FINITE LATTICE DATA: OBSTRUCTIONS AND RECONSTRUCTION BY GRAM THETA SERIES

P. SHI

Abstract. A finite collection of lattice-vector lengths probes a positive-definite Gram matrix through rank-one linear functionals. We use this observation map to give a local criterion for non-identifiability and to construct an exact three-dimensional collision at a closed squared-length cutoff of one: two non-isometric lattices have identical complete truncated vector-length data, although their Gram determinants are different. A signed-cycle construction gives a higher-dimensional family whose complete rank-one length multisets agree at every cutoff below an explicit threshold, with an exact first-shell discrepancy. We then turn to multi-vector data. The full degree-n Gram theta data determines every rank-n Euclidean lattice, by a basis-witness and covolume-index argument. On each compact subset of lattice moduli space, the same argument yields a finite cutoff; a determinant-one cusp family shows that the basis-witness cutoff supplied by this method cannot be uniform on the full moduli space. Finally, a published rank-32 theta-series coincidence, together with a minimum-shell spanning identity, is stabilized by orthogonal direct sums to every rank $n \geq 32$. If $k(n)$ denotes the least degree of full Gram theta data that universally determines rank-n Euclidean lattices, we obtain $4 \leq k(n) \leq n$ for $n \geq 32$. The paper separates proved statements, cited lattice-theoretic input, and computational checks; it does not claim a solution of the exact minimum-degree problem.

Keywords. Euclidean lattices; Gram matrices; finite observations; Siegel theta series; lattice reconstruction; geometry of numbers.
2020 Mathematics Subject Classification. Primary 11H06, 11F27; Secondary 11E20, 15A63.

hexagon:2611.00078v1

Contents

1. Introduction 3
2. Finite observations as a linear map 4
3. An exact three-dimensional collision 6
4. A signed-cycle family with an exact first discrepancy 6
5. A rank-two prefix resolves a rank-one collision 7
6. Full Gram theta data and a basis witness 9
7. Finite reconstruction on compact regions 9
8. A degree-three obstruction stabilized to all ranks at least 32 10
9. Discussion and open problems 11
Acknowledgements 12
References 12

1. Introduction

A positive-definite quadratic form records the squared lengths of all integer vectors. It is natural to ask how much of that information is needed to recover the form, or the lattice it defines. There are at least three distinct data sets in this question:

(i) a finite set of labelled values $v^{\mathsf{T}}Gv$ for prescribed vectors $v \in \mathbb{Z}^n$;
(ii) the unlabelled multiset of all vector lengths below a cutoff;
(iii) the multi-vector Gram data recorded by a Siegel theta series of degree g.

These data sets are related, but they should not be conflated. In particular, a finite-cutoff collision does not imply that the complete ordinary theta series fails to determine a lattice. In rank three, Schiemann proved that positive-definite ternary quadratic forms are determined by their complete theta series [6]. Our three-dimensional example below concerns only a short finite prefix.

The purpose of this paper is to organize a reconstruction problem into complementary local and global statements. The local part is controlled by the span of the rank-one matrices vv^{T}. The global part uses higher-degree Gram data, which records mutual inner products among tuples rather than only individual lengths. A compactness argument converts the global basis-witness proof into a finite cutoff on compact regions of moduli space. Finally, a known rank-32 example gives a stable obstruction to universal degree-three reconstruction in every larger rank.

We use the following threshold notation. Let $k(n)$ be the least positive integer g such that equality of complete degree-g Gram coefficient data forces isometry for every pair of rank-n Euclidean lattices. The coefficient-data convention for arbitrary Euclidean lattices is made precise in section 6; for integral lattices it agrees with the Fourier-coefficient interpretation of the usual Siegel theta series.

The main results are as follows.

Theorem 1.1 (Local finite-observation obstruction)**.** *Let $S \subset \mathbb{Z}^n \setminus \{0\}$ be finite, let $d_n = n(n+1)/2$, and put*

$$r_S = \dim \operatorname{span}_{\mathbb{R}} \{vv^{\mathsf{T}} : v \in S\}.$$

If $r_S < d_n$, then at every positive-definite Gram matrix G there is a local family of positive-definite matrices of dimension at least $d_n - r_S$ that preserves all labelled observations $v^{\mathsf{T}}Gv$ for $v \in S$. Along this family, the determinant is nonconstant. Hence the family contains matrices arbitrarily close to G that are not integrally congruent to G.

Theorem 1.2 (A three-dimensional finite-cutoff collision)**.** *There are positive-definite Gram matrices*

$$G_0 = I_3, \qquad G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 1/3 & 1 \end{pmatrix}$$

whose complete closed rank-one cutoff data at squared-length cutoff $T = 1$ agree: on each side there are exactly six nonzero vectors of squared length 1 and no other vectors of squared length at most 1. Nevertheless,

$$\det G_0 = 1, \qquad \det G_1 = \frac{8}{9},$$

so the corresponding lattices are not isometric.

Theorem 1.3 (Signed-cycle cutoff family)**.** *For every $n \geq 3$ and $0 < \delta \leq 1/3$, let G_+ and G_- be the symmetric cycle Gram matrices with diagonal entries 1, with all cycle-edge entries equal to $+\delta$ for G_+, and with exactly one cycle-edge entry changed to $-\delta$ for G_-. Put $T_0 = n(1-2\delta)$. Then both matrices are positive definite, their complete unlabelled rank-one cutoff multisets agree for every $T < T_0$, and*

$$\#\{v \neq 0 : Q_{G_+}(v) = T_0\} - \#\{v \neq 0 : Q_{G_-}(v) = T_0\} = 2(-1)^n.$$

Moreover,

$$\det G_+ - \det G_- = 4(-1)^{n-1}\delta^n \neq 0.$$

Thus the pair is non-isometric and its first rank-one cutoff discrepancy occurs exactly at T_0*.*

Theorem 1.4 (Degree-n reconstruction)**.** *Let* L, L' *be Euclidean lattices of rank* n*. If their complete degree-*n *Gram theta data agree, then* L *and* L' *are isometric. In particular, the least universal determining degree* $k(n)$ *is well-defined and satisfies* $k(n) \leq n$*.*

Theorem 1.5 (Finite reconstruction on compact moduli regions)**.** *For every compact subset* K *of the moduli space of rank-*n *Euclidean lattices, there exists a finite constant* R_K *such that equality of the degree-*n *Gram theta data truncated to tuples whose individual squared lengths are at most* R_K *implies isometry for lattices represented in* K*.*

Theorem 1.6 (A stable degree-three obstruction)**.** *For every* $n \geq 32$*, there exist non-isometric rank-*n *Euclidean lattices whose Siegel theta series agree in degrees* 1*,* 2*, and* 3*. Consequently,*

$$\boldsymbol{4 \leq k(n) \leq n \qquad (n \geq 32).}$$

The rank-32 input in theorem 1.6 is due to Erokhin and is summarized by Oura and Ozeki: the degree-1, degree-2, and degree-3 Siegel theta series are unique among the even unimodular extremal lattices of rank 32, while there are at least one billion non-isometric such lattices [4, 2]. The minimum-shell identity used to stabilize this obstruction is also recorded by Oura and Ozeki, who attribute it to Venkov. Our contribution in this part is the direct-sum stabilization argument and its consequence for all ranks $n \geq 32$.

Scope. The paper concerns positive-definite Euclidean lattices and their Gram data. It does not establish a topological non-identifiability theorem for arbitrary spaces, nor does it identify finite primal length data with the Laplace spectrum of a general manifold. Such extensions require separate hypotheses and arguments.

2. Finite observations as a linear map

Let $V_n = \mathrm{Sym}_n(\mathbb{R})$, the vector space of real symmetric $n \times n$ matrices. Its dimension is

$$d_n = \dim V_n = \frac{n(n+1)}{2}.$$

A positive-definite matrix $G \in V_n$ defines a quadratic form

$$Q_G(v) = v^{\mathsf{T}} G v, \qquad v \in \mathbb{Z}^n.$$

It is the Gram matrix of the lattice with coordinate group $\mathbb{Z}^n$ and this metric. We write $\mathcal{M}_n = V_n^{++}/\operatorname{GL}_n(\mathbb{Z})$ for the moduli space of rank-n Euclidean lattices, where V_n^{++} is the positive-definite cone and $U \in \operatorname{GL}_n(\mathbb{Z})$ acts by $G \mapsto U^{\mathsf{T}} G U$. Two Gram matrices describe isometric lattices precisely when they are integrally congruent: $G' = U^{\mathsf{T}} G U$ for some $U \in \operatorname{GL}_n(\mathbb{Z})$. The squared covolume is $\det G$, so determinant is an isometry invariant.

Definition 2.1. For a finite labelled observation set $S \subset \mathbb{Z}^n \setminus \{0\}$, define

$$\mathcal{A}_S : V_n \longrightarrow \mathbb{R}^S, \qquad \mathcal{A}_S(H) = (v^{\mathsf{T}} H v)_{v \in S}.$$

The observation rank is

$$r_S = \operatorname{rank} \mathcal{A}_S = \dim \operatorname{span}\{v v^{\mathsf{T}} : v \in S\}.$$

The equality of the two expressions for r_S follows from the trace pairing $\langle A, B \rangle = \operatorname{tr}(AB)$ on V_n, since $v^{\mathsf{T}} H v = \operatorname{tr}(H v v^{\mathsf{T}})$.

Proof of theorem 1.1. By rank-nullity,

$$\dim \ker \mathcal{A}_S = d_n - r_S > 0.$$

Choose $0 \neq H \in \ker \mathcal{A}_S$. Then $v^\mathsf{T} H v = 0$ for every $v \in S$, and hence

$$Q_{G+tH}(v) = Q_G(v) \qquad (v \in S)$$

for every real t. Positive definiteness is an open condition, so $G + tH$ remains positive definite for all sufficiently small $|t|$.

It remains to show that the determinant varies. Set

$$A = G^{-1/2} H G^{-1/2}.$$

The matrix A is nonzero and symmetric. If its nonzero eigenvalues, counted with multiplicity, are $\lambda_1, \dots, \lambda_s$, then

$$\det(G + tH) = \det(G) \det(I + tA) = \det(G) \prod_{j=1}^{s} (1 + t\lambda_j).$$

This is a nonconstant polynomial in t, since its leading coefficient is nonzero. Thus values of t arbitrarily close to zero give a determinant different from $\det G$. Integral congruence preserves determinant, so these matrices cannot all be integrally congruent to G. The affine slice $G + \ker \mathcal{A}_S$ intersects the positive-definite cone in a relatively open neighborhood of G of dimension $d_n - r_S$. This proves the theorem. □

Corollary 2.2 (Closed finite-cutoff data)**.** *Fix $G \in V_n^{++}$ and $T \geq 0$, and define*

$$S_T(G) = \{v \in \mathbb{Z}^n \setminus \{0\} : Q_G(v) \leq T\}.$$

If $\operatorname{span}\{vv^\mathsf{T} : v \in S_T(G)\}$ is a proper subspace of V_n, then there are arbitrarily close positive-definite Gram matrices with the same complete labelled data below the cutoff T and with different determinants.

Proof. The set $S_T(G)$ is finite because G is positive definite. Apply theorem 1.1 to this set. It gives a nonzero H preserving every vector length in $S_T(G)$. We verify that no vector outside this set crosses the cutoff for sufficiently small t.

The set of values $Q_G(v)$ in any bounded interval is finite. Hence there is an $\eta > 0$ such that

$$Q_G(v) > T \quad \Longrightarrow \quad Q_G(v) \geq T + \eta.$$

For sufficiently small $|t|$, continuity of the quadratic forms and compactness of the unit sphere give a uniform comparison

$$Q_{G+tH}(x) \geq (1 - \varepsilon) Q_G(x) \qquad (x \in \mathbb{R}^n)$$

with $\varepsilon > 0$ chosen so that $(1 - \varepsilon)(T + \eta) > T$ (the case $T = 0$ is immediate). Thus every vector outside $S_T(G)$ remains outside the cutoff. Since all vectors in $S_T(G)$ retain their lengths, the full labelled cutoff data is unchanged. The determinant variation follows from theorem 1.1. □

Remark 2.3. The rank condition is sufficient, not necessary, for non-identifiability. If $r_S = d_n$, the linear kernel obstruction disappears, but that fact alone does not imply global uniqueness. It also does not settle uniqueness of unlabelled length multisets.

3. An exact three-dimensional collision

We now prove theorem 1.2. Let

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad G_1 = I_3 + \frac{1}{3}H.$$

The eigenvalues of H are $0, 1, -1$, so the eigenvalues of G_1 are $1, 4/3, 2/3$. In particular G_1 is positive definite, and

$$\det G_1 = 1 \cdot \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}.$$

For $x = (x_1, x_2, x_3)^{\mathsf{T}} \in \mathbb{Z}^3$ we have

$$Q_{G_1}(x) = x_1^2 + x_2^2 + x_3^2 + \frac{2}{3}x_2x_3.$$

The smallest eigenvalue of the lower-right 2×2 block is $2/3$, and the first coordinate has coefficient 1. Therefore

$$Q_{G_1}(x) \geq \frac{2}{3}(x_1^2 + x_2^2 + x_3^2).$$

If $Q_{G_1}(x) \leq 1$ and $x \neq 0$, then

$$x_1^2 + x_2^2 + x_3^2 \leq \frac{3}{2}.$$

The left side is a positive integer, so it equals 1. Thus x is one of $\pm e_1, \pm e_2, \pm e_3$. Each of these six vectors has $Q_{G_1}(x) = 1$. The same conclusion is immediate for $G_0 = I_3$. This proves that the two complete closed cutoff multisets coincide and consist of six copies of the squared length 1.

Since $\det G_0 = 1 \neq 8/9 = \det G_1$, the two Gram matrices cannot be integrally congruent. This proves theorem 1.2.

Remark 3.1 (What the example does and does not show). The example is stronger than a coincidence of a few selected lengths: it gives the entire closed rank-one cutoff multiset at $T = 1$. It is not a collision of the complete rank-one theta series. In fact, Schiemann's theorem states that a positive-definite ternary quadratic form is determined by its complete theta series [6]. The example therefore isolates a finite-information obstruction without contradicting the known complete-data result.

4. A signed-cycle family with an exact first discrepancy

We prove theorem 1.3. Vertex switching for signed graphs is a standard operation; signed-cycle spectral questions have been studied, for example, in [3]. The invariant considered here is different: we compare the unlabelled multiset of values of an integer quadratic form on lattice vectors, truncated by a norm cutoff, rather than the adjacency or Laplacian spectrum of the signed graph. Index the vertices of the cycle by $1, \ldots, n$, with indices read modulo n. Write $s_i^+ = 1$ for every edge and choose signs s_i^- with exactly one value equal to -1 and all others equal to 1. The two quadratic forms are

$$Q_\pm(x) = \sum_{i=1}^{n} x_i^2 + 2\delta \sum_{i=1}^{n} s_i^\pm x_i x_{i+1}, \qquad x_{n+1} = x_1.$$

For every real x, the inequality $2|x_ix_{i+1}| \leq x_i^2 + x_{i+1}^2$ gives

$$Q_\pm(x) \geq (1 - 2\delta)\|x\|^2.$$

Since $0 < \delta \leq 1/3$, the coefficient is positive, so both forms are positive definite.

Let $T_0 = n(1-2\delta)$. If $Q_\pm(x) < T_0$ for a nonzero integer vector x, then the preceding lower bound implies $\|x\|^2 < n$. Thus the support

$$\operatorname{supp}(x) = \{i : x_i \neq 0\}$$

is a proper subset of the vertices of the cycle. The graph induced on a proper subset of a cycle is a forest. On each tree component, one can choose signs $\varepsilon_i \in \{\pm 1\}$ recursively from a root so that

$$s_i^- \varepsilon_i \varepsilon_{i+1} = s_i^+$$

for every edge whose two endpoints belong to the support. The coordinate change $x_i \mapsto \varepsilon_i x_i$ on the support, with zero coordinates left at zero, is a bijection preserving the quadratic-form value. It gives a support-preserving bijection between the vectors for Q_+ and Q_- below T_0. Summing over supports proves equality of the complete unlabelled length multisets for every cutoff $T < T_0$.

At the boundary T_0, every proper-support contribution still matches by the same switching argument. A full-support vector satisfies

$$Q_\pm(x) \geq (1-2\delta)\|x\|^2 \geq n(1-2\delta) = T_0.$$

Equality is possible only if $x_i \in \{\pm 1\}$ for every i and

$$s_i^\pm x_i x_{i+1} = -1 \qquad (1 \leq i \leq n).$$

Multiplying these n equations gives

$$\prod_{i=1}^{n} s_i^\pm = (-1)^n,$$

since $\prod_i x_i^2 = 1$. Conversely, when this sign condition holds, the system has exactly two solutions, related by global sign reversal. The product of the edge signs is $+1$ for G_+ and -1 for G_-. Hence the number of full-support vectors at squared length T_0 is two on the positive side when n is even, and two on the negative side when n is odd. The net shell difference is therefore $2(-1)^n$. This is a net difference: for larger n, additional proper-support vectors may also lie on the shell, but their contributions match exactly.

Finally, consider the Leibniz expansion of the determinant. A nonzero permutation term that is not one of the two orientations of the full cycle uses cycle edges in transpositions or fixed points; every edge then occurs with even multiplicity, so the term is independent of the edge signs. The two oriented full-cycle permutations each contribute $(-1)^{n-1}\delta^n \prod_i s_i$. Therefore

$$\det G_+ - \det G_- = 2(-1)^{n-1}\delta^n(1-(-1)) = 4(-1)^{n-1}\delta^n.$$

This is nonzero, so determinant invariance under integral congruence shows that the two lattices are not isometric. The proof is complete.

Remark 4.1. The first-discrepancy statement concerns the complete unlabelled rank-one length multiset. It is stronger than equality of a selected finite list, but it does not say that the full infinite theta series agrees. The theorem also separates two mechanisms: switching on every proper support explains the coincidence below T_0, while the sign product around the full cycle creates the first unmatched shell.

5. A rank-two prefix resolves a rank-one collision

The preceding example shows that a finite rank-one cutoff can fail to identify a lattice. The next example illustrates a complementary point: retaining mutual inner products of pairs can separate two lattices even when their entire rank-one prefixes agree at the same cutoff.

Put $\delta = 1/20$ and define

$$G_+ = \begin{pmatrix} 1 & \delta & \delta \\ \delta & 1 & \delta \\ \delta & \delta & 1 \end{pmatrix}, \qquad G_- = \begin{pmatrix} 1 & -\delta & \delta \\ -\delta & 1 & \delta \\ \delta & \delta & 1 \end{pmatrix}.$$

Both matrices are positive definite. For example, the Gershgorin bound gives

$$x^{\mathsf{T}} G_\pm x \geq (1 - 2\delta)\|x\|^2 = \frac{9}{10}\|x\|^2.$$

Their determinants are

$$\det G_+ = 1 - 3\delta^2 + 2\delta^3 = \frac{3971}{4000}, \qquad \det G_- = 1 - 3\delta^2 - 2\delta^3 = \frac{3969}{4000}.$$

Thus the corresponding lattices are not isometric.

At the closed squared-length cutoff $T = 21/10$, the lower bound above shows that any nonzero integer vector with $Q_{G_\pm}(x) \leq T$ has $\|x\|^2 \leq 7/3$, hence $\|x\|^2 \leq 2$. Such a vector is either a signed coordinate vector or has exactly two nonzero coordinates, both equal to ± 1. The coordinate vectors contribute six occurrences of 1. For each of the three coordinate pairs, exactly two sign choices give squared length $2 - 2\delta = 19/10$, and two give squared length $2 + 2\delta = 21/10$. Consequently, on both sides the complete rank-one cutoff multiset is

$$\{1^{\,6}, (19/10)^{\,6}, (21/10)^{\,6}\}.$$

In particular, there are exactly 18 nonzero vectors on each side below the cutoff.

For an ordered pair (x, y), record its 2×2 Gram matrix

$$\mathrm{Gram}(x, y) = \begin{pmatrix} Q(x) & (x, y) \\ (x, y) & Q(y) \end{pmatrix}.$$

Consider the pair-Gram witness

$$B = \begin{pmatrix} 19/10 & 0 \\ 0 & 1 \end{pmatrix}.$$

For G_+, every vector x of squared length $19/10$ is supported on one of the three edges and has opposite signs on the two nonzero coordinates. The omitted coordinate vector is orthogonal to x. There are six such vectors x, and for each there are two choices of the signed omitted coordinate vector y. Thus

$$r^{(2)}_{G_+}(B) = 12.$$

For G_-, no vector of squared length $19/10$ is orthogonal to the omitted coordinate vector. Indeed, on the negative edge $(1, 2)$ such a vector has equal signs in coordinates $1, 2$, while on either positive edge its two nonzero coordinates have opposite signs; in each case the two inner-product contributions to the omitted coordinate have the same sign. Hence

$$r^{(2)}_{G_-}(B) = 0.$$

The full finite rank-two prefix therefore distinguishes this pair, although the rank-one prefixes at the same cutoff do not. An independent exact-rational enumeration of the 18 vectors on each side and the 324 ordered pairs on each side is included in the accompanying verification script.

Remark 5.1. This is a pair-specific finite separation result, not a universal theorem that genus two determines ternary lattices. It demonstrates that a finite multi-vector observable can contain information absent from the unlabelled one-vector length multiset at the same cutoff.

6. Full Gram theta data and a basis witness

For a rank-n Euclidean lattice L, and an ordered g-tuple $X = (x_1, \dots, x_g) \in L^g$, write

$$\mathrm{Gram}(X) = \big((x_i, x_j)\big)_{1 \le i,j \le g}.$$

The degree-g Siegel theta series is

$$\Theta_L^{(g)}(Z) = \sum_{(x_1,\dots,x_g) \in L^g} \exp\big(\pi i \, \mathrm{tr}\big(\mathrm{Gram}(X)Z\big)\big), \qquad Z \in \mathfrak{H}_g,$$

where $\mathfrak{H}_g$ is the Siegel upper half-space of degree g. For integral lattices, the Fourier coefficient at a positive-semidefinite half-integral matrix T counts ordered tuples with Gram matrix $2T$; see, for example, [4, Section 2.3]. For arbitrary Euclidean lattices, the reconstruction theorem below uses the formal coefficient function of occurring Gram matrices, without assuming integrality or modularity.

Definition 6.1. The complete degree-g Gram theta data of L is the coefficient function

$$B \longmapsto r_L^{(g)}(B) := \#\{X \in L^g : \mathrm{Gram}(X) = B\},$$

where B ranges over positive-semidefinite symmetric $g \times g$ matrices that occur as Gram matrices of g-tuples in L. Equality of complete degree-g Gram theta data means equality of all these coefficients.

Proof of theorem 1.4. Choose a basis $b_1, \dots, b_n$ of L, and let $B = \mathrm{Gram}(b_1, \dots, b_n)$. Since this tuple occurs in L^n, we have $r_L^{(n)}(B) > 0$. Equality of the degree-n data implies that there is an ordered tuple $y_1, \dots, y_n \in L'$ with Gram matrix B. As B is positive definite, the vectors y_i are linearly independent, so they generate a full-rank sublattice $M \subseteq L'$.

The equality of Gram matrices gives

$$\mathrm{covol}(M) = \mathrm{covol}(L).$$

Since M has finite index $a = [L' : M] \in \mathbb{N}$, the covolume-index formula yields

$$\mathrm{covol}(L) = a \, \mathrm{covol}(L').$$

Repeating the same argument with a basis of L' and using equality of the degree-n data in the reverse direction gives an integer $b \ge 1$ with

$$\mathrm{covol}(L') = b \, \mathrm{covol}(L).$$

Consequently $ab = 1$, and hence $a = b = 1$. Thus $M = L'$, so the tuple $y_1, \dots, y_n$ is a basis of L'. The basis correspondence preserves the Gram matrix and defines an isometry $L \cong L'$. □

Remark 6.2. The proof uses the complete coefficient data at degree n and the fact that a basis Gram matrix is positive definite. It does not show that degree two, degree three, or any fixed lower degree is sufficient in arbitrary rank. It also does not supply a uniform finite cutoff on the noncompact full moduli space.

7. Finite reconstruction on compact regions

Let $\mathcal{M}_n$ denote the moduli space of isometry classes of rank-n Euclidean lattices. For $R > 0$, write $\mathcal{D}_R^{(n)}(L)$ for the degree-n Gram data restricted to ordered tuples $(x_1, \dots, x_n)$ satisfying

$$(x_i, x_i) \le R \qquad (1 \le i \le n).$$

This is finite: a lattice has only finitely many vectors in a bounded ball.

Proof of theorem 1.5. We first record the compactness consequence needed in the argument. For every compact $K \subset \mathcal{M}_n$, there is a constant R_K such that every lattice class in K has a basis $b_1, \dots, b_n$ with $(b_i, b_i) \le R_K$ for all i.

To see this, fix a representative $G \in V_n^{++}$ of a class in K. Choose a bounded open neighborhood U_G of G whose closure is compact and contained in V_n^{++}. The quotient map $q : V_n^{++} \to \mathcal{M}_n$ is open, so $q(U_G)$ is a neighborhood of the chosen moduli class. Every class in $q(U_G)$ has a representative in U_G; for that representative, the standard coordinate basis has squared lengths given by the diagonal entries, which are uniformly bounded on $\overline{U_G}$. These neighborhoods cover K. Compactness gives a finite subcover, and taking the maximum of the finitely many bounds proves the assertion. This argument does not require a local section at orbifold points of the quotient.

Now let $L, L' \in K$ have equal truncated degree-n Gram data at cutoff R_K. Choose a basis of L whose basis-vector squared lengths are all at most R_K, and let B be its Gram matrix. This basis tuple is present in the truncated data of L', so L' contains an ordered tuple with Gram matrix B. The covolume/index argument in the proof of theorem 1.4, applied in both directions, proves $L \cong L'$. This proves the theorem. □

Proposition 7.1 (Divergence of the basis-witness cutoff at a cusp)**.** *For $n \ge 2$, consider the covolume-one lattices with Gram matrices*

$$G_\varepsilon = \operatorname{diag}\left(\varepsilon^2, \varepsilon^{-2/(n-1)}, \dots, \varepsilon^{-2/(n-1)}\right), \qquad 0 < \varepsilon < 1.$$

Every lattice basis contains a vector of squared length at least $\varepsilon^{-2/(n-1)}$. Thus the shortest possible maximum squared length among basis vectors diverges as $\varepsilon \to 0$.

Proof. The determinant of G_ε is 1. Let $u_1, \dots, u_n \in \mathbb{Z}^n$ be the coordinate columns of any lattice basis, so the matrix with these columns lies in $\mathrm{GL}_n(\mathbb{Z})$. Their projections to the last $n-1$ coordinates must span $\mathbb{R}^{n-1}$; otherwise the columns could not span $\mathbb{R}^n$. In particular, at least one basis vector u_j has a nonzero integer projection to the last $n-1$ coordinates. For that vector,

$$u_j^{\mathsf{T}} G_\varepsilon u_j = \varepsilon^2 (u_j)_1^2 + \varepsilon^{-2/(n-1)} \sum_{i=2}^{n} (u_j)_i^2 \ge \varepsilon^{-2/(n-1)}.$$

This proves the lower bound. □

Remark 7.2. theorem 7.1 shows that the basis-witness cutoff used in theorem 1.5 cannot be chosen uniformly over the full noncompact moduli space. It does not prove that every fixed truncated degree-n theta invariant has collisions along every cusp path; it is a statement about the specific reconstruction mechanism based on a short basis witness.

8. A degree-three obstruction stabilized to all ranks at least 32

We now prove theorem 1.6. The literature input is the following.

Proposition 8.1 (Published rank-32 input)**.** *There are non-isometric even unimodular extremal lattices L, L' of rank 32 such that*

$$\Theta_L^{(g)} = \Theta_{L'}^{(g)} \qquad (g = 1, 2, 3),$$

and both have minimum squared norm 4. Moreover, the set of minimum vectors of either lattice spans its ambient real vector space.

Proof. Oura and Ozeki report that Erokhin proved uniqueness of the degree-1, degree-2, and degree-3 Siegel theta series among the even unimodular extremal lattices of rank 32 [1, 4, Abstract and Section 1]. They also report, citing King's Corollary 17, that there are at least one billion non-isometric lattices in this family [4, p. 282]. Hence two non-isometric members have the same three theta series. The minimum squared norm is 4 by extremality [4, p. 283].

For the spanning assertion, Oura and Ozeki record the following second-moment identity (their Proposition 2.1, attributed there to Venkov): for every $\alpha \in L \otimes_{\mathbb{Z}} \mathbb{R}$,

$$\sum_{x \in \Lambda_4(L)} (x, \alpha)^2 = 18360\,(\alpha, \alpha),$$

where $\Lambda_4(L)$ is the set of vectors of squared norm 4 [4, p. 283]. If α is orthogonal to every minimum vector, then the left-hand side is zero, so $(\alpha, \alpha) = 0$ and $\alpha = 0$. Thus the minimum vectors span. The same argument applies to L'. □

Proof of theorem 1.6. For $n = 32$, the assertion follows from theorem 8.1. Suppose $n > 32$, and put $s = n - 32$. Let

$$K = \sqrt{6}\,\mathbb{Z}^s$$

with its standard scaled Euclidean metric, so that its minimum squared norm is 6. Define

$$L_n = L \perp K, \qquad L'_n = L' \perp K.$$

The minimum squared norm of each orthogonal sum is 4, since the auxiliary summand has minimum squared norm 6. Its minimum vectors are exactly the minimum vectors of the rank-32 summand, with zero K-component. By theorem 8.1, these vectors span the embedded real 32-dimensional summand.

Suppose that there were an isometry $F : L_n \to L'_n$. Isometries preserve minimum vectors, so F maps the real span of the minimum shell of L_n onto that of L'_n. Intersecting with the lattices recovers the original summands:

$$L_n \cap (\mathbb{R}^{32} \oplus 0) = L \oplus 0, \qquad L'_n \cap (\mathbb{R}^{32} \oplus 0) = L' \oplus 0.$$

The restriction of F would induce an isometry $L \cong L'$, contrary to the choice of the rank-32 pair. Hence L_n and L'_n are non-isometric.

It remains to compare their theta series. For any two Euclidean lattices A, B and every degree g, orthogonal direct sums satisfy

$$\Theta^{(g)}_{A \perp B}(Z) = \Theta^{(g)}_A(Z)\,\Theta^{(g)}_B(Z).$$

Indeed, every ordered g-tuple in $A \perp B$ decomposes uniquely into its A- and B-components, its Gram matrix is the sum of the two component Gram matrices, and the exponential in the defining series factors. Therefore, for $g = 1, 2, 3$,

$$\Theta^{(g)}_{L_n} = \Theta^{(g)}_L \Theta^{(g)}_K = \Theta^{(g)}_{L'} \Theta^{(g)}_K = \Theta^{(g)}_{L'_n}.$$

This proves that degrees one, two, and three fail to distinguish all rank-n Euclidean lattices for every $n \geq 32$, so $k(n) \geq 4$. The upper bound $k(n) \leq n$ is theorem 1.4. This completes the proof. □

Remark 8.2 (Degree four). Oura and Ozeki later distinguished the degree-4 theta series of five particular rank-32 extremal lattices arising from five binary self-dual extremal codes [5]. This is a result for that specified family; it does not prove that degree four distinguishes every rank-32 lattice. In particular, the exact value of $k(32)$ remains unresolved by the arguments here.

9. Discussion and open problems

The results establish a useful division of labour between rank-one and multi-vector information. The observation map in theorem 1.1 identifies a concrete linear obstruction: if the sampled rank-one matrices fail to span $\mathrm{Sym}_n(\mathbb{R})$, then a positive-dimensional family of Gram matrices preserves the observations while changing covolume. theorem 1.2 shows that this obstruction can survive even after passing from labelled observations to the full unlabelled length multiset below a closed cutoff.

At the other end, the degree-n Gram theta data contains the Gram matrix of a basis as one of its Fourier indices. The covolume-index argument turns this one coefficient into a global isometry. Compactness makes the basis witness uniformly short on compact portions of moduli space, while the determinant-one family in theorem 7.1 demonstrates why this particular cutoff argument degenerates near the cusp.

The rank-32 stabilization supplies a dimension-independent lower bound of four for the universal determining degree in all ranks at least 32. It does not decide whether degree four, or another fixed degree independent of n, suffices in general.

We record several questions.

(1) Determine the exact value of $k(n)$, especially whether $k(32) = 4$.
(2) Find explicit upper bounds for R_K in terms of geometric parameters that control compactness, such as covolume and the successive minima.
(3) Determine whether lower-degree Gram theta data admits a finite reconstruction cutoff on specified compact subsets of moduli space.
(4) For finite rank-one data, characterize when full-rank observation tensors are sufficient for local rigidity beyond the linear-kernel test.

Acknowledgements

The author thanks Professor Max A. Alekseyev for earlier comments on a related manuscript that helped sharpen the distinction between finite primal lattice-length data and other spectral invariants. The author also thanks the authors of the cited works for making their results accessible. The author used generative-AI tools for mathematical exploration, proof review, literature lookup, linguistic editing, and computational-script drafting and review; the author remains responsible for the mathematical content and the final manuscript.

References

[1] V. A. Erokhin, *Theta-series of even unimodular lattices*, *Journal of Soviet Mathematics* **25** (1984), no. 2, 1012–1020.
[2] O. D. King, *A mass formula for unimodular lattices with no roots*, *Mathematics of Computation* **72** (2003), no. 242, 839–863. doi:10.1090/S0025-5718-02-01455-2.
[3] S. Akbari, F. Belardo, E. Dodongeh, and M. A. Nematollahi, *Spectral characterizations of signed cycles*, *Linear Algebra and its Applications* **553** (2018), 307–327. doi:10.1016/j.laa.2018.05.012.
[4] M. Oura and M. Ozeki, *A numerical study of Siegel theta series of various degrees for the 32-dimensional even unimodular extremal lattices*, *Kyushu Journal of Mathematics* **70** (2016), no. 2, 281–314. doi:10.2206/kyushujm.70.281.
[5] M. Oura and M. Ozeki, *Distinguishing Siegel theta series of degree 4 for the 32-dimensional even unimodular extremal lattices*, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **86** (2016), 19–53. doi:10.1007/s12188-016-0120-y.
[6] A. Schiemann, *Ternary positive definite quadratic forms are determined by their theta series*, *Mathematische Annalen* **308** (1997), no. 3, 507–517. doi:10.1007/s002080050086.