Selected character moments, zero-free half-planes for Hecke L-functions over number fields, and Quasi-Riemann Hypothesis

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Abstract

In this paper, we obtain a zero-free half-plane $\operatorname{Re} s > B_2$ for every finite-order Hecke L-function over every number field, where

$$B_2 = \frac{777 - \sqrt{1081}}{852} = 0.873381966613207519929\ldots.$$

We adapt selected character-sum estimates and finite moment inductions to ideal-character sums over arbitrary number fields and restore ordinary sixth powers. Two finite moment inductions give an ordinary product estimate with length condition $n_1 + n_2 + 2z \le M$ for sufficiently fine original prime factors. Applying this estimate to zero detection and the averaged numerator in the cubic-theta comparison gives the stated boundary. We determine the exact infimum of the resulting comparison system and verify its scalar inequalities by rational Bernstein bounds. The proof uses the earlier theta, moment and continuation theorems over number fields.

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1 Introduction

In this paper, we continue to consider the zero-free half-planes for Hecke L-functions over number fields, and improve the bound on the Quasi-Riemann Hypothesis. We write μ_n for the group of n-th roots of unity and $\zeta_n = e^{2\pi i/n}$.

1.1 The main theorem

Let

$$B_2 = \frac{777 - \sqrt{1081}}{852} = 0.873381966613207519929026778642371869 4\ldots. \tag{1.1}$$

It is the smaller root of $1278B^2 - 2331B + 1061 = 0$. Then our main theorem is as follow.

Theorem 1.1. *Let K be a number field and let η be a finite-order Hecke character of K. Then*

$$L_K(s,\eta) \neq 0 \qquad (\operatorname{Re} s > B_2),$$

with the pole of a principal character at $s = 1$ permitted. The numerical boundary is independent of K and η; constants and thresholds may depend on the fixed arithmetic data. No assertion is made on the line $\operatorname{Re} s = B_2$.

The proof is given in Section 7. It uses the ordinary moment of Theorem 4.9 and the strict comparison estimates of Section 6 at each fixed $B > B_2$ sufficiently close to B_2. Theorem 6.9 determines the infimum for the specified comparison system.

1.2 Earlier bounds and the ordinary moment

In this subsection, we compare the earlier bounds with the ordinary moment of Theorem 4.9. The earlier general-field theorem gives $B_\flat = 0.874886647039964\ldots$ [Liu26b, Theorem 17.1]. In [Liu26c, Theorems 1.1–1.2], the enlarged moments give the respective boundaries

$$B_{\mathrm{tw}} = 0.87405850379185900499\ldots, \qquad B_{\mathrm{ex}} = 0.87407284766682790777\ldots.$$

Those arguments improve the moment parameters together with the zero-free boundary. Here we retain the earlier pointwise parameter $\kappa_1 = 0.749774$ and improve the ordinary moment separately.

For the ordinary coefficient families of Section 2, Theorem 4.9 gives the following estimate. For each $\epsilon > 0$ and fixed aggregate length bounds, there is $\eta > 0$, chosen before the arithmetic data and finite factor list, such that

$$z = \sum_i z_i, \qquad 0 < z_i \le \eta$$

implies

$$\sum_u^{\mathrm{ind}(\psi_u)\notin\Theta} \mathcal{W}(u)|S_{n_1,1}(u)S_{n_2,2}(u)Q(u)|^2 \ll Z^{M+\epsilon}\mathfrak{S}_J(\mathcal{W},\mathbf{W})(1+H_{\mathrm{in}})^{J'}, \qquad n_1+n_2+2z \le M. \tag{1.2}$$

Here $S_{n_1,1}, S_{n_2,2}$ are the original smooth ordinary sums, Q is a product of complete original prime factors, and M includes the displayed moving radical. The sum retains ordinary parameter powers and the natural zeros of the characters. Its coefficient families and uniformity are specified in Definition 4.1.

1.3 The method and its precedents

In this subsection, we describe the operator and moment arguments of Sections 3 and 4.

The selected operator of Theorem 3.10 bounds ideal-character sums on parameters where an original prime product is large. Theorem 3.13 restores ordinary sixth powers. On the complementary amplitude classes, removing a complete prime block permits application of an earlier ordinary moment. These two alternatives give the successive length conditions

$$d + 6\kappa_1 z \le M \quad \longrightarrow \quad d + 3z \le M \quad \longrightarrow \quad d + 2z \le M, \qquad d = n_1 + n_2.$$

The first induction proves Corollary 4.4. It supplies the comparison term for the finite iteration proving Theorem 4.9. The required number of steps and the original prime mesh are fixed for each requested exponent loss. The pointwise prime bound retains the parameter κ_1.

The larger ordinary prime length improves both the zero-detector count and the averaged numerator. Displacing the numerator contour adds a second saving. Their combination yields a strict comparison at $B_{\mathrm{rat}} = 0.8733821$ and then at every fixed $B > B_2$ sufficiently close to B_2. The limiting parameters have a unique double contact, computed in Section 6.5.

The selected character-sum operator estimate appears in [Beb26c]. Its combination with the reflected character-sum estimate and the finite moment induction are developed in [Arv26]. The refinement in [Beb26b] permits arbitrarily small fixed positive parameter gaps within the stated ranges. We adapt these estimates and inductions to ideal-character sums over arbitrary number fields. For Dirichlet L-functions, [Beb26d] proves nonvanishing for every positive modulus and every character in $\mathrm{Re}\, s > T_{103}$, with the principal pole permitted, where its explicit rational cutoff satisfies

$$T_{103} = 0.87338196661320751992902677864 2\ldots, \qquad 0 < T_{103} - B_2 < 3\cdot 10^{-254}.$$

This refines the earlier rational cutoff of [Beb26a]. Thus the same limiting boundary B_2 already occurs in the Dirichlet comparison. Here we extend the selected operator, sixth-power restoration and ordinary moment estimates to finite-order Hecke characters over arbitrary number fields, obtaining nonvanishing in $\operatorname{Re} s > B_2$. Theorem 6.9 proves that B_2 is the infimum for the specified positive comparison system.

The optimality theorem of [Liu26c, Theorem 16.1] retains the relation $p \geq 12B_{\mathrm{known}} - 6$ between the ordinary prime coefficient and the known zero-free boundary. Here B_{known} denotes a bound $\beta_*(F) \leq B_{\mathrm{known}}$ already proved before the current iteration step. Theorem 4.9 replaces that moment input. The optimality result below concerns the resulting direct, detector and first-moment comparison system.

Here is the structure of the paper. Section 2 fixes the arithmetic notation and recalls the earlier theorems. Sections 3 and 4 prove the selected operator and the ordinary moment estimates. Section 5 gives the required approximate-functional-equation contours. Section 6 proves the scalar comparison and its optimality, and Section 7 completes the Mellin argument and descent. Appendix A contains the exact scalar bounds; Appendix B verifies the rational choice of the starting moment parameter.

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2 Arithmetic notation and the earlier theorems

In this section, we fix the coefficient families and recall the results of [Liu26b] used in Sections 3–7. The results from its Part II are applied over $F \supset \mu_{12}$ by [loc. cit., Theorem 18.2].

Fix a number field $F \supset \mu_{12}$. It is totally complex. Choose the fixed finite set S of places as in [loc. cit.], including the archimedean places, those above 6 and the different, the fixed coefficient conductors, and all prime ideals of norm at most four. Enlarge S so that $\mathcal{O}_{F,S}$ has the principal and self-dual conventions of that source. Constants may depend on the fixed field and on S.

Write $q_{\mathfrak{a}} = N\mathfrak{a}$ for an exterior ideal norm, $U_S = \mathcal{O}_{F,S}^{\times}$, and $\Gamma = U_S^6$. Allowed generators and their unit classes have precisely the finite ray and compact-quotient interpretation of [loc. cit.]. All prime ideals below lie outside S. Residue characters are extended by zero at nonunits. All quotient presentations retain these zero extensions. All ideal-divisor multiplicities of a fixed degree are bounded by $q_{\mathfrak{a}}^{\epsilon}$, with a constant depending on the fixed field and degree.

Fix a finite group Θ of coefficient characters, closed under conjugation and containing the finite unit characters and all initial finite coefficients. Its exceptional inducing group is classified for each actual returned presentation. A typical orientation is $\psi_u = \nu\overline{\chi_u}$; changing the orientation conjugates the entire polynomial and its test function. Sums over the arithmetic parameter u retain its multiplicities.

2.1 Lengths, coefficient families and common test functions

In this subsection, we fix the coefficient families used in Sections 3 and 4.

The global asymptotic scale is $Z \geq 2$. A polynomial length Z^n refers to the norm of its actual coefficient index. An ordinary central factor has the form

$$S_{n,j}(u) = Z^{-n/2} \sum_{\mathfrak{a}} \psi_u(\mathfrak{a}) W_j(N\mathfrak{a}/Z^n), \tag{2.1}$$

with the source fixed finite character coefficients, common norm modes and common coprimality restrictions understood. The actual normalizer can equivalently be a comparable comparison

scale; its ratio belongs to the same common smooth test function. Each $W_j \in C_c^\infty((0,\infty))$ belongs to a fixed finite derivative family. Coupled compact test functions are separated by a common Mellin or Sobolev measure, retaining its total seminorm. The coefficients are fixed before any parameter amplitude selection.

An original prime position has actual scale $P_i \asymp Z^{z_i}$ and polynomial

$$Q_i(u) = P_i^{-1/2} \sum_{\mathfrak{p} \in \mathcal{P}_i} b_i(\mathfrak{p}) \psi_u(\mathfrak{p}) W_i(N\mathfrak{p}/P_i). \tag{2.2}$$

Here $\mathcal{P}_i$ and b_i are the source ray selector and fixed finite coefficient family. We write $Q = \prod_i Q_i$ and $z = \sum_i z_i$. A complete-factor selection includes or excludes an entire position. Repeated powers used in an operator amplification are explicitly distinguished from the original positions of the ordinary moment.

In a moving presentation let m be the free ordinary parameter exponent and q the full displayed moving radical exponent; put $M = m + q$. The actual ordinary lengths have total $d = n_1 + n_2$. A separate common extra radical is fixed within the current sum and is removed by its exact divisor expansion before an ordinary reflection or recursive call. The displayed exponent q is unchanged by this expansion. A cancelled moving factor keeps its natural zero in the displayed radical. The weights $\mathcal{W}$ are nonnegative common parameter weights with the original Fourier measure.

$\mathfrak{S}_J$ denotes a finite homogeneous seminorm controlling the parameter and coefficient tests, and H_{in} their incoming norm-twist heights. A statement uniform in these test functions means a common finite order J and polynomial height order J'.

2.2 Earlier transformation and moment estimates

In this subsection, we recall the transformation and moment theorems used in Section 4. We use the following results in their stated coefficient families.

Arithmetic and transformations. Allowed-generator reciprocity, the theta coefficient construction and the radial functional equation are [Liu26b, Lemma 3.4 and Propositions 4.1 and 5.2]. Their general-field formulation is [loc. cit., Lemma 18.1 and Theorem 18.2]. Primitive induction retains the exact Euler deletions, conductor and root number. The angular character of the theta coefficients is fixed throughout.

The compact-quotient Poisson formula and the canonical returned families are [loc. cit., Propositions 8.1 and 10.2 and Lemma 10.6]. The full Gauss and correlation identities are [loc. cit., Lemmas 12.9–12.10]. We use them with all prime powers, common divisors, nonunit cases, zero frequencies and the original coefficient-square measures. The equal-product difference is formed before transformation; its principal and finite-character means are combined with the diagonal before absolute integration, as in [loc. cit., proof of Theorem 12.16].

The earlier moments. The inverse moments and the ordinary product moment are [Liu26b, Theorems 10.7, 11.2 and 12.16]. For the two ordinary factors and arbitrary bounded ordinary lengths, the zero-prime case of the last theorem gives

$$\sum_{u}^{\mathrm{ind}(\psi_u)\neq 1} \mathcal{W}(u) |S_{n_1,1}(u) S_{n_2,2}(u)|^2 \ll Z^{M+\epsilon} \mathfrak{S}_J (1 + H_{\mathrm{in}})^{J'}. \tag{2.3}$$

For $z > 0$, the same theorem applies outside Θ under

$$d + 6\kappa z \le M, \qquad 13/18 \le \kappa \le 1,$$

with $\beta_*(F) \le (1+\kappa)/2$ when $\kappa < 1$. Here

$$\beta_*(F) = \sup\big(\{1/2\} \cup \{\operatorname{Re}\rho : L_F(\rho,\nu) = 0,\ \nu \text{ primitive of finite order},\ \rho \text{ nontrivial}\}\big),$$

with principal poles excluded. By [loc. cit., Theorem 17.1], $\beta_*(F) \le B_\flat$. Choose

$$\kappa_1 = 0.749774 < 3/4, \qquad p_1 = 6\kappa_1, \qquad B_\flat < 0.87488664704 < \frac{1+\kappa_1}{2} = 0.874887. \tag{2.4}$$

Appendix B verifies the strict numerical inequality. The pointwise prime estimate [loc. cit., Lemma 12.7] therefore gives

$$|Q|^2 \ll Z^{\kappa_1 z+\epsilon} \qquad \big(\operatorname{ind}(\psi_u) \notin \Theta\big).$$

The complete common coprimality expansion is [loc. cit., Lemma 12.5]. These estimates retain their finite seminorms and polynomial norm-twist height dependence.

2.3 The detector and comparison theorems

In this subsection, we recall the estimates applied in Sections 5 and 6. The simultaneous detector and its Sobolev estimate are [Liu26b, Lemmas 13.7 and 13.9]. They supply ordinary and inverse detected factors with a common length cutoff. The estimates hold for a common finite derivative family before the detecting test function is evaluated at each parameter.

The approximate functional equation and correction strip are [Liu26b, Lemmas 14.2 and 13.10]. Reflections retain the whole character, conductor, root number and smooth weight. In Section 5, we extend the contour interval using the compact-strip argument of [Liu26c, Lemmas 5.1–5.2].

The direct estimate and comparison are [Liu26b, Proposition 13.4 and Lemma 13.11]. The averaged numerator, its longer-length bound and the refined comparison are [loc. cit., Propositions 14.3 and 14.6 and Lemmas 14.7–14.8]. Section 6 records their exponent functions and proves the improved inequalities. Section 7.1 verifies the principal Euler product on the contours used here.

The Mellin identity is [Liu26b, (13.13)]; the continuation argument and general-field descent are [loc. cit., Theorem 13.15 and Proposition 19.1]. Section 7 applies this identity with the new saving and gives the descent explicitly. We use full abelian class-field factorizations and the standard analytic properties of finite-order Hecke L-functions, including the positive simple residue of each Dedekind zeta function [Mil20].

2.4 The order of parameter choices

In this subsection, we specify the order of choices in the moment argument of Section 4 and the comparison in Section 6.

Fix numerical aggregate length bounds and a requested exponent loss. The subsidiary finite recursion depths, positive slope gaps and mesh of the original prime lengths are chosen in that order, before the actual finite factor list. The fixed arithmetic data and finite coefficient and test-function families then determine constants and test orders. At a comparison, fix the actual common test functions and finite presentation before selecting amplitude bins or subsets. The external height exponent is chosen only after those finite orders. Thus the numerical mesh precedes the arithmetic data, while the constants and test orders may depend on them.

The field, finite coefficient characters and angular types are fixed. The ordinary coefficient families are those of (2.1). The operator theorem also allows arbitrary common coefficients, subject to its stated square-mass bound.

Local and global notation. As in [Liu26c], local factor lengths and global comparison parameters are distinguished by their sections:

	ordinary moment calculations	global comparison
w	marked prime length	numerator contour variable
T, S	local principal lengths	$T = 2l_y - h,\ S = S(R)$
R	common-character conductor length	class-count exponent
q	displayed radical length	$N\mathfrak{p}$ in local Euler factors.

The exponents n_i, z, m, q in the moment estimates are measured relative to Z. In the scalar comparison, ordinary and inverse lengths are measured relative to $U = Z^{\rho}$; thus $U^m = Z^{\rho m}$. The averaging lengths are $X = Z^{l_x}$ and $Y = Z^{l_y}$.

3 A selected ideal-character operator

In this section, we prove the selected operator and restore ordinary sixth powers. We use the field and ring of S-integers fixed in Section 2. Ideals in this section are integral exterior ideals, hence are coprime to the fixed set S. Their norms are denoted by $\mathrm{N}\mathfrak{a}$. We use the allowed generators and the zero-extended characters $\chi_{\mathfrak{a}}(\mathfrak{n})$ of [Liu26b, Lemma 3.4]. In particular,

$$\chi_{\mathfrak{a}\mathfrak{b}}(\mathfrak{n}) = \chi_{\mathfrak{a}}(\mathfrak{n})\chi_{\mathfrak{b}}(\mathfrak{n}), \qquad \chi_{\mathfrak{a}}(\mathfrak{b}) = \chi_{\mathfrak{b}}(\mathfrak{a}). \tag{3.1}$$

The identities include natural zeros, with the allowed-generator reciprocity of [Liu26b, Lemma 3.4]. An index is called sixth-power-free if its valuations belong to $\{0, 1, \dots, 5\}$. A natural character with a sixth-power factor still vanishes at the primes of that factor.

All norm lengths and all displayed fixed moduli are bounded by fixed powers of a parameter Z. We suppress arbitrarily small powers of Z, but retain every fixed positive exponent in the argument. Constants may depend on the fixed arithmetic data, finite coefficient families, smooth test functions and requested finite orders. The order in which these objects are fixed is specified below. An operator norm is always a squared ℓ^2 operator norm.

3.1 Fine selection and its inheritance

In this subsection, we define the selected parameter sets used in Theorem 3.10.

Definition 3.1. A fine descriptor is a finite list of common prime polynomials

$$Q_j(\mathfrak{k}) = \sum_{\mathfrak{p}} b_j(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-1/2}\chi_{\mathfrak{k}}(\mathfrak{p})W_j(\mathrm{N}\mathfrak{p}/Z^{z_j}), \qquad 0 < z_j \le \eta.$$

The coefficients are bounded, the test functions have fixed compact support in $(0, \infty)$, and the list is fixed independently of the moving parameter. Fixed characters, common natural deletions and norm twists may be included in b_j. Put $z = \sum_j z_j$. A family $\mathcal{E}$ is selected at slope K by this descriptor if

$$\prod_j |Q_j(\mathfrak{k})|^2 \ge Z^{Kz} \qquad (\mathfrak{k} \in \mathcal{E}). \tag{3.2}$$

We require a fixed positive lower bound for z. Further restrictions on $\mathcal{E}$ are permitted.

Bounded coefficients here may be replaced by any fixed divisor-bounded coefficient class with the same convolution-energy property. The prime formulation suffices for the applications. The constants are uniform in the values of the bounded coefficients. The number of factors and their positive widths are fixed finite data; neither is allowed to vary with Z after the theorem has been applied.

For a selected parameter set define

$$\mathcal{N}_{\mathcal{E}}(X)=\sup_{a\neq 0}\frac{\displaystyle\sum_{\mathfrak{k}\in\mathcal{E}}\left|\sum_{\mathrm{N}\mathfrak{n}\le X}a_{\mathfrak{n}}\chi_{\mathfrak{k}}(\mathfrak{n})\right|^{2}}{\displaystyle\sum_{\mathrm{N}\mathfrak{n}\le X}|a_{\mathfrak{n}}|^{2}}. \tag{3.3}$$

The source includes the unit ideal and every ordinary ideal. No power-free restriction is placed on $\mathfrak{n}$.

Lemma 3.2 (Literal quotient selection)**.** *Fix an ideal $\mathfrak{D}$ and a subfamily of original parameters of the form $\mathfrak{k}=\mathfrak{D}\mathfrak{v}$. Its quotient parameters are selected by the same descriptor with $b_j(\mathfrak{p})$ replaced by $b_j(\mathfrak{p})\chi_{\mathfrak{D}}(\mathfrak{p})$. The widths and the lower bound in (3.2) are unchanged, and coefficient-square masses do not increase.*

Proof. By (3.1),

$$Q_j(\mathfrak{D}\mathfrak{v})=Q_j^{\mathfrak{D}}(\mathfrak{v}),\qquad b_j^{\mathfrak{D}}(\mathfrak{p})=b_j(\mathfrak{p})\chi_{\mathfrak{D}}(\mathfrak{p}),\qquad |b_j^{\mathfrak{D}}(\mathfrak{p})|^2\le|b_j(\mathfrak{p})|^2.$$

This identity retains all natural zeros and inherited restrictions, including those arising from the coprimality expansion. □

Lemma 3.3 (Finite word amplification)**.** *Let the fine mesh be at most η. Uniformly for target widths $0\le w\le w_{\max}$, with $w_{\max}$ fixed, one has, with an arbitrarily small subsidiary loss,*

$$\mathcal{N}_{\mathcal{E}}(X)\ll Z^{\epsilon-Kw+O(\eta)}\sup_{w\le w'\le w+O(\eta)}\mathcal{N}_{\mathcal{E}}(XZ^{w'}). \tag{3.4}$$

This holds for every literal quotient family in Lemma 3.2.

Proof. By (3.2), partition $\mathcal{E}$ into finitely many sets $\mathcal{E}_j$ with

$$|Q_j(\mathfrak{k})|^2\ge Z^{Kz_j}\quad(\mathfrak{k}\in\mathcal{E}_j).$$

Fix j and put $l=\lceil w/z_j\rceil$. Then

$$w\le lz_j<w+z_j,\qquad |Q_j(\mathfrak{k})^l|^2\ge Z^{Klz_j},\qquad \operatorname{supp}(a*\operatorname{coeff}(Q_j^l))\subset\{\mathrm{N}\mathfrak{n}\ll XZ^{lz_j}\}.$$

Let $d_{\mathfrak{p}}$ be the coefficients of Q_j and let $c=a*d^{*l}$. Cauchy's inequality over factorizations gives

$$\begin{aligned}\sum_{\mathfrak{n}}|c_{\mathfrak{n}}|^2&\le\sup_{N\mathfrak{n}\ll XZ^{lz_j}}\tau_{l+1,F}(\mathfrak{n})\sum_{\mathfrak{n}_0,\mathfrak{p}_1,\dots,\mathfrak{p}_l}|a_{\mathfrak{n}_0}|^2\prod_{i=1}^{l}|d_{\mathfrak{p}_i}|^2\\&\ll Z^{\epsilon}\|a\|_2^2\left(\sum_{\mathfrak{p}}|d_{\mathfrak{p}}|^2\right)^{l}\ll Z^{\epsilon}\|a\|_2^2.\end{aligned}$$

Here $\tau_{l+1,F}$ counts ordered ideal factorizations. The degree l is fixed after the positive widths; the divisor bound and the centrally normalized prime square mass give the last line. Apply the longer-source norm and divide by Z^{Klz_j}. Since $0\le lz_j-w<\eta$, this proves (3.4). Quotient coefficients obey the same square-mass bound by Lemma 3.2. □

In the dual Gram matrix, we transform the complete ordinary sum before imposing the selected parameter restriction.

3.2 Primitive summation and the initial norm

In this subsection, we derive the initial estimate used in the reflection argument of Section 3.3.

Fix one finite packet of the data at S and of the allowed generators. The primitive ratio of two distinct sixth-power-free good indices is nonprincipal: at some good prime their inertia digits differ. Any bounded multiplicity arising from presentation data is grouped before forming the parameter family. Common finite characters and norm modes disappear from this ratio.

Lemma 3.4 (Primitive complete-column transformation)**.** *Let ω be a nonprincipal primitive finite-order character whose good conductor has norm Q. Let V be a fixed smooth norm test function, or a nonnegative smooth majorant equal to one near zero and supported in $[0,2]$. The primitive ideal functional equation in [Liu26b, Lemma 18.1(2)] gives*

$$\sum_{\mathfrak{n}} \omega(\mathfrak{n}) V(\mathrm{N}\mathfrak{n}/X) = \frac{X}{\sqrt{Q}} \epsilon(\omega) \sum_{\mathfrak{n}} \overline{\omega}(\mathfrak{n}) \widetilde{V}(\mathrm{N}\mathfrak{n} X/Q). \tag{3.5}$$

Fixed bad-place factors are included in a finite common test function list. The dual test function is rapidly decreasing at infinity and is $O_\epsilon(y^{-\epsilon})$ near zero, with the corresponding finite derivative bounds. If a common natural deletion is expanded at an ideal of norm R, the prefactor and natural dual scale are

$$\frac{X}{R\sqrt{Q}}, \qquad \frac{QR}{X}. \tag{3.6}$$

All its character values and zeros are retained.

Proof. Finite order at the complex places has trivial archimedean type. Write $G_\omega(s)$ for the fixed completion factor, with the factors at S included in the common test functions. The functional equation gives

$$L(s,\omega) = \epsilon(\omega) Q^{1/2-s} \frac{G_{\overline{\omega}}(1-s)}{G_\omega(s)} L(1-s,\overline{\omega}).$$

Mellin inversion and $s \mapsto 1-s$ yield

$$\begin{aligned} \sum_{\mathfrak{n}} \omega(\mathfrak{n}) V(N\mathfrak{n}/X) &= \frac{1}{2\pi i} \int_{(c)} L(s,\omega) \widehat{V}(s) X^s \, ds \\ &= \frac{X\epsilon(\omega)}{\sqrt{Q}} \sum_{\mathfrak{n}} \overline{\omega}(\mathfrak{n}) \widetilde{V}(N\mathfrak{n} X/Q), \\ \widetilde{V}(y) &= \frac{1}{2\pi i} \int_{(c)} \widehat{V}(1-s) \frac{G_{\overline{\omega}}(s)}{G_\omega(1-s)} y^{-s} \, ds, \qquad c > 1. \end{aligned}$$

When $V = 1$ near zero, $\widehat{V}$ has a simple pole at zero. Since $r_F \geq 1$ and the nonprincipal completed function is entire,

$$\operatorname{ord}_{s=0} L(s,\omega) \geq r_F, \qquad \operatorname*{Res}_{s=0} \{ L(s,\omega) \widehat{V}(s) X^s \} = 0.$$

The reciprocal gamma factor likewise cancels the apparent pole of $\widehat{V}(1-s)$ at $s=1$. Contour shifts and Stirling give, for fixed j, A,

$$|(y\partial_y)^j \widetilde{V}(y)| \ll_{j,A,\epsilon} \begin{cases} y^{-\epsilon}, & 0 < y \leq 1, \\ y^{-A}, & y \geq 1. \end{cases}$$

For a fixed squarefree $\mathfrak{d}$,

$$\sum_{(\mathfrak{n},\mathfrak{d})=1} \omega(\mathfrak{n}) V(N\mathfrak{n}/X) = \sum_{\mathfrak{r} \mid \mathfrak{d}} \mu_F(\mathfrak{r}) \omega(\mathfrak{r}) \sum_{\mathfrak{m}} \omega(\mathfrak{m}) V(N\mathfrak{m}/(X/N\mathfrak{r})).$$

Thus $R = N\mathfrak{r}$ replaces X by X/R, giving (3.6), with every zero multiplier retained. □

We record explicitly how test functions will be separated. On a dual annulus $\mathrm{N}\mathfrak{n} \asymp Y$, the conductor is a fixed dyadic scale times compact norm ratios from the two parameter variables. A smooth Mellin expansion in those ratios writes the transformed kernel as a common integral of a bounded source coefficient and two bounded parameter multipliers. Its measure has finite weighted mass for any fixed finite collection of seminorms. If $Y_0 = QR/X$, that mass is bounded by

$$C_{A,\epsilon} Z^{\epsilon} (1 + Y/Y_0)^{-A}. \tag{3.7}$$

The same expansion is made before either of two operator norms is applied. Norm modes then have modulus one, and their annular derivatives produce only fixed powers of logarithms.

Only nonzero ideal annuli $Y \geq 1$ occur. For $b > 0$ and $A > b$,

$$\sum_{Y \geq 1 \text{ dyadic}} Y^b (1 + Y/Y_0)^{-A} \ll_{A,b} \begin{cases} Y_0^A \leq Y_0^b, & Y_0 < 1, \\ Y_0^b, & Y_0 \geq 1. \end{cases}$$

Thus (3.7) gives the same envelope on both ranges.

Lemma 3.5 (Initial bound)**.** *For every subset of natural sixth-power-free parameters of norm at most H, and every $X \geq 1$ in the fixed polynomial range,*

$$\mathcal{N}_{\mathcal{E}}(X) \ll Z^{\epsilon}(X + H^2). \tag{3.8}$$

No selection hypothesis is required.

Proof. Use the nonnegative majorant of Lemma 3.4 and its complete-source Gram matrix G. Then

$$G_{\mathfrak{k},\mathfrak{k}} \ll Z^{\epsilon} X, \qquad Q \ll H^2, \qquad \#\mathcal{E} \ll Z^{\epsilon} H.$$

For each off-diagonal deletion term, put $Y_0 = QR/X$. By (3.7) and ideal counting, including $Y_0 < 1$,

$$|G_{\mathfrak{k},\mathfrak{k}'}| \ll Z^{\epsilon} \sum_{\text{deletions}} \frac{X}{R\sqrt{Q}} Y_0 \ll Z^{\epsilon} \sqrt{Q} \ll Z^{\epsilon} H.$$

The deletion divisors have subpower multiplicity. Schur's bound gives

$$\mathcal{N}_{\mathcal{E}}(X) \leq \|G\| \leq \sup_{\mathfrak{k}} \sum_{\mathfrak{k}'} |G_{\mathfrak{k},\mathfrak{k}'}| \ll Z^{\epsilon}(X + H^2).$$

□

3.3 Squarefree reflection

In this subsection, we prove the squarefree estimate used in Section 3.4.

We first work with squarefree parameters in one dyad of norm H. All estimates in the following lemma are required on the literal inherited selected families of Lemma 3.2. They are also required after further parameter restrictions and bounded parameter multipliers. These do not increase an operator norm.

Lemma 3.6 (Hereditary reflection)**.** *Suppose, for some $q \geq 1 - K$ and $1/2 < K < 1$, that every such squarefree quotient family with parameter bound H' satisfies*

$$\mathcal{N}(Y) \ll Z^{\epsilon}\{Y + (H')^q Y^K\}$$

throughout the fixed polynomial source range. Then, for $X \geq H$,

$$\mathcal{N}_{\mathcal{E}}(X) \ll Z^{\epsilon}\{X + H^{q+2K-1} X^{1-K}\}. \tag{3.9}$$

The loss can be prescribed separately at each use.

Proof. Keep the diagonal of the complete-source Gram matrix separately; it is $O(Z^{\epsilon}X)$. For an off-diagonal pair, write

$$\mathfrak{k} = \mathfrak{D}\mathfrak{a}, \quad \mathfrak{k}' = \mathfrak{D}\mathfrak{b}, \quad (\mathfrak{a},\mathfrak{b}) = 1, \quad D = N\mathfrak{D}, \quad Q \asymp H^2/D^2.$$

The common radical remains a natural deletion. For its divisor of norm R, Lemma 3.4 gives

$$\frac{XD}{RH}, \qquad Y_0 = \frac{H^2R}{XD^2}. \tag{3.10}$$

Coprime CRT and reciprocity give the root-number cross factor

$$\chi_{\mathfrak{a}}(\mathfrak{b})\overline{\chi_{\mathfrak{b}}(\mathfrak{a})} = 1.$$

All remaining root numbers, different factors and finite-packet factors are bounded separate parameter multipliers; cf. [Liu26b, Lemma 12.9].

Separate the common test function first. Extend this expression to all pairs using the declared norm product and the separate multipliers, then insert

$$1_{(\mathfrak{a},\mathfrak{b})=1} = \sum_{\mathfrak{t}\mid\mathfrak{a},\mathfrak{b}} \mu_F(\mathfrak{t}). \tag{3.11}$$

The expansion restores the coprime expression before any inequality; no primitive conductor formula is used on the added pairs. Subtract the omitted pair $(\mathfrak{a},\mathfrak{b}) = (1,1)$ separately; the same estimate below bounds this single term. For each remaining term,

$$\mathfrak{a} = \mathfrak{t}\mathfrak{a}', \quad \mathfrak{b} = \mathfrak{t}\mathfrak{b}', \quad H' \le \frac{H}{DN\mathfrak{t}} \le \frac{H}{D}.$$

Lemma 3.2 gives the descriptor at $\mathfrak{D}\mathfrak{t}\mathfrak{a}'$. The common $\mathfrak{t}$-characters retain their dual-source zeros; no new coprimality restriction is imposed on $\mathfrak{a}'$.

For the restricted original coefficient vectors α_i, β_i, Cauchy and the two inductive operator estimates give

$$|\mathcal{B}_i| \ll Z^{\epsilon}\frac{XD}{RH}\{Y_0 + (H/D)^q Y_0^K\}\|\alpha_i\|_2\|\beta_i\|_2.$$

The linear term is $H/D \le X$; the nonlinear term is

$$H^{q+2K-1}X^{1-K}D^{1-q-2K}R^{K-1}. \tag{3.12}$$

Since $q \ge 1-K$ and $K < 1$,

$$1-q-2K \le -K < 0, \qquad K-1 < 0.$$

The same estimate holds for $Y_0 < 1$ by the kernel tails. For each original parameter, the choices $\mathfrak{D}\mathfrak{t} \mid \mathfrak{k}$ and $\mathfrak{r} \mid \mathfrak{D}$ are divisor-many. Hence

$$\sum_i \|\alpha_i\|_2^2 \ll Z^{\epsilon}\|\alpha\|_2^2, \qquad \sum_i \|\beta_i\|_2^2 \ll Z^{\epsilon}\|\beta\|_2^2, \quad \sum_i \|\alpha_i\|_2\|\beta_i\|_2 \ll Z^{\epsilon}\|\alpha\|_2\|\beta\|_2.$$

Summing the dyadic decompositions and the common Mellin measure proves (3.9). □

Proposition 3.7 (Selected squarefree norm)**.** *Fix a compact interval $[K_0,K_1] \subset (1/2,1)$. For a sufficiently fine original mesh, selected squarefree parameters of norm at most H satisfy*

$$\mathcal{N}_{\mathcal{E}}(X) \ll Z^{\epsilon}\{X + H^{1-K}X^K\}, \qquad K \in [K_0,K_1]. \tag{3.13}$$

The assertion is hereditary in the sense of Lemma 3.2.

Proof. Initially Lemma 3.5 gives the all-source envelope $X + H^2X^K$ for $X \geq 1$. Suppose an all-source envelope with parameter exponent $q \geq 1 - K$ is known, and put $a = q + 2K - 1$. Lemma 3.6 gives $X + H^aX^{1-K}$ for $X \geq H$. Set $L = H^{a/K}$, which is at least H. For $X \geq L$ the reflected bound is $O(Z^\epsilon X)$. For $X < L$, amplify to source length L, within one mesh excess length, by Lemma 3.3. The result is

$$\mathcal{N}_{\mathcal{E}}(X) \ll Z^{\epsilon+O(\eta)}X^K L^{1-K} = Z^{\epsilon+O(\eta)}X^K H^{(1-K)(q+2K-1)/K}.$$

Consequently the parameter exponent is replaced by

$$q_{l+1} = \frac{1-K}{K}(q_l + 2K - 1), \qquad q_l - (1-K) = \left(\frac{1-K}{K}\right)^l (q_0 - (1-K)), \quad q_0 = 2. \tag{3.14}$$

Let $H \leq Z^{C_H}$ and $\vartheta = \max_{K\in[K_0,K_1]}(1-K)/K < 1$. Choose a fixed n with

$$C_H\vartheta^n \max_{K\in[K_0,K_1]}\{2-(1-K)\} < \epsilon/3.$$

Then $H^{q_n-(1-K)} \leq Z^{\epsilon/3}$. All intermediate source lengths are polynomially bounded. Divide the remaining loss among the n reflections and amplifications, and choose the mesh within that allocation.

For the small quotient bounds, choose $\xi > 0$ within the final loss. Trivial matrix estimation gives

$$H' \leq Z^\xi \quad \Longrightarrow \quad \mathcal{N}(Y) \ll Z^{\xi+\epsilon}Y.$$

For $H' > Z^\xi$, the contraction depth is uniform. The mesh is measured in the original Z-units. Fixed positive prime widths give finite word degrees, so the divisor estimates have arbitrarily small exponents with constants depending on those degrees. This proves the stated order of choices. □

3.4 The twenty-five shared cells

In this subsection, we extend the squarefree estimate of Proposition 3.7 to sixth-power-free parameters.

We now pass from squarefree parameters to natural sixth-power-free parameters. Write uniquely

$$\mathfrak{k} = \prod_{i=1}^{5} \mathfrak{k}_i^i, \qquad \mathfrak{k}_i \text{ squarefree}, \qquad (\mathfrak{k}_i, \mathfrak{k}_j) = 1 \quad (i \neq j). \tag{3.15}$$

Partition into common norm boxes for the five colours. Cauchy's inequality over those logarithmically many boxes reduces the norm to a single box on both sides. Write its first-colour cap as U and the product of its other four caps as V. The natural-index bound gives

$$UV^2 \ll H. \tag{3.16}$$

For a pair of parameters fix all ideals $\mathfrak{g}_{ij} = (\mathfrak{k}_i, \mathfrak{k}'_j)$. They are pairwise coprime. The exact residual digit in their primitive ratio is the following table, read modulo six:

$i\backslash j$	1	2	3	4	5
1	0	5	4	3	2
2	1	0	5	4	3
3	2	1	0	5	4
4	3	2	1	0	5
5	4	3	2	1	0

(3.17)

A diagonal cell cancels in the primitive inducing character but remains a common source deletion. Every off-diagonal cell has conductor exponent one, including digits two, three and four. This follows from tame Kummer ramification at the good primes. There is no deletion of its natural zero when the inducing character is formed.

Put

$$G=\prod_{i,j}\mathrm{N}\mathfrak{g}_{ij},\qquad F=\prod_{i\neq j}\mathrm{N}\mathfrak{g}_{ij},\qquad G_L=\prod_{j}\mathrm{N}\mathfrak{g}_{1j},\qquad G_R=\prod_{i}\mathrm{N}\mathfrak{g}_{i1}. \tag{3.18}$$

Then $F\le G$, $G_L,G_R\le G$. The two residual radical scales are comparable to UV/G. Hence the primitive good conductor has norm

$$Q\asymp F(UV/G)^2. \tag{3.19}$$

The numbers of possible residual higher-colour tuples on the left and right are, respectively, at most

$$Z^{\epsilon}VG_L/G,\qquad Z^{\epsilon}VG_R/G. \tag{3.20}$$

Indeed the product of the left higher residual ideals has norm at most a fixed multiple of VG_L/G. Each product ideal has divisor-many allocations to four pairwise coprime colours. The right side is identical. This counts actual higher tuples, not independently chosen parameter families for their individual factors.

Lemma 3.8 (Root phases on a fixed shared table)**.** *After fixing the finite packets and the shared table, the root number of the primitive ratio has no mixed factor between the two residual first-colour variables. If the higher tuples are also fixed, all other factors involving a first-colour variable are bounded multipliers on that variable alone. The remaining higher–higher factor has modulus at most one.*

Proof. For coprime residual components of colours i,j, CRT and reciprocity give

$$C_{ij}(\mathfrak{a},\mathfrak{b})=\chi_{\mathfrak{a}}(\mathfrak{b})^{i}\chi_{\mathfrak{b}}(\mathfrak{a})^{-j}=\chi_{\mathfrak{a}}(\mathfrak{b})^{i-j},\qquad C_{11}=1.$$

The opposite functional-equation convention conjugates this identity. Fix the higher tuples $\mathcal{A}=(\mathfrak{a}_2,\dots,\mathfrak{a}_5)$, $\mathcal{B}=(\mathfrak{b}_2,\dots,\mathfrak{b}_5)$ and the shared factors. For the primitive root number, the CRT factorization therefore has the form

$$\epsilon_*(\mathfrak{k},\mathfrak{k}')=C_{\mathcal{A},\mathcal{B}}\,u_{\mathcal{A},\mathcal{B}}(\mathfrak{a}_1)v_{\mathcal{A},\mathcal{B}}(\mathfrak{b}_1),\qquad |C_{\mathcal{A},\mathcal{B}}|\le 1,\quad |u|,|v|\ll 1.$$

Here $C_{\mathcal{A},\mathcal{B}}$ contains the higher–higher crossings; u,v contain the first–higher crossings, shared factors and separate root numbers. The different, class and bad-place factors belong to the fixed finite packets. No factor depends jointly on $\mathfrak{a}_1,\mathfrak{b}_1$. □

Lemma 3.9 (Natural near range)**.** *For selected natural sixth-power-free parameters of norm at most H,*

$$\mathcal{N}_{\mathcal{E}}(X)\ll Z^{\epsilon}X\qquad (X\ge H), \tag{3.21}$$

uniformly for K in a fixed compact subinterval of $(1/2,1)$.

Proof. Keep the original Gram diagonal separately. For each shared table and deletion divisor of norm R, (3.19) gives

$$\frac{XG}{RUV\sqrt{F}},\qquad T_0=\frac{FU^2V^2R}{G^2X} \tag{3.22}$$

Separate the common kernel in both full residual radical ratios. Fix the higher tuples $\mathcal{A},\mathcal{B}$. By Lemma 3.8, the remaining root factors are bounded separate first-colour multipliers. The parameter bounds and higher-tuple counts are

$$H_L=U/G_L,\qquad H_R=U/G_R,$$
$$N_L\ll Z^{\epsilon}VG_L/G,\qquad N_R\ll Z^{\epsilon}VG_R/G,\qquad \sqrt{N_LN_R}\ll Z^{\epsilon}V\sqrt{G_LG_R}/G.$$

Write the original parameters as $\mathfrak{k} = \mathfrak{D}_L\mathfrak{a}_1$ and $\mathfrak{k}' = \mathfrak{D}_R\mathfrak{b}_1$, with their own higher and shared factors in $\mathfrak{D}_L, \mathfrak{D}_R$. Then

$$Q_j^L(\mathfrak{a}_1) = Q_j(\mathfrak{D}_L\mathfrak{a}_1), \qquad Q_j^R(\mathfrak{b}_1) = Q_j(\mathfrak{D}_R\mathfrak{b}_1).$$

Coprimality with the opposite higher tuple remains a parameter restriction. Expand coprimality between the two first-colour variables by (3.11), as in Lemma 3.6. The quotient bounds only decrease, and all dual-source zeros remain. For each actual dual annulus $Y \geq 1$, Proposition 3.7 gives

$$\sqrt{\mathcal{N}_L(Y)\mathcal{N}_R(Y)} \ll Z^\epsilon\{Y + U^{1-K}Y^K\}.$$

The common kernel measure satisfies (3.7); for $A > 1$,

$$\sum_{Y\geq 1 \text{ dyadic}} (1 + Y/T_0)^{-A}\{Y + U^{1-K}Y^K\} \ll_A T_0 + U^{1-K}T_0^K.$$

Cauchy in the common dual source and then in $\mathcal{A}, \mathcal{B}$ therefore bounds each contribution by

$$Z^\epsilon \frac{X\sqrt{G_L G_R}}{RU\sqrt{F}}\{T_0 + U^{1-K}T_0^K\}\|\alpha\|_2\|\beta\|_2. \tag{3.23}$$

The linear part is

$$UV^2\frac{\sqrt{FG_LG_R}}{G^2} \leq UV^2 \ll H \leq X. \tag{3.24}$$

and the nonlinear part is

$$X^{1-K}(UV^2)^K R^{K-1}\frac{F^{K-1/2}\sqrt{G_LG_R}}{G^{2K}}. \tag{3.25}$$

Indeed,

$$\frac{F^{K-1/2}\sqrt{G_LG_R}}{G^{2K}} \leq G^{1/2-K} \leq 1, \qquad R^{K-1} \leq 1, \qquad X^{1-K}(UV^2)^K \ll X^{1-K}H^K \leq X.$$

The tails in (3.7) include $T_0 < 1$. For every original parameter, the number of shared-table and divisor assignments is at most

$$6^{\omega_F(\operatorname{rad}\mathfrak{k})}\tau_F(\mathfrak{k})^C \ll Z^\epsilon,$$

where ω_F counts distinct prime divisors and τ_F counts ideal divisors. Consequently the restricted original energies satisfy

$$\sum_\iota \|\alpha_\iota\|_2^2 \ll Z^\epsilon\|\alpha\|_2^2, \qquad \sum_\iota \|\beta_\iota\|_2^2 \ll Z^\epsilon\|\beta\|_2^2.$$

Cauchy in ι, followed by the finite-packet and dyadic summations, proves (3.21). □

The preceding argument holds for $1/2 < K < 1$. We state the range needed for the ordinary moments below; their fixed pointwise parameter satisfies $\kappa_1 < 3/4$.

Theorem 3.10 (Selected operator on natural parameters)**.** *Fix polynomial bounds for the parameter, source and displayed fixed moduli, $\epsilon > 0$, and a compact interval $[K_0, K_1] \subset (1/2, 3/4]$. There is an original fine mesh $\eta > 0$ such that the following holds. For a fixed finite descriptor of positive total width, selected natural sixth-power-free parameters of norm at most H satisfy*

$$\sum_{\mathfrak{k}\in\mathcal{E}}\left|\sum_{\mathrm{N}\mathfrak{n}\leq X} a_\mathfrak{n}\chi_\mathfrak{k}(\mathfrak{n})\right|^2 \ll Z^\epsilon\{X + H^{1-K}X^K\}\sum_\mathfrak{n}|a_\mathfrak{n}|^2. \tag{3.26}$$

Here $K \in [K_0, K_1]$, the source coefficients are arbitrary and common, and every further parameter restriction is allowed. The result holds on every literal inherited quotient family with its actual descriptor. Common fixed characters, natural source coprimality conditions and norm modes may be included in the coefficients. There is no principal parameter exclusion in this operator theorem.

Proof. For $X \geq H$, Lemma 3.9 applies. For $1 \leq X < H$, use Lemma 3.3 with $w = \log_Z(H/X)$:

$$H \leq XZ^{w'} \leq HZ^{O(\eta)}, \qquad \mathcal{N}_{\mathcal{E}}(X) \ll Z^{\epsilon+O(\eta)}(X/H)^K H = Z^{\epsilon+O(\eta)} H^{1-K} X^K.$$

Choose the mesh after allocating the loss, as in Proposition 3.7.

The bounded-source and empty-family cases follow directly. All quotient and coefficient assertions have been preserved at each preceding step. □

Remark 3.11 (Uniformity and finite orders). First fix the desired global loss, the compact slope interval and the numerical polynomial bounds. Choose a tiny global parameter-bound threshold and the finite contraction depth, then the original mesh. After that the arithmetic data, finite descriptor dictionary and its positive widths are fixed. Word degrees and the required derivative, Mellin-tail and height orders are then finite. The constants and the sufficiently large lower threshold for Z may depend on these fixed data. If a later application requires a finite height range, it is fixed after these orders, in the order of Section 2.4. The constants may depend on the distance of K from $1/2$, the number of factors and their positive widths. Each application uses a finite iteration with these quantities fixed.

3.5 Actual ordinary products and sixth-power restoration

In this subsection, we deduce the ordinary-parameter estimates used in Section 4 from Theorem 3.10.

We now distinguish the natural operator from the ordinary-parameter terminal used in the moment recursion. Write the moving presentation as

$$\psi_{\mathfrak{k}} = \nu \chi_{\mathfrak{A}} \chi_{\mathfrak{k}},$$

with all literal natural zeros and, if present, a common extra deletion retained. The displayed radical of the moving fixed data has logarithmic size q. The moving parameter bound is Z^m; its displayed allowance is $m+q$. More general finite products of fixed powers are included in $\chi_{\mathfrak{A}}$ with their original displayed radical, not redefined parameter by parameter.

Let S_1, S_2 be the ordinary sums in (2.1), of lengths Z^{n_1}, Z^{n_2}, and put $d = n_1 + n_2$. They are in the same moving-character orientation after a whole-factor conjugation, when one is needed. That conjugation includes its test function and norm mode. Let Q be the retained original prime product of total width z, and put $A = d + z$. Each literal product is a common central ideal polynomial: its coefficients are obtained by multiplicative convolution, have squared mass Z^{ϵ}, and have source length at most a fixed multiple of Z^d or Z^A, respectively. This follows directly from the divisor bound, without modifying either ordinary coefficient family.

The no-prime input used below is precisely [Liu26b, Theorem 12.16, the $z = 0$ clause]: on nonprincipal inducing parameters, the same two ordinary sums, at arbitrary bounded plain lengths, have mean $\ll Z^{u+q+\epsilon}$ for moving parameter bound Z^u.

Lemma 3.12 (An extra sixth-power deletion)**.** *Fix an ordinary sixth-power part $\mathfrak{c}^6$ and write $\mathfrak{k} = \mathfrak{c}^6\mathfrak{v}$, with $\mathfrak{v}$ sixth-power-free. The descriptor and every actual source factor acquire only the common natural deletion at $\mathfrak{c}$. The no-prime mean for the two ordinary factors remains $\ll Z^{u+q+\epsilon}$; the norm of $\mathfrak{c}$ is not added to the displayed radical q. The primitive inducing character and its nonprincipal exclusion are unchanged.*

Proof. The identity $\chi_{\mathfrak{c}^6}(\mathfrak{n}) = 1_{(\mathfrak{c},\mathfrak{n})=1}$ is literal, even if $(\mathfrak{c},\mathfrak{v}) \neq 1$. For each ordinary factor insert complete inclusion–exclusion:

$$S_{\mathfrak{c}}(X) = \sum_{\mathfrak{e} \mid \mathrm{rad}(\mathfrak{c})} \frac{\mu_F(\mathfrak{e})\psi_{\mathfrak{v}}^0(\mathfrak{e})}{\sqrt{\mathrm{N}\mathfrak{e}}} S^0(X/\mathrm{N}\mathfrak{e}).$$

The plain quotient remains unrestricted at the extracted divisor; its original character zeros remain. The parameter multiplier is bounded. The total root mass $\sum_{\mathfrak{e}|\mathrm{rad}(\mathfrak{c})}(\mathrm{N}\mathfrak{e})^{-1/2}$ is Z^{ϵ}. For $c_{\mathfrak{e}} = \mu_F(\mathfrak{e})/\sqrt{N\mathfrak{e}}$, Minkowski and the arbitrary-length no-prime estimate give

$$\|S_{1,\mathfrak{c}}S_{2,\mathfrak{c}}\|_{\ell^2(\mathfrak{v})} \ll Z^{(u+q+\epsilon)/2}\left(\sum_{\mathfrak{e}|\mathrm{rad}(\mathfrak{c})}|c_{\mathfrak{e}}|\right)^2 \ll Z^{(u+q+\epsilon)/2}.$$

The bounded parameter multipliers may be removed before the estimate. This is the expansion of [Liu26b, Lemma 12.5].

Deletion at a prime factor in a descriptor or a retained prime sum is simply a bounded common coefficient. No deleted Euler factor changes the primitive inducing character. □

Theorem 3.13 (High-amplitude estimates on ordinary parameters)**.** *Assume the actual inducing characters are nonprincipal. Retain the fixed test functions, finite norm modes and displayed support above. Let K lie in a fixed compact subinterval of $(1/2, 3/4]$.*

1. *Suppose the retained product Q itself selects the parameters and is on a common amplitude bin $Z^{Kz} \le |Q|^2 \le Z^{Kz+\omega}$, where ω is part of the allocated loss. Then the sum over ordinary parameters $\mathrm{N}\mathfrak{k} \le Z^m$ satisfies*

$$\sum_{\mathfrak{k}\in\mathcal{E}}|S_1S_2Q|^2 \ll Z^{\epsilon+\omega+\max\{(1-K)m+KA,(m+q)/6+5A/6\}}. \tag{3.27}$$

2. *Suppose instead that an arbitrary nonempty original block selects the parameters at slope K. It may be a deleted block and need not be the whole retained Q. If the latter satisfies the pointwise upper bound $|Q|^2 \ll Z^{pz+\epsilon}$ on the selected parameters, then*

$$\sum_{\mathfrak{k}\in\mathcal{E}}|S_1S_2Q|^2 \ll Z^{\epsilon+\max\{(1-K)m+KA,(m+q)/6+5A/6+pz/6\}}. \tag{3.28}$$

   For the original Hecke-compatible prime factors, if in addition $\mathrm{ind}(\psi_{\mathfrak{k}}) \notin \Theta$, one may take $p = \kappa_1$ by [Liu26b, Lemma 12.7]. For the other nonprincipal inducing characters, or for arbitrary bounded common prime coefficients, the choice $p = 1$ is available by absolute counting.

Both assertions permit further parameter restrictions. Their no-prime comparison is used only on the same two actual ordinary factors.

Proof. Write $\mathfrak{k} = \mathfrak{c}^6\mathfrak{v}$ uniquely and put $s = \log_Z \mathrm{N}\mathfrak{c}$, $u = m - 6s$. There are at most $Z^{s+\epsilon}$ choices of $\mathfrak{c}$ in its dyad. For each fixed choice, the selected natural operator applies to $\mathfrak{v}$ with the common deletion described in Lemma 3.12. All its descriptor widths and selection lower bounds remain those of the actual block.

For the first assertion, apply the operator to S_1S_2 and multiply by the upper bound for Q. Apply the no-prime comparison on that same positive parameter subset, and then drop the subset in this comparison alone. The resulting exponent, before optimizing the sixth root, is

$$s + Kz + \min\{\max(d, (1-K)u + Kd), u + q\}. \tag{3.29}$$

For $u \ge d$, use $s = (m-u)/6$ and

$$\frac{d}{du}\{(1-K)u + Kd - u/6\} = 5/6 - K > 0.$$

For $0 \le u \le d$,

$$\min(d, u+q) - u/6 \le 5d/6 + q/6,$$

by splitting at $u = (d-q)_+$. Consequently the two ranges give

$$(1-K)m + K(d+z), \qquad (m+q)/6 + 5d/6 + Kz \le (m+q)/6 + 5(d+z)/6,$$

respectively, since $K \le 3/4 < 5/6$.

For the second assertion, use the literal whole polynomial $S_1 S_2 Q$ as the arbitrary common source in Theorem 3.10. It has length Z^A; its selection descriptor remains the specified original block. On the same positive subset the no-prime comparison, multiplied by the stated upper bound for the retained Q, gives $Z^{u+q+pz+\epsilon}$. The exponent to optimize is now

$$s + \min\{\max(A, (1-K)u + KA), u + q + pz\}. \tag{3.30}$$

The same calculation with (d, q) replaced by $(A, q+pz)$ gives

$$\sup_{0\le u\le m} \left\{ \frac{m-u}{6} + \min\big(\max(A, (1-K)u + KA), u + q + pz\big) \right\} \le \max\{(1-K)m + KA, (m+q+pz)/6 + 5A/6\}.$$

This proves (3.28). Absolute prime counting allows $p = 1$; the fixed factor list and test functions affect only the subsidiary loss. □

The extra term $\kappa_1 z/6$ in the block-selected terminal is essential when only a proper block is known to be heavy. It is not removed by treating that block as if it were the whole retained prime product. Conversely the whole-product amplitude bin in (3.27) supplies its sharper plane without that extra term. Neither terminal is a statement for arbitrary ordinary coefficients in the baseline comparison, nor does either assert an estimate for a selected Gauss-column family. Those distinctions will be maintained in the moment recursion.

4 Ordinary product moments

In this section, we prove the ordinary moment with prime coefficient three and then coefficient two. We use the earlier moments of Section 2.2 and Theorems 3.10 and 3.13. Retain the prime exponent in (2.4); in particular,

$$\frac{1}{2} < \kappa_1 < \frac{3}{4}. \tag{4.1}$$

The bound $|Q|^2 \ll Z^{\kappa_1 z+\epsilon}$, with its polynomial height factor, is [Liu26b, Lemma 12.7] outside Θ. We keep this pointwise parameter fixed throughout the two inductions.

Write m for the free ordinary parameter exponent, q for the full displayed radical exponent, and

$$M = m + q, \qquad d = n_1 + n_2, \qquad A = d + z.$$

In particular A includes the prime positions. The letters used for first or second common divisors below never denote an additional free family of characters. Their norms and their divisor counts are those of [Liu26b, Sections 12.5–12.7].

Definition 4.1. For $0 < B \le 1$ and $L > 1$, let $\mathcal{G}(B, L)$ denote the following assertion on the complete ordinary family of (2.1)–(2.2):

$$d + Lz \le BM \quad \Longrightarrow \quad \sum_{u}^{\mathrm{ind}(\psi_u)\notin\Theta} \mathcal{W}(u)|S_{n_1,1}(u)S_{n_2,2}(u)Q(u)|^2 \ll Z^{M+\epsilon}\mathfrak{S}_J(\mathcal{W}, \mathbf{W})(1 + H_{\mathrm{in}})^{J'}. \tag{4.2}$$

Here all ordinary parameters, including their sixth powers, are counted. The assertion is uniform over bounded ranges of the declared lengths, displayed radicals and common extra coprimality

conditions. For each requested $\epsilon > 0$ there is an original prime-slot mesh, chosen before the arithmetic data and the finite factor list, for which the assertion holds. After those data are fixed there are finite derivative orders J, J', an implied constant, and an asymptotic threshold. The constant may depend on the fixed field, finite coefficients, factor list and test-function seminorms, but not on the actual ordinary parameter or its amplitude cell. At $z = 0$ we also use the arbitrary-length no-prime assertion (2.3), whether or not its lengths satisfy the displayed length inequality.

The only length restriction is $d + Lz \le BM$. The two ordinary factors have the smooth coefficients of (2.1); the selected operator controls the high-amplitude terms. All character factors, including canceled displayed factors, retain their natural zero extensions. Passing from the source orientation $\chi_{\mathfrak{n}}(u)$ to $\nu\overline{\chi_u}$ conjugates the entire factor and its test function, with the finite reciprocity packet retained.

4.1 Ordinary reflection and the exact returned family

In this subsection, we record the returned character families used in Section 4.2.

The following subsection gives the local exponent inequalities.

A common extra radical R_*, fixed within the current parameter sum, is expanded before every ordinary reflection or recursive call. For a centrally normalized factor the expansion has the form

$$S_{u,R_*}(X) = \sum_{\mathfrak{e}|R_*} \frac{\mu(\mathfrak{e})\psi_u^0(\mathfrak{e})}{\sqrt{N\mathfrak{e}}} S_u^0(X/N\mathfrak{e}), \tag{4.3}$$

and a prime position has the corresponding expansion

$$Q_{i,R_*} = Q_i^0 - P_i^{-1/2} \sum_{\mathfrak{p}|R_*} b_i(\mathfrak{p})\psi_u^0(\mathfrak{p}) W_i(N\mathfrak{p}/P_i).$$

The total absolute divisor mass is Z^ϵ. The scalar $\psi_u^0(\mathfrak{e})$ is bounded and may be zero. The plain quotient in (4.3) is unrestricted: a further coprimality condition with $\mathfrak{e}$ would change the identity. This expansion does not permit a cancelled moving character factor to be reclassified as an extra coprimality condition.

For a nonprincipal inducing character, let C_u be its primitive conductor and $R_{0,u}$ its natural redundant radical. By [Liu26b, Lemma 12.6],

$$S_u^0(X; W) = \epsilon_u \sum_{\substack{\mathfrak{e}|R_{0,u} \\ \mathrm{rad}(\mathfrak{f})|R_{0,u}}} \frac{\mu(\mathfrak{e})\psi_u^*(\mathfrak{e})\overline{\psi_u^*(\mathfrak{f})}}{\sqrt{N\mathfrak{e}N\mathfrak{f}}} S^0_{\overline{\psi_u^0}}\left(\frac{C_u N\mathfrak{e}}{XN\mathfrak{f}}; W^\sharp\right). \tag{4.4}$$

The conductor and natural deletion satisfy $C_u N R_{0,u} \ll Z^M$. Thus a factor of exponent n reflects to lengths at most $M - n + \xi$, with total coefficient mass Z^ϵ. Lower dyadic annuli have summable inverse-root mass, and upper tails have arbitrarily rapid decay. We conjugate the whole reflected factor inside its absolute value; this restores the same moving character on the two ordinary factors. Incoming norm modes are absorbed into their test functions before reflection. Conductor-dependent test functions are separated with one common Sobolev measure on the whole product, retaining its total seminorm.

The complete equal-product difference has two scales (X_1, X_2), (Y_1, Y_2) with

$$X_1 X_2 = Y_1 Y_2. \tag{4.5}$$

The same fixed character, natural deletions, norm mode and remaining prime product occur in both rectangles on each arithmetic side. The fixed-character Mellin mean is $c_{\vartheta,R_*} T^{1+it} I_W(t)$ plus the source remainder. Its leading products cancel by (4.5). If all four scales are at least Z^ℓ, the raw centered product is smaller by a factor $Z^{-\ell}$; otherwise its absolute raw-volume bound remains valid [Liu26b, Section 12.5]. Reflection is applied to the whole signed difference.

The first transformed side. Let c_1, d_1 be the exponents of the two full first common divisors, p_1 their radical exponent, and R, E the primitive-difference and complementary radical exponents. Put

$$\widetilde{q} = q + R + E, \qquad K_0 = 2A - c_1 - d_1 + R + E - m, \qquad K \le K_0 + \xi_1. \tag{4.6}$$

Here K is the actual frequency annulus and ξ_1 is an arbitrarily small endpoint buffer. On the C-side the additional allowance is

$$B_C = (3c_1 - 5d_1 - R)_+/6, \tag{4.7}$$

and B_D is obtained by interchanging the sides. The first family count is $Z^{p_1+s_0+\epsilon}$, once, before the common Fourier measure. The full paired exponent is

$$M + \frac{1}{2}\{B_C + B_D - (c_1 + d_1 - 2p_1 - R)\} \le M. \tag{4.8}$$

For completeness, at a prime with valuations $i \ge j \ge 1$ write $r = 1_{6\nmid i-j}$. The available exponent is $i + j - 2 - r$, whereas the nonzero positive part is at most $(3i - 5j - r)_+/6$. If this is positive, six times the difference is $3i + 11j - 12 - 5r \ge 0$; for $r = 1$ one has $i \ge j + 1$. If it is zero, the available exponent is nonnegative directly. Subadditivity of the positive part proves (4.8) after summing the local valuations.

The finite character on one positive side need not equal that on the other. Within a side, however, the source complete-product coefficient remains a single common coefficient. A genuine prime power is assigned either to an ordinary factor or to one original prime position; its actual power divides both comparison scales in that factor. The finite unit packet remains in Θ. The inducing group is classified only after the returned character has been formed.

The auxiliary main term and its three errors. Choose a small positive σ, and an auxiliary prime pool at exponent $\sigma/3$, outside all original prime annuli. The original mesh is smaller than $\sigma/6$. Excluding the fixed support and the current frequency removes only logarithmically many primes. The pool has cardinality $Z^{\sigma/3+o(1)}$. Write $P = \log_Z N\mathfrak{p} = \sigma/3 + o(1)$. The exact Gauss difference has only the error valuations $1, 6, 7$; its squared local factors are respectively $N\mathfrak{p}^{-1}, (1 - N\mathfrak{p}^{-1})^2, N\mathfrak{p}^{-1}$. The normalized pool average is retained on every error. There is no extra pool cardinality after Jensen's inequality.

Let w_a be the ordinary width removed at the auxiliary prime, $w_{a,o}$ its odd-radical width, and ℓ_a the main density saving. Set

$$a_0 = A - c_1 - w_a, \qquad D_0 = d_1 + (K_0 - K), \qquad J_0 = D_0 - c_1 - 2w_a + w_{a,o}.$$

The exact choices are

	w_a	$w_{a,o}$	ℓ_a	$g - J_{0,+}$
main	0	0	$\sigma/3$	2σ
$i = 1$	P	P	0	σ
$i = 6$	$6P$	0	0	σ
$i = 7$	$7P$	P	0	σ

(4.9)

The auxiliary prime is assigned only to ordinary positions. In particular it does not create a new original prime factor. The complete one-sided allowance after its local squared factor is

$$\Lambda_C = a_0 + \widetilde{q} + w_{a,o} + w_a + B_C - s_0. \tag{4.10}$$

The second transformed family. Let c_2, d_2 be the full second common exponents, $\bar{c}_2 = (c_2 + d_2)/2$, p_2 their radical exponent and g_2 their gcd exponent. Among primes with equal valuations $i = j$ and $6 \nmid i$, denote the radical exponents of the unit and nonunit divided-frequency sets by t_2, V, respectively. All other common primes contribute zero to these two exponents. At

a common prime put $P = N\mathfrak{p}$ and write h_0 for the frequency after division by the gcd. For $i \geq j \geq 1$, [Liu26b, Lemma 12.10] gives the following absolute values and the upper bounds used in the exponent calculation:

case	absolute value	upper bound
$i = j,\ 6 \nmid i,\ \mathfrak{p} \nmid h_0$	P^{i-1}	P^{i-1}
$i = j,\ 6 \nmid i,\ \mathfrak{p} \mid h_0$	$P^{i-1}(P-1)$	P^{i}
$i = j,\ 6 \mid i,\ \mathfrak{p} \nmid h_0$	$P^{i-1}(P-2)$	P^{i}
$i = j,\ 6 \mid i,\ \mathfrak{p} \mid h_0$	$P^{i-1}(P-1)$	P^{i}
$i > j,\ 6 \mid j,\ \mathfrak{p} \nmid h_0$	$P^{j-1}(P-1)$	P^{j}

(4.11)

The remaining unequal cases vanish; the case $j > i$ follows by interchanging the columns. In particular, an unequal surviving pair has unit divided frequency. The last column, rather than the exact middle column, supplies the following exponent estimates. The free and displayed widths of the resulting ordinary family are

$$m' = 2a_0 - K - g - g_2 - V, \tag{4.12}$$

$$q' = \widetilde{q} + w_{a,o} + t_2 + V,$$

$$M' = m' + q' = M + J_0 - g - g_2 + t_2. \tag{4.13}$$

For a nonempty annulus,

$$g_2 \geq t_2, \qquad m'_{\mathrm{act}} = \max(0, m' + \xi_2), \qquad \#\mathcal{F} \ll Z^{p_2 - s_0 + \epsilon},$$

where ξ_2 is the endpoint buffer and the family is counted once. Including the exterior factor gives

$$\lambda_0 = K + g - \ell_a - a_0 + p_2 - s_0 + g_2 - t_2 - \bar{c}_2.$$

The identity used for the later ordinary estimate is

$$\Lambda_C - \lambda_0 = M' + \Delta', \qquad \Delta' = \bar{c}_2 - p_2 + w_a + B_C + \ell_a \geq w_a + \ell_a. \tag{4.14}$$

The final artificial coprimality expansion is made on this full product before positivity. At a prime it is the literal identity

$$1_{\mathfrak{p} \mid \prod_j \mathfrak{n}_j} = \sum_{\varnothing \neq I} (-1)^{|I|+1} \prod_{j \in I} 1_{\mathfrak{p} \mid \mathfrak{n}_j}.$$

A selected ordinary quotient is unrestricted; a selected prime position is fixed. Conflicting assignments give zero. There is no Gauss coefficient after this transformation. Thus each positive side is exactly two ordinary factors and a subproduct of the remaining original prime factors, with one common character.

It is useful to state explicitly the normalization at this last expansion. Put $\alpha_1 = a_0 - c_2$, $\alpha_2 = a_0 - d_2$. For a divisor annulus of exponent t_-, let $\widetilde{r}_j$ be its actual ordinary removals together with the nominal lengths of fixed prime positions, and let e_j, ω_j record their endpoint ratios. The exact identities are

$$T'_j \prod_{\text{remaining}} P_i = Z^{\alpha_j + e_j - \widetilde{r}_j}, \qquad \widetilde{r}_j + \omega_j \geq t_-, \qquad |e_j|, |\omega_j|, |e_j + \omega_j| \leq \theta_N.$$

The divisor count is $\mathcal{O}(Z^{t_-})$. Replacing a nonempty subunit ordinary scale by exponent zero changes the paired normalization by at most $2\theta_N$. More precisely, if π_0, π are the source negative-length corrections, then $0 \leq \pi_0 \leq \pi \leq \theta_N$, $r_j^+ = \widetilde{r}_j - (\pi - \pi_0) \geq 0$, the actual length is $\alpha_j + e_j + \pi_0 - r_j^+$, and its exterior factor is $Z^{(-r_j^+ + e_j + \pi_0)/2}$. Pairing the two sides with their single divisor count gives the stated $2\theta_N$ bound. Earlier fixed factors are not counted again at this stage.

4.2 Local inequalities for the returned ordinary bounds

In this subsection, we establish the local inequalities used in Sections 4.3 and 4.6.

We use affine exponents

$$F = sm + tq + \lambda A + \omega z. \tag{4.15}$$

In every application, $0 \le t \le s \le 1$, $\lambda \ge 0$, and the prime slope ω is nonnegative. An extraction of a remaining original prime therefore improves this last term. For the other terms the literal first-common increment, before the nonpositive $-B_C$, is bounded locally by

$$-\lambda i + (1-s)(i-j)_+ + (t-s)(r+e). \tag{4.16}$$

Here r, e indicate the actual primitive and complementary radicals. If one already marked original position is present, the expression has the additional λb, where $b \in \{0,1\}$ and $b \le \min(i,j)$. The other side has the interchanged expression. The coefficient of the global positive part is nonnegative in every application. We may thus bound it by the sum of the local positive parts; we do not replace a negative global $-B_C$ by a sum of negative local terms.

For $s + \lambda = 1$, $0 \le t \le s$, (4.16) is at most

$$-\lambda(\min(i,j) - b) - (s-t)(r+e) \le 0. \tag{4.17}$$

This applies to the baseline M, to $(1-K)m + KA$, and to $M/6 + 5A/6$. It is an individual-side inequality, so that the two sides of Cauchy's inequality may use different terminal bounds.

For the second common divisor, let $i \ge j$ be the two valuations and let $i_\star \in \{i, j\}$ be the valuation removed on the side being estimated. Let t_u, v be the local indicators contributing to t_2, V, respectively; both vanish unless $i = j$ and $6 \nmid i$. The increment is

$$1 + j - t_u - \frac{i+j}{2} - \lambda i_\star - s(j+v) + t(t_u + v). \tag{4.18}$$

The term $-(i+j)/2$ comes from $-\bar{c}_2$ in the exterior factor; $-\lambda i_\star$ comes from the actual column length on that side. If $s + \lambda = 1$, the complete table is

$$\begin{array}{c|c} i = j,\ 6 \nmid i,\ \text{unit} & -i + t \\ i = j,\ 6 \nmid i,\ \text{nonunit} & 1 - i - s + t \\ i = j,\ 6 \mid i & 1 - i \\ i > j,\ 6 \mid j,\ i_\star = i & 1 - j - (1/2 + \lambda)(i-j) \\ i > j,\ 6 \mid j,\ i_\star = j & 1 - j - (i-j)/2 \end{array} \tag{4.19}$$

All entries are nonpositive: in the last two cases $j \ge 6$. Thus the estimate holds separately on each side, for every valuation.

The auxiliary contribution to (4.15), with $\gamma = g - J_{0,+}$, is

$$(1-s)\gamma + (1 - \lambda - 2s)w_a + (t-1)w_{a,o} - \ell_a. \tag{4.20}$$

The main value is $(5/6 - s)\Gamma$, where $\Gamma = 2\sigma$. For $s + \lambda = 1$, each error minus this main value is

$$\frac{\sigma}{3}\{-2 + 3s - si + (t-1)e\} + o(\sigma), \qquad (i,e) = (1,1), (6,0), (7,1).$$

It is nonpositive for $0 \le t \le s \le 1$: for the first pair it is $(-3 + 2s + t)\sigma/3$, and the other two are smaller. The main rates for the two selected bounds are therefore $K - 1/6$ and $2/3$, respectively.

4.3 Amplitude separation and ordinary prime coefficient three

In this subsection, we prove the coefficient-three estimate needed in Section 4.6.

Lemma 4.2 (Complete low subproducts)**.** *Fix the test functions, common norm modes and finite coefficients in a positive ordinary family. On a finite amplitude cell on which $|Q|^2 \le Z^{kz+o(1)}$, one may order the complete prime factors so that every initial complete block of width v has squared amplitude at most $Z^{kv+o(1)}$. A requested width is exceeded by at most one original slot.*

Proof. For nonzero factors write $|Q_i|^2 = Z^{a_i z_i + o(1)}$ and order $a_1 \le \cdots \le a_J$. For every $j \le J$,

$$\frac{\sum_{i\le j} a_i z_i}{\sum_{i\le j} z_i} \le \frac{\sum_{i\le J} a_i z_i}{\sum_{i\le J} z_i} \le k + o(1).$$

For $0 < v_0 \le z$, choose j minimally with $v = \sum_{i\le j} z_i \ge v_0$. Then

$$v_0 \le v < v_0 + \eta, \qquad \left|\prod_{i\le j} Q_i\right|^2 \le Z^{kv+o(1)}.$$

Zero factors contribute nothing. Truncate very negative logarithms using the polynomial bounds for the other factors. The finite bins, orderings and subsets affect only constants. After removal, discard the amplitude restriction before applying the ordinary moment. □

For the later finite iteration it suffices to keep $0 < k - 1/2 \le 1/2000$. This fixed neighborhood determines the upper bound in the next proposition and in Proposition 4.8.

Proposition 4.3. *For every fixed $1/2 < k \le \min(1001/2000, \kappa_1)$, the assertion $\mathcal{G}(1, 6k)$ holds.*

Proof. Put $p = 6k$, $q_k = p - 1 > 2$. First consider a positive ordinary family with $A + q_k z \le M$. A long ordinary factor is reflected by (4.4), after the extra-coprimality condition expansion. This decreases its declared length whenever it is longer than half the conductor allowance, and preserves the length condition, with an arbitrarily small buffer.

If $A > 5M/6$, choose a reference ordinary length $M/4$. When an original factor is shorter, reflect its partner directly. Otherwise form the equal-product comparison with ordinary lengths $M/4, d - M/4$, then reflect the longer factor of that positive comparison only. The resulting total column length satisfies

$$A_{\mathrm{comp}} \le \frac{3}{2}M - A + 2z + \xi.$$

Since $z \le (M - A)/q_k$, we have

$$A_{\mathrm{comp}} \le \left(\frac{2}{3} + \frac{1}{3q_k}\right) M + \xi < 5M/6, \tag{4.21}$$

$$A_{\mathrm{comp}} + q_k z \le \left(\frac{5}{6} + \frac{1}{3q_k}\right) M + \xi < M. \tag{4.22}$$

Both strict inequalities have the gap

$$\gamma_k = \frac{1}{6} - \frac{1}{3(6k-1)} = \frac{2k-1}{2(6k-1)} > 0. \tag{4.23}$$

Thus the reference is an uncentered family at the same parameter scale. It is also balanced. Indeed, in this case

$$z < \frac{M}{6q_k} < \frac{M}{12}, \qquad d = A - z > \frac{3M}{4}, \qquad M - (d - M/4) = \frac{5M}{4} - d < \frac{M}{2}.$$

The other reference length is $M/4$. The strict gaps absorb the reflection buffer, chosen below $\gamma_k M/4$. This comparison uses the finite induction below, not the proposition before it is proved. When $A \le 5M/6$, no centering is required.

Apply the first transformation to the uncentered product or the centered difference. There is a preliminary direct terminal if its completed free width or its formal free width is at most a fixed small $\tau > 0$. Lemma 4.5 and Proposition 4.6 below are consequences of the exact full finite correlation alone; they use no ordinary moment. Here their unmarked specialization has $\mathcal{R}_{\mathrm{loc}} = 0$, $A_* = A$, $M_* = M$, so its relative contribution is at most $A - M + \tau + O(\xi) \le \tau + O(\xi)$. This is a single terminal loss, even when the free width is small because the displayed radical is large. On the complementary range both free widths have fixed positive lower bounds, and the source auxiliary amplification and second transformation apply. Only after its actual ordinary sides and all divisor allocations have been formed do we separate amplitudes. Let A', z', M' be one such side. Before harmless endpoint errors,

$$A' \le A - c_1 - w_a - c_2, \qquad z' \le z, \qquad M' = M + J_0 - g - g_2 + t_2.$$

Since $g_2 \le c_2$, its excess in the length inequality is bounded by

$$\begin{aligned} A' + q_k z' - M' &\le w_a - D_0 - w_{a,o} + g + (g_2 - c_2 - t_2) \\ &\le w_a - D_0 - w_{a,o} + g. \end{aligned} \tag{4.24}$$

The source first-frequency inequality is $-D_0 - w_{a,o} + J_{0,+} \le \xi_1$. On the main branch $g - J_{0,+} = 2\sigma = 6\ell_a$. On an error branch it is σ, and $w_a \ge \sigma/4$ for large Z; hence $w_a + \sigma \le 5w_a \le 6w_a$. The actual-scale normalization already recorded above therefore gives

$$A' + q_k z' - M' \le 6(w_a + \ell_a) + \xi_1 + 2\theta_N. \tag{4.25}$$

On a low-amplitude set, Lemma 4.2 gives a subproduct of width v, where, until exhaustion,

$$D' = (A' + q_k z' - M')_+, \qquad \frac{D'}{6k} \le v < \frac{D'}{6k} + \eta, \qquad |Q_{\mathrm{removed}}|^2 \le Z^{kv + o(1)}.$$

By (4.14) and (4.25),

$$kv \le D'/6 + k\eta \le \Delta' + \xi_1/6 + \theta_N/3 + k\eta + o(1).$$

If the subproduct exhausts the list, use the actual arbitrary-length no-prime theorem. Otherwise discard the amplitude restriction positively and continue on the complete ordinary family. In particular a selected parameter restriction is never passed through Poisson summation.

On a high-amplitude cell, except when the remaining total prime width is below a fixed small floor, its actual slope belongs to $[k, \kappa_1 + v] \Subset (1/2, 3/4)$. Here the small v includes the bin and truncated-height buffers. The whole-product version of Theorem 3.13 gives the two terminal exponents

$$(1 - K)m + KA, \qquad M/6 + 5A/6. \tag{4.26}$$

Both are at most M on a family satisfying the length inequality. Equations (4.17) and (4.19) bound all common-support changes on each side separately. The main and error amplification contributions are bounded by a fixed multiple of σ, by (4.20). This branch terminates, so that this loss is incurred once. A remaining prime product below the fixed floor is also terminal: strip it at the old price κ_1 and apply the no-prime theorem. The sixth powers in the ordinary parameters have already been counted in Theorem 3.13.

We next verify the origins and the exceptional inducing group. The first origin is $\mathcal{O}(Z^{m+\epsilon})$. The second zero frequency is the exact $u = v$ correlation with its $\varphi(u)$ factor; its excess is at most

$$A - M + 5\sigma/3 + \xi_1 + \theta_N + o(1). \tag{4.27}$$

This is terminal and is $\mathcal{O}(\sigma)$ under $A \le M$. For the fixed-character contribution, the formulas of [Liu26b, Sections 12.5–12.7], with the exact exterior, give the excess

$$A - 5M/6 - F_1 - F_2 + 2\theta_N + \xi_2/6 + o(1), \tag{4.28}$$

where

$$F_1 = c_1/6 + 5D_0/6 + w_a/3 + \widetilde{q}/6 + w_{a,o} + B_C - 5g/6 + \ell_a,$$
$$F_2 = 2\bar{c}_2 - 5g_2/6 - p_2 + t_2 + V/6 + f/6.$$

Let $\mathfrak{f}$ be the product of the new equal-valuation-one primes with nonunit divided frequency. On an exceptional inducing parameter,

$$f = 2\log_Z N\mathfrak{f}, \qquad \mathfrak{f}^4 \mid \text{the sixth-power-free frequency}, \qquad (N\mathfrak{f})^4 = Z^{2f}.$$

The ordinary sixth-power part thus has norm at most $Z^{(m'_{\mathrm{act}}-2f)/6}$. The exceptional count is $\ll Z^{(m'_{\mathrm{act}}-2f)/6+\epsilon}$, which gives the displayed $f/6$ term.

With $g = J_{0,+}$, the case $J_0 \ge 0$ gives $F_1 = c_1 + 2w_a + \widetilde{q}/6 + w_{a,o}/6 + B_C + \ell_a$. For $J_0 < 0$, use

$$B_C + \widetilde{q}/6 \ge (3c_1 - 5D_0)_+/6 - 5\xi_1/6$$

to obtain $c_1/6 + 5D_0/6 + B_C + \widetilde{q}/6 \ge 2c_1/3 - 5\xi_1/6$. Restoring $g - J_{0,+} \le 2\sigma$ and $w_a \le 7\sigma/3 + o(1)$ proves

$$F_1 \ge 2(c_1 + w_a)/3 - 3\sigma - 5\xi_1/6 + o(1). \tag{4.29}$$

The local values of F_2 are

case	F_2	$\bar{c}_2$
$i = j,\ 6 \nmid i,$ unit	$7i/6$	i
$i = j,\ 6 \nmid i,$ nonunit	$(7i-5)/6 + 1_{i=1}/3$	i
$i = j,\ 6 \mid i$	$7i/6 - 1$	i
$i > j,\ 6 \mid j$	$i + j/6 - 1$	$(i+j)/2$

Each is at least $2\bar{c}_2/3$. For the second case equality is allowed at $i = 1$; for $i \ge 2$ the difference is $(3i-5)/6 > 0$. The third and fourth cases use $i \ge 6$ and $j \ge 6$, respectively. This proves the assertion for arbitrarily large valuations.

In the uncentered case $A \le 5M/6$, these estimates finish (4.28) with a terminal loss $3\sigma + o(1)$. In the centered case set $v = c_1 + w_a + \min(c_2, d_2)$, and let $r = (M/4 - v)_+$. The complete final divisor allocation still retains the centered saving: its exponent contains

$$t_- - \widetilde{r}_1 - \widetilde{r}_2 - (r - \widetilde{r}_1)_+ = (t_- - \widetilde{r}_2) - \max(r, \widetilde{r}_1) \le \omega_2 - r.$$

Thus its raw volume is at most $Z^{a_0 - \bar{c}_2 - r + 2\theta_N + \epsilon}$. Since $F_1 + F_2 \ge 2v/3 - 3\sigma - o(1)$, the exceptional excess is at most

$$A - 5M/6 - 2v/3 - (M/4 - v)_+ + 3\sigma + o(1) \le (A - M)_+ + 3\sigma + o(1).$$

This uses $\sup_{v \ge 0}\{-2v/3 - (M/4 - v)_+\} = -M/6$. Principal and finite inducing parameters have therefore been bounded before any nonprincipal prime-amplitude estimate is used.

We give the finite-induction details to specify the order of the small parameters. Fix bounded ranges of all length exponents. Choose a terminal parameter floor $\rho_0 > 0$, a prime-width floor, a free-width threshold $\tau > 0$, and $\sigma > 0$, so that every terminal loss above is less than a prescribed fraction of ϵ. At $M \le \rho_0$, length condition implies $z \le \rho_0/(6k)$, so the old prime bound and the no-prime theorem give the required terminal estimate. For larger M, choose the reflection buffer smaller than $\gamma_k \rho_0$. A nonreflection return in (4.13) decreases combined width by at least $\sigma/2$, after all endpoint buffers. At fixed width, ordinary reflection changes an arbitrary-input call to

a balanced call. Give an arbitrary input the flags $(c,p)=(1,1)$, and a balanced input the flags $(c,p)=(1_{A>5M/6},0)$. Use the rank

$$\mathcal{R}(M,c,p)=4\lceil 4M/\sigma\rceil+2c+p.$$

The possible continuing steps satisfy

step	change
ordinary reflection	$p:1\longmapsto 0$
reflected comparison	$(c,p):(1,0)\longmapsto(0,0)$
nonreflection return	$M'\le M-\sigma/2.$

The last step decreases $4\lceil 4M/\sigma\rceil$ by at least eight, whereas $0\le 2c+p\le 3$. Hence every continuing step decreases $\mathcal{R}$, and the depth is bounded in terms of the fixed numerical parameters.

Choose the original mesh smaller than the selected-operator mesh, $\sigma/6$, and the requested loss divided by this depth bound. After the arithmetic and finite factor lists are fixed, choose endpoint, test-function separation and amplitude-bin errors so that their sum along a path is smaller than the remaining loss. Weighted Cauchy and the common Fourier measures give a finite tree of complete ordinary calls. Each continuing low edge contributes only its allocated error; each path has one terminal contribution. This proves (4.2) by the decreasing rank.

Choose the derivative orders backwards through the finite induction. For an internal height order C_H, take $\tau_H>0$ with

$$C_H\tau_H<\text{the allocated loss},\qquad \kappa_1+O(C_H\tau_H)<3/4.$$

Thus the bounded-height argument applies when $H_{\mathrm{in}}\le Z^{\tau_H}$. For larger heights, write the polynomial bound as $Z^{A_H}(1+H_{\mathrm{in}})^{C_0}$ and choose $J'\ge C_0+A_H/\tau_H$. Then

$$H_{\mathrm{in}}>Z^{\tau_H}\quad\Longrightarrow\quad Z^{A_H}(1+H_{\mathrm{in}})^{C_0}\le(1+H_{\mathrm{in}})^{J'}.$$

This final order is chosen after τ_H; constants may depend on k. □

Corollary 4.4. *The ordinary moment assertion $\mathcal{G}(1,3)$ holds.*

Proof. Fix the loss and the bounded range of z. Choose $\delta>0$ small, put $k=1/2+\delta$, $p_\delta=3+6\delta$, and apply Proposition 4.3. Remove a parameter-independent complete subproduct with requested width $v_0=6\delta z/p_\delta$. Unless the product is exhausted,

$$v_0\le v<v_0+\eta,\qquad d+p_\delta(z-v)\le d+3z\le M,$$

and the pointwise estimate contributes at most

$$\kappa_1 v\le\frac{6\kappa_1\delta z}{3+6\delta}+\kappa_1\eta\le 2\kappa_1\delta z+\kappa_1\eta.$$

Choose δ, then η, within the allotted loss before the arithmetic data. At exhaustion apply the arbitrary-length no-prime theorem. All choices are finite, with $k>1/2$. □

4.4 A direct estimate for the complete Gauss polynomial

In this subsection, we prove the estimate for short parameter sums used in Sections 4.3 and 4.6.

The next bound treats the small completed free-parameter and formal-parameter ranges without an ordinary moment call. This is particularly important when a large inherited displayed radical makes the free parameter length small while the combined width stays large. All ideals, allowed generators, finite packets and norm sheets have the meaning of Section 2. The parameter

sum is the complete nonzero orbit sum of [Liu26b, Proposition 8.1 and Section 12.7], with its finite unit representatives and packets.

Write

$$\mathcal{G}(u,h)=q_u^{-1/2}\sum_{x \bmod u}\chi_u(x)e(hx/u)$$

for the full normalized Gauss sum. In particular the columns u in this subsection are ordinary ideals, not just squarefree or sixth-power-free ideals. The finite correlation of [Liu26b, Lemma 12.10] has the normalization

$$\mathcal{F}(u,v;j)=(q_uq_v)^{-1/2}\sum_{h \bmod uv}\mathcal{G}(u,h)\overline{\mathcal{G}(v,h)}e(-jh/(uv)). \tag{4.30}$$

The fixed different and the finite bad-place supplements are included in the corresponding finite packets. They affect only fixed constants and the common smooth kernel in the statements below.

Lemma 4.5 (A complete Gauss mean). *Let $D,H\geq 1$ lie in a fixed polynomial range in Z. Let $B(u)$ be common coefficients supported on $q_u\asymp D$, and let W be a nonnegative smooth norm test function in the source norm sheets. We write $[h]\neq 0$ for its nonzero allowed-generator orbits, and its radial argument is q_h/H. With the usual finite test function seminorm factors,*

$$\sum_{[h]\neq 0}W(q_h/H)\left|D^{-1/2}\sum_u B(u)\mathcal{G}(u,h)\right|^2\ll Z^{\epsilon}\left\{\frac{H}{D}\sum_u|B(u)|^2+\left(\sum_u|B(u)|\right)^2\right\}. \tag{4.31}$$

Consequently, if $s\mid u$ throughout the support and $B(u)$ is divisor-bounded, then

$$\sum_{[h]\neq 0}W(q_h/H)\left|D^{-1/2}\sum_{s\mid u}B(u)\mathcal{G}(u,h)\right|^2\ll Z^{\epsilon}\left\{\frac{H}{q_s}+\frac{D^2}{q_s^2}\right\}. \tag{4.32}$$

The coefficients may include common finite characters, norm modes, literal natural deletions, and the complete equal-product difference of [Liu26b, Section 12.5].

Proof. Use nonnegative annular lifts, each vanishing at the additive origin, as required by the imprimitive orbit Poisson formula. For $H\geq 1$, choose $J_H=\lceil\log_2 H\rceil$ and such lifts with

$$1_{1\leq q_h\leq H}\leq\sum_{j=0}^{J_H}W_j(q_h/2^j),\qquad\sum_{j=0}^{J_H}2^j\ll H,\qquad J_H+1\ll\log(2H).$$

Include the norm-one part at $j=0$. Summing the annular estimates therefore preserves the linear term and multiplies the quadratic term by at most $O(\log(2H))$. Rapidly decreasing parameter tests are treated by summable annular majorants. Only actual nonzero ideal norms $q_h\geq 1$ occur.

We next verify the finite arithmetic bound. Open the two full Gauss sums in (4.30) and perform the sum over h. Writing $c=(u,v)$, the congruence is

$$vx-uy=j\pmod{uv}. \tag{4.33}$$

Its solution count satisfies

$$\#\{(x \bmod u,y \bmod v):vx-uy\equiv j\pmod{uv}\}\leq q_c\,1_{c\mid j}.$$

Each character multiplier has modulus at most one.

The same statement holds in each allowed finite packet, with its fixed supplements retained. Therefore

$$|\mathcal{F}(u,v;j)|\leq q_{(u,v)}1_{(u,v)\mid j}. \tag{4.34}$$

This can also be read from the all-valuation correlation table of [Liu26b, Lemma 12.10], with the natural zeros in x, y retained. At $j = 0$, they force $u = v$: the two unit fractions x/u and y/v have the same exact denominator only in that case. Hence the exact zero-frequency identity is

$$\mathcal{F}(u, v; 0) = 1_{u=v}\varphi(u). \tag{4.35}$$

Open the square in (4.31) and apply the complete second bridge before estimating its terms. Its exterior factor for a column pair is

$$\frac{H}{D\sqrt{q_u q_v}}.$$

By (4.35), the zero frequency contributes at most $Z^{\epsilon}(H/D)\sum_u |B(u)|^2$.

For the nonzero frequencies the common kernel has natural norm scale $q_u q_v/H$. Separate its compact column ratios with the same weighted Fourier measure before either coefficient sum is estimated. The Poisson formula [Liu26b, Proposition 8.1] gives a kernel with arbitrary decay at infinity and at most small-power growth at zero, uniformly for the finite seminorms under consideration. Counting genuine nonzero frequencies divisible by c in the source norm sheets gives

$$\sum_{\substack{j\neq 0\\ c|j}} \left|\mathcal{K}\left(\frac{Hj}{uv}\right)\right| \ll Z^{\epsilon}\frac{q_u q_v}{Hq_c}. \tag{4.36}$$

Here and below the notation for $\mathcal{K}$ includes its fixed archimedean coordinates and finite-packet indices. Put $Y = q_u q_v/(Hq_c)$ and extract $j = cj_0$. Annular ideal counting and the kernel bounds of [Liu26b, Proposition 8.1] give, for $A > 1$,

$$\sum_{j_0\neq 0} |\mathcal{K}(j_0/Y)| \ll Z^{\epsilon}\begin{cases} Y, & Y \geq 1,\\ Y^A \leq Y, & 0 < Y < 1.\end{cases}$$

The finite-packet indices and the unit representatives remain in the common measure. Thus no additional constant term occurs in (4.36).

Multiplying (4.36) by the bound q_c in (4.34) and by the exterior factor gives $O(Z^{\epsilon})$ for each pair, since $q_u, q_v \asymp D$. The nonzero contribution is therefore bounded by $Z^{\epsilon}(\sum_u |B(u)|)^2$. This proves (4.31).

If $s \mid u$, actual ideal counting and the divisor bound give

$$\sum_{s|u} |B(u)|^2 \ll Z^{\epsilon}D/q_s, \qquad \sum_{s|u} |B(u)| \ll Z^{\epsilon}D/q_s.$$

This proves (4.32). The bounded common coefficient factors in the statement only reduce these absolute estimates. For a coupled common test function, apply the same argument under its single finite-mass Fourier measure. A centered coefficient is a difference of two divisor-bounded convolution coefficients, so the argument also covers it without reflecting or discarding one of its summands. □

We apply this estimate in the exact completed normalization of the initial marked transform. To distinguish the two uses of the letter s, let $s_0 = \log_Z q_s$ denote its fixed divisibility width. Write $A_*, M_*, q_*^{\mathrm{disp}}$ for the completed column, combined, and displayed widths, and $m_* = M_* - q_*^{\mathrm{disp}}$ for its free width. On a first side let c_1, d_1 be the full common exponents and $u_1 = R + E$ the primitive and complementary radical exponent. The actual polynomial has

$$\begin{aligned} &\alpha = A_* - c_1, \qquad K_0 = 2A_* - c_1 - d_1 + u_1 - m_*, \qquad K \leq K_0 + \xi,\\ &\Lambda_* = (A_* - c_1) + q_*^{\mathrm{disp}} + u_1 + B_C - s_0, \qquad B_C = (3c_1 - 5d_1 - R)_+/6. \end{aligned} \tag{4.37}$$

The source's initial relative exterior is denoted by $\mathcal{R}_{\mathrm{loc}}$. All fixed-support coefficient-square weights have already been included in that exterior; they are not counted again in Lemma 4.5.

Proposition 4.6 (Tiny completed and formal free widths)**.** *In the actual initial marked family, choose a fixed $\tau > 0$. If $m_* \le \tau$, then the complete Gauss mean, relative to the original allowance, has exponent at most*

$$\mathcal{R}_{\mathrm{loc}} + A_* - M_* + \tau + O(\xi). \tag{4.38}$$

The same holds on the remaining branch if the source formal width

$$m_f = m_* - c_1 + \widehat{d}_1 - u_1, \qquad \widehat{d}_1 = \min(d_1, A_*),$$

is at most τ. Both statements allow an arbitrary inherited displayed radical.

Proof. Nonempty coefficient and parameter annuli with nominal scale below one have scales bounded below by a fixed test function endpoint. Clip such an endpoint to one with its single aggregate $O(\xi)$ allowance; otherwise the range is empty, with genuine high-frequency tails treated by the source decay. In particular nonempty coefficient annuli have $\alpha \ge -o(1)$. Apply (4.32) with $D = Z^{\alpha}$ and the actual parameter bound $H = Z^{K}$. Its first term, after subtracting Λ_* and adding $\mathcal{R}_{\mathrm{loc}}$, is at most

$$\mathcal{R}_{\mathrm{loc}} + A_* - M_* - d_1 - B_C + O(\xi) \le \mathcal{R}_{\mathrm{loc}} + A_* - M_* + O(\xi). \tag{4.39}$$

The second term has relative exponent

$$\mathcal{R}_{\mathrm{loc}} + A_* - M_* + m_* - c_1 - u_1 - B_C - s_0. \tag{4.40}$$

If $m_* \le \tau$, this proves (4.38), including a negative formal free width. No ordinary theorem has been invoked at that negative width.

For the formal branch, the permitted outer-length replacement preserves B_C, $\min(c_1, d_1)$ and all actual summands, and gives

$$\widehat{d}_1 = \min(d_1, A_*), \qquad \widehat{K}_0 = K_0 + d_1 - \widehat{d}_1 = 2\alpha - m_f.$$

Use nonzero annular majorants up to $Z^{\widehat{K}_0 + O(\xi)}$. An empty range contributes zero; a nonempty subunit endpoint contributes only $O(\xi)$ to the exponent. Since $s_0 \ge 0$ and $m_f \le \tau$,

$$\begin{aligned} \max\{\widehat{K}_0 - s_0, 2\alpha - 2s_0\} &= \widehat{K}_0 - s_0 + \max\{0, m_f - s_0\} \\ &\le \widehat{K}_0 - s_0 + \tau. \end{aligned}$$

The same subtraction as in (4.39), with d_1 replaced by $\widehat{d}_1$, proves the assertion. □

The payment of (4.38) is an existing origin inequality, not a new restriction on the inherited radical. The exact marked identities give

$$\begin{aligned} \mathcal{R}_{\mathrm{loc}} + A_* - M_* &= A_t - 1 + 4t_{\mathrm{loc}} - L_1 - S_1 - L_{\mathrm{pow}} - 4v - 4c_{\mathrm{cor}} - S \\ &\le A_t - 1 + 4w, \end{aligned} \tag{4.41}$$

where $A_t = d + z$ is the original total column width and $t_{\mathrm{loc}} \le w$ is the locally removed marked width, including its free nonunit and correction parts. On a fixed face $d + Lz \le B < 1$, $w \le az + \eta$, this is at most $-(1 - B) - (L - 1 - 4a)z + 4\eta$. Choose τ, the mesh, and the endpoint buffer below a fixed fraction of this positive origin margin. The remaining branch has both completed free width and formal width greater than τ and therefore admits the complementary source transformations with fixed positive margins.

The distinction from the old raw estimate is essential. Although $\mathcal{R}_{\mathrm{loc}} + 3A_* - M_*$ is invariant when an inherited radical is transported, the condition that m_* is small is not invariant. Its raw expansion can acquire a $3q_0$ term. Lemma 4.5 replaces that raw bound in both tiny ranges; its payment (4.41) has no such term.

4.5 The initial marked product and its exact support bound

In this subsection, we derive the marked-product identities used in Section 4.6.

We next prepare the passage from one already proved ordinary face to another. The marked product in this passage is a fixed subproduct of the original prime positions. It is chosen before the first common-support decomposition, not from a subsequently selected parameter. Write $Q = Q_0E$, with original total width z and marked width w. The marked positions are consumed in the initial transformation; they do not reappear as additional free prime positions at its ordinary terminal.

Normalize the combined root allowance to one. Suppose

$$d + Lz \le B \le B_0 \le 1, \tag{4.42}$$

and suppose that $\mathcal{G}(B_0, L_0)$ is already known on the complete ordinary presentations. We will use no additional prime-width cap in this parent theorem.

The classification of the marked positions in [Liu26b, Section 12.7] has nonnegative exponents

$$\begin{gathered} w = w_f + f_0 + f_1 + u_f, \qquad t_{\mathrm{loc}} = f_0 + u_f, \\ u_f = g_{\mathrm{good}} + v + c_{\mathrm{cor}}, \qquad L_1 = f_1 + 4T_1 + e_1, \qquad L_{\mathrm{pow}} = v + 4T + e. \end{gathered} \tag{4.43}$$

Here w_f is the marked width on the first common support. For a remaining marked prime, first distinguish whether it divides the actual inner divisor s. Those which do are fixed before the frequency varies. Their ordinary valuation-zero part has width f_0; their positive-valuation part has width f_1. The remaining free positions split into the unit part g_{good}, the actual nonunit part v, and the fifth-power correction part c_{cor}.

For precision, let r be the ordinary valuation at such a positive-valuation prime $\mathfrak{p}$, let $P = N\mathfrak{p}$, and let $\mathfrak{h} \neq 0$ be the current frequency. The full Gauss formulas [Liu26b, Lemma 12.9] give

case	frequency condition	frequency divisor	relative squared factor	$(v, L_{\mathrm{pow}}, T, S)$
$6 \nmid r+1$	$r \ge 1,\ v_{\mathfrak{p}}(\mathfrak{h}) = r$	$\mathfrak{p}^r$	1	$(1, r, 0, 0)$
$6 \mid r+1$	$v_{\mathfrak{p}}(\mathfrak{h}) = r$	$\mathfrak{p}^r$	P^{-1}	$(1, r, 0, 1)$
$6 \mid r+1$	$v_{\mathfrak{p}}(\mathfrak{h}) \ge r+1$	$\mathfrak{p}^{r+1}$	$P(1-P^{-1})^2 \le P$	$(1, r, 1, 0)$.

(4.44)

The last column is in units of $\log_Z P$. For fixed positive-valuation primes the same column reads (f_1, L_1, T_1, S_1). Thus T_1, T are exactly the radical widths of the upper principal cases, and S_1, S those of the lower principal cases. Since an upper principal case has $r \equiv 5 \pmod 6$, $r \ge 5$, the remainders $e_1 = L_1 - f_1 - 4T_1$, $e = L_{\mathrm{pow}} - v - 4T$ are nonnegative. This proves the inequalities implicit in (4.43) for every valuation.

The nonprincipal Gauss magnitude $P^{r/2}$ cancels the ordinary extraction factor $P^{-r/2}$. The remaining central prime coefficient has square P^{-1}, and the principal relative factors in (4.44) are kept as Z^{T-S}, or $Z^{T_1-S_1}$. For free nonunit positions, every actual prime divides the original nonzero frequency. Apply Cauchy's inequality at that frequency before dividing by any prime-dependent divisor. There are divisor-many tuples and valuation allocations; their coefficient-square sum is Z^{ϵ}, not the unrestricted count of all prime tuples. For fixed primes dividing s, the same sum retains the additional square factor P^{-1}. Thus Cauchy's inequality is applied at the original fixed frequency, before frequency division. The local identities used here are those of [Liu26b, Lemma 12.9]; the table and the preceding coefficient calculation give their normalization in the present product.

At a unit prime with $(\mathfrak{p}, n) = 1$, the exact identity is $\mathcal{G}(\mathfrak{p}n, h) = \mathcal{G}(\mathfrak{p}, h)\mathcal{G}(n, h\mathfrak{p}^4)$. When the prime is fixed by s, its coprimality condition is kept as a common extra coprimality condition and the frequency change is injective. For a freely varying unit prime, extending that condition adds only the ordinary valuation five: its normalized coefficient is $P^2P^{-5/2}P^{-1/2} = P^{-1}$. These correction primes are fixed first with their absolute P^{-1} masses. The remaining unit choices

then use one common residual polynomial, with divisor-bounded fibers in the frequency change. A principal inducing digit, including an upper digit divisible by six, still retains its natural zero. This explains the complete local classes in (4.43) and introduces no additional independent parameter count into their exponents.

Before the subsequent first-common minimum is removed, the completed ordinary and displayed widths are

$$A_* = d + z - f_1 - t_{\mathrm{loc}} - L_1 - L_{\mathrm{pow}} - 5c_{\mathrm{cor}}, \tag{4.45}$$

$$M_* = 1 - 6t_{\mathrm{loc}} - f_1 - L_1 - L_{\mathrm{pow}} - 5c_{\mathrm{cor}} + T_1 + 5v + T, \tag{4.46}$$

$$q_*^{\mathrm{disp}} = f_1 + v + c_{\mathrm{cor}}. \tag{4.47}$$

Equivalently,

$$M_* = 1 - 6w + 6w_f + 4(f_1 + v) - 3(T_1 + T) - e_1 - e - 5c_{\mathrm{cor}}. \tag{4.48}$$

The exact local exterior in the original normalization is

$$-f_0 - L_1 + T_1 - S_1 - u_f - L_{\mathrm{pow}} + v - 4c_{\mathrm{cor}} + T - S.$$

After adjoining the baseline ordinary allowance and the common-support normalization, the excess over the root allowance is

$$\mathcal{E}_0 = -w + w_f - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0. \tag{4.49}$$

These are complete exponent identities: the centrally normalized coefficients, the single count of each genuine support and the coefficient-square measure are included. They are the marked version of the first transform in [Liu26b, Section 12.7], including the multiplicities of the contracted positions.

Let $z_{C,\mathrm{old}}$ be the original Q_0 width fixed on first common support, and let $\mu \ge w_f$ be the full common minimum drop on the current side. The original remaining prime width is $z - w - z_{C,\mathrm{old}}$. Define

$$D_{B_0} = A_* + (L_0 - 1)(z - w - z_{C,\mathrm{old}}) - B_0 M_*, \qquad \eta_0 = B - d - Lz \ge 0,$$

and

$$D_{\mathrm{ref}} = (L_0 - L)z + (6B_0 - L_0)w + B - B_0. \tag{4.50}$$

The actual excess in the length inequality after this common minimum is $D_{B_0} - \mu$, with further nonnegative removal losses retained in subsequent stages. Substitution of (4.43)–(4.48) gives the identity

$$\begin{aligned} D_{B_0} - \mu = D_{\mathrm{ref}} &- \eta_0 - 6B_0 w_f - (1 + 4B_0)(f_1 + v) \\ &- (4 - 3B_0)(T_1 + T) - (1 - B_0)(e_1 + e + 5c_{\mathrm{cor}}) \\ &- (L_0 - 1)z_{C,\mathrm{old}} - (\mu - w_f). \end{aligned} \tag{4.51}$$

For example the coefficient of w_f is $-6B_0$, not -6: the parent allowance is $B_0 M_*$. This dependence is essential in the finite face induction.

Lemma 4.7 (Marked low-support inequality)**.** *Put $\chi = k/L_0$, where $k > 1/2$. If*

$$1 < L_0 \le 6B_0 k, \tag{4.52}$$

then the complete low-terminal contribution satisfies

$$\mathcal{E}_0 + \chi(D_{B_0} - \mu)_+ \le \max\{\mathcal{E}_0, -w + \chi D_{\mathrm{ref}}\}. \tag{4.53}$$

Proof. Add χ times (4.51) to (4.49). Relative to $-w+\chi D_{\mathrm{ref}}$, the coefficients of the initial support variables are

$$\begin{array}{c|c} w_f & 1-6B_0\chi \\ f_1, v & -\chi(1+4B_0) \\ T_1, T & -3-\chi(4-3B_0) \\ e_1, e & -1-\chi(1-B_0) \\ c_{\mathrm{cor}} & -4-5\chi(1-B_0) \\ S_1, S & -1 \\ z_{C,\mathrm{old}} & -\chi(L_0-1) \\ \mu-w_f & -\chi \\ \eta_0 & -\chi \end{array} \tag{4.54}$$

They are all nonpositive by (4.52). Since $\mathcal{E}_0 \le 0$, taking the positive part of the defect gives exactly (4.53). It would be incorrect to replace its right side by the signed reference alone: if $w_f = w$ and all other losses vanish, then $\mathcal{E}_0 = 0$ and the signed defect can be negative. □

Later low-support inequalities. The positive-part parent contribution is the maximum of the baseline M and the affine exponent

$$(1-\chi B_0)(m+q)+\chi A+\chi(L_0-1)z. \tag{4.55}$$

This identity follows by adding $\chi[A+(L_0-1)z-B_0M]$ to M. Its first common-side increment, for $c_1 \ge d_1$, before the favorable $-B_C$, is

$$-\chi c_1+\chi B_0(c_1-d_1) = -\chi\{(1-B_0)c_1+B_0d_1\} \le 0. \tag{4.56}$$

If the opposite ordering occurs, the same expression with a positive part is at most $-\chi c_1$; the other side is treated independently. These later edges contain no still-free marked E position: such positions have already been bounded in (4.54).

At equal second valuation i, substitution of $s=t=1-\chi B_0, \lambda=\chi$ in (4.18) gives

$$1-\chi B_0 t_u-[1+\chi(1-B_0)]i \le 0. \tag{4.57}$$

For $i>j$, $6 \mid j$, the side $i_\star = j$ has increment

$$1-(i-j)/2-[1+\chi(1-B_0)]j < 0.$$

The other side has the additional term $-\chi(i-j) \le 0$. The extra prime slope in (4.55) is positive; fixing any remaining original factor only improves these bounds. For the three pool errors, subtracting their main rate $\chi B_0 - 1/6$ in (4.20) gives, in units $\sigma/3$, respectively

$$-\chi-2\chi B_0, \qquad -5-6\chi+9\chi B_0, \qquad -6-7\chi+10\chi B_0. \tag{4.58}$$

They are negative for $0<\chi<1$, $B_0 \le 1$. All these comparisons are on individual positive sides. Thus one side may stop at the parent face while the other uses either selected bound or the baseline.

The high bound when only a deleted block is large. If a particular original block is large, the whole retained prime product need not have that amplitude. The separate-block version of Theorem 3.13 supplies

$$\max\{(1-K)m+KA, M/6+5A/6+\kappa_1 z_{\mathrm{live}}/6\}, \qquad k \le K \le \kappa_1+o(1). \tag{4.59}$$

The last width counts only remaining original Q_0 positions. It contains no contracted E position, including its first-common part w_f. All later extractions decrease this positive term. The other terms satisfy the sidewise inequalities (4.17)–(4.19).

The complete initial high bound can be checked without discarding any of the nonunit corrections. From (4.45)–(4.49), the first plane minus the baseline, including its exterior, is

$$\begin{aligned}\mathcal{E}_0 + K(A_* - M_*) - (1-K)q_*^{\mathrm{disp}} &= K(d+z-1) + (5K-1)t_{\mathrm{loc}} \\ &\quad - (2-K)f_1 - (1+4K)v - (3+K)(T_1+T) \\ &\quad - (5-K)c_{\mathrm{cor}} - S_1 - S - e_1 - e. \end{aligned} \tag{4.60}$$

For the second plane the corresponding exact expression, before the remaining-prime term, is

$$\begin{aligned}\mathcal{E}_0 + \frac{5}{6}(A_* - M_*) &= \frac{5}{6}(d+z-1) + \frac{19}{6}t_{\mathrm{loc}} - f_1 - \frac{25}{6}v \\ &\quad - \frac{23}{6}(T_1+T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e. \end{aligned} \tag{4.61}$$

Since $t_{\mathrm{loc}} \le w$, $z_{\mathrm{live}} \le z - w$, all unlisted common losses are favorable, and $5K - 1 > 0$, these are bounded by

$$K(d+z-1) + (5K-1)w, \qquad \frac{5(d+z-1) + 19w + \kappa_1(z-w)}{6}. \tag{4.62}$$

In particular this calculation allows different high or low choices on the opposite Cauchy side. It does not replace the actual K by its lower endpoint when $A > m$.

An inherited displayed radical. The preceding formulas were normalized with no inherited radical only to simplify their notation. Their transport to the complete presentation is literal. At fixed original total allowance, adding an inherited displayed radical of exponent q_0 lowers the free parameter exponent by q_0. Every nonzero original column is coprime to it, so the new genuine column common supports and the marked slots are also coprime to it. Frequency valuations at those fixed primes need not vanish; they remain fixed-support local sectors with their actual coefficient weights.

At fixed frequency deficiency $K_0 - K$, (4.6) changes by

$$\widetilde{q} \longmapsto \widetilde{q} + q_0, \quad K_0 \longmapsto K_0 + q_0, \quad K \longmapsto K + q_0. \tag{4.63}$$

Hence J_0, g are unchanged. At the second return,

$$m' \longmapsto m' - q_0, \qquad q' \longmapsto q' + q_0. \tag{4.64}$$

The combined M' is unchanged. Both Λ_C and λ_0 increase by q_0, so that (4.14) is unchanged. The original free-parameter factor decreases by the same exponent in the first bridge, and the first origin improves. This transports the complete normalized identities, but does not by itself transport a terminal estimate whose hypothesis bounds the free width: adding q_0 can create new small-free-width cases. Those cases are treated by Proposition 4.6, which is independent of q_0. On the remaining range both actual formal widths are positive, so the source complementary transformations apply. The completed difference and its actual inducing character are formed before the exceptional group is classified. No fixed support is encoded as additional natural moving parameters.

4.6 A step between ordinary length regions

In this subsection, we combine the identities of Section 4.5 with the estimates of Theorem 3.13.

Proposition 4.8. *Suppose $\mathcal{G}(B, L_0)$ is known, and fix parameters satisfying*

$$\frac{1}{2} < k \le \frac{1001}{2000}, \quad k < \kappa_1, \quad 0 < a \le \frac{1}{100}, \quad \frac{5}{6} < B < 1, \quad 2 \le L < L_0 \le 3, \quad L_0 \le 6Bk. \tag{4.65}$$

Assume also

$$\rho = a - \frac{k}{L_0}\{L_0 - L + (6B - L_0)a\} > 0, \qquad 1 - \frac{k(6B - L_0)}{L_0} > 0. \tag{4.66}$$

Then $\mathcal{G}(B, L)$ holds on the same complete ordinary family.

Proof. Fix the requested exponent loss. Tiny combined parameter allowance is handled by the old prime bound and the no-prime theorem. On the remaining compact range normalize $M = 1$, so that $d + Lz \le B$. The unnormalized small parameters are chosen relative to the previously fixed positive parameter floor. If z is below a fixed small floor, strip the whole prime product at the old pointwise price and stop. Otherwise choose a parameter-independent union of complete original positions with

$$az \le w \le az + \eta. \tag{4.67}$$

This is the fixed product E of the preceding subsection, chosen before the ordinary parameter.

The reference is supplied by coefficient three. Put $w_{\max} = az+\eta$. The root second-origin condition is

$$d + z + 4w_{\max} < 1, \tag{4.68}$$

because $L - 1 - 4a \ge 0.96$ and $B < 1$. If $d + z + 3w_{\max} < 5/6$, with a fixed buffer, use the uncentered product. On the overlapping remaining range choose

$$\max\{0, \tfrac{3}{2}(d + z + 3w_{\max} - 5/6)\} < \ell < \min\{1/4, (d - 3z)/2\}. \tag{4.69}$$

We verify the full interval, including its upper endpoint. Ignoring the arbitrarily small mesh buffer temporarily, the need for centering and $d + Lz \le B$ imply

$$z < \frac{B - 5/6}{L - 1 - 3a} \le \frac{1}{6(L - 1 - 3a)} \le \frac{50}{291}.$$

Moreover

$$d - 3z > \frac{5}{6} - (4 + 3a)z \ge \frac{41}{291} > 0.$$

The difference of the two nonconstant endpoints in (4.69) is at least

$$\frac{5}{4} - B + (L - 3 - 9a/2)z \ge \frac{1}{4} - \frac{209}{1164} = \frac{41}{582} > 0. \tag{4.70}$$

The lower endpoint is also at most $3(B - 5/6)/2 < 1/4$, with the separate gap $3(1 - B)/2$. The mesh and branch buffer can therefore be chosen to preserve all these strict inequalities.

If $n_{\min} < \ell$, reflection of its partner gives

$$d_{\mathrm{refl}} \le 1 - d + 2n_{\min} + o(1), \qquad d_{\mathrm{refl}} + 3z < 1,$$

so Corollary 4.4 applies directly. Otherwise subtract the equal-product comparison with lengths $(\ell, d - \ell)$. Its reflection satisfies

$$d_{\mathrm{comp}} \le 1 - d + 2\ell + o(1), \qquad d_{\mathrm{comp}} + 3z < 1.$$

The same corollary bounds this positive comparison. Only the centered difference enters the marked transformations; all four original scales are at least Z^{ℓ}. No use of the new moment is made in this comparison.

Origins, fixed means and short formal parameters. The first origin is bounded by $Z^{m+\epsilon} \le Z^{1+\epsilon}$. With the complete initial exterior the second origin has excess at most $d + z - 1 + 4t_{\mathrm{loc}}$, and is bounded by (4.68). For the centered fixed-character term the exact initial bound is

$$d + z - 5/6 + 3t_{\mathrm{loc}} - \frac{9}{8}(L_1 + L_{\mathrm{pow}}) - \frac{29}{6}c_{\mathrm{cor}} - \frac{2}{3}\big(\ell - (L_1 + L_{\mathrm{pow}}) - 5c_{\mathrm{cor}}\big)_+. \tag{4.71}$$

Put $X = L_1 + L_{\mathrm{pow}}$, $c = c_{\mathrm{cor}}$ and $\Phi = -9X/8 - 29c/6 - 2(\ell - X - 5c)_+/3$. Then

$$\Phi + \frac{2\ell}{3} \le \begin{cases} -11X/24 - 3c/2 \le 0, & X + 5c \le \ell, \\ -2(X + 5c - \ell)/3 \le 0, & X + 5c \ge \ell. \end{cases}$$

Thus the expression in (4.71) is at most $d + z - 5/6 + 3t_{\mathrm{loc}} - 2\ell/3$.

Thus (4.69) bounds this term. In the uncentered range the same inequality without the last saving is bounded by its defining strict condition.

For the small completed or formal free-parameter ranges use Proposition 4.6, including its full natural-zero correlation. By (4.41), its excess is at most

$$d + z - 1 + 4w + \tau + O(\xi). \tag{4.72}$$

Indeed, with $w \le az + \eta$,

$$d + z - 1 + 4w + \tau + O(\xi) \le -(1 - B) - (L - 1 - 4a)z + 4\eta + \tau + O(\xi) < 0,$$

by taking $\tau < (1 - B)/10$ and then smaller mesh and endpoint buffers. This estimate includes inherited displayed radicals and negative formal free widths. Apply the complementary transformation only on

$$m_* > \tau, \qquad m_f > \tau.$$

Principal and finite inducing parameters are therefore bounded before the ordinary amplitude alternatives.

The low alternative uses an already proved parent. Perform the complete marked first transformation, one sufficiently small auxiliary main/error step, and the second correlation. Retain all genuine support and the full artificial divisor expansion before forming each positive ordinary side. For such a side choose a parameter-independent complete block of its remaining original prime positions with width at least

$$v_* = [d' + L_0 z' - BM']_+ / L_0, \tag{4.73}$$

with an excess length of at most one mesh interval, or choose the entire list at exhaustion. A nonpositive defect uses the parent immediately; a block below the fixed width floor uses the old pointwise bound and stops. Otherwise form amplitude sets for this fixed block. On the low-amplitude set,

$$|Q_{\mathrm{block}}|^2 \le Z^{kv + o(1)}, \qquad d' + L_0(z' - v) \le BM'$$

unless the list is exhausted. Remove the block and discard the amplitude restriction before applying $\mathcal{G}(B, L_0)$; at exhaustion use (2.3). In either case the additional exponent is at most

$$\frac{k}{L_0}[d' + L_0 z' - BM']_+ + O(\eta). \tag{4.74}$$

Unlike Lemma 4.2, this step does not require the whole remaining product to be small: the tested block itself was fixed by the actual defect before its amplitude was tested.

Apply Lemma 4.7 with $B_0 = B$. The pure reference part is

$$-w + \frac{k}{L_0}\{(L_0 - L)z + (6B - L_0)w\} \le -\rho z + O(\eta),$$

by (4.66) and (4.67). Hence the complete initial contribution is

$$\le \max\{\mathcal{E}_0, -\rho z + O(\eta)\} \le O(\eta). \tag{4.75}$$

All initial common and nonunit supports have been included in (4.54). Later support extraction is bounded on each side by (4.56)–(4.58). The single auxiliary step has a bounded linear loss; choose its width inside the requested final loss. No strict root saving is asserted on the possible zero-baseline branch of (4.75). This is a finite application of the previously proved parent estimate.

The high alternative terminates on the full polynomial. On a high-amplitude set for the block of (4.73), apply Theorem 3.13 with

$$A' = d' + z', \qquad \text{source} = S_{n_1',1} S_{n_2',2} Q_{\text{remaining}}, \qquad |Q_{\text{block}}|^2 \ge Z^{Kv}.$$

The selecting block is not inserted again in the source. Each positive side retains its own K; ordinary sixth powers and their natural zeros are included in (4.59).

Equations (4.60) and (4.61) reduce the root exponents to (4.62). Since $d + Lz \le B \le 1$, strictly sufficient inequalities are

$$K(L-1) > (5K-1)a \quad (k \le K \le \kappa_1), \qquad 5(L-1) - \kappa_1 > (19 - \kappa_1)a. \tag{4.76}$$

They hold throughout (4.65). For $1/2 \le K \le 3/4$, $L \ge 2$, $a \le 1/100$, the first difference is at least $K(1-5/100)+1/100 \ge 0.485$. The second difference is at least $5-3/4-(19-3/4)/100 = 4.0675$. Thus above the fixed width floor they have a uniform strict margin. All further common supports are sidewise favorable, and the added $\kappa_1 z'/6$ only decreases under prime extraction. The single auxiliary width and the bin buffers are chosen inside these margins. This completes every high branch.

Finally, the transport in (4.63)–(4.64) proves the argument for every inherited displayed radical. The low parent estimate applies to the complete ordinary family, and the high bound restores every ordinary sixth power. Reintroducing the positive reflected comparison therefore returns exactly (4.2). There is no omitted natural-parameter restoration and no auxiliary condition $z' \le cM'$.

Choose the total-parameter and prime-width floors first from the requested loss; then fix the root, reference, retention and high margins. Choose the one auxiliary width, the parent loss and the original mesh below these bounds. The finite arithmetic factor lists, amplitude cells and common-derivative orders follow. The source weighted Fourier measures and the artificial-divisor normalizations above introduce only the allotted losses. Incoming heights are treated by the same small-height and final polynomial-height argument as in Proposition 4.3. This proves the proposition. □

4.7 Finite iteration and prime coefficient two

In this subsection, we iterate Proposition 4.8 to prove the moment theorem used in Section 6.

Theorem 4.9. *Use the arithmetic data and ordinary coefficient families of Section 2. Fix bounded aggregate lengths and $\epsilon > 0$. There is $\eta > 0$, chosen before the arithmetic data and finite factor list, such that every original prime product with*

$$z = \sum_i z_i, \qquad 0 < z_i \le \eta,$$

satisfies

$$\sum_{u}^{\operatorname{ind}(\psi_u)\notin\Theta} \mathcal{W}(u)|S_{n_1,1}(u)S_{n_2,2}(u)Q(u)|^2 \ll Z^{M+\epsilon}\mathfrak{S}_J(\mathcal{W},\mathbf{W})(1+H_{\mathrm{in}})^{J'}, \qquad n_1+n_2+2z\le M. \tag{4.77}$$

The quantifiers, common coefficients, natural zeros, extra-coprimality condition expansion and complete family of ordinary parameters are those of Definition 4.1. At $z=0$ the actual two-factor arbitrary-length no-prime assertion remains available.

Proof. Choose a fixed $0<\delta\le 1/2000$, small enough also that $1/2+\delta<\kappa_1$, and put

$$B=1-\delta, \qquad k=1/2+\delta, \qquad a=1/100, \qquad L_*=\frac{6Bk}{1+k}, \qquad L_0=3. \tag{4.78}$$

Corollary 4.4 supplies $\mathcal{G}(B,3)$. Define a finite initial portion of the sequence

$$L_{n+1}=L_n-\frac{a}{2}\left(\frac{1+k}{k}L_n-6B\right). \tag{4.79}$$

One has $2<L_*<3$: indeed

$$6Bk-2(1+k)=\delta-6\delta^2>0,$$

and $6Bk=3+3\delta-6\delta^2<3(1+k)$. Also $3\le 6Bk$, and

$$L_{n+1}-L_*=\left(1-\frac{a(1+k)}{2k}\right)(L_n-L_*).$$

The multiplier belongs to $(0,1)$. Thus $L_n\downarrow L_*$, with $2<L_n\le 3\le 6Bk$.

For a step from L_n to L_{n+1}, the retention coefficient in (4.66) is exactly

$$\rho_n=\frac{a}{2}\left(1+k-\frac{6Bk}{L_n}\right)>0. \tag{4.80}$$

The excess length coefficient is the same expression without $a/2$, and is positive. Every other condition of Proposition 4.8 is already verified uniformly. Induction therefore proves $\mathcal{G}(B,L_n)$ for every fixed finite n. Given a fixed $L>L_*$, choose n with $L_n<L$; monotonicity in length condition gives $\mathcal{G}(B,L)$.

For the fixed finite number N of stages, choose the parent and auxiliary losses and the mesh so that

$$\sum_{j=1}^{N}(\epsilon_{\mathrm{parent},j}+\epsilon_{\mathrm{aux},j})<\epsilon/2, \qquad \eta\le\min_{1\le j\le N}\eta_j.$$

Include the zero-baseline case of (4.75). These numerical choices precede the arithmetic data; no uniformity as $\delta\downarrow 0$ is required.

Now suppose $d+2z\le M$, in the bounded range of lengths in the statement. Choose δ first, and a finite $L>L_*$ close enough to two, so that the old pointwise price of

$$\frac{(1-B)M+(L-2)z}{L}+\eta \tag{4.81}$$

is smaller than its allotted loss. Remove a parameter-independent complete original subproduct of width v such that

$$v_0\le v<v_0+\eta, \qquad v_0=\frac{(1-B)M+(L-2)z}{L},$$

or exhaust the prime list if its total width is less than v_0. Thus (4.81) bounds the removed width, including an excess length of at most one mesh interval. In the nonexhausted case,

$$d + L(z - v) \le d + 2z - (1 - B)M \le BM,$$

so $\mathcal{G}(B, L)$ applies. In the exhausted case the removed width is no greater than the requested bound and the actual arbitrary-length no-prime theorem applies. The prime bound [Liu26b, Lemma 12.7] bounds the squared modulus of the removed subproduct by $Z^{\kappa_1 v+\epsilon}$. Combining the allocated losses proves (4.77). □

Remark 4.10. Theorem 4.9 is applied to the ordinary detector and numerator moments. Its proof uses the coefficient-three comparison in Proposition 4.8 and finitely many selected operator estimates with slope greater than 1/2. The pointwise prime estimate retains κ_1.

5 The approximate functional equation and its contours

In this section, we extend the compact-strip argument of [Liu26c, Lemmas 5.1–5.2] to the full numerator contour interval. The smooth-weight estimate holds on any fixed positive substrip and is applied in Section 6.

5.1 Smooth weights beyond the 1/10 contour shift

In this subsection, we bound the approximate-functional-equation weights on a compact strip for the comparison in Section 6.

The earlier restriction to shifts of size at most 1/10 came from the strip chosen in the lemma on smooth weights for the approximate functional equation; the holomorphic region for the correction factors permits a larger shift of the original numerator contour.

Lemma 5.1 (Smooth weights on compact substrips)**.** *Fix $0 < \sigma_0 < 1/2$. For a primitive nonprincipal finite-order Hecke character over the fixed totally complex field F, use the original approximate functional equation with*

$$V_\omega(v) = \frac{1}{2\pi i}\int_{(2)} e^{\xi^2}\left(\frac{\Gamma(\omega+\xi)}{\Gamma(\omega)}\right)^{r_F} v^{-\xi}\frac{d\xi}{\xi}, \qquad T_\omega = (1 + |\Im\omega|)^{r_F}, \quad v > 0.$$

Uniformly for $\sigma_0 \le \operatorname{Re}\omega \le 1 - \sigma_0$ and fixed $j, N \ge 0$,

$$|(v\partial_v)^j V_\omega(v)| \ll_{F,\sigma_0,j,N} (1 + v/T_\omega)^{-N}.$$

For $\omega = 1/2 + \theta + it$, $0 \le \theta \le 1/2 - \sigma_0$, put

$$\widetilde{V}_\omega(v) = \left(\frac{\Gamma(1-\omega)}{\Gamma(\omega)}\right)^{r_F} v^{2\theta} V_{1-\omega}(v).$$

After increasing the tail order by one, the same derivative bound holds for $\widetilde{V}_\omega$. Before contour displacement use

$$H_\omega(v) = v^{2\omega-1}\left(\frac{\Gamma(1-\omega)}{\Gamma(\omega)}\right)^{r_F} V_{1-\omega}(v), \qquad H_\omega(v) = v^{2it}\widetilde{V}_\omega(v) \quad (\omega = 1/2 + \theta + it),$$

with the real logarithm of v. Here $\operatorname{Re}\omega = 1/2 + \theta \ge 1/2$, so $\operatorname{Re}(1-\omega) = 1/2 - \theta$. The functions V_ω, H_ω are holomorphic on $0 < \operatorname{Re}\omega < 1$; the real-line factorization is used only after displacement. Their dyadic expansions converge normally on compact substrips for each fixed finite annulus of character parameters.

Proof. Put $T = (1 + |\Im\omega|)^{r_F}$. Euler differentiation inserts powers of ξ in the defining integral. For $v \geq T$, shift to any fixed line $\operatorname{Re}\xi = B > 0$. Stirling's formula and the Gaussian e^{ξ^2} give

$$|(v\partial_v)^j V_\omega(v)| \ll_{F,\sigma_0,j,B} (T/v)^B.$$

For $v \leq T$, shift to $\operatorname{Re}\xi = -\sigma_0/2$. Then

$$\operatorname{Re}(\omega + \xi) \geq \sigma_0/2 > 0, \qquad (v\partial_v)^j V_\omega(v) = \mathbf{1}_{j=0} + O_{F,\sigma_0,j}((v/T)^{\sigma_0/2}).$$

Only $\xi = 0$ is crossed. These two estimates prove the required bound for every fixed j, N.

The completed nonprincipal function is entire, so its functional equation gives the same two exact AFE halves on the enlarged strip. For $\omega = 1/2 + \theta + it$, Stirling gives

$$\left|\frac{\Gamma(1-\omega)}{\Gamma(\omega)}\right|^{r_F} \ll T^{-2\theta}, \qquad 0 \leq 2\theta \leq 1 - 2\sigma_0 < 1.$$

Hence the normalized dual factor is bounded using

$$(v/T)^{2\theta}(1 + v/T)^{-N-1} \leq (1 + v/T)^{-N}.$$

Fixed Euler derivatives obey the same bounds. Before the contour move use the holomorphic factor $v^{2\omega-1}$; afterwards its unitary part is included in the Mellin twist. The exact conductor and gamma factors are retained, and the large-v bounds give normal convergence of the dyadic tails on compact substrips. □

Lemma 5.2 (Extended contours for the dyadic sums)**.** *Use the nonprincipal character family and amplitude classes of Section 2.3, the correction region [Liu26b, Lemma 13.10], and Theorem 4.9. First fix z on its final contour $\operatorname{Re} z = 17/50$, then assign the amplitude classes at that fixed z. No z-contour displacement is made after this assignment. For $11/50 \leq \delta \leq 13/20$, every dyadic sum in the approximate functional equation admits each fixed classwise shift*

$$0 \leq \theta \leq \delta/2, \qquad \operatorname{Re} w = 1/2 - \theta, \qquad \theta_- = 0, \quad \theta_+ = \delta/2.$$

Keep s on its initial right-hand line and move the whole dyadic integrand before the local s-contour displacement, absolute values, conductor separation and conjugation of dual weights. Retain the prescribed positive separation margins. The central-moment comparison exponent changes by

$$(\theta - \theta_0)\{d(1-m) - l_y\}, \qquad \theta_0 = \delta x/p_1,$$

with the same central-line moment, initial prime product and prime-sum coefficients.

Proof. Here $a = (1+\delta)/2 = a_{\mathrm{det}}$ is the detector parameter, and s, w, z are the contour variables of [Liu26b, Lemma 13.10]; this z is not the length exponent of Q_0, and this s is not an inner divisor in the Gauss transformation. The entire swept range satisfies

$$0 \leq \theta \leq \delta/2, \qquad 7/40 \leq \operatorname{Re} w = 1/2 - \theta \leq 1/2, \qquad 1/2 \leq \operatorname{Re}(1-w) \leq 33/40.$$

Lemma 5.1 applies with $\sigma_0 = 1/7$. Indeed, both reflected arguments satisfy

$$\sigma_0 = 1/7 < 7/40 \leq \operatorname{Re} w, \operatorname{Re}(1-w) \leq 33/40 < 1 - \sigma_0.$$

Use V_ω at $\omega = w$ and $\omega = 1 - w$.

The connecting region of [Liu26b, Lemma 13.10] contains

$$\operatorname{Re} s \geq a + 16e, \quad \operatorname{Re} w \geq 1 - a - 6e, \quad \operatorname{Re} z \geq 17/50.$$

On the final local contours,

$$\operatorname{Re} w \geq 1 - a \quad \Longrightarrow \quad \operatorname{Re}(s + w) \geq 1 + 16e.$$

Thus the correction holomorphy and the original unit- and nonunit-prime bounds hold throughout. For each strict tuple,

$$q_p^{\theta} q_p^{-(j_p-1)\theta} = q_p^{-(j_p-2)\theta} \leq 1 \qquad (j_p \geq 2,\ \theta \geq 0);$$

nonstrict errors retain their positive e margins. If S_{fin} denotes the finite places of S, the fixed-place majorant is summable:

$$\sum_{\substack{\mathfrak{a} \text{ integral} \\ \operatorname{supp}(\mathfrak{a}) \subset S_{\mathrm{fin}}}} q_{\mathfrak{a}}^{-1/6} = \prod_{p \in S_{\mathrm{fin}}} (1 - q_p^{-1/6})^{-1} < \infty.$$

The independent AFE weights and derivatives therefore remain in the original M2 families with fixed polynomial height dependence.

Keep finite character and dyadic cutoffs. With s first on its original right-hand line, perform the contour moves in this order:

unsplit numerator	$\operatorname{Re} w:\ c_w \longrightarrow 1/2$
exact two-part AFE	insert on $\operatorname{Re} w = 1/2$
each holomorphic dyadic sum	$\operatorname{Re} w:\ 1/2 \longrightarrow 1/2 - \theta$
local s-contour	$\operatorname{Re} s \geq a + 16e.$

Here c_w is the initial right-hand line. The first move is valid because the numerator is entire, the reciprocal factor is independent of w, and the cited region contains the path. Mellin decay removes horizontal joins; locally uniform bounds permit removal of the cutoffs in the original weighted height order. Only then conjugate the dual half as a whole. Keep each actual conductor, root number and gamma ratio.

The amplitude partition depends only on the character parameter and the fixed z, so the prime coefficients remain fixed during both moves. For $q_u \asymp U = Z^d$, $D = U^m$, $Y = Z^{l_y}$,

$$q_u^{\theta-\theta_0} D^{-(\theta-\theta_0)} Y^{-(\theta-\theta_0)} \asymp Z^{(\theta-\theta_0)\{d(1-m)-l_y\}}.$$

The bounded annular ratios remain in the common smooth test functions. □

Here $a \in [61/100, 33/40]$, within the parameter range of the cited connecting region. The permitted ordinary prime width changes the estimate on the final line; it does not change these holomorphic identities. Section 6 uses the resulting range with the ordinary coefficient-two theorem.

6 The scalar comparison and its exact endpoint

In this section, we apply Theorem 4.9 to the ordinary detector and numerator moments. The pointwise prime bound and inverse moment retain the earlier parameter κ_1.

We shall prove strict comparison estimates at

$$B_{\mathrm{rat}} = \frac{8733821}{10^7} = 0.8733821 \tag{6.1}$$

and at every fixed B sufficiently close above

$$B_2 = \frac{777 - \sqrt{1081}}{852} = 0.873381966613207519929026778642\ldots. \tag{6.2}$$

Section 7 then gives the open half-plane with boundary B_2. The limiting Dirichlet comparison value B_2 is identified in [Beb26a, Beb26d]; here we adapt the comparison to number fields using the ideal-character moments proved above.

6.1 Normalization and the comparison inputs

In this subsection, we normalize the moments of Section 4 for the comparison argument.

Fix the field, excluded places, finite character packets, and test families of Section 2. Write q_a for the ideal norm. A central ordinary polynomial of length U^m is denoted by S_m, and its inverse counterpart by M_r when its length is U^r. Their coefficients are respectively the actual ordinary coefficients and the same coefficients multiplied by μ_F; they are not arbitrary coefficient sequences. Original prime products are denoted by Q. Every displayed character remains zero at its nonunits.

By [Liu26b, Theorem 17.1], we may choose

$$\beta_*(F) \le 0.87488664704 < \frac{1+\kappa_1}{2}, \qquad \kappa_1 = \frac{374887}{500000} = 0.749774, \qquad p_1 = 6\kappa_1 = \frac{1124661}{250000}. \tag{6.3}$$

The strict margin is 0.00000035296. Hence [Liu26b, Lemma 12.7 and Theorems 10.7 and 11.2] give the pointwise prime estimate and the inverse moments at κ_1.

For Theorem 4.9, let m_{par} be the free parameter length. If the full displayed fixed radical has width q, and $M = m_{\mathrm{par}} + q$, then

$$n_1 + n_2 + 2z \le M \quad \Longrightarrow \quad \sum_{\substack{q_k \ll Z^{m_{\mathrm{par}}} \\ \mathrm{ind}(\psi_k)\notin\Theta}} |S_{n_1}(k) S_{n_2}(k) Q(k)|^2 \ll Z^{M+\epsilon}(1+H_{\mathrm{ext}})^C. \tag{6.4}$$

Here z is the total width of the original prime factors. Common test functions, inherited radicals, and ordinary parameter sixth powers have already been restored in that theorem. The arbitrary-length no-prime estimate (2.3) applies when $z = 0$.

For a candidate boundary B and a comparison imbalance b, put

$$\begin{gathered} \ell = \frac{11}{3} - 4B, \qquad l_x = \frac{1-b-\ell}{2}, \qquad l_y = \frac{1+b-\ell}{2}, \\ h = \frac{1+b+3\ell}{2} = l_y + 2\ell, \qquad T = 2l_y - h, \qquad C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}. \end{gathered} \tag{6.5}$$

Thus $l_x + l_y = 1 - \ell$. The comparison parameter scale is $U = Z^\rho$; the largest relevant moderate scale is $\rho = h$, up to a later small upper buffer. The letter T in (6.5) is a length expression, not an external height.

In a buffered amplitude class write

$$\delta = 2a - 1, \qquad g = \delta x, \qquad 0 < \delta \le \frac{5}{6}, \quad 0 \le x \le \frac{1}{2}. \tag{6.6}$$

If an actual original prime subproduct of width z is selected in decreasing amplitude order, its squared magnitude is at least $U^{2gz-o(1)}$. Widths here are relative to U, not Z. If all positive-amplitude factors are exhausted, retaining all of them supplies the same lower bound. A single fractional last factor is discarded; its contribution is controlled by the fixed original mesh. No new prime coefficients depending on the parameter are introduced. Keep the stock and the reference contour

$$\mathscr{L} = \frac{21}{100}, \qquad \theta_0 = \frac{\delta x}{p_1}. \tag{6.7}$$

The following proposition records the detector and comparison formulas of [Liu26b, Lemmas 13.7 and 13.9, Proposition 14.3 and Lemma 14.7] in this normalization.

Proposition 6.1 (Detector and comparison formulas)**.** *Fix all strict contour buffers and a compact family of smooth test functions. After polynomial external-height factors and an arbitrary U^ϵ loss, the following assertions hold.*

1. *For each fixed $1 \le t \le 3/2$, every nonfloor buffered bad parameter has, in one simultaneous presentation and with one common norm twist, a detected pair satisfying*

$$r \le t + o(1), \quad r + m \ge t - o(1), \quad r \ge t - \frac{1}{2} - o(1), \quad m \le \frac{1}{2} + o(1), \tag{6.8}$$

and

$$|M_r|^2 \ge U^{\delta r - o(1)}, \qquad |S_m|^2 \ge U^{\delta m - o(1)}. \tag{6.9}$$

The common pointwise bounds used to obtain these individual lower bounds are $|M_r|^2 \ll U^{\delta r+\epsilon}$ and $|S_m|^2 \ll U^{\delta \min(m,1-m)+\epsilon}$ on the bounded length ranges in question.

2. *The short marked inverse moment is available under $r + 2z < 1$ and $2r + 8z < 3$. The unmarked long inverse moment has exponent*

$$e(r) = \max\left(1, \frac{1+5r}{6}\right). \tag{6.10}$$

The original ordinary product moment has prime coefficient p_1; (6.4) improves that coefficient to two.

3. *Suppose a class has count $\ll U^{R+\epsilon}$. Define*

$$\begin{aligned} E(R) &= C_0 + \frac{\delta}{2} + \delta x\left\{\ell + \frac{h/2 - l_y}{p_1}\right\} - \frac{3h(1-R)}{4}, \\ S(R) &= \frac{3R+1}{4} + \frac{\theta_0}{2} - \frac{17}{50}, \qquad N(R) = \frac{3R+1}{4} - \frac{\theta_0}{2}, \\ D_{\mathrm{sh}}(R) &= R + \left(\frac{1}{2} - \theta_0\right)\frac{3}{40} - N(R). \end{aligned} \tag{6.11}$$

The reference correction exponent, relative to the comparison main term, is $E(R) + (\rho - h)S(R)$. For a numerator block of length U^m, a first-moment saving γ relative to that reference and a contour $0 \le \theta \le \delta/2$ give the exponent

$$E(R) + (\rho - h)S(R) + (\theta - \theta_0)\{\rho(1-m) - l_y\} - \rho\gamma. \tag{6.12}$$

Both halves of the approximate functional equation are included.

4. *The direct comparison estimate, for general nonnegative comparison lengths with $M = l_x + l_y$ and $b = l_y - l_x$, is*

$$\begin{aligned} B_{\mathrm{low}} &= \frac{5}{6} + \frac{l_x}{6} - \frac{\ell}{6} + G + \frac{1}{2}k, \\ G &= \max(0, b/12, b - l_y/2), \\ k &= \max\left(0, \frac{1 + 3\ell - 2M}{4}, M + \ell - 1, \frac{3\ell - 1}{2}\right). \end{aligned} \tag{6.13}$$

A nonempty selected full-index subset of width d only decreases this majorant, by at least $3d/8$. The small-amplitude, floor, small-parameter, and principal estimates have the scalar requirements listed in Lemma 6.5 below.

All statements retain the literal natural zeros and the common finite packets fixed in Section 2.

Proof. Use the truncated inverse identity at the zero, with inverse cutoff U^t and exponential cutoff U^{20}. Its coefficients satisfy

$$a_t(1) = 1, \qquad a_t(\mathfrak{n}) = \sum_{\substack{\mathfrak{d} \mid \mathfrak{n} \\ N\mathfrak{d} \le U^t}} \mu_F(\mathfrak{d}) = 0 \quad (1 < N\mathfrak{n} \le U^t).$$

The character multiplier may vanish; the coefficient identity remains valid. After the Mellin shift and dyadic separation,

$$r \le t + o(1), \qquad r + m \ge t - o(1), \qquad |M_r S_m|^2 \ge U^{\delta(r+m)-o(1)}.$$

Comparing with the simultaneous pointwise bounds gives

$$\begin{aligned} m &\le 1/2 + o(1), \qquad r \ge t - 1/2 - o(1), \\ |M_r|^2 &\ge |M_r S_m|^2 U^{-\delta m - o(1)} \ge U^{\delta r - o(1)}, \\ |S_m|^2 &\ge |M_r S_m|^2 U^{-\delta r - o(1)} \ge U^{\delta m - o(1)}. \end{aligned}$$

The inverse estimates are [Liu26b, Theorems 10.7 and 11.2], with their strict length conditions.

To control the detecting twists simultaneously, fix the discrete presentation and the dyadic lengths first, and enclose all remaining test function parameters in a common box $\mathcal{B}$. Repeated Sobolev gives

$$\sum_u \sup_{\mathbf{t}\in\mathcal{B}} |P_u(\mathbf{t})|^2 \ll \sum_{\alpha\in\{0,1\}^{\dim \mathcal{B}}} \int_{\mathcal{B}^+} \sum_u |\partial^\alpha P_u(\mathbf{t})|^2 \, d\mathbf{t}. \tag{6.14}$$

Logarithmic derivatives of normalized annular test functions have bounded seminorms. Norm-twist differences are put into the original prime test functions; the arithmetic coefficients and prime intervals remain unchanged. Apply the appropriate moment inside the integral. Only then evaluate the parameter-dependent detector. The fixed box volume and derivative bounds give a polynomial height factor.

The comparison assertions follow from [Liu26b, Proposition 14.3 and Lemma 14.7], with the compact-strip contour justification of Lemma 5.2. The three changes in (6.12) are

$$\begin{aligned} \Delta E &= (\theta - \theta_0)(\rho - l_y) - (\theta - \theta_0)\rho m - \rho\gamma \\ &= (\theta - \theta_0)\{\rho(1 - m) - l_y\} - \rho\gamma. \end{aligned}$$

Thus p_1 remains the reference parameter.

The functional equation is applied to the whole numerator factor; its conductor test function is separated before the positive moment, and its root number has modulus one. Formula (6.13) is the four-branch comparison theta and additive-Gram estimate of [Liu26b, Proposition 13.4]. □

6.2 The improved count and the complementary classes

In this subsection, we apply the moment of Theorem 4.9 to the detector of Section 6.1.

Set

$$\mathcal{D} = [11/50, 13/20] \times [6/25, 1/2]. \tag{6.15}$$

For the inverse detector use the limiting admissible width $z_I(r) = (1 - r)/2$. Its saving, divided by δ, is

$$F_I(r) = r + 2xz_I(r) = x + (1 - x)r. \tag{6.16}$$

For the ordinary detector, apply (6.4) to $S_m^2 Q$. This is a fourth moment of the detected ordinary sum. The permitted width and saving are

$$z_2(m) = \frac{1 - 2m}{2}, \qquad F_2(m) = 2m + 2xz_2(m) = x + 2(1 - x)m. \tag{6.17}$$

Widths in these formulas are decreased slightly when applying a strict source inequality.

Lemma 6.2 (One common detector cutoff). *For $(\delta, x) \in \mathcal{D}$, put*

$$\begin{gathered} A = \frac{5}{6} - \delta, \qquad m_d = \frac{5/6 - \delta x}{5/2 - \delta - 2\delta x}, \qquad r_d = 2m_d, \qquad t_d = 3m_d, \\ F_d = x + 2(1 - x)m_d, \qquad R_d = 1 - \delta F_d = \frac{1}{6} + At_d. \end{gathered} \tag{6.18}$$

Then the buffered class has count $\ll U^{R_d+\epsilon}$, with the common height and test function factors specified above. Uniformly on this rectangle,

$$0.35 < m_d < 0.49, \qquad 0.6 < r_d < 1, \qquad 1 < t_d < 1.5. \tag{6.19}$$

Proof. The denominator in (6.18) is positive. The identities follow by balancing $F_I(r_d) = F_2(m_d)$ and the resulting short count with the long count $1/6 + At_d$. Their strict ranges are among the exact polynomial inequalities proved in Proposition 6.4.

Use the single cutoff t_d in Proposition 6.1. For $r \geq 1$,

$$e(r) - \delta r = \frac{1}{6} + Ar \leq \frac{1}{6} + At_d + o(1) = R_d + o(1).$$

Near $r = 1$ or $m = 1/2$, the unmarked inverse or ordinary fourth moment gives

$$1 - \delta < R_d = 1 - \delta F_d \qquad (F_d < 1).$$

Choose these endpoint strips inside this strict difference.

Outside these strips, if $r \geq r_d$ use (6.16). Otherwise (6.8) implies $m \geq m_d - o(1)$, and (6.17) applies. Both savings are increasing in their indicated length. At $m = 0.49$ the ordinary saving is $0.98 + 0.02x > F_d$; it increases up to $m = 0.5$, where the no-prime treatment takes over. Thus no detected factor is chosen from a different detector cutoff to improve either branch.

For the inverse branch, $z_I(r_d) < 0.20 < \mathscr{L}$ and $2r + 8z_I(r) = 4 - 2r < 3$, with a fixed margin by $r > 0.6$. For the ordinary branch, $z_2(m_d) < 0.15 < \mathscr{L}$. Decrease these widths and the detector tolerances within the strict margins. Decreasing-amplitude whole factors then realize the required prime products, with a fixed mesh loss. The common Sobolev estimate (6.14) completes the use of Markov's inequality on the actual buffered class. □

For the complementary classes we need no new detector. To give the count explicitly, put

$$c = \frac{2}{p_1}, \qquad D = 3 - (1 + 2c)x, \qquad P = (2 - 2cx)(1 - x), \qquad J = (5/6 - \delta)D + \delta P,$$
$$R_s = 1 - \delta + \frac{(5/6 - \delta)\delta P}{2J}. \tag{6.20}$$

Here $J > 0$ on $[1/8, 5/6] \times [0, 1/2]$. Indeed the old ordinary saving is $2x/p_1 + (2 - 4x/p_1)m$, whereas the inverse saving is (6.16). Their crossing on $r + m = t$ gives the denominator D in (6.20). Balancing that short count with $1/6 + (5/6 - \delta)t$ gives $t = 1 + \delta P/(2J)$ and the displayed R_s. The same endpoint-strip, stock, and Sobolev argument as above is the count of [Liu26b, Section 13.4].

The closed rectangles

$$\begin{aligned} \mathcal{D}_1 &= [1/8, 5/6] \times [0, 6/25], \\ \mathcal{D}_2 &= [1/8, 11/50] \times [6/25, 1/2], \\ \mathcal{D}_3 &= [13/20, 5/6] \times [6/25, 1/2] \end{aligned} \tag{6.21}$$

satisfy

$$\mathcal{D} \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 = [1/8, 5/6] \times [0, 1/2].$$

On each $\mathcal{D}_i$, the reference exponent alone is negative.

6.3 The reference numerator and the coefficient-two saving

In this subsection, we estimate the numerator using the class counts of Section 6.2.

The following elementary first-moment calculation uses the approximate functional equation [Liu26b, Lemma 14.2] with common test functions.

Lemma 6.3 (A reference valid at all lengths)**.** *Suppose* $\#\mathscr{B} \ll U^{R+\epsilon}$ *and*

$$(p_1 - 2\delta x)\mathscr{L} > R. \tag{6.22}$$

Then the old ordinary moment, the no-prime theorem, and the trivial short-sum bound give the reference exponent $N(R)$ *in* (6.11)*, for every numerator length required by the approximate functional equation.*

Proof. For $0 \le m \le 1/2$ and $z = (1-2m)/p_1 \le \mathscr{L}$, the old moment gives

$$\sum_{u\in\mathscr{B}} |S_m(u)| \le (\#\mathscr{B})^{3/4} \left(\sum_{u\in\mathscr{B}} |S_m(u)|^4 \right)^{1/4} \ll U^{(3R+1)/4 - gz/2 + \epsilon}.$$

Since $\theta_0 = g/p_1$,

$$\frac{3R+1}{4} - \frac{gz}{2} - \theta_0 m = \frac{3R+1}{4} - \frac{\theta_0}{2} = N(R).$$

If $z > \mathscr{L}$, then $m \le m_* = (1 - p_1\mathscr{L})/2$, and the trivial bound gives

$$R + (1/2 - \theta_0)m - N(R) \le \frac{R - (p_1 - 2g)\mathscr{L}}{4} < 0.$$

For $m \ge 1/2$, the arbitrary-length no-prime fourth moment gives

$$\frac{3R+1}{4} - \theta_0 m \le N(R).$$

These estimates apply to each whole reflected factor, with common conductor test functions separated first. Finite partitions of the conductor and of the test function parameters, followed by (6.14), preserve the bound for both halves of the approximate functional equation. □

Define

$$J_N = \frac{5}{2} - \frac{5}{p_1}, \qquad \gamma_2(m) = \frac{\delta x J_N}{10}(1-2m). \tag{6.23}$$

For $0.30 \le m \le 0.47$,

$$z_2(m) = \frac{1-2m}{2} \le 0.20 < \mathscr{L}.$$

Apply (6.4) to $S_m^2 Q$ and use Hölder:

$$\sum_{u\in\mathscr{B}} |S_m(u)| \ll U^{(3R+1)/4 - gz_2(m)/2 + \epsilon}, \qquad \frac{g}{2}\left(\frac{1-2m}{2} - \frac{1-2m}{p_1}\right) = \gamma_2(m).$$

Let

$$m_0 = 1 - l_y/h. \tag{6.24}$$

At top parameter scale $\rho = h$, the contour coefficient in (6.12) vanishes precisely at $m = m_0$. Choose $\theta = 0$ to its left and $\theta = \delta/2$ to its right. The savings on the new band are respectively

$$h\gamma_2(m) + \theta_0 h(m_0 - m), \qquad h\gamma_2(m) + (\delta/2 - \theta_0)h(m - m_0).$$

Denote these two affine functions by $A_-(m)$ and $A_+(m)$. Then

$$A_-'(m) = -h\delta x/2 \le 0, \qquad A_+'(m) = h\delta(1-x)/2 > 0.$$

For $0.30 < m_0 < 0.47$,

$$\min_{0.30\le m\le m_0} A_-(m) = \min_{m_0\le m\le 0.47} A_+(m) = h\gamma_2(m_0).$$

The three required lower bounds are

$$\begin{aligned} G_N &= \frac{\delta x J_N}{10}(2l_y - h), \\ G_L &= \theta_0(0.70h - l_y), \\ G_R &= (\delta/2 - \theta_0)(l_y - 0.53h). \end{aligned} \tag{6.25}$$

Here G_L is the minimum baseline saving with $\theta = 0$ on $3/40 \le m \le 0.30$, and G_R is the baseline saving at $m = 0.47$ with $\theta = \delta/2$.

For clarity, the treatment after 0.47 retains the amplified old reference on $[0.47, 0.5]$. Only at and above 0.5 does the no-prime arbitrary-length clause of Lemma 6.3 supply that reference. At any fixed ρ, the full baseline exponent with $\theta = \delta/2$ decreases with m. Consequently the fixed endpoint 0.47 majorizes the entire longer range, including the reflected blocks; no selected-prime estimate for an unbounded length is being assumed.

For $m < 3/40$, use the trivial short-sum estimate. Relative to $N(R)$ it contributes at most $D_{\mathrm{sh}}(R)$. We have therefore reduced all numerator lengths to the following four tests:

numerator range	required top-scale inequality
$0 \le m < 3/40$	$E(R) + hD_{\mathrm{sh}}(R) < 0$
$3/40 \le m \le 0.30$	$E(R) < G_L$
$0.30 \le m \le 0.47$	$E(R) < G_N$
$m \ge 0.47$	$E(R) < G_R$.

(6.26)

6.4 Exact inequalities on the full parameter range

In this subsection, we prove the inequalities needed for the numerator estimates of Section 6.3. For (6.1) choose

$$b = \frac{499781491}{5 \cdot 10^9}, \qquad \ell = \frac{1298537}{7500000}, \qquad h = \frac{8096855491}{10^{10}}, \qquad l_y = \frac{13902270473}{3 \cdot 10^{10}}. \tag{6.27}$$

Then

$$T = \frac{3513974473}{3 \cdot 10^{10}}, \qquad m_0 = \frac{10388296000}{24290566473} = 0.4276679183891\ldots. \tag{6.28}$$

Proposition 6.4 (The rational comparison inequalities)**.** *Use* (6.3) *and* (6.27)*. On* $\mathcal{D}$*, the following inequalities hold on the entire closed rectangle:*

$$\begin{gathered} G_L - E(R_d) > 10^{-8}, \quad G_N - E(R_d) > 10^{-8}, \quad G_R - E(R_d) > 10^{-8}, \\ \frac{1}{5} < S(R_d) < 1, \qquad E(R_d) < \frac{1}{250}, \qquad (p_1 - 2\delta x)\mathscr{L} > R_d, \\ E(R_d) + hD_{\mathrm{sh}}(R_d) < -10^{-5}, \qquad 0 < S(R_d) + D_{\mathrm{sh}}(R_d) < 2, \\ 0.35 < m_d < 0.49, \quad 0.6 < r_d < 1, \quad 1 < t_d < 1.5, \\ 0.98 + 0.02x > F_d, \qquad \gamma_2(0.30) < 0.02. \end{gathered} \tag{6.29}$$

On $[1/8, 5/6] \times [0, 1/2]$*, one has*

$$J > 0, \qquad 0 < S(R_s) < 1, \qquad (p_1 - 2\delta x)\mathscr{L} > R_s. \tag{6.30}$$

On each $\mathcal{D}_i$ *in* (6.21)*,*

$$E(R_s) < -10^{-5}. \tag{6.31}$$

All denominator signs in these assertions are positive.

Proof. Substitute the displayed definitions into (6.29)–(6.31), subtract the stated positive margins, and clear the positive denominators. Appendix A gives the complete list of resulting functions and their exact bounds.

For a polynomial on a rational rectangle $[a,b]\times[c,d]$, make the substitution $X=a+(b-a)v$, $Y=c+(d-c)w$. If its power coefficients on $[0,1]^2$ are c_{ij} and its separate degrees are M,N, its Bernstein coefficients are

$$b_{kl}=\sum_{i\le k,\ j\le l} c_{ij}\frac{\binom{k}{i}}{\binom{M}{i}}\frac{\binom{l}{j}}{\binom{N}{j}}. \tag{6.32}$$

For the tensor Bernstein basis B_{kl},

$$P=\sum_{k,l} b_{kl}B_{kl},\qquad B_{kl}\ge 0,\qquad \sum_{k,l}B_{kl}=1\quad\Longrightarrow\quad \min_{k,l} b_{kl}\le P\le \max_{k,l} b_{kl}.$$

Bisect at rational midpoints until every coefficient is positive. For a positive numerator P and denominator D,

$$P\ge P_{\min}>0,\quad 0<D\le D_{\max}\quad\Longrightarrow\quad P/D\ge P_{\min}/D_{\max}>0.$$

All coefficients and subdivisions are rational.

The alternating midpoint subdivisions in Appendix A terminate at depth at most thirteen. Every accepted rectangle has positive Bernstein coefficients. The least listed quotient bound after subtracting the stated margins is

$$194222\cdot 10^{-12}>1.9422\cdot 10^{-7}.$$

The same appendix gives the constant inequalities used below. □

Lemma 6.5 (All remaining scalar and comparison ranges)**.** *The parameters above satisfy the direct, stock, small-amplitude, floor, small-parameter, and principal inequalities of [Liu26b, Sections 13.2, 13.5 and Proposition 14.6]. Together with Proposition 6.4, they give a strict correction saving on every retained class and numerator length.*

Proof. First, the following strictly positive geometric differences are obtained by rational substitution:

$$\begin{gathered} b>0,\quad \ell-\frac{1}{6}>0,\quad \frac{1}{5}-\ell>0,\quad l_x-\ell>0,\quad l_y-\ell-\frac{11b}{6}>0,\\ p_1\ell-\frac{3}{4}>0,\quad T>0,\quad \frac{1}{2}-l_y>0,\quad B-0.87>0,\\ \frac{\ell}{h+10^{-9}}>\mathscr{L},\qquad \frac{\ell}{h-1/40}<\frac{2}{9},\qquad 0.30<m_0<0.47,\qquad 0.57<l_y/h<0.59. \end{gathered} \tag{6.33}$$

Since $M=1-\ell$ and $\ell<1/5$, the four branches in (6.13) give $k=0$. Also $G=b/12$, since $l_y>11b/6$. Thus the empty-subset direct exponent is exactly

$$\frac{5}{6}+\frac{M}{12}-\frac{\ell}{6}=\frac{11}{12}-\frac{\ell}{4}=B.$$

Every nonempty subset is smaller. As usual the final direct bound allows an arbitrarily small positive exponent loss.

For main-rectangle parameters with $\rho\le h-1/40$, use the reference instead of the improved numerator. Its exponent is at most

$$\frac{1}{250}-\frac{1}{40}\frac{1}{5}=-\frac{1}{1000}. \tag{6.34}$$

For the complementary rectangles the negative reference exponent improves as ρ decreases, by (6.30). In the remaining interval near h, use (6.26). On bounded numerator bands with $m \ge 3/40$, the full slope in ρ is greater than

$$\frac{1}{5} - 0.073\frac{37}{40} - 0.02 = 0.112475 > \frac{1}{10}, \tag{6.35}$$

because $\theta_0 < 0.073$ and the largest new saving is less than 0.02. Its upper bound is two. The short-source slope is $S + D_{\mathrm{sh}}$, already tested in (6.29). On the long range first majorize by the fixed endpoint $m = 0.47$; its slope is $S + 0.53(\delta/2 - \theta_0)$, positive and less than two. It follows that an upper parameter buffer $h + \zeta$ contributes at most 2ζ. These arguments do not apply an unbounded-length slope bound to the approximate functional equation.

We give the small-amplitude calculation explicitly. For $0 \le \delta \le 1/8$ one may retain the weaker old parameter $\kappa = 3/4$. Put

$$A = 5/6 - \delta, \quad D = 3 - 17x/9, \quad P = (2 - 8x/9)(1 - x), \quad J = AD + \delta P,$$

and $R = 1 - \delta + A\delta P/(2J)$. With $v = 1/2 - x$, the polynomial identity

$$2J - 3AP - \frac{44}{27}xJ = \frac{250x^2}{729} + \frac{\delta}{243}(352v^3 + 948v^2 + 984v + 199) \ge 0 \tag{6.36}$$

implies $R \le 1 - 2\delta/3 - 22\delta x/81$. The source's small-amplitude contour, with $\theta = g\min(\ell/\rho, 2/9)$, is bounded at top scale by

$$C_0 + \frac{\delta}{2} + \ell\delta x\left(\frac{1}{2} + \frac{2\ell}{h}\right) - \frac{3h(1-R)}{4} \le C_0 + \delta\left\{\frac{1-h}{2} + Q_{\mathrm{sm}}x\right\},$$

where

$$Q_{\mathrm{sm}} = \ell\left(\frac{1}{2} + \frac{2\ell}{h}\right) - \frac{11h}{54} < 0, \qquad C_0 + \frac{1-h}{16} < -\frac{1}{250}. \tag{6.37}$$

The positive margin in the latter inequality is exactly 0.001023130452. This covers small amplitudes without extending the main rectangle down to zero. The lowest buffered bins use the trivial count, and the remaining small comparison parameters use the unamplified estimates of [Liu26b, Section 13.5 and Proposition 14.6]. Their requirements are respectively

$$C_0 + \frac{1/2 + 3\ell/2}{50} < -\frac{1}{1000}, \qquad \frac{13h}{75} - \frac{l_y}{2} + \frac{1}{50} < -\frac{1}{20}. \tag{6.38}$$

The positive differences in these inequalities are $21709069/(3 \cdot 10^{10})$ and greater than 0.021359. Bounded nonprincipal unit parameters are included in the second bound. Far parameters use the original right contours and a sufficiently large fixed far-parameter exponent, after all polynomial height orders are fixed.

Finally the actual principal paths lie in

$$\operatorname{Re} s \ge 0.87, \qquad \operatorname{Re} w \ge 0.95, \qquad \operatorname{Re} z \ge 0.165. \tag{6.39}$$

Lemma 7.1 proves the required normal convergence and nonvanishing there from the exact local polynomial, rather than from the older numerical cutoff in a previous comparison. Its five selected-prime error exponents have minimum 0.87, so one can fix $\omega_P = 0.435\min_i \ell_i > 0$ for the original positive-width prime intervals. The principal savings $l_y/20$ and $h/600$ are unchanged. Choose the later contour buffer e so that, for example,

$$l_y/20 - (20 + h)e > 0, \qquad h/600 - 20e > 0, \qquad \omega_P - 20e > 0.$$

Finite inducing and principal parameters therefore retain their own strict estimates; (6.4) is never applied to an inducing character excluded by that theorem. □

Proposition 6.6 (Strict comparison at the rational boundary). *For $B = B_{\mathrm{rat}}$, the parameters (6.27) supply a positive numerical saving in the comparison, uniform over all the amplitude, parameter, numerator-length, and principal decompositions. Constants may depend on the fixed field and tests. The estimates needed for the Mellin argument are*

$$\begin{aligned} |J_\eta(Z)| &\ll_\eta Z^{C(B)+\epsilon_{\mathrm{low}}}, \\ |J_\eta(Z) - f_\eta(Z)| &\ll_\eta Z^{C(\beta)-m_{\mathrm{save}}/2}, \qquad C(s) = s - \frac{4+b}{6}, \end{aligned} \tag{6.40}$$

for some $m_{\mathrm{save}} > 0$, whenever $\beta = \beta_*(F) > B$ and $0 < \epsilon_{\mathrm{low}} < (\beta - B)/4$ is fixed sufficiently small.

Proof. The closed cover (6.15), (6.21), and the small-amplitude range is exhaustive. Lemma 6.2 and Proposition 6.4, together with Lemma 6.5, give a positive margin for each part. Take a common margin smaller than their finite minimum. Use a still smaller portion for strict detector length conditions, prime-width decrements, the original mesh, and the upper parameter buffer. For each requested moment loss, fix the finite face chain in Theorem 4.9, with its selector strictly larger than one half, before the arithmetic data. No endpoint operator estimate at one half is used.

The Sobolev and derivative orders are finite. Choose the small external-height powers only after these orders, so their polynomial contributions fit inside the common margin. Choose the rapid tail orders afterward. At fixed complex variables and fixed parameter dyad the complete holomorphic numerator is moved before absolute values; both functional-equation halves and all natural deletions remain in that expression. This is precisely the contour order of Lemma 5.2 and [Liu26b, Lemma 14.7], with strict inequalities on every class. Summing the logarithmically many dyadic pieces and the rapidly convergent test function tails consumes another fixed fraction of the saving. The direct estimate and the corrected comparison then give (6.40). The continuation of its Mellin transform and the passage from the family supremum to fixed targets are proved in Section 7. □

6.5 Attaining the algebraic boundary as an infimum

In this subsection, we extend the comparison of Section 6.4 to every fixed boundary sufficiently close above B_2.

Set $z_0 = \sqrt{1081}$ and define

$$\delta_2 = \frac{61 - z_0}{66}, \qquad b_2 = -\frac{11}{71} + \frac{595 z_0}{76751}. \tag{6.41}$$

Thus $\delta_2 = 0.4260823568856485\ldots$ and $b_2 = 0.0999562982254378\ldots$. Keep $b = b_2$, the same rectangles, and the same numerator bands. Let

$$\begin{aligned} g(B, \delta, x) &= G_N - E(R_d) \\ &= \frac{1}{4} - \frac{b}{6} - \frac{5\ell}{4} - \frac{\delta}{2} - \delta x \ell + \frac{3h(1 - R_d)}{4} + \frac{\delta x T}{4}. \end{aligned} \tag{6.42}$$

The second equality follows directly from (6.23); every occurrence of p_1 cancels. This is the only scalar margin that vanishes at the exact endpoint.

Lemma 6.7 (The unique double contact). *At $B = B_2$, $g \geq 0$ on $\mathcal{D}$, with equality only at $(\delta, x) = (\delta_2, 1/2)$. Moreover $\partial_B g > 0$ uniformly on $\mathcal{D}$, with $b = b_2$ held fixed.*

Proof. Algebraic substitution gives the exact factorization

$$g(B_2, \delta, 1/2) = \frac{178/213 - (5179/460506) z_0}{5 - 4\delta} (\delta - \delta_2)^2. \tag{6.43}$$

The coefficient in front of the square is greater than 0.11308683 throughout $[0.22, 0.65]$. On $[0.4, 0.46] \times [0.24, 0.5]$, polynomial division at $x = 1/2$ and exact coefficient bounds give

$$\frac{g(B_2, \delta, x) - g(B_2, \delta, 1/2)}{1/2 - x} > 0.00908267, \tag{6.44}$$

where the quotient has its continuous value after cancelling $1/2 - x$ at $x = 1/2$. On the two remaining rectangles $[0.22, 0.4] \times [0.24, 0.5]$ and $[0.46, 0.65] \times [0.24, 0.5]$, the saving is greater than $7.2884 \cdot 10^{-5}$. These facts prove the uniqueness of the zero, including the closed boundaries of the rectangles.

For the derivative there is a particularly simple formula:

$$\partial_B g = \frac{-26\delta^2 x^2 + 14\delta^2 x + 5\delta x - 25\delta + 25}{5 - 2\delta - 4\delta x} > 2.3723473. \tag{6.45}$$

The derivative is independent of B and of b_2.

Use the Bernstein procedure (6.32) over $\mathbb{Q}(z_0)$. For rational $0 < l < u$ with $l^2 < 1081 < u^2$,

$$a + bz_0 \geq \begin{cases} a + bl, & b \geq 0, \\ a + bu, & b < 0. \end{cases}$$

Rational midpoint subdivision gives the stated bounds on the three rectangles and on $\partial_B g$, including their boundaries. □

Proposition 6.8 (Strict comparisons arbitrarily near B_2)**.** *For every fixed $B > B_2$ sufficiently close to B_2, the geometry (6.5) with $b = b_2$ satisfies the conclusions of Proposition 6.6, with constants and saving allowed to depend on $B - B_2$.*

Proof. Let f_j denote the nonactive margins in Proposition 6.4 and Lemma 6.5, on their closed domains $\mathcal{D}_j$. Appendix A gives

$$\min_j \inf_{\mathcal{D}_j} f_j(B_2, b_2) > 0.$$

For the nonactive function entries, the least listed lower bound is $14627855 \cdot 10^{-12} > 1.46278 \cdot 10^{-5}$; the constant entries are also strict. Thus, for some $\eta_B, \mu > 0$,

$$B_2 \leq B \leq B_2 + \eta_B \quad \Longrightarrow \quad \min_j \inf_{\mathcal{D}_j} f_j(B, b_2) \geq \mu.$$

For $B > B_2$, the active margin satisfies, by Lemma 6.7,

$$g(B, \delta, x) > g(B_2, \delta, x) + 2.3723473(B - B_2) \geq 2.3723473(B - B_2).$$

For each fixed $B > B_2$, take the final losses below $\min\{\mu, 2.3723473(B - B_2)\}$. No fixed active margin is required as $B \downarrow B_2$.

Now make the finite choices in the proof of Proposition 6.6. Their constants need not be uniform as B decreases to B_2. In particular the ordinary coefficient two is obtained by a finite choice for each prescribed loss; no infinite induction is applied to the arithmetic data. This proves the required strict comparisons. After Section 7, any point with real part greater than B_2 is covered by choosing such a fixed B below it. No assertion on the boundary line is made. □

6.6 Optimality within the specified positive comparison

In this subsection, we prove the converse inequality for the comparison system of Section 6.1.

We finish by specifying exactly what is optimized. Retain (6.13), the short inverse saving (6.16), the long inverse exponent (6.10), the fourth-moment Hölder return, and the original amplitude information (6.6). Allow arbitrary detector cutoffs in the supplied domain, arbitrary

comparison rectangles, and arbitrary positive comparison lengths with $b = l_y - l_x \geq 0$. For each application to the ordinary detector or numerator, allow the full permitted prime width

$$z(m) \leq \frac{1-2m}{2}. \tag{6.46}$$

The two uses may choose different smaller widths. Grant away all reflection, stock, exceptional-parameter, and width-floor restrictions that could decrease this width. This is an optimistic positive comparison system, not the class of all possible estimates for the underlying arithmetic sums.

Theorem 6.9 (The exact scalar infimum)**.** *The infimum of boundaries supplied by this specified system is B_2. In particular, arbitrary unequal positive comparison lengths with $b \geq 0$, changes of detector cutoff, and a change only in the old reference parameter p_1 do not produce a boundary below B_2.*

Proof. We first derive the necessary count at $x = 1/2$. The short inverse and ordinary savings are $(1+r)/2$ and $1/2+m$. For a cutoff $1 \leq t \leq 3/2$, the admissible split

$$r = 2t/3, \qquad m = t/3$$

equalizes them and forces the supplied short-count exponent $1-\delta(1/2+t/3)$. The separate split $(r,m) = (t,0)$ forces the increasing long-count exponent $1/6+(5/6-\delta)t$. At $\delta = \delta_2$ the ordinary alternative at this second split is no better: its count $1-\delta_2/2$ exceeds the long count by at least $\delta_2 - 5/12 > 0$. A zero-length annulus here means a limit of allowed shrinking length exponents.

The decreasing and increasing bounds cross at

$$t_* = \frac{5-3\delta}{5-4\delta}, \qquad F(\delta) = \frac{25-18\delta}{30-24\delta}, \qquad R(\delta) = 1-\delta F(\delta). \tag{6.47}$$

Lemma 6.2 attains this crossing. In the supplied comparison system, the two adverse splits therefore give

$$\inf_{1\leq t\leq 3/2} \max\left\{1-\delta_2(1/2+t/3),\ \frac{1}{6}+(5/6-\delta_2)t\right\} = R(\delta_2).$$

Even granting both short formulas for $0 < t < 1$ gives

$$1-\delta_2(1/2+t/3) \geq 1-5\delta_2/6 > R(\delta_2).$$

This concerns the stated inequalities; no arithmetic class is asserted to realize an adverse split.

For the balanced geometry the critical numerator length is $m_0 = 1-l_y/h$. At this length the contour coefficient vanishes, so no numerator contour can improve the top-scale value. The necessary reference excess and maximal numerator saving are

$$\begin{aligned} E_* &= -\frac{1}{4}+\frac{b}{6}+\frac{5\ell}{4}+\frac{\delta}{2}+\frac{\delta}{2}\left\{\ell+\frac{h/2-l_y}{p_1}\right\}-\frac{3h\delta F(\delta)}{4}, \\ G_* &= \frac{\delta T}{20}\left(\frac{5}{2}-\frac{5}{p_1}\right). \end{aligned} \tag{6.48}$$

The parameter p_1 cancels in $E_* - G_*$, and

$$\partial_b(E_*-G_*) = \frac{33\delta^2-61\delta+20}{24(5-4\delta)}. \tag{6.49}$$

Its smaller zero is δ_2. At that amplitude

$$E_*-G_* = \frac{221-2\sqrt{1081}}{33}(B_2-B). \tag{6.50}$$

This already excludes $B < B_2$ for all balanced imbalances.

We now remove the balanced total-length assumption. Write

$$M = l_x + l_y, \quad b = l_y - l_x \geq 0, \quad h = 1 - l_x + \ell, \quad l_y = (M+b)/2, \quad T = 2l_y - h.$$

The direct estimate implies the three necessary planes

$$\begin{aligned} B \geq \mathcal{L}_0 &= \frac{5}{6} + \frac{M}{12} - \frac{\ell}{6}, \qquad B \geq \mathcal{L}_K = \frac{1}{3} + \frac{7M}{12} + \frac{\ell}{3}, \\ B &\geq \mathcal{L}_0 + \mathcal{G}_0, \qquad \mathcal{G}_0 = \frac{2b}{3} - \frac{M}{4}. \end{aligned} \tag{6.51}$$

For $\mu = 37/100$ the identity

$$\begin{aligned} h - l_y - \mu h = \frac{11(4-3\mu)}{6} &- \left(\frac{22}{3} - \frac{25\mu}{4}\right)\mathcal{L}_0 - \left(\frac{2}{3} + \frac{\mu}{4}\right)\mathcal{L}_K \\ &- \frac{3\mu}{4}(\mathcal{L}_0 + \mathcal{G}_0) \end{aligned} \tag{6.52}$$

has three positive coefficients on the subtracted planes. The direct plane $\mathcal{L}_K \leq B \leq 0.8741$, together with $\ell \geq 0$ and $0 \leq l_x \leq M$, gives

$$M \leq \frac{12}{7}\left(B - \frac{1}{3}\right) < 1, \qquad h = 1 - l_x + \ell \geq 1 - M + \ell > 0.$$

If $B \leq 0.8741$, the right side of (6.52) is at least $41671/12000000 > 0$. Consequently $m_0 > 0.37$. The trivial short-sum majorant cannot improve the norm at this length, since $(R(\delta_2) - 1)/4 + 0.37/2 > 0$, even before the nonnegative prime terms. If $M \geq 1 + \ell$, then $\mathcal{L}_K \geq 11/12$, already too large. It remains to consider the ordinary numerator range with $M < 1 + \ell$.

Put $C = 1/8$ and $R_2 = R(\delta_2)$. When $T > 0$, the identity $T = 2l_y - h$ gives $m_0 = 1 - l_y/h < 1/2$; hence $0.37 < m_0 < 1/2$ lies in the ordinary fourth-moment range. The zero-contour-coefficient numerator length forces $B \geq H$, where direct substitution of the count and the maximal saving from this width gives

$$\begin{aligned} H &= \frac{1+\delta_2}{2} - \frac{l_y}{2} + h\left(\frac{3R_2+1}{4} - \frac{1}{6}\right) + \frac{\delta_2 - 1}{2}\ell - C\delta_2 T \\ &= 1 + (2C + 1/2)\delta_2 - (1/2 + 2C\delta_2)M + (2C + 1/2)\delta_2\ell. \end{aligned} \tag{6.53}$$

Let $D_+ = 16C\delta_2 + 2\delta_2 + 3 = 4\delta_2 + 3$ and

$$u = \frac{44C\delta_2 + 7\delta_2 + 4}{3D_+}, \qquad v = \frac{4C\delta_2 - \delta_2 + 2}{3D_+}, \qquad w = \frac{1}{D_+}.$$

All three are positive, their sum is one, and coefficient comparison gives

$$u\mathcal{L}_0 + v\mathcal{L}_K + wH = B_2. \tag{6.54}$$

Thus every such positive choice of comparison lengths has $B \geq B_2$.

If $T \leq 0$, use the no-prime endpoint $m = 1/2$, for which the optimal allowed contour is zero. Its plane is $H_0 = H + C\delta_2 T$. Put

$$\begin{aligned} D_- = 19 - (12C + 9)\delta_2, \qquad u_- = \frac{3\{2 - (4C+3)\delta_2\}}{D_-}, \qquad v_- = \frac{13}{D_-}, \\ k_- = \frac{18C\delta_2 + 7\delta_2 + 4}{2D_-}. \end{aligned}$$

These numbers are positive and $u_- + v_- = 1$. The identity

$$u_-\mathcal{L}_K + v_- H_0 + k_- T = \frac{52 + b(44C\delta_2 + 7\delta_2 + 4)}{4D_-} \tag{6.55}$$

and $T \leq 0$, $b \geq 0$ imply $B \geq 13/D_- = 0.8949386587587\ldots > B_2$. The necessary direction is proved without restricting the total comparison length. The sufficient direction, as an infimum, is Proposition 6.8. □

6.7 Unequal ordinary factors and the limit of positive interpolation

In this subsection, we examine unequal ordinary factors at the equality case of Theorem 6.9.

The equal factors in the detector do not conceal a saving from pairing its ordinary detected factor with an unequal numerator factor. At the contact write

$$m_D = \frac{1+\sqrt{1081}}{90}, \qquad r_D = 2m_D, \qquad t_D = 3m_D, \qquad R_2 = \frac{281-\sqrt{1081}}{396}, \qquad \mathcal{C}(v) = \frac{1-v}{2}. \tag{6.56}$$

The ordinary detector lower bound is $|S_{m_D}|^2 \geq U^{\delta_2 m_D - o(1)}$. Use the unequal moment for $S_{m_D} S_n Q$ at width $\mathcal{C}(m_D+n)$, divide by this lower bound, and apply Cauchy–Schwarz on the counted parameters. The resulting numerator first-moment exponent is

$$A_{\mathrm{mix}} = \frac{R_2 + 1 - \delta_2 m_D - \delta_2 \mathcal{C}(m_D+n)}{2} = \frac{3R_2 + 1 - \delta_2 \mathcal{C}(2n)}{4} = A_{\mathrm{sep}}. \tag{6.57}$$

The equality follows from the affine nature of $\mathcal{C}$ and $1 - R_2 = \delta_2\{2m_D + \mathcal{C}(2m_D)\}$. It holds throughout the admitted reflected length range.

A simultaneous scalar model makes the scope of this neutrality precise. On an abstract set of U^{R_2} parameters assign, for $0 \leq y \leq 1/2$,

$$|S_y| = U^{s(y)}, \qquad s(y) = \min\{\delta_2 y/2, \delta_2(y+m_D)/4\}, \qquad |Q_z|^2 = U^{\delta_2 z}. \tag{6.58}$$

For every $y_1, y_2 \in [m_D, 1/2]$,

$$R_2 + 2s(y_1) + 2s(y_2) + \delta_2 \mathcal{C}(y_1+y_2) = 1. \tag{6.59}$$

For either factor below m_D the left side decreases. Also $s(y) \leq \delta_2 y/2 \leq \kappa_1 y/2$, so the pointwise ordinary price is respected. For original lengths $0 \leq y \leq 1$ use the amplitude at the reflected length $\min(y, 1-y)$, and give negligible amplitude beyond that range. Reflection only decreases the original total length, so the original admissible permitted prime width is no larger than that of the reflected pair. The arbitrary-length no-prime moments are therefore respected as well. Equality in (6.59) is claimed only on the indicated reflected rectangle.

For the inverse factors, assign

$$|M_r|^2 = \begin{cases} U^{\delta_2 r}, & r = r_D \text{ or } t_D \leq r \leq 3/2, \\ 1, & \text{otherwise.} \end{cases}$$

The critical inverse estimates are simultaneously saturated:

$$R_2 + \delta_2 r_D + \frac{\delta_2(1-r_D)}{2} = 1, \qquad R_2 + \delta_2 t_D = \frac{1+5t_D}{6}. \tag{6.60}$$

For $t_D \leq r \leq 3/2$, the long inverse margin is

$$e(r) - \delta_2 r - R_2 = (5/6 - \delta_2)(r - t_D) \geq 0.$$

At the other short lengths the prime width is at most $3/8$, and $R_2 + 3\delta_2/8 < 1$. Every detector cutoff in the supplied domain has a simultaneous detected factor:

$$(r,m) = \begin{cases} (r_D, m_D), & 1 \leq t \leq t_D, \\ (t, 0), & t_D \leq t \leq 3/2. \end{cases}$$

These pairs satisfy $r \leq t$, $r + m \geq t$, $r \geq t - 1/2$, and $m \leq 1/2$, with the required lower bounds. The critical numerator length lies between m_D and $1/2$, and its first moment in this model is exactly $U^{R_2 + s(n)} = U^{A_{\mathrm{sep}}}$. Every original prime subproduct has squared-amplitude

slope $\delta_2 < 1/2$; the selected operator at a fixed threshold greater than one half imposes no additional condition on these abstract parameters.

This is a model of the simultaneous inequalities, not a construction of Hecke characters. It excludes a strict improvement obtained solely by positive interpolation of these unequal ordinary moments, the stated inverse bounds, and the same amplitude data. It neither supplies nor rules out a phase-sensitive mixed inverse–ordinary estimate. Nor does Theorem 6.9 assert optimality among all zero-free methods, all possible arithmetic relations between detectors, or comparison geometries outside its stated $b \geq 0$ scope. Changing one of those inputs remains a separate analytic problem.

7 Principal factors, continuation, and arbitrary number fields

In this section, we prove Theorem 1.1. We verify the principal Euler product on the new contours, apply the Mellin identity with the scalar saving, and descend from $K(\zeta_{12})$ to K.

7.1 The principal contour region

In this subsection, we verify the Euler product needed in the Mellin argument of Proposition 7.2.

At a good prime put $q = N\mathfrak{p}$, $T = q^{-s}$, $Y = q^{-w}$, $V = q^{-6z}$, $D = \eta(\mathfrak{p})T$, and

$$R = \overline{\Lambda_F(\mathfrak{p})}^{6}\eta(\mathfrak{p})^6 q^4 T^6 V.$$

The angular factor Λ_F is unitary, with exactly the coefficient orientation of [Liu26b, Section 13.2]. For the principal numerator class, the source local identity is

$$\begin{aligned}(1-R)(1-D)(H_p-1) = {} & -VY + DY + DV(1-Y) \\ & - D(q-1)YV(1-Y) + R - DR \\ & - RY^2(1-V) - R(1-Y)/q.\end{aligned} \tag{7.1}$$

This is the cleared local formula of [Liu26b, (13.9)], in its principal region. It is used with all phases retained.

Lemma 7.1. *After enlarging the fixed excluded set S, the principal Euler product is holomorphic and nonzero throughout*

$$\operatorname{Re} s \geq 87/100, \qquad \operatorname{Re} w \geq 19/20, \qquad \operatorname{Re} z \geq 33/200, \tag{7.2}$$

uniformly in the heights and unitary target phases. The selected prime factor has the principal asymptotic of [Liu26b, Section 13.2], with error $O(q^{-87/100})$.

Proof. On this box, $|D| \leq q^{-0.87}$ and $|R| \leq q^{-2.21}$. In the expansion of (7.1), the following table lists lower bounds for the decay exponents. For the terms containing $q-1$, we use $q - 1 \leq q$.

Term	VY	DY	DV	DVY	$(q-1)DYV$	$(q-1)DY^2V$
Exponent	1.94	1.82	1.86	2.81	1.81	2.76
Term	R	DR	RY^2	RY^2V	R/q	RY/q
Exponent	2.21	3.08	4.11	5.10	3.21	4.16

All exponents are nondecreasing in the three real coordinates. Hence

$$|H_p - 1| \ll q^{-1.81}, \qquad \sum_{p \notin S} \sup|H_p - 1| < \log(3/2), \qquad \sup|H_p - 1| < 1/2$$

after enlarging S. Normal convergence of the logarithms gives

$$\prod_{p\notin S} H_p \neq 0, \qquad \left|\prod_{p\notin S} H_p - 1\right| \leq \exp\left(\sum_{p\notin S} |H_p - 1|\right) - 1 < 1/2.$$

The exact source selected-factor calculation has the five error exponents

$$\operatorname{Re} s, \quad \operatorname{Re} w, \quad 6\operatorname{Re} z, \quad \operatorname{Re} w + 6\operatorname{Re} z - 1, \quad 5\operatorname{Re} s + 6\operatorname{Re} z - 4.$$

At the lower corner these are respectively $0.87, 0.95, 0.99, 0.94, 1.34$, and they increase in every coordinate. Hence $G_p/H_p = -1 + O(q^{-0.87})$ in this principal region. The quotient is taken only where the nonvanishing just proved applies. No division by a nonprincipal H_p is inserted elsewhere in the proof. □

At $w = 1, z = 1/6$, (7.1) also gives the useful exact identity

$$H_p(s, 1, 1/6) = \frac{(1 - q^{-2})(1 - D + D/q - R/q)}{(1 - R)(1 - D)}. \tag{7.3}$$

The product used in the Mellin quotient is $H_\eta(s) = \prod_{p\notin S} H_p(s, 1, 1/6)$, so the lemma proves its nonvanishing for $\operatorname{Re} s > 0.87$. The source residue normalizer is unchanged and nonzero for the original chosen test functions. The principal savings $l_y/20$ and $h/600$ therefore remain available. In particular, one may choose $\omega_P = 0.435 \min_i \ell_i$ for the tuple error, before the later contour and height losses. The principal paths retain the source coordinates $w_- = 19/20$, $w_+ = 1 + e$, $z_- = 33/200$ and $z_+ = 1/6 + e$, with s first on $\beta + 20e$. Thus all three coordinates lie in (7.2) when $B > 0.87$. The buffered savings are $l_y/20 - (20 + h)e$, $h/600 - 20e$ and $\omega_P - 20e$, and are positive after decreasing e.

7.2 The order of the strict losses

In this subsection, we choose the errors in Section 6 before applying the Mellin identity.

Fix either the rational comparison $B = 0.8733821$, or a fixed $B > B_2$ sufficiently close to B_2 as in the exact-contact argument. Choose a positive margin smaller than every strict scalar, stock, slope, principal and geometric margin for this B. The margin is allowed to tend to zero as $B \downarrow B_2$.

Decrease the chosen prime permitted lengths and detector cutoffs within that margin and take a finite closed cover of the retained parameter ranges. For each required ordinary moment loss, fix the interior moment parameters, selected threshold greater than $1/2$, and finite list of moment-induction steps. Allocate the parent and auxiliary losses separately, including the low branches whose baseline excess is zero. Choose the original slot mesh below the finite minimum of those bounds and the final complete-subproduct removal bound. A finite iteration with such choices is used for each fixed B.

At each comparison parameter dyad, fix the presentation, complex variables and common test functions before forming amplitude bins and selecting original prime factors. Move the entire holomorphic dyadic numerator expression before absolute values. Keep both approximate-functional-equation halves and every natural deletion, root number and strict local error tuple. The original conductor deficit absorbs the prime-selection errors. The finite derivative and Fourier orders are then fixed, followed by the small external height exponents and the final tail orders. This makes every polynomial height contribution a prescribed fraction of the positive saving. For large heights use the same polynomial bound with a larger final height power; that power is not fed back into the earlier small-height choice.

For this fixed B, choose one finite numerical list of positive prime widths (ℓ_i), with the required total width and mesh, for every target character. The fixed ray selectors may depend

on the target; their prime-ideal thresholds and implied constants may also depend on it. The numerical widths do not. In particular,

$$\omega_P = \frac{87}{200} \min_i \ell_i > 0$$

is common to the family. Test orders may vary with the target; choose each small-height cutoff so that its contribution is below the same fixed fraction of the numerical margin. Thus the remaining exponent saving is independent of the target.

By Section 6, the upper parameter extension has exponent at most 2ζ. Choose

$$0 < 2\zeta < \tfrac{1}{10} \min\{\text{the fixed positive margins}\}.$$

The resulting $m_{\mathrm{save}} > 0$ is numerical and common to all targets; constants and thresholds may depend on the target.

7.3 The Mellin contradiction

In this subsection, we deduce the fixed-field result from the strict comparisons in Section 6.

Proposition 7.2. *Let $F \supset \mu_{12}$ be a number field. For each fixed $B = B_{\mathrm{rat}}$ or $B > B_2$ sufficiently close to B_2, the scalar construction gives $\beta_*(F) \le B$. Consequently $\beta_*(F) \le B_2$.*

Proof. Suppose $\beta = \beta_*(F) > B$. We use the fixed pointwise parameter κ_1 from (2.4). Choose $0 < \epsilon_{\mathrm{low}} < (\beta - B)/4$ and write $C(s) = s - (4+b)/6$. Propositions 6.6 and 6.8, together with [Liu26b, (13.13) and Lemma 14.7], give, for every fixed target η, the original functions J_η, f_η, H_η with

$$|J_\eta(Z)| \ll_\eta Z^{C(B)+\epsilon_{\mathrm{low}}}, \qquad |J_\eta(Z) - f_\eta(Z)| \ll_\eta Z^{C(\beta)-m_{\mathrm{save}}/2}. \tag{7.4}$$

Set

$$\sigma = \max\{B + \epsilon_{\mathrm{low}}, \beta - m_{\mathrm{save}}/2\} < \beta.$$

Then $f_\eta(Z) \ll_\eta Z^{\sigma-(4+b)/6}$. Writing $a = (4+b)/6$, for $\operatorname{Re} s > \sigma$ we have

$$\int_1^\infty |f_\eta(Z) Z^{a-s}| \frac{dZ}{Z} \ll_\eta \int_1^\infty Z^{\sigma - \operatorname{Re} s} \frac{dZ}{Z} = \frac{1}{\operatorname{Re} s - \sigma}.$$

Thus the large-scale Mellin integral is holomorphic on that half-plane. The integral over $(0,1]$ is entire by the original right shifts. On the initial common domain, [Liu26b, (13.13)] gives

$$e^{(s-5/6)^2} \frac{H_\eta(s)}{L_F^S(s,\eta)}. \tag{7.5}$$

By uniqueness it is the continuation of this quotient. Since $\sigma > B > 0.87$, Lemma 7.1 makes its numerator holomorphic and nonzero throughout that half-plane. Thus the denominator has no zero there. Finite omitted Euler factors have no zero with positive real part.

The exponent σ is common to all targets. Therefore

$$L_F(s,\eta) \neq 0 \quad (\operatorname{Re} s > \sigma,\ \eta \text{ of finite order}) \quad \Longrightarrow \quad \beta_*(F) \le \sigma < \beta_*(F),$$

a contradiction. Constants and thresholds may depend on η.

Finally take a sequence of fixed boundaries $B \downarrow B_2$ for which the exact-contact construction is valid. Each has its own positive margins; their conclusions give $\beta_*(F) \le B_2$. □

7.4 Field dependence and descent

In this subsection, we descend the fixed-field result of Proposition 7.2 to prove Theorem 1.1.

The operator proof uses ideal-character functional equations, finite reciprocal packets, and divisor bounds over the fixed field F. The moment proof uses the exact general-field identities of [Liu26b, Theorem 18.2] over F. In the approximate functional equation, $r_F = [F : \mathbb{Q}]/2$ enters the gamma ratios and the polynomial height powers. Long dyadic lengths are handled by the original arbitrary-length zero-prime moment, and the bounded selected bands keep the scalar cutoffs already verified. Consequently the constants depend on F, whereas the exponents and the final number B_2 do not.

Proof of Theorem 1.1. Let η be primitive of finite order d over K, and let M/K be the cyclic class-field extension cut out by its kernel. Put $F = K(\zeta_{12})$ and $E = MF$. Restriction gives

$$\mathrm{Gal}(E/F) \hookrightarrow \mathrm{Gal}(M/K) \simeq \mathbb{Z}/d\mathbb{Z}, \qquad \mathrm{Gal}(E/M) \hookrightarrow (\mathbb{Z}/12\mathbb{Z})^{\times}.$$

Both groups are abelian.

Their full class-field factorizations give

$$\zeta_E(s) = \prod_{\psi \in \widehat{\mathrm{Gal}(E/F)}} L_F(s, \psi), \tag{7.6}$$

$$\zeta_E(s) = \zeta_M(s) \prod_{\substack{\omega \in \widehat{\mathrm{Gal}(E/M)} \\ \omega \neq 1}} L_M(s, \omega), \tag{7.7}$$

$$\zeta_M(s) = \prod_{j=0}^{d-1} L_K(s, \eta^j). \tag{7.8}$$

For $\mathrm{Re}\, s > B_2$, $s \neq 1$, the first product is nonzero by Proposition 7.2. The nonprincipal factors in the second are entire, so a zero of ζ_M could not be canceled by a pole. The third identity then proves nonvanishing of $L_K(s, \eta)$ away from one.

For $d > 1$, the simple residues at one give

$$\prod_{j=1}^{d-1} L_K(1, \eta^j) = \frac{\mathrm{Res}_{s=1}\, \zeta_M(s)}{\mathrm{Res}_{s=1}\, \zeta_K(s)} > 0.$$

Each finite factor is therefore nonzero. For $d = 1$ the principal pole is permitted. An imprimitive function differs by finitely many Euler factors $1 - \alpha q^{-s}$ with $|\alpha| = 1$ or $\alpha = 0$; their zeros, if any, have real part zero. The full factorizations include ramified places and finite-order sign characters at real places of K. This proves the theorem. □

Corollary 7.3. *Every Dedekind zeta function is nonzero for* $\mathrm{Re}\, s > B_2$, *apart from its pole at one. The conclusion includes the Riemann zeta function, all Dirichlet L-functions, and all finite-order Hecke L-functions over* $\mathbb{Q}(\sqrt{-3})$.

Proof. Take the trivial character in the theorem, or specialize K to the indicated fields. □

A Exact scalar inequalities

In this appendix, we verify the inequalities used in Section 6 by rational Bernstein bounds. The formulas, rectangles, and subdivision rule below specify the finite calculation completely.

Write

$$\Delta = 5 - 2\delta - 4\delta x, \qquad \Omega = [1/8, 5/6] \times [0, 1/2].$$

The rectangles $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are those of (6.15) and (6.21). The two parameter choices are

$$(B, b) = \left(\frac{8733821}{10^7}, \frac{499781491}{5 \cdot 10^9}\right), \qquad (B, b) = (B_2, b_2), \qquad p_1 = \frac{1124661}{250000}.$$

All functions in the tables have their displayed definitions in Section 6; thus each entry specifies a rational function over $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{1081})$.

For a function f on $\mathcal{D}$, expand the polynomial Δf. On Ω or $\mathcal{D}_i$, expand Jf, except that the function $f = J$ is already polynomial. The denominator bounds are

$$\Delta \geq \frac{12}{5} \quad \text{on } \mathcal{D}, \qquad J \geq \frac{4373305}{6747966} \quad \text{on } \Omega.$$

The second bound follows directly from the Bernstein coefficients of J on Ω.

Starting with the stated rectangle at depth zero, compute the Bernstein coefficients by (6.32). Accept a rectangle when every coefficient is positive. Otherwise bisect its δ interval at even depth and its x interval at odd depth, and apply the same rule to both subrectangles. The accepted rectangles form a partition of the original closed rectangle. In the tables, (d, L) gives the maximum depth and number of accepted rectangles. If m is the least numerator coefficient over those rectangles and M is an upper Bernstein bound for the positive multiplier on the original rectangle, then

$$f \geq m/M \geq k\, 10^{-12} > 0.$$

The integer k is listed below; the multiplier is one for $f = J$. The first pair of columns uses the rational parameters and the second pair uses (B_2, b_2).

For the quadratic-field coefficients use the rational enclosure

$$3287856444554719302046/10^{20} < \sqrt{1081} < 3287856444554719302047/10^{20}.$$

Both inequalities follow by squaring positive integers. For each coefficient $a + b\sqrt{1081}$, choose the lower or upper endpoint according to the sign of b, before taking its lower bound.

f	Rectangle	(d, L)	k	(d, L)	k
$G_L - E(R_d) - 10^{-8}$	$\mathcal{D}$	$1, 2$	587138129	$1, 2$	586639331
$G_N - E(R_d) - 10^{-8}$	$\mathcal{D}$	$13, 14$	194222	–	–
$G_R - E(R_d) - 10^{-8}$	$\mathcal{D}$	$1, 2$	1302783884	$1, 2$	1302190830
$S(R_d) - 1/5$	$\mathcal{D}$	$0, 1$	24051095581	$0, 1$	24051095581
$1 - S(R_d)$	$\mathcal{D}$	$0, 1$	416191181453	$0, 1$	416191181453
$1/250 - E(R_d)$	$\mathcal{D}$	$3, 4$	107812329	$3, 4$	107411143
$(p_1 - 2\delta x)\mathscr{L} - R_d$	$\mathcal{D}$	$0, 1$	81252174270	$0, 1$	81252174270
$-E(R_d) - hD_{\mathrm{sh}}(R_d) - 10^{-5}$	$\mathcal{D}$	$0, 1$	2018042319	$0, 1$	2017445082
$S(R_d) + D_{\mathrm{sh}}(R_d)$	$\mathcal{D}$	$0, 1$	90495844449	$0, 1$	90495844449
$2 - S(R_d) - D_{\mathrm{sh}}(R_d)$	$\mathcal{D}$	$0, 1$	1013256914932	$0, 1$	1013256914932
$m_d - 7/20$	$\mathcal{D}$	$0, 1$	1073092960	$0, 1$	1073092960
$49/100 - m_d$	$\mathcal{D}$	$0, 1$	35084007848	$0, 1$	35084007848
$r_d - 3/5$	$\mathcal{D}$	$0, 1$	96884964434	$0, 1$	96884964434
$1 - r_d$	$\mathcal{D}$	$0, 1$	84314446897	$0, 1$	84314446897
$t_d - 1$	$\mathcal{D}$	$0, 1$	50588668138	$0, 1$	50588668138
$3/2 - t_d$	$\mathcal{D}$	$0, 1$	126471670345	$0, 1$	126471670345
$49/50 + x/50 - F_d$	$\mathcal{D}$	$0, 1$	36638459651	$0, 1$	36638459651
$1/50 - \gamma_2(3/10)$	$\mathcal{D}$	$0, 1$	1075495777	$0, 1$	1075495777
J	Ω	$0, 1$	648092328858	$0, 1$	648092328858
$S(R_s)$	Ω	$0, 1$	22188021055	$0, 1$	22188021055
$1 - S(R_s)$	Ω	$0, 1$	250692959516	$0, 1$	250692959516
$(p_1 - 2\delta x)\mathscr{L} - R_s$	Ω	$0, 1$	13927405530	$0, 1$	13927405530
$-E(R_s) - 10^{-5}$	$\mathcal{D}_1$	$9, 11$	15034272	$9, 11$	14627855
$-E(R_s) - 10^{-5}$	$\mathcal{D}_2$	$0, 1$	463076646	$0, 1$	462618241

f	Rectangle	(d,L)	k	(d,L)	k
$-E(R_s)-10^{-5}$	$\mathcal{D}_3$	$0,1$	1764325203	$0,1$	1763978740

The omitted contact entry is replaced by the exact factorization (6.43). For completeness, the following bounds verify all the accompanying contact inequalities. The multipliers for the five functions are respectively $5-4\delta$, $(5-4\delta)\Delta$, Δ, Δ, and Δ. The removable quotients are taken after cancellation.

$$\begin{aligned}\mathcal{D}_0 &= [2/5, 23/50] \times [6/25, 1/2],\\ \mathcal{D}_- &= [11/50, 2/5] \times [6/25, 1/2],\\ \mathcal{D}_+ &= [23/50, 13/20] \times [6/25, 1/2].\end{aligned}$$

f	Rectangle	(d,L)	k
$g(B_2,\delta,1/2)/(\delta-\delta_2)^2$	$\mathcal{D}$	$0,1$	113086830076
$(g(B_2,\delta,x)-g(B_2,\delta,1/2))/(1/2-x)$	$\mathcal{D}_0$	$0,1$	9082672972
$g(B_2,\delta,x)$	$\mathcal{D}_-$	$0,1$	72884214
$g(B_2,\delta,x)$	$\mathcal{D}_+$	$0,1$	147316073
$\partial_B g$	$\mathcal{D}$	$0,1$	2372347314201

Finally, direct substitution gives the following constant inequalities. Each displayed expression is at least $k\,10^{-12}$; the two columns again refer to the rational and algebraic parameters, respectively. They include the geometric conditions, the small-amplitude and small-parameter estimates, and the positive slopes used when the parameter interval is enlarged.

Expression	k	k
b	99956298200	99956298225
$\ell-1/6$	6471600000	6472133547
$1/5-\ell$	26861733333	26861199786
$l_x-\ell$	190314450900	190313650566
$l_y-\ell-11b/6$	107017535733	107016735378
$p_1\ell-3/4$	28887424510	28889824749
T	117132482433	117131148578
$1/2-l_y$	36590984233	36591250994
$B-87/100$	3382100000	3381966613
$\ell/(h+10^{-9})-\mathscr{L}$	3833958930	3834406521
$2/9-\ell/(h-1/40)$	1575535322	1575080419
$m_0-3/10$	127667918389	127668813572
$47/100-m_0$	42332081610	42331186427
$l_y/h-57/100$	2332081610	2331186427
$59/100-l_y/h$	17667918389	17668813572
$-Q_{\mathrm{sm}}$	4321129754	4320642840
$-1/250-C_0-(1-h)/16$	1023130452	1022513534
$-1/1000-C_0-(1/2+3\ell/2)/50$	723635633	722952688
$-1/20-13h/75+l_y/2-1/50$	21359012706	21358740601
$h/600$	1349475915	1349477249
$l_y/20$	23170450788	23170437450
$73/1000-13/(40p_1)$	756008254	756008254
$1/5-(73/1000)(37/40)-1/50-1/10$	12475000000	12475000000

These bounds prove the strict inequalities on their full closed rectangles. At the algebraic parameters only the stated double contact remains; (6.45) makes its inequality strict for every fixed $B>B_2$ sufficiently close to B_2, with $b=b_2$ held fixed.

B The starting moment parameter

In this appendix, we verify the rational upper bound used in (2.4) and (6.3). Theorem 17.1 of [Liu26b] gives $\beta_*(F) \le B_\flat$.

The number $B_\flat$ is the root near 0.8749 of

$$P(B) = 133488B^4 - 376596B^3 + 384534B^2 - 167541B + 26231.$$

For the Sturm sequence $P_0 = P$, $P_1 = P'$, $P_{j+1} = -\operatorname{rem}(P_{j-1}, P_j)$, let $V(t)$ be the number of sign variations at t. Rational substitution gives

t	signs of $(P_0, \dots, P_4)$	$V(t)$
0.8748	$(+,-,-,+,+)$	2
0.87488664703996	$(+,-,-,+,+)$	2
0.87488664704	$(-,-,-,+,+)$	1
0.8749	$(-,-,-,+,+)$	1

Hence

$$0.87488664703996 < B_\flat < 0.87488664704 < \frac{1+\kappa_1}{2}, \qquad \kappa_1 = 0.749774.$$

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