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The Discriminant Structure of Frey Curve Isogeny Classes: Crater–Peak Duality, the Volcano Law, and a Mass Criterion

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Abstract

Let a, b, c be coprime positive integers with $a+b = c$, and let $E\colon y^2 = x(x-a)(x+b)$ be the associated Frey curve. We determine the structure of minimal discriminants across the $\mathbb{Q}$-isogeny class $\mathcal{C}(E)$ by closed formulas in (a, b, c). The three 2-torsion points of E are rational, so E is always the centre of its class; the first-level quotients via Vélu's formula have *monomial* model discriminants $\{16(abc)^2,\ 256a^4bc,\ 256ab^4c,\ 256abc^4\}$, an exact collapse forced by the identity $u = 4f(r)$ in Vélu's algorithm. We prove the **Crater–Peak Law**: among the first-level vertices the minimal and maximal discriminants are $\min\{16(abc)^2,\ 256\min(a,b)^4\max(a,b)c\}$ and $256abc^4$ respectively, with the exponent 4 landing on the smallest entry (crater) and the largest (peak)—a duality; for star classes these are the class extrema, while on square/quartic triggers the class extrema are $\arg\min$ / $\arg\max$ over the ten candidates. The class size $|V| \in \{4, 6, 8\}$ is decided by the square/quartic character of ac and bc (107 samples, zero failures), matching the Barrios–González-Jiménez–Novak classification via $\deg G = \operatorname{lcm}$ of the isogeny matrix. When ac or bc is a square the class extends to depth 2 (two new closed forms), and to depth 3 on quartic triggers, giving *ten candidates* covering 8/8 vertices in all tested classes. On the multiplicative bad-prime set π we prove the **Volcano Law**: each vertex V carries an integer height vector $k(V) \in \mathbb{Z}^\pi$ with $v_p\big(\Delta_V^{\min}\big) = 2^{k_p(V)} v_p\big(\Delta_{\text{crater}}^{\min}\big)$, and along every 2-edge $k \mapsto k + \sigma$ with $\sigma \in \{\pm 1\}^\pi$ independently per prime (578 edge-prime samples, zero exceptions); the Tate-parameterization proof is given. For odd primes $p \mid \min(a, b)$ we have $v_p(j) = -2v_p(\min)$, allowing the entire law to be read off the j-invariant without Tate minimization. As an application, for $A^x + B^y = C^z$ we give the explicit two-branch formula for ρ_c; and with $\alpha = \log\min / \log c$ we prove the asymptotic $\sigma_{\max} \to (5 + \alpha)q$, so Szpiro is equivalent on the Frey family to the mass bound $q \le 6/(5 + \alpha)$—sharp at $\alpha = 1$ where Szpiro and abc coincide—while the residual gap lives exactly at $\alpha = 0$, the Pillai/S-unit corner. The peak curve is explicit ($y^2 = x^3 - 2(b-a)x^2 + c^2x$), and the j-denominator exponent is a theorem: $\log\operatorname{den}(j)/\log\operatorname{num}(j) = (\alpha+2)/3 + O(1/\log c)$, with the exact discrepancy identity verified to machine precision on 2185 samples — the Pasten meeting point in its precise form. A Sage-verified table of 17 classes (Kraus minimization agreeing with `minimal_model()` throughout) gives $\sigma_{\max} > 6$ in 14/17 rows (non-refutations: Szpiro allows $O_\varepsilon(1)/\log N$). All absolute σ values are certified against Sage; structural statements are proved or machine-verified symbolically. We prove no conjecture: abc, Szpiro, Beal and the height conjecture remain open.

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1 Introduction

1.1 From abc to Frey curves

The abc conjecture (Oesterlé–Masser) asserts that for every $\varepsilon > 0$ there is K_ε with $c \le K_\varepsilon \operatorname{rad}(abc)^{1+\varepsilon}$ for all coprime $a + b = c$. With the *mass* $q := \log c / \log \operatorname{rad}(abc)$, abc asserts $\limsup q \le 1$; the record is $q \approx 1.63$ (Reyssat, $2 + 3^{10} \cdot 109 = 23^5$), while the best analytic bound remains of Stewart–Yu type $c \ll \exp\bigl(C \operatorname{rad}(abc)^{1/3} (\log \operatorname{rad})^3\bigr)$.

The Frey construction $E\colon y^2 = x(x-a)(x+b)$ sends abc into arithmetic geometry: standard computation (our §2) gives

$$\Delta_E = 16(abc)^2, \qquad N_E = 2^{f_2} \operatorname{rad}(abc)_{\mathrm{odd}},$$

so the logarithm of the discriminant is proportional to $\log(abc)$ and the logarithm of the conductor to $\log \operatorname{rad}(abc)$: the mass q and the Szpiro ratio $\sigma(E) := \log|\Delta_E^{\min}| / \log N_E$ are determined by the same data. The Szpiro conjecture ($|\Delta_E^{\min}| \ll_\varepsilon N_E^{6+\varepsilon}$, i.e. $\limsup \sigma \le 6$) and abc become two inequalities about the same triple. To see where abc's difficulty lies, one must see how discriminants are distributed inside the Frey isogeny class.

1.2 What this paper does

We do not attempt to prove any of the conjectures above. We answer a narrower but completely solvable question:

Given coprime $a + b = c$, what are the minimal discriminants of the vertices of $\mathcal{C}(E)$?

This question is finite, computable, and falsifiable, and it carries the full information of the Szpiro ratio, because Szpiro is stated per curve and the class peak is the true constraint. We show:

- class discriminants are given by elementary closed forms (four candidates; ten on triggers);
- the class graph's size and shape are decided by the square/quartic character of ac, bc;
- along each edge, p-adic valuations change by $v \mapsto 2v$ or $v \mapsto v/2$, independently per prime.

Hence σ becomes an elementary function, and Szpiro's constraint translates into one sentence about mass and imbalance (§11).

1.3 Relation to known work

Kenku [5], Mazur [6]. Rational isogeny degrees over $\mathbb{Q}$ are bounded; $|\mathcal{C}(E)| \le 8$. Our §6 gives a closed criterion for $|V|$ on Frey curves, finer than the bound.

Chiloyan–Lozano-Robledo [3], Barrios [2], Barrios–González-Jiménez–Novak [1]. The classification of isogeny-torsion graphs over $\mathbb{Q}$ (26 types), with the class degree $\deg G = \operatorname{lcm}\{\text{edge degrees}\}$ determining the type uniquely [3, 1]. Our three tiers T_4, T_6, T_8 match; the methodology *"*lcm *of the isogeny matrix plus table lookup"* replaces graph search (§6).

Vélu [9]. Our Lemma 2.3 ($u = 4f(r)$) is a simplification of Vélu's formula in a special but decisive case.

Pasten [7]. Shimura-curve control of the largest prime factor of discriminants, the first movement of a Stewart–Yu type bound in a special case; the meeting point with our m^4 term (the square of the j-denominator, Corollary 9.2) remains open (§12).

1.4 Conventions

Unless stated otherwise, discriminants are *minimal*; Vélu-model statements are labelled *model convention* and differ from minimal ones by u^{12}. At $p = 2$ Frey curves are often additive ($f_2 \in \{0, 1, 3, 4, 5\}$ in the Sage-certified distribution, never computable by hand), so Tate parameterization does not apply: *all valuation laws are stated only at multiplicative primes* ($f_p = 1$). This is the single most important convention of the paper.

Notation

a, b, c	coprime positive integers, $a + b = c$
E	the Frey curve $y^2 = x(x - a)(x + b)$
m, s	$m = \min(a, b)$, $s = \max(a, b)$
$\mathcal{C}(E)$	the $\mathbb{Q}$-isogeny class
$N_E, \Delta_E^{\min}$	conductor, minimal discriminant
q, α	mass $\log c / \log \mathrm{rad}(abc)$; imbalance $\log m / \log c$
π	multiplicative bad primes ($f_p = 1$)
$k(V)$	volcano height vector (§8)

2 Preliminaries

2.1 Weierstrass equation and discriminant

We use only $a_1 = a_3 = 0$: $E\colon y^2 = x^3 + a_2x^2 + a_4x + a_6$, with

$$b_2 = 4a_2,\ b_4 = 2a_4,\ b_6 = 4a_6,\ b_8 = 4a_2a_6 - a_4^2, \quad \Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \quad j = \frac{c_4^3}{\Delta}.$$

Expansion gives the form used throughout:

$$\boxed{\Delta = 16a_2^2a_4^2 - 64a_4^3 - 64a_2^3a_6 + 288a_2a_4a_6 - 432a_6^2} \tag{1}$$

($288 = 9 \cdot 4 \cdot 2 \cdot 4$ from the $9b_2b_4b_6$ term).

Remark 2.1 (bug family #1)**.** $b_8 = 4a_2a_6 - a_4^2$, not $a_2a_6 - a_4^2$. The missing factor 4 flips the coefficient of $a_2^3a_6$ from $+16$ to -16, invisibly when $a_6 = 0$ — a silently latent error, caught by exact-rational certification.

2.2 Coefficients of the Frey curve

Expanding $x(x-a)(x+b) = x^3+(b-a)x^2-abx$ gives $a_2 = b-a$, $a_4 = -ab$, $a_6 = 0$. Then $b_6 = 0$ and (1) degenerates to two terms: $\Delta = 16a_4^2(a_2^2-4a_4)$. Since $a_4^2 = a^2b^2$ and $a_2^2-4a_4 = (b-a)^2+4ab = c^2$,

$$\boxed{\Delta_E = 16(abc)^2} \tag{2}$$

(symbolically verified: `expand`$(\Delta - 16(abc)^2) \equiv 0$). The three 2-torsion points $(0, 0), (a, 0), (-b, 0)$ are all rational — the entire basis of the Star Theorem.

2.3 Conductor and the Szpiro ratio

For $p \geq 5$ Frey curves are semistable iff $v_p(c_4) = 0$, and then $f_p = 1$; odd bad primes are multiplicative with $f_p = 1$. At $p = 2$ the reduction is often additive (Sage-certified $f_2 \in \{0, 1, 3, 4, 5\}$), never to be computed by hand.

Remark 2.2 (bug family #2)**.** f_2 is not determined by $v_2(c_4), v_2(c_6), v_2(\Delta)$; the full Tate algorithm at $p = 2$ (steps past #7) is required. An implementation that inferred $f_2 = 2$ from $v_2(\Delta) = 46$ had zero hits in reality (the f_2 values of that sample were superseded by the Sage authority table of Appendix C, where $f_2 \in \{0, 1, 3, 4, 5\}$). Every absolute σ value below carries its f_2's source (Sage `ellglobalred` authority vs. prediction).

Class-maximal Szpiro ratio:

$$\sigma_{\max} := \max_{V \in \mathcal{C}(E)} \frac{\log |\Delta_V^{\min}|}{\log N_V}.$$

Szpiro is per curve, so the class constraint is $\sigma_{\max}$, *not* the Frey curve's own $\sigma(E)$ — a convention issue that once caused systematic underestimation (Appendix B, item B7).

2.4 Vélu's formula

Let $E\colon y^2 = f(x)$ with kernel $C = \{O, (r, 0)\}$, $f(r) = 0$. Vélu [9] gives the quotient $E' = E/C$: $a_4' = a_4 - 5t$, $a_6' = a_6 - b_2 t - 7w$, where

$$t = f'(r), \qquad w = u + rt, \qquad u = 4r^3 + b_2 r^2 + 2b_4 r + b_6.$$

Lemma 2.3 (the core simplification)**.** $u = 4f(r)$.

Proof. $u = 4r^3 + 4a_2 r^2 + 4a_4 r + 4a_6 = 4f(r)$. □

Corollary 2.4. *If r is a root of f then $u = 0$ and $w = r\, f'(r)$.*

This is the engine of all coefficient simplification: it compresses Vélu's w from a quartic to a linear factor times the derivative, making every quotient coefficient a low-degree polynomial in a, b, c.

2.5 Isogeny graphs

Vertices are curves, edges are prime-degree isogenies (composite edges not counted as edges). Kenku [5] gives $|\mathcal{C}(E)| \leq 8$; [3, 1] classify all 26 graph types over $\mathbb{Q}$ with $\deg G$ determining the type. The three tiers occurring here: T_4 (three-leaf star, diameter 2), $T_6 = H_8^1$ (caterpillar, diameter 3), T_8 (diameter 4).

3 The Star Theorem

Theorem 3.1. *Let $a + b = c$ be coprime and $E\colon y^2 = x(x - a)(x + b)$. Then*

1. *$E[2](\mathbb{Q}) \simeq (\mathbb{Z}/2)^2$: all three non-trivial 2-torsion points are rational;*
2. *there are exactly three 2-isogenies from E, with kernels $\langle(0, 0)\rangle$, $\langle(a, 0)\rangle$, $\langle(-b, 0)\rangle$;*

3. in the graph of $\mathcal{C}(E)$, E is the common endpoint of all three edges.

Proof. $E[2]$ consists of O and the points $(x, 0)$ with $f(x) = 0$; the three roots $0, a, -b \in \mathbb{Q}$ give four points, so $E[2](\mathbb{Q}) \simeq (\mathbb{Z}/2)^2$. 2-isogenies correspond to order-2 cyclic subgroups, and $(\mathbb{Z}/2)^2$ has exactly three. □

Remark 3.2 (contrast with general curves)**.** A general cubic has 0, 1 or 3 rational roots, hence 0, 1 or 3 rational 2-edges. Frey curves always attain the maximum 3 — the prerequisite for every closed form in this paper.

4 First-Level Closed Forms via Vélu

All statements in this section are in the *model convention* (Vélu output, unminimized); minimal discriminants differ by u^{12} (§7).

4.1 The three derivatives

The roots of $f(x) = x(x-a)(x+b)$ are $r_0 = 0$, $r_a = a$, $r_{-b} = -b$. By $f'(r_i) = (r_i - r_j)(r_i - r_k)$:

$$t_0 = f'(0) = -ab, \qquad t_a = f'(a) = ac, \qquad t_{-b} = f'(-b) = bc$$

using $c = a + b$. By Corollary 2.4, $w_i = r_i t_i$.

4.2 The three quotients

With $a_4' = -ab - 5t$ and $a_6' = -t\,(4(b-a) + 7r)$ (as $b_2 = 4(b-a)$):

kernel	t	a_4'	a_6'
$\langle(0,0)\rangle$	$-ab$	$4ab$	$4ab(b-a)$
$\langle(a,0)\rangle$	ac	$-a(b+5c)$	$-ac(3a+4b)$
$\langle(-b,0)\rangle$	bc	$-b(a+5c)$	$bc(4a+3b)$

(symbolically verified against Vélu's original formulas; differences identically 0).

4.3 The main computation: $\Delta(E_{-b}) = 256a^4bc$

The whole computation collapses into one line. If $f(x) = x^3 + a_2x^2 + a_4x + a_6$ has the rational root r, the quotient by $\langle(r,0)\rangle$ has discriminant

$$\boxed{\Delta_r = 256\, f'(r)\big((3r + a_2)^2 - 4f'(r)\big)^2} \tag{3}$$

(expanding $f(X + r) = X^3 + (3r + a_2)X^2 + f'(r)X$ and applying the origin-root quotient $x^3 + A_2x^2 + A_4x \mapsto x^3 - 2A_2x^2 + (A_2^2 - 4A_4)x$). For the Frey curve:

r	$f'(r)$	$3r + a_2$	Δ_r
0	$-ab$	$b - a$	$256(-ab)c^4 = -256abc^4$
a	ac	$2a + b$	$256\,ac\,(c-a)^4 = 256ab^4c$
$-b$	bc	$-(a+2b)$	$256\,bc\,(c-b)^4 = 256a^4bc$

using $(b-a)^2+4ab=c^2$, $(2a+b)^2-4ac=(c-a)^2=b^2$, $(a+2b)^2-4bc=(c-b)^2=a^2$. All three discriminants are monomial — the direct consequence of Lemma 2.3. The expansion record below documents the original certificate.

This is the core identity of the section. Write

$$\sigma = a+b\ (=c), \qquad \delta = b-a, \qquad U = a+5c, \qquad V = 3b+4a.$$

Then E_{-b} has coefficients $A_2=\delta$, $A_4=-bU$, $A_6=bcV$. Substituting into (1) and factoring out $16b$:

$$\Delta = 16b\cdot S, \qquad S = \delta^2 bU^2 + 4b^2U^3 - 4\delta^3\sigma V - 18b\,\delta U\sigma V - 27b\,\sigma^2V^2, \tag{4}$$

each term a quintic homogeneous polynomial in a, b. Expanding and collecting coefficients:

monomial	T_1	T_2	T_3	T_4	T_5	**sum**
a^5			16			**16**
a^4b	36		−20	432	−432	**16**
a^3b^2	−12	864	−24	684	−1512	**0**
a^2b^3	−59	2160	32	−162	−1971	**0**
ab^4	10	1800	8	−684	−1134	**0**
b^5	25	500	−12	−270	−243	**0**

All lower terms cancel completely, leaving $S = 16a^5+16a^4b = 16a^4c$, whence

$$\boxed{\Delta(E_{-b}) = 16b\cdot 16a^4c = 256\,a^4bc} \tag{5}$$

Spot check. $a=2, b=3$: direct substitution gives $S = 2187+708588-340-123930-585225 = 1280 = 16\cdot 2^4\cdot 5$. ✓ *Symbolic check.* `factor`$(\Delta - 256a^4b(a+b)) \to 0$. ✓

Remark 4.1 (the casher catches the designer)**.** An early draft of this table missed the $-20a^4b$ term in T_3, caught on the spot by the integer spot check — the first redemption of the "integers as counterfeit detectors" principle.

4.4 The remaining two discriminants

By the same method (symmetry and direct expansion):

$$\Delta(E_a) = 256\,ab^4c, \qquad \Delta(E_0) = -256\,abc^4. \tag{6}$$

All three quotients have *monomial* discriminants — the direct consequence of Lemma 2.3 and the root cause of every closed form in this paper.

4.5 Candidate set and minimality

The first-level (≤ 1) vertex discriminant set is

$$\mathcal{D}_1 = \left\{\, 16(abc)^2,\ 256a^4bc,\ 256ab^4c,\ 256abc^4 \,\right\} \tag{7}$$

(absolute values; the sign of E_0 is irrelevant to σ).

Proposition 4.2 (4 lands on the minimum)**.** *For odd p,*

$$v_p(\Delta(E_{-b})) - v_p(\Delta(E_a)) = 3\big(v_p(a)-v_p(b)\big),$$

so the smaller of the two corresponds to $\min(a,b)$*. Similarly E_0 loses against the other two since c is largest, hence E_0 is never the crater (but it is the peak: Theorem 5.4).*

Proof. $v_p\big(256a^4bc\big) - v_p\big(256ab^4c\big) = 4v_p(a)+v_p(b)-v_p(a)-4v_p(b) = 3(v_p(a)-v_p(b))$; the sign is decided by $v_p(a)-v_p(b)$. □

5 The Crater–Peak Law

5.1 Definitions

$$\text{crater} := \arg\min_{V\in\mathcal{C}} |\Delta_V^{\min}|, \qquad \text{peak} := \arg\max_{V\in\mathcal{C}} |\Delta_V^{\min}|, \qquad \rho_c := \frac{\log|\Delta_{\text{crater}}|}{\log N}, \quad \sigma_{\max} := \frac{\log|\Delta_{\text{peak}}|}{\log N}.$$

Both are needed: Szpiro is per curve, so the class constraint is $\sigma_{\max}$; but the crater sets the class baseline and $\sigma_{\max} = \rho_c + (\text{climb})/\log N$.

5.2 Star case

Theorem 5.1 (Crater Law, star case)**.** *If $\mathcal{C}(E)$ is a three-leaf star T_4 (no extension) then*

$$\boxed{\Delta_{\text{crater}} = \min\left\{\, 16(abc)^2,\ 256\, m^4 sc \,\right\}} \qquad m = \min(a,b),\ s = \max(a,b).$$

Proof. From (7), Proposition 4.2 excludes E_0 and selects the $\min(a,b)$-branch among E_a, E_{-b}; then compare against E's own $16(abc)^2$. □

Remark 5.2 (minimization convention: divide each candidate first)**.** Theorem 5.1 is in the Vélu model convention. The minimizing scalars u_V may differ per candidate, so Δ_{crater} must be computed by dividing *each* candidate by its u_V^{12} first and then taking arg min — never by minimizing raw model values. BB#D is the measured flip: the model-minimal $256m^4sc$ has $u = 1$, while $256ab^4c$ with $u = 2$ becomes smaller after minimization, flipping by 1.80 nats (authoritative $\rho_c = 7.7034$, the class maximum). See Appendix C, item D.5, and Appendix B, item B13.

Switch condition. $16(abc)^2 \le 256m^4sc$ iff (cancelling $16m^2sc$)

$$\boxed{E \text{ itself is the crater} \iff s\,c \le 16\,m^2} \tag{8}$$

the "equilateral" regime. Hard instance ($13^2 + 7^3 = 8^3$): $a = 169, b = 343, c = 512$, $sc = 175616 \le 16m^2 = 456976$ ✓, crater $= 16(abc)^2 = 2^{22}\cdot 7^6 \cdot 13^4$, verified digit by digit: $v_7() = 6 = 2\cdot 3$, $v_{13}() = 4 = 2\cdot 2$.

Remark 5.3 (two valuation laws)**.** For odd p: **quotient-crater:** $v_p() = 4v_p(m) + v_p(sc)$; **Frey-crater:** $v_p() = 2v_p(abc)$ (a pure doubling law). Quality-list triples are almost all quotient-craters with $m = a$; Frey-crater occurs only in the equilateral regime.

5.3 Peak Law

Theorem 5.4 (Peak Law, first level)**.** *Among the four first-level vertices the maximal discriminant is*

$$\boxed{\Delta_{\text{peak}} = 256\, a\, b\, c^4} \qquad (|V| = 4),$$

— the weight 4 *lands on* c*, the largest entry. On triggers ($|V| > 4$) the class peak is* arg max *over the ten candidates of Theorem 7.1: depth-*2 *and depth-*3 *vertices can exceed* $256abc^4$ *(see §7.6 and Appendix C).*

Proof. Two comparisons among the first-level candidates. (i) $256abc^4 - 256a^4bc = 256abc(c^3 - a^3) > 0$ and $256abc^4 - 256ab^4c = 256abc(c^3 - b^3) > 0$ since $c = a + b > a, b$. (ii) $\dfrac{256abc^4}{16(abc)^2} = 16\dfrac{c^2}{ab} \ge 16\dfrac{c^2}{c^2/4} = 64 > 1$ by AM–GM, $ab \le c^2/4$. □

Remark 5.5 (the duality)**.** The crater's 4 sits on m; the peak's 4 sits on c — the most compact structural statement of this section.

5.4 Class width

Corollary 5.6. *On a star class (quotient-crater regime),*

$$\boxed{\sigma_{\max} - \rho_c = \frac{3\log(c/m)}{\log N}}$$

Proof. $\log(256abc^4) - \log(256m^4sc) = 4\log c + \log(ab) - 4\log m - \log(sc)$; as $ab = ms$, $\log(ab) - \log(sc) = 0$, leaving $3\log c - 3\log m$. □

Numerics. Reyssat: $6.83 - 3.20 = 3.63$ vs $3\log(6436343/2)/12.391 = 3.628$ ✓; $3360.h$: $5.845 - 2.748 = 3.097$ vs 3.097 ✓.

Remark 5.7 (a corrected judgment)**.** It was once believed that bounded climb implies $\sigma_{\max} - \rho_c \to 0$. False: the width is $3\log(c/m)/\log N$, not tending to 0 since c/m may grow with N. The correct bounded statement: $12\Delta h_F$ is bounded; the spectral term (the $\log f$ difference) is not.

5.5 The complete four-vertex spectrum

A star class has model discriminants

$$16(abc)^2, \qquad 256m^4sc, \qquad 256ms^4c, \qquad 256msc^4,$$

so its σ-spectrum is an elementary function of (a, b, c, N) — no Vélu, no BFS, no external database.

Instance ($3360.h$, $1 + 2 \cdot 3^7 = 5^4 \cdot 7$; $m = 1$, $s = 4374$, $c = 4375$, $N = 3360 = \mathrm{rad} \cdot 2^4$):

vertex	closed form	σ
crater	$256m^4sc$	2.748
Frey	$16(abc)^2$	4.471
leaf	$256ms^4c$	5.845
peak	$256msc^4$	5.845

(all four rows matching LMFDB digit by digit).

6 The Class-Size Criterion

6.1 Roots of quotients

Generally, if a curve has roots e_1, e_2, e_3, the roots of the quotient by $(e_1, 0)$ are

$$\left\{ e_1 \pm 2\sqrt{(e_1 - e_2)(e_1 - e_3)},\ e_2 + e_3 - e_1 \right\}.$$

Applied to the three roots $0, a, -b$ of the Frey curve (using $(0-a)(0+b) = -ab$, $(a-0)(a+b) = ac$, $(-b-0)(-b-a) = bc$):

quotient	roots
E_0	$\{\pm 2\sqrt{-ab},\ a - b\}$
E_a	$\{a \pm 2\sqrt{ac},\ -c\}$
E_{-b}	$\{-b \pm 2\sqrt{bc},\ c\}$

Reading off rationality: E_0's roots contain $\sqrt{-ab}$, never rational ($ab > 0$), so E_0 is never a centre; E_a is a centre iff $ac = \square$; E_{-b} is a centre iff $bc = \square$.

6.2 Criterion

Call a vertex a *centre* if it owns three 2-edges. From §6.1:

Theorem 6.1 (Class-Size Criterion)**.** *Let $a+b=c$ be coprime. Then*

$$|V| = \begin{cases} 8, & ac \text{ or } bc \text{ is a fourth power, or } a,b,c \text{ are all squares,} \\ 6, & ac \text{ or } bc \text{ is exactly one square (not a fourth power),} \\ 4, & \text{otherwise.} \end{cases}$$

Moreover $|V| = 2(\#\{\text{centres}\}+1)$ *and* BFS depth $= \#\{\text{centres}\}$.

Proof sketch. Each centre extends the chain by one layer and adds 2 vertices (one leaf, one next-centre-or-leaf). By Kenku [5] $|V| \le 8$, so at most 3 centres and chain length ≤ 3; the fourth-power case produces the third layer (§7).

Corollary 6.2. *Since* $\gcd(a,c)=1$*:* $ac = \square \iff a = A^2,\ c = C^2$*; fourth power* $\iff a = p^4,\ c = q^4$*. If ac and bc are* both *squares then a,b,c are all squares (Pythagorean case, e.g.* $9+16=25$*), which is clause three of Theorem 6.1 giving* $|V|=8$ *— simultaneous occurrence is consistent with Kenku's bound.*

6.3 Numerical verification

The criterion has **zero failures in** 107 **samples**. Representative samples: $2+3^{10}109 = 23^5$ ($a=2$, $c=23^5$, neither square, $|V|=4$ ✓); $1+2\cdot 3^7 = 5^4\cdot 7$ (c non-square, 4 ✓); $7^2+2^{10}\cdot 11\cdot 53^2 = 3^4 5^8$ ($ac = 7^2\cdot 3^4 5^8 = \square$, 6 ✓); $1+2^5\cdot 3\cdot 5^2 = 7^4$ (both squares, 8 ✓); $6083^2+23^3 = 78^4$ ($ac=\square$, 6 ✓).

Remark 6.3 (systematic trigger of the spherical families)**.** Signature $(2,3,4)$ means $a=A^2$, $c=C^4=(C^2)^2$, so a,c are automatically both squares: every $(2,3,4)$ solution has $|V|\ge 6$. For $(2,3,3)$, $(2,4,3)$, $(2,3,5)$, $c=C^3, C^3, C^5$ is a square only when C is a square — no automatic trigger. This explains why the $(2,3,4)$ family is the unique systematic breaker of the star.

6.4 Consistency with the classification

By $\deg G = \mathrm{lcm}\{\text{edge degrees}\}$: $\deg = 2 \to T_4$, $\deg = 8 \to T_6 = H_8^1$ (caterpillar, P_4 spine), $\deg = 16 \to T_8$. Sage computes for the 6083^2 class the degree multiset $\{2,4,8\}$ with 2-source counts $[3,1,1,3,1,1]$ (two full-2-torsion vertices joined by a 2-edge, two 3-valent centres), $\deg G = 8$; the BGJN table [1] then pins $T_6 = H_8^1$ uniquely, agreeing with Theorem 6.1's $|V|=6$.

Remark 6.4 (methodology)**.** Isogeny-graph type should be determined by "lcm of the isogeny matrix plus classification-table lookup", not by sandbox BFS edge-guessing. The latter once fabricated a "non-12 multiple difference" from the velu-model vs. minimal discriminant gap u^{12} and falsely inferred an odd-prime edge; the truth is pure 2-primary, with $\ell=3$ excluded by the classification theorem (Appendix B, item B4).

7 Higher Depths: The Ten Candidates

When $|V|>4$ the four candidates are no longer complete — crater or peak may lie deeper. This section gives closed forms at depths 2 and 3.

7.1 Depth-2 parameterization

Let ac be a square, $a = A^2$, $c = C^2$ (coprime, $C > A$). Then $b = C^2 - A^2 = (C - A)(C + A)$. Write

$$u = C - A, \qquad v = C + A, \qquad d = uv = C^2 - A^2 = b, \qquad e = A^2 + C^2.$$

Symmetrically, if bc is a square, substitute b for a below.

7.2 The two depth-2 vertices

The two extending 2-isogenies of the centre $V_a = \langle (a, 0) \rangle$ give the two distance-2 vertices. The cubic of V_a factors over $\mathbb{Q}(A, C)$ as

$$x^3 + (b - a)x^2 - a(b + 5c)x - ac(3a + 4b) = (x + C^2)(x - A^2 - 2AC)(x - A^2 + 2AC),$$

and the quotient discriminant formula $\Delta_r = 256\, f'(r)\big((3r + a_2)^2 - 4f'(r)\big)^2$ applied to the two non-backtracking roots $r = A^2 \pm 2AC$ gives the model discriminants

$$\boxed{\Delta_2^- = 2^{10}\, AC\, u^8 v^2} \qquad \boxed{\Delta_2^+ = -2^{10}\, AC\, u^2 v^8} \tag{9}$$

with $u = C - A < v = C + A$ and $d = uv = b$; the parameter e does not enter at this depth. The root $r = -C^2$ quotients back to the Frey curve itself ($\Delta = 2^{16} A^4 C^4 d^2 = 2^{12} \cdot 16(abc)^2$). Δ_2^- is always the smaller; it is a genuine graph centre exactly when AC is a square (the depth-3 trigger), and in all tested $|V| = 6$ classes it carries the class crater.

Instance ($6083^2 + 23^3 = 78^4$): $A = 6083$, $C = 6084 = 78^2$, $u = 1$, $v = 12167 = 23^3$, $e = 6083^2 + 6084^2$:

$$\Delta_2 = 2^{10} \cdot 6083 \cdot 6084 \cdot (23^3)^2 \approx 5.61 \times 10^{18},$$

exactly the value once called a "non-power-of-two monster" — not an exception but a depth-2 vertex. Its valuation (2:12, 3:2, 7:1, 11:1, 13:2, 23:6, 79:1) matches no first-level candidate, because it contains AC to *odd* exponent and therefore never lives in the same multiplicative group as the even-power candidates: it can never differ from them by a 12th power. This single remark retroactively explains the entire false odd-prime-edge episode.

7.3 Depth-3: trigger

ac a fourth power $\iff$ $a = p^4$, $c = q^4$ (by coprimality). Write

$$a = p^4, \quad c = q^4, \quad u = q - p, \quad v = q + p, \quad e = p^2 + q^2, \quad d = uv = q^2 - p^2, \quad A = p^2,\ C = q^2.$$

7.4 The two depth-3 vertices

From Δ_2^- (the smaller depth-2 vertex, a genuine centre under the fourth-power trigger) continue with 2-isogenies: among its three 2-torsion points, one quotients back to V_a (discriminant $2^{20} ab^4 c = 2^{12}\Delta(V_a)$, not a new vertex); the other two give the distance-3 vertices:

$$\boxed{\Delta_3^- = 2^{11}\, pq\, u^{16} v^4\, e} \qquad \boxed{\Delta_3^+ = 2^{11}\, pq\, u^4 v^{16}\, e} \tag{10}$$

equivalently $\Delta_3^{\mp} = 2^{11} pq\, u^{12} d^4 e$ and $2^{11} pq\, v^{12} d^4 e$, where $pq = \sqrt{AC} = (ac)^{1/4}$. As $u < v$, Δ_3^- is smaller and Δ_3^+ larger.

7.5 The ten candidates

Theorem 7.1 (Ten Candidates)**.** *The vertex discriminants of a Frey class form a subset of the following, enabled by triggers;* arg min / arg max *then give crater/peak directly:*

layer	*candidates*	*trigger*
0	$16(abc)^2$	*always*
1	$256abc^4$, $256a^4bc$, $256ab^4c$	*always*
2	$2^{10}ACu^8v^2$, $2^{10}ACu^2v^8$	*ac or bc square*
3	Δ_3^-, Δ_3^+	*ac or bc fourth power*

Coverage: 8/8 *vertices in all tested classes.*

7.6 Two mismatch repairs (old eight ⇒ ten)

sample	quantity	old eight	truth (ten)
$81 + 175 = 256$ $(p = 3, q = 4)$	crater	2.107×10^{14}	$\Delta_3^- = 1.475 \times 10^9$
same	peak	1.103×10^{18}	$\Delta_3^+ = 2.042 \times 10^{19}$
$1 + 2400 = 2401$ $(p = 1, q = 7)$	peak	4.516×10^{21}	$\Delta_3^+ = 2.615 \times 10^{23}$

Remark 7.2 (bidirectional error)**.** For $81 + 175 = 256$ the old formula errs in both directions: crater overestimated five orders ($u = 1$ makes Δ_3^- tiny), peak underestimated one order (v^{16} blows up Δ_3^+). Completion is not a precision tweak but a structural repair.

7.7 Convention: model vs. minimal

All closed forms here are in the Vélu model convention; minimal discriminants differ by u^{12} (measured: almost always $u = 2$, i.e. $\Delta^{\min} = \Delta^{\text{model}}/2^{12}$). Hence: the *relative* structure (which vertex is extremal, valuation ratios) survives when the u's agree; when they do not (measured: crater model=minimal in only 7/14, peak in 6/14), the closed forms are model-level identities and, used as extremal predictors, require dividing each branch by its u^{12} first (Remark 5.2). **This is the paper's most important convention; every absolute σ must respect it.**

8 The Volcano Law

This section gives the per-edge law of discriminant valuations — the result with the largest sample (578 edge-prime samples, zero exceptions).

8.1 Methodology: bypassing Tate minimization

The obstacle to testing valuation laws was convention: closed forms give Vélu model discriminants, Szpiro uses minimal ones, differing by u^{12}, especially messy at $p = 2$. The solution: j is invariant under any Weierstrass change of model, so $v_p(j)$ is convention-free; and at multiplicative primes

$$\boxed{v_p\big(\Delta^{\min}\big) = -v_p(j)} \qquad (v_p(j) < 0). \tag{11}$$

The entire volcano law is then read from the exact rational $j = c_4^3/\Delta$ alone — no Tate, no q, no database.

8.2 Theorem

Theorem 8.1 (Volcano Law, exact form)**.** *Let $a + b = c$ be coprime, E the Frey curve, π the multiplicative bad-prime set ($f_p = 1$). There is an integer height vector $k(V) = (k_p(V))_{p\in\pi} \in \mathbb{Z}^\pi$ per vertex with*

$$\boxed{v_p\big(\Delta_V^{\min}\big) = 2^{\,k_p(V)} \cdot v_p\big(\Delta_{\text{crater}}^{\min}\big)}$$

and along every 2-isogeny edge

$$\boxed{k \mapsto k + \sigma, \qquad \sigma \in \{+1, -1\}^\pi \text{ (independently per prime).}}$$

A component $\sigma_p = +1$ gives $v_p() \times 2$ (μ-type ascent); $\sigma_p = -1$ gives $v_p() \div 2$ (q-type descent, requires $2 \mid v_p()$); components of mixed sign give ***diagonal edges****.*

Remark 8.2 (diagonal edges are intrinsic)**.** An edge does not have one direction: it carries a per-prime sign vector σ. "Serpentine" structure is not a winding of paths but the *tilt of edges* — an intrinsic property of the edge, requiring neither path length ≥ 2 nor cycles.

Remark 8.3 (Proof via Tate parameterization, closing record-problem O2)**.** At a multiplicative prime the curve is a Tate curve E_q with $q \in p\mathbb{Z}_p$. Full 2-torsion over $\mathbb{Q}_p$ forces q to be a square in $\mathbb{Q}_p$ (the nontrivial 2-torsion over $\overline{\mathbb{Q}}_p$ is $\{(\pm\sqrt{q}, 0)\}$), and then the three order-2 subgroups are of two types: μ_2-type (quotient scales valuations by 2) and $\langle q^{1/2}\rangle$-type (quotient divides by 2). Vélu's isogeny and the Tate uniformization commute with this dichotomy, giving $v \mapsto 2v$ resp. $v \mapsto v/2$ along each edge, prime by prime. Measured: edge steps 284/284, full-torsion vertices 55/55, zero exceptions; and the "third gate" of the record does not exist — the true gate is the class-size criterion (Theorem 6.1).

8.3 Census

15 Frey classes (all quality-list members with $q \geq 1.45$):

test	result
$v_p(V)\,/v_p(\text{crater})$ a power of 2	15/15 classes, 0 exceptions
each edge $\Delta k_p \in \{\pm 1\}$	578/578, 0 exceptions
$\max_p \lvert k_p\rvert = \lvert V\rvert/2$	15/15 (empirical)

The last line is the empirical law

$$\boxed{\max_p |k_p| = \frac{|V|}{2} = \text{caterpillar diameter}} \tag{12}$$

volcano depth equals class diameter, matching tier by tier (T_4 diameter 2, T_6 diameter 3, T_8 diameter 4). Unproved; open problem 12.

8.4 The three gates (when descent exists)

$\sigma_p = -1$ (descent) requires:

1. *Unlock:* $2 \mid v_p(\Delta)$;
2. *Kummer:* $j_0 := p^{v_p(\Delta)} j \in \mathbb{Z}_p^\times$ must be an ℓ-th power; for $p \neq \ell$, equivalently $j_0^{(p-1)/g} \equiv 1 \pmod p$, $g = \gcd(\ell, p-1)$;

3. *Galois type:* the kernel must be of $\langle q^{1/\ell}\rangle$-type at p (a global condition).

Measured (gap rate, denominator = Kummer passes):

layer	gap rate
$\ell = 3,\ p \equiv 1\ (3)$ (split)	$0/37 = 0.0\%$
$\ell = 3,\ p \equiv 2\ (3)$ (inert)	$18/101 = 17.8\%$
$\ell = 2,\ p$ odd	$104/504 = 20.6\%$

Descent samples 520/520 all pass the Kummer necessary condition (100%); gap samples pass locally 75.8% — the remaining gap is the third gate (global Galois type).

Remark 8.4 (a falsified prediction)**.** It was predicted that $\ell = 3$, $p \equiv 2$ (3) forces gap rate 0 (all of $\mathbb{Z}_p^\times$ is cubes). Measured: all 18/18 gaps fall in that bucket — opposite to the prediction. The error: a local necessary condition mistaken for sufficient. Kummer guarantees *local existence* of such a subgroup, not that any global $\mathbb{Q}$-rational subgroup attains it. The corrected reading (by denominator) is in the table; the 18/18 concentration is a denominator effect (split layer Kummer rejects 2/3), not a mechanism.

Remark 8.5 (a withdrawn derivation)**.** A "mixed type" $\langle \zeta^c q^{1/\ell}\rangle$ with valuation-preserving quotient was once derived. False: $(\zeta^c q^{1/\ell})^\ell = q$, so the quotient's parameter is again $\zeta^c q^{1/\ell}$ with valuation v/ℓ — mixed type also descends. The correct dichotomy is

$$v_p(\Delta) \in \left\{\ell v,\ \tfrac{v}{\ell}\right\}, \qquad \text{valuation-invariant edges do not exist under multiplicative reduction.}$$

Zero valuation-invariant edges observed in the semistable domain, consistent; the unique observed instance ($N = 49$, $\ell = 2$, $v_7() = 3$) is additive — outside Tate's domain, hence a boundary confirmation.

8.5 Why $p = 2$ escapes the law

Frey curves at $p = 2$ are often additive ($f_2 \in \{0, 1, 3, 4, 5\}$ measured; Appendix C). Tate parameterization (hence the $v \mapsto \ell v$ law) holds only at multiplicative reduction. Hence: the law is stated only at multiplicative primes; additive primes do not participate. At record-level Frey curves $p = 2$ is systematically additive, so v_2 is a noise slot; the halving law is stated for odd primes only, hitting 5/5 across samples. This simultaneously explains why v_2 in §5.5 fails the crater halving relation: in 3360.h the crater's $v_2 = 9$ against the Frey's $v_2 = 6$, ratio $\neq 2$; but the odd primes $(3, 5, 7)$ halve 3/3.

9 The j-Valuation Duality

Proposition 9.1. *Let $a + b = c$ be coprime, p an odd prime with $p \mid m = \min(a, b)$. Then*

$$\boxed{v_p(j) = -2\, v_p(m)}$$

Proof. $j = c_4^3/\Delta$ with $c_4 = b_2^2 - 24b_4 = 16(b - a)^2 + 48ab$. Let p be odd, $p \mid m$; by symmetry say $p \mid a$. Coprimality gives $p \nmid b$ and $p \nmid b - a$, so $16(b - a)^2$ is a p-adic unit and $v_p(c_4) = 0$ (the two terms cannot merge: $v_p(48ab) = v_p(a) > 0$). From (2), $v_p(\Delta) = 2v_p(m)$. Hence $v_p(j) = 3v_p(c_4) - v_p(\Delta) = -2v_p(m)$. □

Numerical check (8 samples, digit by digit): $v(j) = -6 = -2 \cdot 3$, $-4 = -2 \cdot 2$, $-2 = -2 \cdot 1$, all matching.

The odd-prime restriction is essential: at $p = 2$ the formula fails, $v_2(j) = 8 + 3v_2(a^2 + ab + b^2) - 2v_2(abc)$ (e.g. Reyssat: $8 + 0 - 2 = +6 > 0$).

Corollary 9.2. *Combining with the crater valuation law $v_p(\Delta_{\text{crater}}) = 4v_p(m)$ (Theorem 5.1):*

$$4\, v_p(m) = -2\, v_p(j)\,,$$

i.e. the crater discriminant's entire high-valuation structure is decided by the denominator of j.

Remark 9.3 (a falsified criterion)**.** $v_p(j) < 0$ was once proposed as a detector of odd-prime edges. It has no discriminative power: Proposition 9.1 shows $v_p(j) < 0$ holds universally for $p \mid \min$ — an intrinsic property of the Frey construction, unrelated to edge structure. Frey curves are *always potentially multiplicative* at the prime factors of the minimal entry.

9.1 The explicit peak curve and the j-denominator exponent

Proposition 9.4 (Explicit peak curve)**.** *The peak vertex is the quotient $E_0 = E/\langle(0,0)\rangle$, isomorphic to*

$$E'\colon\ y^2 = x^3 - 2(b-a)x^2 + c^2 x, \qquad \Delta(E') = -256\, abc^4.$$

Hence $\sigma_{\max} = \log(256abc^4)/\log N$ is computed without searching the class *(at the first level; on triggers the class peak is the ten-candidate maximum, Theorem 7.1); measured $\sigma_{\max} > \sigma(E)$ in* $2185/2185$ *samples. Since $|\Delta'|/|\Delta| = 16c^2/(ab) \geq 64$ (AM–GM), $\log|\Delta'| \approx (5+\alpha)\log c$ and $\log|\Delta| \approx (2\alpha+4)\log c$; the two mass bounds $q \leq 6/(5+\alpha)$ and $q \leq 3/(2+\alpha)$ are two strengths of one Szpiro on one conductor, with gaps* 0.2 *and* 0.5 *at the same corner — two calibres of one gun, not a contradiction.*

Theorem 9.5 (Denominator exponent)**.** *Write $j = c_4^3/\Delta = 256\, M^3/(abc)^2$ with $M = a^2 + ab + b^2$ (odd part after cancelling the common factor of* 16*), and set*

$$r := \frac{\log \operatorname{den}(j)}{\log \operatorname{num}(j)}.$$

Then on the Frey family

$$\boxed{r = \frac{\alpha+2}{3} + O\Big(\frac{1}{\log c}\Big)}$$

with endpoints $r \to 2/3$ as $\alpha \to 0$ and $r \to 1$ as $\alpha \to 1$.

Proof. Three steps.

Step 1: closed form of j and parity of M. With $b_2 = 4(b-a)$, $b_4 = -2ab$:

$$c_4 = b_2^2 - 24b_4 = 16(b-a)^2 + 48ab = 16\,(a^2 + ab + b^2) = 16M,$$

using $(b-a)^2 + 3ab = M$. Hence

$$j = \frac{c_4^3}{\Delta} = \frac{(16M)^3}{16(abc)^2} = \frac{256\, M^3}{(abc)^2}.$$

M is always odd: if a, b are both odd then $M = \text{odd} + \text{odd} + \text{odd}$ is odd; if one is even, $M = \text{odd} + \text{even} + \text{even}$ is odd. Moreover $\gcd(M, abc) = 1$: $\gcd(M, a) \mid \gcd(b^2, a) = 1$ and $\gcd(M, b) \mid \gcd(a^2, b) = 1$; and $M \equiv a^2 \pmod{c}$ (substituting $b \equiv -a$), so $\gcd(M, c) \mid \gcd(a^2, c) = 1$.

Step 2: the reduced fraction. By Step 1, $\gcd\big(256M^3, (abc)^2\big) = 2^{\min(8,\,2v_2(abc))}$. With $G = \min(8, 2v_2(abc))\log 2$:

$$\log \mathrm{den}(j) = 2(\log A + \log B + L) - G, \qquad \log \mathrm{num}(j) = 8\log 2 + 3\log M - G.$$

In particular $\mathrm{den}(j) = (ABc)^2/2^{2G'}$ (with $2G' = G/\log 2$) is always a square, measured 2185/2185.

Step 3: the identity and the asymptotics. Subtracting $\frac{\alpha+2}{3}$ from $r = \log \mathrm{den}(j)/\log \mathrm{num}(j)$, putting over the common denominator $8\log 2 + 3\log M - G$ and eliminating $\log A$ via $\alpha L = \log A$ gives the identity of Proposition 9.6 (whose proof is the same manipulation written out). Now estimate: $B = c - A$ gives $\log B - L = \log(1 - A/c) = O(A/c)$; $M = c^2 - ab$ gives $\delta = \log M - 2L = \log(1 - ab/c^2) = O(ab/c^2) = O(1)$, and $\delta = O(A/B)$ on the corner; $G \le 8\log 2$; and the denominator $8\log 2 + 3\log M - G = 6L + O(1)$. The numerator is $O(1)$, dominated by the constant $-\frac{8\log 2}{3}(\alpha+2)$. Hence $r = \frac{\alpha+2}{3} + O(1/L)$.

Endpoints. As $\alpha \to 0$, $A = O(1)$, $B/c \to 1$, every $O(A/c)$ term vanishes and $r \to 2/3$. As $\alpha \to 1$, $A \approx B \approx c/2$, $\mathrm{den}(j) \asymp c^6$, $\mathrm{num}(j) \asymp c^6$, and $r \to 1$. □

Proposition 9.6 (Exact discrepancy identity)**.** *Let $A = \min(a,b)$, $B = \max(a,b)$, $L = \log c$, $\alpha = \log A/L$, $M = A^2 + AB + B^2$, $\delta = \log M - 2L$, and $G = \min\big(8,\, 2v_2(ABc)\big)\log 2$ (the exact 2-part of the* gcd *of the reduced numerator $256M^3$ and denominator $(abc)^2$; M is odd, and $\mathrm{den}(j) = (ABc)^2/2^{G/\log 2}$ is always a square). Then, with no approximation,*

$$\log \mathrm{den}(j) = 2(\log A + \log B + L) - G, \qquad \log \mathrm{num}(j) = 8\log 2 + 3\log M - G,$$

and

$$r - \frac{\alpha+2}{3} = \frac{2(\log B - L) - (2+\alpha)\,\delta + \frac{\alpha-1}{3}\,G - \frac{8\log 2}{3}(\alpha+2)}{8\log 2 + 3\log M - G}.$$

The numerator terms satisfy $\log B - L = O(A/c)$, $\delta = O(A/B)$, $G = O(\log 2)$, and $\log M = 2L + O(1)$, which proves Theorem 9.5 and displays the finite-size correction explicitly: the dominant discrepancy comes from the 2^8 numerator constant ($-\frac{8\log 2}{3}(\alpha+2) \approx -3.78$ on the record) and the cancellation term $(\alpha-1)G/3$.

Corollary 9.7 (Leading term at the corner)**.** *Uniformly in α,*

$$r = \frac{\alpha+2}{3} - \frac{4\log 2}{9}\cdot\frac{\alpha+2}{L} + O\Big(\frac{1}{L}\Big),$$

where the $O(1/L)$ term carries the explicit coefficients of Proposition 9.6 (its leading piece is $\frac{(\alpha-1)G}{18L}$, and $O(A/cL)$, $O(\delta/L)$ tails). At the corner $\alpha = 0$: $r = \frac{2}{3} - \frac{8\log 2}{9L} + o(1/L)$; on Reyssat ($L = \log 6436343$) this predicts 0.6274 against the measured 0.6384, the 0.011 gap being the second-order terms of the identity, itemized in §9.1.

Verification (2185 samples, exact `Fraction` arithmetic): the identity of Proposition 9.6 holds with maximal residual 3.9×10^{-16} (machine precision); binned medians track $(\alpha+2)/3$ with the predicted $O(1/\log c)$ sag; Reyssat itemized: $r = 0.638439381$, $(\alpha+2)/3 = 0.681404315$, discrepancy -0.042964935 reproduced term by term ($2(\log B - L) = -6.2 \times 10^{-7}$, $-(2+\alpha)\delta = 6.4 \times 10^{-7}$, $(\alpha-1)G/3 = -0.441667$, $-\frac{8\log 2}{3}(\alpha+2) = -3.778508$).

Remark 9.8 (bug family #8: the gcd must be taken on the *reduced* pair)**.** The 2-part of the gcd is $\min\big(v_2\big(256M^3\big), v_2\big((abc)^2\big)\big) = \min(8, 2v_2(abc)) \cdot \log 2$, taken only *after* cancelling the common factor 16 between c_4^3 and Δ. Computing it on the unreduced pair $(c_4^3, \Delta) = (4096M^3, 16(abc)^2)$ mis-assigns the 2-part by 2^4 and corrupts r at the third decimal — a convention error of exactly the same family as Remark 2.1 (2-adic factors must be expanded explicitly).

9.2 The Pasten mechanism on Frey curves: a negative result

Proposition 9.9 (Δ lives in the j-denominator)**.** *For the Frey curve, in the conventions of §9.1,*

$$\boxed{\Delta_E = 16 \cdot 2^G \cdot \mathrm{den}(j_E)} \qquad G = \min\big(8,\, 2v_2(abc)\big), \quad 2^G \in \{4, 16, 64, 256\}.$$

Measured ratio $\Delta_E/\,\mathrm{den}(j_E) \in \{64, 256, 1024, 4096\}$ *in* 2185/2185 *samples, zero exceptions; the* 2*-adic factor* $16 \cdot 2^G$ *is* 6–12 *bits. In particular* $\mathrm{den}(j_E) \asymp \Delta_E$*: the Frey discriminant lives almost entirely in the* j*-denominator.*

Proof. $\Delta_E = 16(abc)^2$ and, by Proposition 9.6, $\mathrm{den}(j_E) = (abc)^2/2^{2G'}$ with $2G' = G/\log 2$ reduced by exactly $\min(8, 2v_2(abc))$ powers of 2. □

Theorem 9.10 (The Pasten mechanism is information-free on Frey curves)**.** *Pasten's implication [7] runs:* $\mathrm{den}(j_E) \le A(\log \mathrm{num}\, j_E)^B$ *(assumption) together with the height bound* $h(j) \le 16N \log N$ *(Cor. 2.1 there, from Murty–Pasten and Silverman) yields* $\Delta_E \le A\, 16^{B+1} N_E^{B+5} (\log N_E)^B$ *via* $\Delta_E \mid 16\,\mathrm{den}(j_E) N_E^5$ *(Pesenti–Szpiro). On the Frey family:*

1. ***The assumption's domain is empty:*** *no uniform* A *covers an asymptotic family. The required* A *(at* $B = 1$*) has median* 10^{29}*, range* $10^{3.6}$–10^{49}*, with* $\mathrm{corr}(\log A_{\mathrm{req}}, \log c) = +0.979$ *and slope* $+4.21 \approx \partial \log \mathrm{den}(j)/\partial \log c$*;*

2. ***The divisor step collapses:*** *by Proposition 9.9,* $\Delta_E \mid 16\,\mathrm{den}(j_E) N_E^5$ *becomes* $2^G \mid 2^{5f_2}$ *(equivalently* $G \le 5f_2$*; in particular always true for* $f_2 \ge 2$*, which covers all certified samples); the slack has median* 70.7 *nats, minimum* 11.46*, with no near-equality in* 2185 *samples;*

3. ***The power-type repair is self-cancelling:*** *replacing the polylog assumption by the true law* $\mathrm{den}(j) = \mathrm{num}(j)^{c_0 + o(1)}$ *(Theorem 9.5) turns the divisor step into* $\Delta_E \le O(\Delta_E \cdot N_E^5)$*, automatically true, and the power-type slack is* identically *the Cor. 3.4 slack — measured equal entry by entry in* 2185/2185 *samples, the signature of zero information.*

Pasten's mechanism derives its force from $\mathrm{den}(j) \ll \mathrm{num}(j)$*; Frey curves satisfy* $\mathrm{den}(j) \asymp \Delta_E$*, the worst case.*

Remark 9.11 (The height question is already answered — and circles back)**.** The Frey-specialized Cor. 2.1 is *already proved*: $\log \mathrm{num}(j) = 6 \log c + O(1)$ (Proposition 9.6), i.e. $h(j) = 6q \log N + O(1)$, against the general $h(j) \le 16N \log N$ — an improvement of order $N/\log N$. If $h(j) \le C \log N$ with $C \to 6$ could be promoted to C independent of q, Cor. 3.4 would deliver Szpiro with $B = 0$. But $h(j) = 6q \log N + O(1)$ makes $h(j) \le C \log N$ with universal C *equivalent to* $q \le C/6$, i.e. to abc itself: the circle closes. The c_0 theorem's role is thereby fixed: it is the exact meter of *why* the assumption fails (power type $c_0 = (\alpha + 2)/3 \in [2/3, 1]$ against polylog), not an input to any surviving mechanism.

10 High Exponents and the Beal Parameterization

Let A, B, C be coprime, $x, y, z \ge 1$, and consider $A^x + B^y = C^z$ with Frey curve $E\colon y^2 = x(x - A^x)(x + B^y)$. Substituting into Theorem 5.1 with $a = A^x$, $b = B^y$, $c = C^z$:

Theorem 10.1 (explicit ρ_c, two branches)**.**

$$\rho_c = \frac{\min\Big\{ \underbrace{4\ln 2 + 2\big(x\ln A + y\ln B + z\ln C\big)}_{\textit{Frey-crater}},\ \underbrace{8\ln 2 + \min\big(4x\ln A + y\ln B,\ x\ln A + 4y\ln B\big) + z\ln C}_{\textit{quotient-crater}} \Big\}}{\ln\left(2^{f_2}\,\mathrm{rad}(ABC)_{\mathrm{odd}}\right)} \tag{13}$$

The first branch comes from $\Delta = 16(abc)^2$ *(*$\ln 16 = 4\ln 2$ *plus* $2(\ln a + \ln b + \ln c)$*); the second from* $\Delta = 256m^4sc$*. Switch (§5.2): use the first branch when* $s \cdot c \le 16m^2$*.*

Numerical check ($13^2 + 7^3 = 8^3$): the two branches give 37.184496296 (this instance takes the first branch) and the corresponding value; direct closed-form difference 0.

Remark 10.2 (an incomplete formulation)**.** Writing only the second branch miscomputes $13^2 + 7^3 = 8^3$ (there $sc = 175616 < 16m^2 = 456976$ takes the first branch). Both branches are mandatory.

10.1 Darmon–Granville

At fixed exponents (x, y, z), ρ_c has an explicit positive lower bound (numerator contains $4x\ln A$ etc. while the denominator contains only rad), connecting to Darmon–Granville: fixed exponents $\Rightarrow$ finitely many solutions (Faltings). This is a necessary-condition fingerprint, not a criterion: a lower bound on ρ_c cannot exclude solutions. This must be stated plainly to avoid overpromising.

10.2 Fermat–Catalan spectrum

All 9 known non-trivial FC solutions under Theorem 10.1:

$$\rho_c \in [3.97,\ 5.80], \qquad \sigma_{\max} \in [4.45,\ 6.02].$$

All 9 hug the Szpiro boundary 6 without crossing (closest: $1414^3 + 2213459^2 = 65^7$, missing by 0.02). Notably FC masses are only 1.06–1.20, far below the quality list's 1.44+, yet $\sigma_{\max}$ reaches 4.45–6.02: the σ-scanner is an order of magnitude more sensitive to *exponent structure* than the q-scanner.

10.3 Beal in ratio form

The Beal conjecture asserts $A^x + B^y = C^z$ has no coprime solutions with $x, y, z \ge 3$. A counterexample would give, for its whole isogeny class,

$$\sigma_{\max}(E) = \frac{2\big(x\ln A + y\ln B + z\ln C\big) + 4\ln 2}{\ln N}$$

(at the first level; triggers can only enlarge it, per Theorem 7.1), with the numerator containing exponents and the denominator only radicals of the bases — unbounded. Hence:

Beal counterexample = a $\sigma_{\max}$-exploding Frey class.

The class-maximum convention of §2.3 is used throughout; the Frey curve's own $\sigma(E)$ would be a systematic underestimate (Appendix B, item B7).

11 The Mass Criterion and the Szpiro Boundary

11.1 Ceiling formula

From Theorem 5.4 ($\Delta_{\text{peak}} = 256abc^4$) and $N = 2^{f_2}\,\text{rad}_{\text{odd}}$:

$$\boxed{\sigma_{\max} = \frac{8\ln 2 + \ln(ab) + 4\ln c}{f_2 \ln 2 + \ln \text{rad}_{\text{odd}}}} \tag{14}$$

the elementary closed form of $\sigma_{\max}$ (given model minimality and known f_2; on triggers the peak factor is replaced by arg max over (9)–(10)).

11.2 Asymptotic main term

Define the imbalance parameter

$$\alpha := \frac{\ln \min(a,b)}{\ln c} \in [0,1),$$

$\alpha = 0$ when min is bounded ($a = 1, 2$); $\alpha \to 1$ when $a \approx b \approx c/2$. Taking $\ln \text{rad} \to \infty$ with f_2 bounded, from $\ln(ab) = \ln m + \ln s \approx \ln m + \ln c$:

$$\boxed{\sigma_{\max} \longrightarrow (5+\alpha)\,q} \qquad \text{(star classes; on triggers } \sigma_{\max} \text{ uses the ten-candidate peak)} \tag{15}$$

Derivation. Numerator $\approx \ln m + \ln c + 4\ln c = (5+\alpha)\ln c$ (as $\ln m = \alpha \ln c$); denominator $\to \ln \text{rad}$; ratio $= (5+\alpha)q$.

Numerical check (Reyssat): $(5 + 0.044) \times 1.6299 = 8.22$ vs. the true value 6.83 — the gap is the denominator's $+4\ln 2$ and finite-size terms, i.e. the true value is below the asymptote. The asymptotic form is not a numerical estimator at medium size.

11.3 The mass criterion

Szpiro asserts $\limsup \sigma_{\max} \le 6$; substituting (15):

$$\boxed{q \;\le\; \frac{6}{5+\alpha}} \tag{16}$$

Calibre table (same conductor, two curves)

calibre	curve	$\log\lvert\Delta\rvert/\log c$	bound	gap at $\alpha \to 0$
σ_{Frey}	E	$2\alpha + 4$	$q \le 3/(2+\alpha)$	0.5
$\sigma_{\max}$	E' (Proposition 9.4)	$5 + \alpha$	$q \le 6/(5+\alpha)$	0.2

α	meaning	Szpiro gives	abc requires
$\to 1$	balanced ($a \approx b$)	$q \le 1.00$	$q \le 1$ — identical
0.044	Reyssat	$q \le 1.190$	$q \le 1$
$\to 0$	extreme imbalance	$q \le 1.200$	$q \le 1$

Conclusion. The Szpiro–abc gap is a function of α, vanishing at the balance point. And $\alpha = 0 \iff \min(a,b)$ bounded $\iff c$ and $\max(a,b)$ differ by a fixed small amount $\iff$ the Pillai/S-unit corner. Measured: all 14 quality samples exceed $6/(5+\alpha)$, the largest excess Reyssat (+0.44), the smallest +0.26 (BB#C); the closest approach to the line is 6083^2, whose $\sigma_{\max} = 6.0447$ sits on the Szpiro boundary.

Remark 11.1 (not a refutation)**.** Szpiro is $|\Delta^{\min}| \ll_\varepsilon N^{6+\varepsilon}$, allowing $\sigma \le 6 + \varepsilon + O_\varepsilon(1)/\log N$. Hence $\sigma > 6$ at finite N is permitted, and "14/17 rows above 6" (Appendix C) is not a refutation. The true criterion is:

$$\text{does there exist a sequence with } \ln N \to \infty \text{ and } q > \tfrac{6}{5+\alpha}?$$

This is the sharpest falsifiable criterion the machine produces.

11.4 abc supplies the last block

abc gives $q \le 1 + \varepsilon$; substituting:

$$\sigma_{\max} \le (5 + \alpha)(1 + \varepsilon) \le 6 + 6\varepsilon \qquad (\alpha \le 1).$$

On the Frey family: abc $\Rightarrow \limsup \sigma_{\max} \le 6$, saturated as $\alpha \to 1$. The reverse (Szpiro $\Rightarrow$ abc) fails: $\sigma \le 6$ gives only $q \le 6/(5 + \alpha) \ge 1.2$, weaker than abc's 1 by 0.2. This 0.2 gap lives exactly at $\alpha \to 0$ — the Pillai/Beal core.

11.5 The Sage-certified f_2 tiering

Old tierings from 145 generated samples ("$f_2 = 5$ layer, 122 entries") were overturned by the Sage authority table; the whole layer is void (Appendix B, B9; Appendix C, D.3(2)). The final version uses 17 authoritative examples from a full `frey_pipeline.py` run (16 unique):

$$f_2: \quad 0:1, \quad 1:6, \quad 3:1, \quad 4:7, \quad 5:2$$

i.e. $f_2 = 5$ survives only twice (Reyssat, #5). The conductor exponent at $p = 2$ is not computable by hand; it requires `ellglobalred` or the Kraus realizability criterion. The qualitative crossing condition remains "high mass q + small f_2", but its quantitative form must be re-derived from the true tiering before citation (§12, items O5/O6).

Two readings on the authority sample (Appendix C, D.3): the largest N gives the largest $\sigma_{\max}$: #3's $N = 4.688 \times 10^9$ ($\log N = 22.27$) yields 8.4474; on 16 unique points $\log N \in [5.35, 22.27]$, slope $+0.0948$, correlation 0.55; low-N half mean 6.81 vs. high-N half 7.48 — no decay. This further weakens the "pure finite-size" reading, but is not a Szpiro refutation: the sample is quality-selected (selection effect), and Szpiro allows $O_\varepsilon(1)/\log N$.

List anchor (authority values). Reyssat: $\alpha = 0.044$, $f_2 = 5$, $q = 1.6299$, $\rho_c = 3.2017$, $\sigma_{\max} = 6.8295$.

11.6 A convention correction

Early scans used the Frey curve's own $\sigma(E)$ as the class representative: a systematic underestimate. Measured: the Frey curve's σ can be 1.43 below the class $\sigma_{\max}$ (Reyssat class: $\sigma(E) = 5.397$ vs. $\sigma_{\max} = 6.830$). Scanners must take the class max, never the Frey's own.

12 Open Problems

Ordered by solvability. Tier A: pure computation or symbolics, closable in days; Tier B: routes exist, no guarantee; Tier C: genuine walls.

Status update (final version): O1 **closed** (Sage authority f_2 + Kraus minimization, 17 examples, Appendix C); O2 **proved** (Tate-parameterization argument of §8; 284/284 edge steps, 55/55 full-torsion vertices); O4 **closed** (written quotient-discriminant proofs in §7.2–§7.3; the depth-2 forms were also corrected there). Original entries retained for traceability.

O1 (Authority f_2 and minimal-model recompute.) Closed; see Appendix C.

O2 (Proof of the Volcano Law.) Proved in §8.

O3 ($\max|k_p| = |V|/2$.) Volcano depth equals caterpillar diameter, matching tier by tier. A pure combinatorial statement; should follow from the chain structure of Theorem 6.1.

O4 (Symbolic proof of depth-2/3 closed forms.) **Closed**: the quotient-discriminant proofs are given in §7.2 (factorization $(x+C^2)(x-A^2-2AC)(x-A^2+2AC)$ of the centre cubic) and §7.3.

O5 (Ten-candidate full-library regression.) The old 145-sample scan used four/eight candidates; recompute with ten — measured corrections reach $\Delta\sigma = +1.77$ (under) and -2.15 (over).

O6 (Data test of the asymptotic criterion.) σ and the excess $q - 6/(5+\alpha)$ show no decay on $\ln N \in [7.4, 29.2]$, but this is selection effect. Larger, systematic samples needed to decide finite-size vs. persistent.

O7 (Serpentine signature vs. graph type.) How does the distribution of $\sigma \in \{\pm 1\}^\pi$ depend on the $|V|$ tier and α? Diagonal edges appear in $\approx 42\%$ of samples, N-independently.

O8 (Meeting Pasten's mixed method.) Pasten [7] controls the *largest prime factor* of the discriminant via Shimura curves; we control the *overall size*. The meeting point may be the m^4 term (the square of the j-denominator, Corollary 9.2). No construction.

O9 (Odd-prime edges.) The classification excludes $\ell = 3$ in pure 2-primary classes; under which signatures $\ell = 3$ opens remains open; needs database-level isogeny-matrix checks.

O10 (Bound on ε_c.) From $h_F = -2\log 2\pi - \frac{1}{2}\log f + \frac{1}{12}\log|\Delta^{\min}|$ one has $\varepsilon_c \approx \log|j_{\text{crater}}| - 6\log\log|j_{\text{crater}}| + 4.828$. Boundedness of ε_c is equivalent to "Szpiro $\Rightarrow$ Frey height conjecture", a recorded open problem: the discriminant gives only the non-Archimedean part of the Faltings height.

O11 (New mathematics.) Every tool here measures a *difference*; abc demands a *position*. Isogeny works because ℓ-torsion subgroups can be quotiented; at the crater there is no subgroup to quotient, and $v_p(\Delta_{\text{crater}})$ is exactly where Szpiro hides. A new tool must bound $v_p(\Delta_{\text{crater}})$ polynomially in N *without any movement*. We do not know what it is, and do not fabricate one.

Five-minute acceptance line (for any future proposal): any "move-then-compare" proposal is dead (movement yields only $O(1/\log N)$); any extremal/spectral proposal giving only universal constants is dead (constants do not grow with N).

A Symbolic Verification Checklist

All assertions below were verified by `sympy`

(scripts: `verify_crater.py`, `verify_quotient_crater.py`, `verify_beal_rho.py`, `depth3.py`, `volcano_scan.py`):

#	assertion	result
1	$b_8 = 4a_2a_6 - a_4^2$ in (1)	difference $S \equiv 0$
2	$\Delta_E - 16(abc)^2$	0
3	Vélu lemma $u - 4f(r)$	0 (each of the three roots)
4	three quotient coefficient rows	verbatim match with hand computation
5	$\Delta(E_{-b}) - 256a^4bc$	0
6	$S - 16a^4(a+b)$	0
7	$\Delta(E_a) - 256ab^4c$, $\Delta(E_0) + 256abc^4$	0
8	Theorem 10.1 two branches vs. direct closed form	difference 0
9	Δ_2, $\Delta_3^{\pm}$ vs. BFS computation	digit by digit
10	$v_p(j) + 2v_p(\min)$	0 (8 samples)

B Version Archaeology (Retracted Claims)

Claims made during this work and later falsified, recorded against mis-citation. This appendix is a core methodological component: an archive recording only successes systematically misleads.

#	once claimed	falsified by	current value
B1	$\log\lvert\Delta\rvert = 12h_F + 6\log\Omega + 12\log 2\pi$	substituted into RHS $\equiv 0$ while $\log 161051 = 11.989$	correct form involves $\log\lVert\Delta\rVert_{\mathrm{Pet}}$
B2	UV-vertex margin table as independent check	the Δ_{UV} taken is Szpiro's own object	crater-only convention; UV margin = Szpiro itself
B3	$\mathbb{Q}$-rational isogeny graphs are always trees (35/35)	rectangles R_4, R_6, R_8 exist (classification)	three tiers $T_4/T_6/T_8$
B4	6083^2 class contains odd-prime edges	model/minimal discriminant gap u^{12} fakes non-12 differences	pure 2-primary; $\ell = 3$ excluded by classification
B5	"quotient is centre" $\iff$ "point $\in 2E(\mathbb{Q})$"	counterexample: not equivalent	the former is the graph condition (§6.1)
B6	"mixed type" $\langle\zeta^c q^{1/\ell}\rangle$ exists, v invariant	$(\zeta^c q^{1/\ell})^\ell = q$ also descends	$v_p(\Delta) \in \{\ell v, v/\ell\}$
B7	Frey's own σ as class representative	systematic underestimate (Reyssat gap 1.43)	take class $\max\sigma$
B8	$\ell = 3$, $p \equiv 2\ (3) \Rightarrow$ gap rate 0	18/18 gaps in that bucket	local $\neq$ sufficient; stratify by denominator
B9	$f_2 = 5$ constant	Sage authority: 11/14 mismatch	$f_2 \in \{0,1,3,4,5\}$, never by hand
B10	$\lvert V\rvert \equiv 4$	criterion (Theorem 6.1); measured $6, 8$	decided by ac, bc squareness
B11	$v_p(j) < 0$ as odd-prime-edge signal	Proposition 9.1: universal for $p \mid \min$	no discriminative power
B12	bounded climb $\Rightarrow \sigma_{\max} - \rho_c \to 0$	width $= 3\log(c/m)/\log N \not\to 0$	bounded is $12\Delta h_F$; spectral term unbounded

(continued)

#	once claimed	falsified by	current value
B13	four candidates in model convention directly give minimal crater	BB#D: different u per candidate; $u = 2$ branch flips to minimal ($\rho_c = 7.7034$)	divide each candidate by its u^{12} first (Remark 5.2)
B14	"10/14 above 6" and the $f_2 = 5$ σ table	Sage authority, 17 examples	14/17 above 6 (13/16 unique); values per Appendix C
B15	j-denominator gcd taken on unreduced (c_4^3, Δ)	shifts the 2-part by 2^4; corrupts r at 10^{-3}	reduce by the common factor 16 first (Remark 9.8)

Meta-statistics. The work records 24 human-side errors, all caught by integer/exact-rational certification, identity residuals, or external truth (LMFDB, the classification theorem): capture rate 24/24. The 23rd capture targeted the designer of the detector itself (a wrong u formula makes the double-blind test 100% false-negative).

C Sage Authority Data

Source: full `frey_pipeline.py` run (17/17 triples, `SELFTEST PASSED`; Kraus minimization agreeing with Sage `minimal_model()` in every case; class sizes 17/17 matching the criterion; volcano edges 0 violations). This appendix supersedes all numerical values in §4, §7, §5.5.

C.1 Run conclusions

- 17/17 passed, `SELFTEST PASSED`, no assertion failures;
- Kraus realizability criterion vs. Sage `minimal_model()` agree case by case — the minimization blocker of O1 is closed;
- isogeny class sizes 17/17 match Theorem 6.1 ($|V| \in \{4, 6, 8\}$); volcano edge samples: 0 violations;
- three external anchors reproduced: 240672 (Reyssat), 3360 (#5), 53130 (#2).

C.2 Authority master table (17 rows)

sample	N	f_2	$\lvert V\rvert$	ρ_c	$\sigma_{\max}$	note
Reyssat	240672	5	4	3.2017	6.8295	anchor 240672.c
#2	53130	1	4	4.7602	8.3159	anchor 53130
#3	4688222070	1	4	4.9403	8.4474	largest N, largest σ
#4	15010320	4	4	4.4658	7.5864	
#5	3360	5	4	2.7479	5.8454	anchor 3360.h
#13	12026480	4	4	4.3061	7.2318	OCR fix $210 \cdot 374$
BB#D	20214480	4	4	7.7034	8.2171	u^{12}-flip class; max ρ_c
BB#8	4290	1	4	4.9397	7.6860	
BB#1x	2252712	3	4	3.0485	7.0527	unique $f_2 = 3$

(Table continued)

sample	N	f_2	$\lvert V\rvert$	ρ_c	$\sigma_{\max}$	note
BB#C	979440	4	6	4.0343	6.9437	T_6
BB#E	1680	4	8	2.8428	5.9870	duplicate of $1 + 2400$
BB#H	227537520	4	4	4.0132	7.0677	
BB#I	6439290	1	4	7.2649	8.1913	$\rho_c > 6$
6083^2	5456451	0	6	2.2468	6.0447	unique $f_2 = 0$; $N = \mathrm{rad}/2$
$81 + 175$	210	1	8	2.3927	6.7598	T_8; depth-3 repair
$1 + 2400$	1680	4	8	2.8428	5.9870	= BB#E
$13^2 + 7^3$	182	1	4	4.7714	6.1388	Frey-crater instance

C.3 Five verification conclusions

(1) One duplicate. BB#E and $1 + 2400$ are identical ($N = 1680$, $f_2 = 4$, $|V| = 8$, $\rho_c = 2.8428$, $\sigma_{\max} = 5.9870$): the same triple $1 + 2^5 \cdot 3 \cdot 5^2 = 7^4$. Unique samples: 16; $\sigma_{\max} > 6$ counts 14/17 by row, 13/16 unique.

(2) Distribution overturns the old assumption.

$$f_2 : \ 0 : 1, \ 1 : 6, \ 3 : 1, \ 4 : 7, \ 5 : 2$$

$f_2 = 5$ survives only twice (Reyssat, #5). The old "$f_2 = 5$" hypothesis and the "$f_2 = 5$ layer, 122 entries" tier are void; the full-library tiering must be redrawn on true f_2.

(3) Largest N gives largest σ. #3's $N = 4.688 \times 10^9$ ($\log N = 22.27$) yields the maximum $\sigma_{\max} = 8.4474$. On 16 unique points $\log N \in [5.35, 22.27]$: slope +0.0948, correlation 0.55; low-N half mean 6.81 vs. high-N 7.48 — no decay. This further weakens the "pure finite-size" reading but is no refutation: quality-selected sample, and Szpiro allows $O_\varepsilon(1)/\log N$.

(4) Independent consistency for 6083^2. $N = 5456451 = \mathrm{rad}(ABC)/2 = 10912902/2$, verifying the unique $f_2 = 0$ (good reduction at 2) without Sage.

(5) Two classes where the crater itself exceeds 6. $\rho_c > 6$ only for BB#D (7.7034) and BB#I (7.2649) — the extreme of Type II (high-valuation crater). BB#D is the trigger of the u^{12} flip rule.

C.4 A self-correction

An earlier report gave "BB#D flipped $\rho_c \approx 7.40$" using a $\ln \mathrm{rad}$-denominator estimate; the Sage authority is 7.7034. The direction (post-flip class maximum) is unchanged; the value is per the table.

C.5 Final-revision checklist (executed in this version)

location	old	changed to
§4/§7/§5.5	"10/14 above 6" and all σ values	"14/17 (13/16 unique)" + table C.2
§4/§7	$f_2 = 5$ hypothesis, "$f_2 = 5$ layer"	deleted; retiered per (2)
§5.2/§5.5	crater by direct arg min on four candidates	divide each candidate by its u^{12} first (BB#D flip)
§6/Appendix B	"$\lvert V\rvert \equiv 4$"	criterion (Theorem 6.1); measured $6, 8$
§11	$\sigma_{\max} \to (5 + \alpha)q$, $q \le 6/(5 + \alpha)$	retained as asymptotic; sample values per Appendix C
Appendix B	—	two new "claimed → falsified → current" rows

C.6 Boundary

$\sigma_{\max} > 6$ is not a Szpiro refutation ($O_\varepsilon(1)/\log N$ allowed); the sample is quality-selected, slopes not extrapolable; Kraus↔Sage consistency covers 17 cases of 2-primary classes only; odd-prime edges are excluded by the CLR classification; abc, Szpiro, Beal and the height conjecture are all unproved. **This paper is structure theorems and reductions, not a proof.**

Acknowledgments

External anchors: LMFDB (curves 240672.c, 3360.h, 14.a, 15.a, 26.a, 30.a, 19.a; the Manin constant $c = 3$ of 19.a matching our computation is the first external certification of an integer certificate); SageMath `ellglobalred` and PARI `ellminimalmodel`; the classification theorems of [5, 3, 1, 2].

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