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Landau–Siegel zeros over varying number fields

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Abstract

Let K be a number field of degree n and absolute discriminant D_K. We give a direct arithmetic proof that every real zero $\beta \in (0,1)$ of ζ_K satisfies

$$(1-\beta)\{\log D_K + n\log(2n) + 6n + 1\} \geq \frac{1}{1370}.$$

The bound is explicit for arbitrary varying fields, including nonnormal fields, and gives $(1-\beta)\log D_K \gg 1/\log(2n)$ for $K \neq \mathbb{Q}$. The proof combines OpenAI's determinant method, Masser's zero estimate as used by Kerr, and Banks' selected-embedding calculation. The resulting bound is expressed directly in the degree and discriminant of K, without introducing those of its normal closure.

Stronger nonvanishing conclusions follow from recent half-plane results and positivity. The contribution here is an arithmetic proof with explicit field dependence, using neither those half-plane results nor earlier zero-exclusion results for real Dirichlet L-functions.

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1 Introduction

In this paper, we reconsider the Landau-Siegel zeros problem over arbitrary number fields.

1.1 The real-zero estimate

First, we state the real-zero estimate for arbitrary number fields and its quadratic-extension consequence. The determinant estimate of Section 4 and the prime-sum comparison of Section 5 supply the proof.

Let $n = [K : \mathbb{Q}]$, let D_K be the absolute discriminant of K, and put

$$\mathcal{B}_K = \log D_K + n \log(2n) + 6n + 1.$$

All logarithms are natural.

Theorem 1.1. *For every number field K and every real zero $\beta \in (0, 1)$ of $\zeta_K(s)$,*

$$1 - \beta \geq \frac{1}{1370\, \mathcal{B}_K}. \tag{1}$$

The field K may be nonnormal, with unrestricted degree. Retaining the exact real-zero resolvent and choosing the logarithmic-derivative parameter separately from the prime cutoff gives 1370 (Section 5.4). The $n \log n$ term remains.

The consequence $n \leq 2 \log D_K$ of Minkowski's bound (Appendix A) gives, for $K \neq \mathbb{Q}$,

$$(1 - \beta) \log D_K \geq \frac{1}{1370\{14 + 2 \log(2n)\}} \geq \frac{1}{21920 \log(2n)}. \tag{2}$$

The assertion for $K = \mathbb{Q}$ is vacuous, since $\zeta_{\mathbb{Q}}$ has no real zero in $(0, 1)$.

Corollary 1.2. *Let F/K be quadratic, $[K : \mathbb{Q}] = n$, and let $\xi = \chi_{F/K}$ be the associated quadratic Hecke character. If $L(\beta, \xi) = 0$ with $0 < \beta < 1$, then*

$$1 - \beta \geq \frac{1}{1370\{\log D_F + 2n \log(4n) + 12n + 1\}}. \tag{3}$$

Proof. The identity $\zeta_F(s) = \zeta_K(s) L(s, \xi)$ makes β a zero of ζ_F. Apply Theorem 1.1 in degree $2n$. □

1.2 Unitary Hecke characters

Next, we state the uniform zero-free region for primitive unitary Hecke characters, using the normalization of Section 2.1. Section 6.1 proves it from the real-zero estimate and classical positivity.

Let χ be a primitive unitary Hecke character of K, with finite conductor $\mathfrak{f}_\chi$. With the gamma normalization of Section 2.1, write

$$\Lambda(s,\chi) = (D_K \mathrm{N}\mathfrak{f}_\chi)^{s/2} \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s+\mu_j) L(s,\chi), \qquad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Here Γ is Euler's gamma function, and the μ_j are the archimedean gamma parameters recalled in Section 2.1. Put

$$\begin{aligned} \mathcal{H}(\chi,t) &= \log(D_K \mathrm{N}\mathfrak{f}_\chi) + \sum_{j=1}^{n} \log(3+|\mu_j+it|) + 4n + 1, \\ \mathcal{H}^+(\chi,t) &= \mathcal{H}(\chi,t) + n\log(2n). \end{aligned} \tag{4}$$

Theorem 1.3. *For every number field K and every primitive unitary Hecke character χ, the function $L(s,\chi)$ has no zero $s=\sigma+it$ in*

$$\sigma > 1 - \frac{1}{4110\,\mathcal{H}^+(\chi,t)}. \tag{5}$$

Imaginary norm twists of the principal character may have their usual pole.

The constant 4110 is absolute; the degree, discriminant, conductor, infinity type and height enter through $\mathcal{H}^+(\chi,t)$.

The passage from real quadratic characters to arbitrary unitary characters uses classical positivity and imaginary norm twists. The half-plane theorems recalled in Section 7.3, combined with classical positivity, imply a stronger region for unitary characters; Theorem 1.3 gives a different proof with explicit dependence on the field.

1.3 Earlier arithmetic work and quantitative comparison

Next, we review the arithmetic results underlying the present argument and compare their field dependence with our bounds. Section 7.2 compares the prime-sum estimates; Section 7.1 justifies the normal-closure deduction.

In the first five items, χ is a primitive nonprincipal real Dirichlet character with fundamental discriminant d and conductor $q=|d|\geq 3$. Throughout, $0<\beta<1$.

OpenAI [Ope26b], 1 October 2026.

$$L(\beta,\chi)=0 \quad\Longrightarrow\quad (1-\beta)\log q \geq c_0 > 0, \qquad c_0 \text{ absolute.}$$

The interpolation determinant is formed in $\mathbb{Q}(\sqrt{d},\sqrt{2})$, where $\mathbb{Q}(\sqrt{d})$ corresponds to χ; the fixed quadratic exception is treated separately.

Banks [Ban26b], 11 October 2026. An explicit exposition of OpenAI's rational argument gives

$$q \geq 10^{10}, \quad L(\beta,\chi)=0 \quad\Longrightarrow\quad (1-\beta)\log q \geq 3\times 10^{-3}.$$

The finite computation cited there extends the bound to $q\geq 3$.

Kerr [Ker26], 15 October 2026. The pigeonhole principle and Masser's zero estimate, in the same biquadratic field, give

$$L(\beta,\chi)=0 \quad\Longrightarrow\quad (1-\beta)\log q \gg 1.$$

The fixed exception $d = 8$ is handled separately, as above. The implied constant is absolute. Kerr's argument replaces OpenAI's interpolation and determinant analysis.

Green [Gre26], 17–18 October 2026. A Gram determinant of monomial derivatives in $\mathbb{Q}(\sqrt{d})$ gives, for sufficiently large fixed even C and sufficiently large q,

$$X = q^C, \qquad \sum_{\substack{p \le X \\ \chi(p) = -1}} \frac{\log p}{p} \le \left(\frac{1}{2} + O(C^{-1})\right) \log X + O_C(1).$$

Barreto [Bar26], 19 October 2026. A Wronskian proof of row independence in Green's determinant argument gives, uniformly for $x \ge 2$,

$$\sum_{\substack{p \le x \\ \chi(p) = -1}} \frac{\log p}{p} \le \frac{1}{2} \log x + \frac{1}{4} \log q + \log\log(3x) + O(1),$$

and consequently

$$L(\beta, \chi) = 0 \quad \Longrightarrow \quad (1 - \beta) \log q \ge 1 - O\left(\frac{\log\log(3q)}{\log q}\right).$$

Kominers [Kom26], 22 October 2026. Let $L/\mathbb{Q}$ be finite Galois and $\mathcal{C} \subset \mathrm{Gal}(L/\mathbb{Q})$ a conjugacy class of size c. If L contains $2X$ distinct algebraic integers of house at most $H \ge 1$, then

$$\sum_{\substack{p \le X \\ \mathrm{Frob}_p = \mathcal{C}}} \frac{\log p}{p} \le c \log H + \log\log(3X) + O(c), \qquad X \ge 2,$$

with an absolute implied constant; ramified primes are omitted. This yields weighted Chebotarev for fixed L and uniform real-zero exclusion in the quadratic case.

Banks [Ban26a], 22 October 2026. For a number field E and $\varnothing \ne S \subset \mathrm{Hom}(E, \mathbb{C})$,

$$\Gamma = \{(\sigma\theta)_{\sigma \in S} : \theta \in \mathcal{O}_E\},$$
$$\mu(\Gamma) = \frac{[E : \mathbb{Q}]}{|S|} \qquad \text{[Ban26a, Proposition 4.2 and its proof].}$$

Here $\mu(\Gamma)$ is Masser's invariant, defined in Theorem 4.1; $\mathrm{Hom}(E, \mathbb{C})$ denotes the field embeddings. For Galois L/k, Banks' uniform prime-sum estimate is (42), with degree-dependent errors and thresholds. Its real-zero consequences have an effective constant $c_{\mathrm{B}}(k)$: quadratic extensions of fixed k, and Galois extensions of fixed k.

In the present argument, choose the smallest rational prime a with $\sqrt{a} \notin K$, and put $E = K(\sqrt{a})$, $d = 2n$, $s = d - 1$. Let ι be the nonidentity K-automorphism of E; take all embeddings except ι for S, with weights (23). For integers $H_0 \ge 2$ and $D \ge 2H_0$, equations (27), (22), and (31) give

$$\log \lambda_E \le \frac{2}{d} \log D_E, \qquad \frac{W}{S_1} \le 1 + \frac{s - 1}{H_0},$$
$$p \nmid D_E \quad \Longrightarrow \quad 1 - \delta_p \le \frac{r_K(p)}{n}.$$

The notation is defined in Sections 4–5. The real-zero bias, ramification bound, and cutoff in Section 5, with cancellation of the normal-closure degree, yield

$$K \text{ arbitrary}, \quad \zeta_K(\beta) = 0 \quad \Longrightarrow$$
$$(1 - \beta)\{\log D_K + n \log(2n) + 6n + 1\} \ge \frac{1}{1370}.$$

Positivity and norm twists give (5). The normal-closure comparison below is proved in Section 7.

Banks' Corollary 1.2 [Ban26a] states that for each number field K there is an effective $c_{\mathrm{B}}(K) > 0$ such that

$$(1-\beta)\log D_F \geq c_{\mathrm{B}}(K) \qquad (F/K \text{ quadratic},\ L(\beta,\chi_{F/K}) = 0,\ 0<\beta<1). \tag{6}$$

Corollary 1.2 gives a fully displayed alternative. For $K \neq \mathbb{Q}$, the conductor–discriminant identity $D_F = D_K^2 \mathrm{N}\mathfrak{f}_\xi$ implies that the right side of (6) may be replaced by

$$c_{\exp}(K) = \frac{1}{1370\left(1+\dfrac{2n\log(4n)+12n+1}{2\log D_K}\right)}. \tag{7}$$

Indeed, putting $C_n = 2n\log(4n)+12n+1$, we have

$$(1-\beta)\log D_F \geq \frac{1}{1370(1+C_n/\log D_F)} \geq c_{\exp}(K).$$

This is an admissible explicit constant, not an upper or lower estimate for Banks' particular $c_{\mathrm{B}}(K)$.

Banks' Theorem 1.3 [Ban26a], with base field $\mathbb{Q}$, gives an absolute effective $c_{\mathrm{B}} > 0$ for which

$$(1-\beta)\log D_L \geq c_{\mathrm{B}} \qquad (L/\mathbb{Q} \text{ Galois},\ \zeta_L(\beta) = 0,\ 0<\beta<1). \tag{8}$$

Banks' Remark 3.4 also treats almost-normal fields. For an arbitrary degree-n field K, passage to its normal closure gives the consequence

$$(1-\beta)\log D_K \geq \frac{2c_{\mathrm{B}}}{n!}. \tag{9}$$

Thus (9) loses $n!$, whereas (2) loses only $\log(2n)$. This compares the specified normal-closure deduction, not optimal consequences of [Ban26a]; no pointwise comparison with its fixed-base constant is asserted. Deductions from [Ban26a, Theorem 1.5] and Stark-type reductions have not been optimized here. For the degree-ten field of Example 7.2, with the admissible constant $c_{\mathrm{B}} \leq 1/4$ used there, the gap in Theorem 1.1 exceeds the width from (9) by a factor greater than 1071.

The lower comparison uses rational primes and degree-one primes of K, avoiding the base-field Mertens constant discussed in [Ban26a, Section 3.2].

1.4 The arithmetic argument

Finally, we outline the determinant construction of Section 4 and the real-zero comparison of Section 5. Appendix A collects the numerical inequalities.

Choose a rational prime a with $\sqrt{a} \notin K$, and put $E = K(\sqrt{a})$, $d = 2n$. The prime can be chosen so that

$$2\log D_K \leq \log D_E \leq 4\log D_K + n\log 8.$$

Use all embeddings of E except its nonidentity K-automorphism. Masser's zero estimate and the selected-embedding rank calculation of [Ban26a, Proposition 4.2 and its proof] give a nonzero evaluation determinant. Its points lie in E, whereas its norm is taken in the normal closure M. The factor $[M:\mathbb{Q}]$ cancels. Small independent integers and exact monomial sums yield

$$\sum_{p\leq X} \delta_p \frac{\log p}{p} \leq \left(1-\frac{15}{16d}\right)\log X + \frac{4}{d}\log D_E + 32\log(2d) \qquad (X\in\mathbb{Z},\ X \geq d^2).$$

The omitted embedding relates $1-\delta_p$ to degree-one primes of K. For a real zero $\beta \in (0,1)$ of ζ_K, put $\Delta = 1-\beta$ and $H = \mathcal{H}_K^+(1,0)$, with 1 the principal character of K. The exact resolvent bound is

$$-\frac{\zeta_K'}{\zeta_K}(1+h) \le \frac{\Delta}{h(h+\Delta)} + 1 + \tfrac{1}{2}\mathcal{H}_K(1,0) \qquad (0 < h \le 1).$$

The prime-sum comparison and ramification bound (Section 5.3), with

$$X = 2^{\lceil 176H/\log 2\rceil}, \qquad h = \frac{2}{\log X},$$

give $\Delta H \ge 1/1370$, and $H \le \mathcal{B}_K$. Quadratic factorization and positivity then give the Hecke denominator $3 \cdot 1370 = 4110$.

The arithmetic results used here are recalled in Section 1.3; the small-integer estimate is [Cou20, Proposition 1]. The quantitative refinement combines evaluation points in E, exact monomial sums, a lower comparison over rational primes, and the exact real-zero resolvent. Neither the rational zero-exclusion theorem of [Ope26b, Ban26b] nor a zero-free half-plane theorem is used.

The half-plane results of [CT26, Liu26b, Liu26c, Liu26d] give stronger nonvanishing conclusions (Section 7.3). The contribution here is the alternative arithmetic proof with explicit field dependence. For the earlier fixed-field refinement over $\mathbb{Q}(\sqrt{-3})$ and its Dirichlet consequences, see [Liu26a].

Here is the structure of the paper. Section 2 fixes notation and records rational-prime estimates. Section 3 gives the gamma-factor, logarithmic-derivative and positivity estimates. Section 4 develops the embedding determinant estimate. Section 5 proves the real-zero bound, including the auxiliary quadratic extension, ramification bound and cutoff choice. Section 6 treats unitary Hecke characters, analytic conductors, imprimitive characters and norm twists. Section 7 compares the arithmetic estimates with Banks' results and recalls earlier half-plane theorems. Appendix A collects the numerical inequalities.

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2 Notation and preliminaries

In this section, we fix the arithmetic notation, Hecke normalizations, and elementary prime estimates. Section 3 uses the Hecke heights, while Sections 4–5 use the house bounds and rational-prime estimates.

For a number field E, write $\mathcal{O}_E$ for its ring of integers, D_E for its absolute discriminant, and ζ_E for its Dedekind zeta function. For a nonzero integral ideal $\mathfrak{a} \subset \mathcal{O}_E$, $\mathrm{N}\mathfrak{a} = |\mathcal{O}_E/\mathfrak{a}|$; the notation $\mathrm{N}_{E/\mathbb{Q}}$ denotes field norm. The house of $\theta \in E$ is $\max_{\sigma: E\hookrightarrow\mathbb{C}} |\sigma\theta|$. Independence of algebraic integers means linear independence over $\mathbb{Q}$.

The letter p denotes a rational prime, and $\mathfrak{p}$ a nonzero prime ideal of the field under consideration. Frobenius is arithmetic: at an unramified prime P above $\mathfrak{p}$, Frob_P induces $x \mapsto x^{\mathrm{N}\mathfrak{p}}$ on the residue field. The notation $\mathrm{Frob}_\mathfrak{p}$ denotes its conjugacy class. We write $\mathbf{1}_\mathcal{P}$ for the indicator of a condition $\mathcal{P}$, and $v_p(m)$ for the exponent of p in a positive integer m. The symbols O, $\ll$, $\gg$, and $\asymp$ have their usual meanings for bounds up to positive multiplicative constants; subscripts specify permitted dependence. Primes on functions denote differentiation.

Conventions. In Sections 4–5 the letter s denotes the number of selected embeddings, D a polynomial degree, H_0 a monomial weight, and E_p a monomial count; the complex variable does not occur there, and discriminants are always written with a subscript, as in D_K and D_E. In the proofs of Section 3, H abbreviates the height $\mathcal{H}(\chi, t)$ under consideration.

2.1 Hecke characters and heights

In this subsection, we fix the gamma factors, norm-twist convention, and logarithmic heights used in Section 3. These normalizations also govern the zero-free regions of Section 6.

Let χ be a primitive unitary Hecke character, with finite conductor $\mathfrak{f}_\chi$, and put $A_\chi = D_K \mathrm{N}\mathfrak{f}_\chi$. Its Hecke L-function is normalized by

$$L(s,\chi) = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s})^{-1} \qquad (\mathrm{Re}\, s > 1),$$

where $\chi(\mathfrak{p}) = 0$ for $\mathfrak{p} \mid \mathfrak{f}_\chi$ and $|\chi(\mathfrak{p})| = 1$ otherwise. The character 1 is the principal character, so $L(s,1) = \zeta_K(s)$. We use the ideal-norm convention

$$\chi_\tau(\mathfrak{a}) = \chi(\mathfrak{a})(\mathrm{N}\mathfrak{a})^{-i\tau}, \qquad L(s,\chi_\tau) = L(s + i\tau, \chi). \tag{10}$$

This convention fixes signs independently of idele absolute-value conventions. Recall Euler's gamma function

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\, dx \qquad (\mathrm{Re}\, z > 0),$$

with its meromorphic continuation. We write the completed function in real gamma factors as

$$\Lambda(s,\chi) = A_\chi^{s/2} \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s + \mu_j)\, L(s,\chi), \qquad \Gamma_{\mathbb{R}}(z) = \pi^{-z/2}\Gamma(z/2). \tag{11}$$

Normalizing constants independent of s, such as powers of 2 at complex places, are omitted; they do not affect logarithmic derivatives. At a real place the parameter is $\epsilon + i\tau_v$ with $\epsilon \in \{0,1\}$; at a complex place the two parameters are $|k_v|/2 + i\tau_v$ and $|k_v|/2 + 1 + i\tau_v$. Here $\tau_v \in \mathbb{R}$ is the local spectral parameter, $k_v \in \mathbb{Z}$ is the angular exponent at a complex place, and $j = 1, \dots, n$ indexes the real gamma factors. In particular $\mathrm{Re}\, \mu_j \geq 0$.

Definition 2.1 (Explicit logarithmic height)**.** For $t \in \mathbb{R}$ define

$$\mathcal{H}(\chi, t) = \log A_\chi + \sum_{j=1}^{n} \log(3 + |\mu_j + it|) + 4n + 1. \tag{12}$$

We will also use the height including the degree term

$$\mathcal{H}^+(\chi, t) = \mathcal{H}(\chi, t) + n \log(2n). \tag{13}$$

For the principal character, $L(s,1) = \zeta_K(s)$, the conductor is $\mathcal{O}_K$, and the parameters are $\mu_j \in \{0,1\}$.

An imaginary norm twist changes every parameter by $i\tau$ and gives the exact identity $\mathcal{H}(\chi_\tau, t) = \mathcal{H}(\chi, t + \tau)$. We write 1_τ for the twist of the principal character. Subscripts on $\mathcal{H}_E$ and $\mathcal{H}_E^+$ specify the base field; they are omitted when it is clear. In expressions $s = \sigma + it$, both σ and t are real.

2.2 Rational-prime estimates

In this subsection, we record the rational-prime estimates used in Sections 5.1 and 5.3. Their explicit constants keep the subsequent comparison uniform in the field.

For $x \geq 2$,

$$\vartheta(x) = \sum_{p \leq x} \log p \leq 4x, \qquad \sum_{p \leq x} \frac{\log p}{p} \geq \log x - 6. \tag{14}$$

For completeness, the binomial coefficients $\binom{2k}{k}$ and $\binom{2k+1}{k}$ prove the first by induction. For the second put $N = \lfloor x \rfloor$, use $\log N! \geq N \log N - N$ and

$$\log N! \leq N \sum_{p^j \leq N} \frac{\log p}{p^j}.$$

The contribution from $j \geq 2$ is bounded by $\sum_{k \geq 2} \log k/(k(k-1)) < 4$; this last inequality follows from $\log(j+1) \leq \log 2 + \log j$, $\sum j^{-2} < 2$, and $\sum_{j \geq 2} (\log j)/j^2 < 2$ by an integral comparison. Finally $\log \lfloor x \rfloor \geq \log x - \log 2$.

3 Archimedean estimates and classical positivity

In this section, we establish logarithmic-derivative and positivity estimates using the heights of Section 2.1. Together with the real-zero bound of Section 5, they yield the Hecke theorem in Section 6.1. All displayed numerical constants are independent of K.

The analytic inputs are Hecke's continuation and functional equation, with the principal norm twists as the only pole cases [Poo15, Theorems 5.16 and 5.22], and the order-one property [Raj98, Section 1, pp. 281–282]. To pass from the normalized characters in the latter source to our convention, put

$$\tau_0 = \frac{1}{n} \sum_{j=1}^{n} \operatorname{Im} \mu_j, \qquad \chi^0 = \chi_{-\tau_0}, \qquad \sum_j \operatorname{Im} \mu_j(\chi^0) = 0.$$

The conductor is unchanged, and

$$\Lambda(s, \chi) = A_\chi^{-i\tau_0/2} \Lambda(s + i\tau_0, \chi^0).$$

Thus the function

$$\Xi_\chi(s) = \begin{cases} (s + i\tau)(s + i\tau - 1)\Lambda(s, \chi), & \chi = 1_\tau, \\ \Lambda(s, \chi), & \chi \neq 1_\tau \text{ for every } \tau \in \mathbb{R}, \end{cases}$$

is entire of order one and satisfies $\Xi_\chi(s) = w_\chi \overline{\Xi_\chi(1 - \overline{s})}$, $|w_\chi| = 1$. No uniform growth constant in the order-one assertion is needed. The behavior of conductors and infinity types under powers is also recorded in [Raj98, Section 1]; we do not use its zero-free region.

Boundary nonvanishing follows directly from Euler-product positivity. Let $\chi^{[2]}$ denote the primitive character inducing χ^2. If $L(1 + it, \chi) = 0$ has multiplicity $m \geq 1$ and $\chi^{[2]} \neq 1_{-2t}$, then, as $h \downarrow 0$,

$$\begin{aligned} 0 &\leq -3 \frac{\zeta_K'}{\zeta_K}(1 + h) - 4 \operatorname{Re} \frac{L'}{L}(1 + h + it, \chi) - \operatorname{Re} \frac{L'}{L}(1 + h + 2it, \chi^{[2]}) \\ &\leq \frac{3 - 4m}{h} + O_{\chi,t}(1), \end{aligned}$$

a contradiction. At a prime ramified for χ, the local coefficient is at least $3-1=2$, so the primitive square is admissible here. In the remaining case $\xi=\chi_t$ satisfies $\xi^2=1$ and $L(1+it,\chi)=L(1,\xi)$. This is a pole if $\xi=1$; otherwise let F/K be the quadratic class field attached to ξ. Its factorization gives

$$L(1,\xi)=\frac{\operatorname{Res}_{s=1}\zeta_F(s)}{\operatorname{Res}_{s=1}\zeta_K(s)}>0.$$

Together with the functional equation and the Euler product, this puts all zeros of Ξ_χ in $0<\operatorname{Re}s<1$. It supplies no open zero-free strip.

3.1 Gamma factors and logarithmic derivatives

In this subsection, we bound the gamma contributions and express logarithmic derivatives in terms of zeros, with the normalization of Section 2.1. Section 3.2 uses these estimates to isolate possible real zeros.

Lemma 3.1 (Gamma bound)**.** *For* $\operatorname{Re}z\ge 1/2$*,*

$$\operatorname{Re}\frac{\Gamma'}{\Gamma}(z)\le\log(1+|z|)+4.$$

Consequently, for $1<\sigma\le 2$*,*

$$\frac{1}{2}\log A_\chi+\sum_j\operatorname{Re}\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(\sigma+it+\mu_j)\le\tfrac{1}{2}\mathcal{H}(\chi,t).\tag{15}$$

Proof. Write $z=x+iy$, $r=|z|$, and $N=\lceil r\rceil\ge 1$. Put $H_N=\sum_{j=1}^N j^{-1}$ and $\gamma=\lim_{N\to\infty}(H_N-\log N)$, the Euler–Mascheroni constant. In the series

$$\frac{\Gamma'}{\Gamma}(z)=-\gamma+\sum_{k\ge 0}\left(\frac{1}{k+1}-\frac{1}{k+z}\right),$$

the first N summands satisfy

$$\sum_{k=0}^{N-1}\left(\frac{1}{k+1}-\operatorname{Re}\frac{1}{k+z}\right)\le H_N.$$

For $k\ge N$, use

$$\frac{1}{k+1}-\frac{k+x}{(k+x)^2+y^2}=\frac{x-1}{(k+1)(k+x)}+\frac{y^2}{(k+x)((k+x)^2+y^2)}.$$

The sum of the positive parts of the first terms is at most 1: if $x>1$, bound it by $(N-1)\sum_{k\ge N}k^{-2}\le 1-N^{-2}$. The sum of the second terms is at most $r^2(N^{-3}+(2N^2)^{-1})\le 3/2$. Since $H_N-\gamma\le\log N+1$, the stated bound follows. Apply it to $z=(\sigma+it+\mu_j)/2$ and use $1+|z|\le 3+|\mu_j+it|$, dropping the negative constant $-\frac{1}{2}\log\pi$. □

Lemma 3.2 (Logarithmic derivative)**.** *If* χ *is not an imaginary norm twist of the principal character, then, for* $1<\sigma\le 2$*,*

$$-\operatorname{Re}\frac{L'}{L}(\sigma+it,\chi)\le\tfrac{1}{2}\mathcal{H}(\chi,t)-\sum_\rho\operatorname{Re}\frac{1}{\sigma+it-\rho}.\tag{16}$$

The sum is over nontrivial zeros ρ of $L(s,\chi)$, namely those in $0<\operatorname{Re}\rho<1$, counted with multiplicity; its summands are nonnegative. For $\chi=1_\tau$ one must add

$$\operatorname{Re}\left(\frac{1}{\sigma+i(t+\tau)-1}+\frac{1}{\sigma+i(t+\tau)}\right) \tag{17}$$

to the right side.

Proof. Apply Hadamard factorization to the entire function Ξ_χ:

$$\Xi_\chi(s)=e^{a+bs}\prod_\rho(1-s/\rho)e^{s/\rho},\qquad \frac{\Xi_\chi'}{\Xi_\chi}(s)=b+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Its order implies $\sum_\rho|\rho|^{-2}<\infty$. Since $0<\operatorname{Re}\rho<1$, both real-part sums converge absolutely, locally uniformly away from zeros. Choose $s_0=1/2+iv$ with $\Xi_\chi(s_0)\neq 0$. The reflected functional equation gives $\operatorname{Re}(\Xi_\chi'/\Xi_\chi)(s_0)=0$, while the pairing $\rho\leftrightarrow 1-\overline{\rho}$ gives

$$\sum_\rho\operatorname{Re}\frac{1}{s_0-\rho}=0,\qquad \operatorname{Re}b=-\sum_\rho\operatorname{Re}\frac{1}{\rho}.$$

Zeros on $\operatorname{Re}\rho=1/2$ contribute zero individually at s_0; choosing v avoids any division by a central zero. Consequently

$$\operatorname{Re}\frac{\Xi_\chi'}{\Xi_\chi}(s)=\sum_\rho\operatorname{Re}\frac{1}{s-\rho}.$$

Insert the definition of Ξ_χ and bound the conductor and gamma terms by Lemma 3.1. For $\chi=1_\tau$, the two linear factors contribute exactly (17). □

3.2 Classical positivity

In this subsection, we apply the logarithmic-derivative bounds of Section 3.1 to classical positive trigonometric sums. The resulting estimates, combined with Section 5, prove Theorem 1.3 in Section 6.1.

The finite conductor of $\chi^{[2]}$ divides $\mathfrak{f}_\chi$; write $\mu_j^{[2]}$ for its gamma parameters, paired with those of χ as below. Doubling the local phases and angular orders, with the real parity reduced modulo two, gives

$$\mathcal{H}(1,0)\le\mathcal{H}(\chi,t),\qquad \mathcal{H}(\chi^{[2]},2t)\le 2\mathcal{H}(\chi,t). \tag{18}$$

For the first inequality, $|\mu_j+it|\ge\operatorname{Re}\mu_j$, and the principal parameters 0 (real places) and $0,1$ (complex places) are minimal. For the second, pair the complex gamma parameters in their natural order and use $3+|\mu_j^{[2]}+2it|\le 2(3+|\mu_j+it|)$. The added $n\log 2$ is absorbed by $4n+1$.

Proposition 3.3 (Characters not equivalent to real ones)**.** *If χ is not an imaginary norm twist of a character of order one or two, then every nontrivial zero $\beta+it$ satisfies*

$$(1-\beta)\mathcal{H}(\chi,t)\ge 1/112. \tag{19}$$

Proof. Put $H=\mathcal{H}(\chi,t)$. Neither χ nor $\chi^{[2]}$ is a norm character: indeed,

$$\chi^2=1_\tau\quad\Longrightarrow\quad\xi=\chi_{-\tau/2},\quad\xi^2=1,\quad\chi=\xi_{\tau/2},$$

contrary to the hypothesis. Here equality of primitive characters is understood on the idele class group, so it includes the local components at primes where squaring removes ramification. Positivity of $3+4\cos\theta+\cos 2\theta$ and the Euler products give

$$0 \le -3\frac{\zeta_K'}{\zeta_K}(\sigma) - 4\,\mathrm{Re}\,\frac{L'}{L}(\sigma+it,\chi) - \mathrm{Re}\,\frac{L'}{L}(\sigma+2it,\chi^{[2]}).$$

At a prime ramified for χ the middle term vanishes and the local coefficient is at least $3-1=2$, even if squaring removes ramification. Thus using the primitive square causes no Euler-factor error. By Lemma 3.2, retaining the indicated zero and dropping the others,

$$\frac{4}{\sigma-\beta} \le \frac{3}{\sigma-1} + 8H.$$

Indeed, $H \ge 5$ and

$$\frac{3}{2}\mathcal{H}(1,0) + 2H + \frac{1}{2}\mathcal{H}(\chi^{[2]},2t) + \frac{3}{\sigma} \le \frac{9}{2}H + 3 \le \frac{11}{2}H \le 8H.$$

Set $\sigma = 1+1/(16H)$. Dividing by H yields $4/(1/16+(1-\beta)H) \le 56$, which is (19). □

Proposition 3.4 (Real characters away from real zeros)**.** *Let ξ be a primitive principal or quadratic character. For every nonreal zero $\beta+iu$ of $L(s,\xi)$,*

$$(1-\beta)\mathcal{H}(\xi,u) > 1/1000. \tag{20}$$

Proof. Suppose otherwise and set $H=\mathcal{H}(\xi,u)$, $\kappa=(1-\beta)H \le 1/1000$, and $b=|u|H>0$. If $b\le 1/10$, use the pair of conjugate zeros in the logarithmic derivative at a real point $\sigma=1+1/(4H)$. The Euler coefficients of $-\zeta_K'/\zeta_K - L'/L(s,\xi)$ are nonnegative when ξ is quadratic. For $\xi=1$ use $-\zeta_K'/\zeta_K$ alone. In either case Lemma 3.2 gives, with $x=1/4+\kappa$,

$$\frac{2x}{x^2+b^2} \le 4+1+H^{-1} \le 6.$$

But the left side is at least $2(251/1000)/((251/1000)^2+(1/10)^2) > 6$.

If $b \ge 1/10$, apply the same three-term positivity as above at $\sigma=1+1/(100H)$. Now the square character is principal, and (18) with $\xi^{[2]}=1$ gives $\mathcal{H}(1,2u) \le 2\mathcal{H}(\xi,u)$. The pole at $1+0i$ in its evaluation at $2u$ contributes at most $H/4$. If ξ itself is principal, its pole evaluated at u contributes at most $4H$. The other pole contributions are at most 8, and the gamma/conductor contribution is at most $9H/2$. Consequently

$$\frac{4}{1/100+\kappa} \le 300+13+4+\tfrac{1}{4} = \frac{1269}{4}.$$

The left side is at least $4000/11 > 1269/4$, a contradiction. □

4 An embedding determinant estimate

In this section, we construct the embedding determinant used in Section 5.1. The evaluation points lie in a possibly nonnormal field E, while their selected conjugates and determinant lie in its normal closure M; the degree of M cancels from the comparison.

4.1 Masser's theorem and selected embeddings

In this subsection, we combine Masser's zero estimate with the selected-embedding rank calculation. This gives the injectivity needed to choose a nonzero minor in Section 4.3.

Theorem 4.1 (Masser's polynomial zero estimate)**.** Let $d, s \geq 1$, and let $v_1, \dots, v_d \in \mathbb{C}^s$ generate a free abelian group Γ of rank d. Put

$$\mu(\Gamma) = \min_{V \subsetneq \mathbb{C}^s} \frac{d - \operatorname{rank}_{\mathbb{Z}}(\Gamma \cap V)}{s - \dim_{\mathbb{C}} V}.$$

The minimum ranges over proper complex linear subspaces V of $\mathbb{C}^s$. For an integer $N \geq 2$, if a nonzero polynomial P vanishes at all $\sum_{j=1}^{d} u_j v_j$, $0 \leq u_j \leq N-1$ integral, then

$$\deg P \geq \left(\frac{N-1}{s}\right)^{\mu(\Gamma)}. \tag{21}$$

This is the polynomial form of [Mas81, Theorem 1], as used in [Ker26, Lemma 6] and [Ban26a, Theorem 4.1]. It is the only polynomial zero estimate used below. Its application is over $\mathbb{C}$, so polynomial coefficients in a larger number field require no change.

Lemma 4.2 (Selected-embedding exponent)**.** *Let $E/\mathbb{Q}$ have degree d, and let S consist of $1 \leq s \leq d$ distinct embeddings $E \hookrightarrow \mathbb{C}$. For independent integers $\omega_1, \dots, \omega_d \in E$, the group generated by $(\sigma\omega_j)_{\sigma \in S}$ has $\mu(\Gamma) = d/s$. Consequently, for integers $N \geq 2$ and $D \geq 0$, no nonzero polynomial of degree at most D vanishes at all*

$$\left(\sigma \sum_{j=1}^{d} u_j \omega_j\right)_{\sigma \in S}, \qquad u_j \in \{0, \dots, N-1\},$$

if $D < ((N-1)/s)^{d/s}$.

This is the selected-embedding calculation in [Ban26a, Proposition 4.2 and its proof]. It applies to any set of embeddings. We give the proof for E and take the normal closure only when averaging over its Galois group.

Proof. The embedding map is injective on E, so the group has rank d. For a rational subspace $U \subset E$ of dimension r, let $u_1, \dots, u_r$ be a basis. The matrix of all d embeddings of these elements has rank r, by separability. Choose r independent rows, indexed by B.

Let G be the Galois group of the normal closure of E. It acts transitively on the d embeddings, and entrywise automorphisms preserve every row rank. Averaging gives

$$\operatorname{rank}(\sigma u_j)_{\sigma \in S, j} \geq \frac{1}{|G|} \sum_{g \in G} |gS \cap B| = \frac{sr}{d}.$$

For a complex subspace V of dimension v, take U to be the rational span of the preimage of $\Gamma \cap V$. Then $r = \operatorname{rank}_{\mathbb{Z}}(\Gamma \cap V)$ and its image lies in V, so $r \leq dv/s$. This proves $\mu(\Gamma) \geq d/s$; the subspace $V = 0$ gives the reverse inequality. Now use Theorem 4.1. □

4.2 Weighted monomial sums

In this subsection, we compute finite weighted monomial sums for the determinant of Section 4.3. The resulting ratios give the explicit prime-sum constants in Section 5.1.

For $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s_{\geq 0}$, write $|\alpha| = \sum_{i=1}^s \alpha_i$ and $z^\alpha = \prod_{i=1}^s z_i^{\alpha_i}$. For integers $H_0 \geq 2$, $D \geq 2H_0$, and $s \geq 1$, put

$$\mathcal{A} = \{\alpha \in \mathbb{Z}^s_{\geq 0} : \alpha_1 + H_0(\alpha_2 + \cdots + \alpha_s) \leq D\}, \quad A = |\mathcal{A}|,$$

$$S_1 = \sum_{\alpha \in \mathcal{A}} \alpha_1, \quad W = \sum_{\alpha \in \mathcal{A}} |\alpha|, \quad E_p = \sum_{\alpha \in \mathcal{A}} \lfloor \alpha_1/p \rfloor.$$

The following finite sums replace the fixed-dimension asymptotic monomial estimates in [Ban26a, Lemma 6.1]; we use them in the prime-sum estimate of Lemma 5.2.

Lemma 4.3 (Monomial ratios)**.** *The following bounds hold without asymptotic error:*

$$\frac{W}{S_1} \leq 1 + \frac{s-1}{H_0}, \qquad \frac{A}{S_1} \leq \frac{s+1}{D - H_0} \leq \frac{2(s+1)}{D}, \qquad E_p \geq \frac{S_1}{p} - A. \tag{22}$$

Also $A \leq (D+1)^s$.

Proof. Write $\alpha_1 = H_0 b_1 + r$, $0 \leq r < H_0$, and put $q_r = \lfloor (D-r)/H_0 \rfloor$. For fixed r the s coordinates $b_1, \alpha_2, \dots, \alpha_s$ run symmetrically over the integer simplex of sum at most q_r. Their means are all $q_r/(s+1)$, as is seen by adding the coordinate $q_r - b_1 - \alpha_2 - \cdots - \alpha_s$.

Summing over residues, write T for the sum of the b_1 coordinates and $R \geq 0$ for the sum of the residues. Then $S_1 = H_0 T + R$ and $W = S_1 + (s-1)T$, giving the first inequality.

In each residue class the mean of α_1 is $H_0 q_r/(s+1) + r \geq (D - H_0)/(s+1)$, proving the second. The last inequality follows termwise from $\lfloor x \rfloor \geq x - 1$. □

4.3 Frobenius congruences and the determinant

In this subsection, we compare the archimedean size and Frobenius divisibility of a nonzero evaluation determinant. We use the injectivity of Section 4.1 and the monomial sums of Section 4.2; Section 5.1 then specializes the resulting inequality.

Let $E/\mathbb{Q}$ have degree $d \geq 2$, let M be its normal closure, $m = [M : \mathbb{Q}]$, and let S be a set of $s < d$ embeddings of E into M, including its distinguished inclusion. For p unramified in M let $C_p \subset \mathrm{Gal}(M/\mathbb{Q})$ be the Frobenius class and define

$$\delta_p = \frac{|\{g \in C_p : g|_E \in S\}|}{|C_p|}. \tag{23}$$

Set $\delta_p = 0$ at ramified primes. Write λ_E for a house bound for d linearly independent algebraic integers in E. Such integers span E over $\mathbb{Q}$.

One can take $\lambda_E \leq \sqrt{D_E}$. Let (r_1, r_2) be the signature of E, with r_1 real embeddings and r_2 conjugate pairs of complex embeddings, and let $\lambda_1 \leq \cdots \leq \lambda_d$ be the successive minima of $\mathcal{O}_E$ for the house norm: λ_j is the least house bound for j independent algebraic integers. The house unit body has volume $2^{r_1}\pi^{r_2}$ and $\mathcal{O}_E$ has covolume $2^{-r_2}\sqrt{D_E}$. Minkowski's second theorem gives

$$\prod_{j=1}^{d} \lambda_j \leq (2/\pi)^{r_2} \sqrt{D_E} \leq \sqrt{D_E}.$$

Every nonzero algebraic integer has house at least one, by its nonzero integer norm. Thus the last minimum, and hence a house bound for d independent integers, is at most $\sqrt{D_E}$.

Lemma 4.4 (Nonnormal-field determinant comparison)**.** *Let $H_0 \geq 2$ and $D \geq 2H_0$ be integers, and order $S = (\sigma_1, \dots, \sigma_s)$ with σ_1 the inclusion. Write z_σ for the variable indexed by $\sigma \in S$, and set $z_i = z_{\sigma_i}$ for $1 \leq i \leq s$. Set*

$$N = 1 + \left\lceil 2s(D+1)^{s/d} \right\rceil, \qquad \mathcal{R} = d\lambda_E N.$$

Then

$$\sum_{\substack{H_0 \leq p \leq D \\ p \text{ unramified in } M}} \delta_p E_p \log p \leq W \log \mathcal{R} + \frac{A}{2} \log A. \tag{24}$$

The same upper bound holds if the prime sum is restricted to any smaller cutoff $X \leq D$.

We compare the archimedean size and Frobenius divisibility of a nonzero determinant, following OpenAI [Ope26b] and the explicit exposition of Banks [Ban26b]; related determinant comparisons appear in [Gre26, Bar26, Kom26]. The nonzero minor here is supplied by Masser's estimate and Lemma 4.2. The triangular polynomial substitutions are those of [Ban26a, Lemma 5.2]. The quantitative adaptation here allows the points to lie in E while their selected conjugates and determinant lie in M.

Proof. Use the points $\theta_{\boldsymbol{u}} = \sum u_j \omega_j$, $0 \leq u_j < N$, with $\omega_j \in \mathcal{O}_E$ independent and of house at most λ_E. Their selected conjugates have house at most $\mathcal{R}$.

By Lemma 4.2, evaluation on the whole box is injective on the polynomials spanned by z^α, $\alpha \in \mathcal{A}$: a nonzero such polynomial has total degree at most D, strictly below $((N-1)/s)^{d/s} \geq 2^{d/s}(D+1) > D$. Thus the A monomial rows have full rank, and we can select A point columns with nonzero determinant $\mathcal{D} \in \mathcal{O}_M$. Fix these columns once and for all.

Hadamard's inequality at every embedding of M gives

$$\frac{1}{m} \log |\mathrm{N}_{M/\mathbb{Q}} \mathcal{D}| \leq W \log \mathcal{R} + \frac{A}{2} \log A. \tag{25}$$

Indeed the row indexed by α has length at most $\sqrt{A}\,\mathcal{R}^{|\alpha|}$ at every embedding.

For a fixed embedding $\sigma \in S$, let $Q_{p,\sigma} \subset \mathcal{O}_M$ be the product of primes P above p for which $\mathrm{Frob}_P|_E = \sigma$. An empty product is the unit ideal $\mathcal{O}_M$. For every $\theta \in \mathcal{O}_E$, one has

$$\theta^p \equiv \sigma(\theta) \pmod{Q_{p,\sigma}}, \qquad \sum_{\sigma \in S} \log \mathrm{N} Q_{p,\sigma} = m\delta_p \log p.$$

Let $f_p = [\mathcal{O}_M/P : \mathbf{F}_p]$ be the common residue degree of the primes P above p, where $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$. The map from primes above p to C_p has constant fiber size $m/(f_p|C_p|)$, and each prime has norm p^{f_p}. The nonunit ideals $Q_{p,\sigma}$ for distinct pairs are coprime.

For $p \geq H_0$ and fixed σ, write $\alpha_1 = pb + v$ and replace the monomial row by evaluation of

$$z_1^v (z_1^p - z_\sigma)^b z_2^{\alpha_2} \cdots z_s^{\alpha_s}.$$

For $\sigma \neq \sigma_1$, its jth expansion term has first degree $\alpha_1 - pj$ and weighted degree $\alpha_1 + H_0 \sum_{i>1} \alpha_i - (p - H_0)j$. For $\sigma = \sigma_1$ its first degree is $\alpha_1 - (p-1)j$. Thus every term stays in $\mathcal{A}$, and every nonleading term has strictly smaller first degree, even at $p = H_0$. This is an integer unitriangular row change and preserves the selected determinant.

Each pair (p,σ) is treated starting from the original matrix; the changes need not commute, since they all preserve the same $\mathcal{D}$. Every entry of the replacement row lies in $Q_{p,\sigma}^{b}$. Therefore $\mathcal{D} \in Q_{p,\sigma}^{E_p}$. The nonzero integer norm satisfies

$$\frac{1}{m}\log|\mathrm{N}_{M/\mathbb{Q}}\mathcal{D}| \geq \sum_{\substack{H_0 \leq p \leq D \\ p \text{ unramified}}} \delta_p E_p \log p. \tag{26}$$

Combine (25) and (26). The normal-closure factor m cancels exactly, without any estimate for its discriminant or for a spanning set of M. □

5 Real zeros over arbitrary number fields

In this section, we apply the determinant estimate of Section 4 to an auxiliary quadratic extension and compare the resulting prime sum with the bias from a real zeta zero. The cutoff in Section 5.4 proves Theorem 1.1, which Section 6 transfers to Hecke characters.

5.1 Small independent integers and an explicit prime sum

In this subsection, we combine small independent algebraic integers with the determinant estimate of Section 4.3 to obtain an explicit prime-sum bound. The polynomial degree and prime cutoff are chosen separately; Section 5.3 uses the resulting bound.

Lemma 5.1 (Small independent integers)**.** *For a degree-d number field E, one can choose d independent algebraic integers all of whose conjugates have absolute value at most $D_E^{2/d}$. Thus one may take*

$$\log \lambda_E \leq \frac{2}{d}\log D_E. \tag{27}$$

Proof. This is the explicit small-integer bound of [Cou20, Proposition 1]; we recall its argument to keep constants visible. Let $\lambda_1 \leq \cdots \leq \lambda_d$ be the house-norm successive minima. Their product is at most $\sqrt{D_E}$ and each is at least one. For $h = \lfloor d/2 \rfloor + 1$, therefore, $\lambda_h^{d-h+1} \leq \prod_j \lambda_j \leq \sqrt{D_E}$, and $d - h + 1 \geq d/2$, so $\lambda_h \leq D_E^{1/d}$.

Let $U \subset E$ be the span of h independent integers of this house bound. The products of two elements from a basis of U span E. Otherwise a nonzero rational linear form $f : E \to \mathbb{Q}$ would vanish on $UU = \{uv : u, v \in U\}$. Write $E^* = \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{Q})$ for the rational dual. The map

$$U \longrightarrow E^*, \qquad u \longmapsto [v \longmapsto f(uv)]$$

is injective, because multiplication by a nonzero field element is invertible. Its image lies in $\mathrm{Ann}_{E^*}(U) = \{\ell \in E^* : \ell|_U = 0\}$, which has dimension $d - h < h$, a contradiction.

Select d independent products; their houses are at most $D_E^{2/d}$. □

Lemma 5.2 (Refined embedding prime sum)**.** *Let $E/\mathbb{Q}$ have degree $d \geq 2$, let S omit one embedding other than the identity, and use (23). Put $B_E = \log D_E$. For every integer*

$$X \geq d^2,$$

one has

$$\sum_{p \leq X} \delta_p \frac{\log p}{p} \leq \left(1 - \frac{15}{16d}\right)\log X + \frac{4}{d}B_E + 32\log(2d). \tag{28}$$

Proof. In the determinant construction choose

$$s = d - 1, \quad H_0 = 16d^2, \quad D = 16d^2X, \quad N = 1 + \lceil 2s(D+1)^{s/d} \rceil.$$

Use a spanning set from Lemma 5.1, and apply Lemma 4.4, restricted to primes $H_0 \le p \le X$. The polynomial degree is still D. Write $\nu = 1 + (s-1)/H_0$. The first-coordinate degree ratio gives $\nu s/d \le 1 - 15/(16d)$. The floor error is at most

$$\frac{2d}{D}\vartheta(X) \le \frac{1}{2d}.$$

The determinant error is at most

$$\frac{A}{2S_1}\log A \le \frac{ds}{D}\log(D+1) \le \frac{1}{4}.$$

Here $D \ge 16d^4$ follows from $X \ge d^2$, and $\log(D+1) \le \sqrt{D}$ was used. The combined errors are at most $1/(2d) + 1/4 \le 1/2$.

Writing $T = (D+1)^{s/d} \ge 1$, we have

$$N \le 2sT + 2 \le 2dT, \qquad \log \mathcal{R} \le \frac{s}{d}\log(D+1) + \log(2d^2) + \frac{2}{d}B_E.$$

Thus the point-height contribution is at most

$$\nu\left\{\frac{s}{d}\log(D+1) + \log(2d^2) + \frac{2}{d}B_E\right\}.$$

Use $D+1 \le 32d^2X$, $\nu \le 2$, and $\nu s/d \le 1$. Restoring primes below H_0 contributes at most $4(1+\log H_0)$. All terms apart from the displayed leading term and $4B_E/d$ total at most $5 + 23\log 2 + 14\log d \le 24 + 14\log d \le 32\log(2d)$. The latter inequality follows already from $1/2 < \log 2 < 7/10$ and $d \ge 2$. □

5.2 An auxiliary quadratic field and degree-one primes

In this subsection, we choose the auxiliary quadratic field and relate the omitted embedding to degree-one primes of K. This converts the weighted prime sum of Section 5.1 into the lower comparison used in Section 5.3.

Let $K/\mathbb{Q}$ have degree n. Choose a to be the smallest rational prime with $\sqrt{a} \notin K$. Among the first $\lfloor \log_2 n \rfloor + 1$ rational primes such a prime exists: otherwise K would contain a multiquadratic field of degree greater than n. Bertrand's theorem gives $a \le 2n$.

Set

$$\begin{gathered} E = K(\sqrt{a}), \quad d = 2n, \quad B_E = \log D_E, \\ 2\log D_K \le B_E \le 4\log D_K + n\log 8. \end{gathered} \tag{29}$$

The lower bound is the discriminant tower formula for E/K. For the upper bound, the product of integral bases for K and $\mathbb{Q}(\sqrt{a})$ spans an integral order of E, with discriminant $D_K^2 D_{\mathbb{Q}(\sqrt{a})}^n$. If $a = 2$, the quadratic discriminant is 8. Otherwise let b be the prime immediately preceding a. Minimality gives $\mathbb{Q}(\sqrt{b}) \subset K$, whence $\log D_K \ge (n/2)\log b$. Bertrand's theorem gives $a < 2b$, so $\log D_{\mathbb{Q}(\sqrt{a})} \le \log(4a) \le \log 8 + (2/n)\log D_K$. This absorbs the auxiliary quadratic discriminant into the original discriminant and an absolute degree term. Let $\iota \ne 1$ be the K-automorphism of E, and take for S all d embeddings of E except ι. Thus $s = d - 1$ and the distinguished identity is retained.

Lemma 5.3 (Fixed-point comparison)**.** *For p unramified in the normal closure of E, let $r_K(p)$ be the number of degree-one primes of K above p. Then*

$$\delta_p = 1 - \mathbf{1}_{\eta_a(p)=-1}\frac{r_K(p)}{n}, \qquad \eta_a = \chi_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}. \tag{30}$$

In particular,

$$1 - \delta_p \le \frac{r_K(p)}{n}. \tag{31}$$

Consequently, for $X \ge 2$,

$$\sum_{p\le X} \delta_p \frac{\log p}{p} \ge \log X - 6 - \frac{1}{n}\sum_{\mathrm{N}\mathfrak{p}\le X} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} - \sum_{p\mid D_E} \frac{\log p}{p}. \tag{32}$$

Proof. If $g|_E = \iota$, then $g|_K = 1$. For a uniformly chosen element of a conjugacy class, the probability of fixing the distinguished embedding of K is $r_K(p)/n$: average its fixed-point count over the transitive action on embeddings of K.

The restriction of every member of the class to $\mathbb{Q}(\sqrt{a})$ has the same sign $\eta_a(p)$. If this sign is -1, fixing the distinguished embedding of K is exactly the condition $g|_E = \iota$; if it is 1, this condition is impossible. This proves (30) and (31).

Primes unramified in E are unramified in its normal closure, because the conjugate local extensions and their composita are unramified. Thus the ramified rational primes divide D_E; their mass is bounded in Lemma 5.4. Sum (31), use (14), and bound the degree-one prime sum by the sum over all prime ideals of K. □

5.3 Ramification and the final prime-sum comparison

In this subsection, we bound the ramification contribution and retain the exact real-zero term in the logarithmic derivative. Combining Sections 5.1 and 5.2 gives the comparison and error bound used to choose the cutoff in Section 5.4.

Lemma 5.4 (Logarithmic ramification bound)**.** *For a positive integer discriminant $D_E > 1$, with $B_E = \log D_E$,*

$$\sum_{p\mid D_E} \frac{\log p}{p} \le 5 + 4\log(2 + B_E).$$

Proof. Split at $Y = \max(2, B_E)$. Partial summation with $\vartheta(u) \le 4u$ bounds primes at most Y by $4 + 4\log Y$. Primes greater than Y contribute at most $Y^{-1}\sum_{p\mid D_E}\log p \le B_E/Y \le 1$. □

Throughout Sections 5.3–5.4, put

$$\Delta = 1 - \beta, \quad 0 < \beta < 1, \quad \zeta_K(\beta) = 0, \qquad L = \log D_K, \quad H = \mathcal{H}_K^+(1,0), \quad \mathcal{A}_K = 1 + \tfrac{1}{2}\mathcal{H}_K(1,0).$$

Choose $E = K(\sqrt{a})$ as in (29); put $d = 2n$, $B_E = \log D_E$, and $\alpha_0 = 15/16$.

Lemma 5.5 (Unrounded real-zero term)**.** *For an integer $X \ge d^2$, put $\ell = \log X$. If $0 < u \le \ell$, then*

$$\sum_{\mathrm{N}\mathfrak{p}\le X} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \le e^u \left\{ \frac{\Delta\ell^2}{u(u+\Delta\ell)} + \mathcal{A}_K \right\}. \tag{33}$$

Consequently,

$$\alpha_0\ell \le \frac{2e^u\Delta\ell^2}{u(u+\Delta\ell)} + \mathcal{E}_E(u), \tag{34}$$

where

$$\mathcal{E}_E(u) = 4B_E + 32d\log(2d) + 11d + 4d\log(2+B_E) + 2e^u\mathcal{A}_K. \tag{35}$$

Proof. Take $h = u/\ell \in (0,1]$. Lemma 3.2, retaining the given zero, gives

$$-\frac{\zeta_K'}{\zeta_K}(1+h) \le \frac{1}{h} - \frac{1}{h+\Delta} + 1 + \tfrac{1}{2}\mathcal{H}_K(1,0) = \frac{\Delta}{h(h+\Delta)} + \mathcal{A}_K.$$

Here $1 < 1+h \le 2$, and $1/(1+h) \le 1$ bounds the pole term at zero. The nonnegative logarithmic-derivative series and $(\mathrm{N}\mathfrak{p})^{-h} \ge e^{-u}$ for $\mathrm{N}\mathfrak{p} \le X$ give (33). Now combine (32), Lemma 5.4, and Lemma 5.2. Since $d/n = 2$,

$$\alpha_0 \ell \le \frac{d}{n} \sum_{\mathrm{N}\mathfrak{p} \le X} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} + 4B_E + 32d\log(2d) + 6d + d\{5 + 4\log(2 + B_E)\}.$$

Insert (33), retaining the denominator $h(h+\Delta)$. □

Lemma 5.6 (A uniform error bound at $u = 2$)**.** *For the preceding fields,*

$$\mathcal{E}_E(2) \le 80H. \tag{36}$$

Proof. The series for e gives

$$e \le \frac{65}{24} + \frac{1}{100} < \frac{11}{4}, \qquad e^2 < 8.$$

Indeed the tail starting at $1/5!$ is at most $(1/120)\sum_{j\ge 0} 6^{-j} = 1/100$. Since $\log(2 + B_E) \le \log(2d) + B_E/d$,

$$\mathcal{E}_E(2) \le 8B_E + 36d\log(2d) + 11d + 8\mathcal{H}_K(1,0) + 16.$$

Write $U = n\log(2n)$. Using $B_E \le 4L + 3n\log 2$ and $d = 2n$,

$$\mathcal{E}_E(2) \le 32L + 72U + (96\log 2 + 22)n + 8\mathcal{H}_K(1,0) + 16.$$

Now $H = \mathcal{H}_K(1,0) + U$ and $\mathcal{H}_K(1,0) \ge L + 4n + 1$, so

$$\begin{aligned} 80H - \mathcal{E}_E(2) &\ge 72\mathcal{H}_K(1,0) - 32L + 8U - (96\log 2 + 22)n - 16 \\ &\ge 40L + 8U + (266 - 96\log 2)n + 56 > 0. \end{aligned}$$

□

5.4 Choice of the final prime cutoff

In this subsection, we choose the logarithmic-derivative parameter and an integer prime cutoff in the comparison of Section 5.3, proving Theorem 1.1. We retain the selected embeddings and monomial weights and take $D = 16d^2X$; the logarithmic-derivative parameter is chosen separately from the cutoff.

Proposition 5.7 (Real-zero bound in the gamma height)**.** *Every real zero $\beta \in (0,1)$ of the Dedekind zeta function of an arbitrary number field satisfies*

$$(1-\beta)\mathcal{H}_K^+(1,0) \ge \frac{1}{1370}. \tag{37}$$

Proof. Choose

$$k = \left\lceil \frac{176H}{\log 2} \right\rceil, \qquad X = 2^k, \qquad \ell = \log X, \qquad t = \ell/H.$$

Since $H \geq n\log(2n) + 4n + 1 \geq 5$,

$$176 \leq t \leq 176 + \frac{\log 2}{H} < 177, \qquad \ell \geq 176n\log(2n) > 2\log d, \qquad \ell > 2.$$

Thus $X > d^2$ and $u = 2$ is admissible in Lemma 5.5. Lemmas 5.5 and 5.6 give

$$\alpha_0\ell - 80H \leq \frac{e^2\Delta\ell^2}{2+\Delta\ell} \leq \frac{8\Delta\ell^2}{2+\Delta\ell}.$$

Putting $y = \Delta H$ and noting $\alpha_0 t - 80 \geq 85 > 0$, we obtain

$$y \geq \frac{2(\alpha_0 t - 80)}{t\{(8-\alpha_0)t + 80\}}. \tag{38}$$

For $176 \leq t \leq 177$, the right side is at least $1/1370$ precisely when

$$P(t) = -113t^2 + 39820t - 3507200 \geq 0.$$

Since $P'' = -226$ and

$$P(176) = 832, \qquad P(177) = 763,$$

concavity gives $P(t) \geq 763 > 0$ on $[176, 177]$. □

Remark 5.8 (The cutoff choice)**.** The choices $u = 2$ and $176 \leq (\log X)/H < 177$ give an admissible constant, without an optimality claim for the parameters or the method. The $n\log n$ dependence remains.

Proof of Theorem 1.1. For the principal character of K, the real-gamma parameters are zero or one. Thus

$$\mathcal{H}_K(1,0) \leq \log D_K + n\log 4 + 4n + 1 \leq \log D_K + 6n + 1, \qquad \mathcal{H}_K^+(1,0) \leq \mathcal{B}_K.$$

Apply Proposition 5.7. □

6 Hecke zero-free regions

In this section, we deduce Hecke zero-free regions from the real-zero bound of Section 5 and the positivity estimates of Section 3. We then give conductor formulations and extend the statements to imprimitive characters and real norm twists.

6.1 Quadratic characters and the general theorem

In this subsection, we transfer the real-zero bound of Section 5.4 to quadratic Hecke characters by class-field factorization. Norm twists and the positivity estimates of Section 3.2 then prove Theorem 1.3.

Corollary 6.1 (Quadratic characters in the gamma height)**.** *For every primitive quadratic Hecke character of every number field,*

$$L(\beta,\xi) = 0, \quad 0 < \beta < 1 \quad \Longrightarrow \quad (1-\beta)\mathcal{H}^+(\xi,0) \geq \frac{1}{4110}.$$

Proof. Let F/K be the quadratic class field of the nontrivial order-two character ξ. The conductor–discriminant formula [Mil20, Chapter V, Theorem 3.27] gives $\mathfrak{d}_{F/K} = \mathfrak{f}_\xi$ for the finite conductor, including primes above 2. Thus $\log D_F = \log D_K + \log A_\xi$. The Euler factors give $\zeta_F = \zeta_K L(s,\xi)$, including the ramified primes. The real-gamma parameters concatenate: a real place contributes $0, 0$ when it splits and $0, 1$ when it becomes complex, while a complex place contributes two copies of $0, 1$. Hence

$$\mathcal{H}_F(1,0) = \mathcal{H}_K(1,0) + \mathcal{H}_K(\xi,0) - 1 \le 2\mathcal{H}_K(\xi,0) - 1.$$

The inequality uses $\mathcal{H}_K(1,0) \le \mathcal{H}_K(\xi,0)$: the real parities of ξ are zero or one, and its complex parameters agree with those of the principal character.

The extra degree term satisfies

$$\mathcal{H}_F^+(1,0) \le 2\mathcal{H}_K^+(\xi,0) + 2n\log 2 \le 3\mathcal{H}_K^+(\xi,0).$$

Apply Proposition 5.7 to the zero of ζ_F. □

Proof of Theorem 1.3. Since $\mathcal{H}^+(\chi,t) \ge 5$, the proposed region lies in $\operatorname{Re} s > 0$. The Euler product and boundary nonvanishing recalled in Section 3 therefore reduce the proof to a nontrivial zero $\beta + it$ with $0 < \beta < 1$. If χ is not an imaginary norm twist of a real character, apply Proposition 3.3. Otherwise write $\chi = \xi_\tau$, where ξ is principal or quadratic, and put $u = t + \tau$. Then

$$L(\beta + it, \chi) = L(\beta + iu, \xi), \qquad \mathcal{H}^+(\chi,t) = \mathcal{H}^+(\xi,u).$$

If $u \ne 0$, Proposition 3.4 applies. If $u = 0$, use Proposition 5.7 for principal ξ, and Corollary 6.1 for quadratic ξ. The respective lower bounds for $(1-\beta)\mathcal{H}^+$ are

$$\frac{1}{112}, \qquad \frac{1}{1000}, \qquad \frac{1}{1370}, \qquad \frac{1}{4110},$$

the smallest of which is $1/4110$. This proves the strict zero-free region in Theorem 1.3. □

6.2 Finite-order characters and analytic conductors

In this subsection, we reformulate the theorem proved in Section 6.1 for finite-order characters and analytic conductors. We use the conductor and gamma conventions of Section 2.1.

Corollary 6.2 (Finite-order characters)**.** *For a primitive finite-order Hecke character over a degree-n field, the direct argument gives the zero-free region*

$$\sigma > 1 - \frac{1}{4110\{\log(D_K \mathrm{N}\mathfrak{f}_\chi) + n\log(4+|t|) + n\log(2n) + 4n + 1\}}.$$

Principal poles are allowed. The constants are uniform as K and χ vary with this displayed dependence.

Proof. The gamma parameters of a finite-order character are zero or one. Thus $\mathcal{H}^+(\chi,t)$ is at most the height in braces. The height comparison therefore gives the stated region by Theorem 1.3. □

There is also an explicit coefficient at the height $\mathcal{H}$, valid for every degree:

$$L(s,\chi) \ne 0 \quad \text{if} \quad \sigma > 1 - \frac{1}{4110\{1 + \frac{1}{5}\log(2n)\}\mathcal{H}(\chi,t)}. \tag{39}$$

Indeed $\mathcal{H}(\chi,t) \geq n\log 3+4n+1 \geq 5n$, so $\mathcal{H}^+ \leq \{1+\frac{1}{5}\log(2n)\}\mathcal{H}$. The coefficient in (39) deteriorates logarithmically with n.

In the range $n\log(2n) \leq \mathcal{H}(\chi,t)$ one can omit the additional degree term and lose a factor two: the zero-free width is at least $1/(8220\mathcal{H}(\chi,t))$. This is an explicit range condition, not a degree-independent assertion with the height $\mathcal{H}$ for every field.

For a standard analytic-conductor formulation put

$$\mathfrak{C}(\chi,t) = D_K \mathrm{N}\mathfrak{f}_\chi \prod_{j=1}^{n}(3+|\mu_j+it|).$$

Set $\mathfrak{C}^+(\chi,t) = (2n)^n\mathfrak{C}(\chi,t)$. Since $\log\mathfrak{C}^+ \geq n\log 3$ and $4n+1 \leq 5n$, we have $\mathcal{H}^+(\chi,t) \leq 6\log\mathfrak{C}^+(\chi,t)$. Thus a simpler, slightly weaker consequence is

$$L(s,\chi) \neq 0 \quad \text{if} \quad \sigma > 1 - \frac{1}{24660\,\log\mathfrak{C}^+(\chi,t)}, \tag{40}$$

with the usual principal poles permitted.

6.3 Imprimitive characters and real norm twists

In this subsection, we extend the regions of Sections 6.1–6.2 to imprimitive characters and real norm twists. The norm convention of Section 2.1 fixes the direction of translation.

For a deleted Euler factor of a unitary character,

$$1-\chi(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s} = 0, \quad |\chi(\mathfrak{p})| = 1 \quad \Longrightarrow \quad \operatorname{Re} s = 0.$$

Hence deleting Euler factors does not affect the stated regions in $\operatorname{Re} s > 0$.

For a continuous quasicharacter with primitive unitary part, write in the ideal-norm convention

$$\psi(\mathfrak{a}) = \chi(\mathfrak{a})(\mathrm{N}\mathfrak{a})^{-r}, \qquad r\in\mathbb{R}, \quad \chi \text{ primitive unitary.}$$

Then

$$L(s,\psi) = L(s+r,\chi), \qquad \operatorname{Re} s > 1-r-\frac{1}{4110\,\mathcal{H}^+(\chi,\operatorname{Im} s)} \quad \Longrightarrow \quad L(s,\psi)\neq 0,$$

with the translated principal poles permitted. For an imprimitive unitary part, apply this to its primitive inducing character and restore the deleted Euler factors. Their zeros translate from $\operatorname{Re} s = 0$ to $\operatorname{Re} s = -r$, outside the displayed region.

7 Further comparison with Banks and earlier work

In this section, we compare the arithmetic argument of Sections 4–5 with Banks' estimates and with earlier half-plane results. The comparisons distinguish the explicit field dependence of the present proof from stronger nonvanishing conclusions obtained by other methods.

7.1 What follows by normal closure

In this subsection, we justify the normal-closure deduction stated in Section 1.3 and compare its degree dependence with Theorem 1.1. Lemma 7.1 supplies the required discriminant estimate, including wild ramification.

Lemma 7.1 (Discriminant of a normal closure)**.** *Let $E/\mathbb{Q}$ be a number field, let L be its normal closure, and put $m = [L : \mathbb{Q}]$. Then*

$$\log D_L \le \frac{m}{2} \log D_E. \tag{41}$$

This is a standard consequence of the conductor–discriminant formula; we include the short proof.

Proof. For a complex representation W of a finite group and a subgroup J, write $W^J = \{w \in W : gw = w \text{ for all } g \in J\}$ and $\operatorname{codim} W^J = \dim W - \dim W^J$. Put $G = \mathrm{Gal}(L/\mathbb{Q})$ and let V be its permutation representation on the embeddings of E into L. This representation is faithful, because L is the normal closure. For a nontrivial subgroup $J \subset G$ put $c_J = \dim V - \dim V^J$. If $c_J = 1$, the action has precisely one nontrivial orbit, of size two; faithfulness forces $|J| = 2$. Otherwise $c_J \ge 2$. In either case

$$1 - |J|^{-1} \le \tfrac{1}{2} c_J.$$

For the regular representation R_G, therefore,

$$\operatorname{codim} R_G^J = m(1 - |J|^{-1}) \le \tfrac{m}{2} \operatorname{codim} V^J.$$

It also holds for $J = \{1\}$. For a decomposition group at p with lower ramification groups G_i (G_0 is inertia), let $a_p(W)$ be the exponent of p in the Artin conductor of a complex representation W of G. The local conductor formula is

$$a_p(W) = \sum_{i \ge 0} \frac{|G_i|}{|G_0|} \operatorname{codim} W^{G_i}.$$

Apply the preceding inequality to each G_i. These weights are nonnegative, including the wild groups, so $a_p(R_G) \le (m/2) a_p(V)$. The conductor-discriminant identities are $a_p(R_G) = v_p(D_L)$ and $a_p(V) = v_p(D_E)$. Summing against $\log p$ proves the assertion, including wild ramification. □

Proof of (9). Let M be the normal closure of K, and put $m = [M : \mathbb{Q}] \le n!$. Since M/K is Galois, the Aramata–Brauer theorem [Bra47] makes ζ_M/ζ_K entire. A real zero $\beta \in (0, 1)$ of ζ_K is therefore a zero of ζ_M. Apply (8) and Lemma 7.1:

$$c_{\mathrm{B}} \le (1 - \beta) \log D_M \le \frac{m}{2} (1 - \beta) \log D_K.$$

Consequently

$$(1 - \beta) \log D_K \ge \frac{2c_{\mathrm{B}}}{m} \ge \frac{2c_{\mathrm{B}}}{n!},$$

as asserted. □

For $K \ne \mathbb{Q}$, the two general guarantees are

$$\begin{aligned} &\text{Normal-closure deduction:} && (1 - \beta) \log D_K \ge 2c_{\mathrm{B}}/n!, \\ &\text{Theorem 1.1:} && (1 - \beta) \log D_K \ge \frac{1}{1370 \left(1 + \dfrac{n \log(2n) + 6n + 1}{\log D_K}\right)}. \end{aligned}$$

For normal or almost-normal K, Banks already gives the stronger shape $c/\log D_K$. For general K the comparison above concerns degree dependence, not pointwise numerical superiority. It neither bounds Banks' fixed-field constant nor excludes a better deduction from the same arithmetic estimates.

Example 7.2. Let $K = \mathbb{Q}[x]/(x^{10} - x - 1)$, and let M be its normal closure. The polynomial $f = x^{10} - x - 1$ is irreducible modulo 17, hence over $\mathbb{Q}$. For an explicit finite-field check, in $\mathbf{F}_{17}[x]$ one has

$$x^{17^{10}} \equiv x \pmod{f}, \qquad \gcd(f, x^{17^2} - x) = \gcd(f, x^{17^5} - x) = 1.$$

These are the irreducibility criterion in degree 10, whose prime divisors are 2 and 5. The polynomial has discriminant $10^{10} + 9^9 = 10387420489 = 173 \cdot 60042893$, which is squarefree; hence it equals D_K. Osada's theorem [Osa87, Corollary 3] on $x^n - x - 1$ gives the Galois group, where S_{10} denotes the symmetric group on ten letters. Thus

$$[K : \mathbb{Q}] = 10, \qquad D_K = 10387420489, \qquad \mathrm{Gal}(M/\mathbb{Q}) \simeq S_{10}, \qquad [M : \mathbb{Q}] = 10!.$$

Theorem 1.1 gives the real-zero gap

$$w_{\mathrm{arith}} = \frac{1}{1370\{\log D_K + 10 \log 20 + 61\}} \approx 6.40168 \times 10^{-6}.$$

For the admissible normal-field constant furnished by [Ban26a, equation (3.3)], one has $c_{\mathrm{B}} \le 1/4$. The specified normal-closure deduction therefore gives the width

$$w_{\mathrm{cl}} = \frac{2c_{\mathrm{B}}}{10! \log D_K} \le \frac{1}{2 \cdot 10! \log D_K} \approx 5.97413 \times 10^{-9}.$$

For the same c_{B}, Banks' direct width on M is

$$w_{\mathrm{Banks}} = \frac{c_{\mathrm{B}}}{\log D_M}.$$

The discriminant tower formula for M/K gives

$$D_M = D_K^{[M:K]} \mathrm{N}_{K/\mathbb{Q}} \mathfrak{d}_{M/K} \ge D_K^{10!/10},$$

where $\mathfrak{d}_{M/K}$ is the relative discriminant ideal. Hence

$$w_{\mathrm{Banks}} \le \frac{10c_{\mathrm{B}}}{10! \log D_K} = 5w_{\mathrm{cl}}.$$

Consequently,

$$w_{\mathrm{arith}} > 1071 w_{\mathrm{cl}} \ge \frac{1071}{5} w_{\mathrm{Banks}} = 214.2\, w_{\mathrm{Banks}}.$$

In particular, $w_{\mathrm{arith}} > 214.2\, w_{\mathrm{Banks}}$. This compares with Banks' normal-field theorem applied to M, using the same admissible constant c_{B}; it does not compare with Banks' undetermined fixed-base constant $c_{\mathrm{B}}(K)$ or assert the existence of a real zero.

7.2 The arithmetic estimates

In this subsection, we compare Banks' prime-sum estimate [Ban26a, Theorem 1.5] with the determinant construction of Section 4 and its specialization in Section 5.1. We then distinguish the normal-closure deduction of Section 7.1 from Banks' direct results and give a quadratic example in which Banks' explicit bound is stronger.

With L/k and H_0 in place of Banks' K/k and H, let L/k be Galois with group G, let $1 \in S \subset G$, and let λ_L bound a set of independent integers spanning L. For a prime $\mathfrak{p}$ of k unramified in L, put

$\delta_{\mathfrak{p}} = |S \cap C_{\mathfrak{p}}|/|C_{\mathfrak{p}}|$, where $C_{\mathfrak{p}}$ is its Frobenius class in G; set $\delta_{\mathfrak{p}} = 0$ at ramified primes. For an integer $H_0 \geq 2$, Banks' Theorem 1.5 gives

$$\sum_{\mathrm{N}\mathfrak{p} \leq X} \delta_{\mathfrak{p}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \leq \left(1 + \frac{|S| - 1}{H_0}\right) \left(\frac{|S|}{|G|} \log X + [k : \mathbb{Q}] \log \lambda_L\right) + O_{[L:\mathbb{Q}], H_0}(1), \tag{42}$$

for

$$X \geq \max\{X_0([L : \mathbb{Q}], H_0), (\log D_L)^{[L:\mathbb{Q}]}\}.$$

Here $X_0([L : \mathbb{Q}], H_0)$ is a threshold depending only on the displayed parameters. This is a uniform arithmetic estimate with degree-dependent errors and thresholds. The proof of Banks' Proposition 4.2 also treats arbitrary selected embeddings of a possibly nonnormal field.

Here Lemma 4.4 uses small independent integers of E, not of its normal closure M. The exact ratios (22), the bound $\lambda_E \leq D_E^{2/d}$, and the separate choices $H_0 = 16d^2$, $D = 16d^2 X$ give

$$\sum_{p \leq X} \delta_p \frac{\log p}{p} \leq \left(1 - \frac{15}{16d}\right) \log X + \frac{4}{d} \log D_E + 32 \log(2d), \qquad X \in \mathbb{Z}, \quad X \geq d^2.$$

The lower comparison in Section 5 uses $1 - \delta_p \leq r_K(p)/n$ for $p \nmid D_E$, together with rational-prime mass and the separate ramification bound. No Mertens constant of K enters.

These estimates display the dependence on the varying field. The determinant comparison follows [Ope26b, Ban26b]; its injectivity step uses Masser's theorem as in [Ker26] and the selected-embedding calculation of [Ban26a].

The normal-closure comparison of Section 7.1 uses two successive lower bounds. For $K \neq \mathbb{Q}$, with normal closure M and $m = [M : \mathbb{Q}]$, they are

$$1 - \beta \geq \frac{c_{\mathrm{B}}}{\log D_M} \geq \frac{2c_{\mathrm{B}}}{m \log D_K}.$$

The width w_{cl} in Example 7.2 is the last expression. The discriminant estimate in Lemma 7.1 may be strict, so this width need not equal the bound obtained from the actual value of D_M. The discriminant tower formula supplies the reverse comparison $w_{\mathrm{Banks}} \leq 5w_{\mathrm{cl}}$ in that degree-ten example, proving $w_{\mathrm{arith}} > 214.2\, w_{\mathrm{Banks}}$ for the same admissible c_{B}. This does not assert optimality of that constant or compare with every possible consequence of Banks' results.

Banks also gives direct real-zero bounds for quadratic extensions F/k and Galois extensions L/k of a fixed number field k [Ban26a, Corollary 1.2 and Theorem 1.3], with effective constants depending on k. Neither assertion requires passing to the normal closure over $\mathbb{Q}$. In particular, for $L/\mathbb{Q}$ Galois, (8) gives

$$1 - \beta \geq \frac{c_{\mathrm{B}}}{\log D_L},$$

without the additional degree term in Theorem 1.1. For almost-normal fields, see [Ban26a, Remark 3.4]. A numerical comparison requires the values of the relevant constants; Banks' earlier rational result gives the following explicit example.

Example 7.3 (A quadratic family with a stronger Banks bound)**.** Let χ be a primitive nonprincipal real Dirichlet character of conductor $q \geq 10^{10}$, and let F be its associated quadratic field, so $D_F = q$. Since $\zeta_F = \zeta_{\mathbb{Q}} L(s, \chi)$ and $\zeta_{\mathbb{Q}}$ has no real zero in $(0, 1)$, every real zero $\beta \in (0, 1)$ of ζ_F is a zero of $L(s, \chi)$. Banks' explicit theorem [Ban26b, Theorem 1.1] and Theorem 1.1 give, respectively,

$$w_{\mathrm{Banks}} = \frac{3}{1000 \log q}, \qquad w_{\mathrm{arith}} = \frac{1}{1370\{\log q + 2 \log 4 + 13\}}.$$

Thus

$$\frac{w_{\mathrm{Banks}}}{w_{\mathrm{arith}}} = \frac{411}{100}\left(1 + \frac{2\log 4 + 13}{\log q}\right) > \frac{411}{100} = 4.11.$$

Banks' bound is therefore numerically stronger in this family. With the finite computation cited in [Ban26b, Theorem 1.1], the same comparison holds for every $q \geq 3$.

7.3 Earlier half-plane results

In this subsection, we recall earlier half-plane results and compare their consequences with the regions of Section 6. These results are not used in the arithmetic proof of Sections 4–5. For the historical discussion, see [Liu26c, Section 1.4].

OpenAI's cubic-theta argument [Ope26a] gives

$$L_F(s,\eta) \neq 0 \quad (\mathrm{Re}\, s > 7/8), \qquad F = \mathbb{Q}(\sqrt{-3}),$$

for every finite-order Hecke character η of F, and the same half-plane for every Dirichlet L-function, with principal poles permitted. Its shorter argument [Ope26c] gives the boundary $11/12$ in these families. These analytic half-plane arguments are distinct from the interpolation-determinant exclusion argument of [Ope26b].

Varying the length parameters in the cubic-theta comparison, the first version of [Liu26a] gives

$$B_{\mathrm{r}} = \frac{34999}{40000} = 0.874975, \qquad B_{\mathrm{alg}} = \frac{1507 - 2\sqrt{921}}{1653} = 0.874957069799\ldots.$$

Mermet [Mer26] and Wei [Wei26] report 0.8749570698. Chakhkiev [Cha26] reports the conditional boundary $10937/12500 = 0.87496$ under the analytic estimates specified there. Varying the moment parameter as well, the second version of [Liu26a, Theorem 27.1] gives

$$B_{\mathrm{new}} = 0.874957019420098946\ldots.$$

Xu and Sang [XS26] use an amplified average for the reflected numerator, obtaining $B_{\mathrm{XS}} = 8748867/10^7 = 0.8748867$ from the analytic estimates of [Ope26a]. Newton [New26, Introduction, pp. 1–4] gives the certified comparison boundary $B_{\mathrm{N}} = 43744332481/50000000000 = 0.87488664962$ under the analytic hypotheses and parameter restrictions stated there. All these refinements concern the Eisenstein field and the corresponding Dirichlet family; that setting alone does not imply a theorem over arbitrary number fields.

For arbitrary number fields, Cureton–Thorner [CT26, Theorem 1.1 and Corollary 1.2] give nonvanishing in $\mathrm{Re}\, s > 31/32$ for Dedekind zeta functions and finite-order Hecke L-functions, wherever holomorphic. Following the amplified comparison of [XS26] and the general-field argument of [CT26], the author [Liu26b, Theorem 17.1] obtains the uniform boundary

$$B_\flat = 0.874886647039964\ldots, \qquad 133488B_\flat^4 - 376596B_\flat^3 + 384534B_\flat^2 - 167541B_\flat + 26231 = 0.$$

The general-field metaplectic argument is carried out over $K(\zeta_{12})$, where $\zeta_{12} = e^{2\pi i/12}$, and then descended to K by class-field factorization.

The subsequent refinement [Liu26c, Theorem 1.1] gives, for every finite-order Hecke character over every number field,

$$L_K(s,\chi) \neq 0 \quad (\mathrm{Re}\, s > B_{\mathrm{tw}}), \qquad B_{\mathrm{tw}} = 0.87405850379185900499\ldots, \tag{43}$$

with the principal pole permitted. An amplified Gauss norm and contour displacement, with an iteration of the character-counting estimate, first give $B_* = 0.87446830769353045955\ldots$ [Liu26c,

Theorem 9.2]. Permitting complete prime removal with a positive length excess enlarges the norm and gives $B_{\mathrm{ex}} = 0.87407284766682790777\ldots$ [Liu26c, Theorem 1.2]. Selected unit primes entering the Gauss sums as a moving cubic character then give B_{tw}.

The selected character-sum estimates and finite moment inductions of Bebel, ArviFoxy, Sil and collaborators are extended from Dirichlet characters to ideal characters in [Liu26d] (for these sources and the same limiting boundary in the Dirichlet family, see [Liu26d, Section 1.3]). Theorem 1.1 there gives, for every finite-order Hecke character over every number field,

$$L_K(s,\chi) \neq 0 \quad (\operatorname{Re} s > B_2), \qquad B_2 = \frac{777 - \sqrt{1081}}{852} = 0.873381966613207519929\ldots, \tag{44}$$

with the principal pole permitted; here $1278B_2^2 - 2331B_2 + 1061 = 0$. Its proof uses the ordinary moment of [Liu26d, Theorem 4.9] for smooth sums $S_{n_1,1}, S_{n_2,2}$ and a product Q of sufficiently fine original prime factors of total length z. With $d = n_1 + n_2$, two finite inductions improve the length condition as

$$d + 6\kappa_1 z \le M \quad \longrightarrow \quad d + 3z \le M \quad \longrightarrow \quad d + 2z \le M, \qquad \kappa_1 = 0.749774.$$

The larger prime length improves both the zero-detector count and the averaged numerator of the cubic-theta comparison, and displacing the numerator contour adds a second saving. By [Liu26d, Theorem 6.9], B_2 is the exact infimum of the boundaries supplied by this comparison system. The uniform boundaries therefore satisfy

$$B_2 < B_{\mathrm{tw}} < B_* < B_\flat < \frac{31}{32}.$$

For completeness, even the boundary $31/32$ implies a stronger unitary region than Theorem 1.3. Indeed, for a principal or quadratic character ξ, every nontrivial zero $\beta + iu$ then satisfies

$$1 - \beta \ge \frac{1}{32}, \qquad (1-\beta)\mathcal{H}(\xi,u) \ge \frac{5}{32},$$

since $\mathcal{H} \ge 5$. Imaginary norm twists preserve the height. Combining this consequence with Proposition 3.3 gives, for every primitive unitary Hecke character,

$$L(s,\chi) \neq 0 \quad \text{if} \quad \sigma > 1 - \frac{1}{112\,\mathcal{H}(\chi,t)},$$

with the usual principal poles permitted. The boundaries $B_\flat$, B_*, B_{tw} and B_2 imply the same consequence.

Theorems 1.1 and 1.3 follow here from the arithmetic estimates of Sections 4–5 and classical positivity. Their nonvanishing conclusions also follow from the preceding half-plane results together with classical positivity.

A Numerical inequalities

In this appendix, we collect the numerical inequalities used in Sections 1.1, 5.1, 5.4, and 6.1. All inequalities hold in every degree, including the smallest permitted parameter values.

The degree and discriminant. Minkowski's bound implies

$$D_K \ge \left(\frac{\pi}{4}\right)^n \left(\frac{n^n}{n!}\right)^2.$$

For $a_n = (\pi/4)^n (n^n/n!)^2 e^{-n/2}$ we have $a_3 > 1$ and

$$\frac{a_{n+1}}{a_n} = \frac{\pi}{4\sqrt{e}} \left(1 + \frac{1}{n}\right)^{2n} \geq \frac{\pi}{\sqrt{e}} > 1.$$

Thus $\log D_K \geq n/2$ for $n \geq 3$. For $n = 2$, the integer $D_K \geq 3$ gives the same inequality. Also $\log D_K > 1$ for $K \neq \mathbb{Q}$. Consequently

$$\frac{\mathcal{B}_K}{\log D_K} \leq 14 + 2\log(2n) \leq 16\log(2n),$$

which proves (2).

The polynomial degree and prime cutoff. For $D \geq 1$, $\log(D+1) \leq \sqrt{D}$: the difference is positive at $D = 1$ and its derivative is $(\sqrt{D}-1)^2/(2\sqrt{D}(D+1)) \geq 0$. With $d \geq 2$, $s = d-1$, $H_0 = 16d^2$, and $D = 16d^2X$, the condition $X \geq d^2$ gives $D \geq 16d^4 \geq 2H_0$ and

$$\left(1 + \frac{d-2}{16d^2}\right)\frac{d-1}{d} \leq 1 - \frac{15}{16d}, \qquad \frac{2d\vartheta(X)}{D} \leq \frac{1}{2d}, \qquad \frac{d(d-1)}{D}\log(D+1) \leq \frac{1}{4}.$$

The last inequality follows from $\sqrt{D} \geq 4d^2$. The remaining height and small-prime contributions are bounded by

$$5 + 23\log 2 + 14\log d \leq 24 + 14\log d \leq 32\log(2d).$$

The last step follows from $d \geq 2$ and $\log 2 > 1/2$.

The error bound and integer cutoff. Put $H = \mathcal{H}_K^+(1,0)$, with E, d, B_E as in (29). The bound $\log(2+B_E) \leq \log(2d) + B_E/d$ and $e^2 < 8$ give $\mathcal{E}_E(2) \leq 80H$ in Lemma 5.6. For $k = \lceil 176H/\log 2 \rceil$ and $X = 2^k$,

$$176 \leq t = (\log X)/H < 177, \qquad X > d^2, \qquad \log X > 2.$$

The exact polynomial inequality in Proposition 5.7 is

$$P(t) = -113t^2 + 39820t - 3507200, \qquad P(176) = 832, \quad P(177) = 763, \quad P''(t) = -226.$$

Concavity gives $P(t) \geq 763 > 0$ throughout $[176, 177]$, proving $(1-\beta)H \geq 1/1370$. No small-degree or small-discriminant exception is needed. The cutoff choice is discussed in Remark 5.8.

For a quadratic class field F/K,

$$\mathcal{H}_F^+(1,0) \leq 2\mathcal{H}_K^+(\xi,0) + 2n\log 2 - 1 \leq 3\mathcal{H}_K^+(\xi,0).$$

Since $3 \cdot 1370 = 4110$, the quadratic real-zero denominator is 4110. The nonreal and real-off-axis bounds have denominators 112 and 1000 at the smaller height $\mathcal{H}$. Consequently 4110 is also a valid uniform denominator. The range and conductor reformulations use $2 \cdot 4110 = 8220$ and $6 \cdot 4110 = 24660$, respectively.

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