A SHARPER TYPE I/II ESTIMATE FOR PRIMES IN ARITHMETIC PROGRESSIONS TO SMOOTH MODULI, AND $H_M \ll E^{3.8043\,M}$

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Abstract. In Stadlmann's Type I/II equidistribution estimate for x^δ-smooth moduli, one of the three conditions, $64\varpi + 20\delta + 2\sigma < 1$, can be replaced by $48\varpi + 18\delta + 2\sigma < 1$. This weaker condition follows from the second one, $48\varpi + 16\delta + 4\sigma < 1$, whenever $\delta \le \sigma$. The proof follows Stadlmann's argument and makes two changes: the length of a block in one Taylor expansion depends on the pair of shifts, and the bounds are summed over the pairs instead of being multiplied by their number. Combined with Stadlmann's general Harman-sieve construction, this gives a minorant for the primes with exponent of distribution $1/2+2\varpi$ to smooth moduli for every $\varpi < 7/544$, instead of $\varpi = 0.01265$. As a consequence, $\liminf_{n\to\infty}(p_{n+m} - p_n) \ll_c e^{cm}$ for every $c > 3.80428$, and for every $c > 3.80423$ if a numerically computed value of the sieve deficit is used. This improves Stadlmann's exponent 3.8075.

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1. Statement

We use the notation of Stadlmann [Sta23], §3. Let $\varpi, \delta, \sigma > 0$. Let α, β be coefficient sequences at scales M, N with $MN \asymp x$ and $N = x^\gamma$, where β has the Siegel–Walfisz property. The moduli are the $q \le x^{1/2+2\varpi}$ dividing $P(x^\delta)$. Stadlmann proved [Sta23, Prop. 1] that the equidistribution estimate [Sta23, (3.28)] holds for all $\gamma \in [1/2 - \sigma, 1/2]$ whenever

$$72\varpi + 24\delta < 1, \qquad 48\varpi + 16\delta + 4\sigma < 1, \qquad 64\varpi + 20\delta + 2\sigma < 1. \tag{1}$$

Theorem 1. *The conclusion of* [Sta23, Prop. 1] *holds when the third condition in* (1) *is replaced by* $48\varpi+18\delta+2\sigma < 1$. *More precisely, in* [Sta23, Lemma 17] *the lower bound* $\gamma \ge 32\varpi+10\delta+400\varepsilon$ *may be replaced by* $\gamma \ge 24\varpi + 9\delta + 400\varepsilon$.

The new condition follows from the second condition of (1) whenever $\delta \le \sigma$, and this is the case in all the applications below.

Remark 2. The theorem does not improve the exponent of distribution $1/2 + 1/40$ of the primes themselves [Sta23, Thm. 1]. That deduction goes through the Heath-Brown identity and [Sta23, Lemma 1], which require $\sigma > 1/10$. With the second condition of (1), $\sigma > 1/10$ already gives $80\varpi + \frac{80}{3}\delta < 1$ [Sta23, Cor. 2], so the third condition is not the binding one there. It is the binding one in the Harman-sieve application: [Sta23, Lemma 18(II)] needs $0.40481 > 32\varpi$. OpenAI extended [Sta23, Prop. 1], with the same three conditions, to triply densely divisible moduli [OAI, Prop. 2.18 and §A.4]. Their proof follows the same lemmas with the same block length, and we expect the same modification to apply. We have not checked the details and claim nothing for those moduli.

2. Proof of Theorem 1

All references in this section are to [Sta23, §3], whose notation we keep: $z \in Z_5(x)$, $\lambda, \widetilde{\lambda} \sim \Lambda$, w_1, w_2, m, q_0, Δ_1, H, $T(z)$ and the classes W_6, W_7, W_8. We write $s_2 = (w_2, m)$ and $\lambda^* = (\lambda, \widetilde{\lambda})$. Since $\lambda \sim \Lambda$ we have $|\lambda| \le 2|\Lambda|$. We go through Lemmas 7–17 and indicate every change. Everything not mentioned is used unchanged.

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Step 1: a block length depending on the pair (Lemma 10). In the proof of Lemma 10, the variable d, of length Δ_1, is cut by a smooth partition of unity into blocks of length $\Delta^* = \min\{N/(|\Lambda|x^{5\varepsilon}), \Delta_1\}$, which does not depend on $(\lambda, \widetilde{\lambda})$. For a fixed pair we use instead

$$\Delta_\lambda^* := \min\Big\{\frac{\lambda^*}{|\lambda|}\,\frac{N}{x^{5\varepsilon}},\ \Delta_1\Big\}, \qquad N_\lambda := \frac{\lambda^*}{|\lambda|}\,N.$$

Since $\lambda^* \geq 1$ and $|\lambda| \leq 2|\Lambda|$, we have $\Delta^* \leq 2\Delta_\lambda^*$. The block length is used in three places, and each still holds with Δ_λ^*:

(a) the number of blocks is $\ll \Delta_1/\Delta_\lambda^*$, and the cut-off functions τ_m satisfy the same derivative bounds, since $\Delta_\lambda^* \leq \Delta_1$;

(b) the Taylor expansion (3.13) of $\psi_N^*((\lambda/\lambda^*)n_1 + F_k d)$ in d needs $|F_k(d-x_0)|/N \ll x^{-5\varepsilon}$ for $|d-x_0| \ll \Delta_\lambda^*/z_1$. Since $0 < F_k \leq |\lambda|/\lambda^*$,

$$\frac{|F_k(d-x_0)|}{N} \leq \frac{|\lambda|}{\lambda^*}\cdot\frac{\Delta_\lambda^*}{z_1 N} \leq \frac{x^{-5\varepsilon}}{z_1};$$

(c) in Step 2 of the proof of Lemma 11, the expansion (3.15) needs

$$\frac{H_k f_4\,\Delta_\lambda^*/(z_1 f^*)}{(\lambda^*/|\lambda|)N} \ll x^{-5\varepsilon}.$$

Since $H_k f_4 \leq [f_2/f_2^*, f_3/f_3^*] \leq w_1$, $f^* \geq 1$ and $w_1 \mid z_1$, the left side is at most $(w_1/z_1)x^{-5\varepsilon} \leq x^{-5\varepsilon}$.

Thus Lemma 10 holds with Δ_λ^* in place of Δ^*, and Lemma 11 holds with W_8 defined by $\Delta_2 \leq \Delta_\lambda^*$ instead of $\Delta_2 \leq \Delta^*$.

Step 2: the length of the n_1-sum (Lemmas 11 and 14). After the substitution $n = (\lambda/\lambda^*)n_1 + F_k d$ in Lemma 10, the variable n_1 runs over a coefficient sequence at scale $(\lambda^*/\lambda)N$. In Lemma 11 it becomes n_3, at scale $(\lambda^*/(\lambda w_3))N \leq N_\lambda$. The definition of W_8 records only $N_2 \leq N$; we keep $N_2 \leq N_\lambda$. Lemma 13 applied as in the proof of Lemma 14 then gives, for this pair,

$$\Sigma_6(z, w_6, k) \ll_\varepsilon \Big(\big(\tfrac{z_1}{w_1}\big)k + (\lambda - \widetilde{\lambda})B, m\Big)\frac{x^{3\varepsilon}}{q_0}\,F_\lambda, \qquad F_\lambda := \frac{\Delta_1}{\Delta_\lambda^*}\Big(\frac{N_\lambda}{m^{1/2}} + m^{1/2}\Big)\Big(\frac{\Delta_\lambda^*}{m^{1/2}} + m^{1/2}\Big). \tag{2}$$

Lemmas 15 and 16 do not involve the block length. The range of k, the condition $(\cdot, m) \leq T(z)$ and the sequence (ξ_k) are the same as in [Sta23]. They give $\Sigma_6(z, w_6, k) \ll_\varepsilon \xi_k\, x^{4\varepsilon} s_2 T(z) F_\lambda/q_0$ with $(\xi_k) \in \Xi(z)$. The proof of Lemma 9 is linear in the target bound: if $\Sigma_6 \ll \xi_k I$ for all relevant k, then $\Sigma_5(z) \ll q_0(q_0, \ell)I$. Hence

$$\Sigma_5(z) \ll_\varepsilon (q_0, \ell)\, x^{4\varepsilon}\, s_2\, T(z)\, F_\lambda. \tag{3}$$

Expanding F_λ gives four terms:

$$F_\lambda = \frac{\Delta_1 N_\lambda}{m} + \frac{\Delta_1}{\Delta_\lambda^*}N_\lambda + \Delta_1 + \frac{\Delta_1}{\Delta_\lambda^*}m.$$

By $N_\lambda \leq N$ and $\Delta^* \leq 2\Delta_\lambda^*$, the first and the last are at most once and twice the corresponding terms $\Delta_1 N/m$ and $(\Delta_1/\Delta^*)m$ of [Sta23]. For the middle two, consider the two possible values of Δ_λ^*. If $\Delta_\lambda^* = N_\lambda x^{-5\varepsilon}$ then $(\Delta_1/\Delta_\lambda^*)N_\lambda = x^{5\varepsilon}\Delta_1$, and otherwise $(\Delta_1/\Delta_\lambda^*)N_\lambda = N_\lambda$. Hence

$$\frac{\Delta_1}{\Delta_\lambda^*}N_\lambda + \Delta_1 \leq 2x^{5\varepsilon}\Delta_1 + N_\lambda. \tag{4}$$

Step 3: summing over the pairs (Lemma 7). In the proof of Lemma 7, $\Sigma_3 \ll_\varepsilon \Upsilon_T + \sup_q x^\varepsilon \Upsilon_S(q)$, and the bound $\Upsilon_S(q) \ll_\varepsilon x^{-48\varepsilon}N^2$, which includes the diagonal terms, gives the term $O(x^{-47\varepsilon}N^2)$. Here Υ_T is a sum over pairs $(y, \widetilde{y})$ of the inner sums $\Sigma_4(z_1, z_2, z_3, (y, \widetilde{y}))$. Stadlmann bounds it by the number of pairs, $\ll \mathcal{P} := \max\{x^{\delta+10\varepsilon}H^3, H^4\}/s_2$, times the supremum. The condition $(\cdot, m) \leq T$ in Σ_4 does not depend on the pair. So, by the triangle inequality, we may instead sum bounds that depend on the pair: $\Upsilon_T \leq \sum_{(y,\widetilde{y})} |\Sigma_4(\cdot, (y, \widetilde{y}))|$. The terms

$\Upsilon_S(q)$ are not changed. By the proof of Lemma 8, each $\Sigma_4(\cdot,(y,\widetilde{y}))$ is zero or equals $\Sigma_5(z)$ for a $z\in Z_5$ with $y=w_1\lambda$, $\widetilde{y}=w_1\widetilde{\lambda}$ and $\Lambda=Y/w_1$. In particular

$$\frac{\lambda^*}{|\lambda|}=\frac{(y,\widetilde{y})}{|y|}.$$

By (3), the target $\Sigma_3\ll q_0^2(q_0,\ell)^2N^2x^{-27\varepsilon}$ of Lemma 7 (hence of Lemma 8) follows from

$$s_2\,T(z)\sum_{(y,\widetilde{y})}F_\lambda\ll q_0^2N^2x^{-31\varepsilon}. \tag{5}$$

Condition (3.29) of [Sta23] is the same inequality with $\sum F_\lambda$ replaced by $\mathcal{P}\,F$, where F is the expression with Δ^* and N. By Step 2, the contribution of the first and last terms of F_λ is at most twice Stadlmann's. It is therefore covered by (3.30) and (3.32) of [Sta23], which give the conditions $\gamma\le 1/2-4\varpi-2\delta-50\varepsilon$ and $\gamma\ge 1/4+12\varpi+4\delta+100\varepsilon$. It remains to replace (3.31), which concerns the two middle terms and needed $\gamma\ge 32\varpi+10\delta+400\varepsilon$. By (4),

$$\sum_{(y,\widetilde{y})}\Big(\frac{\Delta_1}{\Delta_\lambda^*}N_\lambda+\Delta_1\Big)\le 2x^{5\varepsilon}\Delta_1\,\#\{(y,\widetilde{y})\}+N\sum_{(y,\widetilde{y})}\frac{(y,\widetilde{y})}{|y|}.$$

For fixed $y\neq 0$, the integers $\widetilde{y}\sim Y$ lie in an interval of length $|Y|$, and $|y|\le 2|Y|$. Hence

$$\sum_{\widetilde{y}\sim Y}(y,\widetilde{y})\le\sum_{e|y}e\Big(\frac{|Y|}{e}+1\Big)\le\tau(y)|Y|+\sigma(y)\le 3\tau(y)|Y|.$$

Since $|y|\ge|Y|$, it follows that $\sum_{(y,\widetilde{y})}(y,\widetilde{y})/|y|\ll x^{\varepsilon}\,\mathcal{Y}$, where $\mathcal{Y}$ is the number of admissible y. By the proof of Lemma 7, $\mathcal{Y}\ll\min\{H^2/s_2+H,\ |Y|/s_2+1\}$ and $|Y|\ll x^{\delta+5\varepsilon}H^2/(v_1,v_2)$. We also use

$$T(z)\le\max\{s_2^{-1},H^{-1}\}\frac{x^{\delta+100\varepsilon}H^2N}{(v_1,v_2)\Delta_1},\qquad \Delta_1\gg\frac{N}{x^{\delta+55\varepsilon}H^2},\qquad H\ll\frac{x^{4\varpi+\delta+7\varepsilon}}{q_0}.$$

First contribution. We have $\max\{1,s_2/H\}\mathcal{P}\le\max\{x^{\delta+10\varepsilon}H^3,H^4\}\le x^{\delta+10\varepsilon}H^4$. Hence

$$s_2T(z)\cdot x^{5\varepsilon}\Delta_1\,\mathcal{P}\le x^{2\delta+115\varepsilon}H^6N.$$

This is $\ll q_0^2N^2x^{-31\varepsilon}$ provided $x^{2\delta+146\varepsilon}H^6\ll q_0^2N$, which follows from $\gamma\ge 24\varpi+8\delta+190\varepsilon$.

Second contribution. We claim that $\max\{1,s_2/H\}\,\mathcal{Y}\ll x^{\delta+5\varepsilon}H^2$. If $s_2\le H$, then $\mathcal{Y}\le 2H^2$. If $s_2>H$, we use $s_2\le 2|Y|$: indeed $s_2\mid w_2\mid y$ for every admissible $y\neq 0$, and the sum is empty otherwise. Then $(s_2/H)(|Y|/s_2+1)\le 3|Y|/H\ll x^{\delta+5\varepsilon}H$. Therefore

$$s_2T(z)\cdot x^{\varepsilon}N\mathcal{Y}\ll x^{2\delta+106\varepsilon}\frac{H^4N^2}{\Delta_1}\ll x^{3\delta+161\varepsilon}H^6N,$$

which is $\ll q_0^2N^2x^{-31\varepsilon}$ provided $x^{3\delta+192\varepsilon}H^6\ll q_0^2N$. This follows from $\gamma\ge 24\varpi+9\delta+240\varepsilon$.

Step 4: conclusion (Lemma 17 and Proposition 1). Lemma 17 therefore holds with $\max\{1/4+12\varpi+4\delta+100\varepsilon,\ 24\varpi+9\delta+400\varepsilon\}\le\gamma\le 1/2-4\varpi-2\delta-50\varepsilon$. The final step of the proof of Proposition 1 is unchanged. It covers γ near $1/2$ by [PM8a, Thm. 2.8(ii),(iv)] when $68\varpi+14\delta<1$ and $72\varpi+24\delta+200\varepsilon<1$. Hence the range $\gamma\in[1/2-\sigma,1/2]$ is covered as soon as $1/4+12\varpi+4\delta<1/2-\sigma$, $24\varpi+9\delta<1/2-\sigma$ and $72\varpi+24\delta<1$, with ε small enough. These are the conditions of Theorem 1. □

3. Consequences

We use the general Harman-sieve construction of Stadlmann [Sta26, Def. 9 and Prop. 2], with parameters ξ_1,ξ_2,ξ_3. It is the construction of [Sta23, §4], with the numerical values 0.33856, 0.40481, 0.59519, 0.19, 0.405, 0.595 replaced by ξ_1, ξ_2, $1-\xi_2$, $1-2\xi_3$, ξ_3, $1-\xi_3$. Its proof requires the five inequalities

$$2\xi_1+3\xi_2<2,\qquad \xi_2\le\xi_3,\qquad \xi_1+9\xi_2<4,\qquad 2\xi_1+\xi_2>1,\qquad 17\xi_2<7. \tag{6}$$

Its input is equidistribution, on the family of moduli under consideration, for three types of convolutions:

(I) smooth factors of length $\geq x^{\xi_1}$;
(II) bilinear forms with both factors Siegel–Walfisz and one length in $[x^{\xi_2}, x^{1-\xi_2}]$;
(III) three smooth factors of lengths in $[x^{1-2\xi_3}, x^{\xi_3}]$ with pairwise products $\geq x^{1-\xi_3}$.

In [Sta26] it is stated for the moduli Q^* of that paper; its proof, like that of [Sta23, §4] where the moduli are x^δ-smooth, uses these inputs on the family of moduli concerned. In our application the coefficients of type (I) with lengths in $[x^{\xi_2}, x^{1-\xi_2}]$ are treated by (II), which uses that they satisfy the Siegel–Walfisz property [OAI, Lemma A.8]. We include the corrections of OpenAI [OAI, footnotes 2–4 and Lemmas A.8–A.9], written there for fixed numerical values. The numerical inequalities in their arguments reduce to (6), $6(1-2\xi_2) > 1$, $\xi_2 > 2/5$ and a level of distribution at most $x^{0.53}$; for instance their two grouping gaps are $1-\xi_1-3\xi_2/2$ and $4-\xi_1-9\xi_2$. The minorant ρ has mass $1-c_1(\xi_2)$, where $c_1(\xi_2) = I_1 + I_2$ is the explicit sum of two four-dimensional integrals in [Sta26, Prop. 2]. Every constraint in these integrals becomes stricter when ξ_2 decreases, so c_1 is nondecreasing in ξ_2.

Corollary 3. *Let $0 < \varpi < 7/544$ and let $\delta > 0$ be small enough. Take $\xi_2 = \xi_3$ with*

$$\max\Big\{\tfrac{1}{4} + 12\varpi + 4\delta,\ \tfrac{2}{5}\Big\} < \xi_2 < \min\Big\{\tfrac{4}{9} - \tfrac{28}{9}\varpi - \tfrac{2}{9}\delta,\ \tfrac{55}{136}\Big\},$$

and ξ_1 with $\max\{\frac{1}{4}+7\varpi+2\delta, \frac{1-\xi_2}{2}\} < \xi_1 < \min\{1-\frac{3}{2}\xi_2,\ 4-9\xi_2\}$. These intervals are nonempty. Then the minorant ρ of [Sta26, Prop. 2] *with these parameters has the following properties:*

- *$\rho \leq \mathbf{1}_{\mathbb{P}}$, and every prime factor of an n with $\rho(n) \neq 0$ is at least $x^{1-2\xi_2}$;*
- *ρ has exponent of distribution $1/2 + 2\varpi$ to x^δ-smooth squarefree moduli;*
- *ρ has mass $1 - c_1(\xi_2)$ with $c_1(\xi_2) \leq c_1(55/136)$.*

Proof. The intervals are nonempty. We have $1/4+12\varpi+4\delta < 4/9-28\varpi/9-2\delta/9$ if and only if $544\varpi+152\delta < 7$; the two bounds meet at $\varpi = 7/544$, $\xi_2 = 55/136$. Also $1/4+12\varpi+4\delta < 55/136$ for $\varpi < 7/544$ and δ small, and $2/5 < \min\{4/9 - 28\varpi/9 - 2\delta/9, 55/136\}$ since $\varpi < 1/70$. So the interval for ξ_2 is nonempty. Since $2/5 < \xi_2 < 55/136$, we have $(1-\xi_2)/2 < 3/10$, $1-3\xi_2/2 > 0.393$ and $4-9\xi_2 > 49/136 > 0.3602$, while $1/4+7\varpi+2\delta < 0.3401$; so the interval for ξ_1 is nonempty. The program `check_parameters.py` checks all the inequalities used below in exact rational arithmetic, at sample values of (ϖ, δ) (including values close to $7/544$ and small values of ϖ) and at points close to the ends of the intervals for ξ_1 and ξ_2. In general they follow from the displayed choices of ξ_1, ξ_2 by the elementary verifications below.

Type II. We use Theorem 1 with $\sigma = 1/2-\xi_2+\tau$ for a small $\tau > 0$ (this also absorbs the fixed slack ϵ of [Sta26, Def. 9], which we take small depending on ϖ); both factors are Siegel–Walfisz, so they may be interchanged. Its conditions read $\xi_2-\tau > 1/4+12\varpi+4\delta$, $\xi_2-\tau > 24\varpi+9\delta$ and $72\varpi + 24\delta < 1$, all satisfied for τ small. With (1) instead, one would also need $\xi_2 > 32\varpi + 10\delta$; together with the Type III bound $\xi_2 < 4/9 - 28\varpi/9$ this forces $\varpi < 1/79$ (as $\delta \to 0$), and Stadlmann takes $\varpi = 0.01265 < 1/79$.

Type III. We use [PM8a, Thm. 2.8(v)] with σ_3 strictly between $1/18 + 28\varpi/9 + 2\delta/9$ and $1/2 - \xi_3$, the slack ϵ in the ranges of (III) being smaller than these gaps, exactly as in [Sta23, Lemma 18(III)] and [OAI, proof of Prop. 2.22].

Type I. Smooth factors of length between x^{ξ_1} and $x^{1/2}$ are covered by [OAI, Lemma 2.20], which needs $\xi_1 > 1/4 + 7\varpi + 2\delta$ and holds for singly densely divisible moduli, hence for x^δ-smooth squarefree moduli. The coefficients in the decomposition satisfy the Siegel–Walfisz property [OAI, Lemma A.8]. So lengths in $[x^{\xi_2}, x^{1-\xi_2}]$ are covered by Type II, and lengths above $x^{1-\xi_2} > x^{1/2+2\varpi}$ by [OAI, (2.18)]. This is the argument of [OAI, proof of Prop. 2.22], with $(a_*, b_*, c_*) = (\xi_2, 1-\xi_2, \xi_1)$.

Construction. The inequalities (6) hold. Indeed, $\xi_1 + 9\xi_2 < 4$, $2\xi_1 + 3\xi_2 < 2$ and $2\xi_1 + \xi_2 > 1$ are imposed, $\xi_2 = \xi_3$, and $17\xi_2 < 17 \cdot 55/136 = 6.875 < 7$. Moreover $6(1-2\xi_2) > 1$, $\xi_2 > 2/5$, and the level is $1/2 + 2\varpi < 0.53$. The three properties then follow from the proof of [Sta26, Prop. 2] and of [Sta23, §4], with the corrections of [OAI, §A.6], run on the family of x^δ-smooth squarefree moduli with the inputs above. We have checked that the numerical inequalities used in these proofs reduce to those listed; we have not rewritten the whole construction. Since $\xi_2 < 55/136$, the monotonicity of c_1 gives $c_1(\xi_2) \leq c_1(55/136)$. □

The value $c_1(55/136)$ is computed by the program `deficit_c1.py`, in two ways. Nested Gauss–Legendre quadrature, with each range split where its limits change formula, gives $6.604 \cdot 10^{-6}$. A Monte Carlo integration over the regions written directly from the summation conditions gives $(6.628 \pm 0.020) \cdot 10^{-6}$. Thus $c_1(55/136) < 6.7 \cdot 10^{-6}$; this is a numerical, not a certified, value. As a check, the same program gives $c_1(0.40481) \approx 9.33 \cdot 10^{-6}$, in agreement with Stadlmann's published bound $c_1(0.40481) \le 2 \cdot 10^{-5}$ [Sta23, Prop. 2].

For comparison, the minorant of [Sta23, Prop. 2] and [OAI, Prop. 2.22] has $\varpi = 253/20000 = 0.01265$. Here $7/544 = 0.0128676\ldots$, and the exponent of distribution tends to $1/2 + 7/272 = 0.525735\ldots$.

Corollary 4. *For every $c > 3.80423$, $\liminf_{n\to\infty}(p_{n+m} - p_n) \ll_c e^{cm}$. Using only Stadlmann's bound $c_1(0.40481) \le 2 \cdot 10^{-5}$ in place of the numerical value of $c_1(55/136)$, the same holds for every $c > 3.80428$.*

Proof. By the lemma of Baker and Irving [BI, Lemma 1], [Sta23, Lemma 2], a minorant with the properties of Corollary 3 gives $H_m \ll_\epsilon \exp\big(2(1+\epsilon)m/(\theta(1-c_1))\big)$ with $\theta = 1/2 + 2\varpi$, that is, the exponent $(1+\epsilon)/((1/4+\varpi)(1-c_1))$. We let $\varpi \to 7/544$ and use $c_1(\xi_2) \le c_1(55/136)$, which holds because $\xi_2 < 55/136$. Since $1/(1/4 + 7/544) = 544/143 = 3.8041958\ldots$, the exponent can be taken arbitrarily close to $544/(143(1 - c_1(55/136)))$. With $c_1(55/136) < 6.7 \cdot 10^{-6}$ this is < 3.804222. Since $\xi_2 < 55/136 < 0.40481$, monotonicity also gives $c_1(\xi_2) \le c_1(0.40481) \le 2 \cdot 10^{-5}$, and then the bound is $544/(143(1 - 2 \cdot 10^{-5})) < 3.804272$. □

Files. `anc/check_parameters.py` checks the inequalities of Corollary 3 and the two constants of Corollary 4 in exact arithmetic. `anc/deficit_c1.py` computes c_1. The outputs are in the corresponding `.log` files.

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References

[BI] R. C. Baker, A. J. Irving, Bounded intervals containing many primes, *Math. Z.* **286** (2017), 821–841.

[OAI] OpenAI, Improved short gaps between primes, preprint, 30 August 2026, `https://cdn.openai.com/pdf/51126fac-1b68-4128-9666-c908bcc16033/short_gaps.pdf`.

[PM8a] D. H. J. Polymath, New equidistribution estimates of Zhang type, *Algebra Number Theory* **8** (2014), 2067–2199.

[Sta23] J. Stadlmann, On primes in arithmetic progressions and bounded gaps between many primes, *Adv. Math.* **468** (2025), 110190; arXiv:2309.00425v3.

[Sta26] J. Stadlmann, Bounded gaps between primes, arXiv:2608.31126v1 (2026).

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