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THE p-ADIC VALUATIONS OF THE COEFFICIENTS IN YANG'S EXPANSION OF $(1+1/X)^X$

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Abstract. Yang's coefficients b_n, defined by $(1+1/x)^x = e\big(1-\sum_{n\ge1} b_n(x+1)^{-n}\big)$, appear in refinements of Carleman's inequality, and the literature on them concerns their analytic properties. We study their p-adic valuations for an odd prime p. From the identity $1-\sum_n b_n u^n = \exp\big(-\sum_{k\ge1} u^k/(k(k+1))\big)$ one gets $v_p(b_n) \ge -M(n)$, where $M(n)$ is the optimum of a knapsack problem on the compositions of n. We compute $M(n)$ in closed form and show that, for $n \ge 2$, the optimum is attained by a unique composition exactly when $n = (p-1)m$ with m in an explicit set U_p defined by a digit condition. For these n the bound is an equality: $v_p(b_n) = -(m + v_p(m!))$. The set U_p has density $1/(p-1)$, so the valuation of b_n is determined on a set N_p of indices of density $1/(p-1)^2$. The set N_p contains the indices p^K-1, at which b_{p^K-1} is the unique coefficient of minimal valuation among $b_1,\dots,b_{p^K-1}$. The same statements hold for the coefficients of $e(1-u)^{1/u}$, and the minimality statement for those of $e(1-u)^{1/u-1}$. A script checks all statements numerically in finite ranges.

1. Introduction

Yang [Ya] introduced the rational numbers b_n defined by

$$\Big(1+\frac{1}{x}\Big)^x = e\Big(1-\sum_{n\ge1}\frac{b_n}{(x+1)^n}\Big), \qquad x>0, \tag{1}$$

in a strengthened form of Carleman's inequality [Ca]; the first values are $b_1 = 1/2$, $b_2 = 1/24$, $b_3 = 1/48$, $b_4 = 73/5760$, $b_5 = 11/1280$, $b_6 = 3625/580608$ (the value of b_6 agrees with Finch [Fi]); numerators and denominators are tabulated in OEIS A249276 and A249277 [OEIS]. The work on this sequence concerns its analytic properties; see Hu and Mortici [HM] and the references there. We study its p-adic valuations.

Put $u = 1/(x+1)$, so that $1+1/x = 1/(1-u)$ and $x = (1-u)/u$. Since $\log(1-u) = -\sum_{k\ge1} u^k/k$,

$$x\log\Big(1+\frac{1}{x}\Big) = \frac{1-u}{u}\sum_{k\ge1}\frac{u^k}{k} = 1-\sum_{k\ge1}\frac{u^k}{k(k+1)},$$

so that, with $B(u) = \sum_{n\ge1} b_n u^n$,

$$1-B(u) = \exp\Big(-\sum_{k\ge1}\frac{u^k}{k(k+1)}\Big). \tag{2}$$

We also consider

$$T(u) = \exp\Big(-\sum_{k\ge1}\frac{u^k}{k+1}\Big) = e\,(1-u)^{1/u}, \qquad S(u) = \frac{T(u)}{1-u} = \frac{1}{1-B(u)} = e\,(1-u)^{1/u-1},$$

with coefficients T_n and S_n; thus $S_n = T_0+\cdots+T_n$. The series $1/T(u) = (1-u)^{-1/u}/e$ has the coefficients $a_n = A055505(n)/A055535(n)$ of [OEIS] (see also [Co, p. 293]), and $b_n = a_{n-1} - a_n$. Differentiating (2) gives the recurrence $b_1 = 1/2$, $nb_n = 1/(n+1) - \sum_{j=1}^{n-1} b_j/(n-j+1)$.

Throughout, p is an odd prime, $c = p-2 \ge 1$, and $s_p(m)$ is the sum of the base-p digits of m, so that $v_p(m!) = (m-s_p(m))/(p-1)$ (Legendre). Let

$$U_p = \big\{m\ge1 :\ i + s_p(m+ic) > s_p(m) \text{ for every } i\ge1\big\}, \qquad N_p = (p-1)\,U_p.$$

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It suffices to test $1 \le i < s_p(m)$, since for $i \ge s_p(m)$ the inequality holds because $s_p(m+ic) \ge 1$.

Theorem 1. *Let $n = (p-1)m$ with $m \in U_p$. Then*

$$v_p(b_n) = v_p(T_n) = -\big(m + v_p(m!)\big) = -\frac{pm - s_p(m)}{p-1}.$$

Theorem 2. *For $X \ge p-2$,*

$$-\frac{p-2}{2} \;\le\; \#\big(U_p \cap [1,X]\big) - \frac{X}{p-1} \;\le\; (p-2)\big(1 + \log_p(X+p-2)\big).$$

In particular U_p has density $1/(p-1)$, and N_p has density $1/(p-1)^2$ in the positive integers.

Theorem 3. *Let $K \ge 1$, $R = p^K - 1$, $R_K = R/(p-1) = 1 + p + \cdots + p^{K-1}$ and*

$$D_K = R_K + v_p(R_K!) = \frac{p^{K+1} - p - K(p-1)}{(p-1)^2} = \sum_{j=0}^{K-1} (K-j)p^j.$$

Then $R_K \in U_p$, so $v_p(b_R) = v_p(T_R) = -D_K$, while $v_p(b_n) > -D_K$ and $v_p(T_n) > -D_K$ for $1 \le n < R$. Moreover $v_p(S_R) = -D_K$ and $v_p(S_n) > -D_K$ for $1 \le n < R$. Thus each of b_R, T_R, S_R is the unique coefficient of minimal p-adic valuation in its sequence on $[1,R]$.

We know of no previous result on the p-adic valuations of the b_n. The proofs rest on a knapsack problem (Section 2), which we solve in closed form (Proposition 6), with a criterion for the uniqueness of the optimum (Theorem 7). A related phenomenon, for the powers of $\log(1+x)/x$, was found by Davis [Da], where the coefficients of index $(p-1)m$ also behave like those of a series in x^{p-1}/p. Krattenthaler and Rivoal [KR] give general upper bounds for the denominators of the Taylor coefficients of differentially algebraic power series; the statements here are exact valuations for specific series.

2. The knapsack bound

For $k \ge 1$ put $\alpha(k) = v_p(k) + v_p(k+1)$ and $\alpha'(k) = v_p(k+1)$, so $\alpha'(k) \le \alpha(k)$. For a composition $\kappa = (k_1, \dots, k_m)$ of n (an ordered sequence of positive integers with sum n) define the scores

$$\sigma(\kappa) = \sum_{i=1}^{m} \alpha(k_i) + v_p(m!), \qquad \sigma'(\kappa) = \sum_{i=1}^{m} \alpha'(k_i) + v_p(m!),$$

and let $M(n)$, $M'(n)$ be their maxima over the compositions of n.

Lemma 4. *For $n \ge 1$, $v_p(b_n) \ge -M(n)$ and $v_p(T_n) \ge -M'(n)$. If $M(n)$ is attained by a unique composition, then $v_p(b_n) = -M(n)$; if $M'(n)$ is attained by a unique composition, then $v_p(T_n) = -M'(n)$.*

Proof. Expanding the exponential in (2),

$$-b_n = [u^n] \exp\Big(-\sum_{k \ge 1} \frac{u^k}{k(k+1)}\Big) = \sum_{\kappa} \frac{(-1)^m}{m!} \prod_{i=1}^{m} \frac{1}{k_i(k_i+1)},$$

a finite sum over all compositions $\kappa = (k_1, \dots, k_m)$ of n. The term of κ has valuation exactly $-\sigma(\kappa)$. The valuation of a finite sum is at least the minimum of the valuations of its terms, with equality if this minimum is attained by a single term. The same argument applies to T_n, with $1/(k_i+1)$ in place of $1/(k_i(k_i+1))$. □

Lemma 5. *For $k \ge 1$, $\alpha(k) \le k/(p-1)$, with equality if and only if $k = p-1$.*

Proof. If $\alpha(k) = 0$ the inequality is strict. If $a = \alpha(k) \ge 1$, then, since p divides at most one of the consecutive integers k and $k+1$, p^a divides k or $k+1$, so $k \ge p^a - 1 = (p-1)(1 + p + \cdots + p^{a-1}) \ge (p-1)a$. Both inequalities are equalities if and only if $a = 1$ and $k = p-1$. □

3. The optimum

Proposition 6. *For $n \geq 1$,*

$$M(n) = M'(n) = \max_{0 \leq j \leq n/(p-1)} f_n(j), \qquad f_n(j) = j + v_p\big((n-jc)!\big).$$

More precisely, every composition of n can be transformed, by a finite sequence of the moves below, into a composition all of whose parts are equal to 1 or $p-1$, and no move decreases σ or σ'. A composition with j parts equal to $p-1$ and $i = n-j(p-1)$ parts equal to 1 has $m = n-jc$ parts and $\sigma = \sigma' = f_n(j)$.

The moves replace one part $k \notin \{1, p-1\}$ of a composition, as follows.

(A) If $(p-1) \mid k$ and $k > p-1$: by $q = k/(p-1) \geq 2$ parts equal to $p-1$.
(B) If $k = q(p-1)+r$ with $q \geq 1$ and $1 \leq r \leq p-2$: by q parts equal to $p-1$ and one part equal to r.
(C) If $2 \leq k \leq p-2$: by k parts equal to 1.

Every $k \notin \{1, p-1\}$ falls under exactly one move. Each move increases the number of parts, so any sequence of moves stops, and it stops exactly at a composition with parts in $\{1, p-1\}$.

Proof. The number of parts m never decreases, so $v_p(m!)$ never decreases. The parts $1, \dots, p-2$ have $\alpha = \alpha' = 0$, since k and $k+1$ are then smaller than p, and the part $p-1$ has $\alpha = \alpha' = 1$. In move (A) the weight q of the new parts exceeds $\alpha(k) \geq \alpha'(k)$, by Lemma 5; so the move increases σ and σ' by at least 1. In move (B), Lemma 5 gives $\alpha'(k) \leq \alpha(k) \leq k/(p-1) < q+1$, hence $\alpha'(k) \leq \alpha(k) \leq q$, and the new parts have total weight q. In move (C) the total weight is 0 before and after. The last assertion follows from $\alpha(1) = \alpha'(1) = 0$ and $\alpha(p-1) = \alpha'(p-1) = 1$. □

Theorem 7. *Let $n \geq 2$. The following are equivalent:*

(i) *$M(n)$ is attained by a unique composition of n;*
(ii) *$M'(n)$ is attained by a unique composition of n;*
(iii) *$n = (p-1)m$ with $m \in U_p$.*

In that case the optimal composition consists of m parts equal to $p-1$, and $M(n) = M'(n) = m + v_p(m!)$.

Proof. We write the proof for σ; for σ' it is the same, since Proposition 6 holds for both scores. By Legendre's formula,

$$(p-1) f_n(j) = n + j - s_p(n-jc). \tag{3}$$

Step 1: the composition with all parts equal to 1 is never the unique optimum. If $n \leq p-2$, all parts of all compositions of n are at most $p-2$ and $m \leq n < p$, so every composition of n has score 0, and $n \geq 2$ has at least two compositions. If $n \geq p-1$, let $J = \lfloor n/(p-1) \rfloor \geq 1$ and $n = (p-1)J + \rho$ with $0 \leq \rho \leq p-2$. Then $n - Jc = J + \rho$ and $s_p(J+\rho) \leq J + \rho$, so (3) gives $(p-1)(f_n(J) - f_n(0)) \geq s_p(n) - \rho$. Since $s_p(n) \equiv n \equiv \rho \pmod{p-1}$ and $s_p(n) \geq 1$, we have $s_p(n) \geq \rho$. Hence $f_n(J) \geq f_n(0)$, and the composition with J parts $p-1$ followed by ρ parts 1 scores at least as much as the composition with all parts 1.

Step 2: (i) implies (iii). Let κ be the unique optimal composition. Applying a move to κ would give a different composition with at least the same score, so no move applies, and κ has j parts $p-1$ and i parts 1. If $i \geq 1$ and $j \geq 1$, reordering the parts gives a different composition with the same score. By Step 1, $j = 0$ is excluded. Hence $i = 0$, $n = (p-1)j$, and $m := j$. If $f_n(j') \geq f_n(m)$ for some $j' < m$, a composition with j' parts $p-1$ and $n - j'(p-1) \geq 1$ parts 1 would also be optimal. So $f_n(m) > f_n(m-i)$ for $1 \leq i \leq m$. Since $n - (m-i)c = m + ic$, (3) turns this into $i + s_p(m+ic) > s_p(m)$ for $1 \leq i \leq m$, which is the condition $m \in U_p$ (the values $i > m \geq s_p(m)$ need no test).

Step 3: (iii) implies (i). Let $n = (p-1)m$ with $m \in U_p$. By (3) as in Step 2, $f_n(m) > f_n(j)$ for $0 \leq j < m$, so by Proposition 6 $M(n) = f_n(m) = m + v_p(m!)$. Let κ be any optimal composition, and transform it by moves into a composition κ' with parts in $\{1, p-1\}$. Then κ' is optimal, so it has m parts $p-1$ and no part 1. A move of type (A) increases the score, so none occurs. A move of type (B) or (C) creates a part smaller than $p-1$, and no move decreases the

number of parts smaller than $p-1$ (move (C) replaces such a part by parts 1), so none occurs either. Thus $\kappa = \kappa'$, which is the composition with m parts equal to $p-1$. □

Proof of Theorem 1. Combine Lemma 4 and Theorem 7; the last expression follows from Legendre's formula. □

Remark 8. For $n \geq 2$ outside N_p the optimum is not unique, and $v_p(b_n) > -M(n)$ does occur: for $p = 3$ this happens at 23 of the indices $n \leq 80$, the smallest being $n = 5$. The script checks this count and the inequality of Lemma 4 for $p \leq 13$ and $n \leq 100$.

4. The indices $p^K - 1$

Lemma 9. *Let A be a nonnegative integer whose base-p digits are all 0 or 1. Let $w \geq 1$ and $B = wc$, and let κ be the number of carries in the base-p addition $A + B$. If $\kappa \geq 1$, then $s_p(B) \geq \kappa c + 1$ and $w \geq \kappa$. In all cases, $\kappa \leq w$ and $s_p(A+B) \geq s_p(A) + 1 - \kappa \geq s_p(A) + 1 - w$.*

Proof. A carry out of a position where A has digit 1 requires a digit of B at least $p-2 = c$ if a carry comes in, and at least $p-1 = c+1$ if none comes in. A carry out of a position where A has digit 0 requires the digit $p-1$ of B and an incoming carry. The lowest carry position receives no carry. Hence the digits of B at the κ carry positions are at least c, and the one at the lowest carry position is at least $c+1$; this gives $s_p(B) \geq \kappa c + 1$, and also $B \geq c \sum_{\text{carry positions } t} p^t \geq c\kappa$, that is $w \geq \kappa$. Kummer's theorem gives $s_p(A+B) = s_p(A) + s_p(B) - (p-1)\kappa \geq s_p(A) + 1 - \kappa$, and if $\kappa = 0$ then $s_p(A+B) = s_p(A) + s_p(B) \geq s_p(A) + 1$. □

Proof of Theorem 3. The two expressions of D_K follow from $p^K - 1 = (p-1)R_K$, $s_p(R_K) = K$ and Legendre's formula.

$R_K \in U_p$. Apply Lemma 9 with $A = R_K$ and $B = ic$: for $i \geq 1$, $i + s_p(R_K + ic) \geq i + K + 1 - \kappa \geq K + 1 > K = s_p(R_K)$, since $i \geq \kappa$. By Theorem 1, $v_p(b_R) = v_p(T_R) = -(R_K + v_p(R_K!)) = -D_K$.

$M(p^K - 2) \leq D_K - 1$. Let $n = p^K - 2$. Then $\lfloor n/(p-1) \rfloor = R_K - 1$, and for $0 \leq j \leq R_K - 1$ write $w = R_K - j \geq 1$. Since $R_K c = p^K - 1 - R_K$, we get $n - jc = (R_K - 1) + wc$. The integer $R_K - 1 = p + \cdots + p^{K-1}$ has digit 0 in position 0 and $s_p(R_K - 1) = K - 1$, so Lemma 9 gives $s_p(n - jc) \geq K - w$. By (3),

$$(p-1) f_n(j) = n + j - s_p(n - jc) \leq p^K - 2 + R_K - K = (p-1)D_K - 1,$$

using $(p-1)D_K = (p-1)R_K + R_K - K = p^K - 1 + R_K - K$. Since $f_n(j)$ is an integer, $f_n(j) \leq D_K - 1$, and Proposition 6 gives $M(p^K - 2) = M'(p^K - 2) \leq D_K - 1$ (equality holds in all the cases computed by the script). (For $K = 1$, $n = p - 2$, and the statement is $M(n) = 0$.)

Minimality. Appending a part 1 to a composition does not decrease σ or σ', so M and M' are nondecreasing. For $1 \leq n < R$, Lemma 4 gives $v_p(b_n) \geq -M(n) \geq -M(R-1) \geq -(D_K - 1)$, and the same for T_n. Finally $S_n = T_0 + \cdots + T_n$ with $T_0 = 1$; for $n < R$ every term has valuation $> -D_K$, and for $n = R$ the term T_R is the only one of valuation $-D_K$. Hence $v_p(S_n) > -D_K$ for $n < R$ and $v_p(S_R) = -D_K$. □

5. The density of U_p

Put $\Lambda(y) = y + c\, s_p(y)$ for $y \geq 1$. For $m \geq 1$ and $i \geq 1$, $\Lambda(m + ic) - \Lambda(m) = c\big(i + s_p(m + ic) - s_p(m)\big)$, so

$$m \in U_p \iff \Lambda(m) < \Lambda(y) \text{ for every } y > m \text{ with } y \equiv m \pmod{c}. \tag{4}$$

Lemma 10. *If $m \in U_p$, then $\min_{i \geq 1}\big(\Lambda(m + ic) - \Lambda(m)\big) = c(p-1)$.*

Proof. Let $\Delta = \min_{i \geq 1}\big(i + s_p(m + ic) - s_p(m)\big)$; the minimum exists since the quantity tends to infinity with i, and $\Delta \geq 1$ because $m \in U_p$. Since $s_p(y) \equiv y \pmod{p-1}$, we have $\Lambda(y) \equiv (1 + c)y = (p-1)y \equiv 0 \pmod{p-1}$ for every y, so $c\Delta \equiv 0 \pmod{p-1}$. As $c = p - 2$ and $p - 1$ are coprime, $(p-1) \mid \Delta$, hence $\Delta \geq p - 1$. On the other hand, taking $i = 1$ and using $s_p(m + c) \leq s_p(m) + s_p(c) = s_p(m) + c$, we get $\Delta \leq 1 + c = p - 1$. □

Proof of Theorem 2. Fix a residue class modulo c, with elements $x_0 < x_1 < \dots$, where $1 \le x_0 \le c$ and $x_{k+1} = x_k + c$. Put $L(x) = \min\{\Lambda(y) : y \ge x,\ y \equiv x \pmod c\}$, which exists since $\Lambda(y) \ge y$. Then $x \le L(x) \le \Lambda(x) = x + c\, s_p(x)$. By (4), if $x_k \notin U_p$ then $L(x_k) = L(x_{k+1})$, and if $x_k \in U_p$ then $L(x_k) = \Lambda(x_k)$ and, by Lemma 10, $L(x_{k+1}) - L(x_k) = c(p-1)$. Let $x_\ell \le X < x_{\ell+1}$. Summing over $0 \le k \le \ell$,

$$c(p-1) \cdot \#\big(U_p \cap \{x_0, \dots, x_\ell\}\big) = L(x_{\ell+1}) - L(x_0),$$

which lies between $x_{\ell+1} - x_0 - c\, s_p(x_0)$ and $x_{\ell+1} - x_0 + c\, s_p(x_{\ell+1})$. Over the c classes, the numbers $x_{\ell+1}$ are $X+1, \dots, X+c$ and the numbers x_0 are $1, \dots, c$, so the differences $x_{\ell+1} - x_0$ add up to cX. Since $s_p(x_0) = x_0$ and $s_p(y) \le (p-1)(1 + \log_p y)$, summing and dividing by $c(p-1)$ gives the inequalities of the theorem. The density of N_p follows, since $n \in N_p$ requires $(p-1) \mid n$. □

6. Verification

The ancillary file `verify_carleman_padic.py` (Python 3, no dependency, exact rational arithmetic, about fifteen seconds) computes b_n, T_n, S_n for $n \le 250$ and checks:

- (2) against the recurrence of the introduction; the coefficients of $1/T$ against the sequences A055505 and A055535 for $n \le 17$; $b_n = a_{n-1} - a_n$; $S = 1/(1-B)$;
- for $p \le 13$ and $n \le 100$: $M(n)$ and $M'(n)$ computed by dynamic programming over all compositions against Proposition 6; the uniqueness of the optimum against Theorem 7; the inequalities of Lemma 4;
- for $p \le 31$: Theorem 1 on all $n \in N_p$ with $n \le 250$; Theorem 3 for all $p^K - 1 \le 250$; Lemma 10 and the bounds of Theorem 2 for $X = 60000$.

These computations check the statements; the proofs do not depend on them.

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