A CONGRUENCE MODULO p^2 FOR THE p-TH COEFFICIENT OF $e\,(1-z)^{(1-z)/z}$

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Abstract. Define E_n by $\sum_{n\geq 0} E_n z^n = \exp\bigl(\sum_{k\geq 1} z^k/(k(k+1))\bigr) = e\,(1-z)^{(1-z)/z}$. Equivalently, $e\,(1+1/x)^{-x} = \sum_{n\geq 0} E_n (x+1)^{-n}$, the reciprocal of the expansion of $e^{-1}(1+1/x)^x$ used by Yang in refinements of Carleman's inequality. We prove that for every odd prime p, $E_p \equiv -p/2 \pmod{p^2}$; in particular $v_p(E_p) = 1$. The proof is elementary. With $\Phi(u) = e\,(1+u)^{-1/u}$, it combines the two expressions of E_p through the coefficients of Φ that come from the identities $e(1-z)^{(1-z)/z} = \Phi(-z)/(1-z)$ and $e(1-z)^{(1-z)/z} = \Phi(z/(1-z))$, and it reduces the congruence to the differential recurrence of the E_n read modulo p. A script checks the statement and the intermediate congruences for all odd primes below 200 and a few larger ones.

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1. Introduction

Let

$$G(z) = \exp\Bigl(\sum_{k\geq 1} \frac{z^k}{k(k+1)}\Bigr) = e\,(1-z)^{(1-z)/z} = \sum_{n\geq 0} E_n z^n,$$

so that $E_0 = 1$, $E_1 = 1/2$, $E_2 = 7/24$, $E_3 = 3/16$, $E_4 = 743/5760$. The second expression follows from $\sum_{k\geq 1} z^k/(k(k+1)) = 1 + (1-z)\log(1-z)/z$, obtained from $1/(k(k+1)) = 1/k - 1/(k+1)$. Putting $z = 1/(x+1)$ gives

$$e\Bigl(1+\frac{1}{x}\Bigr)^{-x} = \sum_{n\geq 0} \frac{E_n}{(x+1)^n}.$$

Yang [Ya1, Ya2] wrote $(1+1/x)^x = e\bigl(1 - \sum_{n\geq 1} b_n (x+1)^{-n}\bigr)$ to refine Carleman's inequality, so $\sum_n E_n z^n = 1/(1 - \sum_n b_n z^n)$; the E_n are the coefficients S_n of [NG], where the p-adic valuations of the b_n are studied. Equivalently, $1/G(z) = (1-z)\,e^{-1}(1-z)^{-1/z}$, where $e^{-1}(1+x)^{1/x}$ has the coefficients tabulated in [OEIS]. We have found no previous result on the arithmetic of the E_n; see [HM] for analytic properties of the b_n.

Expanding the exponential, E_n is a sum over the partitions of n. For $n = p$ the only terms whose denominator is divisible by p come from the partitions $(p-1)+1$, p and 1^p, namely $\frac{1}{2(p-1)p}$, $\frac{1}{p(p+1)}$ and $\frac{1}{2^p p!}$. Multiplied by p, they are congruent modulo p to $-1/2$, 1 and $-1/2$ (by Fermat and Wilson), so the poles cancel and E_p is p-integral. The point of the theorem is the next digit.

Theorem 1. *For every odd prime p,*

$$E_p \equiv -\frac{p}{2} \pmod{p^2}.$$

In particular $v_p(E_p) = 1$.

For $p = 2$ the statement fails, since $E_2 = 7/24$. For example $E_3 = 3/16$ and $E_5 = 215/2304$, with $E_3 + 3/2 = 27/16 = 3^3/2^4$ and $E_5 + 5/2 = 5975/2304 = 5^2 \cdot 239/(2^8 \cdot 3^2)$.

The proof (Section 3) uses two expressions of E_p through the coefficients of $\Phi(u) = e\,(1+u)^{-1/u}$, the congruence $\binom{p-1}{j} \equiv (-1)^j(1 - pH_j) \pmod{p^2}$, and the recurrence of the E_n obtained by differentiating G. Throughout, $a \equiv b \pmod{p^j}$ means that $a - b \in p^j \mathbb{Z}_{(p)}$, for rationals a, b, and $H_j = 1 + 1/2 + \cdots + 1/j$, $H_0 = 0$.

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2. Identities

Put

$$\Phi(u) = e\,(1+u)^{-1/u} = \exp\Big(\sum_{n\geq 1} h_n u^n\Big), \qquad h_n = \frac{(-1)^{n+1}}{n+1}, \qquad \Phi(u) = \sum_{k\geq 0} \varphi_k u^k,$$

the second expression coming from $1 - \log(1+u)/u = \sum_{n\geq 1}(-1)^{n+1}u^n/(n+1)$. Thus $\varphi_0 = 1$, $\varphi_1 = 1/2$.

Lemma 2. *As formal power series over $\mathbb{Q}$,*

$$\textit{(a)}\ \ G(z) = \frac{\Phi(-z)}{1-z}, \qquad \textit{(b)}\ \ G(z) = \Phi\Big(\frac{z}{1-z}\Big), \qquad \textit{(c)}\ \ (n+1)E_{n+1} = \sum_{j=0}^{n} \frac{E_j}{n+2-j} \quad (n\geq 0).$$

Consequently, for $n, k \geq 0$ and every integer $p \geq 1$,

$$(1) \qquad E_n = \sum_{k=0}^{n} (-1)^k \varphi_k, \qquad (-1)^k \varphi_k = E_k - E_{k-1}\ (E_{-1} = 0), \qquad E_p = \sum_{k=1}^{p} \binom{p-1}{k-1} \varphi_k.$$

Proof. (a) $\Phi(-z) = e(1-z)^{1/z}$ and $G(z) = e(1-z)^{1/z-1}$. (b) With $w = z/(1-z)$ we have $1+w = (1-z)^{-1}$ and $-1/w = -(1-z)/z$, so $\Phi(w) = e(1-z)^{(1-z)/z} = G(z)$. (c) Differentiate $\log G = \sum_k z^k/(k(k+1))$: $G' = G\sum_{m\geq 0} z^m/(m+2)$, and compare the coefficients of z^n.

The first two identities in (1) follow from (a), since $1/(1-z) = \sum_j z^j$. For the third, $[z^p](z/(1-z))^k = [z^{p-k}](1-z)^{-k} = \binom{p-1}{k-1}$ for $1 \leq k \leq p$, and 0 for $k = 0$ or $k > p$; apply (b). □

Fix an odd prime p. Let ψ_k be the coefficients of $\Psi(u) = \exp\big(\sum_{n=1}^{p-2} h_n u^n\big)$.

Lemma 3. (i) *h_n is p-integral for $n \leq p-2$, and ψ_k is p-integral for $k \leq p-1$.*
(ii) *$\varphi_k = \psi_k$ for $k \leq p-2$; in particular φ_k and E_k are p-integral for $k \leq p-2$.*
(iii) *$\varphi_{p-1} = \psi_{p-1} - 1/p$, and $\varphi_p + 1/p$ is p-integral.*
(iv) *$E_p = E_{p-2} + \varphi_{p-1} - \varphi_p$; hence E_p is p-integral.*

Proof. (i) The denominator of h_n is $n+1 \leq p-1$. The coefficient ψ_k is a sum of terms $\prod_j h_j^{m_j}/m_j!$ over the partitions $\sum_j j m_j = k$ with parts $j \leq p-2$; for $k \leq p-1$ every m_j is less than p.

(ii), (iii) Write $\Phi = \Psi \cdot R$ with $R = \exp\big(\sum_{n\geq p-1} h_n u^n\big)$. Since $2(p-1) \geq p+1$, $R \equiv 1 + h_{p-1}u^{p-1} + h_p u^p \pmod{u^{p+1}}$, where $h_{p-1} = (-1)^p/p = -1/p$ and $h_p = 1/(p+1)$ is p-integral. Hence $\varphi_k = \psi_k$ for $k \leq p-2$, $\varphi_{p-1} = \psi_{p-1} - 1/p$ and $\varphi_p = \psi_p + \psi_1 h_{p-1} + h_p = \psi_p - 1/(2p) + h_p$. In the expansion of ψ_p, all partitions of p into parts $\leq p-2$ have multiplicities less than p except 1^p, which contributes $h_1^p/p! = 1/(2^p p!)$. By Fermat and Wilson, $p/(2^p p!) = 1/(2^p(p-1)!) \equiv 1/(2\cdot(-1)) = -1/2 \pmod p$, so $\psi_p + 1/(2p)$ is p-integral and so is $\varphi_p + 1/p$. By (1), (i) and the first part of (ii), $E_k = \sum_{i\leq k}(-1)^i \varphi_i$ is p-integral for $k \leq p-2$.

(iv) By (1), $E_p = E_{p-2} + (-1)^{p-1}\varphi_{p-1} + (-1)^p \varphi_p$, and p is odd. The right-hand side is p-integral by (ii) and (iii). □

3. Proof of Theorem 1

Step A: reduction to a congruence modulo p. For $0 \leq j \leq p-1$, $\binom{p-1}{j} = \prod_{i=1}^{j}(p-i)/i = (-1)^j \prod_{i=1}^{j}(1-p/i) \equiv (-1)^j(1 - pH_j) \pmod{p^2}$. In the last expression of (1) the terms $k \leq p-2$ have p-integral φ_k, so

$$E_p \equiv \sum_{k=1}^{p-2} (-1)^{k-1}\varphi_k - pH^* + (p-1)\varphi_{p-1} + \varphi_p \pmod{p^2}, \qquad H^* = \sum_{k=1}^{p-2} (-1)^{k-1} H_{k-1}\varphi_k.$$

Here $\sum_{k=1}^{p-2}(-1)^{k-1}\varphi_k = 1 - E_{p-2}$ and $(p-1)\varphi_{p-1} + \varphi_p = p\varphi_{p-1} - (\varphi_{p-1} - \varphi_p)$. Adding Lemma 3(iv),

$$2E_p \equiv 1 + p\varphi_{p-1} - pH^* = p\,(\psi_{p-1} - H^*) \pmod{p^2},$$

by Lemma 3(iii). Since ψ_{p-1} and H^* are p-integral, this already gives $E_p \equiv 0 \pmod p$, and the theorem is equivalent to

$$\psi_{p-1} - H^* \equiv -1 \pmod p. \tag{2}$$

Step B: the residue ψ_{p-1}. From $\Psi' = \Psi \sum_{j=1}^{p-2} j h_j u^{j-1}$, the coefficient of u^{p-2} gives the exact identity $(p-1)\psi_{p-1} = \sum_{j=1}^{p-2} j h_j \psi_{p-1-j}$, and $\psi_{p-1-j} = \varphi_{p-1-j}$ by Lemma 3(ii). Put $k = p-1-j$, so $1 \le k \le p-2$. Modulo p, $p-1 \equiv -1$, $j \equiv -(k+1)$ and $h_j = (-1)^{p-k}/(p-k) \equiv (-1)^k/k$, so $j h_j \equiv -(-1)^k(1+1/k)$ and

$$\psi_{p-1} \equiv \sum_{k=1}^{p-2} (-1)^k \Big(1 + \frac{1}{k}\Big)\varphi_k \pmod p.$$

Since $H^* = -\sum_{k=1}^{p-2} (-1)^k H_{k-1}\varphi_k$ and $H_{k-1} + 1/k = H_k$,

$$\psi_{p-1} - H^* \equiv \sum_{k=1}^{p-2} (-1)^k (1+H_k)\varphi_k = E_{p-2} - 1 + \sum_{k=1}^{p-2} (-1)^k H_k \varphi_k \pmod p.$$

Hence (2) is equivalent to

$$\sum_{k=1}^{p-2} (-1)^k H_k \varphi_k \equiv -E_{p-2} \pmod p. \tag{3}$$

Step C: the recurrence. By (1) and summation by parts,

$$\begin{aligned}\sum_{k=1}^{p-2} (-1)^k H_k \varphi_k &= \sum_{k=1}^{p-2} H_k (E_k - E_{k-1}) = H_{p-2}E_{p-2} - H_1 E_0 - \sum_{k=1}^{p-3} (H_{k+1} - H_k) E_k \\ &= H_{p-2}E_{p-2} - 1 - \sum_{k=1}^{p-3} \frac{E_k}{k+1}.\end{aligned}$$

Since $H_{p-1} \equiv 0 \pmod p$ (pair i with $p-i$), $H_{p-2} = H_{p-1} - 1/(p-1) \equiv 1$. So (3) is equivalent to

$$2E_{p-2} \equiv 1 + \sum_{k=1}^{p-3} \frac{E_k}{k+1} = \sum_{j=0}^{p-3} \frac{E_j}{j+1} \pmod p.$$

This is Lemma 2(c) with $n = p-3$, namely $(p-2)E_{p-2} = \sum_{j=0}^{p-3} E_j/(p-1-j)$, read modulo p: all E_j with $j \le p-2$ are p-integral, $p-2 \equiv -2$ and $1/(p-1-j) \equiv -1/(j+1)$ for $0 \le j \le p-3$. This proves (3), hence (2) and the theorem. □

Remark 4. After Step A only $E_0, \dots, E_{p-2}$ and $\varphi_1, \dots, \varphi_{p-2}$ occur, and the congruence finally needed is the recurrence of the E_n at $n = p-3$. For $p = 3$ all the sums in Steps B and C are empty or reduce to a single term, and the theorem reads $E_3 = 3/16 \equiv -3/2 \pmod 9$.

4. Numerical verification and a remark

The ancillary file `verify_Ep_congruence.py` (Python 3, no dependency, exact rational arithmetic, about thirty seconds) computes E_n by the recurrence of Lemma 2(c) and φ_k from the exponential, and checks, for all odd primes $p < 200$ and for $p = 251$, 307, 401: Lemma 2(a) and (b) coefficientwise up to z^{p+1}, and the third identity of (1); Lemma 3; each displayed congruence and equivalence of Steps A, B and C, and the summation by parts; and Theorem 1.

The next p-adic digit $\big((E_p + p/2)/p^2 \bmod p\big)$ is $0, 1, 3, 9, 0, 12, 14, 3$ for $p = 3, 5, \dots, 23$ and shows no pattern. In the tested range, $E_p \equiv -p/2 \pmod{p^3}$ holds only for $p = 3$ and $p = 13$, which is compatible with a heuristic probability $1/p$ for each p.

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References

[HM] Y. Hu, C. Mortici, On the coefficients of an expansion of $(1 + 1/x)^x$ related to Carleman's inequality, arXiv:1401.2236 (2014).

[NG] D. Niedbala Giraudin, The p-adic valuations of the coefficients in Yang's expansion of $(1+1/x)^x$, preprint hexagon:2611.00074, `https://hexagonmath.org` (2026).

[OEIS] The On-Line Encyclopedia of Integer Sequences, sequences A055505 and A055535, `https://oeis.org`.

[Ya1] X. Yang, On Carleman's inequality, *J. Math. Anal. Appl.* **253** (2001), 691–694.

[Ya2] X. Yang, Approximations for constant e and their applications, *J. Math. Anal. Appl.* **262** (2001), 651–659.

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