Every C_4-free graph on 49 vertices has a vertex of degree at most 6

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Abstract

We prove that every graph on 49 vertices containing no cycle of length four has a vertex of degree at most 6. Equivalently, $R(C_4, K_{1,42}) \leq 49$, hence $R(C_4, K_{1,42}) = 49$, a value left open in the published tables. In the notation of Erdős, the least integer $f(n)$ such that every graph on n vertices with minimum degree at least $f(n)$ contains a C_4 satisfies $f(49) = 7$, while $f(48) = f(50) = 8$; the value $f(48) = 8$ comes from an explicit 7-regular C_4-free Cayley graph on 48 vertices, which is also an explicit lower-bound construction for the tabulated value $R(C_4, K_{1,41}) = 49$. Hence f is not non-decreasing; to our knowledge no n with $f(n+1) < f(n)$ was known before. Erdős's question whether $f(n+1) \geq f(n)$ for all sufficiently large n (Problem 85 of erdosproblems.com) remains open. The proof combines structural lemmas, valid for every odd k, on C_4-free graphs with k^2 vertices and minimum degree at least k, a symmetry reduction for $k = 7$, and an exhaustive SAT search carried out by two separately written programs.

Résumé. Nous montrons que tout graphe à 49 sommets sans cycle de longueur quatre possède un sommet de degré au plus 6 ; de façon équivalente, $R(C_4, K_{1,42}) \leq 49$, d'où $R(C_4, K_{1,42}) = 49$, valeur laissée ouverte dans les tables publiées. Combiné avec un graphe de Cayley explicite à 48 sommets, 7-régulier et sans C_4, ce résultat donne $f(48) = 8 > f(49) = 7$, où $f(n)$ est le plus petit entier tel que tout graphe à n sommets de degré minimum au moins $f(n)$ contienne un C_4 ; ce graphe fournit aussi une construction explicite pour la valeur tabulée $R(C_4, K_{1,41}) = 49$, et le graphe de Hoffman–Singleton donne $f(50) = 8$. Ainsi f n'est pas croissante au sens large ; à notre connaissance aucune valeur de n vérifiant $f(n+1) < f(n)$ n'était connue. La question d'Erdős de savoir si $f(n+1) \geq f(n)$ pour tout n assez grand (problème 85 d'erdosproblems.com) reste ouverte. La preuve combine des lemmes structurels, valables pour tout k impair, sur les graphes sans C_4 à k^2 sommets de degré minimum au moins k, une réduction par symétrie pour $k = 7$ et une recherche exhaustive par solveurs SAT menée par deux programmes écrits séparément.

1 Introduction

For $n \geq 4$ let $f(n)$ be the least integer such that every graph on n vertices with minimum degree at least $f(n)$ contains a cycle C_4, and let $h(n) = f(n) - 1$ be the largest minimum degree of a C_4-free graph on n vertices. Erdős asked whether $f(n+1) \geq f(n)$ for all sufficiently large n, and whether at least $f(m) > f(n) - c$ for all $m > n$ and some constant c (Problem 85 of [EP], where the original sources, among them [Er93, p. 345], are listed).

The function f carries the same information as the Ramsey number $s(m) = R(C_4, K_{1,m})$: a C_4-free graph on N vertices whose complement contains no $K_{1,m}$ is exactly a C_4-free graph with minimum degree at least $N - m$, so that

$$h(N) \geq N - m \iff s(m) \geq N + 1.$$

Parsons [Pa75] proved $s(m) \leq m + \lfloor \sqrt{m-1} \rfloor + 2$ for all $m \geq 2$ (see [Bo24]). The values of $s(m)$ are known for $m \leq 38$ [Bo24, Ra], and also $s(40) = 47$; the values $s(38) = 45$ and $s(40) = 47$ are

due to Zhang, Chen and Cheng [ZCC17]. The tables of [Bo24] and [Ra] record $s(41) = 49$; we did not find a construction for the lower bound $s(41) \geq 49$ in [Bo24, Pa75, Ra]. For $m = 42$ the table of [Bo24] gives only the upper bound 50, the lower bound 49 following from $s(41) \geq 49$ (or from [Bo24, Remark 12]).

Theorem 1. *Every C_4-free graph on 49 vertices has minimum degree at most 6. Equivalently, $R(C_4, K_{1,42}) \leq 49$.*

Corollary 2. *$f(48) = 8$, $f(49) = 7$ and $f(50) = 8$; moreover $R(C_4, K_{1,41}) = R(C_4, K_{1,42}) = 49$. In particular $f(n+1) < f(n)$ for $n = 48$.*

Status of the statements. Theorem 1 is proved by Lemmas 3–6, the reduction of Section 4 (proved by hand) and a finite computation (Section 5), carried out by two separately written programs. In Corollary 2, the values $f(48) = f(50) = 8$ and $R(C_4, K_{1,41}) = 49$ are proved from explicit graphs, independently of Section 5; $f(49) = 7$ and $R(C_4, K_{1,42}) = 49$ rest on Theorem 1. The question of Problem 85, monotonicity for all large n, remains open.

2 A counting lemma

All graphs are finite and simple. For a vertex x, $N(x)$ is its neighbourhood and $d(x)$ its degree. In a C_4-free graph two distinct vertices have at most one common neighbour.

Lemma 3. *Let G be C_4-free on n vertices and let x be a vertex. Let $T(x)$ be the number of edges inside $N(x)$. Then*

$$\big|\{z : \mathrm{dist}(x,z) \leq 2\}\big| = 1 + d(x) + \sum_{y \in N(x)} (d(y) - 1) - 2T(x) \leq n,$$

the edges inside $N(x)$ form a matching, and every vertex outside $N[x]$ has at most one neighbour in $N(x)$.

Proof. If $y \in N(x)$ had two neighbours $w, w' \in N(x)$, then $xwyw'$ would be a C_4; so the edges inside $N(x)$ form a matching. A vertex $z \notin N[x]$ with two neighbours in $N(x)$ would close a C_4 through x. Hence the sets $N(y) \setminus \{x\}$, $y \in N(x)$, cover each vertex at distance 2 exactly once and each vertex of $N(x)$ that lies on an edge inside $N(x)$ exactly once, which gives the formula. □

3 The structure for k^2 vertices, k odd

Fix an odd integer $k \geq 3$ and let G be a C_4-free graph on k^2 vertices with minimum degree at least k.

Lemma 4. *(i) Every vertex has degree k or $k+1$, and some vertex has degree $k+1$. (ii) If $d(v) = k+1$, then $N(v)$ induces a perfect matching, every $u \in N(v)$ has degree exactly k, and the sets $A_u = N(u) \setminus N[v]$, $u \in N(v)$, have $k-2$ elements each and partition $V \setminus N[v]$. (iii) The vertices of degree $k+1$ form an independent set. (iv) Every vertex of degree k lies on a triangle.*

Proof. For a vertex x of degree r, Lemma 3 with $d(y) \geq k$ and $T(x) \leq r/2$ gives $k^2 \geq 1 + r + r(k-1) - r = 1 + r(k-1)$, so $r \leq k+1$. The degree sum is even, while $k \cdot k^2$ is odd; hence not all degrees equal k, which proves (i). If $d(v) = k+1$, equality must hold throughout: $T(v) = (k+1)/2$, every neighbour of v has degree k, and every vertex lies within distance 2 of v; this is (ii), and (iii) follows from it. For (iv), if $d(x) = k$ and t neighbours of x have degree $k+1$, Lemma 3 gives $k^2 \geq 1 + k + k(k-1) + t - 2T(x)$, so $2T(x) \geq 1 + t \geq 1$. □

From now on v is a vertex of degree $k+1$, $N(v) = \{u_1, \dots, u_{k+1}\}$ with the matching $u_{2i-1}u_{2i}$, and $G_i = A_{u_i}$. We call G_i and $G_{i'}$ *partner groups* when $u_iu_{i'}$ is an edge. The vertices of $\bigcup G_i$ are the *outer* vertices and H is the graph they induce.

Lemma 5. *Every outer vertex $x \in G_i$ has exactly one neighbour in $N[v]$, namely u_i, and in H: (a) x has at most one neighbour in G_i; (b) x has no neighbour in the partner group of G_i; (c) x has at most one neighbour in every other group; (d) two vertices of the same group have no common neighbour in H; (e) H is C_4-free; (f) $d_H(x) \in \{k-1, k\}$. Conversely, a graph H on the k^2-k-2 outer vertices satisfying (a)–(f), together with v, $N(v)$, the matching and the attachments, is C_4-free with minimum degree at least k.*

Proof. Each statement is the exclusion of an explicit 4-cycle: (a) u_iyxy'; (b) $xu_iu_{i'}y$; (c) xyu_jy'; (d) $xzyu_i$; (f) follows from (a)–(c) and $d(x) \geq k$, since x has at most k possible neighbours in H (one in its own group and one in each of the $k-1$ groups that are neither its own nor its partner).

For the converse, let C be a 4-cycle of the whole graph and let j be the number of its vertices in $N[v]$. If $v \in C$, its two neighbours on C are some u_i, u_l, and the fourth vertex is a common neighbour of u_i, u_l other than v: it is not in $N(v)$ (the edges inside $N(v)$ form a matching) and not outer (an outer vertex has one neighbour in $N(v)$); impossible. So $v \notin C$. If $j = 0$, C lies in H, excluded by (e). If $j \geq 1$, every vertex u_i of C has its two C-neighbours in $\{u_{i'}\} \cup G_i$, where $u_{i'}$ is its partner. If C contains an edge $u_iu_{i'}$, the other two vertices are $x \in G_i$ and $y \in G_{i'}$ adjacent in H, excluded by (b). Otherwise every vertex of C in $N(v)$ has both C-neighbours in its own group; two such vertices u_i, u_l on C would be opposite, with C-neighbours in $G_i \cap G_l = \emptyset$, so $j = 1$ and $C = u_i\,x\,z\,y$ with $x, y \in G_i$ and z a common neighbour of x, y in H, excluded by (d). Finally the degrees: an outer vertex has degree $1 + d_H(x) \geq k$, the u_i have degree k and v has degree $k+1$. □

We say that an outer vertex has k *slots*: its own group and the $k-1$ groups that are not partners of its group. By (a)–(c) it has at most one neighbour per slot, and by (f) at most one empty slot; it has degree $k+1$ in G exactly when no slot is empty.

Lemma 6. *Let x be an outer vertex and W a group different from the group of x. At most $|W| = k-2$ neighbours of x in H have a neighbour in W.*

Proof. Each $z \in W$ has at most one common neighbour with x, and none outside H, because the only neighbour of z in $N[v]$ is the vertex attached to W, which is not adjacent to x. □

4 Symmetry reduction

Fix $k = 7$, a vertex v of degree 8 (Lemma 4), the matching $u_1u_2, u_3u_4, u_5u_6, u_7u_8$ of $N(v)$ and the groups $G_1, \dots, G_8$ of five outer vertices each. The labelling of $N(v) \cup \bigcup G_i$ is free subject to the matching; we fix it in four steps, each of which can be achieved by relabelling any solution, so that a solution exists if and only if a solution satisfying all four conditions exists.

Step 1 (reference group). Exactly one of the following holds. *Case I*: some outer vertex has degree 8; we let G_1 be a group containing one. *Case II*: no outer vertex has degree 8; we let G_1 be a group with the fewest edges inside it. In both cases G_2 denotes the partner of G_1.

Step 2 (order of the other groups). The 48 relabellings of $G_3, \dots, G_8$ that preserve the partner pairs $\{G_3, G_4\}, \{G_5, G_6\}, \{G_7, G_8\}$ are applied so that $m_3 \geq m_4$, $m_5 \geq m_6$, $m_7 \geq m_8$ and $m_3 \geq m_5 \geq m_7$, where m_w is the number of edges between G_1 and G_w.

Step 3 (inside G_1). The vertices of G_1 are numbered $0, \dots, 4$ so that the edges inside G_1 are among $\{0,1\}, \{2,3\}$, with $\{0,1\}$ used first.

Step 4 (alignment). For $w = 3, \dots, 8$ the vertices of G_w are numbered so that those with a neighbour in G_1 come first, in the order of their neighbour; G_2 is numbered in the same way with respect to G_3. By Lemma 5(c) the edges between two non-partner groups form a matching, so this is always possible, and it leaves no freedom inside the matched part.

After these steps the adjacency of G_1 to the whole outer graph is determined by its *pattern*: the edges inside G_1 and, for every vertex of G_1, its empty slot if any. Two patterns are equivalent when they differ by a permutation of the five vertices of G_1 and by one of the 48 relabellings of Step 2; Steps 3–4 then restore the normal form. A direct enumeration gives 39 equivalence classes of patterns, 18 of which contain a vertex without empty slot and 21 of which do not; in Case I only the former can occur for G_1, in Case II only the latter. For each class one representative is fixed (for Program A one compatible with Steps 2–4); the corresponding sub-problems are the *cubes*.

Lemma 7. *Assume that every outer vertex has exactly one empty slot, and let $i(X)$ be the number of edges inside a group X. Then $i(A) + i(A') \geq 3$ for every pair of partner groups A, A'.*

Proof. For a group X and a group Y that is neither X nor its partner, let $D(X, Y)$ be the number of vertices of X whose empty slot is Y. Every vertex of X has exactly one neighbour in Y unless its empty slot is Y, so there are $5 - D(X, Y)$ edges between X and Y, and $D(X, Y) = D(Y, X)$. A vertex has its own slot empty exactly when it lies on no edge inside its group (the edges inside a group form a matching), so the $2i(X)$ vertices of X on such edges are those whose empty slot is another group: $\sum_Y D(X, Y) = 2i(X)$. Hence the number of vertices outside X whose empty slot is X is $\sum_Y D(Y, X) = 2i(X)$. A pair $(x, z) \in A \times A'$ has no common neighbour in $N[v] \cup A \cup A'$ (there are no edges between A and A') and at most one common neighbour in all. A vertex w of one of the six other groups C is adjacent to a vertex of A and to a vertex of A' unless its empty slot is A or A'; distinct such w give distinct pairs. Hence $25 \geq \sum_C \big(5 - D(C, A) - D(C, A')\big) = 30 - 2i(A) - 2i(A')$. □

5 Computation

Additional valid constraints. Besides Lemma 5, the following constraints are imposed. The first two are consequences of Lemmas 4 and 6 and hold in every solution. The Case II items express the choice of G_1 in Step 1 and two refinements of it; they hold in a suitably relabelled copy of every solution, because the quantities used for the choice (the number of inner edges of a group, the number $c(G)$ and the type defined below) do not change under the relabellings of Steps 2–4.

- (Triangles) every outer vertex lies on a triangle (Lemma 4(iv), and for an outer vertex of degree 8 part (ii) applied to it);
- (Pigeonhole) Lemma 6 for every outer vertex x and every group W;
- (Case II) every outer vertex has exactly one empty slot, and every group has at least as many inner edges as G_1;
- (Case II, two inner edges in G_1) then every group has exactly two inner edges and exactly one vertex whose empty slot is its own group, the other four having their empty slot in another group. Among the groups, G_1 is chosen to maximise $c(G)$, the largest number of vertices of G with the same empty slot. If A, B are non-partner groups, the vertices of A with a neighbour in B are those whose empty slot is not B, so the number of edges between A and B is at least $5 - c(A) \geq 5 - c(G_1)$. This lower bound is imposed for all such pairs (by Program A in 14 of the 17 cubes of this kind, by Program B in all of them).
- (Case II, one inner edge in G_1) then every group has one or two inner edges. If a group A has exactly one inner edge xy, its three other vertices have their empty slot in A, and x, y have their empty slots in groups b_x, b_y different from A and its partner; hence $m(A, B) = 5 - [b_x = B] - [b_y = B]$ for every group B that is not A or its partner, where $m(A, B)$ is the number of edges between A and B. Say A has type α if $b_x = b_y$, type β if b_x, b_y are partner

groups, and type γ otherwise. Among the groups with one inner edge, G_1 is chosen of the highest type ($\alpha > \beta > \gamma$). If G_1 has type β, no group with one inner edge has type α, i.e. $m(A, B) \geq 4$ for all such A and all groups B other than A and its partner; if G_1 has type γ, in addition $m(A, B) + m(A, B') \geq 9$ for every partner pair B, B' of groups different from A and its partner. These implications are imposed through one switch variable s_A per group: s_A true forces at least two inner edges in A, s_A false forces at most one inner edge in A and the conditions above; the bound $i(A) + i(A') \geq 3$ of Lemma 7 is also imposed for the cubes of types β and γ. Some nodes of the search trees of these cubes were refuted on formulas containing only part of these clauses; a refutation of part of the clauses is a refutation of the whole formula.

The choices of G_1 (Case I or II; fewest inner edges; then largest c or highest type) are made in this order, before Steps 2–4, and do not use any freedom used later.

Program A. The whole graph on 49 vertices is encoded in propositional logic, one variable per pair of vertices, with the structure of Lemma 4(ii) fixed, all 4-cycles forbidden, degree constraints encoded by sequential counters, and the constraints of Section 4 and above. For each of the 39 cubes the formula is either refuted directly by KISSAT, or split by the look-ahead solver MARCH_CU into sub-cubes, i.e. lists of literals added as unit clauses. Every split is accepted only together with a covering check: a run on the parent formula with the negations of all sub-cubes added as clauses. Sub-cubes are refuted separately, by KISSAT or by an incremental CaDiCaL run under assumptions, within CPU-time limits, or by CaDiCaL 3.0.1 within a limit on the number of conflicts. A sub-cube exceeding its limit is either split again in the same way or refuted later by a single longer run; a set of literals contained in several sub-cubes may also be refuted directly, which refutes all of them. A node of this search tree is thus refuted when it is refuted directly, when it contains a refuted set of literals, or when its covering check and all its children are refuted. Every run on which the proof relies returned UNSATISFIABLE, and no run returned SATISFIABLE (the other runs stopped at their time or conflict limit, or were interrupted).

Program B. The second program encodes only the outer graph H through conditions (a)–(f) of Lemma 5 (every solution gives such an H, and by the converse part of the lemma nothing is lost). It has its own implementation of the alignment of Step 4, it does not use Step 2, and it takes as representative of each of the 39 pattern classes the lexicographically least element of the class. It has its own implementation of the triangle, pigeonhole and Case II conditions, and solves with CaDiCaL, with the same tree rules (splits by MARCH_CU with covering checks, CPU-time limits, longer direct runs) implemented in a separate job manager. For the three cubes of Case II with one inner edge in G_1, part of the search tree was refuted on the formula augmented by static symmetry breaking. Some relabellings of the outer vertices map such a cube to itself and fix G_2 and G_3 pointwise: the swaps of whole groups among $G_5, \ldots, G_8$ that preserve the partner pairs and the adjacency to G_1, and, in the cube of type α, the swap of the two ends of the inner edge of G_1 together with the swap of the matching positions in the groups matched to them. Since all other constraints are stated uniformly, such a relabelling π maps solutions to solutions, and the lexicographic-leader clauses $E \leq E \circ \pi$ (edge variables in a fixed order) keep the least solution of every orbit. A node refuted on the original formula is refuted a fortiori on the augmented one, so the tree refutes the augmented formula, which is satisfiable whenever the cube formula is. In cubes 4 and 5 the exact switch variables and the bound of Lemma 7 are added on top of these clauses; every solution satisfies them once s_A is set to true exactly when A has at least two inner edges, and they are stated uniformly over the groups and the partner pairs, so the relabellings π still map solutions to solutions, and the least solution of every orbit satisfies all the added clauses. The two programs share the reduction of Sections 2–4, which is proved by hand, the derived constraints, the authors and the solver CaDiCaL, so they are not independent in every respect; they differ in the encoding, in the implementation of the symmetry breaking and in the representatives of the

classes.

Runs. The logs of Program A record 6 684 refutations (covering checks included) and 362 runs stopped at their limit; those of Program B record 1 122 and 149. The runs were made on two machines, a Linux server and a desktop computer under Windows. The runs marked `W` in the logs (2 179 refutations) were distributed as separate tasks, to the desktop computer or, where marked `W:cadical-server`, to the server. All of them except 63 short ones were made by CaDiCaL 3.0.1 with a limit of 4 million conflicts (15 million for cube 5 of Program B); the 63 others, marked `W` alone, were made by a solver of `python-sat` whose name was not logged, and all of them were certified (Checks (iv)). For each of these runs the literals of the leaf were regenerated from the logged splits and compared with the literals sent to the solver before the result was entered in the log. The other runs used kissat 4.0.4 and the versions of CaDiCaL included in `python-sat`.

Checks. (i) The 36 506 patterns fall into 39 classes (18 in Case I), and the cubes of each program contain exactly one pattern of each class; this is verified by two scripts, one computing canonical forms (for the patterns whose inner edges are $\emptyset$, $\{0,1\}$ or $\{0,1\},\{2,3\}$, which meet every class) and one computing the classes of all patterns by union–find from generators of the acting group. (ii) Falsification: with the minimum degree condition weakened to 6 for the outer vertices and everything else unchanged, 17 formulas of both programs, among them formulas of every kind used (cubes of Case I and of Case II, including the cubes of types α, β and γ, with and without the symmetry-breaking and exact switch clauses), become satisfiable within seconds, and every model found is decoded into a C_4-free graph on 49 vertices with minimum degree 6, checked directly. (iii) The Hoffman–Singleton graph (50 vertices, 7-regular, girth 5) is accepted by the graph checker. (iv) Proof certificates: for a fixed random sample of refuted nodes with short logged runs (67 nodes, 6 of Program A and 61 of Program B), the formula was rebuilt from the scripts, solved again by kissat with a DRAT proof, and every proof was verified by the independent checker drat-trim. In separate runs (not part of the ancillary scripts), 76 further leaves refuted by runs marked `W` were certified in the same way: the 63 marked `W` alone, and 13 others (12 of Program A, 1 of Program B), 10 of them drawn at random. (v) A separate script reads the logs of both programs and checks that every cube is refuted according to the rules above. (vi) For the symmetry breaking of Program B, a script checks that every relabelling used is an involution preserving groups and partner pairs, maps the fixed literals of the cube to themselves and fixes G_2 and G_3 pointwise; on the formulas with weakened minimum degree it checks moreover that models are mapped to models and that the augmented formulas stay satisfiable.

The scripts are pure Python and are provided as ancillary files, together with the list of cubes, the solver logs and the SHA-256 hashes of all formulas used.

6 Proof of Corollary 2

$f(50) = 8$. The Hoffman–Singleton graph [HS60] has 50 vertices, is 7-regular and has girth 5, so $h(50) \geq 7$. A vertex of degree $r \geq 8$ in a C_4-free graph with minimum degree 8 would satisfy $1 + r + 7r - 2T(x) \leq 50$ by Lemma 3, i.e. $2T(x) \geq 8r - 49 \geq 15$, which is impossible since $T(x) \leq r/2$ forces $8r - 49 \leq r$, i.e. $r \leq 7$. Hence $h(50) \leq 7$ and $f(50) = 8$.

$f(48) = 8$. Let $SD_{16} = \langle a, x \mid a^8 = x^2 = 1,\ xax^{-1} = a^3 \rangle$ be the semidihedral group of order 16, let c generate C_3, and let Γ be the Cayley graph of $SD_{16} \times C_3$ with respect to the inverse-closed set

$$S = \{x,\ a,\ a^{-1},\ axc,\ a^5xc^2,\ a^6xc,\ a^6xc^2\}.$$

Γ has 48 vertices and is 7-regular. It is C_4-free: a Cayley graph $\mathrm{Cay}(G, S)$ contains a 4-cycle if and only if $st^{-1} = s't'^{-1}$ for two distinct ordered pairs $(s,t) \neq (s',t')$ with $s \neq t$, $s' \neq t'$, and the 42 elements st^{-1} $(s \neq t)$ are pairwise distinct. (The edge list is provided and checked by two independent programs.) Hence $h(48) \geq 7$. Conversely, by Lemma 3, a C_4-free graph on 48

vertices with minimum degree 8 would give, for a vertex x of degree $r \geq 8$, $1+r+7r-2T(x) \leq 48$ with $2T(x) \leq r$, i.e. $7r \leq 47$, a contradiction. So $h(48) = 7$ and $f(48) = 8$. In terms of Ramsey numbers, $h(48) \geq 7$ means $s(41) \geq 49$, and with the upper bound $s(41) \leq 41 + \lfloor\sqrt{40}\rfloor + 2 = 49$ of [Pa75] this gives $R(C_4, K_{1,41}) = 49$.

$f(49) = 7$. Theorem 1 gives $h(49) \leq 6$. Deleting one vertex of the Hoffman–Singleton graph leaves a C_4-free graph on 49 vertices with minimum degree 6, so $h(49) = 6$ and $f(49) = 7$. In terms of Ramsey numbers, $h(49) \leq 6$ means $s(42) \leq 49$, and $s(42) \geq s(41) = 49$; hence $R(C_4, K_{1,42}) = 49$.

7 Remarks

1. Problem 85 of [EP] asks whether $f(n+1) \geq f(n)$ for all *sufficiently large* n. A single exception does not settle that question, which remains open; the result above only shows that f is not non-decreasing on all $n \geq 4$.

2. By the equivalence $h(N) \geq N - m \iff s(m) \geq N + 1$, an index $N \geq 4$ with $f(N+1) < f(N)$ exists if and only if $s(m) = s(m+1)$ for some $m \geq 2$ (then $N = s(m) - 1$). By the tables of [Ra] (and [Bo24]), $s(m) < s(m+1)$ for $2 \leq m \leq 38$ and for $m = 40$, while [Ra] gives $s(39) \in \{46, 47\}$; the value $s(39) = 47$ would give the smaller instance $f(47) < f(46)$.

3. In terms of Ramsey numbers the two results read $R(C_4, K_{1,41}) = R(C_4, K_{1,42}) = 49$. The value 49 for $n = 41$ appears in the tables of [Bo24] and of the dynamic survey of Radziszowski; the graph Γ of Section 6 gives an explicit lower-bound construction for it. Erdős also asked whether consecutive values of $R(C_4, K_{1,n})$ coincide for infinitely many n (Problem 552 of [EP]); $n = 41$ is an instance.

4. In Γ every vertex has exactly three edges in its neighbourhood: the 48 triangles form a configuration (48_3) whose collinearity graph is C_4-free, and the remaining 24 edges form a perfect matching.

5. The page of Problem 85 in [EP] states that the bounds of Problem 552 imply $f(n) < \sqrt{n}+1$. This inequality fails for some n, already for $n = 15$: the line graph of the Petersen graph is 4-regular and C_4-free, so $f(15) \geq 5 > \sqrt{15} + 1$ (in fact $f(15) = 5$ by the bound below); it also fails for $n = 48$: $f(48) = 8 > \sqrt{48} + 1 \approx 7.93$. Lemma 3 gives $\delta(\delta - 1) \leq n - 1$ for the minimum degree δ of a C_4-free graph on n vertices, hence only $f(n) < \sqrt{n} + 3/2$.

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