2+a)\|_q \le C_qH^{121}e^{-cC_yH}. \tag{7.8}$$

Proof. The contour proof of Lemma 3.1 applies to the individual primitive function in (7.1). The contour identity itself uses entireness, order one, the functional equation with the contragredient, and fixed-strip growth; it does not require that this individual function be real on the real axis. In the notation of that theorem,

$$\mathcal{L}_\chi(s) = P_\chi(s) + \sum_{\rho \text{ of } L(\cdot,\chi)} K_\ell(\rho - s) + E_{\mathrm{tr},\chi}(s), \qquad K_\ell(z) = \frac{e^{\ell z} - 2e^{2\ell z} + e^{3\ell z}}{\ell^2 z^3}. \tag{7.9}$$

The trivial-zero progression is that of an even primitive Dirichlet character, and $E_{\mathrm{tr},\chi}(s) \ll \ell^{-2}e^{-\ell\Re s}$ in the region in use. The horizontal contours and the absolutely convergent tails can also be chosen using the zeros of Z_χ, since this multiset contains the zeros of $L(\cdot,\chi)$. More explicitly, subtract the individual canonical-product logarithmic derivative at $2+it$ from that at $\sigma+it$. Its possibly complex exponential constant cancels, and the difference of the terms for distant zeros is summable with inverse-square decay. The unit-height zero bound inherited from Z_χ permits horizontal lines separated from all these zeros by a fixed multiple of $1/\log(q_\chi(|t|+2))$; on them the difference is $O(\log^2(q_\chi(|t|+2)))$. The kernel decays as $|t|^{-3}$. On the left contours (7.1) reduces the logarithmic derivative to the convergent Euler derivative for $\bar\chi$ and the fixed gamma derivative. These bounds prove the contour limits in (7.9) without a pairing convention for the individual zeros.

Write $s = 1/2 + a + it$ with a real, $a \ge a_0 = 4/\ell$. Under GRH,

$$\sum_{\rho \text{ of } L(\cdot,\chi)} |K_\ell(\rho - s)| \le \frac{4e^{-a\ell}}{\ell^2 a a_0} \sum_{\rho \text{ of } Z_\chi} \frac{a_0}{a_0^2 + (t - \Im\rho)^2}. \tag{7.10}$$

Indeed $|a+iu|^{-3} \le [a(a^2+u^2)]^{-1}$ and $a \ge a_0$. The sum on the right is a positive statistic for the real function Z_χ. The positivity identity and anchor absorption in Lemma 3.1 bound it by $C(L + |P_{Z_\chi}(1/2 + a_0 + it)|)$. Since $4/(\ell^2 a a_0) = 1/(a\ell) \le 1/4$, we obtain

$$|\mathcal{L}_\chi(1/2+a+it) - P_\chi(1/2+a+it)| \le Ce^{-a\ell}\{L + |P_{Z_\chi}(1/2+a_0+it)|\}. \tag{7.11}$$

This step uses neither a $\pm\Im\rho$ pairing of the individual complex zeros nor a reality assertion for $\mathcal{L}_\chi$.

For complex a in the stated sector, apply (7.11) to its real and imaginary parts and then use (7.7). The precise normalized bound is

$$\|a(\mathcal{L}_\chi - P_\chi)(1/2+a)\|_q \le C_qe^{-\Re(a)\ell}(|a|L + |a|/a_0).$$

Put $x = \Re(a)\ell$. On these discs $|a| \ll \Re a$, $x \ge cC_yH$, and $L/\ell = H^{120}/C_y$. The last expression is at most $C_qH^{120}xe^{-x}$, and hence at most the right side of (7.8). Keeping the product xe^{-x} is essential to its uniformity over the whole bulk. Pointwise polynomial moments and Cauchy's

formula on a slightly larger fixed circle, followed by Minkowski's inequality, give the supremum and derivative moments. The quantities with bounded norms in (7.7) are aP_χ and $a_0P_{Z_\chi}$; their unnormalized versions have scales $1/|a|$ and ℓ. Nor is the weaker condition $\Re(a)\ell \to \infty$ alone being asserted to suffice for these actual-process moment estimates. □

Lemma 7.5 (Small balls for the arithmetic complex process)**.** *Uniformly for t in (7.3),*

$$\mathbb{P}_X\{|G_\chi(t)| \le H^{-5}\} \ll H^{-10}. \tag{7.12}$$

Proof. Set $r = H^{-5}$ and take the nonnegative test $f_r(x,y) = e^2\exp(-(x^2+y^2)/r^2)$, which majorizes the indicator of $|x|, |y| \le r$. For the convention $\widehat{f}(\tau) = \int e^{-i\tau\cdot x}f(x)\,dx$,

$$(2\pi)^{-2}\|\widehat{f_r}\|_1 = e^2, \qquad (2\pi)^{-2}\int_{\|\tau\|_\infty > T_F} |\widehat{f_r}(\tau)|\,d\tau \ll e^{-c(rT_F)^2}.$$

Use $T_F = H^{50}$, $M = \lceil H^{102}\rceil$, and the cutoff in Lemma 7.4. Then

$$T_F/\sqrt{M} = O(H^{-1}), \qquad M\ell/L = O(H^{-18}), \qquad M\log(2+T_F) = o(L).$$

Fourier inversion and Lemma 7.3 compare the arithmetic and random polynomials with error smaller than every fixed inverse power of H. Lemma 7.4, multiplied by the Lipschitz norm $O(r^{-1})$, does the same for the actual process. The random tail has bound $C(1+|a|\ell)e^{-c\Re(a)\ell} + Ce^{-c\ell}$ after normalization, so it has the same property on the specified discs. This is the tail calculation of Lemma 2.15, applied to the real and imaginary prime blocks: their variance tails are bounded by the same sum of squared ranges, and their mean tails start at exponent three.

Finally $a \le H^{-50}$, and the derivative bounds of f_r are $O(r^{-j})$. Lemma 7.2 therefore gives comparison error $O(H^{-45})$. At one point the limiting vector has law $N_2(0, \frac{1}{8}I_2)$ and bounded density, so $\mathbb{E}f_r \ll r^2$. The modulus ball is contained in the square majorized by f_r. This proves (7.12). □

7.3. **A grid argument and domination by the real product.**

Lemma 7.6 (Bulk exclusion)**.** *The probability that G_χ has a real zero in $I_H = [50\log H, H - 121\log H]$ is $O(H^{-3})$.*

Proof. Cover I_H by $O(H)$ fixed-radius discs. The derivative supremum on each disc has bounded sixteenth moment by Lemma 7.4. Markov's inequality and a union bound give

$$\mathbb{P}_X\{\sup_{t\in I_H}|G'_\chi(t)| > H\} \ll H^{-15}.$$

A deterministic grid of spacing at most $h = H^{-6}$ has $O(H^7)$ points. If $G_\chi(t_0) = 0$ and the derivative supremum is at most H, a grid point within distance h satisfies $|G_\chi(t_j)| \le Hh = H^{-5}$, by integration along the real segment. Lemma 7.5 and another union bound give $O(H^{-15}) + O(H^7H^{-10}) = O(H^{-3})$. Under GRH the denominator $L(1/2+e^{-t},\chi)$ is nonzero for finite real t, so these zeros are exactly the zeros of $L'(1/2+e^{-t},\chi)$ in this bulk. This probability assertion does not integrate the number or multiplicity of zeros on its exceptional event. □

Lemma 7.7 (Deterministic domination on the real axis)**.** *Let f be holomorphic on a neighborhood of a real interval J, put $f^\#(s) = \overline{f(\bar{s})}$, and suppose $(ff^\#)'$ is not identically zero. Then*

$$R^{\mathrm{mult}}_{f'}(J) \le R^{\mathrm{mult}}_{(ff^\#)'}(J). \tag{7.13}$$

Proof. At a real point x with $f(x) \ne 0$, a zero of f' of order k is also a zero of $(f^\#)'$ of order k. Both summands of $(ff^\#)' = f'f^\# + f(f^\#)'$ vanish to order at least k; cancellation can only increase the order. If f has a zero of order $m \ge 1$ at x, then $f^\#$ has the same order, $ff^\#$ has order $2m$, and its derivative has order $2m-1 \ge m-1$, the order of f'. Summing these pointwise inequalities proves the result, including zeros at either endpoint. □

Proof of Theorem 7.1. Take $f = L(\cdot,\chi)$, $f^{\#} = L(\cdot,\bar{\chi})$, and $Z_\chi = ff^{\#}$. Its derivative is not identically zero: at every prime not dividing q_χ, its first logarithmic coefficient is 2 or -1. Proposition 6.7 and Lemma 7.7 thus apply. Counting characters or their two-element conjugate pairs gives exactly the same probability for events concerning Z_χ.

The front interval in the t variable, $[\log 2, 50\log H]$, has length $O(\log H)$. Apply Lemma 3.4 to Z_χ and cover this front interval by $O(\log H)$ fixed discs. On its single retained event $\mathcal{R}_X$,

$$\mathbb{E}_X\big[\mathbf{1}_{\mathcal{R}_X} R^{\mathrm{mult}}_{Z'_\chi}(\mathrm{front})\big] \ll \log H, \qquad \mathbb{P}_X(\mathcal{R}_X^{\mathrm{c}}) \ll H^{-7}.$$

By Markov and domination, the front count for f' exceeds $H^{1/2+\varepsilon}$ with probability at most $C(\log H)H^{-1/2-\varepsilon} + CH^{-7}$. This step uses the uniform density of the full random model at the initial bounded times, not a Gaussian limit that requires $t \to \infty$.

The left-hand interval $0 \le s - 1/2 \le H^{121}/L$ is contained in the interval of Lemma 3.3 for the real augmentation. To display the parameters relevant to the present bound, choose

$$\nu = H^{1/2+\varepsilon/4}, \qquad \ell_c = L/H^{1/2+\varepsilon/8}, \qquad q = \lceil 8/\varepsilon \rceil, \qquad z = H^{-1/2+\varepsilon}.$$

Then $zH = H^{1/2+\varepsilon}$ and $(\nu/(zH))^q \le H^{-6}$. That lemma bounds the combined central and bridge multiplicity by $C_\varepsilon H^{1/2+\varepsilon}$ except on an event of probability $O_\varepsilon(H^{-2}\log H + H^{-6})$. Its reference conductor is q_χ^2, so $\log\log(q_\chi^2) = H_\chi + \log 2$; this bounded change in H only alters fixed constants. Domination transfers the same upper bound to f', including a possible central zero.

The intervening bulk has no real zeros except on the event in Lemma 7.6. Adding the three failure probabilities and allowing overlap at the boundaries proves (7.2), since $H_\chi = H + O(1/\log X)$ on this dyadic family.

For $q_\chi \le X$, discard $q_\chi \le \sqrt{X}$, which has $O(\sqrt{X})$ members by (6.26). On all remaining dyadic blocks $\log\log q_\chi = H + O(1)$, the constants are uniform, and the block cardinalities sum to $O(X)$. Summing their exceptional counts gives the cumulative assertion. Finally, conjugation preserves each real zero and its analytic multiplicity, proving the assertion about pairs. □

Remark 7.8. Theorem 7.1 is a probability bound for the original family. It does not assert absence of real zeros on the whole interval, a bounded total count, or a mean bound obtained by integrating an uncontrolled exceptional tail. In particular its conclusion should not be written as an asymptotic with zero leading constant. The positive intensity $1/(\pi\sqrt{2})$ for Z'_χ in Corollary 6.9 concerns a different function. The proof of the complex approximation uses full cubic GRH; no weaker low-height hypothesis is implicit in this section.

8. Numerical diagnostics and reproducibility

The computations in this section are diagnostics, not inputs to a proof. All plotted coordinates are extracted from saved result files. The source archive contains the CSV files and their SHA–256 provenance manifest. Sampling errors and numerical discretization errors are different quantities; we report both where relevant. A finite arithmetic census has no sampling standard error, but still has numerical evaluation and missed-root risks.

8.1. **The Gaussian crossing calculation.** For the Gaussian process with mean and covariance corresponding to $\theta = 1$, the two-point crossing probability can be integrated directly. An independent conditional Monte Carlo calculation used one million independent standard normal draws, with seed 20261014. Figure 4 compares the two calculations. The same samples were used at different mesh widths, so the displayed estimates are correlated across widths. At $h = 0.005$, quadrature gives 0.13651707788667264; the conditional Monte Carlo estimate is 0.1364148646262289, with standard error 0.00010298793221565028. Comparing 64 and 128 quadrature nodes gives a small numerical discrepancy, not an interval-arithmetic certificate.

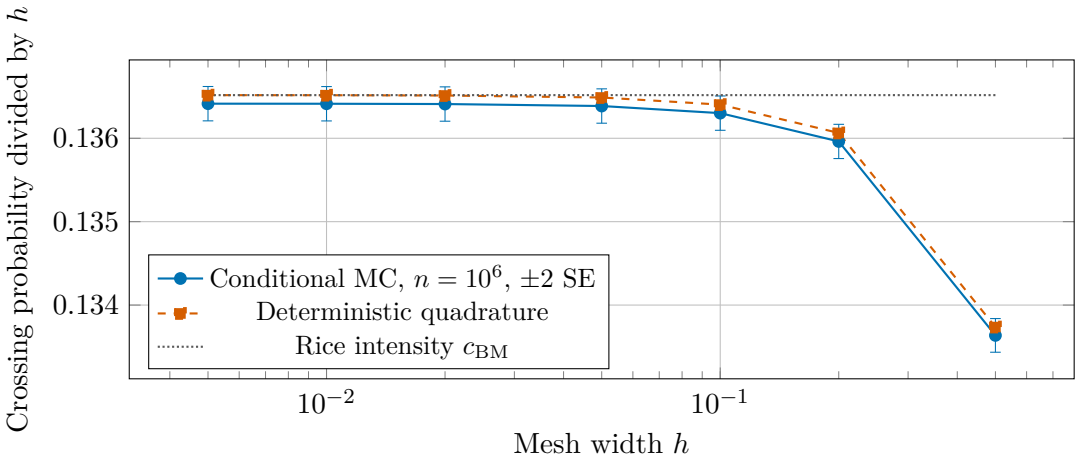


FIGURE 4. A check of the Gaussian two-point crossing calculation. This is Gaussian data, not actual arithmetic or an Euler-product simulation. The finite-mesh bias and Monte Carlo standard error are both visible.

8.2. **Local quadratic distributions and finite root counts.** The quadratic computation evaluated all 60 786 nonprincipal fundamental discriminants with $|D| \le 10^5$. For the local-distribution comparison we conditioned on $\chi_D(2) = 1$, giving 10 125 positive and 10 136 negative discriminants. The random comparison used 20 000 independent Euler samples, primes up to 10^6, and seed 68120261014, with the same condition at 2. No Gaussian tail was added. Figure 5 uses $\delta = 1/2$; the omitted random prime tail has estimated standard deviation 0.00385, which is not a certified error bound. For the random empirical CDF, the pointwise sampling standard error is $\sqrt{\widehat{F}(1-\widehat{F})/20000} \le 0.00354$.

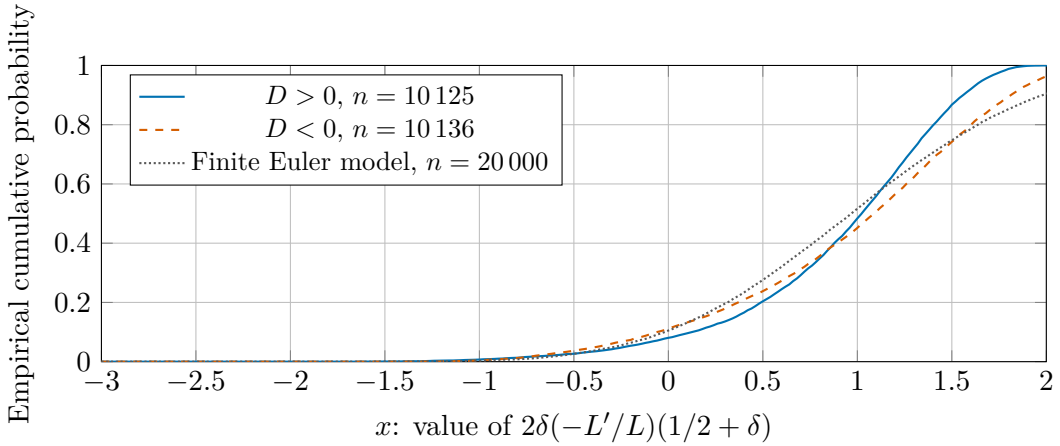


FIGURE 5. Local distributions at $\delta = 1/2$, conditioned on $\chi_D(2) = X(2) = 1$. These finite-conductor distributions need not be Gaussian. Sampling uncertainty applies only to the random curve; its pointwise standard error is at most 0.00354. The comparison tests a local statistic and does not test the real-zero asymptotic.

The empirical root counts in Figure 6 agreed on meshes with 64, 128, and 512 subdivisions and in two winding checks. Selected values were also recomputed at higher precision. These agreements do not certify all roots or their multiplicities. There are 30 394 positive and 30 392

negative discriminants. Their recorded mean counts are 0.27406725 and 0.31248355, respectively. The upper bound $\log\log(10^5)$ is only 2.44347036, so these data cannot distinguish the asymptotic constant from bounded and preasymptotic effects.

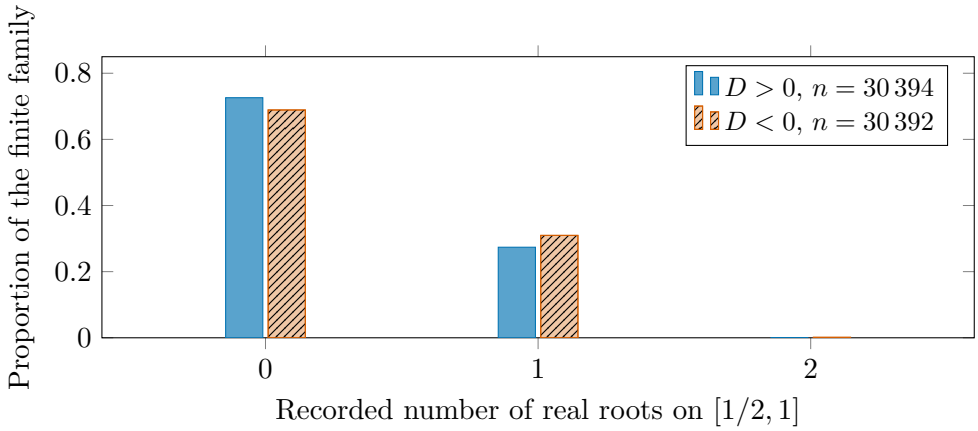


FIGURE 6. Empirical counts for $|D| \le 10^5$. Counts at two roots are 3 and 43 in the positive and negative families and are barely visible at this scale. Sampling SE is inapplicable to the census; missed-root and finite-precision errors have not been interval-certified. The available $\log\log$ range is too small to verify c_{BM}.

8.3. **An actual modular twist.** For the normalized weight-12 form Δ, an independent Mellin-integral calculation produced values of the primitive twists at small signed fundamental discriminants. Figure 7 shows $D = 17$, with conductor $17^2 = 289$. The saved refined computation used 70-digit arithmetic, 20-point Gauss–Legendre panels of maximum width 0.2, and 128 real subintervals. Comparison to 45 digits, 12-point panels of maximum width 0.3, and 64 subintervals gave differences below $2.42 \cdot 10^{-27}$ for L and $6.53 \cdot 10^{-27}$ for L' on common points. Separate high-precision Mellin evaluations checked selected points and bracket endpoints.

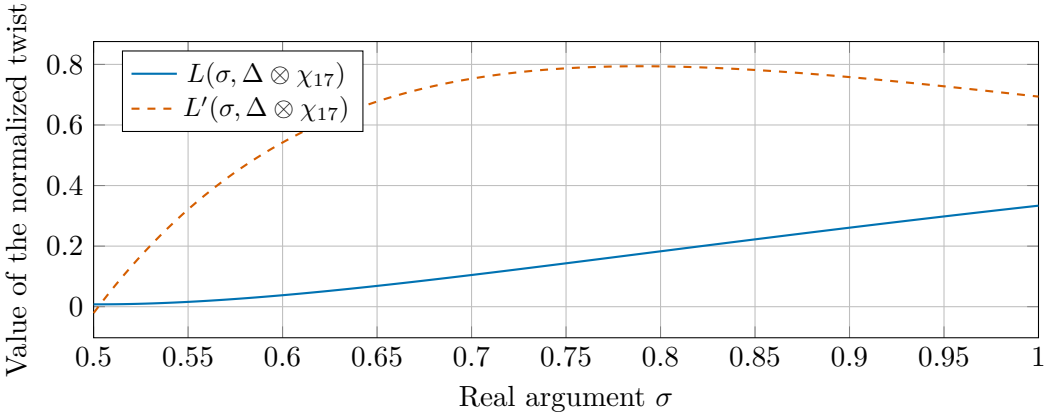


FIGURE 7. An actual modular twist, sampled at 129 real arguments. The derivative changes sign between $1/2$ and $1/2 + 1/128$. This is a numerical bracket, not an interval certificate or a count of all roots. The calculation is deterministic, so sampling SE is inapplicable; precision and quadrature comparisons are given in the text.

8.4. **The cost of eight fixed anchors.** Positive definiteness of the two fixed Gram matrices used in the proof is analytic, but their conditioning explains the size of possible constants. For the central nodes $v_j = -\log(1 + j/9)$ and bridge nodes $v_j = -1/8 + j/36$, $1 \le j \le 8$, the matrix is $(r(v_i - v_j))$. The smallest eigenvalues are approximately $5.2463 \cdot 10^{-16}$ and $5.8477 \cdot 10^{-22}$, respectively. Figure 8 uses the saved 260-digit calculation. Comparison with 180 digits gave relative changes below 10^{-159} in the smallest eigenvalues. The central matrix also has an exact rational LDL^{T} decomposition with positive pivots. Ordinary binary64 arithmetic can incorrectly return a negative smallest bridge eigenvalue. The bridge eigenvalue calculation itself is not an interval certificate.

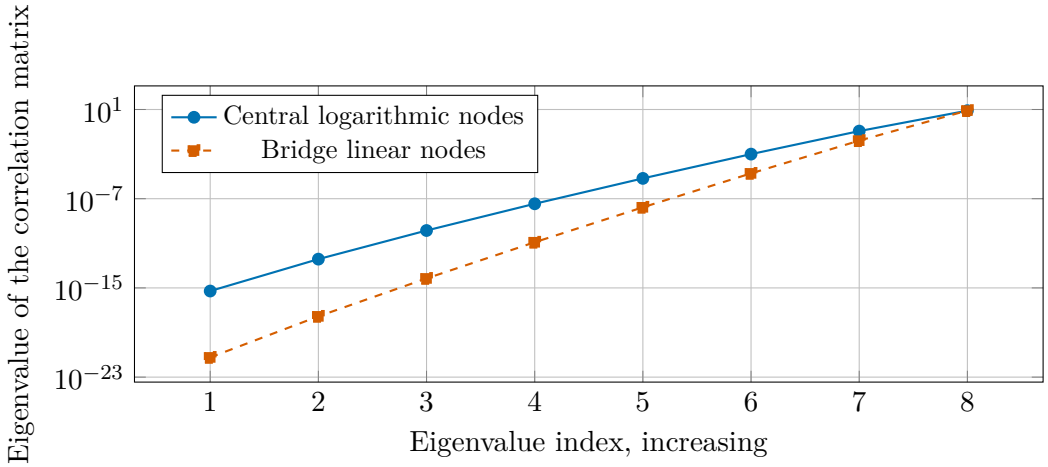


FIGURE 8. Eigenvalues of the two fixed 8×8 Gram matrices. This deterministic calculation has no sampling SE. It illustrates large constants in finite-dimensional small-ball estimates and cannot give an effective conductor threshold for the arithmetic theorems.

As a separate exact algebra check, enumeration of primitive cubic characters up to conductor 5000 gave 1590 oriented characters and 795 fields. The conductor-9 decomposition and mixed local words were checked with exact arithmetic. Such finite checks detect normalization or conjugation errors; they do not prove the asymptotic orthogonality estimate and do not evaluate actual cubic L-function zeros.

APPENDIX A. ASSEMBLY OF THE ABSTRACT ORDER-OF-MAGNITUDE THEOREM

We prove the ideal-family theorem stated with the exact axioms in the transfer section. This argument is included separately because it yields an exponentially small exceptional proportion for order-of-magnitude bounds, whereas the exact arithmetic intensities require the more precise central arguments above. We retain the notation and constants of those axioms. In particular, the moment for a mollified function is an input, and is not inferred from a functional equation.

Let K be the fixed number field. Different prime ideals are kept as different coordinates even when their norms coincide. For ideal polynomials the arithmetic norm is

$$\|P\|_{\mathrm{ar},\lambda} = \sum_{\mathfrak{a}} |c_{\mathfrak{a}}| (N\mathfrak{a})^{\lambda}.$$

It is submultiplicative by ideal convolution. The elementary bound $\#\{\mathfrak{a} : N\mathfrak{a} = n\} \le d_{[K:\mathbb{Q}]}(n)$ gives $\sum_{N\mathfrak{a} \le Y} 1 \le Y(1 + \log Y)^{[K:\mathbb{Q}]-1}$. Thus a Selberg polynomial of logarithmic length ℓ has

arithmetic norm at most $C_K e^{C_{K,\lambda}\ell}$. All arithmetic transfers below have error bounded by $X^{-\kappa_o}$ times this norm.

A.1. **The zero detector and the off-line anchor.** Write $L_X = \log X$, $H = \log L_X$. The mollified axiom says that, for $0 \le v \le 3/2$ and real t,

$$\mathbb{E}|L_D(1/2+v+it)M_D(1/2+v+it) - 1|^2 \ll L_X^{C_m}(1+|t|)^{C_t} e^{-\kappa_m v L_X}.$$

The mollifier is holomorphic in the region in question. If $\rho = 1/2+b+i\gamma$ is a zero, this holomorphic expression equals -1 at ρ. In a band $a \le b \le 2a$, with height at most $T_a = T_* + \ell^{-1}e^{6a\ell}$, submean on a disc of radius $a/4$ bounds the existence indicator by

$$\frac{16}{\pi a^2}\int_{3a/4}^{9a/4}\int_{-T_a-a/4}^{T_a+a/4} |L_D M_D - 1|^2 \, dt\, dv.$$

The rectangle is deterministic and lies in the stated shift range. Taking its expectation gives

$$\ll L_X^{C_m} a^{-1}(1+T_a)^{C_t+1} e^{-3\kappa_m a L_X/4}.$$

For $a \ge L_X^{-1}$, $T_* \le 1$, $8 \le \ell \le c_\ell L_X$, choose $c_\ell \le \kappa_m/[16(C_t+1)]$. Summing dyadic bands proves

$$\mathbb{P}\{\exists \rho = 1/2+b+i\gamma : b > a_*,\ |\gamma| \le T_* + \ell^{-1}e^{3\ell b}\} \ll L_X^{C_m+2} e^{-c_z a_* L_X}. \tag{A.1}$$

Here $L_X^{-1} \le a_* \le 1/2$. The possible number of zeros and their multiplicities are irrelevant to this existence bound.

We record the absorption calculation in the presence of off-line zeros. Suppose $4/\ell \le a \le 1$, $|t| \le 1$, and there is no zero with $b > a/2$ and $|\gamma - t| \le \ell^{-1}e^{3\ell b}$. Pair the two zeros $1/2 \pm b + i\gamma$; both real coefficients and the functional equation are used here. The ratio of their real Hadamard contribution to their positive comparison contribution $\sum a/|1/2+a+it-\rho|^2$ is

$$\frac{a^2 + (t-\gamma)^2 - b^2}{a^2 + (t-\gamma)^2 + b^2} \ge \frac{3}{5}.$$

For $b \le a/2$ this is immediate; otherwise $\ell b > 2$ and the exclusion gives $|\gamma - t| > 2b$. Put $A = \ell a$, $B = \ell b$, $C = \ell(t-\gamma)$. Dividing the absolute Selberg kernel by the positive comparison term gives

$$\frac{e^{B-A} + 2e^{2(B-A)} + e^{3(B-A)}}{A\sqrt{(A-B)^2 + C^2}}.$$

If $B \le A/2$, this is at most $2A^{-2}(e^{-A/2} + 2e^{-A} + e^{-3A/2}) < 1/32$ for $A \ge 4$. If $B > A/2$, use $|C| > e^{3B}$ to bound it by $A^{-1}(e^{-2A} + 2e^{-5A/2} + e^{-3A}) < 1/32$. The zero sum is therefore absorbed against its Hadamard real part. At $s = 1/2 + a_1 + it$, $a \le a_1 \le 3/2$, put $A_i = \ell a_i$, $a_0 = a$, and $D_i^2 = (A_i - B)^2 + C^2$. The ratio of the kernel at a_1 to the positive comparison at a_0 is at most

$$\frac{D_0^2}{A_0 D_1^3}\{e^{B-A_1} + 2e^{2(B-A_1)} + e^{3(B-A_1)}\}.$$

If $B \le A_0/2$, then $D_1 \ge D_0$ and $D_1 \ge A_1/2$. If $B > A_0/2$, then $D_0^2 \le 2C^2$ and $D_1 \ge |C| > e^{3B}$. These two inequalities give

$$|\mathcal{L}_D(s) - P_{\ell,D}(s)| \ll_K e^{-\ell a_1/2}\{\log Q_D + |P_{\ell,D}(1/2+a+it)|\}. \tag{A.2}$$

When $a = 4/\ell$ the exponent $1/2$ can be replaced by 1. For that improvement use $B \le 2$ in the first case. It is not valid for a general moving anchor. The zero-excluded ray is nonvanishing by the exclusion itself.

A.2. **A lower bound without a discrepancy axiom.** In $H/20 \le t_j \le H/10$ and dimension $d \ll H$, choose a rectangle smoothing width $\eta_0 = e^{-H/200}$, logarithmic length $\ell = C_L H e^{H/10}$, frequency cutoff $T = H/\eta_0$, and Taylor half-degree $M = \lceil H^6 e^{H/100} \rceil$. At $\sigma_y = 1/2 + 4/\ell$, orthogonality and the second model moment give $\mathbb{E}|P_y(\sigma_y)|^2 \ll \ell^2$. Hence $|P_y(\sigma_y)| \le \ell e^{K_a H}$ outside probability $O(e^{-2K_a H})$. First fix a required error exponent E, choose $K_a > E/2 + 1$, and then choose C_L sufficiently large. Equations (A.1) and (A.2) give, outside probability $O(e^{-EH})$,

$$\max_j |G_D(t_j) - \delta_j P_y(1/2 + \delta_j)| \le C \max_j \delta_j e^{-\delta_j \ell}(\ell e^{K_a H} + BL_X) \le e^{-E'H},$$

where E' is any prescribed fixed exponent after increasing C_L. The random truncation has the same prescribed L^2 accuracy. The finite-word Taylor error is bounded by

$$X^{-\kappa_o} \exp\{C_{K,\lambda} M(\ell + \log H + H/200 + 1)\} = O(X^{-\kappa_o/2}),$$

because $M\ell = O(H^7 e^{11H/100}) = o(L_X)$. We give the rectangle recovery to keep its dimension cost explicit. Truncate all coordinates to $[-H, H]$. The polynomial moment with order $k = \lfloor c_0 H^2 \rfloor$, where c_0 is sufficiently small, has $k\ell = o(L_X)$ and makes the discarded probability $O(de^{-cH^2}) + O(e^{-EH})$; the model and Gaussian bounds give the same conclusion. For each truncated interval I convolve its indicator with the centered normal density of variance η_0^2, writing $h_I = \mathbf{1}_I * \phi_{\eta_0}$. Then

$$|\widehat{h}_I(\tau)| \le \min(|I|, 2/|\tau|) e^{-\eta_0^2 \tau^2/2}, \qquad (2\pi)^{-1} \|\widehat{h}_I\|_1 \le CH.$$

The product test has normalized Fourier norm at most $(CH)^d$ and tail mass at most $e^{-H^2/8}$ outside $|\tau_j| \le H/\eta_0$. Its specified derivatives satisfy $M_j \le C_j \eta_0^{-j}$. Smooth Gaussian comparison has $b \le e^{-H/20}$, so its error is exponentially small even after the factors d^j and η_0^{-j}. To recover the rectangle, enlarge and shrink each interval by $r = H\eta_0$. Gaussian convolution has pointwise sandwich error $O(de^{-H^2/2})$, and bounded Gaussian marginal densities give boundary cost $O(dr)$. The approximation error above is paid by the product test's Lipschitz norm. Combining these estimates proves discrepancy $O_K(e^{-H/400})$. The truncation treats unbounded sides with the same constants, so no unbounded-interval Fourier norm is used.

Take $J = \lfloor \kappa H \rfloor$ disjoint ordered sign pairs with time separation $\log 4$, lying in this window. The Gaussian covariance has bounded row sum; the smooth oriented-sign count is Lipschitz with squared Lipschitz norm $O(J)$ and has mean at least cJ, uniformly for θ in the fixed model class. Gaussian concentration gives at least $c'J$ oriented crossings except on probability $e^{-c''J}$. Encode the admissible sign configurations by at most 9^J rectangles. Choosing a sufficiently small fixed κ with $\kappa \log 9 < 1/800$ pays the discrepancy sum. Each selected interval lies in the zero-free region for L_D, so either-direction crossing there gives a distinct zero of L_D'. This proves $R_D^{\mathrm{dist}} \ge cH$ outside probability $O(e^{-cH})$.

A.3. **Bulk growth with one fixed choice of constants.** Fix $\delta = AH/L_X$, $T_X = \log(1/\delta)$ and $t_j = \log 2 + j/2$, $0 \le j \le \lceil 2(T_X - \log 2) \rceil$. About t_j use discs V_j, U_j, W_j of radii $1/8, 3/4, 4/5$. Put $\delta_j = e^{-t_j}$, $c_R = e^{-4/5}\cos(4/5)$ and $d_R = e^{4/5}\sin(4/5)$. On W_j the image distance has real part at least $c_R\delta_j$ and imaginary part at most $d_R\delta_j < 1$.

Choose a sufficiently large fixed B_+ and put

$$b_j = B_+(1 + \log^+(\delta_j/\delta)), \qquad \ell_j = b_j/\delta_j.$$

Then $\ell_{\max} \le e^{1/2} B_+/\delta$ and $\sum_j (1 + b_j) = O(H^2)$. Remove the detector events with parameters $(2/\ell_j, \ell_j, d_R\delta_j)$. Their total probability is $O(H \exp((C_m + 2 - cA/B_+)H))$, hence at most e^{-2H} if A is chosen large enough after B_+. The fixed-anchor case of (A.2) gives nonvanishing on neighborhoods of all the discs.

For the normalized prime polynomials on their boundaries, prime-ideal PNT gives a coefficient Gram bound

$$\sum_{\mathfrak{p}} |a_{j,\mathfrak{p}} a_{k,\mathfrak{p}}| \ll_K \frac{\delta_j \delta_k}{(\delta_j+\delta_k)^2} \ll_K e^{-|t_j-t_k|}.$$

Its row sum is bounded. For any real or imaginary sign choice, pairing of independent symmetric prime variables bounds the $2m$th moment of a sum over $d \le 3H$ centers by $(Cmd)^m$. Arithmetic transfer adds $X^{-\kappa_o}\exp(Cm(\log(d+2)+C_{K,\lambda}\ell_{\max}))$. Cauchy's formula from W_j to U_j costs 16; representing a sum of absolute values by real and imaginary sign choices costs at most 4^d. Thus

$$\mathbb{E}\left(\sum_j \max_{U_j} |\mathcal{P}_{j,D}^{\mathrm{prime}}|\right)^{2m} \le 16^{2m}4^d\{(Cmd)^m + X^{-\kappa_o}e^{Cm(\log(d+2)+C_{K,\lambda}\ell_{\max})}\}.$$

Choose $m = \lfloor \kappa_+ H \rfloor$ with a sufficiently small fixed κ_+. Then $m\ell_{\max} \ll \kappa_+(B_+/A)L_X$, so the arithmetic error is a power saving. Markov at $K_P H$ is exponentially small once the fixed K_P is large. Higher prime powers have bounded normalized absolute sum at each center and hence total $O(H)$.

Cover each auxiliary anchor segment by $O(1+b_j)$ discs of radius $1/(2\ell_j)$. On their larger circles the real offset is at least $3/\ell_j$ and the squared coefficient sum of P/ℓ_j is bounded. The same moment calculation and a union over $O(H^2)$ discs give $\sup|P_{\ell_j,D}| \le H\ell_j$ except on an exponentially small event. Consequently

$$\max_{U_j}|G_D - \mathcal{P}_{j,D}| \ll He^{-c_R B_+(1+\log^+ r_j)}\{B_+(1+\log^+ r_j) + CAr_j\}, \qquad r_j = \delta_j/\delta.$$

Because the r_j are geometrically spaced and $c_R B_+ > 2$, the sum is $O_A(H)$. We have proved, on an event $\mathcal{A}_X$ with complement $O(e^{-cH})$,

$$\sum_j M_{j,D} \le K_+(A)H, \qquad M_{j,D} = \max_{U_j}|G_D|. \tag{A.3}$$

A.4. **Retained small values and the tip coefficient vector.** Put $u = \lfloor\sqrt{H}\rfloor$, $\delta_{\mathrm{cut}} = H^8e^{2u}/L_X$, and retain centers with $\delta_j \ge \delta_{\mathrm{cut}}$. For any separated set of retained centers and depths $1 \le n_j \le u$, use the positive test

$$\prod_j \exp(1 - e^{2n_j}x_j^2/4).$$

Writing a factor as $g_n(x)$, one has $g_n(x) \le C_{a_0}e^{-a_0 n}|x|^{-a_0}$, and the product is at least one if all $|x_j| \le 2e^{-n_j}$. Its total gradient norm is at most $Ce^d\sum_j e^{n_j}$. The joint negative moment from the random section bounds its model expectation by $C_s^d e^{-a_0\sum n_j}$. Its normalized Fourier norm is e^d, and the tail outside $|\tau_j| \le C_E\sqrt{H}e^{n_j}$ is at most $Cde^de^{-cC_E^2H}$. Choose $\ell = C_yL_X/(H^7e^{2u})$, $M = \lceil H^4e^{2u}\rceil$. The frequency sum is $O(H^{3/2}e^u) = o(\sqrt{M})$, $M\ell = O(L_X/H^3)$, and $M(u+\log H) = o(L_X)$. The required approximation accuracy is fixed before C_y, so its product with the displayed gradient norm is exponentially small. Detector, anchor, polynomial transfer, and random truncation therefore prove, for every fixed $E > 0$,

$$\mathbb{P}\{\mathcal{A}_X,\ |G_D(t_j)| \le e^{-n_j}\ (j \in I)\} \le C_s^{|I|}e^{-a_0\sum_{j\in I} n_j} + O_E(e^{-EH}). \tag{A.4}$$

At an unretained center t, put $\Delta = e^{-t}$, $S = \Delta L_X$, and $r = \lfloor \varepsilon\sqrt{H}\rfloor$ with a sufficiently small fixed ε. Thus $S > e^{-1/2}AH$ and $\Delta \le e^{-H/2}$ for large X. Set

$$a(z) = \Delta\frac{1+z}{1-z}, \quad \mathcal{H}_D(z) = \frac{\Delta}{(1-z)^2}\mathcal{L}_D(1/2+a(z)), \quad Z_k = \frac{2}{\sqrt{\theta(k+1)}}[z^k]\mathcal{H}_D(z), \quad k < r.$$

For $|z| \le 1/64$, the corresponding $w = t - \log((1+z)/(1-z))$ satisfies $|w-t| < 1/16$. Hence $\max_{V_t} |G_D| \le e^{-u}$ forces $\|Z\| \le C\sqrt{r}\, 64^r e^{-u} \le 1/4$, if $\varepsilon \log 64 < 1/4$.

The analytic covariance calculation in the random section gives, on $|z|, |w| \le 1/2$,

$$\mathbb{E}\mathcal{H}_E(z) = \frac{\theta}{2(1-z^2)} + O(\Delta), \qquad \operatorname{Cov}(\mathcal{H}_E(z), \mathcal{H}_E(w)) = \frac{\theta}{4(1-zw)^2} + O(\Delta^2).$$

This covariance is bilinear. Coefficient extraction shows $\mathbb{E}Z_k = \sqrt{\theta}\mathbf{1}_{2|k}/\sqrt{k+1} + O(\Delta 2^k)$ and $\operatorname{Cov}(Z_k, Z_l) = \mathbf{1}_{k=l} + O(\Delta^2 2^{k+l})$. The mean is not zero; its squared norm is $O(1+\log(r+1))$.

Set $v = r + \log(2+S)$, $\ell = C_y v/\Delta$ and define $Y_{D,k}$ by replacing $\mathcal{L}_D$ with $P_{\ell,D}$. Here $\ell/L_X = C_y v/S = o(1)$. The detector with $(\Delta/100, \ell, 3\Delta)$ fails with probability $O(e^{-2H})$. Outside it, the function is nonvanishing on a neighborhood of the closed Cayley disc, permitting coefficient extraction by Cauchy's formula. For each $|t'| \le 3\Delta$, choose the moving anchor

$$a_D(t') = 2\max(\{2/\ell\} \cup \{\beta - 1/2 : \beta > 1/2,\ |\gamma - t'| \le \ell^{-1} e^{3\ell(\beta-1/2)}\}).$$

It lies between $4/\ell$ and $\Delta/50$. The relevant rectangle is covered by $O(v^2)$ discs of radius $1/(2\ell)$. Cauchy and the second polynomial moment bound $\mathbb{E}(\Delta \sup|P|)^2 \ll v^4 + o(1)$. Thus $\Delta \sup|P| \le e^{bv}$ except on $O(e^{-bv})$, for a large fixed b. On $|z| \le 1/2$, $\Re a(z) \ge \Delta/3$, so the *general* anchor estimate (A.2) gives

$$\sup|\mathcal{H}_D - \mathcal{H}_{\ell,D}| \ll e^{-C_y v/6}(e^{bv} + CS) \le e^{-30v}$$

after C_y is fixed sufficiently large. Therefore $\|Y - Z\| \le 1/4$ outside $O(e^{-cr} + e^{-H})$; the model has an arbitrarily small exponential L^2 error by the random analytic tail.

The finite *model* vector is $Y_E = \mu + \sum_{\mathfrak{p}}[a_{\mathfrak{p}} X(\mathfrak{p}) + b_{\mathfrak{p}}(X(\mathfrak{p})^2 - \theta_{\mathfrak{p}})]$. Its covariance is between $I/2$ and $2I$ for large X. Indeed for each unit vector e the difference of standard deviations is at most $(\mathbb{E}\|Y_E - Z_E\|^2)^{1/2}$. The covariance of Z_E was already computed. The arithmetic vector Y_D is not a sum of independent blocks; it is compared only through its finite polynomial moments. Coefficient Cauchy bounds give

$$\sum_{\mathfrak{p}} \|b_{\mathfrak{p}}\|^2 \ll_K \Delta^2 r 4^r = o(1), \qquad \beta_{\mathfrak{p}} := \|a_{\mathfrak{p}}\| + \|b_{\mathfrak{p}}\| \ll_K \Delta\sqrt{r}\, 2^r (\log N\mathfrak{p})(N\mathfrak{p})^{-1/2-\Delta/3},$$

and so $\sum \beta_{\mathfrak{p}}^3 \le \Delta^3 e^{C_K r}$, $\max \beta_{\mathfrak{p}} \le 1$. Beyond a fixed norm cutoff $\theta_{\mathfrak{p}} \ge \theta_*/2$, the odd coefficient Gram sum is bounded by $2/\theta_*$ times the covariance; the finitely many smaller prime ideals contribute $o(1)$. This verifies the unweighted directional-range hypotheses of the finite-block polynomial test, including primes with $\theta_{\mathfrak{p}} = 0$.

For $m = \lfloor \varepsilon_0 r \rfloor$ and $A_Y = \operatorname{tr}\Sigma + \|\mu\|^2$, the test

$$\mathcal{Q}(y) = \left(\frac{A_Y - \|y\|^2}{A_Y - 1}\right)^{2m}$$

is nonnegative and at least one on the unit ball, and its model expectation is $O(e^{-cr} + \Delta^3 e^{Cr})$. Its use is restricted to sufficiently large r, so $A_Y - 1 \asymp r$ and $m \ge 1$. The arithmetic norm satisfies

$$\log\|\mathcal{Q}(Y)\|_{\mathrm{ar},\lambda} \le 4C_{K,\lambda} m\ell + O_K(mr + m\log(r+1)), \quad \frac{m\ell}{L_X} \ll C_y\left(\frac{\varepsilon^2}{A} + \frac{\varepsilon\log(2+AH)}{A\sqrt{H}}\right).$$

Choose A once, sufficiently large after the preceding constants. Orthogonality then pays this degree-$4m$ polynomial and proves

$$\sup_{j \text{ in tip}} \mathbb{P}\{\mathcal{A}_X, \max_{V_j} |G_D| \le e^{-u}\} \ll e^{-c\sqrt{H}}. \tag{A.5}$$

A.5. **Depth allocation and the bulk upper bound.** Let $v_j = \log^+(1/m_j)$, $m_j = \max_{V_j} |G_D|$. Equations (A.4) and (A.5), summed over $O(H)$ centers, exclude $\max v_j > u$ at cost $O(e^{-c\sqrt{H}})$. This exclusion is made only once. Choose a fixed integer q_0 with $q_0/2$ at least the negative-moment separation, $q_0 > 16$, and $3\log C_s/q_0 < a_0/4$. In one residue class modulo q_0, the number of tip centers is at most $4u/q_0 + O(\log H)$. Hence a capped depth allocation of total $N = \lceil H \rceil$ puts less than $H/2$ depth in the tip. Its retained depth is at least $H/2$, and (A.4) bounds its probability by $e^{-\gamma H}$ for fixed $\gamma > 0$.

Refine to classes modulo $q = q_1 q_0$. Each class has $d_* = (2/q)H + O(\log H)$ centers. The number of weak compositions of N into d_* slots is at most

$$\binom{N + d_* - 1}{d_* - 1} \le \exp((h(2/q) + o(1))H), \qquad h(x) = (1+x)\log(1+x) - x\log x.$$

Take q_1 large enough that $h(2/q) < \gamma/2$. If a class has total real depth greater than $N + d_*$, flooring the capped depths and then decreasing some of them gives an allocation of total N. Union over the compositions and the fixed number of classes therefore proves $\sum_j v_j \le C_v H$ outside $O(e^{-c\sqrt{H}})$.

At a point of maximum modulus in V_j, use Jensen radii $1/2$ and $5/8$. The outer disc lies in U_j and the real interval $[t_j - 1/4, t_j + 1/4]$ lies strictly in the inner disc. The local count is at most $(\log(M_j/m_j))/\log(5/4) \le (M_j + v_j)/\log(5/4)$. The process is nontrivial and holomorphic there, so $m_j > 0$. Summing and using (A.3) proves the upper bound $O(H)$ for the derivative roots from $1/2 + \delta$ to 1.

A.6. **The central interval and root number.** Set $s_\pm = 1/2 \pm \delta$. Use two fixed-anchor cutoffs $\ell_0 = 4/\delta$ and $\ell_1 = B_0/\delta$, where B_0 is a sufficiently large fixed constant chosen before A. Define

$$W_{\ell,D}(s) = \sum_{\mathfrak{p},k\ge 1} \frac{w_{e^\ell}((N\mathfrak{p})^k) a_D(\mathfrak{p})^k}{k(N\mathfrak{p})^{ks}}.$$

Thus $W'_{\ell,D} = -P_{\ell,D}$ and $W_{\ell,D}(s) \to 0$ as real $s \to \infty$. Euler normalization gives $L_D(s) \to 1$ there. The detector gives a nonvanishing real ray. Integrate the fixed-anchor approximation from σ_i to 2, and then use absolute Euler convergence from 2 to infinity. This determines the integration constant and gives

$$\log|L_D(\sigma_i)| = W_{\ell_i,D}(\sigma_i) + O((|P_{\ell_i,D}(\sigma_i)| + \log Q_D)/\ell_i), \quad \sigma_i = 1/2 + 4/\ell_i.$$

The integrated ideal-series tail beyond 2 is at most $\sum_{N\mathfrak{a} > e^{\ell_i}} (N\mathfrak{a})^{-2} \ll_K e^{-\ell_i}(1+\ell_i)^{[K:\mathbb{Q}]-1} \ll_K e^{-\ell_i/2}$. It is absorbed in the stated error.

At real s_+ every even prime power in W is nonnegative; odd powers at least three have a bounded absolute sum. The prime part U and logarithmic derivative polynomial have moments

$$\mathbb{E}|U|^{2m} \le (CmH)^m + X^{-\kappa_o/2}, \qquad \mathbb{E}|P(1/2+v)|^{2m} \le (Cm/v^2)^m + X^{-\kappa_o/2}$$

when $m\ell_i$ is a sufficiently small fixed multiple of L_X. Choose $m = \lfloor \kappa_* H \rfloor$ after fixing A, B_0. Markov bounds $|U|$ by $K_0 H$ and $\delta|P|$ by H outside $O(e^{-cH})$, so $\log|L_D(s_+)| \ge -C_L H$. For the logarithmic derivative, the exact anchor inequality needed is

$$|\mathcal{L}_D(s_+)| \le |P_{\ell_1,D}(s_+)| + Ce^{-B_0}\{|P_{\ell_1,D}(\sigma_1)| + BL_X\}.$$

If $Q_D \ge X^b$, choose B_0 such that the last term involving BL_X is at most $bL_X/16$. Then choose the one fixed A sufficiently large to bound each polynomial term by $bL_X/16$ outside exponentially small events, yielding

$$|L'_D/L_D(s_+)| < \tfrac{b}{4} L_X.$$

The functional equation gives the exact identities

$$\frac{L'_D}{L_D}(s_-) = -\log Q_D - \frac{\gamma'_D}{\gamma_D}(s_-) - \frac{\gamma'_D}{\gamma_D}(s_+) - \frac{L'_D}{L_D}(s_+),$$
$$\log|L_D(s_-)| = \log|L_D(s_+)| + \delta\log Q_D + O_K(\delta).$$

The compact gamma bounds are part of the analytic axiom. The root sign disappears from both expressions. Thus $L'_D(s_-) \neq 0$ and $\log|L'_D(s_-)| \geq -C_*H$. This does not require $L_D(1/2) \neq 0$.

The central derivative second-moment axiom and Cauchy on the circle $|z - s_-| = 7\delta$ imply

$$\mathbb{E}\max_{|s-s_-|\leq 6\delta}|L'_D(s)|^2 \leq 7C_A e^{C_d(A)H}.$$

Choose $2K_c > C_d(A) + 1$ and apply Markov. Jensen with inner radius 3δ and outer radius 6δ then bounds the central derivative count by $(K_c + C_*)H/\log 2$, outside $O(e^{-cH}) + O(X^{-\eta_c})$. The latter term is the small-conductor exception. Central multiple zeros are included.

Combining the lower bound, bulk upper bound and central upper bound gives the abstract theorem with exceptional probability $O(e^{-c\sqrt{H}})$. Fix the field, all axiom and model constants, disc geometry and negative-moment constants first; then $B_+, B_0, \varepsilon, \varepsilon_0, C_y$; then one A meeting all finitely many lower bounds; then the moment proportions and allocation constants; finally increase X_0 to meet the dimension and asymptotic thresholds. There is no later change of A. On $X^b \leq Q_D \leq X^B$, replacing H by $\log\log Q_D$ changes it by a bounded additive quantity. This completes the proof, for the original probability measure, without converting a weighted average into an unweighted one.

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