Nonpolynomial Hall numbers for tame and wild algebras

Qiyue Tang

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Abstract

We give examples showing that fixed-triple Hall polynomials may fail to exist for tame nonhereditary algebras and for wild algebras, both hereditary and nonhereditary.

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Motivation, AI use, and scholarly credit

I wanted to understand whether the Hall-polynomial existence property extends to algebras of tame and wild representation type, beyond the representation-finite existence conjecture.

The constructions and proofs were developed through AI-assisted research recorded in the accompanying research archive. Wolfram Language was used for symbolic calculations, and Python and C++ for exact finite-field and matrix checks. Codex was used to audit the arguments, complete their exposition, and polish the manuscript. The work passed a Rethlas-OSV review [10, 14].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

The classical Hall polynomials count subgroups of finite abelian p-groups with prescribed subgroup and quotient types. Their origins lie in the work of Steinitz and Hall; Hall's algebra of partitions organizes these counts as multiplication constants [6, 12]. Replacing abelian groups by finite-length modules over a discrete valuation ring gives the same polynomial phenomenon, with the cardinality of the residue field as variable. A partition specifies the module type independently of that cardinality.

Ringel extended this construction to modules over finite-dimensional algebras over finite fields [16]. The product of two isomorphism classes counts submodules with those prescribed factors. For representation-directed algebras, he proved the existence of Hall polynomials, and his work on representation-finite hereditary algebras related their specializations at one to the positive part of a semisimple Lie algebra [17]. Twisting Hall multiplication by the Euler form gives the connection with quantized enveloping algebras [18]. These polynomials define a generic Hall algebra and its specializations.

Ringel formulated the conjecture that every representation-finite algebra over a finite field has Hall polynomials [16]. The representation-finite problem is also the subject of [19]. The finite-type problem and its extension to infinite representation type require different treatments of the labels and of conservative extensions. The fixed-triple condition below is the one supplied

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by the tame hereditary theory. We ask whether it persists for tame nonhereditary algebras and for wild algebras.

Green's multiplication–comultiplication formula extended the Hall-algebra realization of quantum groups to hereditary algebras through the composition subalgebra generated by simple modules [5]. It also supplied a way to reduce Hall-polynomial questions by the dimension of the middle module. The composition subalgebra uses particular combinations of isomorphism classes. Polynomial structure constants for that subalgebra and polynomiality of every individual Hall number are separate counting statements.

The tame hereditary case requires a treatment of labels that vary with the field. Besides the discrete preprojective and preinjective families, regular representations occur in tubes indexed by closed points of a projective line. Their residue-field degrees affect scalar extension. Hubery proved Hall-polynomial results for Dynkin, cyclic, and affine quivers using Green's formula; in the affine case the uniform labels are the decomposition classes of Bongartz and Dudek [9]. These classes retain discrete representation data and the degrees of the moving regular parameters.

Deng and Ruan developed this method through coherent sheaves on domestic weighted projective lines [4]. Their decomposition sequences incorporate vector bundles and torsion sheaves, and yield generic Hall algebras and their Drinfeld doubles. Their tame-quiver result also gives the following formulation for individual objects: for a finite field k and fixed kQ-modules M, N, L, one polynomial in $\mathbb{Z}[T]$ computes the Hall numbers after every extension conservative relative to those three modules. This fixed-object formulation is Corollary 6.8 in their original preprint and Corollary 6.7 in the author-hosted revision.

Weighted projective lines connect the hereditary theory with canonical algebras. In particular, the canonical algebras of weight type $(2, 2, 2, 2)$ are tame tubular algebras [1, 7]. Lin and Peng constructed elliptic Lie algebras from their root categories [11]; Burban and Schiffmann studied composition Hall algebras of weighted projective lines, including the tubular case [2]. More recently, Chen, Deng, and Ruan calculated Hall polynomials involving line bundles, extension bundles, and torsion sheaves on weighted projective lines of three weights, in domestic, tubular, and wild types [3]. Those results identify specific polynomial Hall products within categories of infinite representation type.

For wild quivers, geometry gives another reason to examine individual Hall numbers. Reineke and Ringel showed that projective varieties can occur as quiver Grassmannians, and Hille gave constructions using adjacent homogeneous pieces of a projective coordinate ring [15, 20, 8]. A quiver Grassmannian fixes a middle representation and a subrepresentation dimension vector. A Hall number imposes two further isomorphism conditions, on the submodule and its quotient. Hubery's discussion of wild Hall polynomials instead proposes polynomial sums over suitable combinatorial classes [9, Section 9]; here the labels are three fixed modules over one finite base field.

We prove that fixed-triple Hall-polynomial existence fails in each of the following classes: tame nonhereditary algebras, wild hereditary algebras, and wild nonhereditary algebras. In the tame case, every canonical parameter of weight type $(2, 2, 2, 2)$ in characteristic at least five admits an example. Two cross-ratio conditions produce a smooth genus-one curve. In the wild hereditary case, a smooth plane cubic gives an explicit triple for the three-arrow Kronecker quiver. Passing to a local radical-square-zero algebra gives the wild nonhereditary example. In all three cases the remaining genus-one point count prevents rational dependence on field size.

1.1 The fixed-triple question

Let $k = \mathbb{F}_q$ be a finite field with q elements and let A be a finite-dimensional k-algebra. All modules are finite-dimensional, and bound-quiver modules are represented by vector spaces at vertices and maps in the indicated arrow directions. For A-modules M, N, L, define

$$F^L_{M,N}(k) = \#\{U \subseteq L : U \text{ is an } A\text{-submodule},\ U \simeq N,\ L/U \simeq M\}.$$

Thus N specifies the submodule and M the quotient. For a field extension K/k, write $A_K = K \otimes_k A$ and $X_K = K \otimes_k X$ for scalar extension of an A-module X. The extension is *conservative for the triple* if each indecomposable summand of M, N, L remains indecomposable over K. A fixed-triple Hall polynomial is $P(T) \in \mathbb{Z}[T]$ satisfying

$$F^{L_K}_{M_K,N_K}(K) = P(|K|)$$

for every finite extension conservative for the triple. A module is *absolutely indecomposable* if it stays indecomposable after every field extension.

For infinite representation type, this relative condition distinguishes the question from conservativity for every indecomposable of the algebra. For example, choose $d \geq 1$ and a field generator θ of $\mathbb{F}_{q^d}/\mathbb{F}_q$. On the two-arrow Kronecker quiver take $\mathbb{F}_{q^d}$ at both vertices as a k-vector space, with arrows the identity and multiplication by θ. Its endomorphisms are precisely multiplication by elements of $\mathbb{F}_{q^d}$, so it is indecomposable. If $d > 1$ divides n, then over $\mathbb{F}_{q^n}$ the multiplication operator has d distinct eigenvalues and the representation splits into d one-dimensional-at-each-vertex summands. Every $n > 1$ has a prime divisor d, so no nontrivial extension is conservative for every indecomposable of this algebra. The triples constructed here have absolute summands, and all finite extensions are conservative for them.

We use tame and wild representation type after scalar extension to an algebraic closure $\overline{k}$. Tameness means that, for each positive integer d, all but finitely many d-dimensional indecomposable modules occur by specializing finitely many one-parameter families: these families are bimodules free of finite rank on the $\overline{k}[t]$ side, specialized by tensoring with $\overline{k}[t]/(t-c)$ for $c \in \overline{k}$. Wildness means that a bimodule free of finite rank on the $\overline{k}\langle x, y\rangle$ side induces a functor preserving indecomposability and reflecting isomorphisms. Such a functor is called a representation embedding. A finite-dimensional algebra is hereditary if every submodule of a projective module is projective. In particular, a path algebra of a finite acyclic quiver is hereditary. The wild examples below also admit explicit embeddings of pairs of arbitrary linear operators, as checked in Section 5.

1.2 Main results

Theorem 1.1. *Let $k = \mathbb{F}_q$ have characteristic at least five and let $\lambda \in k \setminus \{0, 1\}$. Let A_λ be the algebra with vertices $0, 1, 2, 3, 4, \infty$, arrows*

$$a_i : 0 \longrightarrow i, \qquad b_i : i \longrightarrow \infty \qquad (1 \leq i \leq 4),$$

and relations

$$b_3a_3 = b_1a_1 + b_2a_2, \qquad b_4a_4 = \lambda b_1a_1 + b_2a_2. \tag{1}$$

The algebra A_λ is tame and nonhereditary. There exist A_λ-modules N, L, M of total dimensions $6, 12, 6$, respectively, whose indecomposable summands are absolutely indecomposable and whose Hall numbers over $\mathbb{F}_{q^n}$, $n \geq 1$, are not the values of a rational function in q^n with real coefficients. In particular this triple has no Hall polynomial.

Theorem 1.2. *Let $k = \mathbb{F}_q$ be any finite field. The following wild algebras admit fixed triples with absolutely indecomposable summands and nonrational Hall numbers on the entire finite-extension tower.*

(i) The hereditary algebra $k\mathsf{K}_3$, where K_3 has two vertices and three arrows from the first to the second. There are modules N, L, M of dimension vectors $(3, 6), (12, 18), (9, 12)$ and a smooth plane cubic E/k such that, for every finite extension K/k with $Q = |K|$,

$$F^{L_K}_{M_K,N_K}(K) = Q(Q^2 - 1)(Q^2 + Q + 1 - \#E(K)). \tag{2}$$

(ii) The nonhereditary algebra $B = k\langle r, s, t\rangle/(r, s, t)^2$*, where* $(r, s, t)^2$ *is the ideal generated by all products of two loops. There are modules* $\widehat{N}, \widehat{L}, \widehat{M}$ *of total dimensions* $9, 30, 21$ *such that*

$$F^{\widehat{L}_K}_{\widehat{M}_K,\widehat{N}_K}(K) = Q^{37}(Q^2-1)(Q^2+Q+1-\#E(K)). \tag{3}$$

Here nonrationality means that no $R(T) \in \mathbb{R}(T)$ *represents the sequence as* $R(q^n)$ *for all* $n \geq 1$*.*

Theorem 1.1 is proved in Section 3. Sections 4 and 5 prove the two parts of Theorem 1.2.

2 Genus-one point counts

Lemma 2.1. *Let* $C/\mathbb{F}_q$ *be a smooth projective geometrically integral curve of genus one. For* $n \geq 1$*, put*

$$Q_n = q^n, \qquad a_n = Q_n + 1 - \#C(\mathbb{F}_{q^n}).$$

If $(z_n)_{n\geq 1}$ *is a bounded real sequence, then* $(a_n + z_n)_{n\geq 1}$ *is not eventually the restriction of a real rational function in* Q_n*.*

Proof. The genus-one Hasse–Weil identities give complex numbers α, β with

$$\alpha\beta = q, \quad |\alpha| = |\beta| = \sqrt{q}, \quad a_n = \alpha^n + \beta^n;$$

see [13]. Consequently,

$$|a_n| \leq 2\sqrt{Q_n}, \qquad a_{2n} = a_n^2 - 2Q_n. \tag{4}$$

Suppose $a_n + z_n = R(Q_n)$ for all sufficiently large n. If R is nonzero, write it as a quotient of nonzero polynomials. Its order of growth at infinity is the integer given by numerator degree minus denominator degree. The bound $R(Q_n) = O(Q_n^{1/2})$ forces that integer to be at most zero. Thus $R(Q_n)$ is bounded. The same conclusion holds for $R = 0$. Since z_n is bounded, so is a_n, whereas the second identity in (4) then forces a_{2n} to be unbounded. This is a contradiction. □

In particular, a nonzero rational multiple of a_n added to a rational function of Q_n cannot be rational in Q_n: subtract the rational part and divide by the nonzero multiplier, ignoring its finitely many zeros and poles. This observation applies to the multiplicative factors in (2) and (3).

3 Canonical algebras of type $(2, 2, 2, 2)$

3.1 The algebra and the module labels

Throughout this section $\operatorname{char} k \geq 5$ and $\lambda \in k \setminus \{0, 1\}$. The presentation (1) is a canonical algebra of tubular type $(2, 2, 2, 2)$; the canonical-algebra classification gives its tameness [1, Section 1.4]. It has dimension 16: there are six stationary paths, eight arrows, and a two-dimensional space of length-two paths.

Its nonhereditary property follows directly from the projectives. Let P_v be the representation whose space at a vertex w is spanned by paths from v to w, modulo the relations, and let S_v be the simple representation supported at v. The map $\bigoplus_{i=1}^4 P_i \to \operatorname{rad} P_0$ sends the generator of P_i to a_i. It is an isomorphism at each arm. At the sink its kernel in k^4, ordered by the four arms, has basis

$$(-1, -1, 1, 0), \qquad (-\lambda, -1, 0, 1),$$

the coefficient vectors of the two relations. Thus its kernel is supported at ∞ and is $P_\infty^{\oplus 2}$. Thus

$$0 \longrightarrow P_\infty^{\oplus 2} \longrightarrow \bigoplus_{i=1}^{4} P_i \longrightarrow P_0 \longrightarrow S_0 \longrightarrow 0$$

is a projective resolution. The arrow $i \to \infty$ acts as the identity in P_i, so compatibility forces every map $P_i \to S_\infty$ to be zero. Since $\mathrm{Hom}(P_\infty, S_\infty) = k$, applying $\mathrm{Hom}(-, S_\infty)$ to this resolution gives $\mathrm{Ext}^2(S_0, S_\infty) \simeq k^2$. Hence A_λ is nonhereditary.

We first take $\lambda \notin \{4, -1, -2\}$. Let L have k^2 at every vertex, with $a_i^L = I_2$, the 2×2 identity matrix, and with b_i^L given by

$$B_1 = I_2, \quad B_2 = \mathrm{diag}(1, 2), \quad B_3 = \mathrm{diag}(2, 3), \quad B_4 = \mathrm{diag}(\lambda + 1, \lambda + 2). \tag{5}$$

Here $\mathrm{diag}(c, d)$ denotes the diagonal matrix with entries c, d. These maps satisfy the two relations. All four B_i are invertible.

For $u \in k \setminus \{0, 1\}$ define N_u, and define M, by

$$\begin{aligned} &\dim N_u = (0; 1, 1, 1, 1; 2), && b_i^{N_u} = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}u\\1\end{pmatrix}, \\ &\dim M = (2; 1, 1, 1, 1; 0), && a_i^M = (0\,1),\ (1\,0),\ (1\ -1),\ (1\ -4). \end{aligned}$$

The vertex order is $(0; 1, 2, 3, 4; \infty)$; unlisted arrow maps have zero source or target. In particular both representations satisfy the relations. Their total dimensions are six, while the total dimension of L is twelve.

Lemma 3.1. *For every field extension K/k,*

$$\mathrm{End}((N_u)_K) = K, \qquad \mathrm{End}(M_K) = K, \qquad \mathrm{End}(L_K) = K \times K.$$

The indecomposable summands of the triple are absolutely indecomposable.

Proof. For N_u, an endomorphism at the sink preserves the first three image lines $K(1, 0)$, $K(0, 1)$, $K(1, 1)$. The first two force its matrix to be diagonal, and the third makes its diagonal entries equal. Compatibility with each nonzero column then forces the same scalar at every arm. For M, use its first three kernel lines, which are the same three lines, to obtain the identical conclusion at the source and arms.

For L, the identity a_i maps identify the four arm endomorphisms with the source endomorphism. The map $B_1 = I_2$ identifies the sink endomorphism with it. Commutation with $B_2 = \mathrm{diag}(1, 2)$ forces this common matrix to be diagonal; every diagonal matrix commutes with all B_i. The two coordinate representations give the direct-sum decomposition of L, and each has scalar endomorphisms over every extension. A nontrivial decomposition would give a nontrivial idempotent in its scalar endomorphism ring, which is impossible. □

3.2 The Hall condition as two cross-ratios

The projective line $\mathbb{P}^1(K)$ is the set of one-dimensional subspaces of K^2, written $[X : Z] = K(X, Z)$. We write $\infty = [1 : 0]$ and $c = [c : 1]$ for $c \in K$. For four distinct such lines with nonzero representatives p_i, define

$$d_{ij} = \det(p_i, p_j), \qquad \mathrm{CR}(p_1, p_2, p_3, p_4) = \frac{d_{13} d_{24}}{d_{14} d_{23}}.$$

The quotient is independent of the representatives and invariant under simultaneous invertible linear transformations. For $v \in K \setminus \{0, 1\}$, one has $\mathrm{CR}(\infty, 0, 1, v) = v$. Write $\mathrm{GL}_2(K)$ for the group of invertible 2×2 matrices and $\mathrm{PGL}_2(K)$ for its quotient by nonzero scalar matrices. The first three ordered distinct lines determine a unique element of $\mathrm{PGL}_2(K)$ carrying $(\infty, 0, 1)$ to them. One sees this by choosing representatives of the first two lines as a basis and rescaling them so their sum represents the third. The fourth line is then determined by the cross-ratio. Applied to the columns defining N_u, this proves that distinct values of u give nonisomorphic modules. Over $\overline{k}$ they form an infinite family, so A_λ has infinite representation type.

A submodule $U \subseteq L_K$ of dimension vector $\dim N_u$ has $U_0 = 0$, $U_\infty = K^2$, and a line U_i at each arm. It is a submodule for every such choice. Its sink image lines are B_iU_i, whereas the kernel lines of the four quotient maps at the source are U_i. Changes of basis on a one-dimensional arm absorb each nonzero rescaling of a column or row. The preceding classification therefore gives

$$L_K/U \simeq M_K \iff \mathrm{CR}(U_1, U_2, U_3, U_4) = 4, \tag{6}$$
$$U \simeq (N_u)_K \iff \mathrm{CR}(B_1U_1, B_2U_2, B_3U_3, B_4U_4) = u, \tag{7}$$

with distinctness of the four lines on each side included in each condition.

Every quadruple in (6) is uniquely $g(\infty, 0, 1, 4)$ for $g \in \mathrm{PGL}_2(K)$. The diagonal torus is the subgroup of $\mathrm{PGL}_2(K)$ represented by $\mathrm{diag}(c, 1)$ for $c \in K^\times$. Its left multiplication preserves (7) because it commutes with every B_i. An invertible matrix has two distinct row lines, so these row directions define a map

$$\mathrm{PGL}_2(K) \longrightarrow \{(\ell_1, \ell_2) \in \mathbb{P}^1(K)^2 : \ell_1 \neq \ell_2\}.$$

Every pair has lifts: choose a nonzero representative of each row line. The two representatives can be independently rescaled, and quotienting by common scalar leaves exactly $Q - 1$ choices, where $Q = |K|$. Thus every fiber has $Q - 1$ elements, over the original field K.

Use coordinates $([X : Z], [Y : W])$ for the two row lines. The four image vectors in (7), before taking their projective classes, are

$$(X, Y), \quad (Z, 2W), \quad (2(X + Z), 3(Y + W)), \quad ((\lambda + 1)(4X + Z), (\lambda + 2)(4Y + W)).$$

The row matrix is invertible exactly when $XW - YZ \neq 0$. On the chart $Z = W = 1$, put $x = X/Z$ and $y = Y/W$. The determinants needed for the second cross-ratio are

$$\begin{aligned} d_{13} &= xy + 3x - 2y, & d_{23} &= -4x + 3y - 1, \\ d_{24} &= -8(\lambda + 1)x + 4(\lambda + 2)y - \lambda, & d_{14} &= 4xy + (\lambda + 2)x - (\lambda + 1)y. \end{aligned} \tag{8}$$

In homogeneous coordinates each is a form of degree one in X, Z and degree one in Y, W, obtained from the displayed vectors. Define the curve $C_{\lambda,u} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ by

$$F_{\lambda,u} = d_{13}d_{24} - u\, d_{14}d_{23} = 0. \tag{9}$$

Its equation has bidegree $(2, 2)$, meaning degree two in each of the two coordinate pairs. On this curve, $d_{14}d_{23} \neq 0$ implies that d_{13} and d_{24} are nonzero. The determinant identity

$$d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23} = 0$$

then gives $d_{12}d_{34} = (u - 1)d_{14}d_{23} \neq 0$. Since all B_i are invertible, the four vectors are nonzero whenever the row matrix is invertible. Thus the open conditions

$$XW - YZ \neq 0, \qquad d_{14}d_{23} \neq 0 \tag{10}$$

are exactly the remaining Hall conditions.

3.3 Smooth members over the original field

For a binary quartic

$$D(X, Z) = aX^4 + bX^3Z + cX^2Z^2 + dXZ^3 + eZ^4$$

its discriminant is the integer polynomial defined by

$$\begin{aligned} 27\,\mathrm{Disc}(D) &= 4(12ae - 3bd + c^2)^3 \\ &\quad - (72ace + 9bcd - 27ad^2 - 27b^2e - 2c^3)^2. \end{aligned} \tag{11}$$

The right side is divisible coefficientwise by 27. The discriminant vanishes exactly when the form has a repeated projective zero. For example, over a splitting field write $D = \prod_{i=1}^{4}(r_iX + s_iZ)$; its discriminant is $\prod_{i<j}(r_is_j - r_js_i)^2$, which also shows the stated vanishing criterion. Formula (11) is the expansion of that expression. We use it in characteristics at least five.

Lemma 3.2. *For every finite field k of characteristic at least five and every $\lambda \in k\setminus\{0,1,4,-1,-2\}$ with $|k| > 10$, there is $u \in k \setminus \{0,1,\lambda\}$ for which the discriminant, in the second coordinate, of (9) has four distinct projective zeros in the first coordinate.*

Proof. On the affine chart write $F_{\lambda,u} = Ay^2 + By + C$, where the polynomials in x are

$$\begin{aligned} A &= (4\lambda - 12u + 8)x + 3\lambda u - 8\lambda + 3u - 16, \\ B &= (-8\lambda + 16u - 8)x^2 + (-7\lambda u + 27\lambda - 6u + 40)x - \lambda u + 2\lambda - u, \\ C &= (4\lambda u - 24\lambda + 8u - 24)x^2 + (\lambda u - 3\lambda + 2u)x. \end{aligned}$$

For each coefficient polynomial, its homogeneous form is obtained by $A(X,Z) = Z^2A(X/Z)$, and likewise for B and C. Form the binary quartic $D = B^2 - 4AC$ from these homogeneous forms. Applying (11) gives

$$\mathrm{Disc}(D) = 2^{16}3^4(\lambda-4)^2(\lambda+1)^2(\lambda+2)^2(\lambda-u)^2(u-1)^2u^2H(u,\lambda), \tag{12}$$

where $H = \sum_{i,j=0}^{4} h_{ij}u^i\lambda^j$ and the coefficients, in ascending powers, are

$$(h_{ij})_{0\le i,j\le 4} = \begin{pmatrix} 4096 & 2048 & 9600 & 2336 & 5329 \\ 16384 & -70144 & -87168 & -13192 & -6260 \\ 102912 & 174720 & 90648 & 8472 & 2166 \\ -48128 & -79648 & -30576 & -1816 & -212 \\ 10000 & 10400 & 2904 & 104 & 1 \end{pmatrix}. \tag{13}$$

Substitution of the displayed A, B, C into $B^2 - 4AC$ and (11), followed by collecting the powers $u^i\lambda^j$, gives precisely the matrix (13). The calculation is an identity over $\mathbb{Z}$.

For $h_i(\lambda) = \sum_{j=0}^{4} h_{ij}\lambda^j$, the first and last rows factor as

$$h_0(\lambda) = (73\lambda^2 + 16\lambda + 64)^2, \qquad h_4(\lambda) = (\lambda+2)^2(\lambda+50)^2.$$

If all $h_i(\lambda)$ vanished, then $\lambda = -2$ or $\lambda = -50$. The first case is excluded by

$$h_0(-2) = 324^2 = 2^4 3^8.$$

In the second case direct substitution in the first and fourth rows of (13) gives

$$h_0(-50) = 2^4 3^{10} 11^2 17^2, \qquad h_3(-50) = -2^{15} 3^6 7^2.$$

No prime at least five divides both numbers. Hence the five coefficients have no common zero in any field of the stated characteristic. Therefore $H(u,\lambda)$ is not the zero polynomial in u for any fixed λ. Under the hypotheses on λ, (12) is a nonzero polynomial of degree at most ten in u. It has at most ten roots in k. A field with more than ten elements contains a value with nonzero discriminant, and the factors in (12) exclude $0, 1, \lambda$ for that value. □

The changes $\lambda \mapsto 1/\lambda$ and $\lambda \mapsto 1-\lambda$ come from algebra isomorphisms over k. To see the maps explicitly, write $p_i = b_i a_i$ for the four full paths. The replacements

$$(p_1', p_2', p_3', p_4') = (p_2, p_1, p_3, p_4/\lambda), \qquad (p_1', p_2', p_3', p_4') = (-p_1, p_3, p_2, p_4)$$

give $p_3' = p_1' + p_2'$ and $p_4' = \lambda' p_1' + p_2'$, with $\lambda' = 1/\lambda$ and $1-\lambda$, respectively. Each replacement permutes arms and multiplies one arrow on a specified arm by a nonzero scalar, so it defines an invertible map of the bound quivers.

For $|k| > 10$, every exceptional parameter can thereby be moved outside $\{4, -1, -2\}$. The parameter -1 moves to 2. The parameter -2 moves to $-1/2$; this collides with 4 only in characteristic three and with -2 only in characteristic three, and cannot equal -1. Finally 4 moves to $1/4$; a collision with the exceptional set occurs only in characteristic three or five. In characteristic five, $4 = -1$ and the first change applies. The allowed parameter set $k \setminus \{0, 1\}$ is preserved by both transformations.

The remaining fields are $\mathbb{F}_5$ and $\mathbb{F}_7$. The orbits under the two transformations and direct evaluations of (12) are

k	parameter orbit	representative λ	u	$\mathrm{Disc}(D)$
$\mathbb{F}_5$	$\{2, 3, 4\}$	2	4	4
$\mathbb{F}_7$	$\{2, 4, 6\}$	2	5	1
$\mathbb{F}_7$	$\{3, 5\}$	3	4	5

For example, applying $1-\lambda$ and $1/\lambda$ to 2 gives $4, 3$ in $\mathbb{F}_5$, and $6, 4$ in $\mathbb{F}_7$; in $\mathbb{F}_7$ these transformations interchange $3, 5$. The table therefore covers all allowed parameters. In every case we can construct the modules over an isomorphic algebra with an admissible representative and transport them back over k.

Lemma 3.3. *For the parameters just selected, $C_{\lambda,u}$ is smooth, projective, and geometrically integral of genus one. The complement of the open conditions (10) on this curve is a reduced finite set with at most eight geometric points.*

Proof. Regard the bihomogeneous equation as $AY^2 + BYW + CW^2 = 0$, with A, B, C binary quadratics in X, Z. If they had a common projective zero, then in a local coordinate at that point both B^2 and AC would vanish to order at least two. This would give a repeated zero of $D = B^2 - 4AC$. Thus there is no common zero. The projection to the first projective line has nonzero quadratic equation on every fiber, so it is finite of degree two. Its discriminant is not a square in $\overline{k}(\mathbb{P}^1)$, since it has four simple zeros. The generic quadratic is therefore irreducible, giving a geometrically integral curve.

A fiber with two distinct roots has a nonzero derivative in a local fiber coordinate at each root. At a double root choose a fiber coordinate η in which the quadratic is $a(t)\eta^2 + b(t)\eta + c(t)$ and $a(t)$ is a unit. This is possible because a nonzero binary quadratic with a double root is a nonzero multiple of the square of a linear form. Here t is a local coordinate on the base. The invertible local change $w = 2a(t)\eta + b(t)$ gives

$$w^2 = b(t)^2 - 4a(t)c(t).$$

The right side is a unit times a local representative of D and has a simple zero. Its t derivative is nonzero at the double-root point, proving smoothness. Both coordinates are local coordinates on projective lines, so the argument also covers infinity in either factor. There are exactly four simple branch points. Riemann–Hurwitz for the degree-two map gives $2g - 2 = 2(-2) + 4 = 0$, so $g = 1$. The point $([0:1], [0:1])$ lies on the curve and is k-rational.

Let Δ be the diagonal $[X:Z] = [Y:W]$. Direct substitution gives

$$F_{\lambda,u}|_\Delta = (u-\lambda)XZ(X+Z)(4X+Z),$$
$$d_{14}|_\Delta = X(4X+Z), \qquad d_{23}|_\Delta = -Z(X+Z).$$

Since $u \neq \lambda$, each intersection with the diagonal already lies on $d_{14} = 0$ or $d_{23} = 0$. Each determinant has bidegree $(1,1)$ and has no component in common with the integral curve of bidegree $(2,2)$. The intersection number is $1 \cdot 2 + 1 \cdot 2 = 4$: on $\mathbb{P}^1 \times \mathbb{P}^1$ the two ruling classes have square zero and mutual intersection one. Each determinant divisor consequently meets the curve in at most four geometric points. Their union, taken with reduced structure, is the asserted boundary, of size at most eight. □

Proof of Theorem 1.1. Choose an algebra-isomorphic parameter and u as above. Let $Z_{\lambda,u}$ be the reduced boundary in Lemma 3.3. Equations (6)–(10) and the row-fiber calculation give, for every finite extension K/k,

$$F^{L_K}_{M_K,(N_u)_K}(K) = (|K| - 1)(\#C_{\lambda,u}(K) - \#Z_{\lambda,u}(K)). \tag{14}$$

For $K = \mathbb{F}_{q^n}$ write $z_n = \#Z_{\lambda,u}(K)$ and use the trace a_n of Lemma 2.1. Then $0 \leq z_n \leq 8$ and the count equals

$$(q^n - 1)(q^n + 1 - a_n - z_n).$$

If it were $R(q^n)$ for a rational function, then

$$a_n + z_n = q^n + 1 - \frac{R(q^n)}{q^n - 1}$$

would be rational in q^n, contradicting Lemma 2.1. Lemma 3.1 gives the absolute summands. Transport along the k-algebra isomorphisms preserves submodules, quotient types, endomorphism rings, and scalar extension. It therefore gives the required triple for every original λ. The tame and nonhereditary assertions were proved at the start of the section. □

3.4 An explicit eight-point boundary

Take $\lambda = 2$, $u = 3$, and $\operatorname{char} k \geq 11$. The affine equation in (9) is

$$f(x,y) = 6x - 24x^2 - 5y + 34xy + 24x^2y - 5y^2 - 20xy^2.$$

Its discriminant in y and the discriminant of that quartic are

$$\begin{aligned} D(x) &= 25 - 220x + 916x^2 - 288x^3 + 576x^4, \\ \operatorname{Disc}(D) &= 2^{28}3^8 5^4 7^2. \end{aligned}$$

The homogeneous quartic is squarefree in the stated characteristics, so Lemma 3.3 applies. The boundary consists of the following eight k-rational points:

$$\begin{gathered} (0,0), \quad (-1,-1), \quad (-1/4,-1/4), \quad (\infty,\infty), \\ (1/2,1), \quad (1/4,1/2), \quad (1/8,1/5), \quad (5/4,2). \end{gathered}$$

For $d_{14} = 0$, solving for y gives $y = -4x/(4x-3)$; $x = 3/4$ gives no affine point on this divisor. Direct substitution yields

$$(4x-3)^2 f\left(x, \frac{-4x}{4x-3}\right) = -6x(4x-1)(4x+1)(8x-1).$$

Its four roots give $(0,0)$, $(-1/4,-1/4)$, $(1/4,1/2)$, $(1/8,1/5)$. Similarly, $d_{23} = 0$ gives $y = (4x+1)/3$ and

$$9f\left(x, \frac{4x+1}{3}\right) = -4(x+1)(2x-1)(4x-5).$$

The three roots give $(-1,-1)$, $(1/2,1)$, $(5/4,2)$, and the homogeneous equations also contain (∞,∞). All eight points are distinct in characteristic at least eleven. Each list has four points

and hence exhausts its determinant divisor's intersection with C. The diagonal restriction is $f(x,x) = x(x+1)(4x+1)$, together with its fourth projective root at infinity. Thus, if C denotes the projective curve,

$$F^{L_K}_{M_K,(N_3)_K}(K) = (Q-1)(\#C(K) - 8), \qquad Q = |K|.$$

Over $\mathbb{F}_{11}$, the fiber over each $x \in \mathbb{P}^1(\mathbb{F}_{11})$ has $1 + \chi(D(x))$ points, where $\chi(0) = 0$, $\chi(c) = 1$ for nonzero squares $c \in \mathbb{F}_{11}$, and $\chi(c) = -1$ for nonsquares. At infinity use $D(1,0) = 576$. For $x = 0, 1, \ldots, 10$, the character values are

$$(1, -1, -1, 1, 1, -1, 1, 1, 1, -1, 1),$$

and the value at infinity is 1. Their sum is 4, so $\#C(\mathbb{F}_{11}) = 12 + 4 = 16$ and the Hall number is 80.

4 The three-arrow Kronecker algebra

Choose a smooth plane cubic E/k. Such a curve exists over every finite field: in characteristic two use $y^2z + yz^2 = x^3$, and in other characteristics use $y^2z = x^3 - xz^2$. In the first equation the three partial derivatives are x^2, z^2, y^2, with no common projective zero. For the second equation, in characteristic three the x derivative is $-z^2$, so a singular point would have $z = 0$; its unique point at infinity is $[0:1:0]$, where the z derivative is nonzero. In characteristic at least five, on $z = 1$ a singular point would have $y = 0$, $x^3 - x = 0$, and $3x^2 - 1 = 0$. None of $x = 0, 1, -1$ satisfies the last equation. The point at infinity is again smooth.

The Hasse bound gives

$$\#E(k) \le q + 1 + 2\sqrt{q} < q^2 + q + 1 = \#\mathbb{P}^2(k).$$

Choose a k-point outside E and choose projective coordinates carrying it to $[1:0:0]$. In these coordinates a homogeneous cubic equation $F(x,y,z) = 0$ has a nonzero coefficient of x^3. Divide by that coefficient and set

$$S = k[x,y,z]/(F).$$

The ring is graded by total degree; S_d is the image of homogeneous degree-d polynomials, and $S_d = 0$ for $d < 0$. Division by the monic polynomial F in x identifies S, as a graded $k[y,z]$-module, with

$$k[y,z] \oplus xk[y,z] \oplus x^2k[y,z].$$

For $d = 1$ this gives $\dim S_1 = 3$, and for $d \ge 2$ it gives $(d+1) + d + (d-1) = 3d$. Multiplication by y is injective. Modulo y^mS, the same three basis vectors give a free module over $k[y,z]/(y^m)$, in which multiplication by z is injective, for every $m \ge 1$. These assertions remain true after any field extension. A repeated component would make every point on it singular. Over an algebraic closure, distinct components of a plane cubic would intersect by Bézout's theorem and give a singular point. The smooth cubic is therefore geometrically integral, so S_K is a domain for every extension K/k.

For $t \ge 1$, define the K_3-representation K_t to have source S_t, target S_{t+1}, and three maps given by multiplication by x, y, z. Products and homogeneous components below are taken in this coordinate ring, or in $S_K = K \otimes_k S$ after extension.

Lemma 4.1. *For $t, u \ge 1$ and every field extension K/k, multiplication induces an isomorphism*

$$(S_K)_{u-t} \longrightarrow \mathrm{Hom}_{\mathsf{K}_3}((K_t)_K, (K_u)_K).$$

Proof. We prove the formula over k; the same argument applied to the monic equation over K proves it after any extension. Write a representation morphism as $f_0 : S_t \to S_u$ and $f_1 : S_{t+1} \to S_{u+1}$. Compatibility means $f_1(am) = af_0(m)$ for $a = x, y, z$ and $m \in S_t$. For a monomial $m = y^{t-r}b_1 \cdots b_r$, where $b_j \in \{x, z\}$ and $0 \le r \le t$, the two descriptions of ym give

$$yf_0(m) = b_1 f_0(ym/b_1) \qquad (r \ge 1).$$

Induction on r gives $y^r f_0(m) = b_1 \cdots b_r f_0(y^t)$; multiplying by y^{t-r} yields $y^t f_0(m) = mf_0(y^t)$. In particular

$$y^t f_0(z^t) = z^t f_0(y^t).$$

Since z^t is a nonzero divisor on S/y^tS, this equality implies $f_0(y^t) \in y^tS$. There is a unique $h \in S$ with $f_0(y^t) = y^t h$, and grading gives $h \in S_{u-t}$. If $u < t$ this says $f_0(y^t) = 0$ and $h = 0$. Cancelling the nonzero divisor y^t in $y^t f_0(m) = mf_0(y^t)$ proves $f_0(m) = mh$. Degree-t monomials span S_t. Since $S_{t+1} = xS_t + yS_t + zS_t$, compatibility then forces f_1 also to be multiplication by h. Conversely, every such multiplication pair is a morphism, and its value on y^t determines h, proving bijectivity. □

In particular $\mathrm{End}((K_t)_K) = K$ for every extension, so all K_t are absolutely indecomposable. Set

$$N = K_1, \qquad L = K_2 \oplus K_2, \qquad M = K_3.$$

Their dimension vectors are $(3, 6), (12, 18), (9, 12)$.

Lemma 4.2. *Let K/k be finite. A map $N_K \to L_K$ has cokernel isomorphic to M_K and is injective if and only if its pair of linear forms (s, t) is linearly independent and their common zero $P \in \mathbb{P}^2(K)$ lies outside E.*

Proof. Lemma 4.1 identifies maps $N_K \to L_K$ with pairs $(s, t) \in (S_K)_1^2$, and maps $L_K \to M_K$ with rows $(u, v) \in (S_K)_1^2$. Their composite is zero precisely when $us + vt = 0$ in $(S_K)_2$. Since the relation has degree three, this is an equality in $K[x, y, z]_2$.

The zero pair is not injective. A nonzero dependent pair can be changed to $(s, 0)$, with $s \neq 0$, by an automorphism of the two summands of L_K. Any row annihilating it has $us = 0$ as a quadratic polynomial, hence $u = 0$. Its source map then factors through one copy of $(S_K)_2$ of dimension six and cannot surject onto $(S_K)_3$ of dimension nine. Such a pair cannot have the prescribed cokernel.

Suppose now that s, t are independent. They are relatively prime linear polynomials. The equation $us = -vt$ implies $s \mid v$, so $v = cs$ and $u = -ct$ for $c \in K$. An epimorphism requires $c \neq 0$. Choose a third linear coordinate w. The forms s, t, w are a basis of the original three-dimensional space of linear forms. Choose the representative of their common point P so that $s(P) = t(P) = 0$ and $w(P) = 1$. Restricting the homogeneous cubic to this line gives

$$S_K/(s, t) \simeq K[w]/(F(P)w^3),$$

where changing the representative of P changes $F(P)$ by a nonzero cube and hence does not affect its vanishing. The cokernel of $(-t, s) : (S_K)_2^2 \to (S_K)_3$ is the degree-three component of this quotient. It vanishes exactly when $F(P) \neq 0$. Under that condition its degree-four component also vanishes, so $(-t, s) : (S_K)_3^2 \to (S_K)_4$ is onto as well.

For $P \notin E$, the column (s, t) is injective at both vertices because S_K is a domain and $s \neq 0$. The row $(-t, s)$ is onto at both vertices, and their composite is zero. The dimension equalities

$$2\dim(S_K)_2 - \dim(S_K)_3 = 3 = \dim(S_K)_1, \quad 2\dim(S_K)_3 - \dim(S_K)_4 = 6 = \dim(S_K)_2$$

give equality of the image and kernel at each vertex. Multiplication commutes with x, y, z, so these maps form an exact sequence of representations with quotient exactly $(K_3)_K$. If $P \in E$, the degree-three quotient above is one-dimensional, so no annihilating row is onto. This proves both directions. □

Proof of Theorem 1.2(i). The two-dimensional subspaces of $(S_K)_1$ correspond to points $P \in \mathbb{P}^2(K)$ by taking the common zero of their linear forms. For each point outside E the subspace has

$$|\operatorname{GL}_2(K)| = (Q^2-1)(Q^2-Q)$$

ordered bases. Lemma 4.2 identifies these bases with exactly the injective maps having the specified quotient. Two injections from N_K have the same image precisely when they differ by an automorphism of N_K, and $|\operatorname{Aut} N_K| = Q-1$. Consequently

$$F^{L_K}_{M_K,N_K}(K) = \frac{(Q^2-1)(Q^2-Q)}{Q-1}(\#\mathbb{P}^2(K) - \#E(K)),$$

which is (2). For $Q = q^n$ it equals $Q^5 - Q^3 + (Q^3-Q)a_n$, with a_n as in Lemma 2.1. That lemma proves nonrationality. The absolute summands were proved after Lemma 4.1, and Lemma 5.2 supplies heredity and wildness. □

5 A local wild nonhereditary algebra

For a K_3-representation $X = (V, W; X_1, X_2, X_3)$, define $\widehat{X}$ on $V \oplus W$ by

$$r(v,w) = (0, X_1 v), \qquad s(v,w) = (0, X_2 v), \qquad t(v,w) = (0, X_3 v).$$

Every product of two loops is zero, so this is a module for $B = k\langle r,s,t\rangle/(r,s,t)^2$. The ideal $J = (r,s,t)$ has square zero and $B/J = k$, so it is the Jacobson radical of B. For any B-module Y, its radical is JY, and its socle $\operatorname{soc} Y$ is the subspace annihilated by J. For $X = K_i$ with $i = 1,2,3$, or for $X = K_2^{\oplus 2}$, the images of the arrow maps span W and their common kernel in V is zero, since multiplication by y is injective. Hence

$$J\widehat{X} = \operatorname{soc}\widehat{X} = W. \tag{15}$$

These assertions hold after every field extension.

Lemma 5.1. *For every finite extension K/k with $Q = |K|$,*

$$F^{\widehat{L}_K}_{\widehat{M}_K,\widehat{N}_K}(K) = Q^{36} F^{L_K}_{M_K,N_K}(K).$$

Proof. Work over K and write $\widehat{L}_K = V \oplus W$ with $\dim V = 12$, $\dim W = 18$. Let $X_i : V \to W$, $1 \le i \le 3$, be the arrow maps of L_K. Let $U \subseteq \widehat{L}_K$ be a submodule of the required type and quotient type. Set $W' = U \cap W$ and let V' be its image under projection to V. Since $U \simeq \widehat{N}_K$, equation (15) gives

$$JU \subseteq W' \subseteq \operatorname{soc} U = JU,$$

so $W' = JU$ and $\dim W' = 6$, $\dim V' = 3$. The multiplication maps $V' \to W'$ are the three arrow restrictions. They give a Kronecker representation isomorphic to N_K, by taking the top U/JU and radical JU of the given B-module isomorphism. The quotient similarly has radical W/W' and top V/V', and gives the Kronecker representation M_K. Thus (V', W') is a Kronecker subrepresentation of L_K of type N_K with quotient M_K.

Conversely, fix such a Kronecker subrepresentation (V', W'). A vector subspace of $V \oplus W$ with projection V' and intersection W' is determined uniquely by the graph of a linear map

$$f : V' \longrightarrow W/W'.$$

After quotienting by W', projection onto V' is an isomorphism on the subspace, with inverse $v \mapsto (v, f(v))$. Its preimage in $V \oplus W$ is therefore

$$U_f = \{(v,w) : v \in V',\ w + W' = f(v)\}.$$

It is stable under all three loops because $X_iV' \subseteq W'$. Choose a linear lift $V' \to W$ of f; subtracting that lift from the second coordinate identifies the module U_f with the encoding of (V', W'), hence with $\widehat{N}_K$.

To identify the quotient, extend f linearly to $\widetilde{f} : V \to W/W'$ and define

$$\pi_f : V \oplus W \longrightarrow (V/V') \oplus (W/W'), \qquad \pi_f(v, w) = (v + V',\, w + W' - \widetilde{f}(v)).$$

This map is surjective and has kernel U_f. For $\bar{w} \in W/W'$, give its target the encoded quotient actions, for example $r(v + V', \bar{w}) = (0, X_1v + W')$, and likewise for s, t. These are well-defined because $X_iV' \subseteq W'$. The equality $\pi_f(r(v, w)) = (0, X_1v+W') = r\pi_f(v, w)$, and its two analogues, prove that π_f is a module map. Its target is the encoding of the Kronecker quotient M_K. Thus every graph has exactly the prescribed submodule and quotient types.

The map space has dimension $\dim V' \cdot \dim(W/W') = 3 \cdot 12 = 36$ over K. Each fixed Kronecker subrepresentation therefore has exactly Q^{36} local lifts, and the first construction recovers that subrepresentation uniquely from a lift. □

Proof of Theorem 1.2(ii). The hat construction gives $\widehat{N}, \widehat{L}, \widehat{M}$ of total dimensions $9, 30, 21$. Lemma 5.1 and (2) prove (3). A rational function representing this sequence could be divided by T^{36} to represent the hereditary sequence, so it is nonrational.

An endomorphism of an encoded K_i preserves its radical and therefore has block matrix

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \qquad bX_j = X_ja \ (1 \le j \le 3),$$

where $c : V \to W$ is arbitrary. Its map to the Kronecker endomorphism ring (a, b) is surjective. The kernel consists of the lower-left blocks and has square zero. After any extension K/k, the quotient ring is $\mathrm{End}((K_i)_K) = K$. A ring with a nilpotent ideal and division-ring quotient is local: an element with nonzero quotient has an inverse by lifting an inverse modulo the ideal and using the finite geometric series for the remaining nilpotent error. Thus $\widehat{K_i}$ stays indecomposable. Since $\widehat{L} = \widehat{K_2}^{\oplus 2}$, all summands in the triple are absolute.

Lemma 5.2 proves that B is wild and nonhereditary. □

Lemma 5.2. *The three-arrow Kronecker algebra $k\mathsf{K}_3$ is wild and hereditary. The local algebra B is wild and nonhereditary.*

Proof. Label the two vertices of K_3 by $1, 2$, in arrow order. Its path projectives are $P_1 = (k, k^3)$, with the three coordinate inclusions, and $P_2 = (0, k)$. Evaluation identifies $\mathrm{Hom}(P_1, -)$ and $\mathrm{Hom}(P_2, -)$ with the two vertex spaces, so both functors are exact and P_1, P_2 are projective. For a representation $(V, W; X_1, X_2, X_3)$, the evaluation epimorphism

$$(P_1 \otimes_k V) \oplus (P_2 \otimes_k W) \longrightarrow (V, W; X_1, X_2, X_3)$$

is the identity at vertex 1 and sends (v_1, v_2, v_3, w) to $X_1v_1 + X_2v_2 + X_3v_3 + w$ at vertex 2. Its kernel is zero at vertex 1 and consists of

$$(v_1, v_2, v_3, -X_1v_1 - X_2v_2 - X_3v_3)$$

at vertex 2. It is therefore $P_2 \otimes_k V^{\oplus 3}$. Thus every representation has a projective resolution of length at most one. If $U \subseteq P$ with P projective, dimension shifting gives $\mathrm{Ext}^1(U, -) = \mathrm{Ext}^2(P/U, -) = 0$, so U is projective. This proves heredity.

Associate to a vector space V with two arbitrary operators X, Y the Kronecker representation $(V, V; \mathrm{id}_V, X, Y)$. Its morphisms are exactly the linear maps intertwining both operators: the identity arrow forces source and target maps to agree. This is a fully faithful exact embedding of finite-dimensional $k\langle x, y\rangle$-modules, proving that $k\mathsf{K}_3$ is wild.

Composing $(V, V; \mathrm{id}_V, X, Y)$ with the hat construction gives an exact functor into B-modules. The identity arrow makes the radical and socle both the second copy of V. To verify the two required reflection properties, an isomorphism of encoded modules induces an isomorphism of their tops and radicals, hence of the Kronecker representations. For an indecomposable finite-dimensional representation its endomorphism ring is local: the stable kernel and image decomposition of a sufficiently large power of any endomorphism shows that the endomorphism is either invertible or nilpotent, and in the latter case its difference from the identity is invertible. For the encoded representation the map of endomorphism rings onto the Kronecker endomorphism ring is surjective with square-zero kernel, by the displayed block calculation. An element is invertible exactly when its image is invertible: lift an inverse and correct both products by the finite geometric series for the nilpotent error. The preimage of the nonunits is therefore an ideal, so the encoded endomorphism ring is local. This proves preservation of indecomposability.

Explicitly, take the right module $k\langle x, y\rangle^{\oplus 2}$ and let the three loops act by the right-module-linear matrices

$$r = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}.$$

Their pairwise products are zero. They define a B–$k\langle x, y\rangle$ bimodule free of rank two on the right, whose tensor functor is the construction above. Together with reflection of isomorphisms and preservation of indecomposability, this proves the required representation embedding. This proves that B is wild.

B is four-dimensional and local, with basis $1, r, s, t$. Its unique simple module has dimension one and $\operatorname{rad} B$ is a direct sum of three such simples. Every finite projective over a finite-dimensional local algebra is free: lift a basis of its quotient by the radical and split the resulting surjection from a free module. The chosen ranks make the radical quotient of the kernel zero, and Nakayama's lemma makes the kernel zero. A nonzero projective therefore has dimension divisible by four. The submodule $\operatorname{rad} B$ of the projective B has dimension three and is not projective. This proves nonheredity.

□

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Zhili College, Tsinghua University, Beijing 100084, China
Email address: tangqy24@mails.tsinghua.edu.cn