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Inertia and Elementary Theories of Hermitian Nilpotent Operators

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Abstract

Let V be a finite-dimensional Hermitian space over an algebraically closed field with a nonidentity involution, and suppose its form h is nondegenerate. For a self-adjoint nilpotent operator N, we recover the multiplicity of each signed Jordan block from finite differences of the ranks and signature differences of the forms $h(x, N^j y)$. On a fixed Hermitian space these inertias classify unitary conjugacy. Across scalar fields they classify elementary equivalence in the four-sorted language of scalars, vectors, endomorphisms, and projective points with N named. At fixed dimension each admissible inertia sequence isolates a complete theory. These theories are model complete and decidable, and none has the independence property. A bivariate generating function counts them by ambient signature.

For two self-adjoint square-zero operators with a common image of fixed positive dimension and positive definite quotient forms, simultaneous unitary conjugacy on a fixed Hermitian space is classified by the ordered generalized eigenvalues. The associated Hermitian pencil recovers these eigenvalues even when all operator words have constant ranks and inertias. At fixed signature and rank the family theory is model complete and decidable. Its completions correspond to the ordered-field types of the spectrum, and each completion has the Turing degree of its type. We also characterize the Witt index by square-zero ranks over arbitrary fields with involution. For positive definite spaces we prove that finite-measurement sentences at fixed dimensions and outcome count have the same truth value over $R(\sqrt{-1})$ as over $\mathbb{C}$ for every real closed field R. The least Naimark dilation dimension of a measurement is the sum of its effect ranks.

Keywords. Hermitian form, inertia, self-adjoint nilpotent operator, model completeness, real closed field, quantum measurement.

1 Introduction

The ranks of the powers of a nilpotent operator determine the lengths of its Jordan blocks. If the operator is self-adjoint for an indefinite Hermitian form, each block also has a sign. Even on a hyperbolic plane, two rank-one nilpotent operators can have the same Jordan form and opposite signs. We ask how to read these signs from the operator and the form, and which first-order sentences distinguish the resulting structures.

Let K be an algebraically closed field with a nonidentity involution σ, and let (V, h) be a nondegenerate Hermitian space of dimension n over K. The fixed field $R = K^\sigma$ is real closed, and $K = R(u)$ with $u^2 = -1$. We write (p, q) for the ordered signature of h. The four sorts of our language are the field K, the space V, its full endomorphism algebra, and its projective space. It names neither a basis of V nor a square root of -1.

1.1 One operator and two operators

For a self-adjoint nilpotent operator N, consider the Hermitian forms

$$B_j^N(x, y) = h(x, N^j y), \qquad 0 \le j < n.$$

Write (p_j, q_j) for their positive and negative indices, and set $r_j = p_j + q_j$ and $d_j = p_j - q_j$. The rank r_j equals rank N^j. Second differences of the rank sequence give the number of blocks of each length. The difference $d_{m-1} - d_{m+1}$ gives the excess of positive over negative blocks of length m. Thus if $c_m^{\pm}$ are the multiplicities of the blocks with operator $J_m(0)$ and Gram matrix $\pm F_m$, where F_m is the reversal matrix, then

$$c_m^{\pm} = \frac{1}{2}\left[r_{m-1} - 2r_m + r_{m+1} \;\pm\; (d_{m-1} - d_{m+1})\right], \qquad 1 \le m \le n, \tag{1.1}$$

with $r_n = r_{n+1} = d_n = d_{n+1} = 0$. Theorem 4.3 proves the formula, and Corollary 4.4 determines which sequences of inertias occur. Each inertia is definable in the original language. Consequently, a finite list of inertia conditions defines each nilpotent unitary orbit.

Naming N as an operator constant gives one complete theory for each admissible set of signed block multiplicities (Theorem 5.1). The reason is that, in a signed Jordan basis, both the form and the operator have integral matrices. Every sentence on that branch translates into an ordered-field sentence with no scalar parameters. The translation decides the sentence on each branch, and a canonical basis in a smaller model remains a basis under any embedding. This gives decidability and model completeness in the original language. Each theory is NIP (it does not have the independence property). Corollary 5.3 counts the completions at each signature. When no operator is named there are exactly $n + 1$ completions, one per signature. In both settings the fixed field is stably embedded and has the induced structure of a pure real closed ordered field with parameters.

For two operators, the complete theories can depend on a scalar parameter. Fix $1 \le s \le \min(p, q)$, and let N_1, N_2 be self-adjoint square-zero operators with a common image W of dimension s. The forms $h(x, N_i y)$ descend to nondegenerate forms on $V/W^{\perp}$. When both quotient forms are positive definite, their ordered generalized eigenvalues classify the pair up to simultaneous unitary conjugacy on a fixed Hermitian space (Theorem 6.2). The inertia of

$$h\big(x, (N_2 - tN_1)y\big), \qquad t \in R,$$

recovers each eigenvalue and its multiplicity. The ranks and inertias of all operator words together can be constant for pairs with distinct spectra (Example 6.5).

Two such pairs are elementarily equivalent precisely when their ordered spectra satisfy the same parameter-free ordered-field formulas. At fixed $(p, q; s)$, the family theory is model complete, decidable, and NIP, and has $2^{\aleph_0}$ completions. The Turing degree of each completion is the degree of its spectral type (Theorem 6.4).

1.2 Relation to earlier work

The classification of forms and self-adjoint operators under change of basis is classical [11, 18, 20]. Over $\mathbb{C}$, Mehl [15, Theorem 6.4] gives the signed Jordan blocks used here for the eigenvalue zero. de Seguins Pazzis [8, Section 6.1] describes Hermitian invariants on quotients of kernels. For a nilpotent self-adjoint operator, the signature difference of the ambient form is the sum of the signs of the odd blocks [16, Lemma 1]. Applying this calculation to each power form on V gives (1.1), and the resulting inertia predicates isolate the complete theories.

First-order descriptions of quantum systems depend on which objects the language quantifies over. The lattice approach originates with Birkhoff and von Neumann [4]. For fixed dimension, see Dunn et al. [9] on quantum logics and Dunn et al. [10, Theorem 2.5] on uniform decidability of the corresponding first-order lattice theories. Xu et al. [24, Theorem 6.2] reduce a typed Dirac language in fixed finite dimensions to real closed fields by translating quantification over scalars, vectors, dual vectors, and operators into coordinates. Our translation also uses coordinates. For a distinguished nilpotent it uses the integral signed Jordan matrices, and for a

positive pair it retains the ordered spectrum as field parameters. In a different language, with systems as quantified objects, Brunet [5, Theorem 14] proves completeness of a possibilistic theory for each dimension $d \geq 3$. In the analytic setting, Argoty Pulido [1] studies Hilbert spaces with a distinguished unbounded closed self-adjoint operator.

The role of the scalar field in quantum theory is discussed by Cassinelli and Lahti [7], Gudder and Piron [12]. Algebraic approaches over other scalar fields include Schumacher and Westmoreland [19], Thas [21]. Thas [22, Sections 3.3 and 4.2] describes the real closed fixed field and proposes a quantum Lefschetz principle. When no operator is named, two structures in our class satisfy the same sentences precisely when their signatures agree.

Sections 2 and 3 develop the inertia predicates and the coordinate translation used in the classifications of Sections 4–6. Section 7 gives a square-zero criterion for the Witt index over any field with involution and characterizes the existence of orthogonal eigenbases over algebraically closed fields. Section 8 proves transfer for finite measurements in the positive definite case and computes the least Naimark dilation dimension. Appendix A gives first-order formulas and finite counts by signature.

2 Hermitian structures and their first-order language

Throughout, $n \geq 1$ is a fixed integer. A field with involution is a field K with an automorphism σ satisfying $\sigma^2 = \mathrm{id}$. The involution is allowed to be the identity in the structural definitions. A Hermitian form is conjugate-linear in its first argument and linear in its second, with

$$h(ax, by) = \sigma(a)b\,h(x, y), \qquad h(y, x) = \sigma(h(x, y)).$$

All ambient forms h are nondegenerate. A subspace W is totally isotropic when $h(W, W) = 0$. We define the Witt index $w(h)$ as the maximum dimension of such a subspace. This definition applies also when the characteristic is 2.

2.1 The scalar fields and inertia

Lemma 2.1 (Artin–Schreier)**.** *If K is algebraically closed and $\sigma \neq \mathrm{id}$, then $\operatorname{char} K = 0$, the fixed field $R = \mathrm{Fix}(\sigma)$ is real closed, and*

$$K = R(u), \qquad u^2 = -1, \qquad \sigma(u) = -u.$$

Every $z \in K$ has a unique expression $a + ub$ with $a, b \in R$. Conversely, the complexification of any real closed field, equipped with its conjugation, has these properties.

Proof. The fixed-field theorem gives $[K : R] = 2$. The Artin–Schreier theorem on fields with a proper finite algebraic closure implies that R is real closed of characteristic zero and that its algebraic closure is $R(u)$ with $u^2 = -1$. The nonidentity R-automorphism sends u to $-u$. The converse is the usual characterization of real closed fields [2, 3]. □

In this situation the order on R is unique and definable. We use the field formulas

$$\mathrm{Real}(a) : \ \sigma(a) = a, \tag{2.1}$$

$$\mathrm{Pos}(a) : \ \mathrm{Real}(a) \ \wedge \ \exists b\,(\sigma(b) = b \wedge b \neq 0 \wedge b^2 = a). \tag{2.2}$$

The relations $a > b$ and $a \geq b$, for $a, b \in R$, abbreviate $\mathrm{Pos}(a - b)$ and $\mathrm{Pos}(a - b) \vee a = b$. If $z = a + ub$, then

$$\sigma(z)z = a^2 + b^2,$$

which is positive for $z \neq 0$.

For a possibly degenerate Hermitian form b, its inertia is the triple $\mathrm{In}(b) = (p(b), q(b), z(b))$ of positive, negative, and zero indices. We often write only the positive and negative indices, since $z(b) = n - p(b) - q(b)$ for a form on V. The rank is $p(b) + q(b)$, and the signature difference is $p(b) - q(b)$.

Proposition 2.2. *Over $K = R(u)$, every Hermitian form is congruent to $\mathrm{diag}(I_p, -I_q, 0_z)$, and (p, q, z) is unique. For a nondegenerate form of signature (p, q),*

$$w(h) = \min(p, q).$$

Proof. If a Hermitian form b is nonzero, polarization shows that $b(v, v) \neq 0$ for some v. Indeed, the equalities $b(x + y, x + y) = 0$ and $b(x + uy, x + uy) = 0$, together with vanishing diagonal values, force both the real and imaginary parts of $b(x, y)$ to vanish. Split off the nondegenerate line Kv and proceed by induction. Each nonzero diagonal entry lies in R and can be rescaled to 1 or -1. The remaining subspace is the radical.

In the resulting diagonal representation, projection of a positive definite subspace onto the p positive coordinates is injective. Thus its dimension is at most p, and the coordinate subspace attains this bound. It follows that p is intrinsic. The same argument applies to q, and the radical determines z.

For $h = \mathrm{diag}(I_p, -I_q)$, the vectors $e_i + f_i$, for $1 \leq i \leq \min(p, q)$, span a totally isotropic subspace. Projection of any totally isotropic subspace onto either the positive or the negative coordinates is injective, proving the reverse bound. These are the usual Hermitian inertia and Witt arguments [18]. □

We write $H_{p,q} = \mathrm{diag}(I_p, -I_q)$ for the standard matrix of signature (p, q).

2.2 The four sorts

The language $L_{\mathrm{Q},n}$ has sorts K, V, M, P. The scalar sort carries $0, 1, +, -, \cdot, \sigma$. The vector and operator sorts carry their additive and scalar-multiplication operations. The language also includes the maps

$$M \times M \longrightarrow M, \quad (S, T) \longmapsto ST, \qquad \mathrm{ev} : M \times V \longrightarrow V, \qquad h : V \times V \longrightarrow K,$$

and a relation $\mathrm{Rep} \subseteq V \times P$. We write Tv for $\mathrm{ev}(T, v)$. No basis is named in $L_{\mathrm{Q},n}$.

Definition 2.3. The Hermitian structure associated with (K, σ, V, h) is

$$\mathcal{Q}(K, V, h) = \big(K, V, \mathrm{End}_K(V), \mathbf{P}(V);\, \sigma, h, \mathrm{ev}, \mathrm{Rep}\big),$$

with the natural operations, where $\dim_K V = n$ and

$$\mathrm{Rep}(v, p) \quad \Longleftrightarrow \quad v \neq 0,\ p = Kv.$$

The theory T_n^{str} describes these structures over arbitrary fields with involution. The theory T_n^{inv} adds algebraic closedness of K and $\sigma \neq \mathrm{id}$. Its signature-(p, q) extension, for $p + q = n$, is denoted $T_n^{p,q}$.

Proposition 2.4. *The class in Definition 2.3 is elementary. In particular, T_n^{str} has an effective finite axiomatization.*

Proof. Use the field and involution axioms, the vector-space axioms, and the Hermitian identities and nondegeneracy. The assertion that V has a basis of length n is first-order. On M, impose the vector-space axioms, linearity of evaluation in its vector argument, and

$$(S+T)v = Sv + Tv, \qquad (aT)v = a(Tv), \qquad (-_M T)v = -_V(Tv), \qquad (ST)v = S(Tv).$$

Add extensionality,

$$\forall S\, \forall T\, \big((\forall v\ Sv = Tv) \longrightarrow S = T\big),$$

and the assertion that for some basis $e_1, \dots, e_n$, every list of images $v_1, \dots, v_n$ is realized by an operator, as expressed by

$$\exists \bar{e} \left(\mathrm{Basis}_n(\bar{e}) \wedge \forall v_1 \cdots v_n\ \exists T\ \bigwedge_i Te_i = v_i \right).$$

These axioms identify M with all of $\mathrm{End}_K(V)$.

For P, require each nonzero vector to represent a unique point, every point to have a representative, no zero vector to represent a point, and

$$\mathrm{Rep}(v,p) \wedge a \neq 0 \ \longrightarrow\ \mathrm{Rep}(av,p),$$

$$\mathrm{Rep}(v,p) \wedge \mathrm{Rep}(w,p) \ \longrightarrow\ \exists a\, (a \neq 0 \wedge w = av).$$

Thus the fibers are exactly the nonzero parts of the one-dimensional subspaces. For fixed n these conditions are a finite, effectively given list of axioms. Evaluation identifies the operator sort with $\mathrm{End}_K(V)$, and the representative relation identifies the point sort with $\mathbf{P}(V)$. □

We use I for the unique operator satisfying $\forall v\ Iv = v$. For each $T \in M$, the nondegeneracy of h gives a unique adjoint satisfying

$$h(Tx, y) = h(x, T^\dagger y).$$

In a basis with Gram matrix H, it is

$$T^\dagger = H^{-1} \sigma(T)^{\mathrm{t}} H.$$

Consequently the relations $S = T^\dagger$, $T = T^\dagger$, and $T^\dagger T = TT^\dagger = I$ are first-order. They define adjunction, self-adjointness, and unitarity, respectively.

2.3 Inertia as an intrinsic first-order predicate

Let $g(x, y; \bar{c})$ be a term defining a Hermitian form on V. For nonnegative integers a, b with $a + b \leq n$, let $\mathrm{In}_{a,b}(g)$ assert that some basis $v_1, \dots, v_n$ has Gram matrix

$$\mathrm{diag}(I_a, -I_b, 0_{n-a-b})$$

for g. The basis condition itself has an existential definition over T_n^{str}. It asks for vectors $w_1, \dots, w_n$ satisfying

$$h(w_i, v_j) = \delta_{ij} \qquad (1 \leq i, j \leq n).$$

Pairing a linear relation among the v_j with each w_i shows that all its coefficients vanish. Conversely, nondegeneracy of h gives such a dual basis for every basis of V.

Lemma 2.5. *In T_n^{inv}, the formula $\mathrm{In}_{a,b}(g)$ defines exactly the forms of inertia $(a, b, n-a-b)$. It has an existential definition using vector and scalar variables. In particular the signature sentences $\mathrm{Sig}_{p,q} := \mathrm{In}_{p,q}(h)$ are existential, mutually exclusive, and exhaustive.*

Proof. Proposition 2.2 identifies the three indices from the Gram matrix. The dual-basis condition above uses only existential quantifiers, and the prescribed entries are the scalar terms $0, 1, -1$. □

When a fixed finite-dimensional quotient is used below, its linear independence, basis, and form conditions can likewise be expressed on representatives. Appendix A records the translations.

3 Coordinate interpretations and semialgebraic reduction

An orthogonal coordinate system need not be definable from the form. Once a frame and a square root of -1 are named, all four sorts have coordinates in the ordered fixed field. For a named operator, the only additional data needed are its matrix entries in that frame.

3.1 Frames and projective representatives

Fix a symmetric matrix $H \in GL_n(\mathbb{Z})$. Its entries are scalar terms, and its inverse also has integer entries. An H-frame is a basis $\bar{e} = (e_1, \dots, e_n)$ satisfying $h(e_i, e_j) = H_{ij}$. In the language expanded by $\bar{e}$, the coordinate relations are

$$v = \sum_i x_i e_i, \qquad Te_j = \sum_i a_{ij} e_i. \tag{3.1}$$

Their solutions are unique. In fact vector coordinates are given by terms,

$$x_i = \sum_j (H^{-1})_{ij} h(e_j, v),$$

and the same expression applied to Te_j gives the operator entries.

For projective points, put

$$C_i = \{(x_1, \dots, x_n) \in K^n : x_1 = \cdots = x_{i-1} = 0,\ x_i = 1\}, \qquad C^{\mathrm{proj}} = \bigsqcup_{i=1}^{n} C_i.$$

Each one-dimensional subspace has a unique representative in C^{proj}, obtained by dividing by the first nonzero coordinate. The formula

$$\mathrm{Can}(p, \bar{x}) :\ \bar{x} \in C^{\mathrm{proj}} \ \wedge\ \mathrm{Rep}\left(\sum_i x_i e_i, p\right)$$

therefore specifies a definable bijection between P and this definable subset of K^n.

Theorem 3.1 (Coordinate reduction)**.** *Expand a model of T_n^{inv} by an H-frame and by an element u satisfying $u^2 = -1$ and $\sigma(u) = -u$. Every formula in this expanded language admits an effective translation to an ordered-field formula over $R = \mathrm{Fix}(\sigma)$. The translation uses two real coordinates for each scalar variable, $2n$ for each vector or projective variable, and $2n^2$ for each operator variable.*

With both the frame and u named, the structure is parameter-free bi-interpretable with R. If finitely many operators are also named, the corresponding expansion of R names the real and imaginary parts of their matrix entries. The same assertion holds uniformly when those matrices are specified by polynomial or rational expressions in real parameters named on both sides, with nonzero denominators.

Proof. First replace V, M, and P by K^n, K^{n^2}, and C^{proj}, respectively, using (3.1). Vector operations, matrix multiplication, and evaluation are polynomial. The form is

$$h(x, y) = \sigma(x)^{\mathrm{t}} H y.$$

The representative relation becomes proportionality between a nonzero vector and a canonical representative. Restricted projective quantifiers are handled by the defining formula for C^{proj}.

Next write each scalar x as $a + ub$, where $a, b \in R$, using

$$a = \frac{x + \sigma(x)}{2}, \qquad b = -\frac{u(x - \sigma(x))}{2}.$$

Multiplication and conjugation become

$$(a, b)(c, d) = (ac - bd, ad + bc), \qquad \sigma(a, b) = (a, -b).$$

Translate atomic formulas by these rules and proceed through Boolean operations and quantifiers. This gives the claimed effective translation and coordinate counts.

Conversely, the real field is interpreted on the definable fixed-field subset of K, with the order from (2.2). Starting from an ordered real closed field, form R^2 with the displayed operations and $u = (0, 1)$, then the vector, operator, and canonical projective sorts. The coordinate maps and the scalar decomposition give definable isomorphisms between each structure and the corresponding double interpretation. Named operator matrices are transported by the same maps, including in the parameterized case. □

Real closed fields have effective quantifier elimination in the ordered field language [14, 23]. Hence every formula in Theorem 3.1 has a semialgebraic description in the resulting real coordinates. If an operator is fixed in a canonical matrix form with entries in $\{0, 1, -1\}$, it introduces no scalar parameters. In particular, the signed Jordan matrices used in Section 5 require no expansion of the fixed field.

3.2 Descent and the original language

Proposition 3.2. *Work in the class of models admitting an H-frame. Let $\varphi(\bar{x}; \bar{e})$ be a formula using such a frame. Suppose its truth on a tuple $\bar{x}$ is independent of the chosen H-frame. Then the relation it defines is definable in the original language by*

$$\exists \bar{e}\, (\mathrm{Frame}_H(\bar{e}) \wedge \varphi(\bar{x}; \bar{e})).$$

Proof. An H-frame exists in every structure under consideration. The assumed independence makes the existential choice equivalent to any particular choice. □

The inertia predicates of Lemma 2.5 are already formulas of $L_{\mathrm{Q},n}$. Proposition 5.5 and Remark 5.6 describe which conclusions about coordinates depend on the named constants.

4 Inertia of powers and nilpotent unitary orbits

Let K be an algebraically closed field with a nonidentity involution σ, let $R = K^\sigma$, and let (V, h) be a nondegenerate Hermitian space of dimension n and signature (p, q). Throughout this section, N is an h-self-adjoint nilpotent endomorphism of V. For $0 \le j \le n$, define the Hermitian form on V by

$$B_j^N(x, y) = h(x, N^j y), \qquad B_0^N = h. \tag{4.1}$$

Write (p_j, q_j) for its positive and negative indices, and put

$$r_j = p_j + q_j, \qquad d_j = p_j - q_j. \tag{4.2}$$

The zero index is $n - r_j$. Self-adjointness and nondegeneracy give

$$\operatorname{rad} B_j^N = \ker N^j, \qquad r_j = \operatorname{rank} N^j. \tag{4.3}$$

In particular, $r_0 = n$, $d_0 = p - q$, and $r_n = d_n = 0$. We also set $r_{n+1} = d_{n+1} = 0$.

The signed Jordan form over the complex numbers is classical [15]. We construct its nilpotent blocks over $K = R(u)$, then compute the contribution of one block to each B_j^N. This calculation recovers the signs in any basis.

4.1 Signed cyclic decompositions

Let $J_m = J_m(0)$ be the nilpotent Jordan matrix with ones immediately above the diagonal, and let

$$F_m = (\delta_{a+b,m+1})_{1 \le a,b \le m}$$

be the reversal matrix. Thus $F_m J_m = J_m^{\mathrm{t}} F_m$. The inertia of F_m is $(\lceil m/2 \rceil, \lfloor m/2 \rfloor)$. Each pair of coordinate vectors interchanged by F_m contributes one positive and one negative square, and the middle coordinate contributes a positive square when m is odd.

Lemma 4.1. *Suppose that $N^m = 0$ and $N^{m-1} \neq 0$. There is an N-invariant nondegenerate subspace W of dimension m and a basis of W in which*

$$[N|_W] = J_m, \qquad [h|_W] = \varepsilon F_m, \qquad \varepsilon \in \{1, -1\}.$$

The orthogonal complement $W^\perp$ is also N-invariant.

Proof. Since $N^{m-1} \neq 0$ and h is nondegenerate, B_{m-1}^N is nonzero. By Proposition 2.2 it has a nonzero diagonal value. Choose v with

$$c = h(v, N^{m-1}v) \neq 0.$$

Every moment $\beta_j = h(v, N^j v)$ belongs to R, because N^j is self-adjoint. Replacing v by $|c|^{-1/2}v$ gives $\beta_{m-1} = \varepsilon \in \{1, -1\}$. The vectors $v, Nv, \ldots, N^{m-1}v$ are linearly independent.

To make the Gram matrix a signed reversal matrix, we change the cyclic generator so that its moments of orders $0, \ldots, m-2$ vanish. In the truncated polynomial ring $R[t]/(t^m)$, put

$$C(t) = \varepsilon^{-1} \sum_{a=0}^{m-1} \beta_{m-1-a} t^a.$$

Its constant term is 1. Consequently there is a polynomial $P(t)$ with constant term 1 such that

$$P(t)^2 C(t) \equiv 1 \pmod{t^m}. \tag{4.4}$$

For example, take the truncated binomial expansion

$$P(t) = \sum_{\ell=0}^{m-1} \binom{-1/2}{\ell} (C(t) - 1)^\ell \pmod{t^m}.$$

All its coefficients lie in R. Set $w = P(N)v$. Since $P(N)$ commutes with N, is self-adjoint, and is invertible on V, the vector w generates the same cyclic subspace as v. Writing $P(t)^2 = \sum_a \gamma_a t^a$ modulo t^m gives

$$h(w, N^j w) = \sum_{a=0}^{m-1-j} \gamma_a \beta_{j+a} = \varepsilon\, [t^{m-1-j}](P(t)^2 C(t)) = \varepsilon\, \delta_{j,m-1} \quad (0 \le j < m).$$

Here $[t^a]$ denotes the coefficient of t^a.

For $1 \le a \le m$, put $e_a = N^{m-a} w$. Then $N e_1 = 0$ and $N e_a = e_{a-1}$ for $a > 1$. Moreover,

$$h(e_a, e_b) = h(w, N^{2m-a-b} w) = \varepsilon\, \delta_{a+b,m+1},$$

where moments of index at least m vanish. This proves the two matrix identities and nondegeneracy of the cyclic subspace W. Finally, if $x \in W^\perp$ and $y \in W$, then $h(Nx, y) = h(x, Ny) = 0$, so $Nx \in W^\perp$. □

Theorem 4.2 (Signed Jordan decomposition)**.** *There is an orthogonal decomposition of (V, h, N) into blocks*

$$(K^m,\, \varepsilon F_m,\, J_m), \qquad 1 \le m \le n, \quad \varepsilon \in \{1, -1\}. \tag{4.5}$$

If c_m^+ and c_m^- count the blocks of length m with the indicated sign, respectively, then

$$n = \sum_{m=1}^{n} m(c_m^+ + c_m^-), \tag{4.6}$$

$$p = \sum_{m=1}^{n} \left(\left\lceil \frac{m}{2} \right\rceil c_m^+ + \left\lfloor \frac{m}{2} \right\rfloor c_m^- \right), \tag{4.7}$$

$$q = \sum_{m=1}^{n} \left(\left\lfloor \frac{m}{2} \right\rfloor c_m^+ + \left\lceil \frac{m}{2} \right\rceil c_m^- \right). \tag{4.8}$$

Conversely, any nonnegative integral block multiplicities satisfying (4.7)–(4.8) are realized by an h-self-adjoint nilpotent endomorphism of V.

Proof. Apply Lemma 4.1 with m equal to the nilpotency index, then repeat on the nondegenerate invariant orthogonal complement. The process terminates because the dimension decreases at each step. The signature formulas follow from the inertia of F_m. For the converse, form the indicated orthogonal direct sum. Its form has signature (p, q), so an isometry to (V, h) transports its direct sum of Jordan blocks to the required endomorphism. □

4.2 Recovery and realization of inertia profiles

On a single signed block, taking a power of N shortens the nondegenerate part of the form by one coordinate at each step. Its rank records the remaining length. Its signature difference records the sign whenever that length is odd.

Theorem 4.3 (Recovery from the inertia of powers)**.** *For a self-adjoint nilpotent endomorphism N with signed block multiplicities $c_m^\pm$, one has*

$$r_j = \sum_{m>j} (m - j)(c_m^+ + c_m^-), \tag{4.9}$$

$$d_j = \sum_{\substack{m>j \\ m-j \text{ odd}}} (c_m^+ - c_m^-) \qquad (0 \le j \le n). \tag{4.10}$$

These relations are invertible. More precisely, for $1 \le m \le n$,

$$c_m^{\pm} = \frac{1}{2}\left[r_{m-1} - 2r_m + r_{m+1} \;\pm\; (d_{m-1} - d_{m+1})\right]. \tag{4.11}$$

Consequently the sequence $\big((p_j, q_j)\big)_{j=0}^{n-1}$ *determines the signed Jordan multiplicities uniquely.*

Proof. On a block of length m and sign ε, the Gram matrix of B_j^N is $\varepsilon F_m J_m^j$. If $j < m$, this matrix is

$$\varepsilon F_m J_m^j = 0_j \oplus \varepsilon F_{m-j}. \tag{4.12}$$

For $j \ge m$ it is zero. Thus this block contributes $m - j$ to the rank when $m > j$, and contributes ε to the difference of positive and negative indices exactly when $m - j$ is odd. Additivity of inertia under orthogonal sums proves (4.9)–(4.10).

Set $t_m = c_m^+ + c_m^-$ and $e_m = c_m^+ - c_m^-$. Taking successive differences in (4.9) yields

$$r_{m-1} - r_m = \sum_{\ell \ge m} t_\ell, \qquad t_m = r_{m-1} - 2r_m + r_{m+1}.$$

Similarly, the summands in d_{m+1} are exactly those in d_{m-1} except for e_m, so

$$e_m = d_{m-1} - d_{m+1}.$$

The equalities $c_m^{\pm} = (t_m \pm e_m)/2$ give (4.11). The specified zero boundary values cover $m = n$ and $m = n - 1$ as well. □

Corollary 4.4 (Admissible inertia profiles)**.** *Let* (p_j, q_j), $0 \le j < n$, *be pairs of nonnegative integers with* $(p_0, q_0) = (p, q)$. *Put* $r_j = p_j + q_j$, $d_j = p_j - q_j$ *and append the zero boundary values at* $j = n, n+1$. *This sequence is the inertia profile of an* h*-self-adjoint nilpotent endomorphism if and only if, for every* $1 \le m \le n$, *the integers*

$$t_m = r_{m-1} - 2r_m + r_{m+1}, \qquad e_m = d_{m-1} - d_{m+1}$$

satisfy

$$t_m \ge 0, \qquad |e_m| \le t_m, \qquad t_m \equiv e_m \pmod 2. \tag{4.13}$$

Proof. Necessity follows from Theorem 4.3. Conversely, (4.13) makes $c_m^{\pm} = (t_m \pm e_m)/2$ nonnegative integers. The zero boundary values and telescoping give

$$r_j = \sum_{m > j} (m - j) t_m, \qquad d_j = \sum_{\substack{m > j \\ m-j \text{ odd}}} e_m.$$

At $j = 0$ these identities show that the direct sum of the associated signed blocks has dimension n and signature (p, q). Theorem 4.2 realizes it on (V, h). The same identities at every j recover the prescribed ranks and index differences, hence the prescribed inertias. □

Corollary 4.5 (Admissible Jordan partitions)**.** *A partition* $\lambda = (\lambda_1, \dots, \lambda_a)$ *of* n *occurs as the ordinary Jordan type of an* h*-self-adjoint nilpotent endomorphism if and only if*

$$\sum_{i=1}^{a} \left\lfloor \frac{\lambda_i}{2} \right\rfloor \le \min(p, q). \tag{4.14}$$

Equivalently, the number $o(\lambda)$ *of odd parts of* λ *satisfies* $o(\lambda) \ge |p - q|$.

Proof. Each block contributes at least $\lfloor \lambda_i/2 \rfloor$ to both indices. This proves necessity. Write $b = \sum_i \lfloor \lambda_i/2 \rfloor = (n - o(\lambda))/2$. Every even block has equal positive and negative indices, whereas each odd block contributes one additional index of the sign assigned to that block. If (4.14) holds, assign positive signs to $p - b$ odd blocks and negative signs to $q - b$ odd blocks. Both numbers are nonnegative and their sum is $o(\lambda)$. The signs of even blocks may be chosen arbitrarily. The resulting signed direct sum has signature (p, q). □

4.3 First-order definitions of the unitary orbits

Write $\mathrm{U}(h)$ for the group of invertible endomorphisms preserving h. Its action on endomorphisms is conjugation.

Theorem 4.6. *Two h-self-adjoint nilpotent endomorphisms N and N' are conjugate under $\mathrm{U}(h)$ if and only if*

$$\operatorname{inertia} B_j^N = \operatorname{inertia} B_j^{N'} \qquad (0 \le j < n).$$

Every such orbit is definable without parameters in the language $L_{\mathrm{Q},n}$, uniformly over the algebraically closed fields with nonidentity involution and forms of signature (p,q). There are finitely many orbits, indexed by the nonnegative integral solutions of (4.7)–(4.8).

Proof. If $N' = gNg^{-1}$ with $g \in \mathrm{U}(h)$, then

$$B_j^{N'}(gx, gy) = h(gx, gN^j y) = B_j^N(x,y),$$

so every inertia is preserved. Conversely, equality of the inertias and Theorem 4.3 give equality of all signed block multiplicities. Match the signed block bases supplied by Theorem 4.2. The resulting linear map preserves h and intertwines N with N', proving conjugacy.

For the definability assertion, apply Lemma 2.5 to B_j^T. With $\Delta^{a,b} = \operatorname{diag}(I_a, -I_b, 0_{n-a-b})$ the formula is

$$\mathrm{In}_{a,b}^{(j)}(T) \iff \exists \bar{v} \left(\mathrm{Basis}_n(\bar{v}) \ \wedge \bigwedge_{\alpha,\beta=1}^{n} h(v_\alpha, T^j v_\beta) = \Delta_{\alpha\beta}^{a,b} \right). \tag{4.15}$$

At $j = 0$ we read $T^j v_\beta$ as v_β.

For an admissible profile $\mathcal{I} = ((p_j, q_j))_{j=0}^{n-1}$, define

$$\Phi_{\mathcal{I}}(T): \quad \mathrm{Herm}(T) \ \wedge \ T^n = 0_M \ \wedge \bigwedge_{j=0}^{n-1} \mathrm{In}_{p_j,q_j}^{(j)}(T). \tag{4.16}$$

Here $\mathrm{Herm}(T)$ abbreviates $\forall x \, \forall y \, h(Tx, y) = h(x, Ty)$. The first part of the theorem shows that (4.16) defines exactly one orbit in every model of the fixed signature. All coefficients in the formula are $0, 1, -1$, and all tuple lengths are fixed by n. Finally, there are only finitely many possible signed block multiplicities because their weighted sum is n. □

The disjunction of $\Phi_{\mathcal{I}}(T) \wedge \Phi_{\mathcal{I}}(S)$ over the profiles satisfying (4.13) thus defines unitary conjugacy on the nilpotent self-adjoint operators.

Corollary 4.7 (Square-zero orbits)**.** *For $0 \le s \le \min(p,q)$, there are exactly $s+1$ unitary orbits of self-adjoint square-zero endomorphisms of rank s. They are indexed by*

$$\operatorname{inertia} B_1^N = (a, s-a), \qquad 0 \le a \le s.$$

Their signed multiplicities are

$$c_2^+ = a, \quad c_2^- = s - a, \quad c_1^+ = p - s, \quad c_1^- = q - s.$$

Proof. Square-zero operators have only Jordan blocks of sizes 1 and 2, and rank s means that exactly s blocks have size 2. Each size-2 block contributes $(1,1)$ to the inertia of h, so the size-1 multiplicities are as displayed. Its contribution to B_1^N is a single nonzero square of its assigned sign. Theorem 4.6 now proves both existence and the asserted count. □

Example 4.8. Take $H = F_2$ and $N_+ = J_2$, $N_- = -J_2$. Both operators are H-self-adjoint, have rank 1, and have ordinary Jordan type (2). Their first power forms have matrices

$$HN_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad HN_- = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Their B_1-inertias are $(1,0)$ and $(0,1)$, respectively, so the operators are not unitarily conjugate. The rank sequences agree because they depend only on block lengths.

Example 4.9. On the same Hermitian space with Gram matrix $H = F_4 \oplus F_2$, consider

$$N_1 = J_4 \oplus (-J_2), \qquad N_2 = (-J_4) \oplus J_2.$$

The ambient signature is $(3,3)$. Both operators have ordinary Jordan type $(4,2)$ and rank sequence $(6,4,2,1,0,\ldots)$. Their power-form inertias are

j	0	1	2	3	$j \geq 4$
inertia $B_j^{N_1}$	$(3,3)$	$(2,2)$	$(1,1)$	$(1,0)$	$(0,0)$
inertia $B_j^{N_2}$	$(3,3)$	$(2,2)$	$(1,1)$	$(0,1)$	$(0,0)$

The first power that separates the orbits is $j = 3$. Formula (4.11) recovers a positive length-4 block and a negative length-2 block for N_1, with both signs reversed for N_2.

4.4 The largest nilpotency index

Proposition 4.10. *Let $w = \min(p,q)$. The largest possible nilpotency index of an h-self-adjoint nilpotent endomorphism is*

$$\min(n, 2w+1). \tag{4.17}$$

The nilpotency index of the zero operator is taken to be 1.

Proof. Suppose $N^m = 0$ and $N^{m-1} \neq 0$, and put $a = \lceil m/2 \rceil$. Then

$$h(N^a x, N^a y) = h(x, N^{2a} y) = 0,$$

so $\operatorname{im} N^a$ is totally isotropic. A Jordan chain of length m gives

$$\dim \operatorname{im} N^a \geq m - a = \lfloor m/2 \rfloor.$$

Consequently $\lfloor m/2 \rfloor \leq w$, whence $m \leq 2w+1$. The inequality $m \leq n$ is immediate.

If $p = q = w$, a single block (F_{2w}, J_{2w}) attains the bound. If $p > q$, use one block (F_{2q+1}, J_{2q+1}) and $p-q-1$ positive length-1 blocks. If $q > p$, use $(-F_{2p+1}, J_{2p+1})$ and $q-p-1$ negative length-1 blocks. These constructions include $w = 0$ and, by an isometry, give operators on the prescribed space (V,h). □

5 Elementary classification and definable sets

We use the original multisorted language $L_{\mathrm{Q},n}$, without a named basis or imaginary unit, and the theories T_n^{inv} and $T_n^{p,q}$ from Definition 2.3. In the language $L_{\mathrm{Q},n}(N)$, where N is a constant of the operator sort, put

$$T_{n,\mathrm{nil}} = T_n^{\mathrm{inv}} + \{N = N^\dagger,\ N^n = 0\}.$$

Here the adjoint notation abbreviates its defining relation. Adding the signature sentence $\mathrm{Sig}_{p,q}$ gives $T_{n,\mathrm{nil}}^{p,q}$, where $p+q=n$.

5.1 Defining the signed Jordan branches

Write

$$\Gamma_n = \left\{ \gamma = (c_m^+, c_m^-)_{m=1}^n : c_m^\pm \in \mathbb{Z}_{\geq 0}, \quad \sum_{m=1}^n m(c_m^+ + c_m^-) = n \right\}.$$

For $\gamma \in \Gamma_n$, form the canonical pair (J_γ, H_γ) by taking c_m^+ copies of (J_m, F_m) and c_m^- copies of $(J_m, -F_m)$, in a fixed order, using the reversal matrix F_m of Section 4.1. Thus H_γ is symmetric, integral, and has determinant ± 1. Let (p_γ, q_γ) be its signature. Define

$$\begin{aligned} \mathrm{Can}_\gamma(\bar{e}, N) := & \bigwedge_{a,b=1}^n h(e_a, e_b) = (H_\gamma)_{ab} \\ & \wedge \bigwedge_{b=1}^n N e_b = \sum_{a=1}^n (J_\gamma)_{ab} e_a, \qquad \chi_\gamma := \exists \bar{e}\ \mathrm{Can}_\gamma(\bar{e}, N). \end{aligned}$$

The invertible Gram matrix makes the n vectors in $\bar{e}$ a basis. Set

$$T_{n,\mathrm{nil}}^\gamma = T_{n,\mathrm{nil}} + \{\chi_\gamma\}.$$

Equivalently, one can specify a branch by the power forms $B_j^N(x,y) = h(x, N^j y)$, using the inertia predicates from Lemma 2.5.

If (p_j^γ, q_j^γ) are the inertias of $H_\gamma J_\gamma^j$, Theorems 4.2 and 4.3 give

$$T_{n,\mathrm{nil}} \models \chi_\gamma \iff \bigwedge_{j=0}^{n-1} \mathrm{In}_{p_j^\gamma, q_j^\gamma}(B_j^N). \tag{5.1}$$

Thus χ_γ can be read either as a condition on a single Jordan basis or as a list of basis-independent inertia conditions.

Theorem 5.1 (Elementary classification of nilpotent expansions)**.** *For fixed n, the following statements hold.*

1. *The theories $T_{n,\mathrm{nil}}^\gamma$, for $\gamma \in \Gamma_n$, are exactly the complete extensions of $T_{n,\mathrm{nil}}$. Two models are elementarily equivalent if and only if the inertias of $h(x, N^j y)$ agree for $0 \leq j < n$.*
2. *Every $L_{\mathrm{Q},n}(N)$-sentence is equivalent modulo $T_{n,\mathrm{nil}}$ to a disjunction of some of the sentences χ_γ. The set of branches occurring in this disjunction can be computed from the sentence.*
3. *Each branch, the full theory $T_{n,\mathrm{nil}}$, and each fixed-signature theory $T_{n,\mathrm{nil}}^{p,q}$ are model complete, decidable, and NIP in the original language $L_{\mathrm{Q},n}(N)$.*
4. *After naming a canonical frame and an element u with $u^2 = -1$ and $\sigma(u) = -u$, each branch is parameter-free bi-interpretable with* RCF*.*

Proof. The signed decomposition in Theorem 4.2 provides a γ for each model. The inversion formula (4.11) makes it unique. Hence the sentences χ_γ partition the class of models. Every branch has a model over $\mathbb{C}$. Fix γ and name a canonical frame $\bar{e}$ and an anti-fixed imaginary unit u. Theorem 3.1 identifies the resulting structure with its real-closed fixed field by mutually inverse interpretations. The additional constant N has the fixed integral matrix J_γ, so it introduces no scalar parameter into these interpretations. This proves (4). In particular, each sentence of $L_{\mathrm{Q},n}(N)$ has on this branch a parameter-free ordered-field translation. Completeness of RCF therefore proves completeness of the branch. Together with (5.1), this proves (1).

There are finitely many elements of Γ_n, and they can be enumerated effectively. For a given sentence φ, translate it on each branch and apply the decision procedure for real closed fields. If $\Gamma_n(\varphi)$ denotes the branches on which the translation is true, then

$$T_{n,\mathrm{nil}} \models \varphi \iff \bigvee_{\gamma \in \Gamma_n(\varphi)} \chi_\gamma.$$

The empty disjunction denotes falsity. This proves (2), decidability of each branch, and decidability of the full theory. Restricting the enumeration to γ of signature (p, q) decides the fixed-signature theory.

Model completeness requires compatibility with embeddings. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be an embedding between arbitrary models of $T_{n,\mathrm{nil}}$. Choose a canonical frame $\bar{e}$ in $\mathcal{A}$, of type γ, and an anti-fixed imaginary unit $u \in K_{\mathcal{A}}$. The embedding preserves the Gram and operator equations in $\mathcal{B}$. The matrix H_γ is invertible, so $f(\bar{e})$ is a basis of $V_{\mathcal{B}}$. In particular, both models have the same signed Jordan data.

The restriction of f to the fixed fields is an ordered-field embedding

$$R_{\mathcal{A}} \longrightarrow R_{\mathcal{B}}.$$

Indeed, positive elements of a real closed field are its nonzero squares. This embedding is elementary by model completeness of RCF. Relative to $(\bar{e}, u)$ and $(f(\bar{e}), f(u))$, the coordinates of vectors, operators, and canonical projective representatives commute with f. The effective translation of Theorem 3.1 thus gives, for every formula $\varphi(\bar{x})$ and every tuple $\bar{a}$ from $\mathcal{A}$,

$$\mathcal{A} \models \varphi(\bar{a}) \iff \mathcal{B} \models \varphi(f(\bar{a})).$$

This proves model completeness in the original language, both on each branch and for the full theory. It also proves the assertion for any union of branches, including a fixed ambient signature.

Each framed branch is interpretable in the o-minimal theory RCF. It is therefore NIP, as is its reduct. For any fixed formula, choose an independence bound on each of the finitely many branches. The maximum of these bounds works for the full theory and for every union of branches. This completes (3). □

Corollary 5.2 (The theory without a distinguished operator)**.** *The theory T_n^{inv} has exactly $n + 1$ complete extensions, namely $T_n^{p,n-p}$ for $0 \le p \le n$. Each extension, and the theory T_n^{inv} itself, is model complete, decidable, and NIP. Every sentence is effectively equivalent, modulo T_n^{inv}, to a disjunction of signature sentences. Consequently, two models of T_n^{inv} are elementarily equivalent precisely when their ordered signatures agree.*

Proof. Expand each structure definitionally by $N = 0$. Its signed Jordan data are $c_1^+ = p$, $c_1^- = n - p$, with all remaining multiplicities zero. The preceding theorem applies to these $n + 1$ branches. Their signatures distinguish them first-order, and every nondegenerate form has exactly one such signature. □

The signatures $(n, 0)$ and $(0, n)$ both have Witt index zero. The sentence $\exists v\, h(v, v) = 1$ distinguishes them. In fixed dimension n, a Witt stratum of index $r < n/2$ contains the two complete branches $(r, n - r)$ and $(n - r, r)$. For $n = 2r$ these coincide.

5.2 Counting complete branches at fixed signature

Let $b_{p,q}$ be the number of complete extensions of $T_{p+q,\mathrm{nil}}^{p,q}$, with $b_{0,0} = 1$ for the purpose of the following generating function.

Corollary 5.3. *As an identity of formal power series,*

$$\sum_{p,q\geq 0} b_{p,q}x^p y^q = \prod_{a\geq 0} \frac{1}{(1-x^{a+1}y^a)(1-x^a y^{a+1})} \prod_{a\geq 1} \frac{1}{(1-(xy)^a)^2}. \tag{5.2}$$

Thus $b_{p,q}$ is the coefficient of $x^p y^q$ on the right. In particular,

$$b_{n,0} = b_{0,n} = 1, \qquad b_{1,1} = 3, \qquad b_{2,1} = b_{1,2} = 4, \qquad b_{2,2} = 10.$$

Proof. A positive signed block of odd size $2a+1$ contributes signature $(a+1, a)$, and a negative signed block contributes $(a, a+1)$. Arbitrary multiplicities give the two corresponding geometric-series factors. For even size $2a$, either sign contributes signature (a, a), but the two signs represent distinct signed Jordan blocks and have independent multiplicities. They give the squared factor. Theorem 5.1 identifies precisely these multisets with complete branches. Each coefficient receives contributions from finitely many block sizes, so the product is well-defined formally. The displayed values follow by extracting the corresponding coefficients. □

For signature $(1, 1)$, the three branches consist of $N = 0$ and the two nonzero rank-one square-zero operators with signed blocks (J_2, F_2) and $(J_2, -F_2)$. Their distinguishing invariant is the inertia of $B_1^N(x, y) = h(x, Ny)$, which is respectively $(0, 0)$, $(1, 0)$, and $(0, 1)$. The two nonzero operators therefore have the same ordinary Jordan partition and distinct elementary theories.

5.3 Semialgebraic definable sets and the fixed field

Theorem 5.4 (Relative semialgebraic reduction)**.** *Fix a branch γ, a canonical frame, and an anti-fixed imaginary unit. Every definable set, with parameters, in a finite product of the sorts of $L_{\mathrm{Q},n}$ is carried by the coordinate maps to a semialgebraic set over the real-closed fixed field. The passage from a formula to a Boolean combination of polynomial equalities and strict inequalities is effective. The coefficients lie in the subfield of R generated by the real and imaginary coordinates of the parameters over $\mathbb{Q}$.*

In every model of $T_{n,\mathrm{nil}}$, the fixed field is stably embedded and its induced structure, with parameters, is precisely that of a pure real closed ordered field. Both assertions also hold for T_n^{inv}.

Proof. Using the chosen frame and imaginary unit, the sorts K, V, M are identified with R^2, R^{2n}, R^{2n^2}, respectively. The projective sort is identified with the finite union of charts whose first nonzero complex coordinate is 1. This is a semialgebraic subset of R^{2n}. All domains and primitive symbols have uniform ordered-field definitions, and the matrix of N is the fixed integral matrix J_γ. Theorem 3.1 therefore translates any formula, together with its parameters, to an ordered-field formula over the corresponding real-coordinate tuple. Quantifier elimination for real closed fields yields the stated semialgebraic description and is effective [14, 23]. Denominators introduced by the real–imaginary decomposition can be cleared, so no extra coefficients are needed. Conversely, semialgebraic conditions in these coordinates are definable in the framed structure, using the definable order on R.

To prove stable embeddedness, let $X \subseteq R^d$ be defined in the full structure by a formula with an arbitrary finite tuple of parameters from any of the four sorts. Choose a canonical frame and an imaginary unit, and apply the preceding translation. Since the free variables lie in R, their imaginary coordinates vanish, and the resulting definition uses only ordered-field operations and a finite tuple of parameters from R. Thus every such subset of R^d is definable over R in its pure ordered-field structure. The reverse inclusion follows because the ordering of R is definable by nonzero squares. This proves both stable embeddedness and the description

of the induced structure. Taking the definitional expansion $N = 0$ gives the result without a distinguished operator. □

In the ordered-field coordinates, positivity is atomic. In $L_{\mathrm{Q},n}$ it requires a quantifier, even if the frame and imaginary unit are named.

Proposition 5.5. *No theory $T^{\gamma}_{n,\mathrm{nil}}$ eliminates quantifiers in $L_{\mathrm{Q},n}(N)$, even after a canonical frame and an anti-fixed imaginary unit have been named. The same holds for $T^{p,q}_{n}$ with or without these constants.*

Proof. Consider the definable subset

$$R_{>0} = \{x : \sigma(x) = x \ \wedge \ \exists z\, (\sigma(z) = z \wedge z \neq 0 \wedge z^2 = x)\}.$$

In the framed language, every atomic formula with one free scalar variable, restricted to $x \in R$, becomes a finite system of one-variable polynomial equalities over $R(u)$. To see this, apply induction on terms. The field operations, vector operations, evaluation, composition, and the Hermitian form are polynomial in coordinates, while the named frame and N have fixed integral coordinates. There is no closed term of projective sort, so the representative relation produces no additional atomic condition in this one-variable situation. Each such polynomial equality defines a finite set or all of R. Every quantifier-free formula therefore defines a finite or cofinite subset of R. The set $R_{>0}$ and its complement in R are both infinite, proving the assertion. The same positive cone is definable in the reduct, whose quantifier-free formulas are also quantifier-free in the expansion. Taking $N = 0$ gives the assertion for $T^{p,q}_{n}$. □

Remark 5.6. A frame chosen in the smaller model remains a frame under any embedding, which is enough for model completeness. Bi-interpretability imposes a further constraint because it preserves automorphism groups. In an unframed model, $v \mapsto -v$ on V, acting identically on K, M, P, is an automorphism of order two, also when N is named. An automorphism of an ordered field has no nontrivial finite orbit. Indeed, if it moves an element upwards or downwards, its successive iterates form a strictly monotone sequence. Thus the unframed model cannot be parameter-free bi-interpretable with a real closed field. In the standard complex model, even after a canonical frame is named, complex conjugation in that frame gives an automorphism of order two. Naming u distinguishes it from $-u$, as required by the coordinate interpretation.

6 Pairs of square-zero operators and Hermitian pencils

Two self-adjoint square-zero operators with the same image induce two forms on a common quotient. When both forms are positive definite, their generalized eigenvalues can vary even though the word ranks and inertias remain fixed. Throughout this section, fix $p + q = n$ and $1 \leq s \leq \min(p, q)$.

Definition 6.1. A positive pair of rank s consists of two named operators N_1, N_2 in a model of $T^{p,q}_{n}$ such that

$$N_i = N_i^{\dagger}, \qquad N_i^2 = 0, \qquad \operatorname{rank} N_i = s, \qquad \operatorname{im} N_1 = \operatorname{im} N_2,$$

and

$$N_i x \neq 0 \quad \Longrightarrow \quad \mathrm{Pos}\big(h(x, N_i x)\big) \qquad (i = 1, 2). \tag{6.1}$$

The resulting first-order theory is denoted $T^{p,q;s}_{\mathrm{pair}}$.

Set $W = \operatorname{im} N_1 = \operatorname{im} N_2$. Self-adjointness gives $\ker N_i = (\operatorname{im} N_i)^\perp = W^\perp$, and $N_i^2 = 0$ gives $W \subseteq W^\perp$. Thus

$$N_1 N_2 = N_2 N_1 = 0.$$

On the s-dimensional quotient $E = V/W^\perp$, define

$$q_i([x],[y]) = h(x, N_i y). \tag{6.2}$$

Changing either representative by an element of $W^\perp$ does not change this expression. The forms are Hermitian and nondegenerate. Condition (6.1) says exactly that both are positive definite.

6.1 A simultaneous normal form

Theorem 6.2. *Every positive pair of rank s admits a basis in which*

$$H = \begin{pmatrix} 0 & I_s & 0 \\ I_s & 0 & 0 \\ 0 & 0 & H_{p-s,q-s} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & I_s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & D_{\boldsymbol{\lambda}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.3}$$

where

$$D_{\boldsymbol{\lambda}} = \operatorname{diag}(\lambda_1, \dots, \lambda_s), \qquad 0 < \lambda_1 \le \cdots \le \lambda_s, \quad \lambda_i \in R.$$

The ordered tuple $\boldsymbol{\lambda}$, with repetitions, is unique. Over the same scalar field, it is a complete invariant for simultaneous unitary conjugacy.

Proof. Let $A \in \operatorname{End}_K(E)$ be defined by

$$q_1(x, Ay) = q_2(x, y).$$

The Hermitian symmetry of q_2 makes A self-adjoint for q_1. Algebraic closedness gives an eigenvector $v \neq 0$. Its eigenvalue is $q_2(v,v)/q_1(v,v) > 0$. The q_1-orthogonal complement of Kv is invariant under A. Induction, with normalization in the real closed field, gives a q_1-orthonormal eigenbasis with positive eigenvalues. Order them as $\lambda_1, \dots, \lambda_s$.

Choose representatives $f_i \in V$ of this basis and put $e_i = N_1 f_i$. Then

$$h(f_i, e_j) = \delta_{ij}, \qquad h(e_i, e_j) = 0, \qquad N_2 f_j = \lambda_j e_j.$$

Indeed, both sides of the last identity lie in W and have the same pairings with every f_i. These pairings determine a vector in W because the e_j are a basis of W dual to the f_i under h. Replace each f_j by

$$f_j' = f_j - \frac{1}{2} \sum_{\ell=1}^{s} e_\ell h(f_\ell, f_j).$$

The pairings $h(e_i, f_j') = \delta_{ij}$ are unchanged, and direct expansion gives $h(f_i', f_j') = 0$. Since both operators kill W, their values on the f_j' are also unchanged. Thus the span of the e_i, f_i' is a nondegenerate hyperbolic subspace. Its orthogonal complement has signature $(p-s, q-s)$ and lies in $W^\perp$, so both operators vanish there. This proves (6.3).

A simultaneous isometry induces an isometry of the pair (q_1, q_2), and hence preserves the spectrum of $A = q_1^{-1} q_2$ with multiplicities. Conversely, equality of the ordered spectra identifies the canonical bases in (6.3) and produces a simultaneous isometry. □

The simultaneous reduction of two forms with one positive definite form is the usual definite Hermitian pencil construction [11, 13]. Theorem 6.2 realizes that construction inside the square-zero operator geometry, including the common radical and the ambient orthogonal complement.

Proposition 6.3. *The ordered spectrum $\boldsymbol{\lambda}$ is a uniquely definable tuple in the original language with N_1, N_2. For every $t \in R$, the Hermitian pencil*

$$B_t(x, y) = h(x, (N_2 - tN_1)y)$$

has inertia

$$\mathrm{In}(B_t) = (\#\{i : \lambda_i > t\}, \#\{i : \lambda_i < t\}, n - s + \#\{i : \lambda_i = t\})\,. \tag{6.4}$$

Proof. The tuple is specified by $0 < \lambda_1 \le \cdots \le \lambda_s$, $\sigma(\lambda_i) = \lambda_i$, and the existence of $f_1, \ldots, f_s$ such that

$$h(f_i, N_1 f_j) = \delta_{ij}, \qquad N_2 f_j = \lambda_j N_1 f_j. \tag{6.5}$$

The first condition makes their classes a basis of E, since its dimension is s. The second condition specifies an eigenbasis for A. Theorem 6.2 gives existence and uniqueness of the ordered tuple. In that basis the quotient pencil is $\mathrm{diag}(\lambda_1 - t, \ldots, \lambda_s - t)$. Pulling back to V contributes the $n - s$ additional zero indices from $W^{\perp}$, giving (6.4). In particular, the nullity at t minus $n - s$ is the multiplicity of t. □

6.2 The complete theories of positive pairs

For an ordered-field tuple $\bar{a}$, the notation $\mathrm{tp}_{\mathrm{RCF}}(\bar{a}/\varnothing)$ means the set of parameter-free ordered-field formulas satisfied by $\bar{a}$ in its real closed field.

Theorem 6.4. *Let $\mathcal{A}, \mathcal{B} \models T^{p,q;s}_{\mathrm{pair}}$, with fixed fields R_A, R_B and ordered spectra $\boldsymbol{\lambda}_A, \boldsymbol{\lambda}_B$. Then*

$$\mathcal{A} \equiv \mathcal{B} \quad \Longleftrightarrow \quad \mathrm{tp}_{\mathrm{RCF}}(\boldsymbol{\lambda}_A/\varnothing) = \mathrm{tp}_{\mathrm{RCF}}(\boldsymbol{\lambda}_B/\varnothing). \tag{6.6}$$

The theory $T^{p,q;s}_{\mathrm{pair}}$ is model complete, decidable, and NIP. It has exactly $2^{\aleph_0}$ completions.

Moreover, for each model its complete theory is Turing equivalent to the real-closed-field type of its ordered spectrum.

Proof. For the forward implication in (6.6), interpret the ordered fixed field and use the uniquely definable tuple from Proposition 6.3. This transports any ordered-field formula about the tuple to a sentence of the pair language.

For the reverse implication, choose a canonical basis as in (6.3) and name u. Theorem 3.1 translates any pair-language sentence φ into an ordered-field formula $\theta_\varphi(\boldsymbol{\lambda})$, uniformly in the tuple. Equal types give the same truth value to θ_φ, hence to φ. Every complete type satisfying $0 < \lambda_1 \le \cdots \le \lambda_s$ is realized in some real closed field, and (6.3) constructs a corresponding model.

To prove model completeness, consider an embedding $\mathcal{A} \subseteq \mathcal{B}$. Choose a canonical frame and u in $\mathcal{A}$. Its Gram matrix in $\mathcal{B}$ is the same invertible matrix, so the frame remains a basis over the larger field. The two operator matrices remain exactly those in (6.3), so the embedding preserves the ordered spectrum. The inclusion $R_A \subseteq R_B$ is an embedding of real closed ordered fields, hence elementary, also after naming that tuple. Coordinate reduction therefore makes the original embedding elementary.

For decidability, a sentence φ is a consequence of the family theory precisely when RCF proves

$$\forall \lambda_1 \cdots \lambda_s \, (0 < \lambda_1 \le \cdots \le \lambda_s \longrightarrow \theta_\varphi(\boldsymbol{\lambda})).$$

Quantifier elimination in RCF decides this sentence. It also suffices for the NIP assertion to consider the uniformly interpreted framed family. In each translated formula, regard the spectral coordinates as additional parameter variables. NIP of RCF then gives a bound for

that formula independent of the choice of spectrum. The bound still holds after the frame and u are forgotten.

Over $\mathbb{R}$, take the largest eigenvalue in $(1, 2)$ and every other eigenvalue equal to 1. A rational number strictly between two choices distinguishes their types. This gives $2^{\aleph_0}$ completions. The countable language gives the matching upper bound.

Finally the translations in both directions are effective. An oracle for the real-field type decides θ_φ, hence every sentence about the pair. Conversely, a formula about the ordered spectrum is translated into a sentence using (6.5) and the interpreted fixed-field order. These are the two claimed Turing reductions. □

An individual completion is decidable precisely when its spectral type is decidable. For example, if all eigenvalues are real algebraic, defining polynomials and rational isolating intervals specify the ordered tuple, and quantifier elimination decides its type.

Example 6.5 (Constant word inertias and varying pencil spectra)**.** Take $n = 4$, $(p, q) = (2, 2)$, $s = 2$, and use (6.3) with

$$D_t = \operatorname{diag}(1, t), \qquad t > 1.$$

Here an operator word means a monomial in N_1, N_2. Since $t > 1$, the operators N_1, N_2 are linearly independent. The empty word gives I and hence the ambient form h. Every operator word of length at least two is zero. Both words of length one have rank 2, and both forms $h(x, N_i y)$ have inertia $(2, 0, 2)$. Thus the ranks and inertias of all words in the two operators are independent of t.

The generalized spectrum is $(1, t)$. As t ranges over $(1, \infty) \subset \mathbb{R}$, the structures have pairwise different complete theories by Theorem 6.4. The second jump in the pencil inertia (6.4) occurs at t.

7 Witt index and orthogonal spectral decompositions

A square-zero self-adjoint operator has totally isotropic image. Conversely, a basis of any totally isotropic subspace gives an explicit operator with that image. Neither direction requires an assumption on the scalar field or its characteristic.

Theorem 7.1. *Let h be a nondegenerate Hermitian form on a finite-dimensional vector space over any field with involution. For each integer $s \geq 0$, there exists a self-adjoint operator N with*

$$N^2 = 0, \qquad \operatorname{rank} N = s$$

if and only if $s \leq w(h)$. Every totally isotropic s-dimensional subspace occurs as the image of such an operator.

Proof. If $N = N^\dagger$ and $N^2 = 0$, then

$$h(Nx, Ny) = h(x, N^2 y) = 0,$$

so $\operatorname{im} N$ is totally isotropic.

Conversely, let $W = \operatorname{span}(e_1, \dots, e_s)$ be totally isotropic and put

$$N(x) = \sum_{i=1}^{s} e_i h(e_i, x).$$

This map is linear, and

$$h(Nx, y) = \sum_i h(x, e_i) h(e_i, y) = h(x, Ny).$$

Since $h(e_i, e_j) = 0$, one has $N^2 = 0$. To see that the image is all of W, observe that a relation $\sum_i a_i h(e_i, x) = 0$ for all x gives $h(\sum_i \sigma(a_i)e_i, x) = 0$, hence all $a_i = 0$. Thus the coordinate functionals $h(e_i, \cdot)$ are linearly independent. Their joint map has rank s, so $\operatorname{im} N = W$. For $s = 0$, take $N = 0$. □

For fixed n, this theorem gives first-order sentences

$$\mathrm{HSZ}_s : \ \exists N \, (N = N^\dagger \wedge N^2 = 0 \wedge \operatorname{rank} N = s).$$

In $T_n^{p,q}$, they hold precisely for $s \leq \min(p, q)$. Thus the entire list of square-zero ranks records the Witt index. The proof applies in characteristic 2 as well, with the definition of $w(h)$ used in Section 2.

7.1 The spectral criterion

An operator has a complete orthogonal eigenbasis if there are a basis $v_1, \ldots, v_n$ and scalars $\lambda_1, \ldots, \lambda_n$ such that

$$Tv_i = \lambda_i v_i, \qquad h(v_i, v_j) = 0 \quad (i \neq j).$$

For fixed n, this is a first-order predicate $\mathrm{COEB}(T)$.

Theorem 7.2. *In a model of T_n^{inv}, a self-adjoint operator T has a complete orthogonal eigenbasis if and only if it is diagonalizable and all its eigenvalues belong to $R = \mathrm{Fix}(\sigma)$.*

Proof. For eigenvectors v, w with eigenvalues λ, μ, self-adjointness gives

$$(\sigma(\lambda) - \mu)h(v, w) = 0.$$

If an orthogonal eigenbasis exists, none of its vectors is isotropic, since its Gram matrix is diagonal and nondegenerate. Setting $v = w$ therefore shows $\lambda \in R$.

Conversely, suppose that T is diagonalizable and all its eigenvalues are real. Distinct eigenspaces are orthogonal. Each eigenspace is nondegenerate, because a vector in its radical would be orthogonal to every eigenspace and hence to V. Proposition 2.2 provides an orthogonal basis in each eigenspace. Their union is the required basis. □

Theorem 7.3. *Over an algebraically closed field with any involution, every self-adjoint operator on (V, h) has a complete orthogonal eigenbasis if and only if h is anisotropic. In the nonidentity-involution case, this is equivalent to signature $(n, 0)$ or $(0, n)$.*

Proof. If h is isotropic, Theorem 7.1 gives a nonzero self-adjoint square-zero operator. Such an operator is not diagonalizable.

Assume h is anisotropic and let $T = T^\dagger$. Algebraic closedness gives an eigenvector $v \neq 0$. The equality

$$\sigma(\lambda)h(v, v) = h(Tv, v) = h(v, Tv) = \lambda h(v, v)$$

shows $\sigma(\lambda) = \lambda$. The line Kv is nondegenerate, $V = Kv \perp v^\perp$, and $v^\perp$ is T-invariant. The restricted form is again nondegenerate and anisotropic. Induction on the dimension completes the proof. The last assertion follows from Proposition 2.2. □

For example, over an algebraically closed field of characteristic different from 2 with the identity involution, every nondegenerate symmetric form of dimension at least 2 is isotropic. In each of these dimensions a self-adjoint operator can therefore fail to have an orthogonal eigenbasis.

Example 7.4. Over $K = R(u)$ with its nonidentity involution, consider the hyperbolic plane with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is self-adjoint and square-zero. A different obstruction occurs with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $T^\dagger = T$, $T^2 = -I$, and the eigenvalues are $u, -u$. Their eigenvectors $(1, u), (1, -u)$ are isotropic and have mutual pairing 2. This operator is diagonalizable and has no orthogonal eigenbasis.

Remark 7.5. By Lemma 2.1, an algebraically closed field of positive characteristic admits no nonidentity involution. There is also a restriction inherent in Hermitian symmetry. Applying $h(y, x) = \sigma(h(x, y))$ twice shows that σ^2 fixes every value of h. For $v \neq 0$, nondegeneracy makes the linear functional $h(v, \cdot)$ surjective onto K, so $\sigma^2 = \mathrm{id}$ on all of K.

8 Finite measurements over real closed fixed fields

Work in the positive definite theory $T_n^{n,0}$, with fixed field R. The transition probability between the rays of two nonzero vectors v, w is

$$\mathsf{P}([v], [w]) = \frac{h(v, w)h(w, v)}{h(v, v)h(w, w)}. \tag{8.1}$$

The expression is independent of the representatives. Completing a square,

$$0 \leq h\left(w - v\frac{h(v, w)}{h(v, v)}, w - v\frac{h(v, w)}{h(v, v)}\right) = h(w, w) - \frac{h(w, v)h(v, w)}{h(v, v)},$$

shows $0 \leq \mathsf{P} \leq 1$ in R. Its graph is first-order, after clearing the nonzero denominator. The corresponding rank-one orthogonal projection is

$$\Pi_{[v]}x = v\,\frac{h(v, x)}{h(v, v)}.$$

8.1 States and effects

An operator A is positive semidefinite, written $A \succeq 0$, if

$$A = A^\dagger, \qquad \forall v\ \big(h(v, Av) = 0\ \vee\ \mathrm{Pos}(h(v, Av))\big).$$

A density operator satisfies $\rho \succeq 0$ and $\operatorname{tr}\rho = 1$. An effect satisfies $0 \preceq E \preceq I$. To define the trace of A, quantify over a basis and require the scalar in question to be the sum of the diagonal entries of A in that basis. Invariance under change of basis gives a unique scalar.

Fix a positive integer m. An m-outcome measurement is a tuple $E_1, \ldots, E_m$ with

$$E_i \succeq 0, \qquad \sum_{i=1}^{m} E_i = I. \tag{8.2}$$

These are the finite-dimensional positive-operator-valued measurements (POVMs). For a density operator ρ, the outcome probabilities are

$$\pi_i = \operatorname{tr}(\rho E_i).$$

They lie in $R_{\geq 0}$ and sum to 1. To see nonnegativity, diagonalize ρ in an orthonormal basis. Its eigenvalues a_j are nonnegative, and $\operatorname{tr}(\rho E_i) = \sum_j a_j h(e_j, E_i e_j) \geq 0$. The normalization follows from (8.2). Thus states, effects, measurements, and their finite probability relations are all definable in the same language.

8.2 The exact dilation dimension

For a positive definite space W, a projective measurement is a tuple of self-adjoint projections $P_1, \dots, P_m$ satisfying

$$P_i P_j = 0 \ (i \neq j), \qquad \sum_i P_i = I_W.$$

A dilation of (8.2) consists of an isometry $J : V \to W$ and such a tuple with

$$E_i = J^\dagger P_i J.$$

Over $\mathbb{C}$ this is the finite-dimensional Naimark construction [6]. For the explicit minimal dilation of a qubit POVM, see Pellonpää et al. [17, Section 3]. The rank formula extends to every real closed fixed field.

Theorem 8.1. *Let $K = R(u)$ for a real closed field R, and let $E_1, \dots, E_m$ satisfy (8.2) on an n-dimensional positive definite space. Set $r_i = \operatorname{rank} E_i$. The least possible dimension of a positive definite dilation space is*

$$D_{\min} = \sum_{i=1}^{m} r_i. \tag{8.3}$$

Such a dilation exists in every dimension $d \geq D_{\min}$.

Proof. The positive definite spectral theorem holds over K by Theorem 7.3. Each E_i therefore has an orthonormal eigenbasis and nonnegative real eigenvalues. Every positive eigenvalue has a square root in R. Consequently there is a surjective linear map

$$A_i : V \longrightarrow K^{r_i}, \qquad A_i^\dagger A_i = E_i.$$

In an eigenbasis for E_i, take the coordinates with positive eigenvalues and multiply each by the positive square root of its eigenvalue. When $r_i = 0$, the target is the zero space. Let

$$W = \bigoplus_{i=1}^{m} K^{r_i}, \qquad Jv = (A_1 v, \dots, A_m v).$$

Since $J^\dagger J = \sum_i E_i = I$, the map J is an isometry. Let P_i be the coordinate projection onto the i-th summand. Then $J^\dagger P_i J = A_i^\dagger A_i = E_i$.

Conversely, let $J : V \to W$ and P_i be any dilation. Set $A_i = P_i J$. Positivity of the inner product on W implies

$$\ker(A_i^\dagger A_i) = \ker A_i,$$

because $h(v, A_i^\dagger A_i v) = h(A_i v, A_i v)$. Thus $\dim(P_i J(V)) = \operatorname{rank} A_i = \operatorname{rank} E_i = r_i$. The subspaces $P_i J(V)$ are mutually orthogonal, so $\dim W \geq \sum_i r_i$. This proves minimality. To obtain a larger dimension, adjoin an orthogonal summand to W, extend J by zero into that summand, and assign the new summand to one projection. □

Example 8.2. On K^2, let

$$E_1 = \operatorname{diag}(1, 9/25), \qquad E_2 = \operatorname{diag}(0, 16/25).$$

Their ranks are 2 and 1. A minimal dilation on K^3 is

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 3/5 \\ 0 & 4/5 \end{pmatrix}, \qquad P_1 = \operatorname{diag}(1, 1, 0), \qquad P_2 = \operatorname{diag}(0, 0, 1).$$

The equalities $J^\dagger J = I_2$ and $J^\dagger P_i J = E_i$ hold over every real closed fixed field. Theorem 8.1 shows that dimension 3 is necessary.

Corollary 8.3 (Finite-measurement transfer)**.** *Fix the system dimension n, the number m of outcomes, and the dilation dimension d. A parameter-free first-order sentence about rays and their transition probabilities, density operators, m-outcome POVMs, outcome probabilities, or dilations into dimension d has the same truth value over every $R(u)$ as over $\mathbb{C}$, with the preceding definitions and the standard positive definite forms. This truth value is decidable.*

Proof. All definitions on the n-dimensional space are formulas in $L_{\mathrm{Q},n}$. Since d is fixed, the additional d-dimensional vectors and rectangular maps are encoded by fixed-length scalar tuples. Their Gram, isometry, and projection conditions are polynomial equalities together with the definable order conditions. Theorem 3.1 reduces the resulting sentence to a parameter-free sentence of RCF. Completeness and decidability of RCF, or equivalently Corollary 5.2, give the claim. □

For a particular measurement, the existence of a dilation of dimension d is the finite rank condition $\sum_i \operatorname{rank} E_i \le d$. For fixed n, m, d, this condition is a finite disjunction over the rank tuples satisfying the bound.

A Explicit formulas and finite calculations

A.1 Linear algebra without named bases

For a fixed nonnegative integer r, membership in a finitely generated span is

$$x \in \operatorname{span}(v_1, \dots, v_r) \quad \Longleftrightarrow \quad \exists a_1 \cdots a_r \; x = \sum_i a_i v_i.$$

Linear dependence is

$$\exists a_1 \cdots a_r \left(\bigvee_i a_i \neq 0 \; \wedge \; \sum_i a_i v_i = 0 \right),$$

and independence is its negation. An n-tuple is a basis if and only if its Gram matrix for the nondegenerate ambient form h is invertible. The determinant is a fixed polynomial, so this gives a quantifier-free basis predicate in $L_{\mathrm{Q},n}$. Alternatively, the dual-basis criterion of Section 2 gives an existential formula.

For an operator A, rank $r > 0$ is specified by independent image vectors $Au_1, \dots, Au_r$ spanning every Av. Rank zero is $A = 0_M$. Ranks can also be expressed relative to an existentially chosen basis by the vanishing of all $(r+1)$-minors and the nonvanishing of some r-minor. Nilpotence is $A^n = 0_M$, since the dimension is n. Self-adjointness is the relation

$$\forall x \, \forall y \; h(Ax, y) = h(x, Ay).$$

Equivalently, one can choose a basis $e_1, \dots, e_n$ and check

$$h(Ae_i, e_j) = h(e_i, Ae_j) \qquad (1 \le i, j \le n).$$

Sesquilinearity extends these finitely many identities to all vectors.

Consequently each nilpotent orbit formula in (4.16) has an existential version. For every power form, existentially choose its diagonalizing basis and use the Gram determinant for the basis condition. On one of these bases impose the displayed self-adjointness identities. The condition $A^n = 0_M$ is already atomic.

A.2 Quotient forms on representatives

For a definable subspace $C \leq V$, the cosets $[v_1], \ldots, [v_s]$ are linearly independent in V/C when

$$\forall a_1 \cdots a_s \ \left(\sum_i a_i v_i \in C \ \longrightarrow \ \bigwedge_i a_i = 0 \right).$$

They span the quotient when

$$\forall v \ \exists a_1 \cdots a_s \ \left(v - \sum_i a_i v_i \in C \right).$$

A form on V/C can therefore be specified and diagonalized by formulas on representatives, provided it vanishes whenever either argument lies in C. No additional quotient sort is necessary. In Section 6, $C = \ker N_1 = \ker N_2$. The positive Gram identity $h(f_i, N_1 f_j) = \delta_{ij}$ already gives independence, and the known quotient dimension gives spanning.

A.3 Reproducible finite calculations

Table 1 lists the counts through dimension 8. The program `anc/verify_inertia.py` enumerates all 3,131 signed partitions in dimensions 1 through 12. For each signed block it computes the inertias of its power forms by exact rational congruence elimination. It sums these inertias over each partition and checks that (4.11) recovers the original multiplicities. A separate calculation extracts coefficients of (5.2) and agrees with the enumeration at every signature in this range.

The script also checks the matrix examples, including the pencil at selected rational parameters and the three-dimensional dilation. Run it with Python 3 to regenerate the table and the counts by signature.

n	$b_{p,n-p}$ for $0 \leq p \leq \lfloor n/2 \rfloor$	$\sum_{p=0}^{n} b_{p,n-p}$
1	1	2
2	1, 3	5
3	1, 4	10
4	1, 4, 10	20
5	1, 4, 13	36
6	1, 4, 14, 27	65
7	1, 4, 14, 36	110
8	1, 4, 14, 39, 69	185

Table 1: Numbers of complete theories with one named self-adjoint nilpotent operator. The omitted signatures follow from $b_{p,q} = b_{q,p}$.

References

[1] Camilo Enrique Argoty Pulido. Model theory of a Hilbert space expanded with an unbounded closed selfadjoint operator. *Mathematical Logic Quarterly*, 60(6):403–424, 2014. doi:10.1002/malq.201110043.

[2] Emil Artin and Otto Schreier. Algebraische Konstruktion reeller Körper. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 5(1):85–99, 1927. doi:10.1007/BF02952512.

[3] Emil Artin and Otto Schreier. Eine Kennzeichnung der reell abgeschlossenen Körper. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 5(1):225–231, 1927. doi:10.1007/BF02952522.

[4] Garrett Birkhoff and John von Neumann. The logic of quantum mechanics. *Annals of Mathematics*, 37(4):823–843, 1936. doi:10.2307/1968621.

[5] Olivier Brunet. A formal theory for finite-dimensional possibilistic quantum mechanics, 2026. URL https://arxiv.org/abs/2602.16368. Preprint.

[6] Paul Busch, Pekka Lahti, Juha-Pekka Pellonpää, and Kari Ylinen. *Quantum Measurement*. Theoretical and Mathematical Physics. Springer, Cham, 2016. ISBN 978-3-319-43387-5. doi:10.1007/978-3-319-43389-9.

[7] Gianni Cassinelli and Pekka Lahti. Quantum mechanics: why complex Hilbert space? *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 375(2106):20160393, 2017. doi:10.1098/rsta.2016.0393.

[8] Clément de Seguins Pazzis. Sums of two square-zero selfadjoint or skew-selfadjoint endomorphisms. *Linear Algebra and its Applications*, 693:140–213, 2024. doi:10.1016/j.laa.2023.06.020.

[9] J. Michael Dunn, Tobias J. Hagge, Lawrence S. Moss, and Zhenghan Wang. Quantum logic as motivated by quantum computing. *The Journal of Symbolic Logic*, 70(2):353–359, 2005. doi:10.2178/jsl/1120224716.

[10] J. Michael Dunn, Lawrence S. Moss, and Zhenghan Wang. Editors' introduction: The third life of quantum logic: Quantum logic inspired by quantum computing. *Journal of Philosophical Logic*, 42:443–459, 2013. doi:10.1007/s10992-013-9273-7.

[11] Israel Gohberg, Peter Lancaster, and Leiba Rodman. *Indefinite Linear Algebra and Applications*. Birkhäuser, Basel, 2005. ISBN 978-3-7643-7349-8. doi:10.1007/b137517.

[12] Stanley P. Gudder and Constantin Piron. Observables and the field in quantum mechanics. *Journal of Mathematical Physics*, 12(8):1583–1588, 1971. doi:10.1063/1.1665777.

[13] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, second edition, 2013. doi:10.1017/CBO9781139020411.

[14] David Marker. *Model Theory: An Introduction*, volume 217 of *Graduate Texts in Mathematics*. Springer, New York, 2002. doi:10.1007/b98860.

[15] Christian Mehl. On classification of normal matrices in indefinite inner product spaces. *Electronic Journal of Linear Algebra*, 15:50–83, 2006. doi:10.13001/1081-3810.1220. URL https://journals.uwyo.edu/index.php/ela/article/view/365.

[16] Christian Mehl. On the existence of Schur-like forms for matrices with symmetry structures. *Vietnam Journal of Mathematics*, 48:831–845, 2020. doi:10.1007/s10013-020-00394-3.

[17] Juha-Pekka Pellonpää, Sébastien Designolle, and Roope Uola. Naimark dilations of qubit POVMs and joint measurements. *Journal of Physics A: Mathematical and Theoretical*, 56 (15):155303, 2023. doi:10.1088/1751-8121/acc21c.

[18] Winfried Scharlau. *Quadratic and Hermitian Forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 1985. doi:10.1007/978-3-642-69971-8.

[19] Benjamin Schumacher and Michael D. Westmoreland. Modal quantum theory. *Foundations of Physics*, 42(7):918–925, 2012. doi:10.1007/s10701-012-9650-z.

[20] Vladimir V. Sergeichuk. Classification problems for systems of forms and linear mappings. *Mathematics of the USSR-Izvestiya*, 31(3):481–501, 1988. doi:10.1070/IM1988v031n03ABEH001086.

[21] Koen Thas. General quantum theory—unification of classical and modal quantum theories. *Journal of Physics A: Mathematical and Theoretical*, 53(39):395304, 2020. doi:10.1088/1751-8121/aba307.

[22] Koen Thas. Quantum theory without the axiom of choice, and Lefschetz quantum physics. *Physics*, 5(4):1109–1125, 2023. doi:10.3390/physics5040072.

[23] Lou van den Dries. *Tame Topology and o-Minimal Structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998. doi:10.1017/CBO9780511525919.

[24] Yingte Xu, Gilles Barthe, and Li Zhou. Automating equational proofs in Dirac notation. *Proceedings of the ACM on Programming Languages*, 9(POPL):1227–1259, January 2025. Article 42. doi:10.1145/3704878.